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AN INNOVATIVE ITERATIVE APPROACH TO SOLVING VOLTERRA INTEGRAL EQUATIONS OF SECOND KIND

Mohammed Abdulshareef Hussein[a, b, *], Hassan Kamil Jassim[c], Ali Kareem Jassim[b, d]

[a] *Al-Ayen University, Scientific Research Center, 64001 Nasiriyah, Iraq*
[b] *Ministry of Education, Education Directorate of Thi-Qar, 64001 Nasiriyah, Iraq*
[c] *University of Thi-Qar, Department of Mathematics, 64001 Nasiriyah, Iraq*
[d] *National University of Science and Technology, College of Technical Engineering, 64001 Nasiriyah, Iraq*
[*] corresponding author: mshirq@utq.edu.iq

Abstract. Many scientists have shown great interest in exploring the realm of second-kind integral equations, offering many techniques for solving them, including exact, approximate, and numerical methods. This paper introduces the Hussein-Jassim method (HJ-method) for solving Volterra integral equations of the second kind (VIESKs). The foundation of this approach lies in the principle of Maclaurin expansion. The algorithm of the method was derived, and its convergence was analysed. Furthermore, the method was applied to various Volterra integral equations, encompassing linear, nonlinear, homogeneous, and nonhomogeneous cases. Ultimately, the proposed method successfully addressed these equations, with the approximate solutions converging toward the exact solution.

Keywords: Volterra integral equations, new iterative method, Taylor series, Hussein Jassim method.

1. Introduction

In the realm of quantum mechanics, Volterra integral equations of the second kind are employed to describe scenarios involving quantum interference and fusion of systems. In addition, integral equations find applications in studying electric current, electric charge, and the distribution of electromagnetic fields. In engineering, integral equations are extensively used. In structural analysis, they provide insights into the behaviour of structures under external forces and load distribution. Additionally, in the field of electromagnetic fields, integral equations are valuable tools for analysing the interaction between electric currents and electromagnetic fields [1, 2].

Integral equations have proven their value in fields as diverse as mathematical modelling, image processing, and signal analysis. Offering a versatile framework, these equations prove instrumental in solving complex problems involving boundary or initial conditions. In the field of mathematical modelling, integral equations empower researchers to articulate and comprehend real-world phenomena through mathematical representations. In image processing, they find application in tasks such as image reconstruction and denoising. In addition, integral equations play a pivotal role in signal analysis, where they are used for tasks such as signal deconvolution and system identification [3–7].

The applications of integral equations are vast and varied, spanning fields including physics, engineering, mathematics, image processing, and signal analysis. Their versatility in handling complex problems and incorporating boundary or initial conditions renders them a valuable tool for modelling and understanding diverse phenomena across different disciplines [8].

Volterra integral equations of the second kind are a set of integral equations with the general form:

$$u(x) = \psi(x) + \sigma \int_0^x K(x,t) N(u(t))\, dt,$$

where $u(x)$ is the unknown function to be determined, $\psi(x)$ is the known function, σ is a real number, $K(x,t)$ is the integral kernel that determines the interaction between the variables x and t, and N is a nonlinear operator.

Many complex problems in mathematics, chemistry, biology, astrophysics, and mechanics, such as the problem of radiative energy transfer, the oscillation problems of string and membrane, and the problem of momentum representation in quantum mechanics, can be expressed in terms of the Volterra integral equation. Solving these equations poses a challenge due to their nonlinearity and complex mathematical implications. Solving Volterra integral equations of the second kind requires specialised techniques, such as Laplace transformations, numerical analysis, and numerical approximation methods to obtain accurate and reliable results [9–13].

In recent times, numerous researchers have directed their attention towards investigating solutions for NDEs, employing diverse techniques as exemplified by their work, Adomian decomposition technique (ADM) [14], variational iteration technique (VIM) [15], homotopy perturbation technique (HPM) [16] and Daftardar-Jafari technique (DJM) [17].

This article presents a new iterative technique for solving Volterra integral equations of the second kind. This technique is mainly based on Taylor's expansion. In order to introduce the technique, we need to present the definition of the Maclaurin series.

A Maclaurin series is a series expansion of a function about the origin point. A one-dimensional Maclaurin series is an expansion of a real function $f(x)$ about point $x = 0$, given by [18]:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \ldots .$$

The Hussein-Jassim method is considered as one of the latest analytical approaches to finding approximate solutions to differential and integral equations. The idea of this method is based on the use of Maclaurin's expansion. It has been successfully applied to various types of equations, including linear, non-linear, homogeneous, and non-homogeneous, as well as to both differential and integral equation systems. Ultimately, this method has proven to be efficient and effective in solving these equations.

Following is an outline of the paper. In the second section of the paper, we derive the algorithm for the method. The third section discusses the convergence of the approximate solution of the method to the exact solution. In the fourth section, we apply the method to several illustrative examples. The last section presents the conclusion and the results reached.

2. Analysis of the technique

Consider the following a nonlinear Volterra integral equation of the second kind:

$$u(x) = \psi(x) + \sigma \int_0^x K(x,t)N(u(t))\, dt. \tag{1}$$

Now, we rewrite Equation (1) by using Maclaurin's expansion with respect to x:

$$u(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} D_x^i \left(\psi(x) + \sigma \int_0^x K(x,t)N(u(t))\, dt \right)_{x=0}, \tag{2}$$

where D_x^i is a differential operator of order i. Suppose that $u(x)$ is a solution of Equation (2), which we express as:

$$u(x) = \sum_{i=0}^{\infty} u_i(x), \tag{3}$$

by substituting Equation (3) in Equation (2):

$$\sum_{i=0}^{\infty} u_i(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} D_x^i \left(\psi(x) + \sigma \int_0^x K(x,t)N\left(\sum_{i=0}^{\infty} u_i(t)\right) dt \right)_{x=0}, \tag{4}$$

by putting $i = i + 1$ in the left side of Equation (4):

$$u_0(x) + \sum_{i=0}^{\infty} u_{i+1}(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} D_x^i \left(\psi(x) + \sigma \int_0^x K(x,t)N\left(\sum_{i=0}^{\infty} u_i(t)\right) dt \right)_{x=0}, \tag{5}$$

by comparing the two sides of Equation (5):

$$\begin{cases} u_0 = u(0) = \psi(0), \\ u_1(x) = u'(0) = xD_x \left(\psi(x) + \sigma \int_0^x K(x,t)N(u_0(t))\, dt \right)_{x=0}, \\ u_2(x) = u''(0) = \dfrac{x^2}{2!} D_x^2 \left(\psi(x) + \sigma \int_0^x K(x,t)N(u_0(t) + u_1(t))\, dt \right)_{x=0}, \\ u_3(x) = u'''(0) = \dfrac{x^3}{3!} D_x^2 \left(\psi(x) + \sigma \int_0^x K(x,t)N(u_0(t) + u_1(t) + u_2(t))\, dt \right)_{x=0}, \\ \vdots \\ u_{i+1}(x) = u^{i+1}(0) = \dfrac{x^{i+1}}{(i+1)!} D_x^2 \left(\psi(x) + \sigma \int_0^x K(x,t)N\left(\sum_{j=0}^{i} u_j(t)\right) dt \right)_{x=0}. \end{cases} \tag{6}$$

Therefore, the approximate solution can be formulated as follows:

$$u(x) = u_0 + u_1 + u_3 + \cdots = \sum_{i=0}^{\infty} u_i. \tag{7}$$

3. Convergence of the technique

Theorem 1. The new technique used in solving Equation (1) is equivalent to determining the following sequence:

$$\begin{aligned} \varepsilon_\eta &= u_1 + u_2 + u_3 + \cdots + u_\eta, \\ \varepsilon_0 &= 0 \end{aligned} \tag{8}$$

by using the iterative scheme:

$$\varepsilon_{\eta+1} = \sum_{i=1}^{\eta+1} \frac{x^i}{i!} D_t^i \left(\psi(x) + \sigma \int_0^x K(x,t) N\left(\sum_{j=0}^{i-1} u_j(t) \right) dt \right)_{x=0}. \tag{9}$$

Proof. For $\eta = 0$, from Equation (9), we have:

$$\varepsilon_1 = x D_x \left(\psi(x) + \sigma \int_0^x K(x,t) N(u_0(t))\, dt \right)_{x=0},$$

then:

$$u_1 = x D_x \left(\psi(x) + \sigma \int_0^x K(x,t) N(u_0(t))\, dt \right)_{x=0}.$$

For $\eta = 1$ and by Equation (9):

$$\begin{aligned} \varepsilon_2 &= \sum_{i=1}^{2} \frac{x^i}{i!} \left(\psi(x) + \sigma \int_0^x K(x,t) N\left(\sum_{j=0}^{i} u_j(t) \right) dt \right)_{x=0} \\ &= x D_x \left(\psi(x) + \sigma \int_0^x K(x,t) N(u_0(t))\, dt \right)_{x=0} + \frac{x^2}{2!} D_x^2 \left(\psi(x) + \sigma \int_0^x K(x,t) N(u_0(t) + u_1(t))\, dt \right)_{x=0} \\ &= u_1 + \frac{x^2}{2!} D_x^2 \left(\psi(x) + \sigma \int_0^x K(x,t) N(u_0(t) + u_1(t))\, dt \right)_{x=0}. \end{aligned}$$

According to $\varepsilon_2 = u_1 + u_2$, we get:

$$u_2 = \frac{x^2}{2!} D_x^2 \left(\psi(x) + \sigma \int_0^x K(x,t) N(u_0(t) + u_1(t))\, dt \right)_{x=0}.$$

This theorem will be proved by strong induction. Let's assume that:

$$u_{m+1} = \frac{x^{m+1}}{(m+1)!} D_t^{m+1} \left(\psi(x) + \sigma \int_0^x K(x,t) N\left(\sum_{j=0}^{m} u_j(t) \right) dt \right)_{x=0},$$

where $m = 1, 2, 3, \ldots, \eta - 1$, so:

$$\begin{aligned} \varepsilon_{\eta+1} &= \sum_{i=1}^{\eta+1} \frac{x^i}{i!} D_t^i \left(\psi(x) + \sigma \int_0^x K(x,t) N\left(\sum_{j=0}^{i-1} u_j(t) \right) dt \right)_{x=0} \\ &= x D_x \left(\psi(x) + \sigma \int_0^x K(x,t) N(u_0(t))\, dt \right)_{x=0} + \frac{x^2}{2!} D_x^2 \left(\psi(x) + \sigma \int_0^x K(x,t) N(u_0(t) + u_1(t))\, dt \right)_{x=0} \\ &+ \cdots + \frac{x^{\eta+1}}{(\eta+1)!} D_t^{\eta+1} \left(\psi(x) + \sigma \int_0^x K(x,t) N\left(\sum_{j=0}^{\eta} u_j(t) \right) dt \right)_{x=0} \\ &= u_1 + \cdots + u_\eta + \frac{x^{\eta+1}}{(\eta+1)!} D_t^{\eta+1} \left(\psi(x) + \sigma \int_0^x K(x,t) N\left(\sum_{j=0}^{\eta} u_j(t) \right) dt \right)_{x=0}. \end{aligned}$$

Then, from Equation (8), it can be derived:

$$u_{\eta+1} = \frac{t^{\eta+1}}{(\eta+1)!} D_t^{\eta+1} \left(\psi(x) + \sigma \int_0^x K(x,t) N \left(\sum_{j=0}^{\eta} u_j(t) \right) dt \right)_{x=0},$$

which is the same as the result of Equation (6), and the theorem is proved.

Theorem 2. Let B be a Banach space.

(I) $\sum_{i=0}^{\infty} u_i$ obtained by Equation (6) convergence to $\varepsilon \in B$, if:

$$\exists (0 \le \epsilon \le 1), \text{ such that } (\forall \eta \in N \Rightarrow \|u_\eta\| \le \epsilon \|u_{\eta+1}\|). \tag{10}$$

(II) $\varepsilon = \sum_{\eta=1}^{\infty} u_\eta$, satisfies in:

$$\varepsilon = \sum_{i=1}^{\infty} \frac{x^i}{(i+1)!} D_t^i \left(\psi(x) + \sigma \int_0^x K(x,t) N \left(\sum_{j=0}^{\infty} u_j(t) \right) dt \right)_{x=0}. \tag{11}$$

Proof.

(I) From Equation (10), we prove that $\{\varepsilon\}_{\eta=0}^{\infty}$ is a Cauchy sequence in B.

$$\|\varepsilon_{\eta+1} - \varepsilon_\eta\| = \left\| \sum_{i=0}^{\eta+1} u_i - \sum_{i=0}^{\eta} u_i \right\| = \|u_{\eta+1}\| \le \epsilon \|u_\eta\| \le \epsilon^2 \|u_{\eta-1}\| \le \epsilon^3 \|u_{\eta-2}\| \le \cdots \le \epsilon^{\eta+1} \|u_0\|. \tag{12}$$

For all $\eta, m \in N, \eta \ge m$:

$$\begin{aligned} \|\varepsilon_\eta - \varepsilon_m\| &= \|(\varepsilon_\eta - \varepsilon_{\eta-1}) + (\varepsilon_{\eta-1} - \varepsilon_{\eta-2}) + \cdots + (\varepsilon_{m+1} - \varepsilon_m)\| \\ &\le \|\varepsilon_\eta - \varepsilon_{\eta-1}\| + \|\varepsilon_{\eta-1} - \varepsilon_{\eta-2}\| + \cdots + \|\varepsilon_{m+1} - \varepsilon_m\| \\ &\le \epsilon^\eta \|u_0\| \le \epsilon^{\eta-1} \|u_0\| \le \epsilon^{\eta-2} \|u_0\| \le \cdots \le \epsilon^{m+1} \|u_0\| \\ &\le \epsilon^{m+1} \|u_0\| \left(\epsilon^{\eta-m-1} + \epsilon^{\eta-m-2} + \cdots + 1 \right) \\ &= \frac{1-\epsilon^{n-m}}{1-\epsilon} \epsilon^{m+1} \|u_0\|. \end{aligned} \tag{13}$$

Since $(\epsilon^{n-m-1} + \epsilon^{n-m-2} + \cdots + 1)$ is a geometric series, then:

$$\lim_{\eta,m\to\infty} \|\varepsilon_\eta - \varepsilon_m\| = 0.$$

Thus, $\{\varepsilon_\eta\}$ is Cauchy sequence in B, and it is convergent.

(II) From Equation (9):

$$\begin{aligned} \lim_{\eta\to\infty} \varepsilon_{\eta+1} &= \lim_{\eta\to\infty} \sum_{i=1}^{\eta+1} \frac{x^i}{i!} D_t^i \left(\psi(x) + \sigma \int_0^x K(x,t) N \left(\sum_{j=0}^{i-1} u_j(t) \right) dt \right)_{x=0} \\ &= \sum_{i=1}^{\infty} \frac{x^i}{i!} D_t^i \left(\psi(x) + \sigma \int_0^x K(x,t) N \left(\sum_{j=0}^{i-1} u_j(t) \right) dt \right)_{x=0}. \end{aligned}$$

Since the upper bound of the sum is approaching infinity, we get the following result:

$$\varepsilon = \sum_{i=1}^{\infty} \frac{x^i}{i!} D_t^i \left(\psi(x) + \sigma \int_0^x K(x,t) N \left(\sum_{j=0}^{\infty} u_j(t) \right) dt \right)_{x=0} = \sum_{i=1}^{\infty} u_i.$$

Theorem 3. The following Equation (11):

$$\varepsilon = \sum_{i=1}^{\infty} \frac{x^i}{(i+1)!} D_t^i \left(\psi(x) + \sigma \int_0^x K(x,t) N \left(\sum_{j=0}^{\infty} u_j(t) \right) dt \right)_{t=0},$$

is equivalent to Equation (1):

$$u(x) = \psi(x) + \sigma \int_0^x K(x,t) N(u(t))\, dt.$$

Proof. We rewrite Equation (11) as follows:

$$\psi(0)+\varepsilon=\psi(0)+\sum_{i=1}^{\infty}\frac{x^i}{i!}D_t^i\left(\psi(x)+\sigma\int_0^x K(x,t)N\left(\sum_{j=0}^{\infty}u_j(t)\right)dt\right)_{x=0}$$
$$=\sum_{i=0}^{\infty}\frac{x^i}{i!}D_t^i\left(\psi(x)+\sigma\int_0^x K(x,t)N\left(\sum_{j=0}^{\infty}u_j(t)\right)dt\right)_{x=0}.$$

Thus, we get:

$$u=\varepsilon+\psi(0)=\sum_{i=1}^{\infty}u_i+u_0=\sum_{i=0}^{\infty}u_i.$$

Now, we have:

$$\sum_{i=0}^{\infty}u_i=\sum_{i=0}^{\infty}\frac{x^i}{i!}D_t^i\left(\psi(x)+\sigma\int_0^x K(x,t)N\left(\sum_{j=0}^{\infty}u_j(t)\right)dt\right)_{x=0}. \tag{14}$$

It seems clear that the right side of Equation (14) is Maclaurin's expansion, so Equation (14) can take the following form:

$$\sum_{i=0}^{\infty}u_i(x)=\psi(x)+\sigma\int_0^x K(x,t)N\left(\sum_{j=0}^{\infty}u_j(t)\right)dt, \tag{15}$$

by using Equation (3), Equation (15) becomes as follows:

$$u(x)=\psi(x)+\sigma\int_0^x K(x,t)N(u(t))\ dt. \tag{16}$$

Then, the solution of Equation (11) is the same as the solution of Equation (1).

4. Illustrative examples

This section involves the application of the Hussein Jassim method (HJM) to solve certain second-kind Volterra integral equations. We will examine and analyse the results obtained using this innovative method. Additionally, we will conduct a comparative analysis between the outcomes of this method and those of the Power Series Method (PSM), Reduced Differential Transform Method (RDTM), and Adomian Decomposition Method (ADM). This comparison will be carried out by measuring the absolute error through MATLAB.

The purpose of this comparison is to provide a deeper understanding of the effectiveness of the method used compared to other approaches commonly used to solve this type of equation. We will highlight the relative superiority of the Hussein Jassim method and underscore its ability to address the complex challenges associated with second-type Volterra integral equations, whether linear or nonlinear.

Example 1. Consider the following nonhomogeneous linear VIESK:

$$u(x)=1+\int_0^x(t-x)u(t)\ dt, \tag{17}$$

using the Maclaurin expansion with Equation (17), we obtain:

$$u=\sum_{i=0}^{\infty}\frac{t^i}{i!}D_t^i\left(1+\int_0^x(t-x)u(t)\ dt\right)_{x=0}. \tag{18}$$

Suppose that $u(x)$ is a solution of Equation (17), we obtain:

$$u=\sum_{i=0}^{\infty}u_i, \tag{19}$$

by substituting Equation (19) in Equation (18), we obtain:

$$\sum_{i=0}^{\infty}u_i=\sum_{i=0}^{\infty}\frac{x^i}{i!}D_x^i\left(1+\int_0^x(t-x)\sum_{j=0}^{i-1}u_j(t)\ dt\right)_{x=0}. \tag{20}$$

By comparing both sides of Equation (20), we obtain:

$$\begin{cases} u_0 = 1, \\ u_1 = xD_x\left(1+\int_0^x (t-x)\ dt\right)_{x=0} = 0, \\ u_2 = \frac{x^2}{2!}\ D_x^2\left(1+\int_0^x (t-x)\ dt\right)_{x=0} = -\frac{x^2}{2!}, \\ u_3 = \frac{x^3}{3!} D_x^3\left(1+\int_0^x (t-x)\ dt\right)_{x=0} = 0, \\ \vdots \\ u_{i+1} = \frac{x^{i+1}}{(i+1)!} D_x^{i+1}\left(1+\int_0^x (t-x)\sum_{j=0}^{i} u_j(t)\ dt\right)_{x=0} \\ = (-1)^{\frac{i+1}{2}}\frac{x^{i+1}}{(i+1)!},\ \text{for all } i \text{ odd number.} \end{cases}$$

Thus, the approximate solution of Equation (17) is:

$$u_a(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{i=0}^{\infty} (-1)^i\ \frac{x^{2i}}{(2i)!}, \tag{21}$$

So, the exact solution of Equation (17) is:

$$u_e(x) = \cos x.$$

x	u_a	u_e	Absolute error HJM	RDTM	PSM	ADM
0.1000	0.9950	0.9950	0.0000	0.0000	0.0000	0.0000
0.2000	0.9801	0.9801	0.0000	0.0003	0.0000	0.0000
0.3000	0.9553	0.9553	0.0000	0.0017	0.0000	0.0000
0.4000	0.9211	0.9211	0.0000	0.0053	0.0000	0.0000
0.5000	0.8776	0.8776	0.0000	0.0130	0.0000	0.0000
0.6000	0.8254	0.8253	0.0001	0.0271	0.0001	0.0000
0.7000	0.7650	0.7648	0.0002	0.0502	0.0002	0.0000
0.8000	0.6971	0.6967	0.0004	0.0857	0.0004	0.0000
0.9000	0.6223	0.6216	0.0007	0.1374	0.0007	0.0000
1.0000	0.5417	0.5403	0.0014	0.2097	0.0014	0.0000

TABLE 1. The values of the approximate and exact solutions of Equation (17) for different values of x.

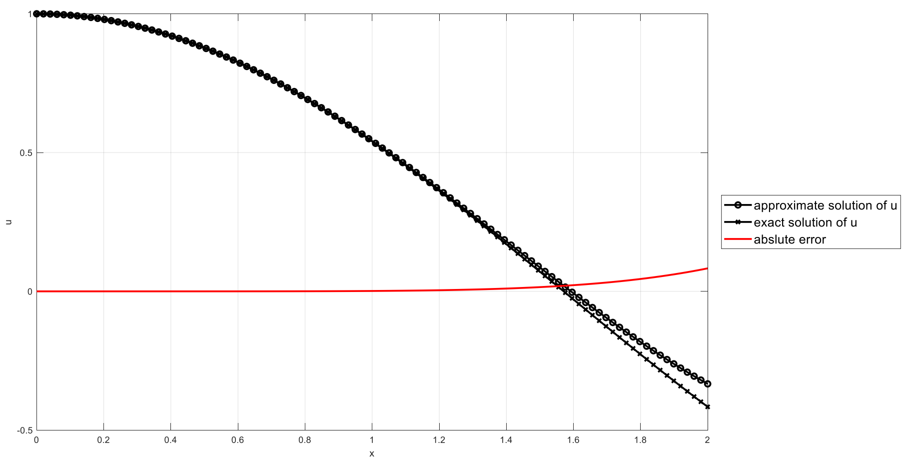

FIGURE 1. The plot of approximate solution, exact solution, and absolute error of Equation (17).

Example 2. Assume the following linear nonhomogeneous VIESK:

$$u(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t - x)u(t)\, dt. \tag{22}$$

By applying the algorithm of the new technique to Equation (22), we obtain:

$$\sum_{i=0}^{\infty} u_i = \sum_{i=0}^{\infty} \frac{t^i}{i!} D_t^i \left(1 - x - \frac{1}{2}x^2 - \int_0^x (t - x) \sum_{j=0}^{i-1} u_j(t)\, dt \right)_{t=0}. \tag{23}$$

By comparing both sides of Equation (23), we obtain:

$$\begin{cases} u_0 = 1, \\ u_1 = x D_x \left(1 - x - \frac{1}{2}x^2 - \int_0^x (t - x)\, dt \right)_{x=0} = -x, \\ u_2 = \frac{x^2}{2!} D_x^2 \left(1 - x - \frac{1}{2}x^2 - \int_0^x (t - x)(1 - t)\, dt \right)_{x=0} = 0, \\ u_3 = \frac{x^3}{3!} D_x^3 \left(1 - x - \frac{1}{2}x^2 - \int_0^x (t - x)(1 - t)\, dt \right)_{x=0} = -\frac{x^3}{3!}, \\ \vdots \\ u_{i+1} = \frac{x^{i+1}}{(i+1)!} D_t^i \left(1 - x - \frac{1}{2}x^2 - \int_0^x (t - x) \sum_{j=0}^{i} u_j(t)\, dt \right)_{x=0} = -\frac{x^{i+1}}{(i+1)!}. \end{cases}$$

Thus, the approximate solution of Equation (22) is:

$$\begin{aligned} u_a(x) &= 1 - x - \frac{x^3}{3!} - \dots \\ &= 1 - \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!}, \text{ for all } i \text{ even number.} \end{aligned}$$

Therefore, the exact solution of Equation (22) is:

$$u_e(x) = 1 - \sinh x.$$

x	u_a	u_e	Absolute error HJM	RDTM	PSM	ADM
0.1000	0.8998	0.8998	0.0000	0.0002	0.0000	0.0000
0.2000	0.7987	0.7987	0.0000	0.0013	0.0000	0.0000
0.3000	0.6955	0.6955	0.0000	0.0045	0.0000	0.0000
0.4000	0.5893	0.5892	0.0001	0.0106	0.0001	0.0000
0.5000	0.4792	0.4789	0.0003	0.0206	0.0003	0.0000
0.6000	0.3640	0.3633	0.0007	0.0353	0.0007	0.0000
0.7000	0.2428	0.2414	0.0014	0.0557	0.0014	0.0000
0.8000	0.1147	0.1119	0.0028	0.0826	0.0028	0.0000
0.9000	−0.0215	−0.0265	0.0050	0.1165	0.0050	0.0001
1.0000	−0.1667	−0.1752	0.0085	0.1581	0.0085	0.0002

TABLE 2. The values of the approximate and exact solutions of Equation (22) for different values of x.

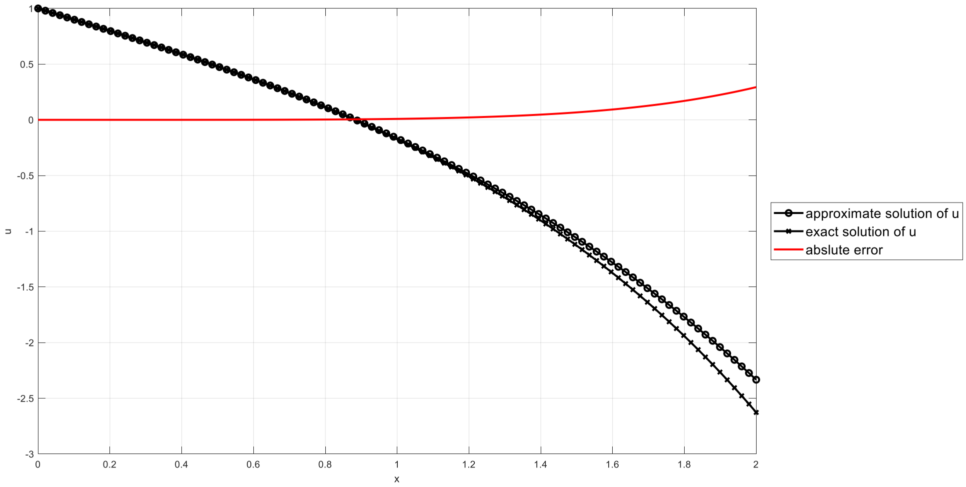

FIGURE 2. The plot of approximate solution, exact solution, and absolute error of Equation (22).

Example 3. Suppose that the nonhomogeneous nonlinear VIESK:

$$u(x) = x + \int_0^x u^2(t)\ dt, \tag{24}$$

by applying the algorithm of the technique to Equation (24), we obtain:

$$\sum_{i=0}^{\infty} u_i = \sum_{i=0}^{\infty} \frac{x^i}{i!} D_x^i \left(x + \int_0^x \left(\sum_{j=0}^{i-1} u_j \right)^2 dt \right)_{x=0}. \tag{25}$$

By comparing both sides of Equation (25), we obtain:

$$\begin{cases} u_0 = 0, \\ u_1 = x D_x \left(x + \int_0^x (0)^2\ dt \right)_{x=0} = x, \\ u_2 = \frac{x^2}{2!} D_x^2 \left(x + \int_0^x t^2\ dt \right)_{x=0} = 0, \\ u_3 = \frac{x^3}{3!} D_x^3 \left(x + \int_0^x t^2\ dt \right)_{x=0} = \frac{x^3}{3}, \\ \vdots \end{cases}$$

Thus, the approximate solution of Equation (24) is:

$$u_a(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots.$$

Therefore, the exact solution of Equation (24) is:

$$u_e(x) = \tan x.$$

x	u_a	u_e	Absolute error HJM	RDTM	PSM	ADM
0.1000	0.1003	0.1003	0.0000	0.0007	0.0000	0.0000
0.2000	0.2027	0.2027	0.0000	0.0053	0.0000	0.0000
0.3000	0.3090	0.3093	0.0003	0.0177	0.0003	0.0000
0.4000	0.4213	0.4228	0.0015	0.0412	0.0015	0.0001
0.5000	0.5417	0.5463	0.0046	0.0787	0.0046	0.0005
0.6000	0.6720	0.6841	0.0121	0.1319	0.0121	0.0018
0.7000	0.8143	0.8423	0.0280	0.2007	0.0280	0.0055
0.8000	0.9707	1.0296	0.0590	0.2824	0.0590	0.0153
0.9000	1.1430	1.2602	0.1172	0.3688	0.1172	0.0384
1.0000	1.3333	1.5574	0.2241	0.4426	0.2241	0.0907

TABLE 3. The values of the approximate and exact solutions of Equation (24) for different values of x.

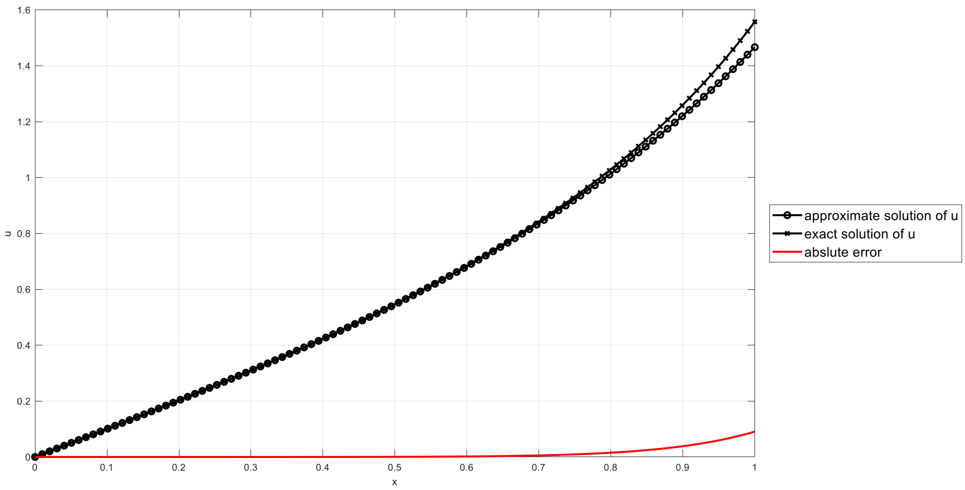

FIGURE 3. The plot of approximate solution, exact solution, and absolute error of Equation (24).

Example 4. Suppose that the nonhomogeneous nonlinear VIESK:

$$u(x) = e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x xu^3(t)\, dt, \tag{26}$$

by applying the algorithm of the new technique to Equation (26), we obtain:

$$\sum_{i=0}^{\infty} u_i = \sum_{i=0}^{\infty} \frac{x^i}{i!} D_x^i \left(e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x x \left(\sum_{j=0}^{i-1} u_j \right)^3 dt \right)_{x=0}. \tag{27}$$

By comparing both sides of Equation (27), we obtain:

$$\begin{cases} u_0 = 1, \\ u_1 = xD_x \left(e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x x\, dt \right)_{x=0} = x, \\ u_2 = \frac{x^2}{2!} D_x^2 \left(e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x x(1+t)^3\, dt \right)_{x=0} = \frac{x^2}{2!}, \\ u_3 = \frac{x^3}{3!} D_x^3 \left(e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x x(1+t)^3\, dt \right)_{x=0} = \frac{x^3}{3!}, \\ \vdots \\ u_{i+1} = \frac{x^{i+1}}{(i+1)!} D_x^{i+1} \left(e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x x \left(\sum_{j=0}^{i} u_j \right)^3 dt \right)_{x=0} = \frac{x^{i+1}}{(i+1)!}. \end{cases}$$

Thus, the approximate solution of Equation (26) is:

$$u_a(x,t) = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

Therefore, the exact solution of Equation (26) is:

$$u_e(x,t) = e^x.$$

x	u_a	u_e	Absolute error HJM	RDTM	PSM	ADM
0.1000	1.1052	1.1052	0.0000	0.0000	0.0000	0.0000
0.2000	1.2213	1.2214	0.0001	0.0001	0.0001	0.0001
0.3000	1.3495	1.3499	0.0004	0.0004	0.0004	0.0004
0.4000	1.4907	1.4918	0.0012	0.0012	0.0012	0.0012
0.5000	1.6458	1.6487	0.0029	0.0029	0.0029	0.0029
0.6000	1.8160	1.8221	0.0061	0.0061	0.0061	0.0061
0.7000	2.0022	2.0138	0.0116	0.0116	0.0116	0.0116
0.8000	2.2053	2.2255	0.0202	0.0202	0.0202	0.0202
0.9000	2.4265	2.4596	0.0331	0.0331	0.0331	0.0331
1.0000	2.6667	2.7183	0.0516	0.0516	0.0516	0.0516

Table 4. The values of the approximate and exact solutions of Equation (26) for different values of x.

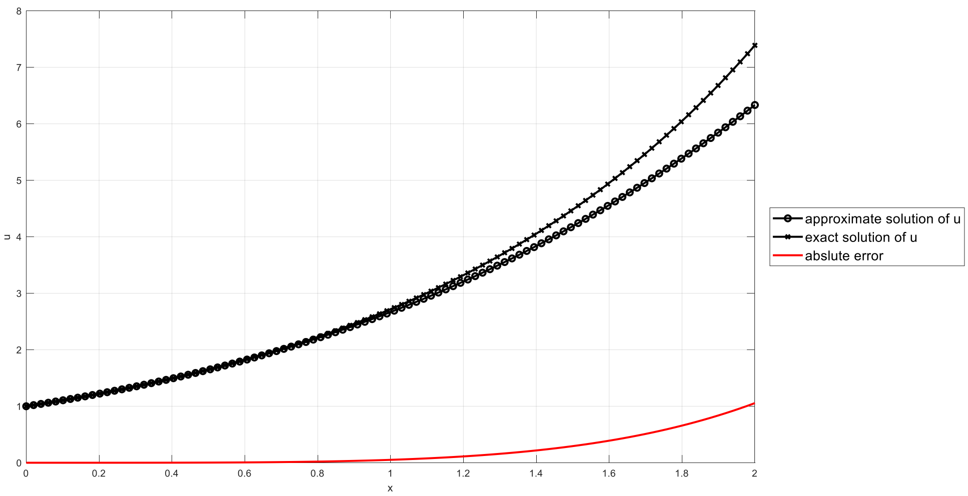

Figure 4. The plot of approximate solution, exact solution, and absolute error of Equation (26).

Example 5. Consider the linear nonhomogeneous system Volterra integral equation of the second kind:

$$\begin{cases} u(x) = e^x - 2x + \int_0^x e^{-t}u(t) + e^t v(t)\ dt, \\ v(x) = e^{-x} - \sinh 2x + \int_0^x e^t u(t) + e^{-t}v(t)\ dt, \end{cases} \tag{28}$$

by applying the algorithm of the new technique to Equation (28), we obtain:

$$\begin{cases} \sum_{i=0}^{\infty} u(x) = e^x - 2x + \int_0^x e^{-t}\sum_{i=0}^{\infty} u(t) + e^t \sum_{i=0}^{\infty} v(t)\ dt, \\ \sum_{i=0}^{\infty} v(x) = e^{-x} - \sinh 2x + \int_0^x e^t \sum_{i=0}^{\infty} u(t) + e^{-t}\sum_{i=0}^{\infty} v(t)\ dt. \end{cases} \tag{29}$$

By comparing both sides of Equation (29), we obtain:

$$\begin{cases} u_0 = 1, \\ v_0 = 1, \\ u_1 = xD_x \left(e^x - 2x + \int_0^x e^{-t} + e^t \, dt \right)_{x=0} = x, \\ v_1 = xD_x \left(e^{-x} - \sinh 2x + \int_0^x e^t + e^{-t} \, dt \right)_{x=0} = -x, \\ u_2 = \frac{x^2}{2!} D_x^2 \left(e^x - 2x + \int_0^x e^{-t}(1+t) + e^t(1-t) \, dt \right)_{x=0} = \frac{x^2}{2!}, \\ v_2 = \frac{x^2}{2!} D_x^2 \left(e^{-x} - \sinh 2x + \int_0^x e^t(1+t) + e^{-t}(1-t) \, dt \right)_{x=0} = \frac{x^2}{2!}, \\ u_3 = \frac{x^3}{3!} D_x^3 \left(e^x - 2x + \int_0^x e^{-t} \left(1 + t + \frac{t^2}{2}\right) + e^t \left(1 - t + \frac{t^2}{2}\right) dt \right)_{x=0} = \frac{x^3}{3!}, \\ v_3 = \frac{x^3}{3!} D_x^3 \left(e^{-x} - \sinh 2x + \int_0^x e^t \left(1 + t + \frac{t^2}{2}\right) + e^{-t} \left(1 - t + \frac{t^2}{2}\right) dt \right)_{x=0} = -\frac{x^3}{3!}, \\ \vdots \\ u_{i+1} = \frac{x^{i+1}}{(i+1)!} D_x^{i+1} \left(e^x - 2x + \int_0^x e^{-t} \left(\sum_{j=0}^{i} u_j \right) + e^t \left(\sum_{j=0}^{i} v_j \right) dt \right)_{x=0} = \frac{x^{i+1}}{(i+1)!}, \\ v_{i+1} = \frac{x^{i+1}}{(i+1)!} D_x^{i+1} \left(e^{-x} - \sinh 2x + \int_0^x e^t \left(\sum_{j=0}^{i} u_j \right) + e^{-t} \left(\sum_{j=0}^{i} v_j \right) dt \right)_{x=0} = (-1)^i \frac{x^{i+1}}{(i+1)!}. \end{cases}$$

Thus, the approximate solution of Equation (28) is:

$$\begin{cases} u_a(x,t) = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}, \\ v_a(x,t) = 1 - x + \frac{x^2}{2!} + \ldots = \sum_{i=0}^{\infty} (-1)^i \frac{x^i}{i!}. \end{cases}$$

Therefore, the exact solution of Equation (28) is:

$$\begin{cases} u_e(x,t) = e^x, \\ v_e(x,t) = e^{-x}. \end{cases}$$

x	u_a	u_e	Absolute error HJM	RDTM	PSM	ADM
0.1000	1.1052	1.1052	0.0000	0.0000	0.0000	0.0000
0.2000	1.2213	1.2214	0.0001	0.0001	0.0001	0.0001
0.3000	1.3495	1.3499	0.0004	0.0004	0.0004	0.0004
0.4000	1.4907	1.4918	0.0012	0.0012	0.0012	0.0012
0.5000	1.6458	1.6487	0.0029	0.0029	0.0029	0.0029
0.6000	1.8160	1.8221	0.0061	0.0061	0.0061	0.0061
0.7000	2.0022	2.0138	0.0116	0.0116	0.0116	0.0116
0.8000	2.2053	2.2255	0.0202	0.0202	0.0202	0.0202
0.9000	2.4265	2.4596	0.0331	0.0331	0.0331	0.0331
1.0000	2.6667	2.7183	0.0516	0.0516	0.0516	0.0516

Table 5. The values of approximate and exact solutions of Equation (28) of u for different values of x.

x	v_a	v_e	Absolute error HJM	RDTM	PSM	ADM
0.1000	0.9048	0.9048	0.0000	0.0000	0.0000	0.0000
0.2000	0.8187	0.8187	0.0001	0.0001	0.0001	0.0001
0.3000	0.7405	0.7408	0.0003	0.0003	0.0003	0.0003
0.4000	0.6693	0.6703	0.0010	0.0010	0.0010	0.0010
0.5000	0.6042	0.6065	0.0024	0.0024	0.0024	0.0024
0.6000	0.5440	0.5488	0.0048	0.0048	0.0048	0.0048
0.7000	0.4878	0.4966	0.0088	0.0088	0.0088	0.0088
0.8000	0.4347	0.4493	0.0147	0.0147	0.0147	0.0147
0.9000	0.3835	0.4066	0.0231	0.0231	0.0231	0.0231
1.0000	0.3333	0.3679	0.0345	0.0345	0.0345	0.0345

Table 6. The values of approximate and exact solutions of Equation (28) of v for different values of x.

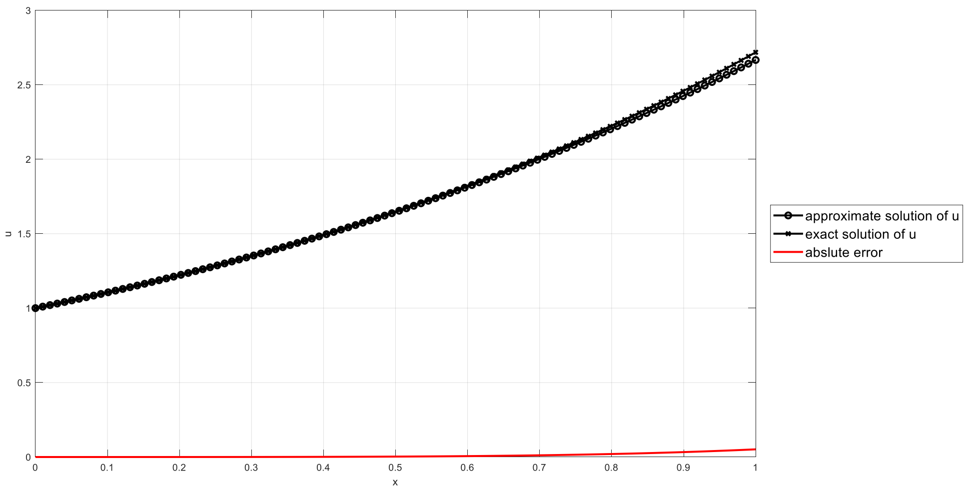

Figure 5. The plot of approximate solution, exact solution, and absolute error of u of Equation (28).

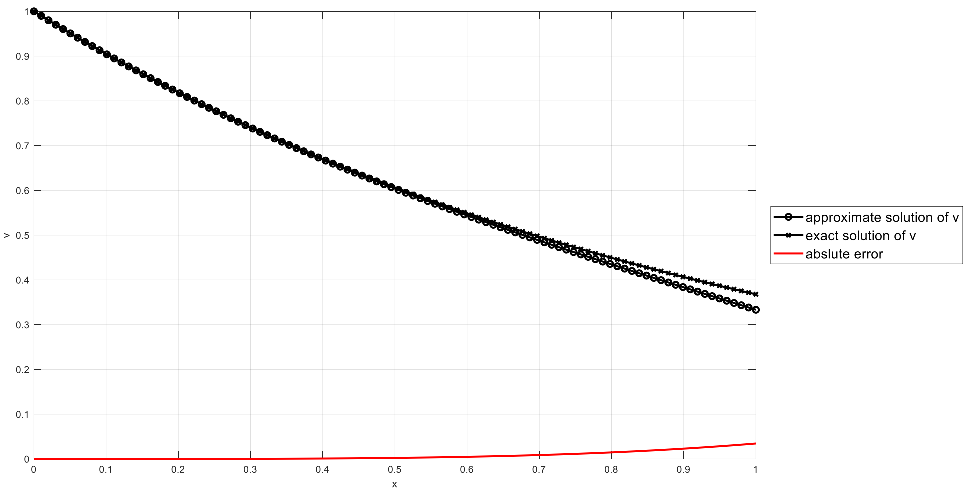

Figure 6. The plot of approximate solution, exact solution, and absolute error of v of Equation (28).

Example 6. Consider the linear nonhomogeneous system VIESK:

$$\begin{cases} u(x) = 1 - x^2 + x^3 + \displaystyle\int_0^x (x-t)u(t) + (x-t)v(t)\ dt, \\ v(x) = 1 - x^3 - \dfrac{1}{10}x^5 + \displaystyle\int_0^x (x-t)u(t) - (x-t)v(t)\ dt, \end{cases} \tag{30}$$

by applying the algorithm of the new technique to Equation (30), we obtain:

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} u(x) = 1 - x^2 + x^3 + \int_0^x (x-t)\sum_{i=0}^{\infty} u(t) + (x-t)\sum_{i=0}^{\infty} v(t)\ dt, \\ \displaystyle\sum_{i=0}^{\infty} v(x) = 1 - x^3 - \frac{1}{10}x^5 + \int_0^x (x-t)\sum_{i=0}^{\infty} u(t) - (x-t)\sum_{i=0}^{\infty} v(t)\ dt. \end{cases} \tag{31}$$

By comparing both sides of Equation (31), we obtain:

$$\begin{cases} u_0 = 1, \\ v_0 = 1, \\ u_1 = xD_x \left(1 - x^2 + x^3 + \displaystyle\int_0^x 2(x-t)\ dt\right)_{x=0} = 0, \\ v_1 = xD_x \left(1 - x^3 - \dfrac{1}{10}x^5 + \displaystyle\int_0^x (0)\ dt\right)_{x=0} = 0, \\ u_2 = \dfrac{x^2}{2!}D_x^2 \left(1 - x^3 - \dfrac{1}{10}x^5 + \displaystyle\int_0^x 2(x-t)\ dt\right)_{x=0} = 0, \\ v_2 = \dfrac{x^2}{2!}D_x^2 \left(1 - x^3 - \dfrac{1}{10}x^5 + \displaystyle\int_0^x (0)\ dt\right)_{x=0} = 0, \\ u_3 = \dfrac{x^3}{3!}D_x^3 \left(1 - x^2 + x^3 + \displaystyle\int_0^x 2(x-t)\ dt\right)_{x=0} = 0, \\ v_3 = \dfrac{x^3}{3!}D_x^3 \left(1 - x^3 - \dfrac{1}{10}x^5 + \displaystyle\int_0^x (0)\ dt\right)_{x=0} = 0, \\ u_4 = \dfrac{x^3}{3!}D_x^3 \left(1 - x^2 + x^3 + \displaystyle\int_0^x 2(x-t)\ dt\right)_{x=0} = x^3, \\ v_4 = \dfrac{x^3}{3!}D_x^3 \left(1 - x^3 - \dfrac{1}{10}x^5 + \displaystyle\int_0^x (0)\ dt\right)_{x=0} = -x^3, \\ \vdots \\ u_{i+1} = \dfrac{x^{i+1}}{(i+1)!}D_x^{i+1} \left(e^x - 2x + \displaystyle\int_0^x e^{-t}(\sum_{j=0}^{i} u_j) + e^t(\sum_{j=0}^{i} v_j)\ dt\right)_{x=0} = 0\ \forall i > 3, \\ v_{i+1} = \dfrac{x^{i+1}}{(i+1)!}D_x^{i+1} \left(e^x - 2x + \displaystyle\int_0^x e^{t}(\sum_{j=0}^{i} u_j) + e^{-t}(\sum_{j=0}^{i} v_j)\ dt\right)_{x=0} = 0\ \forall i > 3. \end{cases}$$

Thus, the approximate solution of Equation (30) is:

$$\begin{cases} u_a(x,t) = 1 + x^3, \\ v_a(x,t) = 1 - x^3. \end{cases}$$

Therefore, the exact solution to Equation (30) is:

$$\begin{cases} u_e(x,t) = 1 + x^3, \\ v_e(x,t) = 1 - x^3. \end{cases}$$

x	u_a	u_e	Absolute error HJM	RDTM	PSM	ADM
0.1000	1.0010	1.0010	0.0000	0.0000	0.0000	0.0000
0.2000	1.0080	1.0080	0.0000	0.0000	0.0000	0.0000
0.3000	1.0270	1.0270	0.0000	0.0000	0.0000	0.0000
0.4000	1.0640	1.0640	0.0000	0.0000	0.0000	0.0000
0.5000	1.1250	1.1250	0.0000	0.0000	0.0000	0.0000
0.6000	1.2160	1.2160	0.0000	0.0000	0.0000	0.0000
0.7000	1.3430	1.3430	0.0000	0.0000	0.0000	0.0000
0.8000	1.5120	1.5120	0.0000	0.0000	0.0000	0.0000
0.9000	1.7290	1.7290	0.0000	0.0000	0.0000	0.0000
1.0000	2.0000	2.0000	0.0000	0.0000	0.0000	0.0000

Table 7. The values of approximate and exact solutions of Equation (30) of u for different values of x.

x	v_a	v_e	Absolute error HJM	RDTM	PSM	ADM
0.1000	0.9990	0.9990	0.0000	0.0000	0.0000	0.0000
0.2000	0.9920	0.9920	0.0000	0.0000	0.0000	0.0000
0.3000	0.9730	0.9730	0.0000	0.0000	0.0000	0.0000
0.4000	0.9360	0.9360	0.0000	0.0000	0.0000	0.0000
0.5000	0.8750	0.8750	0.0000	0.0000	0.0000	0.0000
0.6000	0.7840	0.7840	0.0000	0.0000	0.0000	0.0000
0.7000	0.6570	0.6570	0.0000	0.0000	0.0000	0.0000
0.8000	0.4880	0.4880	0.0000	0.0000	0.0000	0.0000
0.9000	0.2710	0.2710	0.0000	0.0000	0.0000	0.0000
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 8. The values of approximate and exact solutions of Equation (30) of v for different values of x.

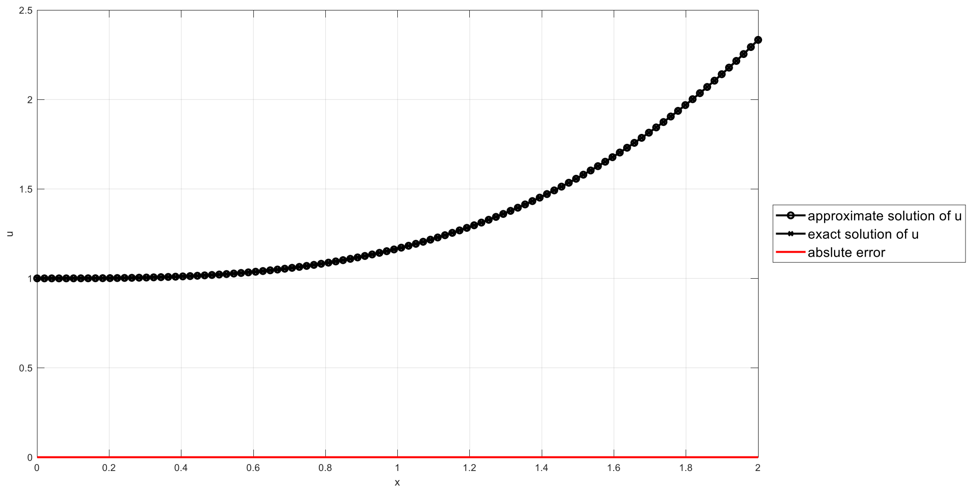

Figure 7. The plot of approximate solution, exact solution, and absolute error of u of Equation (30).

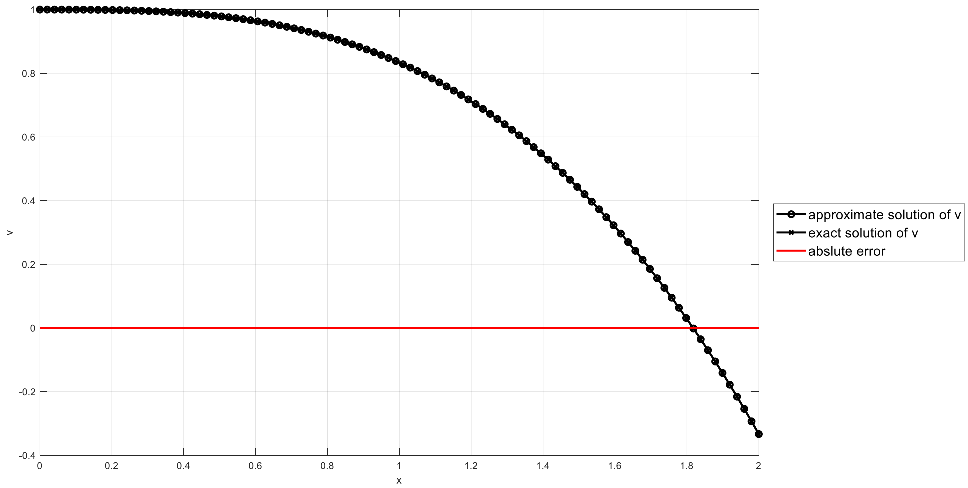

Figure 8. The plot of approximate solution, exact solution, and absolute error of v of Equation (30).

5. Results and Conclusion

In conclusion, the development of an innovative approximation method for solving second-kind Volterra integral equations, including linear, nonlinear, homogeneous, and nonhomogeneous cases, marks a significant breakthrough in integral equation solving. This method introduces a versatile and robust approach that effectively addresses a wide range of Volterra integral equations. Incorporating innovative techniques and numerical algorithms, this novel approach accurately estimates solutions to both linear and nonlinear Volterra equations, providing a comprehensive solution framework capable of handling different scenarios.

Effectively navigating the complexities inherent to Volterra integral equations, including nonlinearity and diverse nature, this method opens avenues for solving real-world problems across disciplines, such as physics, engineering, biology, and economics. Researchers and practitioners now have a powerful tool with a wide range of applications. While further research and validation is needed to fully explore the capabilities and limitations of the method, its potential to revolutionise the field of integral equations and improve our understanding of complex systems is evident.

In summary, the analysis of the results in Figures 1–8 and Tables 1–8 shows that the Hussein Jassim Method (HJM) outperforms the Reduced Differential Transform Method (RDTM) in terms of accuracy and has a lower absolute error. In addition, accuracy of the HJM is comparable to that of the Power Series Method (PSM), highlighting its outstanding effectiveness. However, the HJM exhibits a higher absolute error as compared to the Adomian Decomposition Method (ADM). These findings contribute to a deeper understanding of the HJM's efficacy, emphasizing its relative superiority in solving second-kind Volterra integral equations, whether linear or nonlinear.

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