Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Discrete polyharmonic operator with complex potential

Diskrétní polyharmonický operátor s komplexním potenciálem

Master's thesis

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Academic year: 2023/2024

ZADÁNÍ DIPLOMOVÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

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Zadávající katedra/ústav: **Katedra matematiky**

Studijní program: **Matematické inženýrství**

II. ÚDAJE K DIPLOMOVÉ PRÁCI

Název diplomové práce:

Diskrétní polyharmonický operátor s komplexním potenciálem

Název diplomové práce anglicky:

Discrete polyharmonic operator with complex potential

Pokyny pro vypracování:

1) Zaveďte operátor, který je obecnou kladnou mocninou diskrétního Laplaceova operátoru, a vyšetřete jeho spektrální vlastnosti.
2) Diskutujte optimální lokalizaci spektra diskrétního biharmonického operátoru s komplexním l^1 - potenciálem.
3) Pokuste se zobecnit bod (2) i na diskrétní polyharmonický operátor.
4) Pokuste se o analýzu existence vlastních hodnot vložených v esenciálním spektrum operátoru z bodu (2), příp. (3).
5) Diskutujte kritikalitu a proveďte rešerši optimálních Hardyho nerovností pro kladnou mocninu diskrétního Laplaceova operátoru.
6) Pokuste se o analýzu slabých vazeb pro diskrétní biharmonický operátor, příp. pro kritický případ kladné mocniny diskrétního Laplaceova operátoru.

Seznam doporučené literatury:

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Jméno a pracoviště vedoucí(ho) diplomové práce:

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Jméno a pracoviště druhé(ho) vedoucí(ho) nebo konzultanta(ky) diplomové práce:

Datum zadání diplomové práce: **31.10.2023** Termín odevzdání diplomové práce: **10.05.2024**

Platnost zadání diplomové práce: **31.10.2025**

Ing. František Štampach, Ph.D. podpis vedoucí(ho) práce	prof. Ing. Zuzana Masáková, Ph.D. podpis vedoucí(ho) ústavu/katedry	doc. Ing. Václav Čuba, Ph.D. podpis děkana(ky)

III. PŘEVZETÍ ZADÁNÍ

Diplomant bere na vědomí, že je povinen vypracovat diplomovou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací. Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v diplomové práci.

9.11.2023 — Datum převzetí zadání

Podpis studenta

Acknowledgment:
I would like to thank and express my gratitude to my supervisor Ing. František Štampach, PhD. for his expert guidance.

Author's declaration:
I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, May 10, 2024 Tomáš Hrdina

Hrdina

Název práce: *Diskrétní polyharmonický operátor s komplexním potenciálem*

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Druh práce: Diplomová práce

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Abstrakt: Cílem této práce je definovat obecnou kladnou mocninu diskrétního Laplaceova operátoru na diskrétní přímce a vyšetřit její spektrální vlastnosti. Zvolili jsme přístup využívající teorii Laurentových operátorů. Dále jsme zkoumali spektrum diskrétního polyharmonického operátoru s komplexním potenciálem. Za využití Birmanova–Schwingerova principu jsme nalezli takzvané spektrální obálky a lokalizovali spektrum. Nalezené obálky jsou za předpokladu platnosti jisté hypotézy optimální. Tato hypotéza byla analyticky dokázána pro bilaplaceův operátor. Dále jsme se zabývali kritikalitou kladné mocniny Laplaceova operátoru a Hardyho nerovnostmi. Byl stanoven přesný rozsah mocniny Laplaceova operátoru tak, aby byl kritický.

Klíčová slova: Birmanův–Schwingerův princip, diskrétní polyharmonický operátor, Hardyho nerovnosti, jaderná porucha, kritkalita, Laurentovy operátory, lokalizace spektra

Title: *Discrete polyharmonic operator with complex potential*

Author: Tomáš Hrdina

Abstract: The aim of this thesis is to define the general positive power of the discrete Laplace operator over integers and analyze its spectral properties. We have chosen an approach utilizing the theory of Laurent operators. Furthermore, we analyzed the spectrum of the discrete polyharmonic operator with a complex potential. Having used the Birman-Schwinger principle, we found so-called spectral enclosures and localized the spectrum. The given enclosures are optimal under the assumption of validity of a certain conjecture. This conjecture was analytically proven for the bilaplace operator. We conclude by analyzing the criticality of the positive power of the discrete Laplace operator and Hardy's inequalities. An exact range of the Laplace operator's power was determined to make it critical.

Key words: Birman–Schwinger principle, criticality, discrete Polyharmonic operator, Hardy's Inequalities, Laurent operators, spectral enclosures, trace-class perturbation

Contents

Introduction

The discrete Laplace operator Δ on $\ell^2(\mathbb{Z})$ is a linear second-order difference operator given by

$$(\Delta v)_n := -v_{n-1} + 2v_n - v_{n+1}, \quad \forall v \in \ell^2(\mathbb{Z}).$$

It is an approximation of the continuous 1-D Laplace operator which is acting like

$$\Delta f := \frac{d^2 f}{dx^2}, \quad \forall f \in C^2(\mathbb{R}).$$

The continuous Laplace operator arises in various areas of mathematics, for example differential equations or mathematical physics. Therefore it has been a long-standing subject of study in scientific publications, including the spectral theory.
In recent years, there has been a number of works dealing with discrete versions of Laplace operator and problems inspired by the continuous cases. One of them should be this master's thesis.

The introduced discrete Laplace operator is a representative of the class of Laurent operators. In the first chapter, we use the theory of Laurent operators to define the general power of the discrete Laplace operator Δ^α, $\alpha > 0$ and analyze it's spectral properties. Later in the thesis, we define a negative power of the discrete Laplace operator in a similar way. We also study the resolvent operator of the discrete polyharmonic operator. The resolvent is obtained in the form of the Green kernel (matrix element) using a modified Joukowski transform of the spectral parameter.

In the second chapter, we study the spectrum of the discrete polyharmonic operator with a complex potential. We consider a trace-class operator V. The operator is associated with a sequence $v \in \ell^1(\mathbb{Z})$. Inspired by a method from [9], we use the Birman-Schwinger principle to obtain so called spectral enclosures. The spectral enclosure is a subset of the complex plane which contain the discrete spectrum of the discrete polyharmonic operator perturbed by the potential V. In addition, we are able to localize the whole spectrum of the perturbed operator.
In order to obtain the spectral enclosures, we need to estimate the absolute value of the Green kernel of the polyharmonic operator. We introduce two estimates. One of them is non-optimal but analytically proved. The other is optimal. Unfortunately, we only have the proof of the optimality for the case of the discrete bilaplace operator. The proof uses methods of the complex analysis. The validity of the estimate for higher integer powers of

the discrete Laplace operator was numerically tested and it has not been disproved. We also discuss the absence of eigenvalues in the essential spectrum of the perturbed bilaplace operator.

Finally, we are dealing with criticality of the positive power of the discrete Laplace operator. The main result is that Δ^α is critical if and only if $\alpha \geq 1/2$. We use an approach introduced in [4] for the positive power of the discrete Laplacian over the positive integers. The operator Δ^α is subcritical for $\alpha \in (0, 1/2)$. This is equivalent with the existence of Hardy's inequalities which were analyzed in [1]. We made a recherche of their proof with an emphasis on linking their semigroup approach with ours.

Chapter 1

Positive power of the discrete Laplace operator and polyharmonic operator

A crucial object of this thesis is the discrete Laplace operator, since now denoted by T, which is acting on the Hilbert space

$$\ell^2(\mathbb{Z}) = \left\{ u : \mathbb{Z} \to \mathbb{C} : \sum_{j\in\mathbb{Z}} |u_j|^2 < +\infty \right\}$$

with the orthonormal basis

$$\mathcal{E} = \{e_j\}_{j\in\mathbb{Z}}, \text{ where } \forall j \in \mathbb{Z} : \quad e_j = \{\delta_{j,k}\}_{k\in\mathbb{Z}}$$

and $\delta_{j,k}$ is the Kronecker delta. We consider the Euclidean inner product on $\ell^2(\mathbb{Z})$ to be linear in the second argument.

Operator T is defined by its action on vectors of $\mathcal{E}$ as follows:

$$Te_n = -e_{n-1} + 2e_n - e_{n+1}, \quad \forall n \in \mathbb{Z}. \tag{1.1}$$

It is usual to associate the operator with its matrix with respect to the standard basis $\mathcal{E}$, which is doubly infinite. It reads

$$T = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & \boxed{2} & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & & \end{pmatrix}.$$

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Spectral properties of the operators T and T^2 were studied in my Bachelor's degree project [10]. In this thesis we would like to study a general positive power of T, i.e. T^α, $\alpha > 0$. Proper definitions and spectral properties will be introduced and analyzed using the theory of Laurent operators.

1.1 A brief introduction to Laurent operators

In this section, we will introduce the class of Laurent operators. It consists of operators whose matrix with respect to standard basis has a certain structure. We state that any operator of this class is uniquely associated with a complex valued function, which, in a certain sense, carries almost all the information about spectral properties of the operator. We find it easier to use this method to do a basic spectral analysis for the polyharmonic operator, which is, indeed, in the Laurent class, instead of the direct approach which consists of solving the equation for eigenvalues etc.
All the propositions are based on [5] and can be found in Section 3.1. Some details might be found in [10] as well, where the same theory was used for discrete Laplacian and bilaplacian. Since this is a brief introduction, all the propositions will be introduced without proofs, although they are not complicated.

Definition 1.1. A bounded operator A on $\ell^2(\mathbb{Z})$ is called Laurent operator if its matrix element (kernel) $A_{m,n}$ depends only on the difference $m - n$.

Definition 1.1 says that the matrix representation of the operator A has constant diagonals.

Let us now consider a unitary mapping $\mathcal{U} : \ell^2(\mathbb{Z}) \to L^2([-\pi, \pi])$ given by the action on the vectors of $\mathcal{E}$

$$(\mathcal{U}e_n)(t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}nt} =: f_n(t), \quad \forall t \in [-\pi, \pi],\ \forall n \in \mathbb{Z}. \tag{1.2}$$

$\mathcal{U}$ maps the orthonormal basis of $\ell^2(\mathbb{Z})$ onto an orthonormal basis of $L^2([-\pi, \pi])$. Therefore $\mathcal{U}$ us unitary.
The inverse of $\mathcal{U}$ is nothing but the discrete Fourier transform,

$$\mathcal{U}^{-1} f_n = e_n, \quad \forall n \in \mathbb{Z}.$$

Definition 1.2. Let $\phi_A : [-\pi, \pi] \to \mathbb{C}$ be a bounded, measurable function. We call ϕ_A the symbol of a Laurent operator A if it holds

$$A = \mathcal{U}^{-1} M_{\phi_A} \mathcal{U}, \tag{1.3}$$

where M_{ϕ_A} is the operator of multiplication by ϕ_A, which is, under these assumptions, bounded on $L^2([-\pi, \pi])$.

Since the diagonals of the matrix representation of A are constant, it is convenient to denote $\forall m, n \in \mathbb{Z}: \; A_{m,n} = a_{n-m}$. Thus, the matrix of A is of following form

$$T = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & & \\ a_{-2} & a_{-1} & a_0 & a_1 & a_2 & & \\ \dots & a_{-2} & a_{-1} & \boxed{a_0} & a_1 & a_2 & \dots \\ & & a_{-2} & a_{-1} & a_0 & a_1 & \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & \end{pmatrix}.$$

Now, provided that $a \in \ell^1(\mathbb{Z})$, it is easy to verify that the function

$$\phi_A(t) = \sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{-\mathrm{i}nt}, \quad t \in [-\pi, \pi] \tag{1.4}$$

is the symbol of a Laurent operator with a kernel $A_{m,n}$. Indeed, for every basis function $f_m \in L^2([-\pi, \pi])$ it holds

$$\begin{aligned}(\mathcal{U}A\mathcal{U}^{-1} f_m)(t) = (\mathcal{U}Ae_m)(t) &= \left(\mathcal{U}\sum_{n\in\mathbb{Z}} a_n e_{m-n}\right)(t) = \sum_{n\in\mathbb{Z}} a_n f_{m-n}(t) = \\ &= \left(\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{-\mathrm{i}nt}\right) f_m(t) = \phi_A(t) f_m(t),\end{aligned}$$

using the continuity of $\mathcal{U}$ and the fact that A is a Laurent class operator. Hence the desired formula

$$\mathcal{U}A\mathcal{U}^{-1} = M_{\phi_A}$$

directly follows. Moreover, for a given symbol ϕ_A, one can find matrix elements of the corresponding operator A as

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_A(t) \mathrm{e}^{\mathrm{i}nt} dt, \quad \forall n \in \mathbb{Z}.$$

The relation (1.3) is called diagonalization of the Laurent operator A. It follows that every Laurent operator is uniquely related to its symbol ϕ_A defined by (1.4) and corresponding multiplication operator M_{ϕ_A}. the spectra of these operators are identical including the spectral classification (point, continuous and residual spectrum). Thus, it is easy to study spectral properties of Laurent operator A using ϕ_A.

Theorem 1.3. (**Inverse operator**) Let A be a Laurent operator with a continuous symbol ϕ_A. Then A is invertible if and only if $\phi_A(t) \neq 0, \; \forall t \in [-\pi, \pi]$. If so, A^{-1} is Laurent operator with symbol $1/\phi_A$ and its the matrix representation kernel is

$$(A^{-1})_{m,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}(n-m)t}}{\phi_A} dt, \quad \forall m, n \in \mathbb{Z}.$$

Theorem 1.4. (**Operator norm of a Laurent operator**) Let A be a Laurent operator with a continuous symbol ϕ_A. Then A is bounded and

$$\|A\| = \max_{t\in[-\pi,\pi]} |\phi(t)|.$$

Theorem 1.5. (**Spectral properties of a Laurent operator**) Let A be a Laurent operator with a continuous symbol ϕ_A. The spectrum of A coincides with the set $\overline{\mathrm{Ran}(\phi_A)}$. Moreover, the point spectrum of A consists of all points $\lambda \in \mathbb{C}$ such that there exists a Borel set B in $\mathbb{R}$ satisfying $\mu(B) \neq 0$ & $\phi_A(t) = \lambda, \forall t \in B$, where μ is the Lebesgue measure.

At this time, we should mention that a slight abuse of terminology was made. The term *symbol* is usually used for a complex function ϕ_A defined on the unit circle. Such a function is also uniquely associated with the Laurent operator and has the same properties. One can easily obtain the complex function from the defining equation (1.4) putting $z = \mathrm{e}^{\mathrm{i}t} \in \mathbb{C}$, then

$$\phi_A(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n}, \quad z \in \mathbb{T},$$

where $\mathbb{T}$ stands for the unit circle.

1.2 Definition of the positive power of discrete Laplacian and basic properties

The results from the previous section will be applied now. Let us make the terminology and notation clear. By the discrete polyharmonic operator it will be understood operator T^α where $\alpha \in \mathbb{Z}^+$ and T is the discrete Laplacian. We will strictly use *positive power of discrete Laplacian* if $\alpha \in (0, +\infty)$. One can find terms like *fractional discrete Laplacian* for $\alpha \in (0, 1)$ which will not be used here. The definition of discrete Laplacian itself might be different too. We are strictly using the definition given by (1.1) which is a convention used to guarantee positiveness of the spectrum of the Laplacian. There is often used a definition corresponding to the operator $2I - T$, which has the spectrum symmetrically located around zero. This definition is used for example in [9].

The discrete Laplacian T is clearly in the Laurent class. The definition of the discrete polyharmonic operator is clear and since the class of Laurent operators is closed under composition, it is again a Laurent operator. For the definition of the positive power of T, we use the diagonalization of Laurent operator (1.3).
First, let us determine the symbol ϕ_T. Using equation (1.1) and the relation for a symbol (1.4) we obtain

$$\phi_T(t) = -\mathrm{e}^{\mathrm{i}t} + 2 - \mathrm{e}^{-\mathrm{i}t} = 2 - 2\cos(t), \quad t \in [-\pi, \pi].$$

Definition 1.6. Let $\mathcal{U}$ be the unitary operator defined by (1.2) and $\alpha \in (0, +\infty)$. The positive power of discrete Laplacian is defined as follows

$$T^\alpha := \mathcal{U}^{-1} M_{(\phi_T)^\alpha}\, \mathcal{U}.$$

The matrix element of T^α is for $n \geq m \in \mathbb{Z}$ given by

$$\begin{aligned}(T^\alpha)_{m,n} &= \langle e_m, T^\alpha e_n\rangle_{\ell^2(\mathbb{Z})} = \langle \mathcal{U} e_m, \mathcal{U} T^\alpha e_n\rangle_{L^2([-\pi,\pi])} = \Big\langle f_m, \underbrace{\mathcal{U} T^\alpha \mathcal{U}^{-1}}_{M_{2^\alpha(1-\cos(\cdot))^\alpha}} f_n \Big\rangle_{L^2([-\pi,\pi])} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2^\alpha (1-\cos(t))^\alpha \mathrm{e}^{\mathrm{i}(n-m)t} dt.\end{aligned}$$

To compute this integral, we use the following proposition, which could be found in [7], 3.631 eq. 8.

Proposition 1.7. Let $s \in \mathbb{Z}$, $\nu \in \mathbb{C}$, $\operatorname{Re}(\nu) > 0$, it holds

$$\int_0^{\frac{\pi}{2}} \sin^{\nu-1}(t)\cos(2st)dt = (-1)^s \frac{\pi\Gamma(\nu)}{2^\nu \Gamma(\frac{\nu+1}{2}+s)\Gamma(\frac{\nu+1}{2}-s)}.$$

Rewriting the $\mathrm{e}^{\mathrm{i}(m-n)t}$ using sines and cosines in addition with some basic trigonometric identities one has

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} 2^\alpha(1-\cos(t))^\alpha \mathrm{e}^{\mathrm{i}(n-m)t}dt = \frac{2^\alpha}{2\pi}\int_{-\pi}^{\pi}(1-\cos(t))^\alpha\cos((n-m)t)dt = \\ &= \frac{4^\alpha}{\pi}\int_0^\pi \sin^{2\alpha}(\frac{t}{2})\cos((n-m)t)dt = \frac{2\cdot 4^\alpha}{\pi}\int_0^{\frac{\pi}{2}}\sin^{2\alpha}(\tau)\cos(2(n-m)\tau)d\tau.\end{aligned}$$

Hence, using Proposition 1.7, it immediately follows that

$$\begin{aligned}(T^\alpha)_{m,n} &= (-1)^{n-m}\frac{2\cdot 4^\alpha \pi\Gamma(2\alpha+1)}{\pi 2^{2\alpha+1}\Gamma(\alpha+1+(m-n))\Gamma(\alpha+1-(m-n))} = \\ &= (-1)^{n-m}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1+(m-n))\Gamma(\alpha+1-(m-n))}.\end{aligned}$$

Recall that $1/\Gamma(z) = 0$ for $z \in \mathbb{Z}_0^-$.

From the symmetry of m and n one immediately obtains

$$\forall m,n \in \mathbb{Z}: \ (T^\alpha)_{m,n} = (-1)^{n-m}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1+(m-n))\Gamma(\alpha+1-(m-n))}. \tag{1.5}$$

Since the kernel $(T^\alpha)_{m,n}$ depends only on the difference of its indices, the operator is in the Laurent class.

It is clear, that a bounded continuous function $\phi_{T^\alpha}(t) = 2^\alpha(1-\cos(t))^\alpha$ is the symbol of the operator T^α. One can immediately get its spectral properties. This function is real-valued, thus the T^α is self-adjoint. The range of this function is

$$\mathrm{Ran}(\phi_{T^\alpha}) = [0, 4^\alpha].$$

And since the sets of zero derivative of ϕ_{T^α} consist of at most three points $-\pi, \pi$ and 0, then using Theorem 1.5, one arrives at following proposition.

Proposition 1.8. Let $\alpha \in (0, +\infty)$, then the operator T^α is self-adjoint Laurent operator. It's spectrum is

$$\sigma(T^\alpha) = [0, 4^\alpha]$$

and is purely essential and purely continuous.

To be more concrete, in Figure 1.1, one can see graphs of the symbols of T^α for few selected values of $\alpha > 0$.

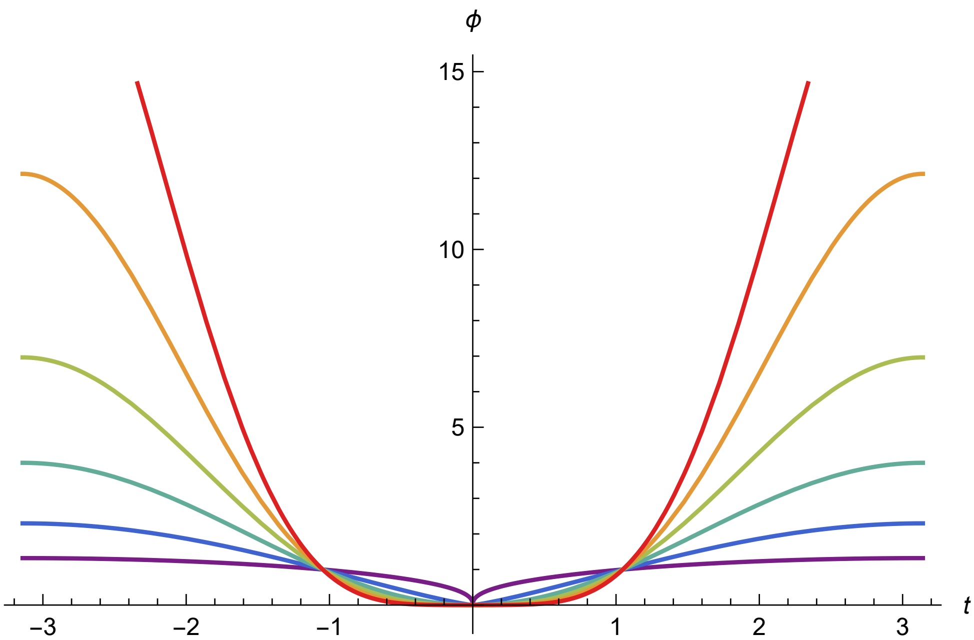


Figure 1.1: Graphs of ϕ_{T^α} for $\alpha \in \{0.2, 0.6, 1.0, 1.4, 1.8, 2.2\}$.

1.3 Polyharmonic operator

Spectral properties of the discrete polyharmonic operator follow directly from the previous section. In what follows, we would like to find the resolvent of the polyharmonic operator. Considering general $\alpha \in (0, +\infty)$, this problem will turn out to be more complicated. The key step is solving a complex contour integral, what I was not able to do so in general, thus we will later restrict on a positive integer α. But let us at first proceed in the most general way.

According to Theorem 1.3, the resolvent operator $(T^\alpha - \lambda)^{-1}$ is in the Laurent class and its symbol is

$$\phi_{(T^\alpha-\lambda)^{-1}} = \frac{1}{\phi_{T^\alpha} - \lambda}.$$

Thus, the matrix element of the resolvent for $\lambda \in \mathbb{C} \setminus [0, 4^\alpha]$ is

$$\forall m \le n \in \mathbb{Z}: \quad (T^\alpha - \lambda)^{-1}_{m,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}(n-m)t}}{\phi_{T^\alpha} - \lambda} dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}(n-m)t}}{(2 - \mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t})^\alpha - \lambda} dt. \quad (1.6)$$

The matrix element of the resolvent is called the *Green kernel* of T^α.
Using the substitution $z = \mathrm{e}^{\mathrm{i}t}$, we can transform this integral into complex contour integral over the unit circle $\mathbb{T}$, which can be simplified through the Residue theorem if $\alpha \in \mathbb{Z}^+$. For positive integer powers, the polynomial in the denominator can be factorized quickly, unfortunately, for general rational or even the real powers, this procedure does not apply. First of all, we need to transform the spectral parameter λ by a parameter from the unit disk $\mathbb{D}$. In the following we put $N := \alpha \in \{1, 2, 3, ...\}$.

1.3.1 Transformation of the spectral parametr

Working with discrete Laplacians and their resolvents, it is natural and convenient to transform the spectral parameter λ using the *Joukowski transform*, it was used e.g. in [9] or [6], where the standard notation for the new parameter is k. It was also used in [10], where a modification of the *Joukowski transform* fitting for T^2 was introduced. Recall that the Joukowski transform is bijection from $\mathbb{D} \setminus \{0\}$ onto $\mathbb{C} \setminus [-2, 2]$ defined by

$$\lambda(k) = k + k^{-1}.$$

We introduce a modification of this transformation which satisfies our demands on the target set which is $\mathbb{C} \setminus [0, 4^N]$. Let us define

$$\zeta_N(k) := (-(k - k^{-1})^2)^N, \quad \text{for } k \in \mathbb{D} \setminus \{0\}. \quad (1.7)$$

Let us now find a proper subset of $\mathbb{D}$ which is mapped by ζ on $\mathbb{C} \setminus [0, 4^N]$ bijectively. It is clear, that the transform ζ could be written as a composition of following mappings

$$\zeta_i(u) := u - \frac{1}{u}, \quad \zeta_{ii}(u) := u^2, \quad \zeta_{iii}(u) := -u, \quad \text{and} \quad \zeta_{iv}(u) := u^N$$

as $\zeta_N = \zeta_{iv} \circ \zeta_{iii} \circ \zeta_{ii} \circ \zeta_i$. Let us denote $\zeta' := \zeta_{iv} \circ \zeta_i \circ \zeta_{ii}$ and

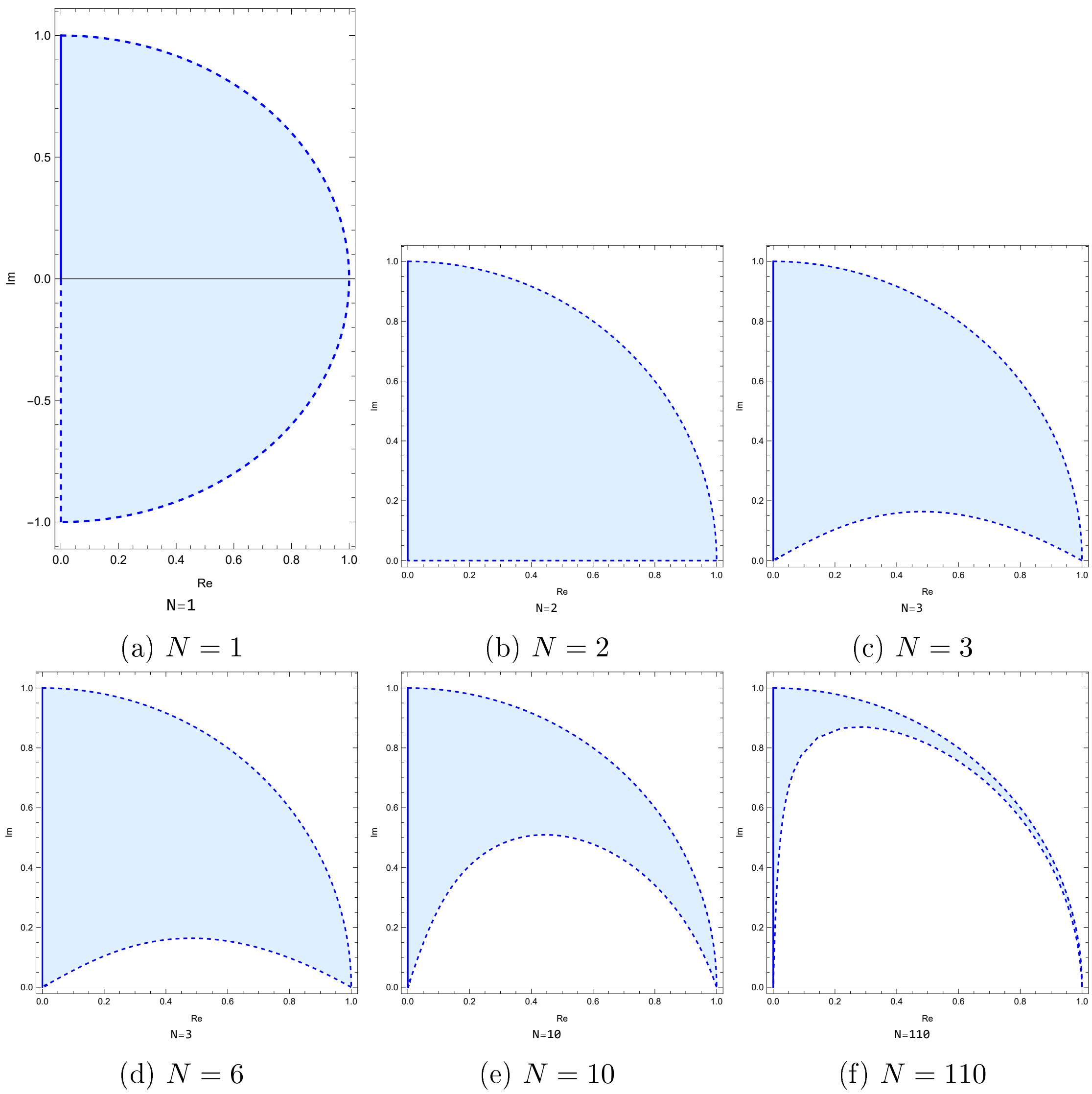
$$H_N := \left\{ w = r\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C} : \ r \in (0, +\infty), \ \varphi \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{N}\right) \right\} \setminus \mathrm{i}\,[0, 2],$$

which is in fact a sector with the segment from 0 to 2i missing.
One can easily see that
$$\zeta'(H_N) = \mathbb{C} \setminus [0, 4^N]$$
and the mapping is injective.

To finish this, we need to describe the set $\zeta_i^{-1}(H_N)$, we will denote it by $\mathbb{D}_N^\zeta \subset \mathbb{D}$. Again, we want the ζ_i to be bijection from $\mathbb{D}_N^\zeta$ onto H_N. For $N = 1$ and $N = 2$ one can easily see that
$$\begin{aligned}
\mathbb{D}_1^\zeta &= \zeta_i^{-1}(H_1) = \{w \in \mathbb{D} : \ \mathrm{Re}(w) > 0\} \cup \mathrm{i}(0,1), \\
\mathbb{D}_2^\zeta &= \zeta_i^{-1}(H_2) = \{w \in \mathbb{D} : \ \mathrm{Re}(w) > 0, \mathrm{Im}(w) > 0\} \cup \mathrm{i}(0,1).
\end{aligned}$$
Indeed, consider $k = r\mathrm{e}^{\mathrm{i}t}$ for $r \in (0, +\infty)$ and $t \in (-\pi, \pi]$. Then
$$\zeta_i(k) = (r - \frac{1}{r})\cos(t) + \mathrm{i}(r + \frac{1}{r})\sin(t)$$
and it is not hard to verify that once $r \in (0,1)$ and $t \in (-\pi/2, \pi/2]$ or $t \in (0, \pi/2]$, $\zeta_i(k)$ is in H_1 or H_2 respectively. Especially, the segment $\mathrm{i}(0,1)$ is mapped onto $\mathrm{i}(2, +\infty)$. Moreover, for fixed $r \in (0,1)$ and t moving in given interval, the generated curves are parts of ellipses which do not intersect each other for any two r, thus the bijectivity follows. To conclude, we have found domains $\mathbb{D}_1^\zeta, \mathbb{D}_2^\zeta$, where the transform ζ_N is bijective, for $N = 1, 2$.

The situation is no longer "straight" considering $N > 2$. We know how $\mathrm{i}(0,1)$ would be mapped. Next, one can describe the segment in $\mathbb{D}$ mapping on the one boundary segment of H_N, which is not lying on the imaginary axis. Let us denote it by l_N . It is not a straight line. We have
$$l_N = \zeta_i^{-1}\left(\left\{r\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\frac{\pi}{N})} : \ r \in (0, +\infty)\right\}\right).$$
Thus, one can find the explicit formula for $k \in \mathbb{D}$ as a function of $r \in (0, +\infty)$ solving the equation
$$k - \frac{1}{k} = r\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\frac{\pi}{N})}.$$
Since $k \neq 0$ our result is one solution of previous quadratic equation and it reads
$$l_N = \left\{\frac{r\mathrm{e}^{i(\frac{\pi}{2}+\frac{\pi}{N})} + \sqrt{r^2\mathrm{e}^{\mathrm{i}2(\frac{\pi}{2}+\frac{\pi}{N})} + 4}}{2} : \ r \in (0, +\infty)\right\}. \tag{1.8}$$
By $l_N(a)$ we understand a point of l_N corresponding to the parameter a $\in (0, +\infty)$. Now it is clear that for positive integer $N > 1$ we can define domain of the transformation ζ_N as follows
$$\mathbb{D}_N^\zeta :=$$
$$\{w \in \mathbb{D} : \mathrm{Re}(w) \geq 0, \ \mathrm{Im}(w) > 0, \ \forall a > 0, \ \mathrm{Re}(w) = \mathrm{Re}(l_N(a)) \Rightarrow \mathrm{Im}(w) > \mathrm{Im}(l_N(a))\}.$$
The set $\mathbb{D}_N^\zeta$ is a part of the unit disk bounded by the imaginary axis and the curve l_N. As an illustration, some of the sets $\mathbb{D}_N^\zeta$ are in Figures 1.2(a)-(f).

Remark 1.9. It is not necessary to restrict the transform ζ only to integer $N > 0$. One can now easily see that if we have defined ζ_α for $\alpha \in (0, +\infty)$, all the properties would have continuously preserved. Considering just integer parameter N, it is easier to explain and understand, moreover, the general case with α would not be useful for us.



(a) $N = 1$ (b) $N = 2$ (c) $N = 3$

(d) $N = 6$ (e) $N = 10$ (f) $N = 110$

Figure 1.2: Domains $\mathbb{D}_N^\zeta$ for selected N.

1.3.2 Resolvent

As it was mentioned, the Green kernel obeys the formula (1.6). One can still consider real $\alpha > 0$. Using the substitution $z = \mathrm{e}^{\mathrm{i}t/2}$ one has, for $n \geq m \in \mathbb{Z}$,

$$(T^\alpha - \lambda)^{-1}_{m,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}(n-m)t}}{(2-\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{-\mathrm{i}t})^\alpha - \lambda}dt = \frac{1}{2\pi\mathrm{i}}\oint_{\mathbb{T}} \frac{z^{2(n-m)-1}}{(2-z^2-z^{-2})^\alpha - \lambda}dz.$$

Now we put $\lambda = \zeta_\alpha(k)$, hence

$$\begin{aligned}(T^\alpha - \lambda)^{-1}_{m,n} =& \frac{1}{2\pi\mathrm{i}}\oint_{\mathbb{T}} \frac{z^{2(n-m)-1}}{(-(z-z^{-1})^2)^\alpha - (-(k-k^{-1})^2)^\alpha}dz = \\ =& \frac{(-1)^\alpha}{2\pi\mathrm{i}}\oint_{\mathbb{T}} \frac{z^{2(n-m)-1}}{((z-z^{-1})^2)^\alpha - ((k-k^{-1})^2)^\alpha}dz. \end{aligned} \tag{1.9}$$

We will integrate using the Residue theorem provided that α is positive integer. From now on, until the end of this chapter, we will consider only integer powers of the discrete Laplacian $N := \alpha \in \mathbb{Z}^+$.

We multiply the fraction in the integral in (1.9) by z^{2N}/z^{2N}, thus we arrive at

$$(T^N - \lambda(k))^{-1}_{m,n} = \frac{(-1)^N}{2\pi\mathrm{i}}\oint_{\mathbb{T}} \frac{z^{2(n-m)+2N-1}}{(z^2-1)^{2N} - z^{2N}(k-k^{-1})^{2N}}dz. \tag{1.10}$$

Let us denote the denominator of the fraction in integral (1.10) as $p_N = p_N(z)$. We have the polynomial

$$p_N(z) = (z^2-1)^{2N} - z^{2N}(k-k^{-1})^{2N} = z^{2N}\left((z-z^{-1})^{2N} - (k-k^{-1})^{2N}\right),$$

where we are giving two different forms, because they are both useful. The polynomial p_N is clearly of degree $4N$ and thus has $4N$ roots (counting the multiplicity). Recall that $k \in \mathbb{D}^\zeta_N$. From the reciprocal form of p_N on can see that once $z \neq 0$ is a root of p_N, $1/z$ is also a root of p_N. Since any $z \in \mathbb{T}$ such that $p(z) = 0$ would imply k outside $\mathbb{D}^\zeta_N$, half of the roots are in the unit circle $\mathbb{D}$ and half of them are in $\overline{\mathbb{D}}^C$. We are interested in roots in $\mathbb{D}$ which is the interior of the integration curve. Moreover, since $p_N(z) = p_N(-z)$, it follows that half of the roots are in the left part of $\mathbb{D}$ and half in the right part of $\mathbb{D}$ or on the corresponding parts of the imaginary axis. It is also clear that k is always a root of p_N. We can work out even a better description of the roots of the polynomial p_N in the unit circle.

Let us denote $z_1, ..., z_N$ the roots of p_N in the right half of $\mathbb{D}$ (i.e $\mathrm{Re}(z_j) \geq 0, \forall j \in \{1, ..., N\}$) and $z_{N+1}, ..., z_{2N}$ the roots in the left half (i.e $\mathrm{Re}(z_j) \leq 0, \forall j \in \{N+1, ..., 2N\}$). We do it in the way that $z_j = -z_{N+j}$, $\forall j \in \{1, ..., N\}$. The zeros of the polynomial p_N in $\mathbb{D}$ fulfill

$$(z_j - z_j^{-1})^{2N} = (k-k^{-1})^{2N}, \quad \text{for } j \in \{1, ..., 2N\},$$

from which we obtain

$$z_j - z_j^{-1} = \mathrm{e}^{2\pi\mathrm{i}\frac{j-1}{2N}}(k - k^{-1}), \quad \text{for } j \in \{1, ..., 2N\}. \tag{1.11}$$

This equation determine the numbering of the roots z_j. Later, we will be able to derive exact regions of the roots in $\mathbb{D}$ using (1.11).

Next, the derivative of p_N reads

$$p_N'(z) = 4Nz(z^2 - 1)^{2N-1} - 2Nz^{2N-1}(k - k^{-1})^{2N}.$$

For every $j \in \{1, ..., 2N\}$, it holds

$$\begin{aligned} p_N'(z_j) &= 2Nz_j^{2N-1}(2z_j(z_j - z_j^{-1})^{2N-1} - (k - k^{-1})^{2N}) = \\ &= 2Nz_j^{2N-1}(2z_j(z_j - z_j^{-1})^{2N-1} - \mathrm{e}^{-2\pi\mathrm{i}(j-1)}(z_j - z_j)^{2N}) = \\ &= 2Nz_j^{2N-1}(z_j - z_j^{-1})^{2N-1}(z_j - z_j^{-1}) \neq 0, \end{aligned}$$

where we used (1.11).
Thus, every described root z_j for $j \in \{1, ..., 2N\}$ generates a simple-pole-type singularity. We can proceed from (1.10) using the residue theorem as follows

$$\begin{aligned} (T^N - \lambda(k))^{-1}_{m,n} =&(-1)^N \sum_{j=1}^{N} \frac{z_j^{2(m-n)+2N-1}}{p_N'(z_j)} + \frac{z_{N+j}^{2(n-m)+2N-1}}{p_N'(z_{N+j})} = \\ =&\frac{(-1)^N}{N} \sum_{j=1}^{N} \frac{z_j^{2(n-m)+2N}}{2z_j^2(z_j^2 - 1)^{2N-1} - z_j^{2N}(k - k^{-1})^{2N}} = \\ =&\frac{(-1)^N}{N} \sum_{j=1}^{N} \frac{z_j^{2(n-m)+2N}}{2z_j^{2N+1}(z_j - z_j^{-1})^{2N-1} - z_j^{2N}(k - k^{-1})^{2N}} = \\ =&\frac{(-1)^N}{N} \sum_{j=1}^{N} \frac{z_j^{2(n-m)}}{2z_j(k - k^{-1})^{2N-1}\mathrm{e}^{-2\pi\mathrm{i}\frac{j-1}{2N}} - (k - k^{-1})^{2N}} = \\ =&\frac{(-1)^N}{N} \sum_{j=1}^{N} \frac{\mathrm{e}^{2\pi\mathrm{i}\frac{j-1}{2N}} z_j^{2(n-m)}}{2z_j(k - k^{-1})^{2N-1} - \mathrm{e}^{2\pi\mathrm{i}\frac{j-1}{2N}}(k - k^{-1})^{2N}} = \\ =&\frac{(-1)^N}{N(k - k^{-1})^{2N-1}} \sum_{j=1}^{N} \frac{z_j^{2(n-m)}\mathrm{e}^{2\pi\mathrm{i}\frac{j-1}{2N}}}{z_j + z_j^{-1}}, \end{aligned}$$

where the second line was obtained using $z_j = -z_{N+j}$, fourth and sixth line were obtained repeatedly using (1.11). It is necessary to mind that $z_j = z_j(k)$, for $j \in \{1, ..., 2N\}$.

The green kernel of T^N could be obtained using the symmetry of the matrix kernel $T^N_{m,n}$ with respect to m and n. One immediately has the following proposition.

Proposition 1.10. Let T^N be a discrete polyharmonic operator, $\lambda(k) = \zeta_N(k)$ defined in (1.7) and $z_j = z_j(k)$ a root of p_N in $\mathbb{D}$ given by (1.11). Then

$$\forall m, n \in \mathbb{Z}: \quad (T^N - \lambda(k))^{-1}_{m,n} = \frac{(-1)^N}{N(k-k^{-1})^{2N-1}} \sum_{j=1}^{N} \frac{z_j^{2|n-m|} \mathrm{e}^{2\pi \mathrm{i} \frac{j-1}{2N}}}{z_j + z_j^{-1}}. \tag{1.12}$$

Let us point out the known cases $N = 1$ and $N = 2$. For discrete Laplace operator one has

$$\forall m, n \in \mathbb{Z}: \quad (T - \lambda(k))^{-1}_{m,n} = -\frac{k^{2|m-n|}}{k^2 - k^{-2}},$$

where the fact that $z_1 = k$ was used. Recall that, in [10] and [9], it appears the well known formula

$$\forall m, n \in \mathbb{Z}: \quad (H - \lambda(\xi))^{-1}_{m,n} = \frac{\xi^{|m-n|}}{\xi - \xi^{-1}},$$

where $H := 2I - T$ and $\lambda(\xi) = \xi + \xi^{-1}$ is the classical Joukowski transform. We can see a notably similar structure of the formulas, whereas the differences are caused by another transform and slightly modified operator. Finally, we propose a similar form for the Green kernel for $N = 2$, we will later work with. The formula (1.11) for $j = 2$ is now of the form $(z_2 - z_2^{-1}) = \mathrm{i}(k - k^{-1})$. We abuse the notation slightly by denoting $z_1 = k$ and $z_2 = z$, which is still a function of k. From (1.12), we have

$$\begin{aligned}(T^2 - \lambda(k))^{-1}_{m,n} &= \frac{1}{2(k-k^{-1})^3}\left[\frac{k^{2|m-n|}}{k+k^{-1}} + \mathrm{i}\frac{z^{2|m-n|}}{z+z^{-1}}\right] = \\ &= \frac{k^2}{(k^2-1)^2}\left[\frac{k^{2|m-n|}}{k^2-k^{-2}} + \mathrm{i}\frac{z^{2|m-n|}}{(z+z^{-1})(k-k^{-1})}\right].\end{aligned}$$

Now, we use (1.11) and arrive at

$$\forall m, n \in \mathbb{Z}: \quad (T^2 - \lambda(k))^{-1}_{m,n} = \frac{k^2}{2(k^2-1)^2}\left[\frac{k^{2|m-n|}}{k^2-k^{-2}} - \frac{z^{2|m-n|}}{z^2-z^{-2}}\right]. \tag{1.13}$$

Chapter 2

Spectral enclosures for the discrete polyharmonic operator

In this section we would like to improve upon the main result from [10], which is the localization of the spectrum of the discrete bilaplace operator T^2 perturbed by a complex potential. It is well known and solved problem for the discrete Laplace operator, even with the optimal result, see [9]. On the other hand, for the discrete bilaplacian T^2, only non-optimal result were found. A Conjecture on optimal enclosures has been formulated. It will be proved afterwards.
We will introduce a general approach for the discrete polyharmonic operator T^N and formulate a generalized conjecture on optimal result. This conjecture will be proved for $N = 2$. Recall that N is a positive integer.

Let us start with some general results describing changes of spectrum of a bounded self-adjoint linear operator on a Hilbert space under a compact perturbation.

Theorem 2.1. (**Birman–Schwinger principle**) Let L be a bounded self-adjoint operator and V be a compact operator on a Hilbert space $\mathcal{H}$ and let A, B be bounded operators on $\mathcal{H}$ such that $V = AB$, then for $\lambda \in \rho(L)$:

$$\lambda \in \sigma_p(L + V) \iff -1 \in \sigma_p(K(\lambda)),$$

where $K(\lambda) := B(L - \lambda)^{-1}A$ is the Birman–Schwinger operator.

Proof of Theorem 2.1 could be found in [10]. A more general version of the Birman–Schwinger principle is discussed in [8]. One can immediately deduce on important corollary, which is a necessary condition for λ from the resolvent set of L to be in a point spectrum of the operator $L + V$.

Corollary 2.2. Let L be a bounded self-adjoint operator and V be a compact operator on a Hilbert space $\mathcal{H}$ and let A, B be bounded operators on $\mathcal{H}$ such that $V = AB$, then following implication holds true

$$\lambda \in \sigma_p(L+V) \Longrightarrow \|K(\lambda)\| \geq 1,$$

for λ in $\rho(L)$ and $K(\lambda)$ the Birman–Schwinger operator.

We proceed by a theorem which deals with the essential spectrum of a bounded self-adjoint operator with a compact perturbation. Let us recall that we use a definition of the essential spectrum from [13].
Let A be a bounded operator on a Hilbert space H. The *algebraic multiplicity* of an isolated $\lambda \in \sigma_{\mathrm{p}}(A)$ is defined as

$$\nu_a(\lambda) := \dim(\mathrm{Ran}(P_\lambda)),$$

where

$$P_\lambda := \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_\lambda} (A-z)^{-1} dz, \quad \gamma_\lambda := \{w \in \mathbb{C} : \ |w-\lambda| = \varepsilon\}.$$

The $\varepsilon > 0$ is small enough that the interior and the boundary of γ_λ do not contain any other point of the spectrum of A. We define the *discrete spectrum* and the *essential spectrum* of A as follows:

$$\begin{aligned}\sigma_{\mathrm{disc}}(A) &:= \{\lambda \in \sigma_{\mathrm{p}}(A) : \ \lambda \text{ is isolated } \& \ \nu_a(\lambda) < +\infty\}, \\ \sigma_{\mathrm{ess}}(A) &:= \sigma(A) \setminus \sigma_{\mathrm{disc}}(A).\end{aligned}$$

Theorem 2.3. Let A be a bounded operator on a Hilbert space $\mathcal{H}$ and V be a compact operator on $\mathcal{H}$. And let the following hold true

1. The interior of $\sigma(A)$ in the topology of $\mathbb{C}$ is empty.
2. $\forall\ C$ connected component of $\rho(A)$: $C \cap \rho(A+V) \neq \emptyset$.

Then $\sigma_{\mathrm{ess}}(A+V) = \sigma_{\mathrm{ess}}(A)$.

Proof of Theorem 2.3 could be found in [13] in section XIII 4.

2.1 Spectral enclosures

Let us introduce the problem. We consider discrete polyharmonic operator with a complex potential $T^N + V$, where V is determined by a sequence $v \in \ell^1(\mathbb{Z})$, i.e.

$$Ve_n = v_n e_n, \quad \forall n \in \mathbb{Z}, e_n \in \mathcal{E}. \tag{2.1}$$

One can easily show that operator V is compact, even trace-class, see [10].

Our aim is to localize the spectrum of T^N perturbed by the potential V. The following theorem localizes the essential spectrum of $T^N + V$.

Proposition 2.4. Let V be a potential defined by $\ell^1(\mathbb{Z})$ sequence v as in (2.1), then

$$\sigma_{\text{ess}}(T^N + V) = \sigma_{\text{ess}}(T^N) = [0, 4^N].$$

Proof. It is clear that the interior of $\sigma(T^N) = [0, 4^N]$ in the topology of $\mathbb{C}$ is empty. Moreover, the operator T^N is bounded as well as V, thus the resolvent set $\rho(T^N + V)$ is unbounded and it surely intersects the only connected component of $\rho(T^N)$ which is $\mathbb{C} \setminus [0, 4^N]$. Theorem 2.3 concludes the proof. $\square$

Since the essential spectra of perturbed and unperturbed polyharmonic operator T^N are the same, only eigenvalues can appear due to perturbation V. We will localize the eigenvalues using the Birman–Shwinger principle and we will find the sets containing the discrete spectrum of $T^N + V$ which only depend on the ℓ^1-norm of generating sequence $v \in \ell^1(\mathbb{Z})$. Let us start with the decomposition of the potential V.

We define two operators $|V|^{1/2}$ and $V_{1/2}$ playing the role of operators A and B in the Birman–Schwinger principle (Theorem 2.1). They act as follows,

$$\forall n \in \mathbb{Z}: \quad |V|^{1/2} e_n := |v_n|^{1/2} e_n, \quad V_{1/2} e_n := |v|^{1/2} \operatorname{sgn}(v_n) e_n,$$

where $e_n \in \mathcal{E}$, $v \in \ell^1(\mathbb{Z})$ is the generating sequence of V and sgn is the complex sign function defined as

$$\operatorname{sgn}(z) := \begin{cases} \frac{z}{|z|} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

It is clear that

$$V = |V|^{1/2} V_{1/2}$$

and we define the Birman–Schwinger operator as

$$K(\lambda) := V_{1/2}(T^N - \lambda)^{-1} |V|^{1/2}. \tag{2.2}$$

Proposition 2.5. Let $K(\lambda)$ be defined by (2.2) and let number $E(\lambda) \in (0, +\infty)$ be such that $\forall m, n \in \mathbb{Z}: \ \left|(T^N - \lambda)^{-1}_{m,n}\right| \leq E(\lambda)$, then

$$\|K(\lambda)\| \leq E(\lambda) \|v\|_{\ell^1(\mathbb{Z})}.$$

Proof. The proposition immediately follows from the next estimate. For every $u \in \ell^2(\mathbb{Z})$ it holds

$$\begin{aligned} \|K(\lambda)u\|^2_{\ell^2(\mathbb{Z})} &= \sum_{m \in \mathbb{Z}} |\sum_{n \in \mathbb{Z}} K(\lambda)_{m,n} u_n|^2 \leq \sum_{m \in \mathbb{Z}} (\sum_{n \in \mathbb{Z}} |v_m|^{1/2} |(T^2 - \lambda I)^{-1}_{m,n}| |v_n|^{1/2} |u_n|)^2 \\ &\leq E(\lambda)^2 \sum_{m \in \mathbb{Z}} |v_m| (\sum_{n \in \mathbb{Z}} |v_n|^{1/2} |u_n|)^2 \overset{C-S}{\leq} E(\lambda)^2 \sum_{m \in \mathbb{Z}} |v_m| \sum_{n \in \mathbb{Z}} |u_n|^2 \sum_{k \in \mathbb{Z}} |v_k| \\ &= E(\lambda)^2 \|v\|^2_{\ell^1(\mathbb{Z})} \|u\|^2_{\ell^2(\mathbb{Z})}, \end{aligned}$$

where the Cauchy-Schwarz inequality was used. $\square$

The exact form of the estimate $E(\lambda)$ of the Green kernel of T^N will be introduced later. First, let us finish the general process of searching for the enclosures.

Proposition 2.6. Let $K(\lambda)$ be defined by (2.2) and let number $E(\lambda) \in (0, +\infty)$ be such that $\forall m, n \in \mathbb{Z} : \ \left|(T^N - \lambda)^{-1}_{m,n}\right| \leq E(\lambda)$, then

$$\sigma_p(T^N + V) \subset \left\{\lambda \in \rho(T^N) : \ \frac{1}{E(\lambda)} \leq \|v\|_{\ell^1(\mathbb{Z})}\right\} \cup [0, 4^N].$$

Proof. For $\lambda \in \rho(T^N)$ we use Corollary 2.2 which imply that if $\lambda \in \sigma_p(T^N + V)$ then

$$\|K(\lambda)\| \geq 1 \quad \Longrightarrow \quad 1 \leq \|K(\lambda)\| \leq E(\lambda) \, \|v\|_{\ell^1(\mathbb{Z})} \,,$$

where the result of Proposition 2.5 was used. Hence

$$\lambda \in \sigma_p(T^N + V) \Longrightarrow (E(\lambda))^{-1} \leq \|v\|_{\ell^1(\mathbb{Z})} \,.$$

We have localized the discrete spectrum of $T^N + V$, it is a subset of

$$\left\{\lambda \in \rho(T^N) : \ E(\lambda)^{-1} \leq \|v\|_{\ell^1(\mathbb{Z})}\right\}.$$

Now, we make a union of the essential spectrum of T^N, which is $[0, 4^N]$, with this set, because of the possible eigenvalues in essential spectrum. This concludes the proof. $\square$

From Proposition 2.6, we know that there is no eigenvalue of perturbed polyharmonic operator outside the given set, except the interval $[0, 4^N]$. By a *spectral enclosure* we understand a set

$$\mathrm{O}_N(\|v\|) := \left\{\lambda \in \rho(T^N) : \ \frac{1}{E(\lambda)} \leq \|v\|_{\ell^1(\mathbb{Z})}\right\}$$

which, as it was mentioned, depends only on the norm of the sequence $v \in \ell^1(\mathbb{Z})$ and the estimate $E(\lambda)$.

There exists more than one possibility how to get the estimate E of the absolute value of green kernel. Naturally, we would like to get the optimal one which is the supremum over $m, n \in \mathbb{Z}$ of $|(T^N - \lambda)^{-1}|$. It turns out that it is a non-trivial problem for $N > 1$. But it is still worth the effort since the optimal estimate induces, in a certain sense, optimal enclosures. Let us first propose one of the immediate but non-optimal estimates.

Proposition 2.7. Let $\lambda = \zeta_N(k) \in \mathbb{C} \setminus [0, 4^N]$, then

$$\forall m, n \in \mathbb{Z} : \quad |(T^N - \lambda(k))^{-1}_{m,n}| \leq \frac{1}{|N(k - k^{-1})^{2N-1}|} \sum_{j=1}^{N} \frac{1}{|z_j + z_j^{-1}|}.$$

Proof. The proof follows directly from Porposition 1.10 and the fact that all the roots $z_j = z_j(k)$ are in $\mathbb{D}$. $\square$

We have a conjecture on the optimal estimate, which will be proved later for one special case $N = 2$, the proof is easy for $N = 1$ and in this case, the estimate coincides with the one obtained from Proposition 2.7.

Conjecture 2.8. Let $\lambda = \zeta_N(k) \in \mathbb{C} \setminus [0, 4^N]$, then

$$\forall m, n \in \mathbb{Z}: \ |(T^N - \lambda(k))^{-1}_{m,n}| \leq |(T^N - \lambda(k))^{-1}_{0,0}| = \frac{1}{|N(k - k^{-1})^{2N-1}|} \left| \sum_{j=1}^{N} \frac{e^{2\pi i \frac{j-1}{2N}}}{z_j + z_j^{-1}} \right|.$$

This allows us to summarize our result in the following theorem.

Theorem 2.9. (**partly conjectural**) Let N be positive integer and V be a trace class potential generated by a sequence $v \in \ell^1(\mathbb{Z})$. We can localize the spectrum of the perturbed discrete polyharmonic operator T^N as follows

1. $\sigma(T^N + V) \subset$

$$\left\{ \lambda = \zeta_N(k) \in \mathbb{C} \setminus [0, 4^N] : \left[\frac{1}{|N(k - k^{-1})^{2N-1}|} \sum_{j=1}^{N} \frac{1}{|z_j + z_j^{-1}|} \right]^{-1} \leq \|v\|_{\ell^1(\mathbb{Z})} \right\} \cup [0, 4^N].$$

Similarly, provided that Conjecture 2.8 holds true, then

2. $\sigma(T^N + V) \subset$

$$\left\{ \lambda = \zeta_N(k) \in \mathbb{C} \setminus [0, 4^N] : \left[\frac{1}{|N(k - k^{-1})^{2N-1}|} \left| \sum_{j=1}^{N} \frac{e^{2\pi i \frac{j-1}{2N}}}{z_j + z_j^{-1}} \right| \right]^{-1} \leq \|v\|_{\ell^1(\mathbb{Z})} \right\} \cup [0, 4^N].$$

Proof. Part 1. follows directly from Propositions 2.4, 2.7 and 2.6. Part 2. follows from the same propositions, we just consider Conjecture 2.8 instead of Proposition 2.7. □

2.2 Proof of Conjecture 2.8 for $N = 2$

In this section we prove Conjecture 2.8 for $N = 2$. Afterwards, we discuss a possible method of proving the conjecture for $N \geq 3$.

Firstly, let us recall a well-known result of the complex analysis.

Theorem 2.10. (**Maximum Modulus Principle**).
Let Ω be a connected open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ holomorphic on Ω and $f \neq$ const. Then $|f|$ cannot exhibit a strict local maximum in Ω.

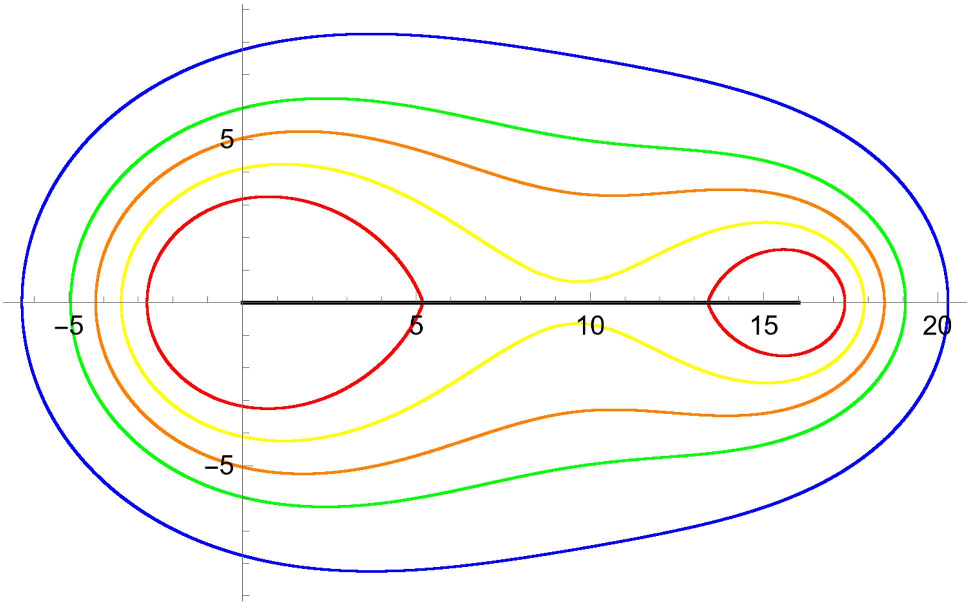


Figure 2.1: The optimal spectral enclosures for the discrete bilaplacian with a complex potential determined by a sequence $v \in \ell^1(\mathbb{Z})$ for $\|v\|_{\ell^1(\mathbb{Z})} \in \{8, 10, 12, 14, 18\}$.

Corollary 2.11. Let Ω be a bounded connected open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ holomorphic on Ω and continuous on $\bar{\Omega}$. Then

$$\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|.$$

We use the following form of the Green kernel of T^2 which was given in (1.13) and reads

$$\forall m, n \in \mathbb{Z}: \quad (T^2 - \lambda(k))^{-1}_{m,n} = \frac{k^2}{2(k^2-1)^2} \left[\frac{k^{2|m-n|}}{k^2 - k^{-2}} - \frac{z^{2|m-n|}}{z^2 - z^{-2}} \right].$$

Or aim is to show

$$\forall m, n \in \mathbb{Z} : |((T^2 - \lambda(k))^{-1})_{m,n}| \leq |((T^2 - \lambda(k))^{-1})_{0,0}|, \;\; k \in \mathbb{D}_2^{\zeta}.$$

Recall that $z = z(k)$ is the unique solution of (1.11) in $\mathbb{D}$ where $z := z_2$. Let us denote $s := |m - n|$ which is nonnegative integer. We can divide both sides of the inequality by $|k^2/2(k^2 - 1)^2|$, slightly manipulate the other terms from which we obtain

$$\forall s \in \mathbb{Z}_0^+ \quad | \, k^{2s} \left(z^2 - z{-}2\right) - z^{2s} \left(k^2 - k^{-2}\right) | \leq | \left(z^2 - z^{-2}\right) - \left(k^2 - k^{-2}\right) | \, .$$

Let us now analyze the position of $z = z(k)$.

Lemma 2.12. Let $k \in \mathbb{D}_2^{\zeta}$, then for $z = z(k)$, which is given by (1.11) as $z - z^{-1} = \mathrm{i}(k - k^{-1})$, holds

$$z \in \{w \in \mathbb{C} : \;\; |w| < 1, \mathrm{Re}(w) > 0, \mathrm{Im}(w) < 0 \;\} \cup (0, \sqrt{2} - 1).$$

Proof. We use the results of Subsection 1.3.1. Hence for $k \in \mathbb{D}_2^\zeta$ one has

$$\zeta_i(k) = k - k^{-1} \in \left\{ w = r\mathrm{e}^{\mathrm{i}t} \in \mathbb{C} : \ r \in (0, +\infty), \ t \in [\frac{\pi}{2}, \pi) \right\} \setminus \mathrm{i}[0,2].$$

Since $\zeta_i(z) = \mathrm{i}\zeta_i(k)$, we see that $\zeta_i(z)$ is located in the same sector which is only rotated by an angle $+\pi/2$ (it is equivalent to the multiplication by i). The proof is concluded by applying the inverse transformation ζ_i^{-1}. □

Due to Lemma 2.12 we see that once $k \in \mathbb{D}_2^\zeta$, z is, up to a part of the boundary, in the symmetric region in the lower half of $\mathbb{D}$ with respect to the real axis (i.e. $\mathrm{Im}(z) \le 0$). One can see that k occurs only as a second power k^2 in our problem, as well as z . Thus, we can immediately determine its regions as

$$\begin{aligned} k^2 &\in \mathbb{D}_{k^2} := \{w \in \mathbb{C} : \ |w| < 1, \mathrm{Im}(w) > 0\} \cup (-1, 0), \\ z^2 &\in \mathbb{D}_{z^2} := \{w \in \mathbb{C} : \ |w| < 1, \mathrm{Im}(w) < 0\} \cup (0, (\sqrt{2} - 1)^2). \end{aligned}$$

The following proposition is in fact more general than the original hypothesis. We simply get rid of the dependence of z on k and formulate the problem for two independent complex numbers in the regions $\mathbb{D}_{k^2}$ and $\mathbb{D}_{z^2}$.

Proposition 2.13. Let $s \in \mathbb{Z}_0^+$, then

$$\forall u \in \mathbb{D}_{k^2}, \forall v \in \mathbb{D}_{z^2} : \quad | \, u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right) | \le | \left(v - v^{-1}\right) - \left(u - u^{-1}\right) | \, .$$

It is clear, that to prove the Conjecture 2.8, it is sufficient to prove this proposition. Moreover, we restrict the regions for numbers u and v.

Proposition 2.14. Let $s \in \mathbb{Z}_0^+$, then the following statements are equivalent:

1. $\forall u \in \mathbb{D}_{k^2}, \forall v \in \mathbb{D}_{z^2} : \quad | \, u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right) | \le | \left(v - v^{-1}\right) - \left(u - u^{-1}\right) |$,
2. $\forall u \in \partial\mathbb{D}_{k^2}, \forall v \in \partial\mathbb{D}_{z^2} : \quad | \, u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right) | \le | \left(v - v^{-1}\right) - \left(u - u^{-1}\right) | \, .$

Proof. We consider a function $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined as

$$f_s(u, v) := \frac{u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right)}{\left(v - v^{-1}\right) - \left(u - u^{-1}\right)} = \frac{u^{s+1} \left(v^2 - 1\right) - v^{s+1} \left(u^2 - 1\right)}{(1 + uv)(v - u)},$$

where $s \in \mathbb{Z}_0^+, \ u \in \mathbb{D}_{k^2}, \ v \in \mathbb{D}_{z^2}$. Part 1 of the proposition is then equivalent to

$$|f_s(u, v)| \le 1, \forall u \in \mathbb{D}_{k^2}, \forall v \in \mathbb{D}_{z^2}.$$

We choose a fixed $v \in \mathbb{D}_{z^2}$ arbitrarily and look closer to the function $f_s(\cdot, v)$, which is analytic on $\mathbb{D}_{k^2}$ and continuous on $\overline{\mathbb{D}}_{k^2}$. Indeed, it holds that the limit

$$\lim_{u \to v} f_s(u, v)$$

exists and is finite. The second factor in the denominator of f_s could generate singularity if $1+uv=0$, but in this case $u=-1/v$, thus $u \notin \overline{\mathbb{D}_{k^2}}$, since $|v|<1$ and $|u|=|1/v|>1$. Using the Maximum Modulus Principle (MMP) we obtain

$$|f_s(u,v)| \leq 1, \forall u \in \partial\mathbb{D}_{k^2}, \forall v \in \mathbb{D}_{z^2}.$$

Now, we take fixed $u \in \partial\mathbb{D}_{k^2}$ and analyze the function $f_s(u,\cdot)$. Function f_s satisfies the assumptions of MMP. Indeed, the only term in the definition of function f_s which could generate a singularity is $(1+uv)$ if $v=-1/u$ and $u \in \partial\mathbb{D}_{k^2}$. We look closer on following situations:

$$\begin{aligned}
&u = e^{\mathrm{i}\phi}, \phi \in (0,\pi): \quad v = -\frac{1}{u} = e^{\mathrm{i}\phi} \notin \mathbb{D}_{z^2},\\
&u \in (-1,1): \quad v = -\frac{1}{u} \notin \mathbb{D}_{z^2} \text{ since } \left|\frac{1}{u}\right| > 1,\\
&u = \pm 1: \quad v = -\frac{1}{u} = \mp 1.
\end{aligned}$$

Functions $f_s(\pm 1, v)$ do not have singularities at $v=\mp 1$. The proof is finished by using MMP again. $\square$

Following theorem concludes the proof of Conjecture 2.8.

Theorem 2.15. Let $s \in \mathbb{Z}_0^+$, then

$$\forall u \in \partial\mathbb{D}_{k^2}, \forall v \in \partial\mathbb{D}_{z^2}: \quad |\, u^s\left(v-v^{-1}\right) - v^s\left(u-u^{-1}\right) | \leq | \left(v-v^{-1}\right) - \left(u-u^{-1}\right) | \,. \quad (2.3)$$

Proof. Let us mention that the boundaries $\partial\mathbb{D}_{k^2}$ and $\partial\mathbb{D}_{z^2}$ are simply the upper and the lower half of the unit circle respectively, and interval $[-1,1]$.

Theorem 2.15 will be proven using following strategy. We divide the boundaries of the upper and the lower half of the unit circle into parts and then we will prove the theorem considering all the positions of u and v step by step.

In the following, we denote

$$u = r_u e^{\mathrm{i}\phi_u}, \quad v = r_v e^{\mathrm{i}\phi_v}.$$

1. We consider the case, when both u and v are real. Without loss of generality, let us assume that $0<r_u<r_v<1$. If $r_u>r_v$ we can use the symmetry of u,v in this problem. If $0>r_u>r_v>-1$ we can get the same problem as if the numbers are positive because of the absolute value in (2.3).

   We have $u=r_u, v=r_v$, thus the inequality is in the form

$$\begin{aligned}
\left|r_u^s\left(r_v-r_v^{-1}\right) - r_v^s\left(r_u-r_u^{-1}\right)\right| &\leq \left|\left(r_v-r_v^{-1}\right) - \left(r_u-r_u^{-1}\right)\right|,\\
\left|r_u^{s+1}\left(1-r_v^2\right) - r_v^{s+1}\left(1-r_u^2\right)\right| &\leq \left|r_u\left(1-r_v^2\right) - r_v\left(1-r_u^2\right)\right|.
\end{aligned}$$

Let us denote RHS $:= r_v\,(1-r_u^2) - r_u\,(1-r_v^2)$ and
LHS $:= r_v^{s+1}\,(1-r_u^2) - r_u^{s+1}\,(1-r_v^2)$. Since RHS ≥ 0 and LHS ≥ 0, $\forall s \in \mathbb{Z}_0^+$, the problem is as follows

$$0 \leq \text{ RHS } - \text{ LHS } = r_v\left(1-r_u^2\right)(1-r_v^s) - r_u\left(1-r_v^2\right)(1-r_u^s),$$
$$0 \leq \frac{r_v\,(1-r_v^s)}{1-r_v^2} - \frac{r_u\,(1-r_u^s)}{1-r_u^2}.$$

Consider the real function
$$\xi(x) = \frac{x\,(1-x^s)}{1-x^2},$$
which is increasing on interval $(0,1)$. Hence we get $0 \leq \xi(y)-\xi(x)$ for any $x,y \in (0,1), y > x$. If we put $x := r_u$ and $y := r_v$ the lemma is proved. It also follows from this proof that the inequality holds $\forall r_u, r_v \in (-1,1)$. Indeed, for odd s we can use directly previous part of this proof. For s even, the function ξ is odd and we can also prove it the same way. It remains to prove that the function ξ is increasing.

Indeed, for $s=0$ it is clear, for $s \geq 1$ it holds
$$\xi(x) = \frac{x}{1+x}\sum_{j=0}^{s-1} x^j$$
and thus
$$\xi'(x) = \frac{1}{(1+x)^2}\left((1+x)\sum_{j=0}^{s-1}(j+1)x^j - \sum_{j=0}^{s-1} x^{j+1}\right) =$$
$$= \frac{1}{(1+x)^2}\left(\sum_{j=0}^{s-1} jx^j + \sum_{j=0}^{s-1} jx^{j+1} + \sum_{j=0}^{s-1} x^j\right) > 0,$$
for $x \in (0,1)$. Which was to be proved.

2. Now consider u and v such that $r_u = r_v = 1$. Let us denote

$$\mathrm{g}(s) :=\mid u^s\left(v-v^{-1}\right) - v^s\left(u-u^{-1}\right)\big|^2$$

We have $u = e^{\mathrm{i}\phi_u}$ and $v = e^{\mathrm{i}\phi_v}$, $\phi_u \in (0,\pi)$, $\phi_v \in (-\pi, 0)$ and using the definition of absolute value of the complex number and some trigonometric identities we obtain

$$\mathrm{g}(s) = \left|e^{\mathrm{i}s\phi_u}2\mathrm{i}\sin\phi_v - e^{\mathrm{i}s\phi_v}2\mathrm{i}\sin\phi_u\right|^2 =$$
$$= 4\left(\left(\sin(s\phi_u)\sin(\phi_v) - \sin(s\phi_v)\sin(\phi_u)\right)^2 - \right.$$
$$\left.\left(\cos(s\phi_u)\sin(\phi_v) - \cos(s\phi_v)\sin(\phi_u)\right)^2\right) =$$
$$= 4\left(\sin^2(\phi_u) + \sin^2(\phi_v) - \right.$$
$$\left.2\sin(\phi_u)\sin(\phi_v)\underbrace{\left(\cos(s\phi_u)\cos(s\phi_v) + \sin(s\phi_u)\sin(s\phi_v)\right)}_{\cos(s(\phi_u-\phi_v))}\right).$$

Now it is not hard to verify that $\mathrm{g}(s) \leq \mathrm{g}(0)$. Let us analyze

$$\mathrm{g}(0) - \mathrm{g}(s) = -8 \underbrace{\sin(\phi_u)}_{\geq 0} \underbrace{\sin(\phi_v)}_{\leq 0} \underbrace{\left(1 - \cos\left(s\left(\phi_u - \phi_v\right)\right)\right)}_{2\sin^2\left(\frac{s}{2}(\phi_u - \phi_v)\right)} \geq 0.$$

Thus, the inequality (2.3) holds for the considered range of u and v.

3. Now consider number v to be real and positive, u to be complex. It is $r_u = 1$, $r_v \in [0, 1]$ and $\phi_u \in (0, \pi), \phi_v = 0$. Denote

$$\mathrm{h}(s) := \left| u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right)\right|^2$$

As in the previous parts, we have $u = e^{\mathrm{i}\phi_u}$ and $v = r_v$ and we obtain

$$\begin{aligned} \mathrm{h}(s) &= \left|e^{\mathrm{i}s\phi_u}\left(r_v - r_v^{-1}\right) - r_v^s 2\mathrm{i}\sin\phi_u\right|^2 = \\ &= \left(r_v - r_v^{-1}\right)\cos^2\left(s\phi_u\right) + \left(\left(r_v - r_v^{-1}\right)\sin\left(s\phi_u\right) - 2r_v^s \sin\left(\phi_u\right)\right)^2 \\ &= \left(r_v - r_v^{-1}\right)^2 - 4r_v^s\left(r_v - r_v^{-1}\right)\sin\left(s\phi_u\right)\sin\left(\phi_u\right) + 4r_v^{2s}\sin^2\left(\phi_u\right). \end{aligned}$$

In fact, the difference $\mathrm{h}(0) - \mathrm{h}(s)$ is nonnegative,

$$\mathrm{h}(0) - \mathrm{h}(s) = 4\sin^2\left(\phi_u\right)\left(1 - r_v^{2s}\right) + 4r_v^s\left(r_v - r_v^{-1}\right)\sin\left(s\phi_u\right)\sin\left(\phi_u\right) \stackrel{?}{\geq} 0.$$

Dividing both sides by positive terms $4\sin^2\left(\phi_u\right), r_v^{-1} - r_v, r_v^s$, we get

$$0 \leq \frac{r_v^{-s} - r_v^s}{r_v^{-1} - r_v} - \frac{\sin\left(s\phi_u\right)}{\sin\left(\phi_u\right)}. \tag{2.4}$$

Since the function

$$\eta(x) = \frac{x^{-s} - x^s}{x^{-1} - x}$$

is decreasing on $(0, 1)$, $\lim_{x\to 1^-} \eta(x) = s$ (see the end of the proof) and

$$\max_{\phi_u \in (0,\pi)} \left(\frac{\sin\left(s\phi_u\right)}{\sin\left(\phi_u\right)}\right) = s$$

(see the end of the proof), the inequality (2.4) holds. We can easily prove that the inequality (2.3) holds for $u \in \partial\mathbb{D}_{k^2}, v \in \partial\mathbb{D}_{z^2}$ such that $r_v = 1, r_u \in [0, 1]$ and $\phi_v \in (-\pi, 0), \phi_u = 0$. It is enough to use the symmetry of u and v in the problem and the fact that

$$\max_{\phi \in (0,\pi)} \left(\frac{\sin\left(s\phi\right)}{\sin\left(\phi\right)}\right) = s.$$

4. Now consider similar case, but $r_u = 1, r_v \in [-1, 0]$ and $\phi_u \in (0, \pi), \phi_v = 0$. Denote

$$\mathrm{h}(s) := \left| u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right)\right|^2.$$

Using exactly the same method as in the previous case we get

$$\mathrm{h}(0)-\mathrm{h}(s)=4\sin^2(\phi_u)\left(1-r_v^{2s}\right)+4r_v^s\left(r_v-r_v^{-1}\right)\sin(s\phi_u)\sin(\phi_u)\geq 0. \tag{2.5}$$

Now we have to separately discuss two situations. Firstly, consider s is odd. In this case, we divide both sides of the inequality (2.5) by the same terms as in the proof of previous part, i.e. $4\sin^2(\phi_u)>0, r_v^{-1}-r_v<0, r_v^s<0$. Now we have the same inequality for different range of parameters

$$0\leq\frac{r_v^{-s}-r_v^s}{r_v^{-1}-r_v}-\frac{\sin(s\phi_u)}{\sin(\phi_u)}.$$

Since both functions are even in this case, the inequality follows directly from the proof of previous case.

To finish the proof, we assume s even. Again, we divide both sides of the inequality (2.5) by $4\sin^2(\phi_u)>0$, $r_v^{-1}-r_v<0, r_v^s<0$ and obtain

$$0\geq\frac{r_v^{-s}-r_v^s}{r_v^{-1}-r_v}+\left(-\frac{\sin(s\phi_u)}{\sin(\phi_u)}\right).$$

Now $\eta(x)=\frac{x^{-s}-x^s}{x^{-1}-x}$ is odd function and $\frac{\sin(s\phi_u)}{\sin(\phi_u)}$ is odd. It is easy to see that the inequality holds.
One can easily prove (2.3) for $u\in\partial\mathbb{D}_{k^2}, v\in\partial\mathbb{D}_{z^2}$ such that $r_v=1, r_u\in[-1,0]$ and $\phi_v\in(-\pi,0), \phi_u=0$. It follows directly from symmetry of u and v in (2.3) and properties of the functions η and $\sin(sx)/\sin(x)$.

Now it remains to prove that the function η is decreasing. Moreover, η is odd function for even s and even function for odd s. Indeed, the oddness/evenness follows directly from the definition and the monotonicity is easy for $s\in\{0,1\}$. For $s\geq 2$, we have

$$\eta(x)=\frac{x^{-s}-x^s}{x^{-1}-x}=x^{-s+1}\frac{x^{2s}-1}{x^2-1}=x^{-s+1}\sum_{k=0}^{s-1}x^{2k}=\sum_{k=0}^{s-1}x^{2k-s+1}.$$

Hence

$$\lim_{x\to 1^-}\eta(x)=s \quad \& \quad \lim_{x\to -1^+}\eta(x)=\begin{cases} s: & s \quad \text{odd,}\\ -s: & s \quad \text{even}\end{cases}$$

and also

$$\eta'(x)=\sum_{k=0}^{s-1}(2k-s+1)x^{2k-s}.$$

Now we look closer to the sum. There are s terms in the sum for every even $s\geq 2$ and $s-1$ terms for odd $s>2$. There are also $\lfloor s/2\rfloor$ positive terms and $\lfloor s/2\rfloor$ negative terms. It holds that for any $x\in(0,1)$

$$\forall s\geq 2, \forall l\in\{0,1,\ldots,\lfloor s/2\rfloor-1\}:$$
$$\left|(2l-s+1)x^{2l-s}\right|\geq\left|(2(s-1-l)-s+1)x^{2(s-1-l)-s}\right|.$$

Indeed, using standard algebraic manipulations we get

$$\begin{aligned}
\left|(2l-s+1)x^{2l-s}\right| &\geq \left|(2(s-1-l)-s+1)x^{2(s-1-l)-s}\right|, \\
\left|(2l-s+1)x^{2l-s}\right| \mid &\geq \left|-(2l-s+1)x^{-2l+s-2}\right|, \\
x^{(2l-s)-(-2l+s-2)} &= x^{4l-2s+2} \geq 1.
\end{aligned}$$

It holds, because $4l-2s+2 \leq 0, \forall s \geq 2, \forall l \in \{0,1,\ldots,\lfloor s/2 \rfloor -1\}$ and x is from $(0,1)$.

Last proposition to show to finish the proof is that for every $s \in \mathbb{Z}_0^+$,

$$\max_{\phi \in (0,\pi)} \frac{\sin(s\phi)}{\sin(\phi)} = s.$$

It is easy for $s=0$. It holds for $s>0$, because the function

$$\frac{\sin(s\phi)}{\sin(\phi)} = U_{s-1}(\cos\phi),$$

where $U_n(x)$ is the second kind Chebyshev polynomial which has extreme values at ± 1. The proof can be found in the first chapter in [12]. It is clear that the value of the function $U_{s-1}(\cos\phi)$ at $\phi = 0$ is s.

□

2.2.1 A remark on validity of the conjecture for $N \geq 3$

The proof of Conjecture 2.8 in general would be the most valuable result for us. Let us now discuss a method of proving this conjecture analytically. Let us mention, that the proof is not done and this is just an idea of a possible method.
Our approach is the same as in the previous part. First, we need to localize the regions for the roots of polynomial

$$p_N(z) = (z^2-1)^{2N} - z^{2N}(k-k^{-1})^{2N},$$

in the unit disk $\mathbb{D}$ which are occurring in the formula for the Green kernel. The following part is a brief reminder of a part in the first chapter. Recall that we consider $\lambda = \lambda(k) = \zeta_N(k)$ and element of the Green kernel (1.12) reads

$$\forall m,n \in \mathbb{Z}: \quad (T^N - \lambda(k))^{-1}_{m,n} = \frac{(-1)^N}{N(k-k^{-1})^{2N-1}} \sum_{j=1}^{N} \frac{z_j^{2|n-m|} e^{2\pi i \frac{j-1}{2N}}}{z_j + z_j^{-1}}.$$

Thus, roots $z_1, \ldots, z_N$ are important for us. We know that they are related via (1.11) as follows

$$z_j - z_j^{-1} = e^{2\pi i \frac{j-1}{2N}}(k-k^{-1}), \quad \text{for } j \in \{1,\ldots,N\}.$$

Hence it follows that $z_1 = k$ and since we know the region for k which is $\mathbb{D}_N^\zeta$, we can localize the rest of the roots.

Let us denote $s := |m-n| \geq 0$, Conjecture 2.8 is then of the form

$$\forall s \in \mathbb{Z}_0^+,\ \forall N \in \mathbb{Z}^+ : \quad \left| \sum_{j=1}^{N} \frac{z_j^{2s} \mathrm{e}^{2\pi \mathrm{i} \frac{j-1}{2N}}}{z_j + z_j^{-1}} \right| \leq \left| \sum_{j=1}^{N} \frac{\mathrm{e}^{2\pi \mathrm{i} \frac{j-1}{2N}}}{z_j + z_j^{-1}} \right|.$$

Having this, we know that the N regions for the roots $z_1, ..., z_N$ fill the right half of $\mathbb{D}$. Let us denote them $\mathbb{D}_{z_1}, ..., \mathbb{D}_{z_N}$. Then we need to get rid of the mutual relations of the roots and formulate the more general problem for N independent parameters $u_1 \in \mathbb{D}_{z_1}, ..., u_N \in \mathbb{D}_{z_N}$. Then, we could possibly use the MMP and reduce the regions onto the boundaries. Since our method is based on proving the inequalities for each pair of the boundaries, there arise first problem. The number of boundaries part increases and thus the number of pairs increases too, even though we can take advantage of the symmetry of this problem with respect to cyclic permutations of the parameters.
Unfortunately, I was not able to proceed this way even for $N = 3$. Moreover, there is another problem which is the shape of the boundaries for higher N. Namely l_N given by (1.8) is one of the boundary parts which complicates the analytic approach.

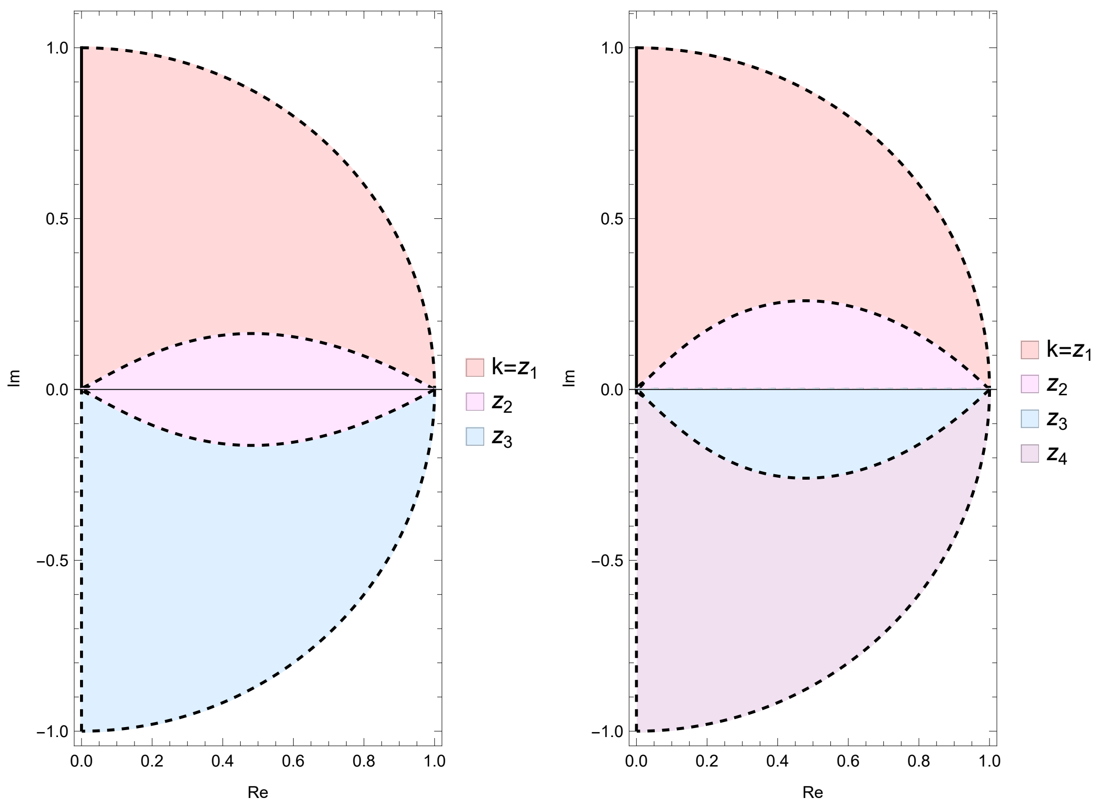
Despite all the problems mentioned, we can try to numerically disprove the conjecture. Using $k \in \mathbb{D}_N^\zeta$ and (1.11), we made a grid covering the regions $\mathbb{D}_{z_1}, ..., \mathbb{D}_{z_N}$. We tested the conjecture on the grid for $0 \leq s \leq 40$ and $N \leq 10$ considering the dependence of $z_1, ..., z_N$. The result is that we cannot disprove the conjecture.
The regions of the roots $z_1, ..., z_N$ for $N = 3$ and $N = 4$ are in Figures 2.2a and 2.2b respectively.

2.3 Optimality

In [9], it was shown that the enclosures obtained using our method for the discrete Schrodinger operator H are optimal. The optimality is understood in the following sense. Take $\|v\|_{\ell^1(\mathbb{Z})}$ fixed, then for every boundary point z of the spectral enclosure $\mathrm{O}_1(\|v\|)$ not belonging to $\sigma_{\mathrm{ess}}(T) = [0, 4]$, there exist a potential V such that z is an eigenvalue of $T + V$.
Similarly, in [10], same method was used for T^2 considering the optimal estimate of the Green kernel, which is now proved in Conjecture 2.8 for $N = 2$. Naturally, one can easily generalize this procedure for the discrete polyharmonic operator.

We define a special compact diagonal potential δ_k on $\ell^2(\mathbb{Z})$ by giving its kernel

$$(\delta_k)_{m,n} := \begin{cases} 1 & m = n = k, \\ 0 & \text{otherweise.} \end{cases} \tag{2.6}$$



(a) $N = 3$ (b) $N = 4$

Figure 2.2: Regions of the first N roots of polynomial p_N for selected N in $\mathbb{D}$.

This potential is generated by sequence $\{\delta_{k,j}\}_{j\in\mathbb{Z}}$. It holds $\delta_k = \langle e_k, \cdot\, \rangle\, e_k$ for $e_k \in \mathcal{E}$.

Recall that the conjectural spectral enclosure from Theorem 2.9 reads

$$\mathrm{O}_N(\|v\|) = \left\{ \lambda = \zeta_N(k) \in \mathbb{C}\setminus[0,4^N] : \left[\frac{1}{|N(k-k^{-1})^{2N-1}|} \left| \sum_{j=1}^{N} \frac{\mathrm{e}^{2\pi\mathrm{i}\frac{j-1}{2N}}}{z_j + z_j^{-1}} \right| \right]^{-1} \leq \|v\|_{\ell^1(\mathbb{Z})} \right\}. \tag{2.7}$$

Assuming the validity of Conjecture 2.8 for all $N > 0$, the following proposition proves the optimality of the enclosures $\mathrm{O}_N(\|v\|)$.

Proposition 2.16. (**conjectural**) Let $v \in \ell^1(\mathbb{Z})$ be a generating sequence of potential V and let $\mathrm{O}_N(\|v\|)$ be the conjectural spectral enclosure given by (2.7). Then

$$\forall z \in \partial\mathrm{O}_N(\|v\|)\setminus[0,4^N],\ \exists \omega\in\mathbb{C}, |\omega| = \|v\|_{\ell^1(\mathbb{Z})} : \quad z \in \sigma_{\mathrm{p}}(T^2 + \omega\delta_0).$$

Proof. Take $z \in \partial\mathrm{O}_N(\|v\|)\setminus[0,4^N]$ arbitrary and fixed. We transform this point using ζ_N and we obtain a unique $k \in \mathbb{D}_N^{\zeta}$ such that $z = \zeta_N(k)$. Now put

$$r(k) := (T^N - \lambda(k))^{-1}_{0,0} = \frac{-1}{N(k-k^{-1})^{2N-1}} \sum_{j=1}^{N} \frac{\mathrm{e}^{2\pi\mathrm{i}\frac{j-1}{2N}}}{z_j + z_j^{-1}}.$$

It follows from the definition of the spectral enclosure and from the fact z is on the boundary of the enclosure that $|1/r(k)| = \|v\|_{\ell^1(\mathbb{Z})}$. Now we consider a potential $\omega\delta_0$ for $\omega \in \mathbb{C}$ and δ_0 given by (2.6). This potential is generated by sequence $\{\omega\delta_{0,n}\}_{n\in\mathbb{Z}}$, which has a norm equal to $|\omega|$. The kernel of Birman–Schwinger operator for this potential is

$$K(z(k))_{m,n} := \begin{cases} r(k)\omega & m = n = 0, \\ 0 & \text{otherweise.} \end{cases}$$

It is clear that $r(k)\omega$ is an eigenvalue of $K(z(k))$. Thus, according to Birman–Schwinger principle, $z = \zeta_N(k) \in \partial\mathrm{O}_N(\|v\|)\setminus[0,4^N]$ is an eigenvalue of $T^N + \omega\delta_0$ if and only if

$$r(k)\omega = -1 \iff \omega = -\frac{1}{r(k)}.$$

So we set

$$\omega := \frac{-1}{r(k)}.$$

Proof is concluded by noticing that $|\omega| = |-1/r(k)| = \|v\|_{\ell^1(\mathbb{Z})}$. $\square$

2.4 Absence of eigenvalues in the essential spectrum of the discrete bilaplacian

In previous parts of this chapter, it was found that only eigenvalues can appear due to the perturbation of T^N by a trace-class potential V. Considering the most explored case $N = 1$, i.e. discrete Laplace operator T or the operator $H = 2I - T$, it was shown in [6], that the eigenvalue cannot appear in the interior of essential spectrum of H ($\sigma_{\text{ess}}(H)^\circ$) , i.e. in interval $(-2, 2)$, perturbed by a trace class perturbation. A direct method of solving the eigenvalue equation was used in the proof. It is based on a tridiagonal structure of the matrix of discrete Laplace operator.
We would like to get a similar result for $N = 2$, but I was not able to generalize this method. Thus, we will use a less general result, which allows us to exclude eigenvalues at least outside the spectral enclosure $\mathrm{O}_2(\|v\|)$. In fact, the spectral enclosure contains the whole essential spectrum of T^2 for the potentials with $\|v\|$ large and thus this method is satisfactory for small potentials only. It is a discrete version of a result from [3].

Proposition 2.17. Let $v \in \ell^1(\mathbb{Z})$ be a generating sequence of a potential V and $\lambda \in (0, 16) \equiv \sigma_{\text{ess}}(T^2)^\circ$ such that $(T^2 + V)u = \lambda u$, for $0 \neq u \in \ell^2(\mathbb{Z})$. Then

$$\forall n \in \mathbb{Z}: \quad \big((I + K(\lambda + \mathrm{i}\varepsilon))\, |V|^{1/2} u\big)_n \longrightarrow 0, \quad \varepsilon \to 0^+,$$

where $K(\mu) = |V|^{1/2}(T^2 - \mu)^{-1} V_{1/2}$, $\mu \in \rho(T^2)$ is the Birman–Schwinger operator.

Proof. Take $n \in \mathbb{Z}$ fixed, then

$$\begin{aligned}(K(\lambda + \mathrm{i}\varepsilon)|V|^{1/2}u)_n &= \sum_{m\in\mathbb{Z}} \underbrace{|v_n|^{1/2}(T^2 - \lambda - \mathrm{i}\varepsilon)^{-1}_{n,m}}_{(G_n(\varepsilon))_m} v_m u_m = \\ &= -\sum_{m\in\mathbb{Z}} (G_n(\varepsilon))_m (T^2 u)_m + \lambda \sum_{m\in\mathbb{Z}} (G_n(\varepsilon))_m u_m,\end{aligned}$$

where the assumption that $Vu = -T^2 u + \lambda u$ was used. Using T^2 is self-adjoint and $G_n(\varepsilon) \in \ell^2(\mathbb{Z})$, we can continue as follows,

$$(K(\lambda + \mathrm{i}\varepsilon)|V|^{1/2}u)_n = \lambda \langle G_n(\varepsilon), \overline{u}\rangle - \left\langle G_n(\varepsilon), T^2\overline{u}\right\rangle = -\left\langle (T^2 - \lambda)G_n(\varepsilon), \overline{u}\right\rangle.$$

Since it holds that $(T^2 - \lambda - \mathrm{i}\varepsilon)G_n(\varepsilon) = |v_n|^{1/2} e_n$, where $e_n \in \mathcal{E}$, we have

$$(K(\lambda + \mathrm{i}\varepsilon)|V|^{1/2}u)_n = -|v_n|^{1/2}\langle e_n, \overline{u}\rangle - \mathrm{i}\varepsilon \langle G_n(\varepsilon), \overline{u}\rangle.$$

Hence

$$\forall n \in \mathbb{Z}: \quad \big((I + K(\lambda + \varepsilon)|V|^{1/2}u)\big)_n = -\mathrm{i}\varepsilon \langle G_n(\varepsilon), \overline{u}\rangle. \tag{2.8}$$

Let us now analyze the absolute value of $\langle G_n(\varepsilon), \overline{u}\rangle$ | on the right hand side. Using the Cauchy–Schwarz inequality we obtain $|\langle G_n(\varepsilon), \overline{u}\rangle| \leq \|G_n(\varepsilon)\|\,\|u\|$. Only part dependent on ε is $\|G_n(\varepsilon)\|$. To get it asymptotic behavior as $\varepsilon \to 0^+$, we proceed as follows

$$\begin{aligned}\|G_n(\varepsilon)\|^2 &= |v_n| \sum_{m\in\mathbb{Z}} \left|(T^2-\lambda-\mathrm{i}\varepsilon)^{-1}_{n,m}\right|^2 = \\ &= |v_n| \sum_{m\in\mathbb{Z}} \left|\frac{k^2}{2(k^2-1)^2}\left[\frac{k^{2|m-n|}}{k^2-k^{-2}} - \frac{z^{2|m-n|}}{z^2-z^{-2}}\right]\right|^2,\end{aligned}$$

where (1.13) and the transform $\lambda + \mathrm{i}\varepsilon = \zeta_2(k)$ were used.
Now one can verify that

$$\|G_n(\varepsilon)\| = \mathcal{O}(\frac{1}{\sqrt{\varepsilon}}), \quad \varepsilon \to 0^+. \tag{2.9}$$

Indeed, we can get the asymptotic expansion of k and k^{-1} from the equation $\lambda + \mathrm{i}\varepsilon = \zeta_2(k) = (k-k^{-1})^4$ as follows

$$k(\varepsilon, \lambda(\varphi)) = \mathrm{e}^{\frac{\mathrm{i}\varphi}{2}}\left(1 + \frac{\varepsilon}{8(2\cos(\varphi)-2)\sin(\varphi)}\right) + \mathcal{O}(\varepsilon^2), \quad \varepsilon \to 0^+, \tag{2.10}$$

where $\lambda = (2-2\cos(\varphi))^2 \in (0,16)$, considering $\varphi \in (0,\pi)$. The expansion for $z = z(k)$ can be obtained from (1.11) which reads $z - z^{-1} = \mathrm{i}(k-k^{-1})$. Consider $z(\varepsilon) = z_0 + z_1\varepsilon + \mathcal{O}(\varepsilon^2)$, $\varepsilon \to 0^+$ and $k(\varepsilon, \lambda(\varphi)) = k_0 + k_1\varepsilon + \mathcal{O}(\varepsilon^2)$, $\varepsilon \to 0^+$, where $k_0 = \mathrm{e}^{\mathrm{i}\varphi/2}$ and k_1 are from (2.10). We put this into (1.11) and obtain

$$(z_0 - \frac{1}{z_0}) + (z_1 + \frac{z_1}{z_0^2})\varepsilon + \mathcal{O}(\varepsilon^2) = \mathrm{i}(k_0 - \frac{1}{k_0}) + \mathrm{i}(k_1 + \frac{k_1}{k_0^2})\varepsilon + \mathcal{O}(\varepsilon^2).$$

By comparing of terms with ε^0 and considering $k = \mathrm{e}^{\mathrm{i}\varphi/2}$ we obtain

$$z_0^2 + 2z_0\sin(\varphi/2) - 1 = 0.$$

Since $|z| < 1$ we choose $z_0 = \sqrt{\sin^2(\varphi/2)+1} - \sin(\varphi/2)$, $\varphi \in (0,\pi)$. One can see that $\forall\varphi \in (0,\pi):\ |z_0| \neq 1$. Although we could obtain z_1 by comparing terms with ε^1, we do not need explicitly have it, since it is sufficient to know z_0.

Having this, we can estimate

$$\begin{aligned}\|G_n(\varepsilon)\|^2 \leq& |v_n| \left|\frac{k^2}{2(k^2-1)^2}\right|^2 \left(2\sum_{m\in\mathbb{Z}}\left|\frac{k^{2|m-n|}}{k^2-k^{-2}}\right|^2 + 2\sum_{m\in\mathbb{Z}}\left|\frac{z^{2|m-n|}}{z^2-z^{-2}}\right|^2\right) = \\ &\underbrace{\frac{|v_n|}{2}\left|\frac{k^2}{(k^2-1)^2}\right|^2}_{=:A}\left(\underbrace{\frac{1}{|k^2-k^{-2}|^2}\frac{1+|k|^4}{1-|k|^4}}_{=:B} + \underbrace{\frac{1}{|z^2-z^{-2}|^2}\frac{1+|z|^4}{1-|z|^4}}_{=:C}\right).\end{aligned}$$

Term $A = \mathcal{O}(1)$ since $k^2 \neq \pm 1$. One can also easily see that $(1+|k|^4)/|k^2-k^{-2}|^2 = \mathcal{O}(1)$, from the same reason. Thus, the asymptotic behavior of term B is equivalent to

$$\frac{1}{1-|k|^4} = \frac{1}{1-\left(1+\frac{\varepsilon}{8(2\cos(\varphi)-2)\sin(\varphi)} + \mathcal{O}(\varepsilon^2)\right)^4} = \mathcal{O}(\frac{1}{\varepsilon}).$$

Situation is similar considering term C. We obtain that

$$C \sim \frac{1}{1-|z|^4} = \frac{1}{1-|z_0|^4 + \mathcal{O}(\epsilon)} = \mathcal{O}(1),$$

since $|z_0| \neq 1$. Thus, $\|G_n(\varepsilon)\|^2 = \mathcal{O}(1/\varepsilon)$. The proof is concluded combining results (2.9) and (2.8) while $\varepsilon \to 0^+$. $\square$

Theorem 2.18. Let $v \in \ell^1(\mathbb{Z})$ be the generating sequence of a potential V and λ be in $(0,16) \cap \mathrm{O}_2(\|v\|)^C$. Then $\lambda \notin \sigma_{\mathrm{p}}(T^2+V)$.

Proof. The theorem clearly holds for $V=0$, so we can assume $V \neq 0$. For a contradiction, let us assume the existence of $\lambda \in (0,16) \cap \mathrm{O}_2(\|v\|)^C$ in the point spectrum of perturbed operator T^2. Hence, there exists a vector $u \neq 0 \in \ell^2(\mathbb{Z})$ such that $(T^2+V)u = \lambda u$. Since the spectrum of T^2 is purely continuous, we can assume $V \neq 0$ and thus $|V|^{1/2} \neq 0$. The result of Proposition 2.17 says that for any $e_n \in \mathcal{E}$: $\left\langle e_n, (I + K(\lambda + \mathrm{i}\varepsilon))|V|^{1/2}u \right\rangle \to 0$, as $\varepsilon \to 0^+$. Hence,

$$\forall x \in \ell^2(\mathbb{Z}) : \quad |\left\langle x, K(\lambda+\mathrm{i}\varepsilon)|V|^{1/2}u \right\rangle| \overset{\varepsilon\to 0^+}{\longrightarrow} |\left\langle x, |V|^{1/2}u \right\rangle|.$$

The vector $u \neq 0$ since it is an eigenvector of T^2+V. Recall that we assume $V \neq 0$ and thus $|V|^{1/2}u \neq 0$. Putting $x = |V|^{1/2}u \neq 0$, one has

$$\lim_{\varepsilon\to 0^+} |\left\langle |V|^{1/2}u, K(\lambda+\mathrm{i}\varepsilon)|V|^{1/2}u \right\rangle| = \left\| |V|^{1/2}u \right\|^2.$$

On the other hand, for small $\varepsilon > 0$, $\lambda + \mathrm{i}\varepsilon \in \mathrm{O}_2(\|v\|)^C$ which is an open set. Thus,

$$\|K(\lambda+\mathrm{i}\varepsilon)\| \leq q < 1.$$

Hence we have

$$\lim_{\varepsilon\to 0^+} |\left\langle |V|^{1/2}u, K(\lambda+\mathrm{i}\varepsilon)|V|^{1/2}u \right\rangle| \leq \lim_{\varepsilon\to 0^+} \|K(\lambda+\mathrm{i}\varepsilon)\| \left\| |V|^{1/2}u \right\|^2 \leq q \left\| |V|^{1/2}u \right\|^2.$$

Combining the two previous results we obtain

$$\left\| |V|^{1/2}u \right\|^2 \leq q \left\| |V|^{1/2}u \right\|^2,$$

which is a contradiction since $q < 1$. $\square$

Chapter 3

Criticality and Hardy's inequalities

In this chapter we would like to build upon the results introduced in [4], where the criticality of the positive power of discrete Laplacian on $\ell^2(\mathbb{N})$ was discussed and it was shown that the operator is critical if and only if $\alpha \geq 3/2$. Using their method on $\mathbb{Z}$, we will show that T^α is critical if and only if $\alpha \geq 1/2$.
The subcriticality for $\alpha \in (0, 1/2)$ is connected with Hardy's inequalities analyzed in [1].

In this section, we consider real valued potentials only. Thus, operator-type inequalities are understood as inequalities of the corresponding quadratic forms, i.e.

$$\forall A, B \text{ bounded, self-adjoint on } \ell^2(\mathbb{Z}): \quad A \geq B \iff \forall x \in \ell^2(\mathbb{Z}): \ \langle x, Ax \rangle \geq \langle x, Bx \rangle \,.$$

Recall one important result of the spectra analysis. It could be found in [13], chapter $XIII$, section 1. We can consider bounded operators only.

Theorem 3.1. (Min-Max principle) Let A be bounded self-adjoint operator on a Hilbert space H. We put

$$\lambda_n(A) := \sup_{x_1,\dots,x_{n-1} \in H} \inf \left\{ \langle u, Au \rangle : \ u \in \{x_1, \dots, x_{n-1}\}^\perp, \ \|u\| = 1 \right\}.$$

Then $\forall n \in \mathbb{Z}^+$, either

1. there are n eigenvalues (counting degenerate eigenvalues a number of times equal to their multiplicity) bellow the bottom of the essential spectrum, and $\lambda_n(A)$ is the nth eigenvalue counting multiplicity,

or

2. $\lambda_n(A) = \inf \sigma_{\mathrm{ess}}(A)$ and in that case $\lambda_n(A) = \lambda_{n+1}(A) = \lambda_{n+2}(A) = \dots$ and there are at most $n-1$ eigenvalues (counting multiplicity) below $\lambda_n(A)$.

As usual, we consider positive power of the discrete Laplace operator T^α, $\alpha > 0$. Let us start with the key definition.

Definition 3.2. Let V be a bounded diagonal operator on $\ell^2(\mathbb{Z})$, Then

- operator T^α is called *critical* if $\quad \forall V \geq 0: \quad T^\alpha - V \geq 0 \Longrightarrow V = 0$,
- operator T^α is called *subcritical* if it is not critical.

Remark 3.3. The criticality of T^α could be equivalently characterized as

$$\forall V \geq 0,\ V \neq 0,\ \text{bounded} \ : \quad \inf \sigma(T^\alpha - V) < 0.$$

This follows from the fact that nonnegative self-adjoint operator has nonnegative spectrum.

Our aim is to determine for which real $\alpha > 0$ is the operator T^α critical. Answer to this problem is given by following theorem.

Theorem 3.4. Operator T^α on $\ell^2(\mathbb{Z})$ is critical $\Longleftrightarrow \alpha \geq 1/2$.

We divide proof of this theorem into few parts. the implication ($\Leftarrow$) will be shown directly. Let us start with a proposition describing the eigenvalues of operator T^α with a δ_n potential given by (2.6).

Proposition 3.5. Let $\alpha \geq 1/2$ and $c > 0$. Then operator $T^\alpha - c\delta_n$ has an unique negative eigenvalue for any $n \in \mathbb{Z}$.

Proof. Take an arbitrary $n \in \mathbb{Z}$ and let $\lambda \in \rho(T^\alpha)$. Then, using the Birman–Schwinger principle, one has

$$\lambda \in \sigma_{\mathrm{p}}(T^\alpha - c\delta_n) \quad \Longleftrightarrow \quad -1 \in \sigma_{\mathrm{p}}(K(\lambda)).$$

Since $\delta_n = \delta_n^2$, the Birman–Schwinger operator is of following form

$$K(\lambda) = -c\delta_n(T^\alpha - \lambda)^{-1}\delta_n$$

and it has an eigenvalue equal to $-c(T^\alpha - \lambda)^{-1}_{n,n}$. Let us denote

$$\mu(\lambda) := (T^\alpha - \lambda)^{-1}_{n,n}$$

and consider it as a function $\mu : \ (-\infty, 0) \to \mathbb{R}$. This function has following properties:

i. $\lim_{\lambda \to 0^-} \mu(\lambda) = \infty$,

ii. $\lim_{\lambda \to -\infty} \mu(\lambda) = 0$,

iii. μ is continuous,

iv. μ is strictly increasing.

ad *i.* Consider $\alpha > 0$, the diagonal element of the Green kernel of T^α is given by (1.6) taking $m = n$. Thus

$$(T^\alpha - \lambda)^{-1}_{n,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2^\alpha(1-\cos(t))^\alpha - \lambda}dt = \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2^\alpha(1-\cos(t))^\alpha - \lambda}dt,$$

using the substitution $x = \cos(t)$ we arrive at

$$\mu(\lambda) = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{2^\alpha(1-x)^\alpha - \lambda}\frac{1}{\sqrt{1-x^2}}dx. \tag{3.1}$$

Considering $\lambda = 0$ we formally have

$$\frac{1}{2^\alpha\pi}\int_{-1}^{1} \frac{1}{(1-x)^\alpha(1-x)^{1/2}(1+x)^{1/2}}dx.$$

This integral is surely divergent for $\alpha \geq 1/2$ since there are singularities at $x = \pm 1$. For given range of α, the function $1/(1-x)^{1/2+\alpha}$ is not integrable in the neighborhood of $+1$. By means of the monotone convergence theorem we obtain

$$\lim_{\lambda\to 0^-}\frac{1}{\pi}\int_{-1}^{1} \frac{1}{2^\alpha(1-x)^\alpha - \lambda}\frac{1}{\sqrt{1-x^2}}dx = +\infty.$$

ad *ii.* The result can be obtained by taking the limit $\lambda \to -\infty$ in the integral (3.1) and using the monotone convergence.

ad *ii.* This point follows from the fact that operator-valued function $\lambda \to (T^\alpha - \lambda)^{-1}$ is real-analytic on the $\rho(T^\alpha) = \mathbb{C} \setminus [0, 4^\alpha]$.

ad *iv.* The property follows directly from the integral (3.1).

Properties *i.*–*iv.* are sufficient to state that there exists a unique $\lambda \in (-\infty, 0)$ such that $-c\mu(\lambda) = -1$, i.e. λ is an eigenvalue of $T^\alpha - c\delta_n$, by the Birman-Schwinger principle. □

The proposition is slightly more general than we need. Since the existence of the eigenvalue is sufficient for us, we do not need the uniqueness.
The implication $(\Leftarrow)$ in Theorem 3.4 is an immediate consequence of following proposition and Remark 3.3.

Proposition 3.6. Let $\alpha \geq 0$ and $V \geq 0$, $V \neq 0$, bounded diagonal operator on $\ell^2(\mathbb{Z})$, then

$$\inf \sigma(T^\alpha - V) < 0.$$

Proof. Let us denote V_n the $n-$th element of the diagonal of V. Since $V \neq 0$ and nonnegative, there exist such n that $V_n > 0$. Hence, clearly $V \geq V_n \delta_n$ and using a direct consequence of the Min-Max principle, we obtain

$$\inf \sigma(T^\alpha - V) \leq \inf \sigma(T^\alpha - V_n \delta_n).$$

Together with Proposition 3.5, this inequality concludes the proof. $\square$

Now, the question is how to prove the opposite implication ($\Rightarrow$) in Theorem 3.4. The subcriticality can be characterized as

$$\exists V \geq 0,\ V \neq 0 \text{ diagonal operator on } \ell^2(\mathbb{Z}): \quad T^\alpha \geq V. \tag{3.2}$$

Such a potential was introduced in [1] for $\alpha \in (0, 1/2)$. This fact proves the following proposition, thus Theorem 3.4 is proved.

Proposition 3.7. Let $\alpha \in (0, 1/2)$, then there exist a bounded diagonal operator on $\ell^2(\mathbb{Z})$ such that

$$V \geq 0,\ V \neq 0 \ \&\ T^\alpha \geq V.$$

Details of the proof follow in the next section.

3.1 Hardy's inequality for the positive power of discrete Laplace operator

In this section, we would like to summarize key parts of the approach used in [1], where the discrete Hardy's inequality for positive power of the discrete Laplace operator was introduced. In some propositions, especially those concerning the quadratic forms, we can consider real-valued sequences from $\ell^2(\mathbb{Z})$ only. The results can be easily extended for $\ell^2(\mathbb{Z})$ sequences with values in $\mathbb{C}$ using properties of the inner product.

Definition 3.8. Inequalities of type (3.2) are called discrete *Hardy's inequalities.*

Hardy's inequality for the positive power of discrete Laplacian is described in following theorem. Moreover, it was shown that introduced Hardy's weight is optimal, see [11].

Theorem 3.9. Let $\alpha \in (0, 1/2)$ and let $u \in \ell^2(\mathbb{Z})$ be a real-valued with compact support. Then

$$\sum_{n\in\mathbb{Z}} u_n (T^\alpha u)_n \geq 4^\alpha \left(\frac{\Gamma(\frac{1+2\alpha}{4})}{\Gamma(\frac{1-2\alpha}{4})} \right)^2 \sum_{n\in\mathbb{Z}} u_n \frac{\Gamma\left(|n| + \frac{1-2\alpha}{4}\right) \Gamma\left(|n| + \frac{3-2\alpha}{4}\right)}{\Gamma\left(|n| + \frac{3+2\alpha}{4}\right) \Gamma\left(|n| + \frac{1+2\alpha}{4}\right)} u_n. \tag{3.3}$$

To make the notation easier, let us denote

$$\forall n \in \mathbb{Z}: \quad (w_\alpha)_n := \frac{\Gamma\left(|n| + \frac{1-2\alpha}{4}\right)\Gamma\left(|n| + \frac{3-2\alpha}{4}\right)}{\Gamma\left(|n| + \frac{3+2\alpha}{4}\right)\Gamma\left(|n| + \frac{1+2\alpha}{4}\right)},$$

w_α is a real-valued sequence. Moreover, if we define an operator W_α like

$$\forall n \in \mathbb{Z}: \quad W_\alpha e_n := (w_\alpha)_n e_n, \quad e_n \in \mathcal{E},$$

the inequality (3.3) becomes

$$\langle u, T^\alpha u\rangle \geq 4^\alpha \left(\frac{\Gamma(\frac{1+2\alpha}{4})}{\Gamma(\frac{1-2\alpha}{4})}\right)^2 \langle u, W_\alpha u\rangle .$$

Key proposition of this section is the following pointwise identity. It was proved in [2], where the authors use a semigroup approach of defining T^α and the result then follows almost directly.

Proposition 3.10. Let $\alpha \in (0,1)$, then $\forall f \in \ell^2(\mathbb{Z})$:

$$(T^\alpha f)_n = \sum_{m\in\mathbb{Z},\ m\neq n} (f_n - f_m) K_\alpha(n-m),$$

where

$$K_\alpha(s) = \begin{cases} \frac{4^\alpha}{\sqrt{\pi}} \frac{\Gamma(|s|-\alpha)\Gamma(1/2+\alpha)}{\Gamma(|s|+1+\alpha)|\Gamma(-\alpha)|}, & s \in \mathbb{Z}\setminus\{0\}, \\ 0, & s = 0. \end{cases}$$

Proof. Recall that we know the matrix element of T^α, which was derived in (1.5). It reads

$$\forall m, n \in \mathbb{Z}:\ T^\alpha_{m,n} = (-1)^{n-m} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1+(m-n))\Gamma(\alpha+1-(m-n))}.$$

Thus, it is sufficient to verify that

$$\begin{aligned} \sum_{m\in\mathbb{Z}} T^\alpha_{n,m} f_m &= \sum_{m\in\mathbb{Z},\ m\neq n} (f_n - f_m) K_\alpha(n-m), \qquad (3.4) \\ &= \underbrace{\left(\sum_{m\in\mathbb{Z},\ m\neq n} K_\alpha(n-m)\right)}_{A_1:=} f_n - \underbrace{\sum_{m\in\mathbb{Z},\ m\neq n} K_\alpha(n-m) f_m}_{B_1:=} . \end{aligned}$$

On the other hand, we can also obtain that the left hand side of (3.4) is equal to

$$\underbrace{\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}}_{A_2:=} f_n + \underbrace{\sum_{m\in\mathbb{Z},\ m\neq n} \frac{(-1)^{n-m}\Gamma(2\alpha+1)}{\Gamma(\alpha+1+(m-n))\Gamma(\alpha+1-(m-n))} f_m}_{B_2:=} .$$

It is enough to show that $A_1 = A_2$ and $B_1 = -B_2$.

- $A_1 = A_2$: According to the definition of K_α, the symmetry $K_\alpha(m) = K_\alpha(-m)$ and by denoting $s := m - n$, we have

$$A_1 = 2\sum_{s=1}^{+\infty} K_\alpha(s) = 2\frac{4^\alpha\Gamma(1/2+\alpha)}{\sqrt{\pi}|\Gamma(-\alpha)|}\sum_{s=1}^{+\infty}\frac{\Gamma(s-\alpha)}{\Gamma(s+1+\alpha)} \overset{\boxed{1}}{=} \frac{4^\alpha\Gamma(1/2+\alpha)}{\sqrt{\pi}|\Gamma(-\alpha)|}\frac{-\Gamma(-\alpha)}{\Gamma(1+\alpha)}$$

$$= \frac{4^\alpha\Gamma(1/2+\alpha)}{\sqrt{\pi}\Gamma(1+\alpha)} \overset{\boxed{2}}{=} \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} = A_2,$$

where the forth equality was obtained using $\forall\alpha \in (0,1) :\ |\Gamma(-\alpha)| = -\Gamma(-\alpha)$. Let us now clarify the identity $\boxed{1}$. It follows from

$$\sum_{s=1}^{+\infty}\frac{\Gamma(s-\alpha)}{\Gamma(s+1+\alpha)} = -\frac{1}{2}\frac{-\Gamma(-\alpha)}{\Gamma(1+\alpha)}.$$

For $\alpha \in \mathbb{R}$ we define the Pochhammer symbol

$$\forall n \in \mathbb{Z}^+ : \quad (\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

It holds

$$\sum_{s=1}^{+\infty}\frac{\Gamma(s-\alpha)}{\Gamma(s+1+\alpha)} = \sum_{s=0}^{+\infty}\frac{\Gamma(s+1-\alpha)}{\Gamma(s+2+\alpha)} = \frac{\Gamma(1-\alpha)}{\Gamma(2+\alpha)}\sum_{s=0}^{+\infty}\frac{(1-\alpha)_s\ (1)_s}{(2+\alpha)_s\ s!}.$$

The last series coincides with the Gauss hypergeometric function $F(a,b,c,z)$, see [15], eq. 15.2.1. One has

$$\sum_{s=0}^{+\infty}\frac{(1-\alpha)_s\ (1)_s}{(2+\alpha)_s\ s!} = F((1-\alpha),1,(2+\alpha),1) = \frac{\Gamma(2+\alpha)\Gamma(2\alpha)}{\Gamma(1+2\alpha)\Gamma(1+\alpha)},$$

which could be found in [15], eq. 15.4.20. Using this result, one arrives at

$$\sum_{s=1}^{+\infty}\frac{\Gamma(s-\alpha)}{\Gamma(s+1+\alpha)} = \frac{\Gamma(1-\alpha)}{\Gamma(2+\alpha)}\frac{\Gamma(2+\alpha)\Gamma(2\alpha)}{\Gamma(1+2\alpha)\Gamma(1+\alpha)} = \frac{\Gamma(1-\alpha)\ (1+\alpha)}{\Gamma(2+\alpha)\ 2\alpha}.$$

It was enough to use the notorious identity $\Gamma(z+1) = z\Gamma(z), \forall z \in \mathbb{C}$. Hence the desired identity directly follows.

The identity $\boxed{2}$ is an immediate consequence of

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}}2^{2z-1}\Gamma(z)\Gamma(z+1/2), \tag{3.5}$$

which holds for all $z \in \mathbb{C}\setminus\mathbb{Z}_0^-$, see [15], eq. 5.5.5.

- $B_1 = -B_2$: Let us denote $s = m - n$ and since $m \neq n$ we consider $s \neq 0$ and due to the symmetry we can consider $s > 0$. To prove the identity, it is enough to show that
$$\forall s \in \mathbb{Z}: \quad K_\alpha(s) = -\frac{(-1)^s \Gamma(2\alpha+1)}{\Gamma(\alpha+1+s)\Gamma(\alpha+1-s)}.$$
Indeed, for $\alpha \in (0,1)$ we have
$$K_\alpha(s) = \frac{4^\alpha \Gamma(1/2+\alpha)}{-\sqrt{\pi}\Gamma(-\alpha)} \frac{\Gamma(s-\alpha)}{\Gamma(\alpha+1-s)}.$$
Using (3.5) and
$$\forall z \in \mathbb{C} \setminus \mathbb{Z}: \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{3.6}$$
see [15], eq. 5.5.3, we immediately obtain
$$K_\alpha(s) = -\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(-\alpha)} \frac{\Gamma(s-\alpha)}{\Gamma(s+1-\alpha)} = \frac{\sin(\pi\alpha)}{\sin(\pi(s-\alpha))} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1+s)\Gamma(\alpha+1-s)}.$$
One can easily verify that
$$\frac{\sin(\pi\alpha)}{\sin(\pi(s-\alpha))} = (-1)^{s+1},$$
which gives the desired result.

$\square$

Let us now make a first step towards proving Theorem 3.9. Following proposition is taken from [1].

Proposition 3.11. Let $\alpha \in (0,1)$ and $f \in \ell^2(\mathbb{Z})$ be a real-valued with compact support, then
$$\sum_{n\in\mathbb{Z}} f_n (T^\alpha f)_n = a_\alpha \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},\ m\neq n} (f_n - f_m)^2 \frac{\Gamma(|m-n|-\alpha)}{\Gamma(|m-n|+\alpha)+1},$$
where
$$a_\alpha := \frac{4^\alpha \Gamma(1/2+\alpha)}{\sqrt{\pi}|\Gamma(-\alpha)|}.$$

Proof. Using Proposition 3.10 we immediately obtain
$$\sum_{n\in\mathbb{Z}} f_n (T^\alpha f)_n = \sum_{m\in\mathbb{Z},\ m\neq n} \sum_{n\in\mathbb{Z}} (f_n - f_m) f_n K_\alpha(n-m).$$
Since the kernel K_α is symmetric, we have
$$\sum_{n\in\mathbb{Z}} f_n (T^\alpha f)_n = \sum_{m\in\mathbb{Z},\ m\neq n} \sum_{n\in\mathbb{Z}} (f_m - f_n) f_m K_\alpha(n-m).$$

We add these two equations and obtain

$$\sum_{n\in\mathbb{Z}} f_n(T^\alpha f)_n = \frac{1}{2}\sum_{m\in\mathbb{Z},\ m\neq n}\sum_{n\in\mathbb{Z}}(f_m-f_n)^2K_\alpha(n-m).$$

Now it is sufficient just to interchange the order of summation and use the definition of kernel K_α. $\square$

Negative power of the discrete Laplacian

In what follows we need to make a brief comment on the negative power of discrete Laplacian. Let $\beta\in(0,1/2)$. Based on the results of Section 1.1, we put

$$T^{-\beta} := \mathcal{U}^{-1}M_{\phi_T^{-\beta}}\mathcal{U},$$

where ϕ_T is the symbol of Laurent operator T. Since the function

$$\phi_T^{-\beta} = \frac{1}{(2-2\cos(t))^\beta}$$

is unbounded for $\beta\in(0,1/2)$, the operator of multiplication is unbounded and thus $T^{-\beta}$ is unbounded too. In this thesis, it is enough to consider

$$\mathrm{Dom}(T^{-\beta}) = \left\{u\in\ell^2(\mathbb{Z}):\ \mathrm{supp}(u) \text{ is compact}\right\}.$$

One can still search for a matrix element of the operator. We have

$$T^{-\beta}_{m,n} = \frac{1}{2\pi}\int_\pi^\pi \frac{\mathrm{e}^{\mathrm{i}(n-m)t}}{(2-2\cos(t))^\beta} = \frac{2}{\pi 4^\beta}\int_{\pi/2}^0 \sin^{-2\beta}(t)\cos(2(m-n)t).$$

This integral can be solved the same way as the one for the positive power of T. Using Proposition 1.7 we arrive at

$$T^{-\beta}_{m,n} = \frac{(-1)^{m-n}\,\Gamma(1-2\beta)}{\Gamma(1-\beta+m-n)\Gamma(1-\beta+n-m)}.$$

The authors of [1] used the semigroup approach, where so called Riesz kernel associated with $T^{-\beta}$ appears. For $\beta\in(0,1/2)$, it reads

$$R_\beta(n) := \frac{1}{\Gamma(\beta)}\int_0^{+\infty}\mathrm{e}^{-2t}I_n(2t)t^{\beta-1}dt,$$

where I_n is the modified Bessel function of order n, which is defined for example in the fifth chapter of [14]. For $\beta\in(0,1/2)$, we can get an explicit value of $R_\beta(n)$, it holds

$$R_\beta(n) = \frac{4^{-\beta}\Gamma(1/2-\beta)}{\sqrt{\pi}\Gamma(\beta)}\frac{\Gamma(|n|+\beta)}{\Gamma(|n|+1+\beta)}, \tag{3.7}$$

see [2].

The Riesz kernel is related to our matrix element of $T^{-\beta}$. In fact, they are equal in following sense

$$\forall m, n \in \mathbb{Z}: \quad T^{-\beta}_{m,n} = R_\beta(m-n).$$

Indeed, denote $s := m-n$. Then it holds

$$\frac{1}{4^\beta}\frac{\Gamma(1/2-\beta)}{\sqrt{\pi}} = \frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)},$$

which is a consequence of (3.5) and from (3.6) we obtain

$$\Gamma(|s|+\beta) = \frac{\pi}{\sin(\pi(|s|+\beta))}\frac{1}{\Gamma(1-\beta-|s|)}.$$

Hence, due to the symmetry of $T^{-\beta}_{m,n}$, it follows

$$\forall s \in \mathbb{Z}: \; R_\beta(s) = \frac{\Gamma(1/2-\beta)}{4^\beta\sqrt{\pi}\Gamma(\beta)}\frac{\Gamma(|s|+\beta)}{\Gamma(|s|+1+\beta)} = \frac{(-1)^s\,\Gamma(1-2\beta)}{\Gamma(1-\beta+s)\Gamma(1-\beta-s)},$$

which was to be shown.

What we need is the following lemma, which is an immediate consequence of the definitions of T^α, $T^{-\beta}$ and its relation to $R_\beta(n)$, see [1], Lemma 2.3.

Lemma 3.12. Let $\beta > 0,\ \alpha < 1/2,\ \beta-\alpha \in (0,1/2)$ and let $u \in \ell^2(\mathbb{Z})$ real-valued with compact support We define real-valued sequences f and g in $\ell^2(\mathbb{Z})$ as follow

$$\forall n \in \mathbb{Z}: \quad g_n := R_\beta(n) \text{ and } f_n := u_n^2/R_\beta(n).$$

Then

$$\langle T^\alpha f, g\rangle = \sum_{n\in\mathbb{Z}} u_n^2 \frac{R_{\beta-\alpha}(n)}{R_\beta(n)}.$$

Proof of Theorem 3.9

We finish the proof exactly the same way as the authors of [1], including some notation. For all $u \in \ell^2(\mathbb{Z})$ real-valued with compact support, we denote

$$H_\alpha[u] := \sum_{n\in\mathbb{Z}} u_n(T^\alpha u)_n - 4^\alpha\left(\frac{\Gamma(\frac{1+2\alpha}{4})}{\Gamma(\frac{1-2\alpha}{4})}\right)^2 \sum_{n\in\mathbb{Z}} u_n \frac{\Gamma\left(|n|+\frac{1-2\alpha}{4}\right)\Gamma\left(|n|+\frac{3-2\alpha}{4}\right)}{\Gamma\left(|n|+\frac{3+2\alpha}{4}\right)\Gamma\left(|n|+\frac{1+2\alpha}{4}\right)} u_n.$$

Let us denote

$$C_\alpha := 4^\alpha\left(\frac{\Gamma(\frac{1+2\alpha}{4})}{\Gamma(\frac{1-2\alpha}{4})}\right)^2.$$

and recall that a_α is given in Proposition 3.11.

Proposition 3.13. Let $\alpha \in (0, 1/2)$. Let F be a real-valued sequence with compact support and put

$$\forall n \in \mathbb{Z}: \; G_n := \frac{F_n}{R_{\frac{1+2\alpha}{4}}(n)}.$$

Then

$$H_\alpha[F] = a_\alpha \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}, m\neq n} (G_n - G_m)^2 \frac{\Gamma(|n-m|-\alpha)}{\Gamma(|n-m|+\alpha+1)} R_{\frac{1+2\alpha}{4}}(n) R_{\frac{1+2\alpha}{4}}(m). \tag{3.8}$$

Proof. Consider $f, g \in \ell^2(\mathbb{Z})$ real-valued with compact support. Then Proposition 3.11 implies that

$$\sum_{n\in\mathbb{Z}} g_n (T^\alpha f)_n = a_\alpha \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}, m\neq n} (f_n - f_m)(g_n - g_m) \frac{\Gamma(|m-n|-\alpha)}{\Gamma(|m-n|+\alpha)+1}. \tag{3.9}$$

We consider $\beta \in (0, 1/2)$ and put

$$\forall n \in \mathbb{Z}: \quad f_n := \frac{F_n^2}{g_n}.$$

The right hand side of (3.9) is equal to

$$a_\alpha \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}, m\neq n} \left((F_m - F_n)^2 - g_m g_n \left(\frac{F_n}{g_n} - \frac{F_m}{g_m} \right)^2 \right) \frac{\Gamma(|m-n|-\alpha)}{\Gamma(|m-n|+\alpha+1)} =$$
$$= \sum_{n\in zet} F_n (T^\alpha F)_n - a_\alpha \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}, m\neq n} g_m g_n \left(\frac{F_n}{g_n} - \frac{F_m}{g_m} \right)^2 \frac{\Gamma(|m-n|-\alpha)}{\Gamma(|m-n|+\alpha+1)}.$$

Indeed, the first equality follows from Proposition 3.10 and it holds that the term

$$\begin{aligned}(f_n - f_m)(g_n - g_m) =& F_n^2 + F_m^2 - \frac{g_m}{g_n} F_n^2 - \frac{g_n}{g_m} F_m^2 = \\ =& (F_m - F_n)^2 - g_m g_n \left(\frac{F_n}{g_n} - \frac{F_m}{g_m} \right)^2.\end{aligned}$$

Now, we put $g_n := R_\beta(n)$ and $\beta := (1+2\alpha)/4$. Assuming the same choice of f, the left hand side of (3.9) reads

$$\sum_{n\in\mathbb{Z}} g_n (T^\alpha f)_n = \sum_{n\in\mathbb{Z}} F_n^2 \frac{R_{\beta-\alpha}(n)}{R_\alpha(n)} = C_\alpha \sum_{n\in\mathbb{Z}} F_n^2 \frac{\Gamma\left(|n| + \frac{1-2\alpha}{4}\right) \Gamma\left(|n| + \frac{3-2\alpha}{4}\right)}{\Gamma\left(|n| + \frac{3+2\alpha}{4}\right) \Gamma\left(|n| + \frac{1+2\alpha}{4}\right)}.$$

The first equality follows from Lemma 3.12, then we have used the exact value of $R_\beta(n)$ from (3.7).

To conclude the proof, recall that we consider $\beta := (1+2\alpha)/4$, then putting the new forms of the left and right hand side together, we obtain

$$\underbrace{\sum_{n\in zet} F_n(T^\alpha F)_n - C_\alpha \sum_{n\in\mathbb{Z}} F_n^2 \frac{\Gamma\left(|n|+\frac{1-2\alpha}{4}\right)\Gamma\left(|n|+\frac{3-2\alpha}{4}\right)}{\Gamma\left(|n|+\frac{3+2\alpha}{4}\right)\Gamma\left(|n|+\frac{1+2\alpha}{4}\right)}}_{H_\alpha[F]} =$$

$$= a_\alpha \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}, m\neq n} \left(G_n - G_m\right)^2 R_{\frac{1+2\alpha}{4}}(m) R_{\frac{1+2\alpha}{4}}(n) \frac{\Gamma(|m-n|-\alpha)}{\Gamma(|m-n|+\alpha+1)}.$$

This concludes the proof. □

Theorem 3.9 is an immediate consequence of Proposition 3.13 since the right hand side of (3.8) is nonnegative. Indeed, a_α, $(G_n - G_m)$ and $\Gamma(|m-n|-\alpha)/\Gamma(|m-n|+\alpha+1)$ are clearly nonnegative $\forall m, n \in zet,\ m \neq n$, $\alpha \in (0, 1/2)$. The nonegativity of $R_{\frac{1+2\alpha}{4}}(m)$ and $R_{\frac{1+2\alpha}{4}}(n)$ follows from (3.7).

Conclusion

In this thesis, we focused on the spectral analysis of the general positive power of the discrete Laplace operator on $\ell^2(\mathbb{Z})$. The definition of the operator itself and the basic properties were obtained using the theory of Laurent operators. Afterwards we localized the spectrum of the polyharmonic operator with a trace class perturbation and derived the spectral enclosures, which are sets containing the discrete spectrum of the perturbed operator. Moreover, we formulated a conjecture on optimal enclosures, which was proved for discrete bilaplace operator. We also showed the absence of eigenvalues in the interior of the essential spectrum outside the spectral enclosure for the discrete bilaplacian. At the end, we discussed the criticality of the positive power of the discrete Laplacian and introduced Hardy's inequalities for the subcritical case.

At the very end, I would like to outline one of the possible extensions of our results. It is the weak-coupling analysis for the discrete bilaplace operator. We consider the operator $T^2 + V$, where V is a small real-valued potential and we want to analyze the existence and uniqueness of eigenvalues of $T^2 + V$. In this setting, the operator $T^2 + V$ is self-adjoint. The analysis was done for the continuous Schrödinger operators in one and two dimensions, see [16]. It was also done for the the discrete version of the Schrödinger operator in [17].

In the papers, the Birman–Schwinger operator is used and the main idea is to decompose the operator onto, in a certain sense, a singular and regular part. We associate the potential V with a sequence v, which is on the diagonal of it's matrix. We put $\lambda = \zeta_2(k)$, defined in (1.7), the matrix element of the Birman–Schwinger operator is of the form

$$\forall m, n \in \mathbb{Z}: \quad K(\lambda(k))_{m,n} = \sqrt{|v_m|}\frac{k^2}{2(k^2-1)^2}\left[\frac{k^{2|m-n|}}{k^2-k^{-2}} - \frac{z^{2|m-n|}}{z^2-z^{-2}}\right]\sqrt{|v_n|}\operatorname{sgn}(v_n).$$

We want to analyze the behavior of the operator as $\lambda \to 0^-$ and $\lambda \to 16^+$, i.e. λ is near the boundary points of the essential spectrum. Now consider the right neighborhood of 16 only. We can see from the definition of the transform ζ_N for $N = 2$ that $\lambda \to 16^+ \iff k \to \mathrm{i}$. Since $k \in \mathbb{D}_2^{\zeta}$, it tends to i from below along the imaginary axis. We put $K(\lambda) = M(\lambda) + L(\lambda)$, where, $\forall m, n \in \mathbb{Z}$:

$$L_{m,n}(\lambda(k)) := \sqrt{|v_m|}\frac{k^2}{2(k^2-1)^2}\left[\frac{(-1)^{|m-n|}}{k^2-k^{-2}} - \frac{(-1)^{|m-n|}}{z^2-z^{-2}}\right]\sqrt{|v_n|}\mathrm{sgn}(v_n),$$

$$M_{m,n}(\lambda(k)) :=$$

$$\sqrt{|v_m|}\frac{k^2}{2(k^2-1)^2}\left[\frac{k^{2|m-n|}-(-1)^{|m-n|}}{k^2-k^{-2}} - \frac{z^{2|m-n|}-(-1)^{|m-n|}}{z^2-z^{-2}}\right]\sqrt{|v_n|}\mathrm{sgn}(v_n).$$

This decomposition was made in such a way that the operator $L(\lambda)$ is rank one and the operator $M(\lambda)$ converges to the operator $M(16)$ in the operator norm as $\lambda \to 16^+$. The situation is slightly different in the left neighborhood of 0. If $\lambda \to 0^-$ then $k \to 1$ along the curve c, which is given by

$$c := \left\{\frac{1}{2}\left(a\mathrm{e}^{\mathrm{i}\frac{3}{4}\pi} + \sqrt{4-\mathrm{i}a^2}\right) : \ a \in (0, +\infty)\right\} \subset \mathbb{D}_2^{\zeta}.$$

The definition of the operators $M(\lambda)$ and $L(\lambda)$ would be different in this case.

In the future, I would like to extend the result of this thesis in many ways, including the weak-coupling analysis. The mentioned ideas are merely an introduction of the approach used in cited articles. The successful completion of at least proving of the existence of eigenvalues requires a significant amount of further work.

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