Czech Technical University
in Prague

Faculty of Nuclear Sciences and Physical
Engineering

DOCTORAL THESIS

Special Functions and Polynomials of Affine Weyl Groups and Corresponding Fourier Methods

Prague 2023 Ing. Adam Brus

Bibliografický záznam

Název práce:	**Speciální funkce a polynomy afinních Weylových grup a příslušné Fourierovy metody**
Autor:	Ing. Adam Brus České vysoké učení technické v Praze, Fakulta jaderná a fyzikálně inženýrská, Katedra fyziky
Studijní program:	Aplikace přírodních věd
Studijní obor:	Matematické inženýrství, Matematická fyzika
Školitel:	doc. Ing. Jiří Hrivnák, Ph.D. České vysoké učení technické v Praze, Fakulta jaderná a fyzikálně inženýrská, Katedra fyziky
Školitel specialista:	Bc. Lenka Motlochová, Ph.D. České vysoké učení technické v Praze, Fakulta jaderná a fyzikálně inženýrská, Katedra fyziky
Akademický rok:	2023/2024
Počet stran:	168
Klíčová slova:	Diskrétní transformace, Fourier–Weylovy transformace, kubaturní vzorce, speciální funkce, Weylovy grupy

Bibliographic entry

Title:	**Special Functions and Polynomials of Affine Weyl Groups and Corresponding Fourier Methods**
Author:	Ing. Adam Brus Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Department of Physics
Degree programme:	Application of Natural Sciences
Field of study:	Mathematical Engineering, Mathematical Physics
Supervisor:	doc. Ing. Jiří Hrivnák, Ph.D. Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Department of Physics
Supervisor specialist:	Bc. Lenka Motlochová, Ph.D. Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Department of Physics
Academic year:	2023/2024
Number of Pages:	168
Keywords:	Discrete transforms, Fourier–Weyl transforms, cubature formulas, special functions, Weyl groups

Acknowledgments

I would like to thank my supervisors Jiří Hrivnák and Lenka Motlochová for their continuous support and motivation as well as benevolence during the past years. Furthermore, I would like to thank everyone who contributed to the present work by any remark, advice or consultation. I would like to express special gratitude to my family which has been supporting me with inspiration, motivation and permissiveness for the entire time I have known them. Additionally, I would like to gratefully acknowledge financial support from Czech Science Foundation (GAČR) Grant No. 19-19535S and student grants of the Grant Agency of Czech Technical University in Prague Grant No. SGS16/239/OHK4/3T/14 and SGS19/183/OHK4/3T/14.

Abstrakt

Tato disertační práce shrnuje příspěvek autora v oboru speciálních funkcí a polynomů afinních Weylových grup a příslušných Fourierových metod. Obsažený výzkum byl vypracován během autorova doktorského studia. Víceproměnné (anti)symetrické diskrétní sinové transformace jsou odvozeny společně s příslušnými interpolačními vzorci, kubaturními vzorci a novými třídami vícerozměrných polynomů Chebyshevova typu. Víceproměnné (anti)symetrické sinové funkce potřebné pro odvození transformací jsou příkladem speciálních funkcí spřízněných s afinní Weylovou grupou. Propojení zmíněných transformací s Weylovou grupou se objevuje ve formě vztahu k specifické třídě duálních kořenových Fourier–Weylových transformací. Paralela mezi těmito dvěma přístupy přináší výhody pro oba formalismy. Dále se disertační práce věnuje aplikacím teorie duálních kořenových a duálních váhových Fourier–Weylových transformací a poskytuje nové třídy modelů kvantových částic na mřížích indukovaných Weylovou grupou. Čtyři články publikované v impaktovaných časopisech prezentované v této práci shrnují studovanou problematiku a obsahují autorův originální přínos do příslušného oboru.

Abstract

This thesis summarizes the contribution of the author to the field of special functions and polynomials of affine Weyl groups and corresponding Fourier methods. The included research was performed during the author's doctoral study. The multivariate (anti)symmetric discrete sine transforms, together with associated interpolation formulas, cubature formulas and new classes of multivariate Chebyshev-like polynomials, are developed. The multivariate (anti)symmetric sine functions necessary to derive the transforms are an example of special functions related to an affine Weyl group. The connection of these transforms to the Weyl group emerges by a one-to-one correspondence to specific class of dual root lattice Fourier–Weyl transforms. The parallel between these two approaches yields benefits for both formalisms. Further, the thesis focuses on the applications of the dual root and dual weight lattice Fourier–Weyl transforms theory and provides new classes of quantum particle models on Weyl group induced lattices. The 4 articles published in impacted journals presented in this thesis summarize the studied topic and include original contributions of the author to the related field.

Contents

Introduction

The classical univariate sine and cosine functions can be generalized in the form of (anti)symmetric multivariate trigonometric functions [42, 46, 60]. This straightforward generalization allows to reexamine various classical mathematical methods to multivariate settings, e.g. discrete sine and cosine transforms (DSTs and DCTs) [8,96]. The DSTs and DCTs are associated with effective integer approximations and various numeric methods, with a notable advantage being the existence of fast algorithms to perform these transforms. With the multivariate generalizations of sine and cosine functions, similar suitable properties could be studied for the (anti)symmetric multivariate discrete sine and cosine transforms (AMDSTs, SMDSTs, AMDCTs and SMDCTs). These multivariate transforms are deduced from remarkable properties of the (anti)symmetric sine and cosine functions and the univariate DSTs and DCTs. The DCTs are generalized to the multivariate settings in [37], obtaining eight AMDCTs and eight SMDCTs. For the sine cases, only the generalizations of the DST-I is done in [60]. The remaining seven AMDSTs and seven SMDSTs are developed in [A1]. Furthermore, the antisymmetric sine and cosine transforms can be viewed as a special case of Fourier transforms of the generalized Schur polynomials [23].

Another well studied topic related to the univariate sine and cosine functions are the Chebyshev polynomials [31, 89]. This particular case of orthogonal polynomials of single variable [15] is connected to effective numerical methods for approximation and integration [11, 14, 16, 68, 69]. The univariate trigonometric definition of these classical orthogonal polynomials serves as a starting point to generalize the Chebyshev polynomials to the multivariate orthogonal polynomial setting [15, 19, 22, 25, 65, 88]. The multivariate Chebyshev-like polynomials related to the cosine functions and their application are studied in [37]. The generalizations of Chebyshev polynomials associated to the sine functions were previously not studied and are introduced and investigated in [A1].

The combination of the (anti)symmetric multivariate discrete sine and cosine transforms, together with the corresponding Chebyshev-like polynomials, yields associated cubature formulas [17, 32, 43, 84, 85, 94]. Cubature formulas allow to replace integration over specific domain by finite summing of polynomial values evaluated on specific point set. These formulas hold exactly for polynomials which do not exceed a maximum degree, known as a degree of precision. The cubature formulas, associated with the multivariate Chebyshev-like polynomials of the first and third kind, are studied in [37], obtaining four cubature formulas for each generalized Chebyshev-like polynomial. Furthermore, for each class of polynomials one of these formulas is the optimal Gaussian formula [4], which for given polynomial degree requires the minimum amount of points to be evaluated at. Similar approach is followed for the multivariate Chebyshev-like polynomials related to the (anti)symmetric multivariate sine functions in [A1]. Moreover, the above mentioned topics analyzed in [A1] are further studied for the two dimensional cases in [9], providing exact form of the polynomial recurrence formulas and listing first few polynomials for each class of the bivariate Chebyshev-like polynomials related to the bivariate (anti)symmetric sine functions.

The (anti)symmetric multivariate sine and cosine functions are multivariate special functions [5,33,46,49,57–59,62,86], with another closely connected class of special functions being the Weyl orbit functions [34,47,48,50,51,63,64] arising from the Weyl group theory [54], which is associated to the Lie groups and algebras [6,7,53,98]. These functions have many remarkable properties, from which many are based on the symmetries of the Weyl group. Moreover, in the case of the Lie algebra C_n, the Weyl orbit functions coincide with the (anti)symmetric multivariate sine and cosine functions [40]. For each Weyl group, there exist up to four classes of Weyl orbit functions, based on the sign homomorphisms on the Weyl group.

Each of the Weyl orbit functions induces discrete Fourier–Weyl transforms [18,20,38, 45,70] on the closure of the Weyl alcove, a simplex in the Euclidean space representing a conveniently selected fundamental domain of the affine Weyl group. The dual root lattice Fourier–Weyl transforms use the Weyl orbit functions as kernels of the transforms and the point sets are given by rescaled admissibly shifted dual root lattices intersected with the signed fundamental domains of the affine Weyl group [18]. The dual weight lattice Fourier–Weyl transforms use as kernels the Weyl orbit functions and the point sets are given by the intersection of the signed fundamental domains of the affine Weyl group and rescaled admissibly shifted dual weight lattices [18].

The comparison of the dual root lattice Fourier–Weyl transforms of the algebras A_1 and C_n and the (anti)symmetric multivariate sine and cosine transforms of the corresponding dimensions is based on the up-to constant correspondence between the Weyl orbit functions and the (anti)symmetric multivariate sine and cosine functions [40]. The exact parallel concerning the point and label sets as well as the normalization and weight factors of these transforms leads to the up-to constant coincidence of the corresponding unitary matrices that are explicitly given in [A2].

One of the applications of the generalized discrete Fourier–Weyl transforms is to describe a special class of the discrete quantum billiard systems [29,52,55,56,75,80,81, 91,95], where the boundary conditions [2,73,74] are controlled via specific Neumann and Dirichlet walls. These boundary conditions are induced by the sign homomorphisms and admissible shifts of the dual root or dual weight lattice Fourier–Weyl transforms. The quantum particle propagates inside the closure of Weyl alcove on the rescaled and shifted dual root or dual weight lattice and the boundaries are presented by Neumann walls as perfect mirrors and Dirichlet walls as ideal barriers. The orthonormal position bases of the finite dimensional Hilbert space are given by ordered point sets of the dual root or dual weight lattice Fourier–Weyl transforms. The amplitudes of the quantum particle propagation on the rescaled and shifted dual root or dual weight lattice are provided by complex-valued hopping function. The hopping operators on the particle Hilbert space are then constructed using the non-zero dominant values of the hopping function. The hopping operator matrix elements incorporate the effect of the boundary walls using the symmetrization of the χ-function, thus modifying the amplitudes of possible jumps inside the Weyl alcove. The summation over all hopping operators provides the Hückel-type Hamiltonian [72, 77] of resulting model. Furthermore, the hopping functions are constrained to be Hermitian to assure the hermiticity of the discrete Hamiltonian.

The stationary states of the described model are provided by the time-independent Schrödinger equation [27,28] and are acquired by inverse of the dual root or dual weight lattice Fourier–Weyl transforms [20,38] of the orthonormal position bases. The eigenenergies of the quantum particle systems are fully determined as a linear combinations of the Weyl orbit functions [64]. Classes of discrete quantum models of a free non-relativistic quantum particle on the dual root lattice in Weyl alcove, together with the corresponding Fourier–Weyl transforms theory summarization, are described in [A3]. The dual weight lattice models produce similarly different classes of discrete quantum models. The models induced by the dual weight lattice Fourier–Weyl transforms are intently stated in [A4]. The dual root and dual weight approaches are further summarized in [10].

The ambition of this work is to summarize the contributions of the author to the special functions and polynomials of affine Weyl groups and corresponding Fourier methods, and provide an consolidation text for the articles [A1–A4]. The aim of these articles is to:

- develop (anti)symmetric multivariate discrete sine transforms and corresponding interpolation methods based on multivariate trigonometric functions
- generalize the classical Chebyshev polynomials of second and fourth kind into multivariate Chebyshev-like polynomials with use of multivariate trigonometric functions
- obtain cubature formulas from combination of the (anti)symmetric multivariate discrete sine transforms and multivariate Chebyshev-like polynomials
- establish connection between the (anti)symmetric multivariate discrete sine transforms and dual root Fourier-Weyl transforms
- develop a family of discrete quantum models describing a non-relativistic quantum particle propagating on a rescaled and shifted dual root or weight lattices based on the theory of Fourier-Weyl transforms.

The contribution of the author of the thesis to the original results, contained in the included articles [A1–A4], ranges from theoretical research to concept and reviews of the manuscripts [A1–A4], preparation of figures and visualizations [A1–A4] and numerical computations [A1–A4]. The most significant results, which were obtained with the essential contribution of the author, encompass

- derivation of the (anti)symmetric multivariate discrete sine transforms [A1, §3]
- formulating recurrence relations for the multivariate Chebyshev-like polynomials of the second and fourth kind [A1, §4]
- obtaining cubature formulas related to multivariate Chebyshev-like polynomials of second and fourth kind [A1, Theorems 1-5]
- verifying the relations between the weight functions of (anti)symmetric multivariate discrete sine and cosine transforms and dual root lattice Fourier–Weyl transforms [A2, Theorem 4]

- verifying and mathematical modeling of the dual root lattice model of C_2 in Wolfram Mathematica [A3, §4.2]
- visualization of the examples [A3, §4]
- verifying and mathematical modeling of the dual weight lattice model of C_2 in Wolfram Mathematica [A4, §4.1]
- visualization of the examples [A4, §4].

The progress achieved in [A1–A4] was obtained using analytical and numerical methods. The models were calculated numerically using Wolfram Mathematica and visualizations were produced using Wolfram Mathematica and LaTeX.

The thesis is organized as follows. Chapter 1 is dealing with the study of (anti)symmetric multivariate discrete sine transforms and corresponding background. Chapter 2 is focused on the multivariate Chebyshev-like polynomials of the second and third kind and related cubature formulas. Chapter 3 is summarizing the Weyl group theory and discrete Fourier–Weyl transforms needed in the following chapters. Chapter 4 is connecting the (anti)symmetric multivariate sine and cosine transforms with the dual root lattice Fourier–Weyl transforms. Chapter 5 is summarizing the development of dual root and weight lattice models of quantum particle in Weyl alcove. In Conclusion, the concluding remarks and follow up problems are presented. In the Included Publications section, the four included articles [A1–A4] are listed.

Chapter 1

Generalization of Discrete Trigonometric Transforms

This section deals with the (anti)symmetric multivariate generalizations of sine and cosine functions [60], that are employed to generalize the classical univariate discrete trigonometric transforms [8], concentrating mainly on the progress achieved in [A1]. Section 1.1 focuses on the definition and properties of the (anti)symmetric multivariate sine and cosine functions, Section 1.2 summarizes the essential information about the classical discrete sine and cosine transforms and Section 1.3 is focusing on the multivariate development accomplished in [A1].

1.1 Multivariate Sine and Cosine Functions

The multivariate sine and cosine functions [60] are straightforward generalizations of the classical sine and cosine functions using determinants and permanents of matrices with univariate sine or cosine entries. For any $n \in \mathbb{N}$ the (anti)symmetric multivariate sine and cosine functions are defined by

$$
\begin{aligned}
\sin^-_\lambda(x) =& \det \begin{pmatrix} \sin(\pi\lambda_1 x_1) & \sin(\pi\lambda_1 x_2) & \cdots & \sin(\pi\lambda_1 x_n) \\ \sin(\pi\lambda_2 x_1) & \sin(\pi\lambda_2 x_2) & \cdots & \sin(\pi\lambda_2 x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(\pi\lambda_n x_1) & \sin(\pi\lambda_n x_2) & \cdots & \sin(\pi\lambda_n x_n) \end{pmatrix}, \\
\cos^-_\lambda(x) =& \det \begin{pmatrix} \cos(\pi\lambda_1 x_1) & \cos(\pi\lambda_1 x_2) & \cdots & \cos(\pi\lambda_1 x_n) \\ \cos(\pi\lambda_2 x_1) & \cos(\pi\lambda_2 x_2) & \cdots & \cos(\pi\lambda_2 x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\pi\lambda_n x_1) & \cos(\pi\lambda_n x_2) & \cdots & \cos(\pi\lambda_n x_n) \end{pmatrix},
\end{aligned} \tag{1.1}
$$

and

$$
\begin{aligned}
\sin^+_\lambda(x) =& \operatorname{perm} \begin{pmatrix} \sin(\pi\lambda_1 x_1) & \sin(\pi\lambda_1 x_2) & \cdots & \sin(\pi\lambda_1 x_n) \\ \sin(\pi\lambda_2 x_1) & \sin(\pi\lambda_2 x_2) & \cdots & \sin(\pi\lambda_2 x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(\pi\lambda_n x_1) & \sin(\pi\lambda_n x_2) & \cdots & \sin(\pi\lambda_n x_n) \end{pmatrix}, \\
\cos^+_\lambda(x) =& \operatorname{perm} \begin{pmatrix} \cos(\pi\lambda_1 x_1) & \cos(\pi\lambda_1 x_2) & \cdots & \cos(\pi\lambda_1 x_n) \\ \cos(\pi\lambda_2 x_1) & \cos(\pi\lambda_2 x_2) & \cdots & \cos(\pi\lambda_2 x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\pi\lambda_n x_1) & \cos(\pi\lambda_n x_2) & \cdots & \cos(\pi\lambda_n x_n) \end{pmatrix},
\end{aligned} \tag{1.2}
$$

where the variable $x = (x_1, x_2, \dots, x_n)$ and parameter $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

Properties of Multivariate Trigonometric Functions

The multivariate (anti)symmetric sine and cosine functions have the following explicit form:

$$
\begin{aligned}
\sin^-_\lambda(x) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\sin(\pi\lambda_{\sigma(1)}x_1)\sin(\pi\lambda_{\sigma(2)}x_2)\cdots\sin(\pi\lambda_{\sigma(n)}x_n),\\
\cos^-_\lambda(x) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\cos(\pi\lambda_{\sigma(1)}x_1)\cos(\pi\lambda_{\sigma(2)}x_2)\cdots\cos(\pi\lambda_{\sigma(n)}x_n),
\end{aligned}
\tag{1.3}
$$

for the antisymmetric case and

$$
\begin{aligned}
\sin^+_\lambda(x) &= \sum_{\sigma\in S_n} \sin(\pi\lambda_{\sigma(1)}x_1)\sin(\pi\lambda_{\sigma(2)}x_2)\cdots\sin(\pi\lambda_{\sigma(n)}x_n),\\
\cos^+_\lambda(x) &= \sum_{\sigma\in S_n} \cos(\pi\lambda_{\sigma(1)}x_1)\cos(\pi\lambda_{\sigma(2)}x_2)\cdots\cos(\pi\lambda_{\sigma(n)}x_n),
\end{aligned}
\tag{1.4}
$$

for the symmetric case, where S_n denotes the permutation group and sgn is the sign homomorphism on S_n.

These functions inherit many remarkable properties from the one dimensional trigonometric functions and from the properties of determinants and permanents of matrices [A1, §2.1].

For further study of these functions, integer sets $P_1^\pm$ are defined as

$$
\begin{aligned}
P_1^+ &= \{(k_1,k_2,\dots,k_n)\in\mathbb{Z}^n | k_1\geq k_2\geq\cdots\geq k_n\geq 1\},\\
P_1^- &= \{(k_1,k_2,\dots,k_n)\in\mathbb{Z}^n | k_1> k_2>\cdots> k_n\geq 1\}
\end{aligned}
\tag{1.5}
$$

and the parameters of $\sin^\pm_\lambda(x)$ are restricted to $\lambda=k$ or $\lambda=k-\rho$, where $k\in P_1^\pm$ and $\rho=\left(\frac12,\frac12,\dots,\frac12\right)$.

Taking into account these limitations, it is sufficient to consider the functions $\sin^\pm_\lambda(x)$ on the closure of the fundamental domain $F(\widetilde{S}_n^{\mathrm{aff}})$,

$$
F(\widetilde{S}_n^{\mathrm{aff}}) = \{(x_1,x_2,\dots,x_n)\in\mathbb{R}^n | 1\geq x_1\geq x_2\geq\ldots\geq x_n\geq 0\}\,. \tag{1.6}
$$

Certain parts of the boundary of the domain $F(\widetilde{S}_n^{\mathrm{aff}})$ are further omitted for specific cases of $\sin^\pm_\lambda(x)$ [A1, §2.1]. Moreover, the (anti)symmetric sine functions are pairwise continuously orthogonal within each family, $\sin^-_k, \sin^-_{k-\rho}, \sin^+_k, \sin^+_{k-\rho}$, when integrated over $F(\widetilde{S}_n^{\mathrm{aff}})$ [A1, §2.2].

1.2 Discrete Trigonometric Transforms

Discrete sine transforms (DSTs) and discrete cosine transforms (DCTs) emerge as solutions of an undamped harmonic oscillator equation with various homogenous boundary conditions [8]. The DSTs are obtained when Dirichlet boundary condition at start of an interval is considered, while the DCTs arise from Neumann boundary condition. In total, there exist eight types of DSTs and eight types of DCTs deduced from various combination of boundary conditions applied at the grid or mid-grid points.

For example, the DST of type VIII is derived when Dirichlet condition is applied at mid-grid point at the start of an interval and Neumann condition is applied at the end grid point. For any $n \in \mathbb{N}$ and any discrete function $f(s)$ defined on the grid

$$s \in \left\{ \frac{2\left(r+\frac{1}{2}\right)}{2N-1} \mid r = 0, 1, \dots, N-1 \right\}, \tag{1.7}$$

the DST-VIII has the following form:

$$f(s) = \sum_{k=0}^{N-1} A_k \sin\left(\pi\left(k+\frac{1}{2}\right)s\right), \tag{1.8}$$

$$A_k = \frac{4c_{k+1}}{2N-1} \sum_{r=0}^{N-1} c_{r+1} f\left(\frac{2\left(r+\frac{1}{2}\right)}{2N-1}\right) \sin\left(\frac{2\pi\left(k+\frac{1}{2}\right)\left(r+\frac{1}{2}\right)}{2N-1}\right), \tag{1.9}$$

where

$$c_r = \begin{cases} \frac{1}{2} & \text{if } r = 0 \text{ or } r = N, \\ 1 & \text{otherwise.} \end{cases} \tag{1.10}$$

The whole set of the classical one-dimensional discrete trigonometric transforms can be found in [8, §2.6 and 2.7].

1.3 Discrete Multivariate Trigonometric Transforms

A convenient combination of properties of the multivariate sine and cosine functions and the univariate discrete trigonometric transforms allows to generalize the discrete trigonometric transforms to multivariate setting, obtaining eight transforms for each case of antisymmetric and symmetric multivariate sine and cosine function. For example, the symmetric multivariate discrete sine transform of type VI is described bellow.

For $N \in \mathbb{N}, N > n$ we consider the symmetric sine functions $\sin_k^+(s)$ labeled by the index set $k \in D_{1,N}^+$ given by

$$D_{1,N}^+ \equiv \{(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n \mid N \geq k_1 \geq k_2 \geq \dots \geq k_n \geq 1\} \tag{1.11}$$

and restricted to the finite set of points $F_N^{\mathrm{VI},+}$ defined by

$$F_N^{\mathrm{VI},+} \equiv \left\{ \left(\frac{2\left(r_1+\frac{1}{2}\right)}{2N+1}, \dots, \frac{2\left(r_n+\frac{1}{2}\right)}{2N+1} \right) \mid (r_1, \dots, r_n) \in D_{1,N}^+ \right\}. \tag{1.12}$$

The functions $\sin_k^+(s)$ are pairwise discretely orthogonal for $k, k' \in D_{1,N}^+$, i.e.

$$\sum_{s \in F_N^{\mathrm{VI},+}} H_s^{-1} \sin_k^+(s) \sin_{k'}^+(s) = \left(\frac{2N+1}{4}\right)^n \delta_{kk'}, \tag{1.13}$$

where H_s denotes the order of the stabilizer subgroup $\mathrm{Stab}_{S_n}(s)$ of S_n with respect to the point $s \in \mathbb{R}^n$. Therefore, any function $f : F_N^{\mathrm{VI},+} \to \mathbb{R}$ is expanded in terms of symmetric multivariate sine functions as

$$f(s)=\sum_{k\in D^{+}_{1,N}} A_k \sin^{+}_k(s),\quad A_k=H_k^{-1}\left(\frac{4}{2N+1}\right)^n \sum_{s\in F_N^{\mathrm{VI},+}} H_s^{-1} f(s)\sin^{+}_k(s). \tag{1.14}$$

The full set of the discrete multivariate cosine transforms is deduced in [37], the multivariate generalization of DST-I is elaborated in [60]. The entire set of (anti)symmetric multivariate discrete sine transforms is derived in [A1, §3.1 and 3.2], resulting in 14 novel multivariate discrete transforms. Furthermore, the orthogonal matrices for the normalized discrete sine transforms are introduced and a trivariate example is numerically calculated by Wolfram Mathematica in [A1, §3.4]. One of direct applications of the multivariate sine discrete transforms is interpolation of a real-valued function over the fundamental domain $F(\widetilde{S}_n^{\mathrm{aff}})$. The total of 16 interpolation polynomials are developed. For a trivariate example, the interpolation is, for various N, performed in Wolfram Mathematica and the graph cuts of the interpolation polynomials are depicted in [A1, Figures 6 and 7]. Moreover, the integral error estimates are numerically calculated in [A1, Table 2]. For the fixed ordering of the label sets $D_N^{\star,c,\pm}$ and $D_N^{\star,s,\pm}$ and point sets $F_N^{\star,c,\pm}$ and $F_N^{\star,s,\pm}$, the unitary transform matrices $C_N^{\star,\pm}$ of the (anti)symmetric multivariate discrete cosine transforms of types $\star\in\{\mathrm{I},\dots,\mathrm{VIII}\}$ are given by their entries as

$$\begin{aligned}
\left(C_N^{\star,+}\right)_{yx} &= \sqrt{\frac{\varepsilon_x^{\star,c}}{h_y^{\star,c} H_y H_x}}\cos^{+}_y(x), && y\in D_N^{\star,c,+},\ x\in F_N^{\star,c,+},\\
\left(C_N^{\star,-}\right)_{yx} &= \sqrt{\frac{\varepsilon_x^{\star,c}}{h_y^{\star,c}}}\cos^{-}_y(x), && y\in D_N^{\star,c,-},\ x\in F_N^{\star,c,-}.
\end{aligned} \tag{1.15}$$

The unitary transform matrices $S_N^{\star,\pm}$ of the (anti)symmetric multivariate discrete sine transforms of types $\star\in\{\mathrm{I},\dots,\mathrm{VIII}\}$ are defined by their entries as

$$\begin{aligned}
\left(S_N^{\star,+}\right)_{yx} &= \sqrt{\frac{\varepsilon_x^{\star,s}}{h_y^{\star,s} H_y H_x}}\sin^{+}_y(x), && y\in D_N^{\star,s,+},\ x\in F_N^{\star,s,+},\\
\left(S_N^{\star,-}\right)_{yx} &= \sqrt{\frac{\varepsilon_x^{\star,s}}{h_y^{\star,s}}}\sin^{-}_y(x), && y\in D_N^{\star,s,-},\ x\in F_N^{\star,s,-}.
\end{aligned} \tag{1.16}$$

The explicit form of the point sets $F_N^{\star,c,\pm}$ and $F_N^{\star,s,\pm}$ and label sets $D_N^{\star,c,\pm}$ and $D_N^{\star,s,\pm}$,$\star\in\{\mathrm{I},\dots,\mathrm{VIII}\}$, together with the normalization functions $h_y^{\star,c}$ and $h_y^{\star,s}$ and weight functions $\varepsilon_x^{\star,c}$ and $\varepsilon_x^{\star,s}$, are listed in [A2, Table 2].

For the univariate case, the antisymmetric and symmetric cases coincide and correspond to the matrix transforms of DCTs and DSTs.

Chapter 2

Multivariate Chebyshev-like Polynomials of the (Anti)symmetric Multivariate Sine Functions

The orthogonal polynomials [15] are widely used in mathematics and physics, with one of their main applications being numerical methods for approximation and integration. This section is mostly describing the development and application of the multivariate Chebyshev-like polynomials of the (anti)symmetric sine functions done in [A1, Chapter 4]. Replacing the trigonometric functions in the defining relations of the classical Chebyshev polynomials by their multivariate generalizations leads to the multivariate Chebyshev-like polynomials. These multivariate generalizations inherit many exceptional properties from the classical Chebyshev polynomials, and thus, deserve observation.

2.1 Classical Chebyshev Polynomials

The Chebyshev polynomials [31, 89] are well known class of orthogonal polynomials defined via trigonometric functions. The first and third kind are connected to the cosine function, the second and fourth kind are related to the sine function. The standard form of these polynomials for variable $x = \cos(\theta)$, $x \in [-1, 1]$ follows:

$$\begin{aligned} \mathcal{P}_n^{\mathrm{I}}(x) &= T_n(x) = \cos(n\theta), & \mathcal{P}_n^{\mathrm{III}}(x) &= V_n(x) = \frac{\cos\left(\left(n+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{1}{2}\theta\right)}, \\ \mathcal{P}_n^{\mathrm{II}}(x) &= U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, & \mathcal{P}_n^{\mathrm{IV}}(x) &= W_n(x) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}. \end{aligned} \tag{2.1}$$

Properties of the Chebyshev Polynomials of the Second Kind

The main properties of the polynomials follow directly from trigonometric identities and are presented for the case of Chebyshev polynomials of the second kind. The continuous orthogonality of the Chebyshev polynomials of the second kind come from their trigonometric definition in the form:

$$\int_0^{\pi} \sin\left((n+1)\,\theta\right)\sin\left((m+1)\,\theta\right)d\theta = 0, \quad n \neq m, \tag{2.2}$$

which is rewritten for the variable x as

$$\int_{-1}^{1} U_n(x)U_m(x)\left(1-x^2\right)^{\frac{1}{2}}dx = 0, \quad n \neq m. \tag{2.3}$$

Additionally, the trigonometric definition yields directly the following expressions for the first two polynomials,

$$U_1(x) = 1, \qquad U_2(x) = 2\cos\left(\theta\right) = 2x. \tag{2.4}$$

For every univariate class of orthogonal polynomials, there exists a recurrence relation connecting any three successive polynomials. In the case of the Chebyshev polynomials of the second kind, the following trigonometric identity,

$$\sin\left((n+1)\theta\right) + \sin\left((n-1)\theta\right) = 2\cos\left(\theta\right)\sin\left(n\theta\right), \tag{2.5}$$

results into the recurrence formula for polynomials $U_n(x)$:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \qquad n = 2, 3, \ldots. \tag{2.6}$$

Formula (2.6), together with the first two polynomials (2.4), provides a method for generating any Chebyshev polynomial of the second kind. Similar relations are obtained for the remaining three classes of the Chebyshev polynomials in [31].

2.2 Multivariate Chebyshev-like Polynomials

For the multivariate Chebyshev-like polynomials, the functions $X_1, X_2, \ldots, X_n$, given by

$$X_1 = \cos^+_{(1,0,\ldots,0)}, \quad X_2 = \cos^+_{(1,1,\ldots,0)}, \quad \ldots, \quad X_n = \cos^+_{(1,1,\ldots,1)}, \tag{2.7}$$

are viewed as variables of the multivariate Chebyshev-like polynomials.

Introducing the following parameters

$$\begin{aligned}
\rho_1^- &= \left(n-1, n-2, \ldots, 0\right), \\
\rho_2^- &= \left(n, n-1, \ldots, 1\right), \\
\rho_2^+ &= \left(1, 1, \ldots, 1\right), \\
\rho_4^- &= \left(n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{1}{2}\right), \\
\rho_4^+ &= \rho = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right),
\end{aligned} \tag{2.8}$$

the multivariate Chebyshev-like polynomials are defined for $k \in P^+$

$$P^+ = \left\{(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n | k_1 \geq k_2 \geq \cdots \geq k_n \geq 0\right\}, \tag{2.9}$$

as

$$
\begin{aligned}
&\mathcal{P}_k^{\mathrm{I},+}\left(X_1(x),\dots,X_n(x)\right)=\cos_k^+(x), &&\mathcal{P}_k^{\mathrm{I},-}\left(X_1(x),\dots,X_n(x)\right)=\frac{\cos^-_{k+\rho_1^-}(x)}{\cos^-_{\rho_1^-}(x)},\\
&\mathcal{P}_k^{\mathrm{II},+}\left(X_1(x),\dots,X_n(x)\right)=\frac{\sin^+_{k+\rho_2^+}(x)}{\sin^+_{\rho_2^+}(x)}, &&\mathcal{P}_k^{\mathrm{II},-}\left(X_1(x),\dots,X_n(x)\right)=\frac{\sin^-_{k+\rho_2^-}(x)}{\sin^-_{\rho_2^-}(x)},\\
&\mathcal{P}_k^{\mathrm{III},+}\left(X_1(x),\dots,X_n(x)\right)=\frac{\cos^-_{k+\rho_4^+}(x)}{\cos^-_{\rho_4^+}(x)}, &&\mathcal{P}_k^{\mathrm{III},-}\left(X_1(x),\dots,X_n(x)\right)=\frac{\cos^-_{k+\rho_4^-}(x)}{\cos^-_{\rho_4^-}(x)},\\
&\mathcal{P}_k^{\mathrm{IV},+}\left(X_1(x),\dots,X_n(x)\right)=\frac{\sin^+_{k+\rho_4^+}(x)}{\sin^+_{\rho_4^+}(x)}, &&\mathcal{P}_k^{\mathrm{IV},-}\left(X_1(x),\dots,X_n(x)\right)=\frac{\sin^-_{k+\rho_4^-}(x)}{\sin^-_{\rho_4^-}(x)}.
\end{aligned}
\tag{2.10}
$$

The generalizations of the Chebyshev-like polynomials corresponding to the cosine functions together with their properties are developed in [37]. The polynomials related to the sine functions are novel and established in [A1, Chapter 4]. These rational functions are well-defined on the interior of $F(\widetilde{S}_n^{\mathrm{aff}})$ [A1, Corollary 1]. Furthermore, each set of the polynomials is totally ordered by the lexicographic ordering $>$ on P^+.

2.3 Recurrence Relations

As in the classical case, the Chebyshev-like polynomials connected to the (anti)-symmetric sine functions are constructed recursively from a generalized trigonometric product-to-sum identity,

$$
\sin^\pm_\lambda(x)\cos^+_\mu(x)=\frac{1}{2^n}\sum_{\sigma\in S_n}\sum_{\left\{\substack{a_i=-1,1\\ i=1,\dots,n}\right\}}\sin^\pm_{\left(\lambda_1+a_1\mu_{\sigma(1)},\dots,\lambda_n+a_n\mu_{\sigma(n)}\right)}(x),
\tag{2.11}
$$

valid for any $\lambda,\mu\in\mathbb{R}^n$. Furthermore, the functions $\mathcal{P}_k^{\mathrm{II},\pm},\mathcal{P}_k^{\mathrm{IV},\pm}$, $k\in P^+$ are polynomials of degree k_1 in variables $X_1,X_2,\dots,X_n$ [A1, Prop.2]. The proposition further implies that the domain of these functions can be further extended to the whole $F(\widetilde{S}_n^{\mathrm{aff}})$ [A1, Remark 3]. The recursive algorithm described in [A1, §4.1] is based on (2.11) and the set of lower trivariate polynomials, together with the explicit form of the recurrence relations, is provided in [A1, Example 4]. Furthermore, the first ten polynomials with degree at most two are listed for all four kinds of Chebyshev-like polynomials of the (anti)symmetric sine functions in [A1, Tables 3-6]. Moreover, the bivariate case of these polynomials is further comprehensively studied in [9].

2.4 Cubature Formulas

One of the most useful applications of classical Chebyshev polynomials are quadrature formulas, replacing integration by finite summing, thus, playing significant role in numerical analysis. In multidimensional setting, such formulas are known as cubature formulas. The cubature formulas equate a weighted integral of polynomial functions with a linear combination of polynomial values in specific points as long as the polynomials do not exceed certain degree. The maximum degree of polynomials for which the formula holds exactly is called degree of precision. The fundamental property to derive the cubature formulas related to the Chebyshev-like polynomials is the continuous orthogonality of the polynomials (2.10).

Weight Functions

The transform $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$\varphi\left(x_1, x_2, \dots, x_n\right) = \left(X_1(x), X_2(x), \dots, X_n(x)\right), \tag{2.12}$$

maps the domain $F(\widetilde{S}_n^{\mathrm{aff}})$ to the integration domain $\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})$. The Jacobian of this transform is, in absolute value, given by

$$|J(x_1, \dots, x_n)| = \pi^n \left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}} \left(\prod_{i=1}^{n} (n-i)!i!\right) \left|\sin^-_{\rho_2^-}(x_1, x_2, \dots, x_n)\right|. \tag{2.13}$$

The absolute value $|J(x_1, \dots, x_n)|$ is expressed as a function $\mathcal{J}$ in the polynomial variables $(X_1, X_2, \dots, X_n)$ (see [37, §4.13]),

$$\mathcal{J}(X_1, \dots, X_n) = |J(x_1, \dots, x_n)|. \tag{2.14}$$

Introducing the functions $\mathcal{J}^{\mathrm{II},\pm}$ and $\mathcal{J}^{\mathrm{IV},\pm}$ by

$$\begin{aligned}
\mathcal{J}^{\mathrm{II},-}\left(X_1(x), \dots, X_n(x)\right) &= \sin^-_{\rho_2^+}(x) \sin^-_{\rho_2^+}(x), \\
\mathcal{J}^{\mathrm{II},+}\left(X_1(x), \dots, X_n(x)\right) &= \sin^+_{\rho_2^+}(x) \sin^+_{\rho_2^+}(x), \\
\mathcal{J}^{\mathrm{IV},-}\left(X_1(x), \dots, X_n(x)\right) &= \sin^-_{\rho_4^-}(x) \sin^-_{\rho_4^-}(x), \\
\mathcal{J}^{\mathrm{IV},+}\left(X_1(x), \dots, X_n(x)\right) &= \sin^+_{\rho_4^+}(x) \sin^+_{\rho_4^+}(x),
\end{aligned} \tag{2.15}$$

the four weight functions related to the Chebyshev-like polynomials are defined by

$$w^{\mathrm{II},\pm} = \frac{\mathcal{J}^{\mathrm{II},\pm}}{\mathcal{J}}, \qquad w^{\mathrm{IV},\pm} = \frac{\mathcal{J}^{\mathrm{IV},\pm}}{\mathcal{J}}. \tag{2.16}$$

Continuous Orthogonality

The continuous orthogonality relations for the Chebyshev-like polynomials of the (anti)symmetric sine functions are given by

$$\begin{aligned}
\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{\mathrm{II},-}(X) \mathcal{P}_{k'}^{\mathrm{II},-}(X) w^{\mathrm{II},-}(X)\, dX \quad &= \quad 2^{-n} \delta_{kk'}, \\
\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{\mathrm{II},+}(X) \mathcal{P}_{k'}^{\mathrm{II},+}(X) w^{\mathrm{II},+}(X)\, dX \quad &= \quad 2^{-n} H_k \delta_{kk'}, \\
\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{\mathrm{IV},-}(X) \mathcal{P}_{k'}^{\mathrm{IV},-}(X) w^{\mathrm{IV},-}(X)\, dX \quad &= \quad 2^{-n} \delta_{kk'}, \\
\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{\mathrm{IV},+}(X) \mathcal{P}_{k'}^{\mathrm{IV},+}(X) w^{\mathrm{IV},+}(X)\, dX \quad &= \quad 2^{-n} H_k \delta_{kk'}.
\end{aligned} \tag{2.17}$$

Furthermore, the explicit formulas of the weight functions for the trivariate case are given in [A1, Example 5].

Cubature Formulas

The continuous orthogonality relations and the properties of the AMDSTs and SMDSTs yield the cubature formulas.

The φ-images of the point sets $F_N^{*,\pm}$ of multivariate discrete sine transforms, explicitly given in [A1, Table 1], are denoted as $\mathfrak{F}_N^{*,\pm}$, i.e.

$$\mathfrak{F}_N^{*,\pm} = \left\{ \varphi(a) \in \mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}}) | a \in F_N^{*,\pm} \right\}. \tag{2.18}$$

Four cubature formulas related to the polynomials $\mathcal{P}_k^{\mathrm{II},-}$ are derived as:

$$\begin{aligned}
\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y) w^{\mathrm{II},-}(Y)\, dY &= \left(\frac{1}{N+1}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{I},-}} f(Y) \mathcal{J}^{\mathrm{II},-}(Y), \\
\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y) w^{\mathrm{II},-}(Y)\, dY &= \left(\frac{1}{N}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{II},-}} f(Y) \mathcal{J}^{\mathrm{II},-}(Y), \\
\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y) w^{\mathrm{II},-}(Y)\, dY &= \left(\frac{2}{2N+1}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{V},-}} f(Y) \mathcal{J}^{\mathrm{II},-}(Y), \\
\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y) w^{\mathrm{II},-}(Y)\, dY &= \left(\frac{2}{2N+1}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{VI},-}} f(Y) \mathcal{J}^{\mathrm{II},-}(Y).
\end{aligned} \tag{2.19}$$

The formulas for the remaining polynomials $\mathcal{P}_k^{\mathrm{II},+}$ and $\mathcal{P}_k^{\mathrm{IV},\pm}$ as well as the degrees of precision of the formulas are formulated in [A1, Theorems 1-4]. Furthermore, each class of the cubature formulas contains an optimal Gaussian cubature formula, which requires minimal number of points to be evaluated at [A1, Theorem 5]. In the bivariate case, the cubature formulas, together with the explicit form of the weight functions, are deduced in [9].

Chapter 3

Discrete Fourier–Weyl Transforms

Discrete Fourier–Weyl transforms [20] are closely connected to the multivariate (anti)symmetric discrete sine and cosine transforms. These transforms are defined by finite groups of geometrical symmetries known as Weyl groups and associated with semisimple Lie algebras and their Coxeter-Dynkin diagrams [53, 54]. The aim of this section is to describe the notation used in the articles [A2–A4]. The notation is established in [20, 38, 44, 47], expanding the theory from [53, 54].

3.1 Weyl Groups and Root Systems

Each simple Lie algebra belongs to one of the four series $A_n\,(n \geq 1)\,, B_n\,(n \geq 3)\,, C_n\,(n \geq 2)\,, D_n\,(n \geq 4)$ or one of the exceptional cases E_6, E_7, E_8, F_4, G_2 and is associated with the set of the simple roots $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. For the simple Lie algebras which have two different root-lengths (cases B_n, C_n, F_4, G_2), the set of simple roots is disjointly divided into the set of short simple roots Δ_s and set of long simple roots Δ_l, i.e.

$$\Delta = \Delta_s \cup \Delta_l, \quad \Delta_s \cap \Delta_l = \emptyset. \tag{3.1}$$

The vector set Δ forms a basis for an Euclidean space $\mathbb{R}^n$ equipped with the standard scalar product $\langle \cdot, \cdot \rangle$. To each simple root $\alpha_i \in \Delta$ corresponds a reflection r_i given by formula:

$$r_i a = a - 2\frac{\langle a, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i, \quad a \in \mathbb{R}^n. \tag{3.2}$$

The set of reflections $\{r_1, r_2, ..., r_n\}$ corresponding to the simple root set Δ generates the Weyl group W which produces the entire root system Π of the Lie algebra by

$$\Pi = W\Delta. \tag{3.3}$$

Dual simple root $\alpha_i^\vee$ is connected to the simple root $\alpha_i \in \Delta$ by the following relation:

$$\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}. \tag{3.4}$$

The set of dual simple roots $\Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\}$ is also associated with a Lie algebra which is dual to the Lie algebra linked to the simple root set Δ. Furthermore, the Weyl group generated from dual simple roots $\Delta^\vee$ contains the same reflections as the Weyl group obtained from the simple roots Δ. The Weyl group applied on the dual simple root system $\Delta^\vee$ spans the entire dual root system $\Pi^\vee$ of the corresponding dual Lie algebra, i.e.

$$\Pi^\vee = W\Delta^\vee. \tag{3.5}$$

There exist a unique highest root $\xi \in \Pi$ and highest dual root $\eta \in \Pi^\vee$ given by

$$\xi = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n, \tag{3.6}$$

$$\eta = m_1^\vee\alpha_1^\vee + m_2^\vee\alpha_2^\vee + \cdots + m_n^\vee\alpha_n^\vee, \tag{3.7}$$

where the marks $m_1, m_2, ..., m_n$ and dual marks $m_1^\vee, m_2^\vee, ..., m_n^\vee$ can be found in [47, Table 1] for each simple Lie algebra.

For each Lie algebra, the fundamental weights ω_i and dual fundamental weights $\omega_i^\vee$ are given by

$$\begin{aligned}\langle \omega_i, \alpha_j^\vee \rangle &= \delta_{ij},\\ \langle \omega_i^\vee, \alpha_j \rangle &= \delta_{ij}.\end{aligned} \tag{3.8}$$

Weyl Group Invariant Lattices

Each Weyl group leads to four Weyl group invariant lattices. The root lattice Q is given as a $\mathbb{Z}$-span of the simple roots Δ and the dual weight lattice $P^\vee$ is defined as $\mathbb{Z}$-dual of the root lattice, i.e

$$\begin{aligned}Q =& \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \cdots + \mathbb{Z}\alpha_n,\\ P^\vee =& \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee + \cdots + \mathbb{Z}\omega_n^\vee.\end{aligned} \tag{3.9}$$

Furthermore, the cone of positive dual weights is given by

$$P^{\vee+} = \mathbb{Z}^{\geq 0}\omega_1^\vee + \mathbb{Z}^{\geq 0}\omega_2^\vee + \cdots + \mathbb{Z}^{\geq 0}\omega_n^\vee. \tag{3.10}$$

Analogously, the dual root lattice $Q^\vee$ is the $\mathbb{Z}$-span of the dual simple roots $\Delta^\vee$ and the weight lattice P is $\mathbb{Z}$-dual to the dual root lattice $Q^\vee$, giving the lattices in the form:

$$\begin{aligned}Q^\vee =& \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee + \cdots + \mathbb{Z}\alpha_n^\vee,\\ P =& \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \cdots + \mathbb{Z}\omega_n.\end{aligned} \tag{3.11}$$

The order of the quotient groups P/Q and $P^\vee/Q^\vee$ as well as the index of connection of the root system Π are given by the determinant of Cartan matrix $C_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$,

$$c = \det C = |P/Q| = |P^\vee/Q^\vee|. \tag{3.12}$$

Admissible Shifts of the Weyl Group Invariant Lattices

Shift by a vector $\varrho \in \mathbb{R}^n$ which satisfies the invariance condition

$$W(\varrho + P) = \varrho + P \tag{3.13}$$

is called an admissible shift of the weight lattice. Equivalently, a shift by a vector $\nu^\vee \in \mathbb{R}^n$ which satisfies similar condition for the dual root lattice:

$$W(\nu^\vee + Q^\vee) = \nu^\vee + Q^\vee \tag{3.14}$$

represents an admissible shift of the dual root lattice. Correspondingly, a shift by a vector $\rho^\vee \in \mathbb{R}^n$ satisfying

$$W\left(\rho^\vee + P^\vee\right) = \rho^\vee + P^\vee \tag{3.15}$$

is called an admissible shift of the dual weight lattice. The admissible shifts which lead to the same lattice are said to be equivalent. All possible shifts of dual root lattice are listed in [20, Table 1].

Sign Homomorphisms

Any homomorphism $\sigma : W \to U_2$ from the Weyl group W to the multiplicative group $U_2 = \{\pm 1\}$ is called a sign homomorphism [44]. There exist up to four types of sign homomorphisms on the Weyl group. The identity 1 and determinant σ^e exist on any Weyl group W and are given on the reflections which generate the Weyl group as follows:

$$\begin{aligned} 1\left(r_i\right) &= 1, \\ \sigma^e\left(r_i\right) &= -1. \end{aligned} \tag{3.16}$$

For the Lie algebras with two lengths of the roots, the short σ^s and long σ^l sign homomorphisms are additionally defined on the set of the short Δ_s and long Δ_l roots as:

$$\sigma^s(r_i) = \begin{cases} -1, & \alpha_i \in \Delta_s, \\ 1, & \alpha_i \in \Delta_l, \end{cases} \tag{3.17}$$

$$\sigma^l(r_i) = \begin{cases} 1, & \alpha_i \in \Delta_s, \\ -1, & \alpha_i \in \Delta_l. \end{cases} \tag{3.18}$$

3.2 Affine and Dual Affine Weyl Groups

Notation in this section summarizes the notation used in [A3].

Affine Weyl Group

A semidirect product of the Weyl group W and the group of translations $Q^\vee$ is called an affine Weyl group:

$$W^{\mathrm{aff}} = Q^\vee \rtimes W. \tag{3.19}$$

Any element $z = T\left(q^\vee\right) w \in W^{\mathrm{aff}}$ of the affine Weyl group, where $w \in W$ and $q^\vee \in Q^\vee$, acts canonically on $a \in \mathbb{R}^n$ as

$$z \cdot a = wa + q^\vee. \tag{3.20}$$

The fundamental domain $F \subset \mathbb{R}^n$ of W^{aff} is given explicitly as

$$F = \left\{a_1 w_1^\vee + \ldots + a_n w_n^\vee | a_0 + m_1 a_1 + \ldots + m_n a_n = 1, a_i \geq 0, i = 0, \ldots, n\right\}. \tag{3.21}$$

The mapping $\tau : W^{\mathrm{aff}} \to Q^\vee$ and retraction homomorphism $\psi : W^{\mathrm{aff}} \to W$ are defined by

$$\tau\left(z\right) = q^\vee, \quad \psi\left(z\right) = w. \tag{3.22}$$

The dual shift homomorphism $\theta_\varrho : W^{\mathrm{aff}} \to U_2$ is, for any admissible shift ϱ of the weight lattice P, induced in [20] by the mapping τ

$$\theta_\varrho(z) = e^{2\pi \mathrm{i} \langle \tau(z), \varrho \rangle}. \tag{3.23}$$

For any sign homomorphism σ and any admissible shift ϱ of the weight lattice, the homomorphism $\gamma_\varrho^\sigma : W^{\mathrm{aff}} \to U_2$ is determined for $z \in W^{\mathrm{aff}}$ as:

$$\gamma_\varrho^\sigma(z) = \theta_\varrho(z) \left[\sigma \circ \psi(z)\right]. \tag{3.24}$$

The stabilizer $\mathrm{Stab}_{W^{\mathrm{aff}}}(a)$ is formed by the elements of W^{aff} that stabilize $a \in \mathbb{R}^n$, and the associated ε-function $\varepsilon : \mathbb{R}^n \to \mathbb{N}$ is defined by

$$\varepsilon(a) = \frac{|W|}{|\mathrm{Stab}_{W^{\mathrm{aff}}}(a)|}. \tag{3.25}$$

For any $a \in \mathbb{R}^n$, there exist precisely one point $a' \in F$ and $z[a] \in W^{\mathrm{aff}}$ such that

$$a = z[a] \cdot a'. \tag{3.26}$$

The function $\chi_\varrho^\sigma : \mathbb{R}^n \to \{-1, 0, 1\}$ is defined for any $a \in \mathbb{R}^n$ as:

$$\chi_\varrho^\sigma(a) = \begin{cases} \gamma_\varrho^\sigma(z[a]), & \gamma_\varrho^\sigma(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)) = 1, \\ 0, & \gamma_\varrho^\sigma(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)) = U_2. \end{cases} \tag{3.27}$$

For each sign homomorphism σ and admissible shift ϱ of the weight lattice, there exists a signed fundamental domain $F^\sigma(\varrho) \subset F$ given by

$$F^\sigma(\varrho) = \left\{ a \in F \mid \chi_\varrho^\sigma(a) = 1 \right\}. \tag{3.28}$$

Two subsets $H^\sigma(\varrho)$ and $B^\sigma(\varrho)$ of the boundary ∂F are defined by

$$H^\sigma(\varrho) = \left\{ a \in \partial F \mid \chi_\varrho^\sigma(a) = 0 \right\}, \tag{3.29}$$

$$B^\sigma(\varrho) = \left\{ a \in \partial F \mid \chi_\varrho^\sigma(a) = 1 \right\}, \tag{3.30}$$

which allows to decompose the fundamental domain F and its boundary ∂F [20, Prop. 2.7] as follows:

$$\begin{aligned} F &= F^\sigma(\varrho) \cup H^\sigma(\varrho), \\ \partial F &= B^\sigma(\varrho) \cup H^\sigma(\varrho). \end{aligned} \tag{3.31}$$

Dual Affine Weyl Group

The dual affine Weyl group W_Q^{aff} is a semidirect product of the Weyl group W and group of shifts from the root lattice Q,

$$W_Q^{\mathrm{aff}} = Q \rtimes W. \tag{3.32}$$

The dual affine Weyl group W_Q^{aff} is denoted as $\widehat{W}^{\mathrm{aff}}$ in [A4].

For any translation $q \in Q$ and any $w \in W$, the element $y = T(q)\, w \in W_Q^{\mathrm{aff}}$ of the dual affine Weyl group acts canonically on $b \in \mathbb{R}^n$ by formula:

$$y \cdot b = wb + q, \quad b \in \mathbb{R}^n. \tag{3.33}$$

The dual fundamental domain $F_Q \subset \mathbb{R}^n$ of W_Q^{aff} is given by

$$F_Q = \{b_1 w_1 + \ldots + b_n w_n | b_0 + m_1^\vee b_1 + \ldots + m_n^\vee b_n = 1, b_i \geq 0, i = 0, \ldots, n\}. \tag{3.34}$$

The subgroup of W_Q^{aff} that consists of elements which stabilize $b \in \mathbb{R}^n$ is denoted by $\mathrm{Stab}_{W_Q^{\mathrm{aff}}}(b)$ and the associated discrete function $h_{Q,M} : \mathbb{R}^n \to \mathbb{N}$ is given by

$$h_{Q,M}(b) = \left| \mathrm{Stab}_{W_Q^{\mathrm{aff}}} \left(\frac{b}{M} \right) \right|. \tag{3.35}$$

The mapping $\widehat{\tau} : W^{\mathrm{aff}} \to Q$ and dual retraction homomorphism $\widehat{\psi} : W_Q^{\mathrm{aff}} \to W$ are defined as follows,

$$\widehat{\tau}(z) = q, \quad \widehat{\psi}(z) = w. \tag{3.36}$$

As in the case of the affine Weyl group, the mapping $\widehat{\tau}$ induces, for any admissible shift $\varrho^\vee$ of the dual weight lattice $P^\vee$, a shift homomorphism $\widehat{\theta}_\varrho : W_Q^{\mathrm{aff}} \to U_2$ defined in [20] as

$$\widehat{\theta}_{\varrho^\vee}(y) = e^{2\pi \mathrm{i} \langle \widehat{\tau}(y), \varrho^\vee \rangle}. \tag{3.37}$$

For any sign homomorphism σ and admissible shift $\varrho^\vee$ of the dual weight lattice $P^\vee$, the homomorphism $\widehat{\gamma}^\sigma_{\varrho^\vee} : W_Q^{\mathrm{aff}} \to U_2$ is constructed for $y \in W_Q^{\mathrm{aff}}$ as the following product:

$$\widehat{\gamma}^\sigma_{\varrho^\vee}(y) = \widehat{\theta}_{\varrho^\vee}(y) \left[\sigma \circ \widehat{\psi}(y) \right]. \tag{3.38}$$

The associated signed dual fundamental domain $F_Q^\sigma(\varrho^\vee) \subset F_Q$ is given by

$$F_Q^\sigma(\varrho^\vee) = \left\{ b \in F_Q \,|\, \widehat{\gamma}^\sigma_{\varrho^\vee}(\mathrm{Stab}_{W_Q^{\mathrm{aff}}}(b)) = 1 \right\}. \tag{3.39}$$

Extended Dual Affine Weyl Group

The extended dual affine Weyl group is a semidirect product of the Weyl group W and shifts from the weight lattice P given by:

$$W_P^{\mathrm{aff}} = P \rtimes W. \tag{3.40}$$

For any translation $p \in P$ and any $w \in W$, the element $y = T(p)\, w \in W_P^{\mathrm{aff}}$ of the extended dual affine Weyl group acts canonically on $b \in \mathbb{R}^n$ by formula:

$$y \cdot b = wb + p. \tag{3.41}$$

To each b from the dual fundamental domain F_Q are assigned its Kac coordinates $[b_0, \ldots, b_n]$ obtained from the defining relation (3.34). Lexicographical ordering is introduced for any $b, b' \in F_Q$ as $b >_{\mathrm{lex}} b'$, if for the first $b_i \neq b'_i$ holds $b_i > b'_i$.

The extended dual affine Weyl group W_P^{aff} is expressed as a semidirect product of the dual affine Weyl group W_Q^{aff} and the finite abelian subgroup Γ that consists of elements which leave the dual fundamental domain F_Q invariant,

$$\Gamma = \left\{ y \in W_P^{\mathrm{aff}} \,|\, y \cdot F_Q = F_Q \right\}. \tag{3.42}$$

The fundamental domain $F_P \subset F_Q$ of W_P^{aff} is defined as a set containing the lexicographically highest point from each Γ-orbit of F_Q, i.e.

$$F_P = \{ b \in F_Q \,|\, b = \max_{>_{\mathrm{lex}}} \Gamma b \}. \tag{3.43}$$

The discrete function $h_{P,M}(b) : \mathbb{R}^n \to \mathbb{N}$ is for $M \in \mathbb{N}$ defined employing the stabilizer $\mathrm{Stab}_{W_P^{\mathrm{aff}}}(b)$, which is a subgroup of W_P^{aff} stabilizing $b \in \mathbb{R}^n$, as:

$$h_{P,M}(b) = \left| \mathrm{Stab}_{W_P^{\mathrm{aff}}} \left(\frac{b}{M} \right) \right|. \tag{3.44}$$

The dual retraction homomorphism $\widehat{\psi} : W_P^{\mathrm{aff}} \to W$ and the mapping $\widehat{\tau} : W_P^{\mathrm{aff}} \to P$ are defined for any $y = T(p)w \in W_P^{\mathrm{aff}}$ as an extension of homomorphism $\widehat{\psi}$ and mapping $\widehat{\tau}$ (3.36) to W_P^{aff},

$$\widehat{\psi}(y) = w, \quad \widehat{\tau}(y) = p. \tag{3.45}$$

The shift homomorphism $\widehat{\theta}_{\nu^\vee} : W_P^{\mathrm{aff}} \to U_c$ from the extended dual affine Weyl group to the multiplicative group of $c-$th roots of unity U_c, corresponding to an admissible shift $\nu^\vee$ of the dual root lattice $Q^\vee$, is defined in [20] as:

$$\widehat{\theta}_{\nu^\vee}(y) = e^{2\pi \mathrm{i} \langle \widehat{\tau}(y),\, \nu^\vee \rangle}. \tag{3.46}$$

The multiplicative group U_c^* is introduced as:

$$U_c^* = \begin{cases} U_c, & c \text{ even}, \\ U_{2c}, & c \text{ odd}. \end{cases} \tag{3.47}$$

For any admissible shift $\nu^\vee$ of the dual root lattice and sign homomorphism σ, the homomorphism $\widehat{\gamma}^\sigma_{\nu^\vee} : W_P^{\mathrm{aff}} \to U_c^*$ is induced by

$$\widehat{\gamma}^\sigma_{\nu^\vee}(y) = \widehat{\theta}_{\nu^\vee}(y) \left[\sigma \circ \widehat{\psi}(y) \right]. \tag{3.48}$$

The signed fundamental domain $F_P^\sigma(\nu^\vee) \subset F_P$ is then defined by

$$F_P^\sigma(\nu^\vee) = \left\{ b \in F_P \;\middle|\; \widehat{\gamma}^\sigma_{\nu^\vee} \left(\mathrm{Stab}_{W_P^{\mathrm{aff}}}(b) \right) = 1 \right\}. \tag{3.49}$$

3.3 Weyl Orbit Functions

Weyl orbit functions [61, 64] are a class of special functions defined as (anti)symmetrized sums of exponential functions, where the (anti)symmetrization is performed with respect to the Weyl group. These functions have many remarkable properties and are necessary for the construction of kernels for the discrete Fourier–Weyl transforms.

The Weyl orbit functions $\varphi_b^\sigma : \mathbb{R}^n \to \mathbb{C}$ are determined for an argument $a \in \mathbb{R}^n$ and label $b \in \mathbb{R}^n$ by the relation

$$\varphi_b^\sigma(a) = \sum_{w \in W} \sigma(w)\, e^{2\pi \mathrm{i} \langle wb,\, a \rangle}. \tag{3.50}$$

The Weyl orbit functions satisfy the following product-to-sum decomposition formula of the form

$$\varphi_b^\sigma(a)\varphi_b^{\mathbf{1}}(a') = \sum_{w \in W} \varphi_b^\sigma(a + wa'). \tag{3.51}$$

For discretized labels $\lambda \in \varrho + P$ are the Weyl orbit functions W^{aff}-(anti)symmetric and the following formula holds for any $a \in \mathbb{R}^n$ and $z \in W^{\mathrm{aff}}$

$$\varphi_\lambda^\sigma(z \cdot a) = \gamma_\varrho^\sigma(z)\, \varphi_\lambda^\sigma(a). \tag{3.52}$$

Furthermore, the Weyl orbit functions φ_λ^σ vanish on the boundary $H^\sigma(\varrho)$,

$$\varphi_\lambda^\sigma(a') = 0, \quad a' \in H^\sigma(\varrho). \tag{3.53}$$

For any $a' \in F$ and $a \in W^{\mathrm{aff}} a'$, the properties (3.52) and (3.53) allow to obtain following equation with the use of the χ_ϱ^σ-function

$$\varphi_\lambda^\sigma(a) = \chi_\varrho^\sigma(a)\varphi_\lambda^\sigma(a'). \tag{3.54}$$

When the summation in relation (3.50) is restricted to the Weyl orbit of the label b, the C-functions [64] arise for the sign homomorphism $\mathbf{1}$:

$$C_b(a) = \sum_{\mu \in Wb} e^{2\pi \mathrm{i} \langle \mu,\, a \rangle}. \tag{3.55}$$

3.4 Discretization of Weyl Orbit Functions

Based on the properties of Weyl orbit functions the dual weight and dual root lattice Fourier–Weyl transforms are derived. The label and point sets for these transforms are finite discrete subsets obtained as intersections of the fundamental domains F^σ, F_Q^σ and F_P^σ with rescaled dual root or dual weight lattices. Furthermore, the unitary matrices of these transforms are formulated.

Dual Root Lattice Fourier–Weyl Transforms

For any sign homomorphism $\sigma \in \left\{\mathbf{1}, \sigma^e, \sigma^s, \sigma^l\right\}$, admissible shift of the weight lattice ϱ, admissible shift of the dual root lattice $\nu^\vee$ and $M \in \mathbb{N}$, the finite set of points $F_{Q^\vee,M}^\sigma\left(\varrho, \nu^\vee\right)$ inside the fundamental domain $F^\sigma\left(\varrho\right)$ is defined as a subset of rescaled and shifted dual root lattice

$$F_{Q^\vee,M}^\sigma(\varrho, \nu^\vee) = \tfrac{1}{M}(\nu^\vee + Q^\vee) \cap F^\sigma(\varrho). \tag{3.56}$$

The finite set of labels $\Lambda_{Q^\vee,M}^\sigma\left(\varrho, \nu^\vee\right)$ of orbit functions is given as an intersection of shifted weight lattice and rescaled fundamental domain $F_P^\sigma(\nu^\vee)$

$$\Lambda_{P,M}^\sigma(\varrho, \nu^\vee) = (\varrho + P) \cap M F_P^\sigma(\nu^\vee). \tag{3.57}$$

The cardinalities of the label and point sets coincide [20, Theorem 2]:

$$\left|\Lambda^{\sigma}_{P,M}(\varrho,\nu^{\vee})\right| = \left|F^{\sigma}_{Q^{\vee},M}(\varrho,\nu^{\vee})\right|.$$

The discrete orthogonality of the Weyl orbit functions on the point set $F^{\sigma}_{Q^{\vee},M}(\varrho,\nu^{\vee})$ holds for any labels $\lambda,\lambda' \in \Lambda^{\sigma}_{Q^{\vee},M}(\varrho,\nu^{\vee})$, [20, Theorem 4],

$$\sum_{a\in F^{\sigma}_{Q^{\vee},M}(\varrho,\nu^{\vee})} \varepsilon(a)\,\varphi^{\sigma}_{\lambda}(a)\varphi^{\sigma *}_{\lambda'}(a) = |W|\,M^n h_{P,M}(\lambda)\,\delta_{\lambda,\lambda'}. \tag{3.58}$$

The unitary matrices $\mathbb{I}^{\sigma}_{Q^{\vee},M}(\varrho,\nu^{\vee})$ of the generalized discrete dual root lattice Fourier–Weyl transforms are, for fixed ordering of the point and label sets $F^{\sigma}_{Q^{\vee},M}(\varrho,\nu^{\vee})$ and $\Lambda^{\sigma}_{P,M}(\varrho,\nu^{\vee})$, determined by the entries $\lambda\in\Lambda^{\sigma}_{P,M}(\varrho,\nu^{\vee})$ and $a\in F^{\sigma}_{Q^{\vee},M}(\varrho,\nu^{\vee})$ as follows:

$$\mathbb{I}^{\sigma}_{Q^{\vee},M}(\varrho,\nu^{\vee})_{\lambda a} = \sqrt{\frac{\varepsilon(a)}{|W|\,M^n h_{P,M}(\lambda)}}\varphi^{\sigma *}_{\lambda}(a). \tag{3.59}$$

Dual Weight Lattice Fourier–Weyl Transforms

For any sign homomorphism $\sigma\in\{\mathfrak{1},\,\sigma^e,\,\sigma^s,\,\sigma^l\}$ and admissible shifts of the weight and dual weight lattices ϱ and $\varrho^{\vee}$, the finite set of points $F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$ inside the fundamental domain $F^{\sigma}(\varrho)$ contains the points from the rescaled and shifted dual weight lattice,

$$F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee}) = \tfrac{1}{M}(\varrho^{\vee}+P^{\vee})\cap F^{\sigma}(\varrho). \tag{3.60}$$

The finite set of labels $\Lambda^{\sigma}_{Q,M}(\varrho,\varrho^{\vee})$ of Weyl orbit functions is defined by

$$\Lambda^{\sigma}_{Q,M}(\varrho,\varrho^{\vee}) = (\varrho+P)\cap MF^{\sigma\vee}(\varrho^{\vee}). \tag{3.61}$$

The cardinalities of the point and label sets coincide [18, Thm. 3.4],

$$\left|\Lambda^{\sigma}_{Q,M}(\varrho,\varrho^{\vee})\right| = \left|F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})\right|. \tag{3.62}$$

Restricted to the finite point sets $F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$, the Weyl orbit functions with any labels $\lambda,\lambda'\in\Lambda^{\sigma}_{Q,M}(\varrho,\varrho^{\vee})$ are discretely orthogonal [18, Theorem 4], i.e.

$$\sum_{a\in F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})} \varepsilon(a)\,\varphi^{\sigma}_{\lambda}(a)\varphi^{\sigma *}_{\lambda'}(a) = c\,|W|\,M^n h^{\vee}_{M}(\lambda)\,\delta_{\lambda,\lambda'}. \tag{3.63}$$

For any points $a,a'\in F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$, the corresponding Plancherel formulas [18] lead to the following complementary orthogonality relations

$$\sum_{\lambda\in\Lambda^{\sigma}_{Q,M}(\varrho,\varrho^{\vee})} \left(h^{\vee}_{M}(\lambda)\right)^{-1}\,\varphi^{\sigma}_{\lambda}(a)\varphi^{\sigma *}_{\lambda}(a') = c\,|W|\,M^n\varepsilon^{-1}(a)\,\delta_{a,a'}. \tag{3.64}$$

Assuming a fixed ordering of the label and point sets $\Lambda^{\sigma}_{Q,M}(\varrho,\varrho^{\vee})$ and $F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$, the unitary matrices of the dual weight lattice Fourier–Weyl transforms are determined by the entries for $\lambda\in\Lambda^{\sigma}_{Q,M}(\varrho,\varrho^{\vee})$ and $a\in F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$ as

$$\mathbb{I}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})_{\lambda a} = \sqrt{\frac{\varepsilon(a)}{c\,|W|\,M^n h^{\vee}_{M}(\lambda)}}\varphi^{\sigma *}_{\lambda}(a). \tag{3.65}$$

Chapter 4

Connecting Multivariate Sine and Cosine Transforms to Fourier–Weyl Transforms

The aim of this chapter is to show link between the (anti)symmetric multivariate sine and cosine transforms and the dual root lattice Fourier–Weyl transforms defined in Chapters 1 and 3. Since the Weyl orbit functions of A_1 coincide with classical sine and cosine functions, the connection of Fourier–Weyl transforms to the transforms (1.15) and (1.16) is anticipated. This section is summarizing the advancements achieved in [A2], coupling together the label and point sets as well as the normalization and weight functions and unitary matrices of the (anti)symmetric multivariate sine and cosine transforms and dual root lattice Fourier–Weyl transforms of algebras A_1 and $C_n, n \geq 2$.

4.1 Discrete Fourier–Weyl Transforms of $\mathcal{C}_n$ Series

$\mathcal{C}_n$ Series

This section further focuses on the crystallographic root systems A_1 and $C_n, n \geq 2$ which are for simplicity of notation unified under the $\mathcal{C}_n, n \in \mathbb{N}$ series:

$$\mathcal{C}_n = \begin{cases} A_1, & n = 1, \\ C_n, & n \geq 2 \end{cases} \tag{4.1}$$

and are connected to the (anti)symmetric multivariate sine and cosine transforms.

In order to simply describe the properties of the $\mathcal{C}_n$ series, the ordered set of indices I_n, together with its extension $\widehat{I}_n$, is introduced as

$$\begin{aligned} I_n &= \{1, \dots, n\}, \\ \widehat{I}_n &= \{0, \dots, n\}. \end{aligned} \tag{4.2}$$

The root system of the $\mathcal{C}_n$ series is characterized by the lengths of the simple roots,

$$\langle \alpha_i, \alpha_i \rangle = 1, \quad i \in I_{n-1}, \qquad \langle \alpha_n, \alpha_n \rangle = 2, \tag{4.3}$$

and for $n \geq 2$ by the relative angles between them,

$$
\begin{aligned}
\langle \alpha_i,\, \alpha_{i+1}\rangle &= -\tfrac{1}{2}, \quad i \in I_{n-2},\\
\langle \alpha_{n-1},\, \alpha_n\rangle &= -1,\\
\langle \alpha_i,\, \alpha_j\rangle &= 0, \quad |i-j|>1,\, i,j \in I_n.
\end{aligned} \tag{4.4}
$$

The highest root ξ and highest dual root η are for $n=1$ of the form $\xi = \alpha_1$ and $\eta = \alpha_1^\vee$ and for $n \geq 2$ of the form

$$
\begin{aligned}
\xi &= 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n,\\
\eta &= \alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_n^\vee.
\end{aligned} \tag{4.5}
$$

Due to the lengths of the simple roots (4.3), the dual simple roots and fundamental weights obey the following relations:

$$
\begin{aligned}
\alpha_i^\vee &= 2\alpha_i, \quad i \in I_{n-1}, \qquad \alpha_n^\vee = \alpha_n,\\
\omega_i^\vee &= 2\omega_i, \quad i \in I_{n-1}, \qquad \omega_n^\vee = \omega_n.
\end{aligned} \tag{4.6}
$$

The admissible shift ϱ of the weight lattice P (3.13) and the admissible shift $\nu^\vee$ of the dual root lattice $Q^\vee$ (3.14) are limited for the $\mathcal{C}_n$ series [20, Table 1]:

$$
\varrho \in \left\{0, \tfrac{1}{2}\omega_n\right\}, \qquad \nu^\vee \in \left\{0, \omega_n^\vee\right\}. \tag{4.7}
$$

Dual Root Fourier–Weyl Transforms of the $\mathcal{C}_n$ Series

For describing the dual root Fourier–Weyl label sets $\Lambda^\sigma_{P,M}(\varrho,\nu^\vee)$ of the $\mathcal{C}_n$ series, the symbols $\lambda_i^{\sigma,\varrho}$ are for $i \in I_{n-1}$ introduced by

$$
\lambda_i^{1,\varrho} \in \mathbb{Z}^{\geq 0}, \quad \lambda_i^{\sigma^e,\varrho} \in \mathbb{N}, \quad \lambda_i^{\sigma^s,\varrho} \in \mathbb{N}, \quad \lambda_i^{\sigma^l,\varrho} \in \mathbb{Z}^{\geq 0} \tag{4.8}
$$

and $\lambda_n^{\sigma,\varrho}$ is given by

$$
\lambda_n^{\sigma,\varrho} \in \begin{cases} \mathbb{Z}^{\geq 0}, & \varrho = 0,\, \sigma \in \{1, \sigma^s\},\\ \mathbb{N}, & \varrho = 0,\, \sigma \in \left\{\sigma^e, \sigma^l\right\},\\ \tfrac{1}{2} + \mathbb{Z}^{\geq 0}, & \varrho = \tfrac{1}{2}\omega_n,\, \sigma \in \left\{1, \sigma^e, \sigma^s, \sigma^l\right\}. \end{cases} \tag{4.9}
$$

The precise form of the label sets $\Lambda^\sigma_{P,M}(\varrho,\nu^\vee)$ (3.57) of the $\mathcal{C}_n$ series is given for $\sigma \in \{1, \sigma^s\}$ as

$$
\begin{aligned}
\Lambda^\sigma_{P,M}(\varrho,0) &= \{\lambda_1^{\sigma,\varrho}\omega_1 + \cdots + \lambda_n^{\sigma,\varrho}\omega_n \mid \lambda_0^{\sigma,\varrho} + \lambda_1^{\sigma,\varrho} + 2\lambda_2^{\sigma,\varrho} + \cdots + 2\lambda_n^{\sigma,\varrho} = M,\, \lambda_0^{\sigma,\varrho} \geq \lambda_1^{\sigma,\varrho}\},\\
\Lambda^\sigma_{P,M}(\varrho,\omega_n^\vee) &= \{\lambda_1^{\sigma,\varrho}\omega_1 + \cdots + \lambda_n^{\sigma,\varrho}\omega_n \mid \lambda_0^{\sigma,\varrho} + \lambda_1^{\sigma,\varrho} + 2\lambda_2^{\sigma,\varrho} + \cdots + 2\lambda_n^{\sigma,\varrho} = M,\, \lambda_0^{\sigma,\varrho} > \lambda_1^{\sigma,\varrho}\}
\end{aligned} \tag{4.10}
$$

and for $\sigma \in \{\sigma^e, \sigma^l\}$ as

$$
\begin{aligned}
\Lambda^\sigma_{P,M}(\varrho,0) &= \{\lambda_1^{\sigma,\varrho}\omega_1 + \cdots + \lambda_n^{\sigma,\varrho}\omega_n \mid \lambda_0^{\sigma,\varrho} + \lambda_1^{\sigma,\varrho} + 2\lambda_2^{\sigma,\varrho} + \cdots + 2\lambda_n^{\sigma,\varrho} = M,\, \lambda_0^{\sigma,\varrho} > \lambda_1^{\sigma,\varrho}\},\\
\Lambda^\sigma_{P,M}(\varrho,\omega_n^\vee) &= \{\lambda_1^{\sigma,\varrho}\omega_1 + \cdots + \lambda_n^{\sigma,\varrho}\omega_n \mid \lambda_0^{\sigma,\varrho} + \lambda_1^{\sigma,\varrho} + 2\lambda_2^{\sigma,\varrho} + \cdots + 2\lambda_n^{\sigma,\varrho} = M,\, \lambda_0^{\sigma,\varrho} \geq \lambda_1^{\sigma,\varrho}\}.
\end{aligned} \tag{4.11}
$$

The symbols $s_0^{\sigma,\varrho}$ are introduced by

$$s_0^{\sigma,0} \in \begin{cases} \mathbb{Z}^{\geq 0}, & \sigma \in \{\imath, \sigma^s\}, \\ \mathbb{N}, & \sigma \in \left\{\sigma^e, \sigma^l\right\}, \end{cases} \tag{4.12}$$

$$s_0^{\sigma,\frac{1}{2}\omega_n} \in \begin{cases} \mathbb{N}, & \sigma \in \{\imath, \sigma^s\}, \\ \mathbb{Z}^{\geq 0}, & \sigma \in \left\{\sigma^e, \sigma^l\right\} \end{cases} \tag{4.13}$$

and the symbols $s_i^{\sigma,\rho}$, $i \in I_n$ by

$$\begin{aligned} & s_i^{\imath,\varrho} \in \mathbb{Z}^{\geq 0}, && i \in I_n, && \\ & s_i^{\sigma^e,\varrho} \in \mathbb{N}, && i \in I_n, && \\ & s_i^{\sigma^s,\varrho} \in \mathbb{N}, && i \in I_{n-1}, && s_n^{\sigma^s,\varrho} \in \mathbb{Z}^{\geq 0}, \\ & s_i^{\sigma^l,\varrho} \in \mathbb{Z}^{\geq 0}, && i \in I_{n-1}, && s_n^{\sigma^l,\varrho} \in \mathbb{N}, \end{aligned} \tag{4.14}$$

giving the dual root Fourier–Weyl point sets $F^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ (3.56) of the $\mathcal{C}_n$ series in form:

$$F^\sigma_{Q^\vee,M}(\varrho,0) = \left\{ \frac{s_1^{\sigma,\varrho}}{M}\omega_1^\vee + \dots + \frac{s_n^{\sigma,\varrho}}{M}\omega_n^\vee \mid s_0^{\sigma,\varrho} + 2s_1^{\sigma,\varrho} + \dots + 2s_{n-1}^{\sigma,\varrho} + s_n^{\sigma,\varrho} = M,\ s_n^{\sigma,\rho} = 0 \bmod 2 \right\},$$

$$F^\sigma_{Q^\vee,M}(\varrho,\omega_n^\vee) = \left\{ \frac{s_1^{\sigma,\varrho}}{M}\omega_1^\vee + \dots + \frac{s_n^{\sigma,\varrho}}{M}\omega_n^\vee \mid s_0^{\sigma,\varrho} + 2s_1^{\sigma,\varrho} + \dots + 2s_{n-1}^{\sigma,\varrho} + s_n^{\sigma,\varrho} = M,\ s_n^{\sigma,\varrho} = 1 \bmod 2 \right\}. \tag{4.15}$$

Taking into account that the order of the Weyl group $|W|$ of the $\mathcal{C}_n$ series equals $2^n n!$, the unitary matrix of the dual root lattice Fourier–Weyl transform (3.59) is given by its entries for $\lambda \in \Lambda^\sigma_{P,M}(\varrho,\nu^\vee)$ and $a \in F^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ as

$$\left(\mathbb{I}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)\right)_{\lambda a} = \sqrt{\frac{\varepsilon(a)}{2^n n! M^n h_{P,M}(\lambda)}}\varphi_\lambda^{\sigma *}(a). \tag{4.16}$$

4.2 Connecting Transforms

The (anti)symmetric multivariate sine and cosine functions are connected to the Weyl orbit functions of the $\mathcal{C}_n$ series. In univariate settings, the antisymmetric and symmetric sine and cosine functions coincide with the classical sine and cosine functions. For the Weyl orbit functions of $\mathcal{C}_1$ only the homomorphisms $\imath$ and σ^e are present, leading to the classical sine and cosine functions. The connection in the multivariate case is examined and the relation between the (anti)symmetric multivariate discrete trigonometric transforms and dual root lattice Fourier–Weyl transforms of the $\mathcal{C}_n$ series is shown.

Expressing the Orbit Functions as (Anti)symmetric Multivariate Trigonometric Functions

To provide the relation between the two types of functions, an orthogonal basis $\mathcal{F} = \{f_1,\dots,f_n\}$ is introduced by

$$\omega_i = f_1 + \dots + f_i, \quad i \in I_n, \tag{4.17}$$

evaluating the points and labels in the $\mathcal{F}$-basis as:

$$\begin{aligned} a &= a_1 f_1 + \cdots + a_n f_n, \\ b &= b_1 f_1 + \cdots + b_n f_n. \end{aligned} \tag{4.18}$$

The Weyl orbit functions φ^σ, $\sigma \in \left\{\mathbb{1}, \sigma^e, \sigma^s, \sigma^l\right\}$ and the (anti)symmetric multivariate sine and cosine functions $\sin^\pm$ and $\cos^\pm$ are for variable $(a_1, \dots, a_n)$ and label $(b_1, \dots, b_n)$ connected as follows:

$$\begin{aligned} \varphi_b^{\mathbb{1}}(a) &= 2^n \cos^+_{(b_1,\dots,b_n)}(a_1, \dots, a_n), \\ \varphi_b^{\sigma^s}(a) &= 2^n \cos^-_{(b_1,\dots,b_n)}(a_1, \dots, a_n), \\ \varphi_b^{\sigma^l}(a) &= (2\mathrm{i})^n \sin^+_{(b_1,\dots,b_n)}(a_1, \dots, a_n), \\ \varphi_b^{\sigma^e}(a) &= (2\mathrm{i})^n \sin^-_{(b_1,\dots,b_n)}(a_1, \dots, a_n). \end{aligned} \tag{4.19}$$

For connecting the (anti)symmetric multivariate discrete sine and cosine transforms with the dual root lattice Fourier–Weyl transforms of the $\mathcal{C}_n$ series, the following relations deduced from (4.19) are further considered:

$$\begin{aligned} c,+ &\longleftrightarrow \mathbb{1}, \\ c,- &\longleftrightarrow \sigma^s, \\ s,+ &\longleftrightarrow \sigma^l, \\ s,- &\longleftrightarrow \sigma^e. \end{aligned} \tag{4.20}$$

Label and Point Sets

The link between the (anti)symmetric multivariate sine and cosine label sets $D_N^{\star,c,\pm}$ and $D_N^{\star,s,\pm}$ and the dual root Fourier–Weyl label sets $\Lambda^\sigma_{P,M}(\varrho, \nu^\vee)$, $\sigma \in \{\mathbb{1}, \sigma^s, \sigma^l, \sigma^e\}$ formulated in [A2, Theorem 1] is given by the expressions:

$$\begin{aligned} \Lambda^{\mathbb{1}}_{P,M}(\varrho, \nu^\vee) &= \left\{\lambda_1 f_1 + \cdots + \lambda_n f_n \,|\, (\lambda_1, \dots, \lambda_n) \in D_N^{\star,c,+}\right\}, \\ \Lambda^{\sigma^s}_{P,M}(\varrho, \nu^\vee) &= \left\{\lambda_1 f_1 + \cdots + \lambda_n f_n \,|\, (\lambda_1, \dots, \lambda_n) \in D_N^{\star,c,-}\right\}, \\ \Lambda^{\sigma^l}_{P,M}(\varrho, \nu^\vee) &= \left\{\lambda_1 f_1 + \cdots + \lambda_n f_n \,|\, (\lambda_1, \dots, \lambda_n) \in D_N^{\star,s,+}\right\}, \\ \Lambda^{\sigma^e}_{P,M}(\varrho, \nu^\vee) &= \left\{\lambda_1 f_1 + \cdots + \lambda_n f_n \,|\, (\lambda_1, \dots, \lambda_n) \in D_N^{\star,s,-}\right\}, \end{aligned} \tag{4.21}$$

where the the magnifying factor $M \in \mathbb{N}$ and admissible shifts ϱ and $\nu^\vee$ are for specific type $\star \in \{\mathrm{I}, \dots, \mathrm{VIII}\}$ and magnifying factor $N \in \mathbb{N}$ determined by [A2, Table 3].

The point sets $F_N^{\star,c,\pm}$ and $F_N^{\star,s,\pm}$ of the (anti)symmetric multivariate sine and cosine transforms [A2, Table 2] are connected to the point sets $F^\sigma_{Q^\vee,M}(\varrho, \nu^\vee)$, $\sigma \in \{\mathbb{1}, \sigma^s, \sigma^l, \sigma^e\}$ of the dual root lattice Fourier–Weyl transforms (3.56) by [A2, Theorem 2] as follows:

$$\begin{aligned} F^{\mathbb{1}}_{Q^\vee,M}(\varrho, \nu^\vee) &= \left\{s_1 f_1 + \cdots + s_n f_n \,|\, (s_1, \dots, s_n) \in F_N^{\star,c,+}\right\}, \\ F^{\sigma^s}_{Q^\vee,M}(\varrho, \nu^\vee) &= \left\{s_1 f_1 + \cdots + s_n f_n \,|\, (s_1, \dots, s_n) \in F_N^{\star,c,-}\right\}, \\ F^{\sigma^l}_{Q^\vee,M}(\varrho, \nu^\vee) &= \left\{s_1 f_1 + \cdots + s_n f_n \,|\, (s_1, \dots, s_n) \in F_N^{\star,s,+}\right\}, \\ F^{\sigma^e}_{Q^\vee,M}(\varrho, \nu^\vee) &= \left\{s_1 f_1 + \cdots + s_n f_n \,|\, (s_1, \dots, s_n) \in F_N^{\star,s,-}\right\}, \end{aligned} \tag{4.22}$$

where the correspondence between the magnifying factor $M \in \mathbb{N}$ and admissible shifts ϱ and $\nu^\vee$ is for specific type $\star \in \{\mathrm{I}, \dots, \mathrm{VIII}\}$ and magnifying factor $N \in \mathbb{N}$ determined by [A2, Table 3].

Normalization and Weight Functions

For the full correspondence of the (anti)symmetric multivariate discrete sine and cosine transforms with the dual root lattice Fourier–Weyl transforms, the link between the normalization and weight functions of the transforms has to be established.

The connections of the normalization functions $h^{\star,c}$ and $h^{\star,s}$ of the (anti)symmetric multivariate discrete sine and cosine transforms and normalization functions $h_{P,M}$ of the dual root lattice Fourier–Weyl transforms are for $\lambda = \lambda_1 f_1 + \cdots + \lambda_n f_n \in \Lambda^{\sigma}_{P,M}(\varrho, \nu^{\vee})$ given by [A2, Theorem 3] as:

$$\begin{aligned}
h^{\star,c}_{(\lambda_1,\ldots,\lambda_n)} H_{(\lambda_1,\ldots,\lambda_n)} &= \left(\tfrac{M}{4}\right)^n h_{P,M}(\lambda), && (\lambda_1,\ldots,\lambda_n) \in D^{\star,c,+}_N,\\
h^{\star,c}_{(\lambda_1,\ldots,\lambda_n)} &= \left(\tfrac{M}{4}\right)^n h_{P,M}(\lambda), && (\lambda_1,\ldots,\lambda_n) \in D^{\star,c,-}_N,\\
h^{\star,s}_{(\lambda_1,\ldots,\lambda_n)} H_{(\lambda_1,\ldots,\lambda_n)} &= \left(\tfrac{M}{4}\right)^n h_{P,M}(\lambda), && (\lambda_1,\ldots,\lambda_n) \in D^{\star,s,+}_N,\\
h^{\star,s}_{(\lambda_1,\ldots,\lambda_n)} &= \left(\tfrac{M}{4}\right)^n h_{P,M}(\lambda), && (\lambda_1,\ldots,\lambda_n) \in D^{\star,s,-}_N.
\end{aligned} \tag{4.23}$$

The links of the weight functions $\varepsilon^{\star,c}$ and $\varepsilon^{\star,s}$ of the (anti)symmetric multivariate discrete sine and cosine transforms and weight $\varepsilon-$functions of the Fourier–Weyl transforms are for $s = s_1 f_1 + \cdots + s_n f_n \in F^{\sigma}_{Q^{\vee},M}(\varrho, \nu^{\vee})$ provided by [A2, Theorem 4] as follows:

$$\begin{aligned}
\varepsilon^{\star,c}_{(s_1,\ldots,s_n)} H^{-1}_{(s_1,\ldots,s_n)} &= \frac{\varepsilon(s)}{2^n n!}, && (s_1,\ldots,s_n) \in F^{\star,c,+}_N,\\
\varepsilon^{\star,c}_{(s_1,\ldots,s_n)} &= \frac{\varepsilon(s)}{2^n n!}, && (s_1,\ldots,s_n) \in F^{\star,c,-}_N,\\
\varepsilon^{\star,s}_{(s_1,\ldots,s_n)} H^{-1}_{(s_1,\ldots,s_n)} &= \frac{\varepsilon(s)}{2^n n!}, && (s_1,\ldots,s_n) \in F^{\star,s,+}_N,\\
\varepsilon^{\star,s}_{(s_1,\ldots,s_n)} &= \frac{\varepsilon(s)}{2^n n!}, && (s_1,\ldots,s_n) \in F^{\star,s,-}_N.
\end{aligned} \tag{4.24}$$

In both cases, the magnifying factor $M \in \mathbb{N}$ and admissible shifts ϱ and $\nu^{\vee}$ are for specific type $\star \in \{\mathrm{I}, \ldots, \mathrm{VIII}\}$ and magnifying factor $N \in \mathbb{N}$ given by [A2, Table 3].

Correspondence of Unitary Matrices

The links between the functions (4.19), label and point sets (4.21) and (4.22) as well as the connection between the normalization and weight functions (4.23) and (4.24) yield the correspondence of the unitary matrices of the (anti)symmetric multivariate discrete sine and cosine transforms (1.15) and (1.16) and the dual root lattice Fourier–Weyl transforms (4.16) of the $\mathcal{C}_n$ series [A2, Theorem 5] as follows:

$$\begin{aligned}
C^{\star,+}_N &= \mathbb{I}^{1}_{Q^{\vee},M}(\varrho, \nu^{\vee}),\\
C^{\star,-}_N &= \mathbb{I}^{\sigma^s}_{Q^{\vee},M}(\varrho, \nu^{\vee}),\\
S^{\star,+}_N &= \mathrm{i}^n\, \mathbb{I}^{\sigma^l}_{Q^{\vee},M}(\varrho, \nu^{\vee}),\\
S^{\star,-}_N &= \mathrm{i}^n\, \mathbb{I}^{\sigma^e}_{Q^{\vee},M}(\varrho, \nu^{\vee}),
\end{aligned} \tag{4.25}$$

where the connection between the magnifying factor $M \in \mathbb{N}$ and admissible shifts ϱ and $\nu^{\vee}$ with type $\star \in \{\mathrm{I}, \ldots, \mathrm{VIII}\}$ and magnifying factor $N \in \mathbb{N}$ is given by [A2, Table 3].

Chapter 5

Quantum Particle on Weyl Group Invariant Lattices in Weyl Alcove

In this section, several types of discrete quantum models of a free non-relativistic quantum particle propagating on rescaled and shifted dual weight or dual root lattices inside the closure of Weyl alcove are derived. The amplitude of the particle's propagation is regulated by a Weyl group invariant, complex-valued hopping function and the boundary conditions are enforced by Dirichlet and Neumann walls representing ideal barriers and perfect mirrors. The propagation of the particle on the finite set of points and interacting with Dirichlet and Neumann walls merge into a class of the discrete quantum billiard systems [26, 82]. For this special class of quantum systems associated with the point sets of the generalized Fourier–Weyl transforms, the explicit solution is obtained, providing uniform methods to describe significant class of discrete quantum billiard systems [26, 66, 82, 83, 90]. The section mainly focus on the topic developed in [A3, A4], providing a brief description of the dual root and dual weight lattice models.

5.1 Quantum Particle on Dual Root Lattice

Dual-Root Hopping Function

The dual-root hopping function controlling the amplitudes of particle propagating on the dual root lattice is given as a discrete complex-valued function $\mathcal{Q}^\vee : Q^\vee \to \mathbb{C}$, which is assumed to be non-zero only on the finite number of points on the dual root lattice $Q^\vee$, i.e.

$$\begin{aligned} \operatorname{supp}(\mathcal{Q}^\vee) &\subset Q^\vee, \\ |\operatorname{supp}(\mathcal{Q}^\vee)| &< +\infty. \end{aligned} \tag{5.1}$$

Furthermore, the hopping function is constrained to be W-invariant and Hermitian. Hence, for any $q^\vee \in Q^\vee$ and $w \in W$ it follows:

$$\begin{aligned} \mathcal{Q}^\vee(wq^\vee) &= \mathcal{Q}^\vee(q^\vee), \\ \mathcal{Q}^\vee(-q^\vee) &= \mathcal{Q}^{\vee *}(q^\vee). \end{aligned} \tag{5.2}$$

The finite set $\operatorname{supp}^+(\mathcal{Q}^\vee)$ is defined as a subset inside the cone of positive dual weights $P^{\vee+}$ as

$$\operatorname{supp}^+(\mathcal{Q}^\vee) = \operatorname{supp}(\mathcal{Q}^\vee) \cap P^{\vee+}. \tag{5.3}$$

The properties of the dual-root hopping function guarantee that it is fully defined by its values on the cone of positive dual weights $q^\vee \in \operatorname{supp}^+(\mathcal{Q}^\vee)$. Moreover, the dual-root hopping function $\mathcal{Q}^\vee$ is for the cases A_1, $B_n (n \geq 3)$, $C_n (n \geq 2)$, $D_{2k} (k \geq 2)$, E_7, E_8, F_4 and G_2 real-valued [A3, §3.1].

Dual Root Point Sets

Any ordered set of points in $F^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ determines an orthonormal basis $|a\rangle, a \in F^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ of the finite-dimensional complex Hilbert space $\mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$. Furthermore, there exists an orthonormal basis of $|\lambda\rangle \in \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ corresponding to the label set $\lambda \in \Lambda^\sigma_{P,M}(\varrho,\nu^\vee)$ and related to the $|a\rangle$-basis as

$$|\lambda\rangle = \sum_{a \in F^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)} |a\rangle\langle a|\lambda\rangle, \tag{5.4}$$

where the amplitudes are given by the inverse of the discrete Fourier–Weyl transform matrix $\mathbb{I}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ (3.59) from Section 4,

$$\langle a|\lambda\rangle = \sqrt{\frac{\varepsilon(a)}{|W|\, M^n h_{P,M}(\lambda)}}\varphi^\sigma_\lambda(a). \tag{5.5}$$

The dimension of the Hilbert space $\mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ corresponds to the point and label sets as follows:

$$\dim \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee) = \left|F^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)\right| = \left|\Lambda^\sigma_{P,M}(\varrho,\nu^\vee)\right|. \tag{5.6}$$

The rescaled and shifted dual root lattice $Q^\vee_{l,M}(\nu^\vee)$ is for a length factor $l \in \mathbb{R}$ and scaling factor $M \in \mathbb{N}$ defined by

$$Q^\vee_{l,M}(\nu^\vee) = \tfrac{l}{M}(\nu^\vee + Q^\vee), \tag{5.7}$$

providing the possible positions for the non-relativistic quantum particle. The jumping of the particle between the points $x, x' \in Q^\vee_{l,M}(\nu^\vee)$ is specified by the dual-root hopping function $\mathcal{Q}^\vee$ with the amplitude $\mathcal{I}_M(x,x') \in \mathbb{C}$ per unit of time given by

$$\mathcal{I}_M(x,x') = \tfrac{\mathrm{i}}{\hbar}\mathcal{Q}^\vee\left(\tfrac{M}{l}(x'-x)\right). \tag{5.8}$$

Dirichlet and Neumann boundaries representing the ideal barriers and perfect mirrors, respectively, trap the quantum particle inside the rescaled fundamental domain lF, thus, the position of the quantum particle is limited to the dual-root dot $D^\sigma_{Q^\vee,l,M}(\varrho,\nu^\vee)$ defined by

$$D^\sigma_{Q^\vee,l,M}(\varrho,\nu^\vee) = l\, F^\sigma_{Q^\vee,M}(\varrho,\nu^\vee). \tag{5.9}$$

The particle positioned at the point $la \in D^\sigma_{Q^\vee,l,M}(\varrho,\nu^\vee)$ is represented by the vector $|a\rangle \in \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$, $a \in F^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$.

Dual Root Models

Each dual root from the dominant support of the dual-root hopping function $q^\vee \in \mathrm{supp}^+(\mathcal{Q}^\vee)$ induces, for any two points $a, a' \in F^\sigma_{Q^\vee,M}(\varrho, \nu^\vee)$, corresponding dual root coupling set $N_{q^\vee,M}(a, a')$ as follows:

$$N_{q^\vee,M}(a,a') = W^{\mathrm{aff}} a' \cap \left(a + \frac{1}{M} W q^\vee \right). \tag{5.10}$$

The dual-root hopping operator $\widehat{A}^\sigma_{q^\vee,M}(\varrho,\nu^\vee) : \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee) \to \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$, which takes into account the interactions with the boundary walls is given via its matrix elements as

$$\left\langle a \,\middle|\, \widehat{A}^\sigma_{q^\vee,M}(\varrho,\nu^\vee) \,\middle|\, a' \right\rangle = -\varepsilon^{\frac{1}{2}}(a)\varepsilon^{-\frac{1}{2}}(a')\mathcal{Q}^\vee(q^\vee) \sum_{N_{q^\vee,M}(a,a')} \chi^\sigma_\varrho(d). \tag{5.11}$$

The Hamiltonian $\widehat{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee) : \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee) \to \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ of the quantum particle on the dot $D^\sigma_{Q^\vee,l,M}(\varrho,\nu^\vee)$ is determined by the dual-root hopping function $\mathcal{Q}^\vee$ as follows:

$$\widehat{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee) = \sum_{q^\vee \in \mathrm{supp}^+(\mathcal{Q}^\vee)} \widehat{A}^\sigma_{q^\vee,M}(\varrho,\nu^\vee). \tag{5.12}$$

The time evolution of states $|\psi(t)\rangle \in \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$ is provided by the Schrödinger equation:

$$\mathrm{i}\hbar \frac{d}{dt} |\psi(t)\rangle = \widehat{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)|\psi(t)\rangle, \tag{5.13}$$

which induces the the time-independent Schrödinger equation with the eigenenergies $E^\sigma_{Q^\vee,\lambda,M}(\varrho,\nu^\vee)$:

$$\widehat{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)|\lambda\rangle = E^\sigma_{Q^\vee,\lambda,M}(\varrho,\nu^\vee)\,|\lambda\rangle. \tag{5.14}$$

The vectors of the orthonormal basis $|\lambda\rangle \in \mathcal{H}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)$, $\lambda \in \Lambda^\sigma_{P,M}(\varrho,\nu^\vee)$ (5.4) satisfy the time-independent Schrödinger equation (5.14) and the eigenenergies $E^\sigma_{Q^\vee,\lambda,M}(\varrho,\nu^\vee)$ are fully determined by the dual-root hopping function and C-functions (3.55) in [A3, Theorem 3.1] as

$$E^\sigma_{Q^\vee,\lambda,M}(\varrho,\nu^\vee) = -\sum_{q^\vee \in \mathrm{supp}^+(\mathcal{Q}^\vee)} \mathcal{Q}^\vee(q^\vee)\, C_{q^\vee}\left(\tfrac{\lambda}{M}\right). \tag{5.15}$$

The time-independent probabilities $P^{\sigma,\varrho,\nu^\vee}_{Q^\vee,M}[\lambda](a)$, $\lambda \in \Lambda^\sigma_{P,M}(\varrho,\nu^\vee)$ of finding the particle at $la \in D^\sigma_{Q^\vee,l,M}(\varrho,\nu^\vee)$ are given by

$$P^{\sigma,\varrho,\nu^\vee}_{Q^\vee,M}[\lambda](a) = |\langle a|\lambda\rangle|^2 = \left|\mathbb{I}^\sigma_{Q^\vee,M}(\varrho,\nu^\vee)_{\lambda a}\right|^2 = \frac{\varepsilon(a)}{|W|\, M^n h_{P,M}(\lambda)} \left|\varphi^\sigma_\lambda(a)\right|^2. \tag{5.16}$$

Examples of dual root lattice models of A_2 and C_2 for the nearest and next-to-nearest neighbour coupling are thoroughly examined in [A3, §4], where in the case of C_2 the calculations were also performed using the (anti)symmetric multivariate discrete sine transform formalism.

5.2 Quantum Particle on Dual Weight Lattice

Dual-Weight Hopping Function

The amplitudes of particle propagation on the dual weight lattice are given by a discrete complex-valued dual-weight hopping function, $\mathcal{P}^\vee : P^\vee \to \mathbb{C}$, which is considered to be non-zero only on a finite subset of the dual weight lattice $P^\vee$:

$$\begin{aligned} \operatorname{supp}(\mathcal{P}^\vee) &\subset P^\vee, \\ |\operatorname{supp}(\mathcal{P}^\vee)| &< +\infty. \end{aligned} \tag{5.17}$$

The hopping function has to be Hermitian and W-invariant, hence it follows for any $p^\vee \in P^\vee$ and $w \in W$:

$$\begin{aligned} \mathcal{P}^\vee(-p^\vee) &= \mathcal{P}^{\vee *}(p^\vee), \\ \mathcal{P}^\vee(wp^\vee) &= \mathcal{P}^\vee(p^\vee). \end{aligned} \tag{5.18}$$

Furthermore, a dominant support $\operatorname{supp}^+(\mathcal{P}^\vee)$ is defined as a subset of $\operatorname{supp}(\mathcal{P}^\vee)$ belonging to the cone of positive dual weights $P^{\vee +}$,

$$\operatorname{supp}^+(\mathcal{P}^\vee) = \operatorname{supp}(\mathcal{P}^\vee) \cap P^{\vee +}. \tag{5.19}$$

The properties (5.17) and (5.18) guarantee that the dual-weight hopping function is fully described by values on a finite set of points inside the dominant support. Moreover, for the cases A_1, $B_n (n \geq 3)$, $C_n (n \geq 2)$, $D_{2k} (k \geq 2)$, E_7, E_8, F_4 and G_2, conditions (5.18) further guarantee the realness of the dual-weight hopping function $\mathcal{P}^\vee$, and for the remaining cases, the dual-weight hopping function satisfies additional admissibility conditions [A4, §3.1].

Dual Weight Point Sets

Any ordered set of points from $F^\sigma_{P^\vee,M}(\varrho, \varrho^\vee)$ generates an orthonormal position basis $|a\rangle$, $a \in F^\sigma_{P^\vee,M}(\varrho, \varrho^\vee)$ of the finite dimensional Hilbert space $\mathcal{H}^\sigma_{P^\vee,M}(\varrho, \varrho^\vee)$. The dimension of the Hilbert space $\mathcal{H}^\sigma_{P^\vee,M}(\varrho, \varrho^\vee)$ coincide with the cardinality of the point $F^\sigma_{P^\vee,M}(\varrho, \varrho^\vee)$ and label $\Lambda^\sigma_{Q,M}(\varrho, \varrho^\vee)$ sets:

$$\dim \mathcal{H}^\sigma_{P^\vee,M}(\varrho, \varrho^\vee) = \left| F^\sigma_{P^\vee,M}(\varrho, \varrho^\vee) \right| = \left| \Lambda^\sigma_{Q,M}(\varrho, \varrho^\vee) \right|. \tag{5.20}$$

The possible positions of the non-relativistic particle are represented by the points of rescaled and shifted dual weight lattice $P^\vee_{l,M}(\varrho^\vee)$, which for a length factor $l \in \mathbb{R}$ and scaling factor $M \in \mathbb{N}$ is defined by

$$P^\vee_{l,M}(\varrho^\vee) = \tfrac{l}{M}(\varrho^\vee + P^\vee). \tag{5.21}$$

The particle jumps between positions $x, x' \in P^\vee_{l,M}(\varrho^\vee)$ are given by amplitude $\mathcal{I}_M(x, x')$ per unit time, which is determined by the dual-root hopping function $\mathcal{P}^\vee$ as

$$\mathcal{I}_M(x, x') = \tfrac{\mathrm{i}}{\hbar} \mathcal{P}^\vee \left(\tfrac{M}{l}(x' - x) \right). \tag{5.22}$$

Since the boundary conditions are represented by the perfect mirrors and ideal barriers on the boundary of the scaled fundamental domain lF, the position of the quantum particle is restricted to the dual-weight dot $D^\sigma_{P^\vee,l,M}(\varrho, \varrho^\vee)$,

$$D^\sigma_{P^\vee,l,M}(\varrho, \varrho^\vee) = l\, F^\sigma_{P^\vee,M}(\varrho, \varrho^\vee). \tag{5.23}$$

The particle at the point $la \in D^{\sigma}_{P^{\vee},l,M}(\varrho,\varrho^{\vee})$ is represented by the position vector $|a\rangle \in \mathcal{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$, $a \in F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$. An orthonormal momentum basis $|\lambda\rangle \in \mathcal{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$ corresponding to the label set $\lambda \in \Lambda^{\sigma}_{Q,M}(\varrho,\varrho^{\vee})$ is then given by

$$|\lambda\rangle = \sum_{a\in F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})} |a\rangle\langle a|\lambda\rangle, \tag{5.24}$$

where the amplitudes $\langle a|\lambda\rangle$ are given by the inverse of the unitary matrix $\mathbb{I}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$ (3.65) of the generalized discrete dual weight lattice Fourier–Weyl transform:

$$\langle a|\lambda\rangle = \sqrt{\frac{\varepsilon(a)}{c\,|W|\,M^n h^{\vee}_M(\lambda)}}\,\varphi^{\sigma}_{\lambda}(a). \tag{5.25}$$

(Note that even thought the bases $|a\rangle$ and $|\lambda\rangle$ in this section are produced by similar approach as in Section 5.1, they are in general different.)

Dual Weight Models

Dual weights from the dominant support of the dual-weight hopping function $p^{\vee} \in \operatorname{supp}^{+}(\mathcal{P}^{\vee})$ induce, for any two points $a, a' \in F^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$, a dual weight coupling set $N_{p^{\vee},M}(a,a')$ given by

$$N_{p^{\vee},M}(a,a') = W^{\mathrm{aff}} a' \cap \left(a + \frac{1}{M} W p^{\vee}\right). \tag{5.26}$$

The dual-weight hopping operator $\widehat{A}^{\sigma}_{p^{\vee},M}(\varrho,\varrho^{\vee}) : \mathcal{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee}) \to \mathcal{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$, due to the interaction of the particle with the boundary walls via the χ-functions and summing over the coupling set $N_{p^{\vee},M}(a,a')$, is determined by its matrix elements in the position basis as

$$\left\langle a \,\middle|\, \widehat{A}^{\sigma}_{p^{\vee},M}(\varrho,\varrho^{\vee}) \,\middle|\, a' \right\rangle = -\varepsilon^{\frac{1}{2}}(a)\varepsilon^{-\frac{1}{2}}(a')\mathcal{P}^{\vee}(p^{\vee}) \sum_{d\in N_{p^{\vee},M}(a,a')} \chi^{\sigma}_{\varrho}(d)\,. \tag{5.27}$$

The Hamiltonian $\widehat{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee}) : \mathcal{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee}) \to \mathcal{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$ of the quantum particle propagating on the dual-weight dot $D^{\sigma}_{P^{\vee},l,M}(\varrho,\varrho^{\vee})$ is then given as the sum of all dual-weight hopping operators,

$$\widehat{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee}) = \sum_{p^{\vee}\in \operatorname{supp}^{+}(\mathcal{P}^{\vee})} \widehat{A}^{\sigma}_{p^{\vee},M}(\varrho,\varrho^{\vee}). \tag{5.28}$$

The time evolution of the state vectors $|\psi(t)\rangle \in \mathcal{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})$ is controlled by the standard Schrödinger equation

$$\mathrm{i}\hbar\,\frac{d}{dt}\,|\psi(t)\rangle = \widehat{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})|\psi(t)\rangle, \tag{5.29}$$

which leads to the time-independent Schrödinger equation:

$$\widehat{H}^{\sigma}_{P^{\vee},M}(\varrho,\varrho^{\vee})|\lambda\rangle = E^{\sigma}_{P^{\vee},\lambda,M}(\varrho,\varrho^{\vee})|\lambda\rangle, \tag{5.30}$$

where the $E^{\sigma}_{P^\vee,\lambda,M}(\varrho,\varrho^\vee)$ are the eigenenergies of the Hamiltonian $\widehat{H}^{\sigma}_{P^\vee,M}(\varrho,\varrho^\vee)$ and are real-valued and fully determined for any dual-weight hopping function $\mathcal{P}^\vee$ by summing the C-functions (3.55) over the dominant support $\mathrm{supp}^+(\mathcal{P}^\vee)$ as

$$E^{\sigma}_{P^\vee,\lambda,M}(\varrho,\varrho^\vee) = -\sum_{p^\vee \in\, \mathrm{supp}^+(\mathcal{P}^\vee)} \mathcal{P}^\vee(p^\vee)\, C_{p^\vee}\left(\tfrac{\lambda}{M}\right). \tag{5.31}$$

Furthermore, the orthonormal basis $|\lambda\rangle \in \mathcal{H}^{\sigma}_{P^\vee,M}(\varrho,\varrho^\vee)$, $\lambda \in \Lambda^{\sigma}_{Q,M}(\varrho,\varrho^\vee)$ corresponds to the eigenvectors of the Hamiltonian $\widehat{H}^{\sigma}_{P^\vee,M}(\varrho,\varrho^\vee)$ [A4, Theorem 1]. The time-independent probability $P^{\sigma,\varrho,\varrho^\vee}_{P^\vee,M}[\lambda](a)$ of finding the particle at position $la \in D^{\sigma}_{P^\vee,l,M}(\varrho,\varrho^\vee)$ is provided by the amplitude (5.25) as:

$$P^{\sigma,\varrho,\varrho^\vee}_{P^\vee,M}[\lambda](a) = |\langle a|\lambda\rangle|^2 = \left|\mathbb{I}^{\sigma}_{P^\vee,M}(\varrho,\varrho^\vee)_{\lambda a}\right|^2 = \frac{\varepsilon(a)}{c\,|W|\,M^n h^\vee_M(\lambda)}\,|\varphi^{\sigma}_{\lambda}(a)|^2\,. \tag{5.32}$$

The examples of dual root models of C_2 and G_2 are in-depth examined for the next and next-to-nearest neighbour interactions in [A4, §4], providing the exact listing of point and label sets, as well as matrices of the dual-weight hopping operators and its eigenenergies calculated in Wolfram Mathematica.

Conclusion

The sixteen (anti)symmetric multivariate discrete sine transforms are developed and further studied in [A1]. With the discrete transforms provided, the inherited interpolation methods are formulated. Furthermore, examples of such interpolations and corresponding integral errors are calculated using Wolfram Mathematica and a matrix form examples for some of the transforms are computed. Using the definition of multivariate sine functions, the classical Chebyshev polynomials are generalized to the multivariate Chebyshev-like polynomials of the second and fourth kind. The recurrence formulas for these polynomials are explicitly calculated for a three-dimensional case. In the three dimensional setting, the first few polynomials are explicitly given for all four Chebyshev-like polynomials connected to the multivariate sine functions. The continuous orthogonality relations of the polynomials and their inherent weight functions are presented. The combination of the continuous orthogonality of the polynomials and the discrete orthogonality of the (anti)symmetric multivariate discrete sine functions yields sixteen cubature rules, including four Gaussian rules, presented in [A1].

The bivariate case of the (anti)symmetric multivariate discrete sine transforms and the related bivariate Chebyshev-like polynomials are further studied in [9]. Based on trigonometric identities, the first three polynomials and the recurrence relations are explicitly deduced for all four classes of the Chebyshev-like polynomials, providing a method to compute any Chebyshev-like polynomial related to the bivariate sine functions. Furthermore, the bivariate cubature formulas are explicitly computed and the corresponding weight functions are specified.

The connection between the (anti)symmetric multivariate discrete transforms and the dual root lattice Fourier–Weyl transforms of the crystallographic C_n series presented in [A2] conveniently contributes to further advancements and method transfer between the two formalisms. The analogy of the point sets, label sets as well as the weight and normalization functions which utilize the Coxeter-Dynkin diagrams [A2, Theorems 1-4] and the connection of the transform unitary matrices [A2, Theorem 5] provide an analogue to the dual root lattice Fourier–Weyl transforms of C_n series independent on Lie theory. The possibility of existence of such analogues to other crystallographic series requires further study.

The set of 32 cubature formulas are developed in [37] and [A1]. Eight of the formulas are Gaussian rules, which provide the highest precisions. Using the functional substitution, together with weight function and Chebyshev node conversion, yields cubature formulas in the Lie theoretical background. Such straightforward comparison suggests an existence of similar Gaussian rules connected to the remaining crystallographic series and poses an open problem.

Similarly to the (anti)symmetric multivariate trigonometric functions, the multivariate (anti)symmetric exponential functions serve as variants of special functions induced by the permutation group S_n. A similar form of the point and label sets of the discrete Fourier transforms associated with the (anti)symmetric multivariate exponential functions [58] to the presented dual root lattice Fourier Weyl transforms of the C_n series indicates the possibility of existence of novel types of orbit functions and induced discrete Fourier transforms connected to the Fourier–Weyl transforms of all crystallographic systems. Based on the good convergence of 3D and 2D numerical interpolation tests in both (anti)symmetric trigonometric [9, A1] and (anti)symmetric exponential [5, 46] cases, a further study in this direction could provide new effective interpolation methods.

The dual root and dual weight single particle quantum systems detailed in [A3, A4] and summarized in Chapter 5 describe a non-relativistic particle propagating on the dual root $D^{\sigma}_{Q^\vee,l,M}(\varrho,\nu^\vee)$ or dual weight $D^{\sigma}_{P^\vee,l,M}(\varrho,\varrho^\vee)$ dots, which are composed as finite point sets of the shifted and rescaled dual root or dual weight lattice inside the closure of the Weyl alcove $lF \in \mathbb{R}^n$. The Weyl alcove is bounded by the Neumann and Dirichlet walls $lB^{\sigma}(\varrho)$ and $lH^{\sigma}(\varrho)$, which represent perfect mirrors and ideal barriers trapping the quantum particle inside the Weyl alcove. The predetermined hopping function, which incorporates the amplitudes of propagation to the neighboring positions, then via the corresponding Fourier–Weyl transforms explicitly provides the eigenenergies (5.15) and (5.31) of the quantum billiard systems. The solutions of the time-independent Schrödinger equations (5.14) and (5.30) are further given as vectors of the orthonormal momentum basis $|\lambda\rangle \in \mathcal{H}^{\sigma}_{Q^\vee,M}(\varrho,\nu^\vee)$, $\lambda \in \Lambda^{\sigma}_{P,M}(\varrho,\nu^\vee)$ or $|\lambda\rangle \in \mathcal{H}^{\sigma}_{P^\vee,M}(\varrho,\varrho^\vee), \lambda \in \Lambda^{\sigma}_{Q,M}(\varrho,\varrho^\vee)$, which are determined independently on the hopping function.

The Weyl group symmetries and hermiticity of hopping functions induce Hermitian Hamiltonian [75] with real-valued eigenenergies (5.15) and (5.31). For the cases $A_n (n \geq 2)$, $D_{2k+1} (k \geq 2)$ and E_6, the hopping function may have non-zero imaginary parts, which in case of A_n leads to the energy degeneracy of the stationary states, which is similar to the degeneracy of energy of one-particle quantum systems produced on equilateral triangle [28, 29, 39, 56, 90]. Thus, the complex hopping function deserves further consideration. Moreover, the hopping functions in the presented models require finite support, considering an infinite support requires formulating convergence criterion for the sums (5.15) and (5.31) compatible with the proofs in [A3, A4] and hence additional study is required.

One of the applications of the quantum particle models is electron propagation in a crystal lattice, providing a novel class of tight binding models [77], where the electron is propagated between atoms positioned at the points of the dual root (5.9) or dual weight (5.23) dots. The hopping function values can be fine-tuned from experimental data or derived via theoretical considerations leading to the hopping integrals [30] related to the coupled neighboring atoms. Both the dual root and dual weight lattice models correspond to the inductively developed electron propagation in a crystal lattice [27], however the dual root and dual weight lattice Fourier–Weyl transforms lead to boundary-dependent alternatives to periodic exponential solutions [27].

The possible physical applications of the studied cubature formulas in [A1] incorporate calculations in micromagnetic simulations [14], quantum dynamics [67], laser optics [21], stochastic dynamics [99], liquid crystal colloids [97], porous materials [78, 79] and electromagnetic wave propagation [92]. Important application of the approach of [A3, A4] is to modify the models to describe honeycomb lattices [12, 35, 36, 41], thus providing models for study of graphene [24,30,39,82]. Furthermore, the presented transforms and models could find a potential application in solid state physics [3], electric properties of 2D and 3D materials [1, 87], image processing [13, 93], data hiding [71], face recognition [76].

Bibliography

[1] Y. Alhassid, *The statistical theory of quantum dots*, Rev. Mod. Phys. **72** (2000), 895–968.

[2] V. Alonso and S. De Vincenzo, *General boundary conditions for a Dirac particle in a box and their non-relativistic limits*, J. Phys. A: Math. Gen. **30** (1997), 8573–8585.

[3] N. W. Ashcroft and N. Mermin, *Solid state physics*, Brooks/Cole, Melbourne (1976).

[4] H. Berens, H. J. Schmid and Y. Xu, *Multivariate Gaussian cubature formulae*, Arch. Math. **64** (1995), 26–32.

[5] A. Bezubik, J. Hrivnák, J. Patera and S. Pošta, *Three–variable symmetric and antisymmetric exponential functions and orthogonal polynomials*, Math. Slovac. **67** (2017), 427–446.

[6] N. Bourbaki, *Groupes et Algèbres de Lie, Chapiters IV, V, VI*, Hermann, Paris (1968).

[7] M. Bremner, R. Moody and J. Patera, *Tables of dominant weight multiplicities for representations of simple Lie algebras*, Marcel Dekker, New York (1985).

[8] V. Britanak, K. Rao, and P. Yip, *Discrete cosine and sine transforms: general properties, fast algorithms and integer approximation*, Elsevier/Academic Press, Amsterdam (2007).

[9] A. Brus, J. Hrivnák and L. Motlochová, *Discretization of Generalized Chebyshev Polynomials of (Anti)symmetric Multivariate Sine Functions*, J. Phys.: Conf. Ser. **1416** (2019), 012007.

[10] A. Brus, J. Hrivnák and L. Motlochová, *Quantum Particle on Lattices in Weyl Alcoves*, Lie Theory and Its Applications in Physics, Sofia, Bulgaria, June 2021; Dobrev, V., Ed.; Springer: Singapore (2022), 501–507.

[11] M. Caliari, S. de Marchi and M. Vianello, *Hyperinterpolation in the cube*, Comput. & Math. with Appl. **55** (2008), 2490–2497.

[12] J. Cserti and G. Tichy, *A simple model for the vibrational modes in honeycomb lattices*, Eur. J. Phys. **25** (2004), 723–736.

[13] G. Chadzitaskos, L. Háková and O. Kájínek, *Weyl Group Orbit Functions in Image Processing*, Appl. Math. **5** (2014), 501–511.

[14] D. Chernyshenko, H. Fangohr, *Computing the demagnetizing tensor for finite difference micromagnetic simulations via numerical integration*, J. Magn. Magn. Mater. **381** (2015), 440–445.

[15] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New York (1978).

[16] C.W. Clenshaw and A. R. Curtis, *A method for numerical integration on an automatic computer*, Numer. Math. **2** (1960), 197–205.

[17] R. Cools, *An encyklopedia of cubature formulas*, Journal of Complexity **19** (2003), 445–453.

[18] T. Czyżycki and J. Hrivnák, *Generalized discrete orbit function transforms of affine Weyl groups*, J. Math. Phys. **55** (2014), 113508.

[19] T. Czyżycki, J. Hrivnák and J. Patera, *Generating functions for orthogonal polynomials of A_2, C_2 and G_2*, Symmetry **10** (2018), 354.

[20] T. Czyżycki, J. Hrivnák, L. Motlochová, *Generalized Dual-Root Lattice Transforms of Affine Weyl Groups*, Symmetry **12** (2020), 1018.

[21] I. R. Çapoğlu, A. Taflove and V. Backman, *Computation of tightly-focused laser beams in the FDTD method*, Opt. Express **21** (2013), 87–101.

[22] E. V. Damaskinsky, P. P. Kulish and M. A. Sokolov, *On calculation of generating functions of Chebyshev polynomials in several variables*, J. Math. Phys. **56** (2015), 063507.

[23] J. F. van Diejen and E. Emsiz, *Discrete Fourier transform associated with generalized Schur polynomials*, Proc. Am. Math. Soc. **146** (2018), 3459–3472.

[24] L. B. Drissi, E. H. Saidi and M. Bousmina, *Graphene, Lattice Field Theory and Symmetries*, J. Math. Phys. **52** (2011), 022306.

[25] C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Encyclopedia Math. Appl. 81, Cambridge University Press, Cambridge (2001).

[26] V. Fernández-Hurtado, J. Mur-Petit, J. J. García-Ripoll and R. A. Molina, *Lattice scars: surviving in an open discrete billiard*, New J. Phys. **16** (2014), 035005.

[27] R. P. Feynman, R. B. Leighton and M. Sand, *The Feynman Lectures on Physics: Volume III*, Basic Books, New York (2010).

[28] W. Gaddah, *A Lie group approach to the Schrödinger equation for a particle in an equilateral triangular infinite well*, Eur. J. Phys. **34** (2013), 1175.

[29] W. Gaddah, *Exact solutions to the Dirac equation for equilateral triangular billiard systems*, J. Phys. A: Math. Theor. **51** (2018), 385304.

[30] A. D. Güçlü, P. Potasz, M. Korkusinski and P. Hawrylak, *Graphene Quantum Dots*, Springer, Berlin (2014).

[31] D. C. Handscomb and J. C. Mason, *Chebyshev polynomials*, Champman & Hall/CRC, Boca Raton, FL (2003).

[32] L. Háková, J. Hrivnák and L. Motlochová, *On cubature rules associated to Weyl group orbit functions*, Acta Polytechnica **56** (2016), 202–213.

[33] G. Heckman and H. Schlichtkrull, *Harmonic analysis and special functions on symmetric spaces*, Elsevier Press, San Diego (1994).

[34] J. Hrivnák and M. Juránek, *On E-Discretization of Tori of Compact Simple Lie Groups. II*, J. Math. Phys. **58** (2017), 103504.

[35] J. Hrivnák and L. Motlochová, *Discrete cosine and sine transforms generalized to honeycomb lattice*, J. Math. Phys. **59** (2018), 063503.

[36] J. Hrivnák and L. Motlochová, *Discrete cosine and sine transforms generalized to honeycomb lattice II. Zigzag boundaries*, J. Math. Phys. **62** (2021), 093504.

[37] J. Hrivnák and L. Motlochová, *Discrete transforms and orthogonal polynomials of (anti)symmetric multivariate cosine functions*, SIAM J. Numer. Anal. Vol. 52, No. 6 (2014), 3021–3055.

[38] J. Hrivnák and L. Motlochová, *Dual-root lattice discretization of Weyl orbit functions*, J. Fourier Anal. Appl. **25** (2019), 2521–2569.

[39] J. Hrivnák and L. Motlochová, *Graphene Dots via Discretizations of Weyl-Orbit Functions*, Lie Theory and Its Applications in Physics, Varna, Bulgaria, June 2019; Dobrev, V., Ed.; Springer: Singapore (2020), 407–413.

[40] J. Hrivnák and L. Motlochová, *On connecting Weyl-orbit functions to Jacobi polynomials and multivariate (anti)symmetric trigonometric functions*, Acta Polytechnica **56** (2016), 283-290.

[41] J. Hrivnák and L. Motlochová, *On electron propagation in triangular graphene quantum dots*, J. Phys. A: Math. Theor. **55** (2022), 125201.

[42] J. Hrivnák, L. Motlochová and J. Patera, *Two dimensional symmetric and antisymmetric generalization of sine functions*, J. Math. Phys. **51** (2010), 073509.

[43] J. Hrivnák, L. Motlochová and J. Patera, *Cubature formulas of multivariate polynomials arising from symmetric orbit functions*, Symmetry **8** (2016), 63.

[44] J. Hrivnák, L. Motlochová and J. Patera, *On discretization of tori compact simple Lie groups II*, J. Phys. A: Math. Theor. **45** (2012), 255201.

[45] J. Hrivnák, M. Myronova and J. Patera, *Central Splitting of A_2 Discrete Fourier–Weyl Transforms*, Symmetry **12** (2020), 1828.

[46] J. Hrivnák and J. Patera, *Two dimensional symmetric and antisymmetric generalization of exponential and cosine functions*, J. Math. Phys. **51** (2010), 023515.

[47] J. Hrivnák and J. Patera, *On discretization of tori compact simple Lie groups*, J. Phys. A: Math. Theor. **42** (2009), 385208.

[48] J. Hrivnák and J. Patera, *On E-discretization of tori compact simple Lie groups*, J. Phys. A: Math. Theor. **43** (2010), 165206.

[49] J. Hrivnák, J. Patera and S.Pošta, *Three variable exponential functions of the altering group*, J. Phys. A: Math. Theor. **45** (2012), 045201.

[50] J. Hrivnák and M.A. Walton, *Discretized Weyl-orbit functions: modified multiplication and Galois symmetry*, J. Phys. A: Math. Theor. **48** (2015), 175205.

[51] J. Hrivnák and M. A. Walton, *Weight-lattice discretization of Weyl-orbit functions*, J. Math. Phys. **57** (2016), 083512.

[52] P. J. Howard and P. F. O'Mahony, *The behaviour of resonances in Hecke triangular billiards under deformation*, J. Phys. A: Math. Theor. **40** (2007), 9275.

[53] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York (1978).

[54] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced mathematics, 29, Cambridge university press, Cambridge (1990).

[55] S. R. Jain and R. Samajdar, *Nodal portraits of quantum billiards: Domains, lines, and statistics*, Rev. Mod. Phys. **89** (2017), 045005.

[56] S. R. Jain and R. Samajdar, *Exact eigenfunction amplitude distributions of integrable quantum billiards*, J. Math. Phys. **59** (2018), 012103.

[57] A. U. Klimyk and J. Patera. *Alternating Group and Multivariate Exponential Functions*, Groups and Symmetries, From Neolithic Scots to John McKay; Harnad, J., Winternitz, P., Eds.; American Mathematical Society: Providence, RI, USA (2009) 233–246.

[58] A. U. Klimyk and J. Patera, *Alternating multivariate trigonometric functions and corresponding Fourier transforms*, J. Phys. A: Math. Theor. **41** (2008), 145205.

[59] A. U. Klimyk and J. Patera, *(Anti)symmetric multivariate exponential functions and corresponding Fourier transforms*, J. Phys. A: Math. Theor. **40** (2007), 10473–10489.

[60] A. U. Klimyk and J. Patera, *(Anti)symmetric multivariate trigonometric functions and corresponding Fourier transforms*, J. Math. Phys. **48** (2007), 093504.

[61] A. U. Klimyk and J. Patera, *Antisymmetric orbit functions*, SIGMA **3** (2007), 023.

[62] A. U. Klimyk and J. Patera, *Altering group and multivariate exponential functions*, J. Math. Theor. **41** (2008), 145205.

[63] A. U. Klimyk and J. Patera, *E-orbit functions*, SIGMA **4** (2008), 002.

[64] A. U. Klimyk and J. Patera, *Orbit functions*, SIGMA **2** (2006), 006.

[65] T. H. Koornwinder, *Two-variable analogues of the classical orthogonal polynomials*, Theory and Application of Special Functions, edited by R. A. Askey, Academic Press, New York (1975), 435–495.

[66] D. O. Krimer and R. Khomeriki, *Realization of discrete quantum billiards in a two-dimensional optical lattice*, Phys. Rev. A **84** (2011), 041807.

[67] D. Lauvergnata and A. Nauts, *Quantum dynamics with sparse grids: A combination of Smolyak scheme and cubature.Application to methanol in full dimensionality*, Spectrochim. Acta, Part A **119** (2014), 18–25.

[68] H. Li, J. Sun and Y. Xu, *Discrete Fourirer analysis and Chebyshev polynomials with G_2 group*, SIGMA **8** (2012), 067.

[69] H. Li, J. Sun and Y.Xu, *Discrete Fourier analysis cubature and interpolation on hexagon and triangle*, SIAM J. Numer. Anal. **46** (2008), 1653–1681.

[70] H. Li and Y. Xu, *Discrete Fourier analysis on fundamental domain and simplex of A_d lattice in d-variables*, J. Fourier Anal. Appl. **16** (2010), 383–433.

[71] Y.-K. Lin, *High Capacity reversible data hiding scheme based upon discrete cosine transformation*, J. Systems Software **85** (2012), 2395–2404.

[72] D. M. Lindsay, Y. Wang and T. F. George, *The Hückel model for small metal clusters. IV. Orbital properties and cohesive energies for model clusters of up to several hundred atoms,* J. Clust. Sci. **1** (1990), 107–126.

[73] M. G. E. da Luz, A. S. Lupu-Sax and E. J. Heller, *Quantum scattering from arbitrary boundaries* Phys. Rev. E **56** (1997), 2469–2507.

[74] A. C. Maioli and A. G. M. Schmidt, *Two-dimensional scattering by boundary-wall and linear potentials* Phys. Scr. **95** (2020), 035227.

[75] A. Mandarino, T. Linowski and K. Życzkowski, *Bipartite unitary gates and billiard dynamics in the Weyl chamber*, Phys. Rev. A **98** (2018), 012335.

[76] K. Manikantan, V. Govindarajan, V. Sasi and S. Ramachandran, *Face recognition using block-based DCT feature extraction*, J. Adv. Comput. Sci. Tehnol. **1** (2012), 266–283.

[77] M. Manninen, *Models of Metal Clusters and Quantum Dots,* In: Guet C., Hobza P., Speigelman F., David F. (eds), *Atomic clusters and nanoparticles*, **73** Springer Berlin Heidelberg, Berlin (2001).

[78] M. Marin, *Weak Solutions in Elasticity of Dipolar Porous Materials*, Math. Probl. Eng. **2008** (2008), 158908.

[79] M. Marin and O. Florea, *On temporal behaviour of solutions in thermoelasticity of porous micropolar bodies*, Analele Universitatii Ovidius Constanta Seria Matematica **22** (2014), 169–188.

[80] M. F. C. Martins Quintela and J. M. B. Lopes dos Santos, *A polynomial approach to the spectrum of Dirac–Weyl polygonal Billiards*, J. Phys.: Condens. Matter **33** (2021), 035901.

[81] S. W. McDonald and A. N. Kaufman, *Wave chaos in the stadium: Statistical properties of short-wave solutions of the Helmholtz equation*, Phys. Rev. A **37** (1988), 3067–3086.

[82] F. Miao, S. Wijeratne, Y. Zhang, U. C. Coskun, W. Bao and C. N. Lau, *Phase-Coherent Transport in Graphene Quantum Billiards*, Science **317** (2007), 1530–1533.

[83] S. Montangero, D. Frustaglia, T. Calarco and R. Fazio, *Quantum billiards in optical lattices*, Europhys. Lett. **88** (2009), 30006.

[84] R. V. Moody, L. Motlochová and J. Patera, *Gaussian cubature arising from hybrid characters of simple lie groups*, J. Fourier Anal. Appl. **20** (2014), 1257–1290.

[85] R. V. Moody and J. Patera, *Cubature formulae for orthogonal polynomials in terms of elements of finite order of compact simple Lie groups*, Adv. Appl. Math. **47** (2011), 509–535.

[86] R. V. Moody and J. Patera, *Orthogonality within families of C-, S- and E- functions of any compact semisimple Lie group*, SIGMA **2** (2006), 076.

[87] N. Mounet, M. Gibertini, P. Schwaller, et al., *Two-dimensional materials from high-throughput computational exfoliation of experimentally known compounds*, Nature Nanotech. **13** (2018), 246–252.

[88] H. Munthe-Kaas and B. N. Ryland, *On multivariate Chebyshev polynomials and spectral approximations on triangles*, Lecture Notes in Computational Science and Engineering **76** (2010), 19–41.

[89] T. J. Rivlin, *The Chebyshev Polynomials*, John Wiley & Sons, New York (1990).

[90] A. V. Rozhkov and F. Nori, *Exact wave functions for an electron on a graphene triangular quantum dot*, Phys. Rev. B **81** (2010), 155401.

[91] H. Ch. Schachner and G. M. Obermair, *Quantum billiards in the shape of right triangles*, Z. Phys. B **95** (1994), 113–119.

[92] I. Sfevanovic, F. Merli, P. Crespo-Valero, W. Simon, S. Holzwarth, M. Mattes and J. R. Mosig, *Integral Equation Modeling of Waveguide-Fed Planar Antennas*, IEEE Antennas Propag. Mag. **51** (2009), 82–92.

[93] M. M. Siddeq and M. A. Rodrigues, *DCT and DST Based Image Compression for 3D Reconstruction*, 3D Res. **8** (2017), 5.

[94] A. Sommariva, M. Vianello and R. Zanovello, *Nontensorial Clenshaw-Curtis cubature*, Numer. Algorithms **49** (2008), 409–427.

[95] M. E. Spina, M. Saraceno, *Quantum spectra of triangular billiards on the sphere*, J. Phys. A: Math. Gen. **34** (2001), 2549.

[96] G. Strang, *The discrete cosine transform.* SIAM Rev. **41** (1999), 135–147.

[97] M. Tasinkevych, N. M. Silvestre and M. M. Telo de Gama, *Liquid crystal boojum-colloids*, New J. Phys. **14** (2012), 073030.

[98] E. B. Vinberg and A. L. Onishchik, *Lie groups and Lie algebras*, Springer, New York (1994).

[99] J. Xu, J. Chen and J. Li, *Probability density evolution analysis of engineering structures via cubature points*, Comput. Mech. **50** (2012), 135–156.

Included Publications

List of Included Articles

[A1] A. Brus, J. Hrivnák and L. Motlochová, *Discrete Transforms and Orthogonal Polynomials of (Anti)symmetric Multivariate Sine Functions,* Entropy **20** (2018), 938.
doi:10.3390/e20120938

[A2] A. Brus, J. Hrivnák and L. Motlochová, *Connecting (Anti)Symmetric Trigonometric Transforms to Dual-Root Lattice Fourier–Weyl Transforms,* Symmetry **13** (2020), 61.
doi:10.3390/sym13010061

[A3] A. Brus, J. Hrivnák and L. Motlochová, *Quantum Particle on Dual Root Lattice in Weyl Alcove,* J. Phys. A: Math. Theor. **54** (2021), 095202.
doi:10.1088/1751-8121/abdc80

[A4] A. Brus, J. Hrivnák and L. Motlochová, *Quantum Particle on Dual Weight Lattice in Weyl Alcove,* Symmetry **13** (2021), 1338.
doi:10.3390/sym13081338