Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Critical exponent and asymptotic critical exponent

Kritický a asymptotický kritický exponent

Diploma thesis

Author: **Bc. Daniela Opočenská**
Supervisor: **doc. Ing. L'ubomíra Dvořáková, Ph.D.**

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ZADÁNÍ DIPLOMOVÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

Příjmení: **Opočenská** Jméno: **Daniela** Osobní číslo: **494701**

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Zadávající katedra/ústav: **Katedra matematiky**

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II. ÚDAJE K DIPLOMOVÉ PRÁCI

Název diplomové práce:

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Název diplomové práce anglicky:

Critical exponent and asymptotic critical exponent

Pokyny pro vypracování:

1) Nastudujte známá fakta o repeticích a kritickém exponentu nekonečných slov.
2) Rozšiřte známé výsledky týkající se (asymptotického) kritického exponentu balancovaných slov.
3) Zkoumejte kritický a asymptotický kritický exponent některých významných tříd nekonečných slov (např. slov "chudých na palindromy", "large complement avoiding" slov, 2-balancovaných slov, morfických slov atd.).
4) Navrhujte a implementujte algoritmy pro výpočet (asymptotického) kritického exponentu.

Seznam doporučené literatury:

1) J. D. Currie, L. Mol and N. Rampersad, The repetition threshold for binary rich words. Discrete Mathematics & Theoretical Computer Science 22(1), 2020, 6
2) D. Krieger, On critical exponents in fixed points of k-uniform binary morphisms, RAIRO - Theoretical Informatics and Applications 43(1), 2009, 41–68.
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Jméno a pracoviště vedoucí(ho) diplomové práce:

doc. Ing. Lubomíra Dvořáková, Ph.D. katedra matematiky FJFI

Jméno a pracoviště druhé(ho) vedoucí(ho) nebo konzultanta(ky) diplomové práce:

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Dvořáková Masáková V. Č.

doc. Ing. Lubomíra Dvořáková, Ph.D.
podpis vedoucí(ho) práce

prof. Ing. Zuzana Masáková, Ph.D.
podpis vedoucí(ho) ústavu/katedry

doc. Ing. Václav Čuba, Ph.D.
podpis děkana(ky)

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Diplomantka bere na vědomí, že je povinna vypracovat diplomovou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací. Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v diplomové práci.

9.11.2023
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Author's declaration:
I declare that this thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, May 2, 2024 Daniela Opočenská

Název práce:

Kritický a asymptotický kritický exponent

Autor: Bc. Daniela Opočenská

Obor: Matematické inženýrství

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Vedoucí práce: doc. Ing. Ľubomíra Dvořáková, Ph.D.,
Katedra matematiky, Fakulta jaderná a fyzikálně inženýrská, České vysoké učení technické v Praze, Trojanova 13, 120 00 Praha 2

Abstrakt: Jednou z hlavních otázek kombinatoriky na slovech jsou repetice ve slovech, které popisuje kritický a asymptotický kritický exponent. Kritický exponent (CE) říká, jaký je maximální počet opakování slova hned po sobě v daném nekonečném slově. Asymptotický kritický exponent (ACE) pak uvažuje pouze opakování faktorů, jejichž délka jde limitně do nekonečna.

Práce je dělena do dvou samostatných celků. V první části se zabýváme otázkou minimálního ACE balancovaných slov nad d-písmennou abecedou. Napřed proto definujeme základní pojmy a popíšeme bispeciály a návratová slova v balancovaných slovech a jak je lze využít pro hledání minimálního ACE. Dále teorii rozšíříme, abychom dokázali pro danou mez najít balancované slovo s danými parametry, které nabývá ACE nižší, než je daná mez (pokud existuje). S využitím teorie navrhneme a implementujeme algoritmus, který nám umožní najít hodnotu minimálního ACE pro balancovaná slova nad d-písmennou abecedou pro $d \in \{3, 4, \ldots, 10\}$.

Druhá část této práce se zabývá nekonečnými slovy nad binární abecedou, která mají malý kritický exponent a zároveň obsahují pouze konečně mnoho palindromů, resp. komplementárních faktorů. Hlavní přínos této části je popis procesu hledání kritického exponentu pro morfické obrazy nekonečného slova, které je pevným bodem zadaného morfismu.

Tato práce navazuje na výzkum v mé bakalářské práci a výzkumném projektu. Dále jsou výsledky této práce součástí tří článků, z toho jeden je již publikovaný [16] a dva odeslané [14, 8].

Klíčová slova: asymptotický kritický exponent, balancované slovo, bispeciál, komplementární faktor, kritický exponent, morfismus, návratové slovo, palindrom, pevný bod morfismu

Title:
Critical exponent and asymptotic critical exponent

Author: Bc. Daniela Opočenská

Abstract: One of the main questions in combinatorics on words concerns repetitions in sequences, this notion is described by (asymptotic) critical exponent. The critical exponent (CE) of a sequence tells us what the maximum number of consecutive repetitions of any non-empty word in the sequence is. The asymptotic critical exponent (ACE) then considers repetitions of words of length that tends to infinity.

The thesis consists of two independent parts. In the first part, we examine the minimal ACE of balanced sequences over a d-letter alphabet. To do so, we recall the notions of bispecial factors and return words in balanced sequences and how to use them to calculate the ACE of a balanced sequence. Next, we expand this theory to allow us to find a balanced sequence with given parameters that attains ACE lower than a given bound (if such a sequence exists). Using the theory, we create and implement an algorithm and then use it to find the minimal ACE of balanced sequences over a d-letter alphabet for $d \in \{3, 4, \dots, 10\}$.

The second part of the thesis deals with binary sequences having small CE while containing either a limited number of palindromes or a limited number of complemented factors. The main contribution of this part is the method of computation of CE of morphic images of the fixed point of a given morphism.

This thesis follows up on research conducted in my bachelor thesis and research project. The results are also part of one published [16] and two submitted [14, 8] papers.

Key words: asymptotic critical exponent, balanced sequence, bispecial factor, complementary factor, critical exponent, fixed point of a morphism, morphism, palindrome, return word

Contents

Introduction

Introduction

The first works in combinatorics on words date back to the 1900's when a Norwegian mathematician Axel Thue studied the consecutive repetition of factors in infinite sequences. He affirmatively answered the following two questions. Is there an infinite sequence over a binary alphabet that contains no cubes, i.e., no factor repeating consecutively three times? Is there an infinite sequence over a ternary alphabet that contains no squares, that is, no factor repeating twice consecutively? The *critical exponent* $E(\mathbf{u})$ of an infinite sequence $\mathbf{u} = u_0u_1u_2\cdots$ captures the maximal possible repetition rate of the factors that occur in $\mathbf{u}$. Therefore, the sequences found as answers to the previous questions have a critical exponent less than or equal to three, resp. less than or equal to two. A natural question to ask is: What is the smallest possible critical exponent for a sequence over a given alphabet? The answer to the question is called *repetition threshold* and it has been found by Dejean's theorem [11], which says that the infimum of critical exponents of sequences over a d-letter alphabet is:

- 2 for $d = 2$
- $\frac{7}{4}$ for $d = 3$
- $\frac{7}{5}$ for $d = 4$
- $\frac{d}{d-1}$ for $d \geq 5$.

For $d = 2$, the minimum is reached by the Thue-Morse sequence, the case for $d = 3$ was proven by Dejean and the rest has been proven step by step by many people [32, 7, 10, 30, 34] over the course of almost 40 years.

Dejean's theorem shows us the repetition threshold for classes C_d of infinite sequences over a d-letter alphabet, where $d \geq 2$. The repetition threshold has also been investigated for other classes of sequences.

- *Factor complexity* of a sequence $\mathbf{u}$ is function $C_{\mathbf{u}} : \mathbb{N} \to \mathbb{N}$ counting the number of distinct factors of a sequence $\mathbf{u}$ of length n. If we consider $\mathbf{u}$ periodic, we can see that the factor complexity is bounded while the critical exponent is $+\infty$.

  For aperiodic sequences, the smallest complexity is $C_{\mathbf{u}}(n) = n+1$ obtained by Sturmian sequences. In [3], it was proven that the repetition threshold for Sturmian sequences is $\frac{5+\sqrt{5}}{2}$ and the bound is attained by the Fibonacci sequence. In [23], the authors showed that the repetition threshold is the same even if we expand the class to any binary sequence with factor complexity of the form $C_{\mathbf{u}}(n) = n + c$, where $c \geq 1$. The authors also showed that the repetition threshold for binary sequences can be lowered to $\frac{5}{2}$ if we allow factor complexity $\leq 2n$.

- A sequence $\mathbf{u}$ is balanced if for any two factors u, v of the same length and any letter b of the alphabet, the number of occurrences of b in u and v differs at most by one. Aperiodic balanced sequences over binary alphabet coincide with Sturmian sequences.

  In [33], the authors tried to recreate Dejean's theorem for balanced sequences over a d-letter alphabet. They found the repetition threshold for d-letter alphabets, where $d \in \{3, 4\}$, and found the sequences that reach the bound. Furthermore, they formulated a conjecture that the repetition threshold for balanced sequences over d-letter alphabet is $\frac{d-2}{d-3}$ for $d \geq 5$ and showed that it cannot be lower for $d \leq 10$. They also provided a list of sequences $\mathbf{x}_d$ for $5 \leq d \leq 10$, which they suspected attain the bound $\frac{d-2}{d-3}$. This proposition has been proven for $5 \leq d \leq 8$ in [6, 1].

  In [12, 19, 18], the authors proposed a different method, which allows to calculate the (asymptotic) critical exponent from the knowledge of the lengths of bispecial factors and their return words, and they showed how to find these values in balanced sequences. In my bachelor thesis [31], we used this approach to first verify that the critical exponents of the sequences $\mathbf{x}_9$ and $\mathbf{x}_{10}$ attain the values $\frac{d-2}{d-3}$ and then we used the programs to refute the conjecture from [33] for bigger alphabets. In [17, 31], we showed that the repetition threshold for balanced sequences over a d-letter alphabet, where $d \geq 12$ even, is $\frac{d-1}{d-2}$ and that for d odd, it cannot be lower. It remains an open problem to prove our new conjecture for all d-letter alphabets, where d is odd and greater than 11.

- The *complement* of a word u over binary alphabet $\{0, 1\}$ is $\bar{u}$ obtained by changing $0 \mapsto 1$ and $1 \mapsto 0$. The only binary sequences avoiding complements of all factors are the sequences $(0)^\omega = 0000000\cdots$ and $(1)^\omega = 1111111\cdots$. Now, we can define the classes

$$\mathrm{CAL}_\ell = \{\mathbf{u} \text{ binary sequence} : (\forall u \text{ factor of } \mathbf{u})(\bar{u} \text{ factor of } \mathbf{u} \Rightarrow |u| < \ell)\}$$
$$\mathrm{CAN}_n = \{\mathbf{u} \text{ binary sequence} : \#\{u \text{ factor of } \mathbf{u} : \bar{u} \text{ factor of } \mathbf{u}\} \leq n\},$$

  i.e. CAL_ℓ limits the length of complementary factors, and CAN_n limits the number of complementary factors in $\mathbf{u}$.

  The authors in [8] found the repetition threshold for these two classes. To finish the proofs, it was necessary to calculate the critical exponent of given morphic images of a sequence $\mathbf{p}$ described in Chapter 8. We were able to do so, and the computation can be found in Chapters 11 and 12 and as a part of submitted paper [8].

- Another studied property is the number of palindromes that can be found in the sequence. A word u of length n is called *rich* if it contains exactly n distinct non-empty factors that are palindromes (i.e. v such that $v_0v_1\cdots v_{|v|-1} = v_{|v|-1}\ldots v_1v_0$). A sequence $\mathbf{u}$ is rich if all factors of $\mathbf{u}$ are rich.

  In [2], the authors constructed a binary rich sequence with critical exponent $2+\frac{\sqrt{2}}{2}$ and conjectured that this is the repetition threshold for the class of binary rich sequences. This proposition has been later proven in [22]. The authors of [2] also showed that the repetition threshold for the class of ternary (i.e. over 3-letter alphabet) rich sequences is greater than or equal to $\frac{9}{4}$. It is known that for d-letter alphabets the critical exponent is less than or equal to the critical exponent of d-ary episturmian sequences (generalization of Sturmian sequences over bigger alphabets) which equals $2 + \frac{1}{t_d-1}$, where $t_d > 1$ is the unique positive root of the polynomial $x^d - x^{d-1} - \cdots - x - 1$ (see [27]). For d growing to infinity, the bound $2 + \frac{1}{t_d-1}$ tends to 3.

  A natural question is: What happens to the repetition threshold if we limit the number of palindromes that can appear in a sequence? For binary sequences, the answer was given in [14]. Similarly to the previous item, it was necessary to calculate the critical exponent of some morphic images of $\mathbf{p}$ and the calculations are in Chapters 9 and 10 and as a part of submitted paper [14].

- The Thue-Morse sequence is an example of a binary sequence avoiding cubes. Similarly, Solomon Arshon gave an algorithm that for any $n \geq 3$ constructs a square-free sequence $\mathbf{a}_n$ over the alphabet $\mathcal{A} = \{0, 1, \ldots, n-1\}$. The construction defines two morphisms and alternates between them at odd/even positions in the sequence. From the construction, the critical exponent of $\mathbf{a}_n$ is ≤ 2. In [25], the authors showed that the critical exponent of $\mathbf{a}_3$ is $\frac{7}{4}$. In [26], the authors proved that for any $n \geq 3$, the critical exponent of $\mathbf{a}_n$ is equal to $\frac{3n-2}{2n-2}$ and the factor that achieves the value of the critical exponent starts at position 1.

Our approach to calculating the repetition threshold of (balanced) sequences relied heavily on our ability to compute the asymptotic critical exponent, which considers only the repetition of the factors as their length tends to infinity. Although we realized that computing asymptotic critical exponent is an integral part of computing the critical exponent of a given sequence, there was not much known about its properties in general. There have been some studies focusing on the asymptotic critical exponent of Sturmian sequences in the past; however, our study expanding it to balanced sequences, which are colourings of Sturmian sequences, is pioneering.

Before describing the content, let us mention that the results of this thesis are part of the published paper [16] and the submitted papers [8, 14].

At first, during my research project, we asked whether we can recreate Dejean's theorem for the asymptotic critical exponent. Cassaigne [21] constructed a binary sequence with the asymptotic critical exponent equal to 1. It follows that the minimum of asymptotic critical exponents over a d-letter alphabet is 1 for all $d \geq 2$. In [16], we proved that the infimum of asymptotic critical exponents of d-ary uniformly recurrent sequences is also 1.

Next, we focused on determining the minimal asymptotic critical exponent of balanced sequences over a d-letter alphabet. For this, we introduced a new tool – a *graph of admissible tails*. Using the theory explained in detail in [16] and in this thesis, I designed and implemented the algorithms for generating and searching the graph to determine the balanced sequences for which the minimum is attained. Using the program, we were able to determine the minimal asymptotic critical exponent for $d \in \{3, 4, \ldots, 10\}$ and find sequences for which the minimum is attained.

The thesis is structured as follows. First, we recall the necessary definitions in Chapter 1 and define the asymptotic critical exponent in Chapter 2. In Part I, we try to determine the minimal asymptotic critical exponent in balanced sequences over a d-letter alphabet. In more detail, in Chapter 3, we focus on some important properties of Sturmian sequences, especially on the forms and lengths of bispecial factors and their return words. In Chapter 4, we show how to generate balanced sequences from Sturmian sequences and focus on the relationship between bispecial factors in a balanced sequence and the underlying Sturmian sequence. In Chapter 5, we define the graph of admissible tails and show how it can be used to determine balanced sequences with the minimal asymptotic critical exponent. We also discuss possible reductions of the graph to significantly reduce the number of vertices and edges that we need to consider. The algorithms used to generate the graphs of admissible tails are described in Appendix A. In Chapter 6, we use the program to determine the minimal asymptotic critical exponent for sequences over a d-letter alphabet for $d \leq 10$. In Part II, Chapter 7 we abandon balanced sequences and focus on the repetition threshold of sequences avoiding palindromes or complements.

At first, we examine the form of bispecial factors in the sequence $\mathbf{p}$ in Chapter 8 and then examine four different morphic images of $\mathbf{p}$ in Chapters 9, 10, 11, and 12. As mentioned above, our contribution in this section is the calculation of the critical exponents of given morphic images of sequence $\mathbf{p}$. In the end, we summarize our results and mention open problems.

1 Combinatorics on words

An *alphabet* $\mathcal{A}$ is a finite set of symbols that are called *letters*. A concatenation of a finite number of letters is called a *word*, formally $u = u_0u_1 \cdots u_n$, where $u_i \in \mathcal{A}$ for all $i \in \{0, 1, \ldots, n\}$. The *length* of a word u is denoted by $|u|$ and it is the number of letters contained in u, so $|u_0u_1 \cdots u_n| = n + 1$. For each letter a in $\mathcal{A}$, we denote the number of occurrences of a in u by $|u|_a$.

The neutral element of concatenation is called the *empty word*, denoted by ε. The set of all words over an alphabet $\mathcal{A}$ is denoted by $\mathcal{A}^*$. If $u = xyz$ for some words x, y, z (they can be empty words) over an alphabet $\mathcal{A}$, then x is a *prefix*, z is a *suffix*, and y is a *factor* of u.

The *reverse* of a word $u = u_0u_1 \cdots u_{n-2}u_{n-1}$ is a word $u^R = u_{n-1}u_{n-2} \cdots u_1u_0$. A word u is a *palindrome*, if $u = u^R$, i.e. it is the same when read in the opposite direction.

A *sequence* over $\mathcal{A}$ is an infinite sequence of letters from the alphabet, we write $\mathbf{u} = u_0u_1u_2 \cdots$, where $u_n \in \mathcal{A}$ for all $n \in \mathbb{N}_0$. Sequences will always be written in bold. A *factor* of a sequence is an arbitrary finite sequence of consecutive letters, which means that u is a factor of $\mathbf{u} = u_0u_1 \cdots$ if $u = u_iu_{i+1} \cdots u_{j-1}$, where $i, j \in \mathbb{N}_0, i \leq j$. The index i is the *occurrence* of u in $\mathbf{u}$. The *language* of $\mathbf{u}$, denoted by $\mathcal{L}(\mathbf{u})$, is a set of all factors of $\mathbf{u}$.

To a word u over a d-letter alphabet $\mathcal{A}$, we assign its *Parikh vector* $\vec{V}(u) \in \mathbb{N}_0^d$, where $(\vec{V}(u))_a = |u|_a$ for all $a \in \mathcal{A}$. We say that a sequence $\mathbf{u}$ is *recurrent* if each of its factors has infinitely many occurrences in $\mathbf{u}$. If for each factor the distances between its consecutive occurrences are bounded, we say that $\mathbf{u}$ is *uniformly recurrent*.

A sequence $\mathbf{u}$ is *eventually periodic* if there exist words $u, v \in \mathcal{L}(\mathbf{u})$ such that $\mathbf{u} = vuuuuu \cdots = v(u)^\omega$, where exponent ω means an infinite repetition of the word u. The factor v is called a *preperiod* and u is a *period* of $\mathbf{u}$. If the length of u is minimal possible, then we denote the period of $\mathbf{u}$ as $\mathrm{Per}(\mathbf{u}) = |u|$. Let us note that period means two different things – a word u or the length of u. Usually, it is clear from the context which one is meant. The sequence $\mathbf{u}$ is *periodic* if $v = \varepsilon$. If $\mathbf{u}$ is not eventually periodic, we say that $\mathbf{u}$ is *aperiodic*.

Definition 1.1. *A sequence* $\mathbf{u}$ *over an alphabet* $\mathcal{A}$ *is* balanced*, if for any two of its factors* $u, v \in \mathcal{L}(\mathbf{u})$ *of the same length and any letter* $c \in \mathcal{A}$*, the number of occurrences of* c *in* u *and in* v *differs at most by one, i.e.,* $||u|_c - |v|_c| \leq 1$.

A *return word* to a factor w in $\mathbf{u}$ is a word $v \in \mathcal{L}(\mathbf{u})$ such that $vw \in \mathcal{L}(\mathbf{u})$ and w occurs in vw only as a prefix and as a suffix. In other words, if i, j, such that $i < j$, are two consecutive occurrences of w in $\mathbf{u}$, then a return word to w in $\mathbf{u}$ is the factor $v = v_iv_{i+1} \cdots v_{j-1}$. The set of all return words to w in $\mathbf{u}$ is denoted by $\mathcal{R}_{\mathbf{u}}(w)$.

The factor u of $\mathbf{u}$ is called a *right special factor* of $\mathbf{u}$ if there exist two different letters $a, b \in \mathcal{A}$ such that $ua, ub \in \mathcal{L}(\mathbf{u})$ – the factor has two different right extensions. We define a *left special factor* symmetrically – it has two different left extensions. A *bispecial factor* is a factor that is both right and left special.

Definition 1.2. *Let* $\mathbf{u}$ *be a sequence over* $\mathcal{A}$ *and* $u \in \mathcal{L}(\mathbf{u})$ *be a bispecial factor. Let*

$$B(u) = \#\{(a, b) \in \mathcal{A} \times \mathcal{A} : aub \in \mathcal{L}(\mathbf{u})\} - \#\{a \in \mathcal{A} : au \in \mathcal{L}(\mathbf{u})\} - \#\{b \in \mathcal{A} : ub \in \mathcal{L}(\mathbf{u})\} + 1.$$

We say that u *is a* weak bispecial factor *if* $B(u) < 0$ *and it is an* ordinary bispecial factor *if* $B(u) = 0$.

The *letter density* of a letter $a \in \mathcal{A}$ in a sequence $\mathbf{u}$ over $\mathcal{A}$ is the limit (if it exists)

$$\rho_a = \lim_{n \to +\infty} \frac{|u_0u_1 \cdots u_{n-1}|_a}{n}.$$

For example, in periodic sequences, the letter densities always exist and are rational. If $\mathbf{u}$ can be written as $\mathbf{u} = v(u)^\omega$, then for any $a \in \mathcal{A}$, the density of a in $\mathbf{u}$ can be calculated as $\frac{|u|_a}{|u|}$.

For Sturmian sequences, the letter densities always exist and are irrational ([28]).

Example 1.3. Let us choose a ternary alphabet $\mathcal{A} = \{a, n, s\}$. A word over this alphabet is, for example, *ananas*. Its Parikh vector (in alphabetical order of the letters) is

$$\vec{V}(ananas) = \begin{pmatrix} |ananas|_a \\ |ananas|_n \\ |ananas|_s \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The prefixes are, for instance, a, ana and suffixes are $anas, as$. If we consider the periodic sequence $\mathbf{u} = (ananas)^\omega = ananasananasana\cdots$, we can see that the return words to the factor na are $\mathcal{R}_\mathbf{u}(na) = \{na, nasa\}$. If we consider the factor sa, we will find only one return word, which is $sanana$. A return word can even be shorter than its factor. It happens when the occurrences of a factor overlap: for example, $\mathcal{R}_\mathbf{u}(ana) = \{an, anas\}$.

We can also observe left and right special factors. A bispecial factor is, for example, the factor a, because $an, as, sa, na \in \mathcal{L}(\mathbf{u})$. A longer bispecial factor is ana.

We can also see that, for eventually periodic words, there exists only a finite number of bispecial factors because we cannot find a bispecial factor longer than the sum of lengths of preperiod and period.

We can also determine the letter densities: $\rho_a = \frac{3}{6} = \frac{1}{2}, \rho_n = \frac{2}{6} = \frac{1}{3}$, and $\rho_s = \frac{1}{6}$.

Morphisms

Let $\mathcal{A}, \mathcal{B}$ be alphabets. A *morphism* χ is a map $\chi : \mathcal{A}^* \to \mathcal{B}^*$ such that $\chi(uv) = \chi(u)\chi(v)$ for all words $u, v \in \mathcal{A}^*$. Therefore, the image of a word by morphism χ can be constructed by applying the morphism χ to the individual letters of the word. Using this remark, we can apply the morphism χ even on sequences by setting $\chi(u_0u_1u_2\cdots) := \chi(u_0)\chi(u_1)\chi(u_2)\cdots$.

A sequence $\mathbf{v}$ is a *fixed point* of a morphism χ if $\chi(\mathbf{v}) = \mathbf{v}$. We say that the morphism χ is *non-erasing* if $\chi(c) \neq \varepsilon$ for each $c \in \mathcal{A}$.

A morphism $\chi : \mathcal{A}^* \to \mathcal{B}^*$ is *injective*, if for any $u, v \in \mathcal{A}^*$, we get $\chi(u) = \chi(v) \Rightarrow u = v$. We say that a morphism χ is *primitive*, if there exists $n \in \mathbb{N}$ such that

$$(\forall a \in \mathcal{A})(\forall b \in \mathcal{B})(|\chi^n(a)|_b > 0),$$

i.e. there exists $n \in \mathbb{N}$ such that image of any letter from $\mathcal{A}$ under χ^n contains every letter from $\mathcal{B}$ at least once.

If we consider a word u over a binary alphabet $\{0, 1\}$, then the *complement* of u, denoted by $\bar{u}$ is obtained as an image of u under the morphism that maps $0 \to 1$ and $1 \to 0$.

Example 1.4. A well-known aperiodic sequence is the *Fibonacci* sequence. It is the fixed point of the morphism

$$\begin{aligned} \varphi_F : \mathtt{a} &\to \mathtt{b}, \\ \mathtt{b} &\to \mathtt{ba}. \end{aligned}$$

We can see that φ_F is non-erasing, injective and also from the form of φ_F^2, we can say that φ_F is also primitive. We would like to note here that the predominant form of φ_F in the literature often uses different letters (usually $\mathtt{0}, \mathtt{1}$), or switches $\mathtt{a}, \mathtt{b}$. This does not have any impact on the properties we study.

Prefixes of the Fibonacci sequence of arbitrary length can be generated by recursive application of φ_F to the letter $\mathtt{b}$.

$$\mathtt{b} \to \varphi_F(\mathtt{b}) = \mathtt{ba} \to \varphi_F(\mathtt{ba}) = \mathtt{bab} \to \varphi_F(\mathtt{bab}) = \mathtt{babba} \to \varphi_F(\mathtt{babba}) = \mathtt{babbabab} \quad \cdots$$

2 Repetitions in words and sequences

In the previous chapter, we have already used the notation $(v)^\omega$ for the infinite repetition of the word v. We will now define an *exponent*.

Given a natural number $n \in \mathbb{N}$ and a word s, then s^n is a concatenation of n instances of s, for example, $s^4 = ssss$. Specially, $s^2 = ss$ is often called a *square* and s^3 is called a *cube*. The exponent can also be a rational number; if we take $\ell = |s|$, then for any $\frac{k}{\ell} \in \mathbb{Q}$, we obtain $s^{\frac{k}{\ell}}$ by taking the prefix of length k of the sequence s^ω.

Reversely, if z is a prefix of u^ω, where u is the shortest possible period, then we can write $z = u^e$, with $e = \frac{|z|}{|u|}$ called *exponent*.

Example 2.1. We will once again use the word *ananas*:

$$\begin{aligned}(ananas)^3 &= ananasananasananas,\\ (ananas)^{\frac{1}{6}} &= a,\\ (ananas)^{\frac{5}{3}} = (ananas)^{\frac{10}{6}} &= ananasanan.\end{aligned}$$

Another example is $pulp = (pul)^{\frac{4}{3}}$ or $onion = (oni)^{\frac{5}{3}}$.

The *critical exponent* of a sequence $\mathbf{u}$ is defined as

$$\mathrm{E}(\mathbf{u}) = \sup\{e \in \mathbb{Q} : \text{ there exist nonempty factors } x, y \in \mathcal{L}(\mathbf{u}) \text{ such that } x^e = y\}.$$

If the critical exponent is $+\infty$, then the *asymptotic critical exponent* is also $+\infty$. Otherwise, we define the asymptotic critical exponent as

$$\mathrm{E}^*(\mathbf{u}) = \lim_{n\to+\infty} \sup \{e \in \mathbb{Q} : \text{ there exist nonempty factors } x, y \in \mathcal{L}(\mathbf{u}), \text{ where } |x| > n, \text{ and } x^e = y\}.$$

It is obvious that for any sequence $\mathbf{u}$, we have $\mathrm{E}^*(\mathbf{u}) \leq \mathrm{E}(\mathbf{u})$. Also, if we consider an eventually periodic sequence $\mathbf{u} = v(u)^\omega$, we get $\mathrm{E}^*(\mathbf{u}) = \mathrm{E}(\mathbf{u}) = +\infty$, because $u^n \in \mathcal{L}(\mathbf{u})$ for all $n \in \mathbb{N}$. Therefore, we are interested only in aperiodic sequences.

The next theorem shows that we do not need to take into account all possible factors, only the bispecial factors and their shortest return words.

Theorem 2.2 ([15])**.** *Let* $\mathbf{u}$ *be a uniformly recurrent aperiodic sequence. Let* (w_n) *be a sequence of all bispecial factors ordered by their length. For every* $n \in \mathbb{N}$*, let* v_n *be a shortest return word to* w_n *in* $\mathbf{u}$*. Then*

$$\mathrm{E}(\mathbf{u}) = 1 + \sup_{n\in\mathbb{N}} \left\{\frac{|w_n|}{|v_n|}\right\} \qquad \textit{and} \qquad \mathrm{E}^*(\mathbf{u}) = 1 + \limsup_{n\to+\infty} \frac{|w_n|}{|v_n|}. \tag{2.1}$$

Part I

Minimal asymptotic critical exponent of balanced sequences

3 Sturmian sequences

Sturmian sequences form one of the most known classes of sequences over a binary alphabet. Therefore, there exist many equivalent definitions.

Definition 3.1 ([29])**.** Sturmian sequences *are aperiodic balanced sequences over a binary alphabet.*

In the thesis, we will consider Sturmian sequences over the alphabet $\mathcal{A} = \{\mathtt{a}, \mathtt{b}\}$.

For an arbitrary Sturmian sequence $\mathbf{u}$ and any $n \in \mathbb{N}_0$, there are exactly $n + 1$ different factors of length n. This also means that for any $n \in \mathbb{N}_0$, there is exactly one left and one right special factor of length n in $\mathbf{u}$. Another equivalent definition can be found in [35]: a sequence $\mathbf{u}$ is Sturmian if and only if each factor of $\mathbf{u}$ has exactly two return words.

The Sturmian sequences are recurrent and can therefore be written as a concatenation of return words to any prefix. Formally, let u be a prefix of $\mathbf{u}$ and denote r, resp. s, its return word, which is, resp. is not, a prefix of $\mathbf{u}$. Then $\mathbf{u}$ can be written as a sequence over an alphabet $\mathcal{A} = \{r, s\}$. This sequence is called a *derived sequence* of $\mathbf{u}$ to a factor u, denoted $\mathrm{d}_{\mathbf{u}}(u)$. We can also construct a derived sequence to a factor u that is not a prefix of $\mathbf{u}$ by omitting the prefix up to the first occurrence of u in $\mathbf{u}$.

For any factor u of a Sturmian sequence $\mathbf{u}$, the derived sequence $\mathrm{d}_{\mathbf{u}}(u)$ is also Sturmian, see [4].

In the following chapters, we shall keep the notation that the first element of the Parikh vector is the number of occurrences of the letter $\mathtt{a}$, resp. $\mathtt{s}$, and the second element is the number of occurrences of the letter $\mathtt{b}$, respectively the letter $\mathtt{r}$.

Definition 3.2. *A Sturmian sequence* $\mathbf{u}$ *is* standard *if each prefix of* $\mathbf{u}$ *is a left special factor.*

Theorem 3.3 ([28])**.** *If* $\mathbf{u}$ *is a Sturmian sequence, then there exists a standard Sturmian sequence* $\mathbf{u}'$ *such that* $\mathcal{L}(\mathbf{u}) = \mathcal{L}(\mathbf{u}')$.

We want to compute the asymptotic critical exponent, therefore we are interested only in the language of $\mathbf{u}$. Because of that, we will consider only standard Sturmian sequences in the rest of the thesis.

We will define two morphisms to help us work with standard Sturmian sequences:

$$G : \mathtt{a} \to \mathtt{a}, \quad \mathtt{b} \to \mathtt{ab}, \qquad\qquad D : \mathtt{a} \to \mathtt{ba}, \quad \mathtt{b} \to \mathtt{b}.$$

Both of morphisms G, D are non-erasing, injective, but not primitive, since, for example, for any $n \in \mathbb{N}$, $G^n(\mathtt{a}) = \mathtt{a}$, i.e. it does not contain $\mathtt{b}$.

Remark. We can see that for any word w over an alphabet $\{\mathtt{a}, \mathtt{b}\}$, we get $|G(w)|_{\mathtt{a}} = |w|_{\mathtt{a}} + |w|_{\mathtt{b}} = |w|$ and $|G(w)|_{\mathtt{b}} = |w|_{\mathtt{b}}$. Similarly for D, we get $|D(w)|_{\mathtt{b}} = |w|_{\mathtt{a}} + |w|_{\mathtt{b}} = |w|$ and $|D(w)|_{\mathtt{a}} = |w|_{\mathtt{a}}$.

Theorem 3.4 ([24])**.** *For every standard Sturmian sequence* $\mathbf{u}$*, there is a uniquely given* directive sequence $\mathbf{\Delta} = \Delta_0\Delta_1\Delta_2\cdots \in \{G, D\}^{\mathbb{N}_0}$ *of morphisms and a sequence* $(\mathbf{u}^{(n)})_{n\in\mathbb{N}}$ *of standard Sturmian sequences such that*

$$\mathbf{u} = \Delta_0\Delta_1\cdots\Delta_{n-1}\left(\mathbf{u}^{(n)}\right) \quad \text{for every } n \in \mathbb{N}.$$

Both G and D occur in the sequence $\mathbf{\Delta}$ *infinitely often.*

Since G and D occur infinitely often, we can write the directive sequence in one of the forms $\mathbf{\Delta} = D^{a_1}G^{a_2}D^{a_3}\cdots$ or $\mathbf{\Delta} = G^{a_1}D^{a_2}G^{a_3}\cdots$, where for every $i \in \mathbb{N}$, $a_i \in \mathbb{N}$. Following from the previous remark, if the directive sequence of $\mathbf{u}$ starts with D, then $\mathtt{b}$ is the most frequent letter in $\mathbf{u}$.

Definition 3.5. *If* $\mathbf{u}$ *is a standard Sturmian sequence with the directive sequence* $\mathbf{\Delta} = D^{a_1}G^{a_2}D^{a_3}\cdots$ *or* $\mathbf{\Delta} = G^{a_1}D^{a_2}G^{a_3}\cdots$, *we say that the* slope *of* $\mathbf{u}$ *is* $\theta \in (0,1)$ *determined by the continued fraction expansion in the form*

$$\theta = [0, a_1, a_2, a_3, a_4, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}.$$

Given continued fraction expansion $\theta = [0, a_1, a_2, a_3, a_4, \dots]$, we define the sequences $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$, which will be useful for the next calculations. The sequences satisfy the following recurrence relation

$$X_{n+1} = a_{n+1}X_n + X_{n-1} \qquad \text{for } n \geq 0, \tag{3.1}$$

where X_n represents either p_n or q_n, with initial conditions $p_{-1} = 1, p_0 = 0$, $q_{-1} = 0, q_0 = 1$. We also define a sequence $(Q_n)_{n\geq -1}$ by $Q_n = p_n + q_n$. This sequence can also be computed using the recurrence relation (3.1) with $Q_{-1} = 1 = Q_0$. It is well known that $\theta = \lim_{n\to\infty} \frac{p_n}{q_n}$.

Example 3.6. We will consider the Fibonacci sequence which is the fixed point of the morphism

$$\begin{aligned} \varphi_F : \mathtt{a} &\to \mathtt{b}, \\ \mathtt{b} &\to \mathtt{ba}. \end{aligned}$$

This morphism does not correspond to any of the morphisms G and D. However, the morphism φ_F^2 can be written as a concatenation of these two morphisms.

$$\begin{aligned} \varphi_F^2 : \mathtt{a} &\to \mathtt{ba}, & \qquad (DG) : \mathtt{a} &\to D(\mathtt{a}) \to \mathtt{ba}, \\ \mathtt{b} &\to \mathtt{bab}. & \mathtt{b} &\to D(\mathtt{ab}) \to \mathtt{bab}. \end{aligned}$$

Therefore, $\mathbf{\Delta} = (DG)^\omega$, $\theta = [0, (1)^\omega]$, and the sequences (p_n) and (q_n) satisfy the relations

$$p_n = p_{n-1} + p_{n-2}, \qquad q_n = q_{n-1} + q_{n-2}$$

with $p_0 = 0$, $p_1 = 1$ and $q_0 = 1$, $q_1 = 1$. Therefore, $p_n = F_n$ and $q_n = F_{n+1}$, where F_n denotes the Fibonacci numbers.

The elements of Fibonacci sequence can be calculated as

$$F_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

therefore,

$$\theta = \lim_{n\to+\infty} \frac{F_n}{F_{n+1}} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2} = \frac{1}{\tau},$$

where τ is the golden ratio.

The slope θ of $\mathbf{u}$ has more interesting properties, the most important ones for us are summarized in the next theorem.

Theorem 3.7. *Let* $\mathtt{b}$ *be the most frequent letter of a standard Sturmian sequence* $\mathbf{u}$ *with slope* θ*. Then*

1. *([28])* $\theta = \frac{\rho_\mathtt{a}}{\rho_\mathtt{b}}$,
2. *([12]) There exists* $u \in \mathcal{L}(\mathbf{u})$ *such that* $|u|_\mathtt{a} = \ell$ *and* $|u|_\mathtt{b} = k$ *if and only if* $k, \ell \in \mathbb{N}_0$ *and*

$$|\ell - k\theta| < (1 + \theta).$$

3.1 Bispecial factors and their return words

Considering a standard Sturmian sequence $\mathbf{u}$, we know that each left special factor is a prefix of $\mathbf{u}$. Therefore, the bispecial factors, which are both left and right special factors, are also prefixes of $\mathbf{u}$. The Sturmian words are closed under reversal, i.e. if u is a factor of $\mathbf{u}$, then u^R is also a factor of $\mathbf{u}$, see [36]. In particular, this means that if $u \in \mathcal{L}(\mathbf{u})$ is a left special factor, then $u^R \in \mathcal{L}(\mathbf{u})$ is a right special factor. Therefore, the bispecial factors in $\mathbf{u}$ are exactly the palindromic prefixes of $\mathbf{u}$. Each bispecial factor has exactly two return words, one of them is a prefix of $\mathbf{u}$, denoted by r, and the second one is denoted by s.

Theorem 3.8 ([15])**.** *Suppose that* $\mathbf{u}$ *is a standard Sturmian sequence whose directive sequence begins with D. Let* $\theta = [0, a_1, a_2, \dots]$ *be the slope of* $\mathbf{u}$ *and* z *be a bispecial factor of* $\mathbf{u}$. *Let* r *(resp.* s*) denote the return word to* z *which is (resp. is not) a prefix of* $\mathbf{u}$. *Then*

1. *there exists a unique pair* $(N, m) \in \mathbb{N}_0^2$ *with* $0 \le m < a_{N+1}$ *such that the Parikh vectors of* r, s *and* z *are respectively*

$$\vec{V}(r) = \begin{pmatrix} p_N \\ q_N \end{pmatrix}, \qquad \vec{V}(s) = \begin{pmatrix} mp_N + p_{N-1} \\ mq_N + q_{N-1} \end{pmatrix}, \qquad \vec{V}(z) = \vec{V}(r) + \vec{V}(s) - \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \tag{3.2}$$

2. *the slope of the derived sequence* $d_{\mathbf{u}}(z)$ *is*

$$\theta' = [0, a_{N+1} - m, a_{N+2}, a_{N+3}, \dots].$$

Remark. Let z be the n-th bispecial of a standard Sturmian sequence when ordered by length. Then, using the same notation as in Theorem 3.8, the pair (N, m) assigned to the bispecial factor z is defined by the following relation:

$$n = a_0 + a_1 + \cdots + a_N + m, \text{ where } a_0 = 0 \text{ and } m < a_{N+1}.$$

This assignment is therefore bijective.

Example 3.9. Let us continue with Example 3.6. We already know that $p_n = F_n$ and $q_n = F_{n+1}$, where F_n denotes the Fibonacci number.

Since $0 \le m < a_n = 1$ for all $n \in \mathbb{N}$, we get $m = 0$ for every bispecial factor. So, the Parikh vectors of the N-th bispecial factor u_N and its return words r, s are

$$\vec{V}(r) = \begin{pmatrix} p_N \\ q_N \end{pmatrix} = \begin{pmatrix} F_N \\ F_{N+1} \end{pmatrix},$$
$$\vec{V}(s) = \begin{pmatrix} p_{N-1} \\ q_{N-1} \end{pmatrix} = \begin{pmatrix} F_{N-1} \\ F_N \end{pmatrix},$$
$$\vec{V}(u_N) = \begin{pmatrix} F_N + F_{N-1} - 1 \\ F_{N+1} + F_N - 1 \end{pmatrix} = \begin{pmatrix} F_{N+1} - 1 \\ F_{N+2} - 1 \end{pmatrix}.$$

The prefix of the Fibonacci sequence is

$$\mathtt{babba}\underline{\mathtt{bab\ bab}}\mathtt{ba\ babbabab\ }\underline{\mathtt{babbab}}\mathtt{ab\ babba\ babbabab\ babba\ babbabab.}$$

So, for example, the third bispecial is babbab (as can be seen from the underlined occurrences) and its length is indeed $F_5 + F_4 - 2 = 5 + 3 - 2 = 6$.

4 Balanced sequences

We have already said that Sturmian sequences are aperiodic balanced sequences over a binary alphabet. Even more interesting is that recurrent aperiodic balanced sequences over an arbitrary alphabet can be derived from Sturmian sequences by a process called colouring. Before we get to it, we will need some definitions.

Definition 4.1. *A sequence* $\mathbf{y}$ *is a* constant gap sequence *if for each letter* c *occurring in* $\mathbf{y}$ *there is a positive integer denoted by* $\mathrm{gap}_{\mathbf{y}}(c)$ *such that the distance between any two consecutive occurrences of* c *in* $\mathbf{y}$ *is* $\mathrm{gap}_{\mathbf{y}}(c)$.

We can see that any constant gap sequence is periodic, the minimum length of the period is denoted by $\mathrm{Per}(\mathbf{y})$ and it is equal to the least common multiple of $\mathrm{gap}_{\mathbf{y}}(c)$ for all $c \in \mathcal{A}$.

Definition 4.2. *Let* $\mathbf{u}$ *be a sequence over* $\{\mathtt{a}, \mathtt{b}\}$*,* $\mathbf{y}$ *and* $\mathbf{y}'$ *be arbitrary sequences. The colouring of* $\mathbf{u}$ *by* $\mathbf{y}$ *and* $\mathbf{y}'$ *is the sequence* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ *obtained from* $\mathbf{u}$ *by replacing the subsequence of all* $\mathtt{a}$*'s with* $\mathbf{y}$ *and the subsequence of all* $\mathtt{b}$*'s with* $\mathbf{y}'$.

Theorem 4.3 ([20])**.** *A recurrent aperiodic sequence* $\mathbf{v}$ *is balanced if and only if* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ *for some Sturmian sequence* $\mathbf{u}$ *and constant gap sequences* $\mathbf{y}$ *and* $\mathbf{y}'$ *over two disjoint alphabets.*

If $\mathcal{A}$ and $\mathcal{B}$ are the two disjoint alphabets, we will use the notation π for a "discolouration map", i.e., $\pi : \mathcal{A} \cup \mathcal{B} \to \{\mathtt{a}, \mathtt{b}\}$ is a morphism mapping

$$\begin{aligned} a &\mapsto \mathtt{a} \quad \text{for all } a \in \mathcal{A}, \\ b &\mapsto \mathtt{b} \quad \text{for all } b \in \mathcal{B}. \end{aligned}$$

Let $\mathbf{u}$ be a Sturmian sequence, $\mathbf{y}$ and $\mathbf{y}'$ two constant gap sequences over disjoint alphabets and let $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. Then $\mathbf{u} = \pi(\mathbf{v})$, resp. for any $v \in \mathcal{L}(\mathbf{v})$, $\pi(v) \in \mathcal{L}(\mathbf{u})$.

This theorem allows us to work only with a Sturmian sequence and two constant gap sequences when studying a balanced sequence. If we consider only standard Sturmian sequences (and the following theorem allows us to do so), we can describe any aperiodic balanced sequence by the slope θ and two finite words representing periods of constant gap sequences $\mathbf{y}$ and $\mathbf{y}'$.

Theorem 4.4 ([12])**.** *Let* $\mathbf{u}$*,* $\mathbf{u}'$ *be two Sturmian sequences such that* $\mathcal{L}(\mathbf{u}) = \mathcal{L}(\mathbf{u}')$ *and* $\mathbf{y}$ *and* $\mathbf{y}'$ *be two constant gap sequences over disjoint alphabets. Let* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ *and* $\mathbf{v}' = \mathrm{colour}(\mathbf{u}', \mathbf{y}, \mathbf{y}')$*. Then* $\mathcal{L}(\mathbf{v}) = \mathcal{L}(\mathbf{v}')$.

Example 4.5. The sequence $(01)^\omega$ is balanced. On the other hand, the sequence $\mathbf{v} = (0110)^\omega$ is not because $00, 11 \in \mathcal{L}(\mathbf{v})$ has the same length and the number of ones differs by 2.

The sequence $(010203)^\omega$ is a constant gap sequence over a 4-letter alphabet. For 0, the gap is 2, and it is 6 for the other letters. We can therefore see that 0 is the longest bispecial factor in the sequence.

On the other hand, the sequence $(101)^\omega$ is not a constant gap sequence.

Now we will consider a standard Sturmian sequence $\mathbf{u}$ with the directive sequence $\mathbf{\Delta} = (DG^2)^\omega$. Its prefix can be constructed by repeated application of the morphisms DG^2 on $\mathtt{b}$:

$$\begin{aligned} DG^2 : \mathtt{a} &\to DG(\mathtt{a}) \;\;= D(\mathtt{a}) \quad\;\; = \mathtt{ba}, \\ \mathtt{b} &\to DG(\mathtt{ab}) = D(\mathtt{aab}) = \mathtt{babab}. \end{aligned}$$

Therefore, the following word is a prefix of $\mathbf{u}$:

$$DG^2DG^2(\mathtt{b}) = DG^2(\mathtt{babab}) = \mathtt{babab\ ba\ babab\ ba\ babab}.$$

Next, we will show the process of colouring the prefix by two constant gap sequences $\mathbf{y} = (34)^\omega$ and $\mathbf{y}' = (0102)^\omega$. We will obtain a prefix of a balanced sequence $\mathbf{v} = \mathrm{colour}(\mathbf{u}, (34)^\omega, (0102)^\omega)$, which is a sequence over a 5-letter alphabet $\mathcal{A}\dot{\cup}\mathcal{B} = \{0, 1, 2, 3, 4\}$.

$\mathbf{u}$ =	b	a	b	a	b	b	a	b	a	b	a	b	b	a	b	a	b	a	b	⋯
colouring of a:		3		4			3		4		3			4		3		4		⋯
colouring of b:	0		1		0	2		0		1		0	2		0		1		0	⋯
$\mathbf{v}$ =	0	3	1	4	0	2	3	0	4	1	3	0	2	4	0	3	1	4	0	⋯

Factors of length 5 are, for example, 03140, 31402, 23041, 30240, 02403. The factor 03140 contains two 0's, therefore, since the sequence $\mathbf{v}$ is balanced, there is no factor $w \in \mathcal{L}(\mathbf{v})$ of length 5 such that $|w|_0 = 0$. Since there are other factors of length five containing only one 0, there is no factor $w \in \mathcal{L}(\mathbf{v})$ of length 5 such that $|w|_0 \geq 3$.

We can also see that the factor bab is coloured by many possible combinations of letters from $\mathbf{y}$ and $\mathbf{y}'$, mainly 031, 140, 230, 041, 130, 240. Therefore, we can see that the return words to a factor $w \in \mathcal{L}(\mathbf{v})$ will be longer than the return words to the factor $\pi(w)$ in $\mathbf{u}$ and not every occurrence of $\pi(w)$ in $\mathbf{u}$ forces occurrence of w in $\mathbf{v}$.

4.1 Bispecial factors and their return words

Using Theorem 4.3, we can obtain many characteristics of balanced sequences from the characteristics of Sturmian sequences.

Theorem 4.6 ([12])**.** *Let $\mathbf{u}$ be a standard Sturmian sequence, $\mathbf{y}$ and $\mathbf{y}'$ constant gap sequences over disjoint alphabets, and let $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. Let $w \in \mathcal{L}(\mathbf{v}), u \in \mathcal{L}(\mathbf{u})$ be factors such that $\pi(w) = u$. Then,*

1. *if u is a bispecial factor of $\mathbf{u}$, then w is a bispecial factor of $\mathbf{v}$.*

2. *If w is a bispecial factor of $\mathbf{v}$ and $|u|_{\mathtt{a}} \geq \mathrm{Per}(\mathbf{y})$ and $|u|_{\mathtt{b}} \geq \mathrm{Per}(\mathbf{y}')$, then u is a bispecial factor of $\mathbf{u}$.*

This theorem combined with Theorem 3.8 gives us a formula for the lengths of bispecial factors in a recurrent aperiodic balanced word. Now, we need a formula for the length of a shortest return word to a given bispecial factor.

Let $\mathbf{u}, \mathbf{y}, \mathbf{y}', \mathbf{v}$ be the same as in Theorem 4.6. Let x be a bispecial factor in $\mathbf{u}$, y be a bispecial factor in $\mathbf{v}$ such that $\pi(y) = x$ and we want to find the necessary conditions for z to be a return word to y in $\mathbf{v}$. Let r, s be the two return words to x in $\mathbf{u}$.

1. Let i, j be the occurrences of y in $\mathbf{v}$. Then i, j are also occurrences of x in $\mathbf{u}$ and the factor $u_i u_{i+1} \dots u_{j-1}$ is a concatenation of return words r, s, because $u_i u_{i+1} \dots u_{j-1} x$ has x as a prefix and as a suffix, although it can appear somewhere in the middle, too.

2. Moreover, y can be obtained only by colouring the factor x in $\mathbf{u}$, and for the factors $u_i u_{i+1} \dots u_{i+|x|-1}$ and $u_j u_{j+1} \dots u_{j+|x|-1}$ to be coloured by the same letters in $\mathbf{y}$ and $\mathbf{y}'$ (so that we get y and not some other factor w such that $\pi(w) = x$), we have to be at the same positions in the periods of $\mathbf{y}$ and $\mathbf{y}'$. Therefore, for x "long enough" the number of a's and b's in $u_i u_{i+1} \dots u_{j-1}$ must be divisible by $\mathrm{Per}(\mathbf{y})$ and $\mathrm{Per}(\mathbf{y}')$, respectively.

Theorem 4.7 ([12]). *Let* $\mathbf{u}$ *be a Sturmian sequence,* $\mathbf{y}$ *and* $\mathbf{y}'$ *be two constant gap sequences over disjoint alphabets, and* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. *Let* $u \in \mathcal{L}(\mathbf{u})$ *be a bispecial factor such that* $|u|_{\mathtt{a}} \geq \mathrm{Per}(\mathbf{y})$ *and* $|u|_{\mathtt{b}} \geq \mathrm{Per}(\mathbf{y}')$. *Let r, resp. s be the return word to u in* $\mathbf{u}$ *which is, resp. is not, a prefix of* $\mathbf{u}$. *We denote* $\mathcal{S}(u) = \mathcal{S}_1(u) \cap \mathcal{S}_2(u) \cap \mathcal{S}_3$, *where*

$$\mathcal{S}_1(u) = \left\{ \begin{pmatrix} \ell \\ k \end{pmatrix} : k\vec{V}(r) + \ell\vec{V}(s) = \begin{pmatrix} 0 & \mod \mathrm{Per}(\mathbf{y}) \\ 0 & \mod \mathrm{Per}(\mathbf{y}') \end{pmatrix} \right\},$$

$$\mathcal{S}_2(u) = \left\{ \begin{pmatrix} \ell \\ k \end{pmatrix} : \begin{pmatrix} \ell \\ k \end{pmatrix} \text{ is a Parikh vector of a factor of derived sequence } \mathrm{d}_{\mathbf{u}}(u) \right\},$$

$$\mathcal{S}_3 = \left\{ \begin{pmatrix} \ell \\ k \end{pmatrix} : 1 \leq k + \ell \leq \mathrm{Per}(\mathbf{y})\mathrm{Per}(\mathbf{y}') \right\}.$$

Then a shortest return word $v \in \mathcal{L}(\mathbf{v})$ *to any bispecial factor* $w \in \mathcal{L}(\mathbf{v})$*, where* $\pi(w) = u$*, has length*

$$|v| = \min \left\{ k|r| + \ell|s| : \begin{pmatrix} \ell \\ k \end{pmatrix} \in \mathcal{S}(u) \right\}.$$

Theorem 4.7 says that the conditions mentioned above are sufficient. The set $\mathcal{S}_2(u)$ ensures that the concatenation of return words r, s are allowed in $\mathbf{u}$, i.e. Item 1 above is fulfilled. The set $\mathcal{S}_1(u)$ then ensures that after reading this concatenation, we are at the same positions in $\mathbf{y}$ and $\mathbf{y}'$, as discussed in Item 2 above. The set $\mathcal{S}_3$ only limits the number of pairs $(\ell, k)^T$ we need to consider. And for v to be a shortest return word, we consider the shortest factor complying with the restrictions above.

Example 4.8. Let us recall the prefix of the balanced sequence generated in Example 4.5, we obtained $\mathbf{v} = 03140\underline{230}41\underline{30}2403140\cdots$ as a colouring of $\mathbf{u} = \mathtt{bababbababababbababab}\cdots$ by the constant gap sequences $\mathbf{y} = (34)^\omega$ and $\mathbf{y}' = (0102)^\omega$.

If we consider the factor 30, we can see from the underlined occurrences that it is a bispecial factor of $\mathbf{v}$.

However, $\pi(30) = \mathtt{ab}$ is not a bispecial in $\mathbf{u}$ because the letter $\mathtt{a}$ is always preceded by $\mathtt{b}$ in $\mathbf{u}$. The conditions of Theorem 4.6 were not fulfilled.

Now we will choose the bispecial factor $u = \mathtt{bababbabab}$. To find the return words, we will need a longer prefix of $\mathbf{u}$ and of $\mathbf{v}$. It can be found by applying DG^2 on the prefix we have already found and then colouring it.

$\mathbf{u} = DG^2(\mathtt{babab\ ba\ b}\cdots)$	$= \mathtt{babab}$	$\mathtt{ba}$	$\mathtt{babab}$	$\mathtt{ba}$	$\mathtt{babab}$	$\mathtt{babab}$	$\mathtt{ba}$	$\mathtt{babab}$	$\cdots$
$\mathbf{v} = \mathrm{colour}(\mathbf{u}, (34)^\omega, (0102)^\omega)$	$= 03140$	23	04130	24	03140	23041	03	24031	$\cdots$

We can find the return words in the prefix, they are $r = \mathtt{bababba}$ and $s = \mathtt{babab}$. Now the set $\mathcal{S}_1(u)$ has the form

$$\mathcal{S}_1(u) = \left\{ \begin{pmatrix} \ell \\ k \end{pmatrix} : k\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \ell\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & \mod 2 \\ 0 & \mod 4 \end{pmatrix} \right\}.$$

For example, $k = 2, \ell = 0$ is in the set $\mathcal{S}_1(u)$.

The prefix of the derived sequence $\mathrm{d}_{\mathbf{u}}(u)$ can be constructed from $\mathbf{u}$ and we get $\mathrm{d}_{\mathbf{u}}(u) = \mathtt{rrsr}\cdots$. Therefore, $k = 2, \ell = 0$ is also in the set $\mathcal{S}_2(u)$.

Since $2 + 0 < 2 \cdot 4$, the pair $k = 2, \ell = 0$ is also in the set $\mathcal{S}_3$ and therefore $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \in \mathcal{S}(u)$.

If we set $k \in \{0, 1\}$, then the second equation for $\mathcal{S}_1(u)$ forces $\ell = 4a$ for $a \in \mathbb{N}$ and these pairs are not in the set $\mathcal{S}_2(u)$ since the sequence $\mathrm{d}_{\mathbf{u}}(u)$ is Sturmian with $\mathtt{r}$ more frequent letter.

Therefore, the pair $k = 2, \ell = 0$ defines the length of a shortest return word to $z \in \mathcal{L}(\mathbf{v}), \pi(z) = u$. The length can be computed as $k|r| + \ell|s| = 2 \cdot 7 = 14$. For example, the shortest return word to $z = 03140\ 23\ 041$ is $03140\ 23\ 04130\ 24$.

5 Construction of graph of admissible tails

Notation. In this chapter, we will always denote $\mathbf{u}$ a standard Sturmian sequence with slope in the form $\theta = [0, a_1, a_2, a_3, \dots]$, where the directive sequence starts with D. Hence, the letter b is the most frequent in $\mathbf{u}$. The sequences $\mathbf{y}$ and $\mathbf{y}'$ will be two constant gap sequences over disjoint alphabets and $\mathbf{v} := \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. We will also use the symbols $P = \mathrm{Per}(\mathbf{y})$ and $P' = \mathrm{Per}(\mathbf{y}')$.

For given $N \in \mathbb{N}$, we denote $A_N := \left(\begin{smallmatrix} p_{N-1} & p_N \\ q_{N-1} & q_N \end{smallmatrix}\right)$ and $\delta_N = [a_{N+1}, a_{N+2}, \dots]$.

Remark. The recurrence relation between A_N and A_{N+1} can be written in the form

$$A_{N+1} = A_N \begin{pmatrix} 0 & 1 \\ 1 & a_{N+1} \end{pmatrix} = A_N \begin{pmatrix} 0 & 1 \\ 1 & \lfloor \delta_N \rfloor \end{pmatrix}. \tag{5.1}$$

Convention: We will write

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix}$$

if

$$a_1 \equiv b_1 \mod P \quad \wedge \quad a_2 \equiv b_2 \mod P',$$

and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix}$$

if

$$\begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix} = \begin{pmatrix} b_{1i} \\ b_{2i} \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix} \qquad \forall i \in \{1, 2\}.$$

If we consider a bispecial factor u with assigned pair (N, m), we can combine Theorem 4.7 with Theorems 3.8 and 3.7, and obtain the following definition.

Definition 5.1. *Let $N \in \mathbb{N}$ and $0 \le m < a_{N+1}$. Then to a pair (N, m), we assign the sets*

$$\begin{aligned}
\mathcal{S}_1(N, m) &= \left\{ \begin{pmatrix} \ell \\ k \end{pmatrix} \in \mathbb{N}_0^2 : A_N \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} \ell \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix} \right\}, \\
\mathcal{S}_2(N, m) &= \left\{ \begin{pmatrix} \ell \\ k \end{pmatrix} \in \mathbb{N}_0^2 : |\ell(\delta_N - m) - k| < \delta_N - m + 1 \text{ and } k + \ell > 0 \right\}, \\
\mathcal{S}_3 &= \left\{ \begin{pmatrix} \ell \\ k \end{pmatrix} \in \mathbb{N}_0^2 : 1 \le k + \ell \le PP' \right\}, \\
\mathcal{S}(N, m) &= \mathcal{S}_1(N, m) \cap \mathcal{S}_2(N, m) \cap \mathcal{S}_3.
\end{aligned}$$

Combining this with Theorem 2.2 results in the following theorem.

Theorem 5.2 ([12])**.** *Let $\mathbf{u}$ be a standard Sturmian sequence with slope $\theta = [0, a_1, a_2, a_3, \dots]$ and the directive sequence starting with D. Let $\mathbf{y}$ and $\mathbf{y}'$ be two constant gap sequences over disjoint alphabets, and $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. We define the sequence $(\Phi_N)_{N \in \mathbb{N}}$ as*

$$\Phi_N := \max \left\{ \frac{1 + m + \frac{Q_{N-1}}{Q_N}}{k + \ell m + \ell \frac{Q_{N-1}}{Q_N}} : \begin{pmatrix} \ell \\ k \end{pmatrix} \in \mathcal{S}(N, m) \text{ and } 0 \le m < a_{N+1} \right\}. \tag{5.2}$$

Then $\mathrm{E}^*(\mathbf{v}) = 1 + \limsup_{n \to +\infty} \Phi_N$.

This theorem shows us that the asymptotic critical exponent of $\mathbf{v}$ depends only on the periods of $\mathbf{y}$ and $\mathbf{y}'$, not on their structure. The same is not true for the critical exponent.

5.1 Equivalence on unimodular matrices

We will now focus on the set $\mathcal{S}_1(N, m)$ and try to reduce the number of matrices A_N necessary for the computation of $\mathrm{E}^*(\mathbf{v})$. Let us start with some notation.

Notation. For given periods P and P', we will use the notation H, L, Y, Y' such that

$$H = \gcd(P, P'), \qquad P = HY, \qquad P' = HY', \qquad L = \operatorname{lcm}(P, P') = YY'H. \tag{5.3}$$

We always consider $L > 1$ because for $L = 1$ the sequence $\mathbf{u}$ and $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ have the same asymptotic critical exponent (they are the same up to letter symbols) and the (minimal) asymptotic critical exponent for Sturmian sequences is already known.

Lemma 5.3. *For all $N \in \mathbb{N}_0$, the matrix A_N is unimodular and $A_N^{-1} \in \mathbb{Z}^{2\times 2}$.*

Proof. We can see that $\det A_N = p_{N-1}q_N - p_N q_{N-1}$. For A_0, we obtain $\det A_0 = p_{-1}q_0 - p_0 q_{-1} = 1\cdot 1 - 0 = 1$. Let us consider $N \geq 0$. Then

$$\begin{aligned} \det A_{N+1} &= p_N q_{N+1} - p_{N+1} q_N = p_N(a_{N+1}q_N + q_{N-1}) - q_N(a_{N+1}p_N + p_{N-1}) \\ &= p_N q_{N-1} - q_N p_{N-1} = -\det A_N. \end{aligned}$$

Therefore $\det A_N = (-1)^N$. Moreover, the inverse of a matrix in $\mathbb{Z}^{2\times 2}$ can be computed as

$$\begin{pmatrix} p_{N-1} & p_N \\ q_{N-1} & q_N \end{pmatrix}^{-1} = \frac{1}{p_{N-1}q_N - p_N q_{N-1}} \begin{pmatrix} q_N & -p_N \\ -q_{N-1} & p_{N-1} \end{pmatrix} = (-1)^N \begin{pmatrix} q_N & -p_N \\ -q_{N-1} & p_{N-1} \end{pmatrix} \in \mathbb{Z}^{2\times 2}.$$

□

Using the lemma above, we get the following statement. It will help us create classes of equivalence for matrices A_N such that $\mathcal{S}_1(N, m)$ will be the same for each representative of the class $[A_N]_{\equiv}$.

Lemma 5.4 ([16])**.** *Let $A \in \mathbb{Z}^{2\times 2}$ be a unimodular matrix and $\ell, k \in \mathbb{Z}$. Then* $A\begin{pmatrix} \ell \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix}$ *if and only if there exist $\lambda, \varkappa \in \mathbb{Z}$ such that*

$$\begin{pmatrix} \ell \\ k \end{pmatrix} = H\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix} \qquad \text{and} \qquad A\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} Y \\ Y' \end{pmatrix}.$$

Therefore, solving equation $A\begin{pmatrix} \ell \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix}$, where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, is equivalent to finding the solution of $M_A\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} Y \\ Y' \end{pmatrix}$ with matrix

$$M_A = \begin{pmatrix} a_{11} \mod Y & a_{12} \mod Y \\ a_{21} \mod Y' & a_{22} \mod Y' \end{pmatrix} \quad \text{and setting} \quad \begin{pmatrix} \ell \\ k \end{pmatrix} = H\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix}.$$

Remark. Since there is a bijection between solutions of

$$A\begin{pmatrix} \ell \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix} \quad \text{and of} \quad M_A\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} Y \\ Y' \end{pmatrix},$$

we will work only with the representation of a matrix by M_A in the program and in the figures in this thesis.

In fact, even two matrices A, B, where $M_A \neq M_B$, can have identical sets of solutions of equation $M_A \left(\begin{smallmatrix} \lambda \\ \varkappa \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \mod \left(\begin{smallmatrix} Y \\ Y' \end{smallmatrix}\right)$ as shown in the following example.

Example 5.5. If we set the periods $P = Y = 4$ and $P' = Y' = 1$, we can check that the matrices $A = M_A = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = M_B = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 1 \end{smallmatrix}\right)$ have the same set of solutions of the equations

$$A\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} Y \\ Y' \end{pmatrix}, \qquad B\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} Y \\ Y' \end{pmatrix}.$$

Since $Y' = 1$, we need to satisfy only the first row because the second row holds for any integers $\lambda, \varkappa$. The equation for A gives us

$$0 \equiv 1\lambda + 2\varkappa \mod 4\,.$$

If we multiply both sides by 3, which is coprime to 4 (so the multiplication does not change the solution), we get

$$0 \equiv 3\lambda + 6\varkappa \equiv 3\lambda + 2\varkappa \mod 4\,,$$

which is the same as the first equation for B.

We will therefore use the following equivalence to group such matrices M_A, M_B together into the same class of equivalence.

Definition 5.6. *Let A and B be unimodular matrices in $\mathbb{Z}^{2\times 2}$. We say that A is equivalent to B, we write $A \equiv B$, if there exist $c \in \mathbb{Z}$ coprime with Y and $c' \in \mathbb{Z}$ coprime with Y' such that*

$$\begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix} A = B \mod \begin{pmatrix} Y \\ Y' \end{pmatrix}. \tag{5.4}$$

The class of equivalence containing A will be denoted $[A]_\equiv$.

Remark. The relation $\equiv$ is an equivalence. Obviously $A \equiv A$. Also, the set of elements $\{c \in \mathbb{Z} : c \text{ coprime with } z\}$ is closed under multiplication and inversion modulo z for any $z \in \mathbb{Z}$, giving us the two remaining properties, i.e., transitivity and symmetry, required for $\equiv$ to be an equivalence. It is true for any $z \in \mathbb{Z}$, especially for $z \in \{Y, Y'\}$.

Remark. We can also combine the definition with the previous lemma because if $A \equiv B$, then $M_A \equiv M_B$.

The reason for the definition of $\equiv$ in this way is revealed in the following lemma.

Lemma 5.7 ([16])**.** *Let $\ell, k \in \mathbb{N}$ and let A and B be unimodular matrices in $\mathbb{Z}^{2\times 2}$ such that $A \equiv B$. Then*

- $A\begin{pmatrix} \ell \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix}$ *if and only if* $B\begin{pmatrix} \ell \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} P \\ P' \end{pmatrix}$,
- $AC \equiv BC$ *for any unimodular matrix $C \in \mathbb{Z}^{2\times 2}$.*

5.2 Lower bound on the asymptotic critical exponent

The next step will be to use the equivalence $\equiv$ to determine the lower bound on the asymptotic critical exponent for all balanced sequences $\mathbf{v} = \text{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$, where $\mathbf{u}$ is arbitrary Sturmian, but P and P' are fixed. Since for a given size of the alphabet we have only a finite number of possible periods of constant gap sequences over the alphabet, this approach will allow us to find the lower bound.

Definition 5.8 ([16]). *Let $A \in \mathbb{Z}^{2\times 2}$ be unimodular and $\beta > 0$. We say that $\delta > 1$ is* $(1+\beta)$-forcing *for the class $[A]_\equiv$ if there exist $m, k, \ell \in \mathbb{N}_0$, $\ell + k > 0$, such that*

$\mathfrak{P}1$*:* $A\begin{pmatrix}1 & 0\\ m & 1\end{pmatrix}\begin{pmatrix}\ell\\ k\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \mod \begin{pmatrix}P\\ P'\end{pmatrix}$;

$\mathfrak{P}2$*:* $m + 1 < \delta$ *and* $|\ell(\delta - m) - k| < \delta - m + 1$;

$\mathfrak{P}3$*:*
- *if $k = \ell$, then $\frac{1}{k} > \beta$;*
- *if $k > \ell$, then $\frac{1+m}{k+m\ell} \geq \beta$;*
- *if $k < \ell$, then $\frac{2+m}{k+(m+1)\ell} \geq \beta$.*

The set of $(1+\beta)$-forcing δ's for the class $[A]_\equiv$ is denoted $\mathcal{F}(\beta, A)$.

Lemma 5.7 tells us that the definition is correct because only $\mathfrak{P}1$ depends on the matrix A and we know that the solutions are the same for equivalent matrices. We can also see that for $A = A_N$, the fulfillment of $\mathfrak{P}1$ means $(\ell, k)^T \in \mathcal{S}_1(N, m)$ and the condition $\mathfrak{P}2$ means $(\ell, k)^T \in \mathcal{S}_2(N, m)$. The reason behind $\mathfrak{P}3$ and the term $(1+\beta)$-*forcing* will be explained in the next theorem.

Theorem 5.9 ([16]). *Let* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$*, where* $\mathbf{u}$ *is a Sturmian sequence with the slope* $\theta = [0, a_1, a_2, a_3, \ldots]$*, and let β be a fixed positive number.*

Assume that there exist infinitely many $N \in \mathbb{N}$ such that δ_N is $(1+\beta)$-forcing for the class $[A_N]_\equiv$. Then $\mathrm{E}^*(\mathbf{v}) > 1 + \beta$*.*

5.3 Admissible tails of continued fraction expansions

In this section, we want to find all balanced sequences $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ such that the asymptotic critical exponent of $\mathbf{v}$ is $\mathrm{E}^*(\mathbf{v}) \leq 1 + \beta$ for given $\beta > 0$. Theorem 5.9 shows us that the best way

to proceed is to guarantee that δ_N is not $(1+\beta)$-forcing for the class $[A_N]_\equiv$ for any N large enough.

Remark. In Definition 5.8, only the property $\mathfrak{P}2$ depends on δ. Hence, if a triplet (m, k, ℓ) satisfies the properties $\mathfrak{P}1$ and $\mathfrak{P}3$, the property $\mathfrak{P}2$ means that the interval

$$\begin{cases} (m + k - 1, +\infty) \cap (m + 1, +\infty) & \text{if } \ell = 0;\\ (m + \frac{k-1}{2}, +\infty) \cap (m + 1, +\infty) & \text{if } \ell = 1;\\ (m + \frac{k-1}{\ell+1}, m + \frac{k+1}{\ell-1}) \cap (m + 1, +\infty) & \text{if } \ell \geq 2; \end{cases} \tag{5.5}$$

belongs to the set $\mathcal{F}(\beta, A)$. Therefore, the set $\mathcal{F}(\beta, A)$ is a union of several open intervals. Since we want to consider δ as a tail of the continued fraction expansion of slope θ, it cannot be rational. This means that even the boundaries of the intervals in (5.5) are not suitable candidates.

This leads to the following definition.

Definition 5.10 ([16]). *Let $\beta > 0$ and $A \in \mathbb{Z}^{2\times 2}$ be unimodular. We denote*

$$\mathcal{D}(\beta, A) = \{\delta > 1 : \delta \textit{ is NOT in the closure of } \mathcal{F}(\beta, A)\}.$$

Using this notation, we can see that $\mathrm{E}^*(\mathbf{v}) \leq 1+\beta$ implies that for any N large enough, $\delta_N \in \mathcal{D}(\beta, A_N)$.

In the computer program, we do not want to work with unbounded intervals. The following lemma shows that we do not have to.

Lemma 5.11 ([16]). *Let $A \in \mathbb{Z}^{2\times 2}$ be a unimodular matrix. Let $\beta > 0$ and* $L = \mathrm{lcm}(P, P') > 1$*. Then the set $\mathcal{D}(\beta, A)$ is a subset of $(1, \lceil L(1+\beta)\rceil - 2)$. In particular, $\mathcal{D}(\beta, A)$ is bounded for each equivalence class $[A]_\equiv$.*

Not only this lemma ensures that we will not have to work with unbounded intervals, but it gives us also a bound on m. If $m \geq \lceil L(1+\beta)\rceil - 2$, then all intervals in Equation (5.5) lie in the complement of $(1, \lceil L(1+\beta)\rceil - 2)$ and therefore will not change $\mathcal{D}(\beta, A)$.

Moreover, since we have to consider only a finite number of triplets m, k, ℓ to generate $\mathcal{D}(\beta, A)$, the set $\mathcal{D}(\beta, A)$ consists of a finite number of intervals.

Example 5.12. Consider $P = 2$, $P' = 12$ and $\beta = 0.14095$. So $Y = 1$, $Y' = 6$, $H = 2$ and $L = 12$.

For this setting, one of the matrices we need to consider has $M_A = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 1 \end{smallmatrix}\right)$.

From Lemma 5.11, we have $\mathcal{D}(\beta, A) \subset (1, \lceil 12 \cdot 1.14095\rceil - 2) = (1, 12)$. We want to find the solutions (m, k, ℓ) such that

- $m < 12$
- $\begin{pmatrix} \ell \\ k \end{pmatrix} = 2\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix}$, where
$$\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}\begin{pmatrix} \lambda \\ \varkappa \end{pmatrix} = \begin{pmatrix} 0 & \bmod 1 \\ 0 & \bmod 6 \end{pmatrix}$$
- and
  - if $k = \ell$, then $\frac{1}{k} > 0.14095$, i.e., $k \leq 7$;
  - if $k > \ell$, then $\frac{1+m}{k+m\ell} \geq 0.14095$;
  - if $k < \ell$, then $\frac{2+m}{k+(m+1)\ell} \geq 0.14095$.

There are several solutions, we will list only the ones that impact the resulting $\mathcal{D}(\beta, A)$.

(m,k,ℓ)	Adding to $\mathcal{F}(\beta, A)$		Modified $\mathcal{D}(\beta, A)$
$(0,4,4)$	$\left(0+\frac{4-1}{4+1}, 0+\frac{4+1}{4-1}\right) \cap (1+0, +\infty)$	$= \left(1, \frac{5}{3}\right)$	$\left(\frac{5}{3}, 12\right)$
$(1,6,2)$	$\left(1+\frac{6-1}{2+1}, 1+\frac{6+1}{2-1}\right) \cap (1+1, +\infty)$	$= \left(\frac{8}{3}, 8\right)$	$\left(\frac{5}{3}, \frac{8}{3}\right) \cup (8, 12)$
$(1,6,6)$	$\left(1+\frac{6-1}{6+1}, 1+\frac{6+1}{6-1}\right) \cap (1+1, +\infty)$	$= \left(2, \frac{12}{5}\right)$	$\left(\frac{5}{3}, 2\right) \cup \left(\frac{12}{5}, \frac{8}{3}\right) \cup (8, 12)$
$(2,16,2)$	$\left(2+\frac{16-1}{2+1}, 2+\frac{16+1}{2-1}\right) \cap (2+1, +\infty)$	$= (7, 19)$	$\left(\frac{5}{3}, 2\right) \cup \left(\frac{12}{5}, \frac{8}{3}\right)$

Table 5.1: The triplets (m, k, ℓ) that determine the set $\mathcal{D}(\beta, A)$.

The final set $\mathcal{D}(\beta, A) = \left(\frac{5}{3}, 2\right) \cup \left(\frac{12}{5}, \frac{8}{3}\right)$.

5.4 Graph of admissible tails

Now, we have all we need to construct the graph of admissible tails.

We can see that the number of classes of equivalence $\equiv$ for unimodular matrices in $\mathbb{Z}^{2\times 2}$ is finite. Also, we are able to construct these classes, the exact algorithm can be found in Appendix A.

Now we will use the relation between $[A_N]_\equiv$ and $[A_{N+1}]_\equiv$ to construct an oriented graph that will help us determine whether there exists a recurrent aperiodic balanced sequence $\mathbf{v}$ such that $\mathrm{E}^*(\mathbf{v}) \leq 1+\beta$ and if yes, find at least one such.

Definition 5.13. *Let $\beta > 0$ and P, $P' \in \mathbb{N}$. An oriented graph (V, E) is called the* graph of $(1+\beta)$-admissible tails*, and denoted Γ_β, if*

- *the set of vertices V consists of classes of equivalence $\equiv$;*

- *a pair $([A]_\equiv, [B]_\equiv)$ labeled by interval $(a,b) \neq \emptyset$ belongs to the set E of oriented edges if $(a,b) = \mathcal{D}(\beta, A) \cap (k, k+1)$ for some $k \in \mathbb{N}$ and $B \in [A \left(\begin{smallmatrix} 0 & 1 \\ 1 & \lfloor a \rfloor \end{smallmatrix}\right)]_\equiv$.*

Example 5.14. Let us generate the graph of admissible tails for $P = 2$, $P' = 4$ and $\beta = \frac{1}{2}$. For this, we have three classes of equivalence for matrices. The classes are represented by matrices M_A, M_B and M_C as discussed in the remark after Lemma 5.4.

$$M_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with} \quad \mathcal{D}(\beta, A) = (1,4)\,;$$

$$M_B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{with} \quad \mathcal{D}(\beta, B) = \emptyset\,;$$

$$M_C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{with} \quad \mathcal{D}(\beta, C) = (1,4)\,.$$

So, the graph of admissible tails will have 3 vertices and 6 edges.

The first edge going from M_A will have the label $(1,4) \cap (1,2) = (1,2)$. Since

$$\left[M_A \begin{pmatrix} 0 & 1 \\ 1 & \lfloor 1 \rfloor \end{pmatrix}\right]_\equiv = \left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right]_\equiv = [M_C]_\equiv,$$

this edge will go from M_A to M_C. We can find the other edges similarly, and, in the end, we will obtain the graph depicted in Figure 5.1.

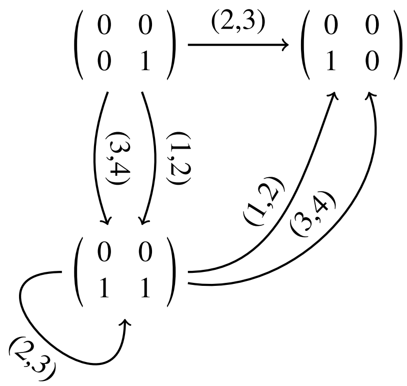

Figure 5.1: The graph of $(1+\beta)$-admissible tails before reductions for $P = 2$, $P' = 4$ and $\beta = \frac{1}{2}$.

The graph Γ_β is finite. We have already discussed why the number of vertices is finite. Moreover, it follows from Lemma 5.11 that for each vertex, the number of outgoing edges is finite.

We are able to rephrase Theorem 5.9 using the graph Γ_β.

Theorem 5.15 ([16]). *Let $\beta > 0$ and $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$, where $\mathbf{u}$ is a standard Sturmian sequence with slope $\theta = [0, a_1, a_2, a_3, \ldots]$. Assume that $E^*(\mathbf{v}) \leq 1 + \beta$. Then there exists an infinite oriented path $v_0, e_0, v_1, e_1, v_2, e_2, \ldots$ in Γ_β and $N_0 \in \mathbb{N}$ such that for every $N \in \mathbb{N}$*

1. *$[A_{N+N_0}]_\equiv = v_N$;*

2. *$a_{N+N_0+1} = \lfloor c_N \rfloor$, where c_N is the left point of the edge label $e_N = (c_N, d_N)$.*

We are interested only in paths on which the vertices occur infinitely many times (we have finitely many vertices and want to find an infinite path and any finite prefix can be included in the preperiod of θ). Therefore, we will consider only the strongly connected components of graph Γ_β, i.e., subgraphs with an oriented path from each vertex to each vertex. In particular, if a component consists of exactly one vertex, it has to contain a loop. The components not having an oriented cycle will be deleted from Γ_β.

Remark. In particular, if the graph Γ_β contains no oriented cycle for given $\beta > 0$ and $P, P' \in \mathbb{N}$, then $E^*(\mathbf{v}) > 1 + \beta$ for every colouring $\mathbf{v}$ of a Sturmian sequence by constant gap sequences of periods P and P'.

Example 5.16. If we continue with the graph in Example 5.14, Figure 5.1, we can see that no edge enters the matrix $M_A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and that no edge leaves the matrix $M_B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. We can therefore delete all edges except the loop and obtain the graph depicted in Figure 5.2.

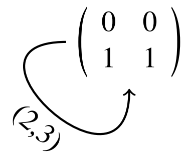

Figure 5.2: The graph of $(1 + \beta)$-admissible tails for $P = 2$, $P' = 4$ and $\beta = \frac{1}{2}$ after reduction to strongly connected component.

5.5 Graph reductions

We have already talked about the reduction to strongly connected components. Now, we will show two more simplifications – forward and backward reductions.

5.5.1 Forward reduction

The forward reduction works with the relation between δ_N and δ_{N+1}, that is,

$$\delta_N = a_{N+1} + \frac{1}{\delta_{N+1}}.$$

Therefore, if a class $[A_N]_\equiv$ has an edge to $[A_{N+1}]_\equiv$ determined by the interval (a, b), where $a \geq 1$ and $b \leq \lfloor a \rfloor + 1$, then

$$\begin{aligned} & \delta_N = \lfloor a \rfloor + \frac{1}{\delta_{N+1}} \\ \delta_N \in (a, b) \quad \Rightarrow \quad & \frac{1}{\delta_{N+1}} \in (a - \lfloor a \rfloor, b - \lfloor a \rfloor)\,. \end{aligned}$$

$$\text{If } a \notin \mathbb{N}, \quad \text{then} \quad \delta_{N+1} \in \left(\frac{1}{b - \lfloor a \rfloor}, \frac{1}{a - \lfloor a \rfloor}\right), \tag{5.6}$$

$$\text{if } a \in \mathbb{N}, \quad \text{then} \quad \delta_{N+1} \in \left(\frac{1}{b - \lfloor a \rfloor}, +\infty\right). \tag{5.7}$$

Remark. If both a and b are integers, we will obtain $\delta_{N+1} \in (1, +\infty)$, which does not bring any new information nor any constraints.

The forward reduction has two parts. In the first part, we can delete edges that cannot be extended into a suitable path in the graph; in the second, we change the labels of the edges.

5.5.1.1 Edge deletion

Finding a suitable path in the graph means going recursively through the graph. In each step, we will choose an edge and generate a new interval from it. Then we move along this edge to the next vertex, intersect the interval with edge labels (but do not modify the edge labels in the graph) and check whether we can get at least one non-empty intersection.

- If yes, we repeat the process recursively with this edge and the new interval as starting point up to the specified recursion depth. If we reach the specified recursion depth, then we stop the process and choose another edge as a new starting point.
- If not, we will return to the previous vertex and check another edge.

If we cannot reach the specified recursion depth because there does not exist a path, then we can delete the edge that started this process. The algorithm can be found in Subsection A.6.1. After edge deletion, we can again reduce the graph to strongly connected components.

5.5.1.2 Edge modification

The other part of the forward reduction allows us to modify the edges. As we have shown, every edge going into the class $[A_N]_\equiv$ sets the boundaries on δ_N. Let us consider i-th edge going into the class $[A_N]_\equiv$ labeled by (a_i, b_i). This interval generates the interval (c_i, d_i) using equation (5.6). Then δ_N must lie in the union $\bigcup_{i=1}^n (c_i, d_i)$, where n is the number of edges going into $[A_N]_\equiv$. Therefore, any label of an edge going from $[A_N]_\equiv$ must be a subset of $\bigcup_{i=1}^n (c_i, d_i)$.

For each vertex, we can find all edges going into $[A_N]_\equiv$, find $\bigcup_{i=1}^n (c_i, d_i)$ and modify all labels going from $[A_N]_\equiv$ accordingly. It is possible that this process will delete some edges, but even the modification can help reduce the graph significantly when combined with edge deletion described above.

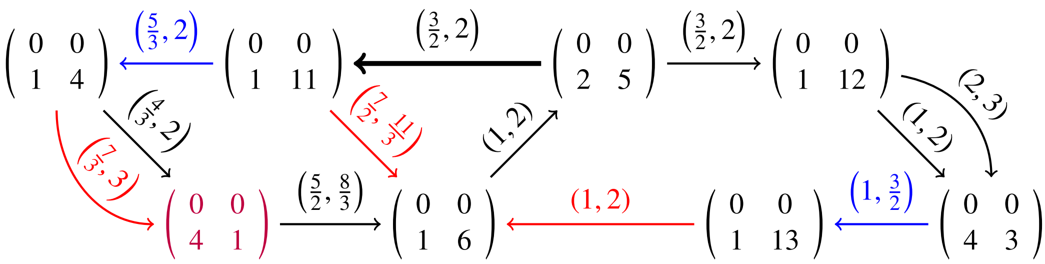

Figure 5.3: Strongly connected component of the graph of admissible tails for $P = 1, P' = 16$ and $\beta = \frac{25}{104}$.

Example 5.17. Consider the graph in Figure 5.3 and the bold edge. The bold edge is the only edge entering the vertex $\begin{pmatrix} 0 & 0 \\ 1 & 11 \end{pmatrix}$ and its label $\left(\frac{3}{2}, 2\right)$ forces any outgoing edge to be in the interval $(1, 2)$. Therefore, the red edge with the label $\left(\frac{7}{2}, \frac{11}{3}\right)$ can be completely removed.

If we consider the purple vertex $\begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix}$, it has two entering edges. Both edges, with labels $\left(\frac{7}{3}, 3\right)$ and $\left(\frac{4}{3}, 2\right)$, force the outgoing labels to be in the interval $(1, 3)$. They do not modify the edge $\left(\frac{5}{2}, \frac{8}{3}\right)$.

This process can be repeated for all edges. For example, the blue edge $\left(\frac{5}{3}, 2\right)$ forces the outgoing edges to be in $\left(1, \frac{3}{2}\right)$, therefore the edge $\left(\frac{7}{3}, 3\right)$ is removed and the edge $\left(\frac{4}{3}, 2\right)$ must be modified to $\left(\frac{4}{3}, \frac{3}{2}\right)$. And the blue edge $\left(1, \frac{3}{2}\right)$ forces the deletion of the next edge in the cycle.

Combining this, we obtain only one cycle depicted in Figure 5.4.

$$\begin{tikzcd}
\begin{pmatrix}0&0\\1&4\end{pmatrix} \arrow[dr, "\left(\frac{4}{3},\frac{3}{2}\right)"] & \begin{pmatrix}0&0\\1&11\end{pmatrix} \arrow[l, "\left(\frac{5}{3},2\right)"'] & & \begin{pmatrix}0&0\\2&5\end{pmatrix} \arrow[ll, "\left(\frac{3}{2},2\right)"'] \\
& \begin{pmatrix}0&0\\4&1\end{pmatrix} \arrow[r, "\left(\frac{5}{2},\frac{8}{3}\right)"] & \begin{pmatrix}0&0\\1&6\end{pmatrix} \arrow[ur, "(1,2)"] &
\end{tikzcd}$$

Figure 5.4: Strongly connected component of the graph of admissible tails for $P = 1, P' = 16$ and $\beta = \frac{25}{104}$ after forward reduction.

5.5.2 Backward reduction

The forward reduction worked with how δ_N affects δ_{N+1}. On the other hand, the backward reduction works with knowledge of the first p coefficients of δ_{N-p} and modifies the process used in the proof of Theorem 5.9 to obtain more accurate boundaries.

Consider a path in the graph Γ_β corresponding to $\theta = [0, a_1, a_2, \dots]$. Let us focus on a vertex $[A_N]_\equiv$ and denote $x_N = \frac{Q_{N-1}}{Q_N}$. It follows from the recurrence relations that $\frac{Q_{N-1}}{Q_N} = [0, a_N, a_{N-1}, \dots, a_1]$.

In Theorem 5.9, we only used that $x_N \in [0,1]$ to prove that if (5.8) is satisfied and m, k, ℓ satisfy $\mathfrak{P}1$ from Definition 5.8, then $\Phi_N > 1 + \beta$.

$$\left.\begin{array}{ll} \frac{1}{k} > \beta & \text{if } k = \ell\,; \\[2ex] \frac{1+m}{k+\ell m} \geq \beta & \text{if } k > \ell\,; \\[2ex] \frac{2+m}{k+\ell(m+1)} \geq \beta & \text{if } k < \ell\,; \end{array}\right\} \Rightarrow \frac{1+m+x_N}{k+\ell m+\ell x_N} \geq \min\left\{\frac{1+m+x}{k+\ell m+\ell x} : x \in [0,1]\right\} \geq \beta. \tag{5.8}$$

Let us refine the lower bound from (5.8). Using the recurrence relation (3.1) and choosing $0 \leq p < N$, we can rewrite x_N as

$$x_N = \frac{Q_{N-1}}{Q_N} = \frac{Q_{N-1}}{a_N Q_{N-1} + Q_{N-2}} = \frac{1}{a_N + x_{N-1}},$$

$$x_N = \cfrac{1}{a_N + \cfrac{1}{a_{N-1} + \cfrac{1}{\ddots + \cfrac{1}{a_{N-p} + x_{N-p-1}}}}}, \qquad \text{where } x_{N-p-1} \in [0,1].$$

Therefore, with the knowledge of $a_N, \dots, a_{N-p}$, we can obtain better bounds on x_N. Namely

$$x_N \in \left\{\frac{q'_{N-1} + p'_{N-1}\cdot x}{q'_N + p'_N \cdot x} : x \in [0,1]\right\} =: [X, Y], \quad \text{where} \quad \frac{p'_N}{q'_N} = [0, a_{N-p}, \dots, a_N].$$

Then we can claim that if (5.9) is satisfied and m, k, ℓ satisfy $\mathfrak{P}1$, then $\Phi_N > 1 + \beta$.

$$\left.\begin{array}{ll} \frac{1}{k} > \beta & \text{if } k = \ell\,; \\[2ex] \frac{1+m+X}{k+\ell(m+X)} \geq \beta & \text{if } k > \ell\,; \\[2ex] \frac{1+m+Y}{k+\ell(m+Y)} \geq \beta & \text{if } k < \ell\,. \end{array}\right\} \Rightarrow \frac{1+m+x_N}{k+\ell m+\ell x_N} \geq \min\left\{\frac{1+m+x}{k+\ell m+\ell x} : x \in [X,Y]\right\} \geq \beta. \tag{5.9}$$

We cannot use it directly for graph generation because we do not know the coefficients $a_{N-p}, \ldots, a_N$ at the beginning. Even with the graph in hand, there might be more paths of length p to a vertex $[A_N]_\equiv$, and the coefficients $a_{N-p}, \ldots, a_N$ are therefore not uniquely determined.

We will now demonstrate how to determine the left edge of the interval $[X, Y]$ in such a way that the above inequality holds for any choice of the path ending in $[A_N]_\equiv$. Let us denote by a'_N the maximal value of $\lfloor a \rfloor$, where (a, b) is a label of an edge entering $[A_N]_\equiv$. Let us denote by $[B]_\equiv$ the vertex which is the starting point of the edge (a, b). Let a'_{N-1} be the *minimal* value of $\lfloor c \rfloor$, where (c, d) is an edge that enters $[B]_\equiv$. Denote $[C]_\equiv$ the vertex from which this edge leaves. We repeat the process from the beginning, starting from $[C]_\equiv$ and choosing *maximal* a'_{n-2} until we reach the desired number of coefficients. Then we have $x_N \geq \min_{x \in [0,1]}[0, a'_N, a'_{N-1}, \ldots, a'_{N-p} + x]$.

So to find a better lower bound X on x_N, we have to go backwards through the graph and find alternately the maximal or the minimal label of an edge going into a given vertex.

Similarly if we wanted to find a better upper bound Y on x_N, we would walk backwards through the graph and find alternately the minimal or the maximal label (starting with the minimal) of an edge going into a given vertex.

After we find the lower bound X and the upper bound Y and all m, k, ℓ satisfying the property $\mathfrak{P}1$ and the inequality from (5.9), these m, k, ℓ enlarge $\mathcal{F}(\beta, A_N)$ as in Definition 5.5, resp. restrict $\mathcal{D}(\beta, A_N)$ to $\mathcal{D}'(\beta, A_N)$ and change the labels of the edges that go from $[A_N]_\equiv$ in the graph. This process usually does not delete edges, but can change some labels. Therefore, after backward reduction, we will repeat forward reduction to get new labels.

Example 5.18. Let us consider the graph of $(1+\beta)$-admissible tails for $P = 2$, $P' = 12$ and $\beta = 0.14095$. The graph after the reduction to strongly connected components is depicted in Figure 5.5. The forward reduction was then preformed and the resulting graph can be found in Figure 5.6. We can see that the forward reduction simplified the graph to only two connected cycles.

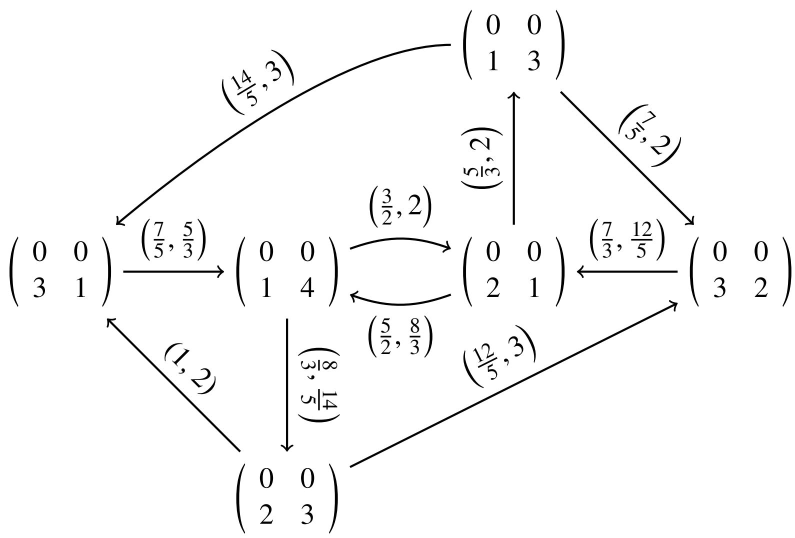

Figure 5.5: The graph of admissible tails after reduction to strongly connected components for $P = 2$, $P' = 12$ and $\beta = 0.14095$.

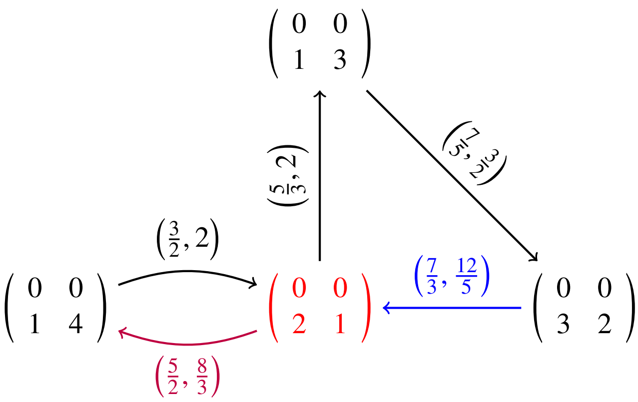

Figure 5.6: The graph of admissible tails after forward reductions for $P = 2$, $P' = 12$ and $\beta = 0.14095$.

Let us show, how to determine the left side of the interval (X, Y) for $p = 3$.

If we consider the red vertex $\left(\begin{smallmatrix} 0 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, we can see that it has two entering edges, one with the label $\left(\frac{7}{3}, \frac{12}{5}\right)$ and one with the label $\left(\frac{3}{2}, 2\right)$. Our first (resp. last) coefficient will therefore be $a'_N = \max\left\{\lfloor\frac{7}{3}\rfloor, \lfloor\frac{3}{2}\rfloor\right\} = 2$.

Now we move in reverse through the blue edge to the vertex $\left(\begin{smallmatrix} 0 & 0 \\ 3 & 2 \end{smallmatrix}\right)$. We can see that there is only one edge entering this vertex and so the next coefficient is $a'_{N-1} = \lfloor\frac{7}{5}\rfloor = 1$. Similarly, the last (resp. first) coefficient is $a'_{N-2} = \lfloor\frac{5}{3}\rfloor = 1$.

So, we set $\frac{p'_N}{q'_N} = [0, 1, 1, 2] = \frac{3}{5}$ and $\frac{p'_{N-1}}{q'_{N-1}} = [0, 1, 1] = \frac{1}{2}$. Therefore, $x_N \geq \min\left\{\frac{2+x}{5+3x} : x \in [0, 1]\right\} = \frac{3}{8}$.

As we have seen in Example 5.12, the interval for this matrix is $\mathcal{D}(\beta, A) = \left(\frac{5}{3}, 2\right) \cup \left(\frac{12}{5}, \frac{8}{3}\right)$. It is different from $\left(\frac{5}{3}, 2\right) \cup \left(\frac{5}{2}, \frac{8}{3}\right)$ in Figure 5.6 because the forward reduction changed the second interval.

If we now try to compute the interval $\mathcal{D}'(\beta, A_N)$, we can see that the triplet $(m, k, \ell) = (0, 8, 2)$ satisfies the property $\mathfrak{P}1$ and the second inequality from (5.9), but not the second inequality from (5.8). This triplet adds a new interval $\left(\frac{7}{3}, 9\right)$ to $\mathcal{F}(\beta, A)$ and therefore

$$\mathcal{D}'(\beta, A_N) \subset \left(\left(\frac{5}{3}, 2\right) \cup \left(\frac{12}{5}, \frac{8}{3}\right)\right) \cap \left(\left(1, \frac{7}{3}\right) \cup (9, 12)\right) = \left(\frac{5}{3}, 2\right).$$

The purple edge is therefore completely removed by backward reduction.

6 Results

In this chapter, we will consider $\mathbf{y}$ a constant gap sequence over an alphabet $\mathcal{A}$ with period P and $\mathbf{y}'$ a constant gap sequence over $\mathcal{B}$ with period P', where $\mathcal{A} \cap \mathcal{B} = \emptyset$. Also, $\mathbf{u}$ will always be a standard Sturmian sequence.

The program, with pseudocode described in Appendix A, works with the parameters P, P' and $\alpha = \frac{1}{\beta}$.

Our goal is to find a standard Sturmian sequence $\mathbf{u}$ and periods of $\mathbf{y}$ and $\mathbf{y}'$ such that the balanced sequence $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ has asymptotic critical exponent lower than or equal to $1 + \frac{1}{\alpha}$.

The following theorem will help us reduce the number of combinations of P and P' that we need to consider.

Theorem 6.1 ([16])**.** *Let* $\mathbf{y}$ *and* $\mathbf{y}'$ *be two constant gap sequences over disjoint alphabets and* $\mathbf{u} \in \{\mathtt{a}, \mathtt{b}\}^{\mathbb{N}}$ *be a Sturmian sequence with* $\mathtt{b}$ *being the most frequent letter in* $\mathbf{u}$*. Then there exists a Sturmian sequence* $\widetilde{\mathbf{u}} \in \{\mathtt{a}, \mathtt{b}\}^{\mathbb{N}}$ *with* $\mathtt{b}$ *being the most frequent letter in* $\widetilde{\mathbf{u}}$ *such that*

$$\mathrm{E}^*(\mathrm{colour}(\widetilde{\mathbf{u}}, \mathbf{y}', \mathbf{y})) = \mathrm{E}^*(\mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}'))\,.$$

It means that, without loss of generality, we can consider $|\mathcal{A}| \leq |\mathcal{B}|$. We can further reduce the number of combinations using the following two lemmas.

Lemma 6.2 ([16])**.** *Let* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ *and* $\hat{\mathbf{v}} = \mathrm{colour}(\mathbf{u}, \hat{\mathbf{y}}, \hat{\mathbf{y}}')$*. If* $\mathrm{Per}(\hat{\mathbf{y}})$ *is divisible by* $\mathrm{Per}(\mathbf{y})$ *and* $\mathrm{Per}(\hat{\mathbf{y}}')$ *is divisible by* $\mathrm{Per}(\mathbf{y}')$*, then* $\mathrm{E}^*(\mathbf{v}) \geq E^*(\hat{\mathbf{v}})$.

Lemma 6.3 ([12])**.** *Let* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$*. Then*

$$\mathrm{E}^*(\mathbf{v}) \geq 1 + \frac{1}{\mathrm{Per}(\mathbf{y}) \cdot \mathrm{Per}(\mathbf{y}')}.$$

Therefore we need to consider only P, P' such that $P \cdot P' \geq \alpha$.

We will now go step by step through balanced words over the alphabet with 3 to 12 letters. We will list the considered combinations of P and P' after using Theorem 6.1 and Lemma 6.2 and if the graph is not empty after reductions, we will show its structure. From the cycle, we can deduce the form of θ which might have a low asymptotic critical exponent as follows: If A_N is a vertex in a cycle, then we can find a preperiod of θ of length h such that $\begin{pmatrix} p_{h-1} & p_h \\ q_{h-1} & q_h \end{pmatrix} \in [A_N]_{\equiv}$. Then we add infinite repetition of coefficients obtained from going through the cycle. The asymptotic critical exponent depends only on the structure of the cycle, not on the first matrix we step into the cycle. For any matrix, there are infinitely many preperiods that will generate the same matrix, and we can choose any matrix from the cycle, so we try to select the simplest form of θ. This also means that there are infinitely many balanced sequences that have the same value of the asymptotic critical exponent, although they often differ only in the preperiod and not period of θ.

Example 6.4. For example, if we consider the cycle in Figure 5.4, the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 4 \end{pmatrix}$ can be generated by the preperiod of length 1 equal to 4. Then following the cycle starting from this matrix, we obtain a period in the form $\left(\lfloor\frac{4}{3}\rfloor, \lfloor\frac{5}{2}\rfloor, 1, \lfloor\frac{3}{2}\rfloor, \lfloor\frac{5}{3}\rfloor\right)^{\omega} = (1, 2, 1, 1, 1)^{\omega}$. Combining this, we obtain $\theta = [0, 4, (1, 2, 1, 1, 1)^{\omega}]$, which in fact has the lowest asymptotic critical exponent with $P = 1$ and $P' = 16$.

We then use the program I made for my bachelor thesis [31] to assess the exact value of the asymptotic critical exponent for the given P, P' and θ.

The parameter α is chosen mostly by trial and error. We start with theoretical estimation such as the minimal value of the critical exponent of balanced sequences for a given alphabet (which is greater than the minimal asymptotic critical exponent) found in [31] and if we obtain a nonempty graph for given α and a possible combination of P and P', we use my program to calculate the asymptotic critical exponent of the resulting cycles. If at least one of the cycles has an asymptotic critical exponent lower than $1 + \frac{1}{\alpha}$, we modify α to reflect it, i.e., we set new α as approximately $\frac{1}{\mathrm{E}^*(\mathbf{v})-1}$ for this $\mathbf{v}$ and let the program generate the graph for this new α. In the following text, we will show only the optimal α and the results for it, not the whole testing process.

To be able to test all possible combinations of P and P', we need to know what periods of constant gap sequences are possible for a given size of an alphabet. The periods of constant gap sequences over alphabet of size d are as follows [16]:

d	possible period of a constant gap sequence
1	1
2	2
3	3, 4
4	4, 6, 8
5	5, 6, 8, 9, 12, 16
6	6, 8, 10, 12, 16, 18, 24, 32
7	7, 8, 9, 10, 12, 15, 16, 18, 20, 24, 27, 32, 36, 48, 64
8	8, 10, 12, 14, 16, 18, 20, 24, 30, 32, 36, 40, 48, 54, 64, 72, 96, 128
9	9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32, 36, 40, 45, 48, 54, 60, 64, 72, 80, 81, 96, 108, 128, 144, 192, 256

6.1 3-letter alphabet

The only possible periods to consider are $P = 1$ and $P' = 2$. If we choose $\alpha = 0.5857$, i.e., $\beta \doteq 1.7074$, we obtain a single cycle with $\theta = [0, 1, (2)^{\omega}]$ depicted in Figure 6.1. This sequence has the asymptotic critical exponent equal to $\frac{4+\sqrt{2}}{2} \doteq 2.7071$.

$$\begin{tikzcd} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \arrow[loop right, "{(2,3)}"] \end{tikzcd}$$

Figure 6.1: The strongly connected component of the graph of admissible tails for $P = 1, P' = 2$ and $\alpha = 0.5857$.

6.2 4-letter alphabet

We have 3 possible combinations of P and P'. We set $\alpha = 1.236$, i.e., $\beta \doteq 0.8091$. If we consider $P = 1$ and $P' \in \{3, 4\}$, the graph of admissible tails does not have a strongly connected component. For $P = 2 = P'$, we obtain one cycle with $\theta = [0, (1)^{\omega}]$, as shown in Figure 6.2. This sequence has the asymptotic critical exponent equal to $\frac{5+\sqrt{5}}{4} \doteq 1.8090$.

$$\begin{tikzcd} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \arrow[loop right, "{(1,2)}"] \end{tikzcd}$$

Figure 6.2: The strongly connected component of the graph of admissible tails for $P = 2, P' = 2$ and $\alpha = 1.236$.

6.3 5-letter alphabet

Now we set $\alpha = 2$, i.e., $\beta = 0.5$. The combinations $P = 1$ and $P' \in \{4, 6, 8\}$ and $P = 2, P' = 3$ do not create a strongly connected component. The combination $P = 2$ and $P' = 4$ generates one cycle depicted in Figure 5.2 with $\theta = [0, 1, (2)^\omega]$, and the asymptotic critical exponent equal to $\frac{3}{2}$.

6.4 6-letter alphabet

We set $\alpha = 4.16953$, i.e., $\beta \doteq 0.239835$. The results for given P, P' are given in Table 6.1. We can see that for $P = 4 = P'$, the forward reduction was necessary. The only cycle exists for $P = 1$ and $P' = 16$; it is shown in Figure 5.4. The sequence has $\theta = [0, 4, (1, 2, 1, 1, 1)^\omega]$ and its asymptotic critical exponent is equal to $\frac{75+3\sqrt{65}}{80} \doteq 1.2398$.

Table 6.1: Tested combinations of P and P' with $\alpha = 4.16953$ for 6-letter alphabet and their graphs.

P	P'	is empty?	why?
1	5	Yes	no component
1	6	Yes	no component
1	8	Yes	no component
1	9	Yes	no component
1	12	Yes	no component
1	**16**	**No**	
2	4	Yes	no component
2	6	Yes	no component
2	8	Yes	no component
3	3	Yes	no component
3	4	Yes	no component
4	3	Yes	no component
4	4	Yes	forward reduction

6.5 7-letter alphabet

We set $\alpha = 7.0946$, i.e., $\beta \doteq 0.140952$. The results for given P, P' are in Table 6.2. We can see that for some combinations of P, P', both forward and backward reductions were necessary.

For $P = 1$ and $P' = 32$, the strongly connected component of the graph contains 2 cycles, where one of them can be deleted using forward reduction.

The resulting graph can be seen in Figure 6.3. Then the sequence with $\theta = [0, 5, 1, (1, 1, 1, 5, 2)^\omega]$ has the asymptotic critical exponent equal to $\frac{49+\sqrt{577}}{64} \doteq 1.140950$.

Table 6.2: Tested combinations of P and P' with $\alpha = 7.0946$ for 7-letter alphabet and their graphs.

P	P'	is empty?	why?
1	10	Yes	no component
1	18	Yes	no component
1	24	Yes	no component
1	**32**	**No**	
2	5	Yes	no component
2	9	Yes	no component
2	12	Yes	cycle of reductions
2	16	Yes	cycle of reductions
3	6	Yes	no component
3	8	Yes	no component
4	6	Yes	cycle of reductions
4	8	Yes	cycle of reductions

$$\begin{tikzcd}
\begin{pmatrix} 0 & 0 \\ 4 & 3 \end{pmatrix} \arrow[r, "\left(\frac{5}{2}, \frac{8}{3}\right)"] & \begin{pmatrix} 0 & 0 \\ 1 & 14 \end{pmatrix} \arrow[r, "\left(\frac{3}{2}, 2\right)"] & \begin{pmatrix} 0 & 0 \\ 2 & 9 \end{pmatrix} \arrow[dl, "\left(\frac{7}{4}, 2\right)"'] \\
\begin{pmatrix} 0 & 0 \\ 1 & 28 \end{pmatrix} \arrow[u, "\left(\frac{16}{3}, \frac{27}{5}\right)"] & \begin{pmatrix} 0 & 0 \\ 1 & 19 \end{pmatrix} \arrow[l, "\left(1, \frac{4}{3}\right)"'] &
\end{tikzcd}$$

Figure 6.3: The strongly connected component of the graph of admissible tails after all reductions for $P = 1, P' = 32$ and $\alpha = 7.0946$.

6.6 8-letter alphabet

We set $\alpha = 20.94$, i.e., $\beta \doteq 0.047755$. The results for given P, P' are given in Table 6.3. We can see that for some combinations of P, P', forward reduction was necessary.

For $P = 8$ and $P' = 8$, the graph of admissible tails contains 1 vertex and 1 edge. The edge is modified during forward reduction, but not deleted. The resulting graph is shown in Figure 6.4. The sequence with $\theta = [0, (1)^\omega]$ has the asymptotic critical exponent equal to $\frac{19-\sqrt{5}}{16} \doteq 1.047746$.

$$\begin{tikzcd}
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \arrow[loop right, "\left(\frac{3}{2}, \frac{5}{3}\right)"]
\end{tikzcd}$$

Figure 6.4: The strongly connected component of the graph of admissible tails after reductions for $P = 8$, $P' = 8$ and $\alpha = 20.94$.

Table 6.3: Tested combinations of P and P' with $\alpha = 20.94$ for 8-letter alphabet and their graphs.

P	P'	is empty?	why?
1	20	Yes	no component
1	27	Yes	no component
1	36	Yes	no component
1	48	Yes	no component
1	64	Yes	forward reduction
2	18	Yes	no component
2	24	Yes	no component
2	32	Yes	forward reduction
3	9	Yes	no component
3	12	Yes	no component
3	16	Yes	no component
4	9	Yes	no component
4	12	Yes	no component
4	16	Yes	forward reduction
6	4	Yes	no component
6	6	Yes	no component
6	8	Yes	no component
8	4	Yes	no component
8	6	Yes	no component
8	**8**	**No**	

6.7 9-letter alphabet

We set $\alpha = 30.31$, i.e., $\beta \doteq 0.0329924$. The results for given P, P' are given in Table 6.4. We can see that for some combinations of P, P', both reductions were necessary.

The only nonempty graph is for $P = 8$ and $P' = 16$. After forward reduction, only one vertex and one edge remain, as shown in Figure 6.5. The sequence with $\theta = [0, 1, (4)^{\omega}]$ has the asymptotic critical exponent equal to $\frac{21-\sqrt{20}}{16} \doteq 1.0329915$.

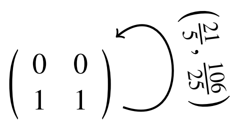

Figure 6.5: The strongly connected component of the graph of admissible tails after all reductions for $P = 8, P' = 16$ and $\alpha = 30.31$.

6.8 10-letter alphabet

We set $\alpha = 68.48$, i.e., $\beta \doteq 0.0146028$. The results for given P, P' are in Table 6.5. We can see that for some combinations of P, P', both reductions were necessary.

The only non-empty graph is for $P = 4$ and $P' = 64$. After finding the correct asymptotic exponent, we modified α to 64.485 to obtain less edges in the final graph. The final graph contains two connected cycles as shown in Figure 6.6.

Table 6.4: Tested combinations of P and P' with $\alpha = 30.31$ for 9-letter alphabet and their graphs.

P	P'	is empty?	why?
1	40	Yes	no component
1	54	Yes	no component
1	72	Yes	forward reduction
1	96	Yes	forward reduction
1	128	Yes	forward reduction
2	20	Yes	no component
2	27	Yes	no component
2	32	Yes	no component
2	36	Yes	forward reduction
2	48	Yes	forward reduction
2	64	Yes	cycle of reductions
3	18	Yes	no component
3	24	Yes	forward reduction
3	32	Yes	forward reduction
4	10	Yes	no component
4	18	Yes	forward reduction
4	24	Yes	forward reduction
4	32	Yes	cycle of reductions
6	9	Yes	no component
6	12	Yes	forward reduction
6	16	Yes	forward reduction
8	5	Yes	no component
8	9	Yes	forward reduction
8	12	Yes	forward reduction
8	**16**	**No**	

Table 6.5: Tested combinations of P and P' with $\alpha = 68.48$ (resp. $\alpha = 68$ for some graphs that ended up empty before changing α) for the 10-letter alphabet and their graphs.

P	P'	is empty?	why?
1	80	Yes	no component
1	81	Yes	no component
1	108	Yes	no component
1	144	Yes	forward reduction
1	192	Yes	forward reduction
1	256	Yes	forward reduction
2	40	Yes	no component
2	54	Yes	no component
2	72	Yes	forward reduction
2	96	Yes	forward reduction
2	128	Yes	both reductions
3	27	Yes	no component
3	36	Yes	no component
3	48	Yes	forward reduction
3	64	Yes	forward reduction
4	20	Yes	no component
4	27	Yes	no component
4	36	Yes	no component
4	48	Yes	forward reduction
4	**64**	**No**	
6	18	Yes	no component
6	24	Yes	forward reduction
6	32	Yes	forward reduction
8	10	Yes	no component
8	18	Yes	forward reduction
8	24	Yes	forward reduction
8	32	Yes	forward reduction
5	16	Yes	no component
9	9	Yes	no component
9	12	Yes	no component
9	16	Yes	forward reduction

$$\begin{tikzcd}
\begin{pmatrix} 0 & 0 \\ 1 & 12 \end{pmatrix} \arrow[d, "\left(\frac{21}{13},\frac{55}{34}\right)"] & & \begin{pmatrix} 0 & 0 \\ 2 & 5 \end{pmatrix} \arrow[ll, "\left(\frac{144}{55},\frac{55}{21}\right)"'] \arrow[r, "\left(\frac{41}{26},\frac{71}{45}\right)"] & \begin{pmatrix} 0 & 0 \\ 1 & 11 \end{pmatrix} \arrow[r, "\left(\frac{45}{26},\frac{71}{41}\right)"] & \begin{pmatrix} 0 & 0 \\ 1 & 4 \end{pmatrix} \arrow[d, "\left(\frac{41}{30},\frac{67}{49}\right)"] \\
\begin{pmatrix} 0 & 0 \\ 4 & 3 \end{pmatrix} \arrow[r, "\left(\frac{34}{21},\frac{47}{29}\right)"] & \begin{pmatrix} 0 & 0 \\ 1 & 13 \end{pmatrix} \arrow[r, "\left(\frac{29}{18},\frac{79}{49}\right)"] & \begin{pmatrix} 0 & 0 \\ 1 & 6 \end{pmatrix} \arrow[u, "\left(\frac{29}{21},\frac{47}{34}\right)"] \arrow[u, bend right=60, "\left(\frac{49}{30},\frac{67}{41}\right)"'] & & \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix} \arrow[ll, "\left(\frac{128}{47},\frac{79}{29}\right)"']
\end{tikzcd}$$

Figure 6.6: The strongly connected component of the graph of admissible tails after reductions for $P = 4$, $P' = 64$ and $\alpha = 68.485$.

If we consider the left or right cycles separately, they both have too large asymptotic critical exponent. The infinite path must therefore alternate between the cycles.

If we do the forward reduction "by hand" starting from the red edge, the condition given by Equation (5.6) on the next edge is $\left(\frac{29}{21},\frac{47}{34}\right)$, which means that we must continue along the straight edge to $\left(\begin{smallmatrix} 0 & 0 \\ 2 & 5 \end{smallmatrix}\right)$. This edge gives us the condition on the next edge being $\left(\frac{34}{13},\frac{21}{8}\right)$. Therefore, the only possible way is to go through the blue edge.

On the other hand, if we start from the green edge, the condition given by Equation (5.6) on the next edge is $\left(\frac{49}{30},\frac{18}{11}\right)$ and we must continue along the curved edge to $\left(\begin{smallmatrix} 0 & 0 \\ 2 & 5 \end{smallmatrix}\right)$. The curved edge restricts the next edge to $\left(\frac{41}{26},\frac{30}{19}\right)$ and we can continue only through the violet edge.

Therefore, the only candidate for low asymptotic critical exponent corresponds to an infinite path going alternatively through the left and right cycle without going through one cycle twice consecutively.

Moreover, both edges going from $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 6 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 & 0 \\ 2 & 5 \end{smallmatrix}\right)$ generate the same coefficient for θ because $\lfloor\frac{49}{30}\rfloor = 1 = \lfloor\frac{29}{21}\rfloor$.

Therefore, we obtain $\theta = [0, 6, (1, 1, 1, 1, 2, 1, 2, 1, 1, 1)^{\omega}]$ and the asymptotic critical exponent of this sequence is equal to $\frac{364-21\sqrt{7}}{304} \doteq 1.0146027$.

6.9 Asymptotic behaviour

We were able to determine the minimal asymptotic critical exponent of balanced sequences only for alphabets up to ten letters. Increasing the alphabet size has many difficulties. First, the number of combinations of P and P' grows fast. Second, the time necessary to run the program increases significantly, since we have more matrices, larger α, which means checking more triplets m, k, ℓ, and therefore often many more vertices and edges.

To expand the results to bigger alphabets, we added some modifications to the program. Mainly, the first part, i.e. finding the set $\mathcal{D}(\frac{1}{\alpha}, A)$ for every matrix in the list, is independent and therefore can be done in parallel. The same modification was done for back reduction, where we calculate the modified set $\mathcal{D}'(\frac{1}{\alpha}, A_N)$. Another modification was done to speed up the calculation of $\mathcal{D}(\frac{1}{\alpha}, A)$ when we already have the set $\mathcal{D}(\frac{1}{\alpha_0}, A)$ for $\alpha_0 < \alpha$. In this case, we can take the set $\mathcal{D}(\frac{1}{\alpha_0}, A)$ as a starting set and only consider intersection for the triples (m, k, ℓ) which satisfy all the items in Definition 5.8 for $\frac{1}{\alpha}$, but not for $\frac{1}{\alpha_0}$.

Even using these modifications, the computation is time-consuming, and we were not able to expand the results in time to incorporate them into the thesis.

Another problem is to choose the correct α. If the initial α is chosen too high, we will not find any non-empty graph and we will need to run all of the combinations of P and P' again for smaller α. On the other hand, if α is chosen too low, then the resulting graphs will contain many interlinked cycles and it will be hard to find a cycle that maximalizes α for the next iterations.

The greatest difficulty can be seen even for the alphabet of 10 letters. We have shown that, in this case, the graph contains two interlocked cycles, where each of them has an asymptotic critical exponent too high. Only manual intervention showed that we obtain an unambiguous optimal solution when we go alternatively through the first and the second cycle. The situation is similar for the case in which we choose α too small which happens when we do not have any theoretical approximation of the minimal value.

The same problem also occurs for some combinations of P and P', even though α is chosen maximal possible for this combination to give non-empty graph. One of the examples for this case could be the combination of periods $P = 6, P' = 108$. This combination with $\alpha = 135.186$ generates the graph shown in Figure 6.7, where we use only $\lfloor a \rfloor$ as the edge label and we omit the multiplicity of the edges. When α is modified to $\alpha' = 135.187$, the graph of admissible tails for the same combination of P and P' is empty. We are therefore quite lucky that the graphs for optimal combinations of P, P' and α are actually quite simple.

So, in the future, to find balanced sequences with minimal asymptotic critical exponent over larger alphabets, we will either have to enhance our graph reductions or find a completely different approach.

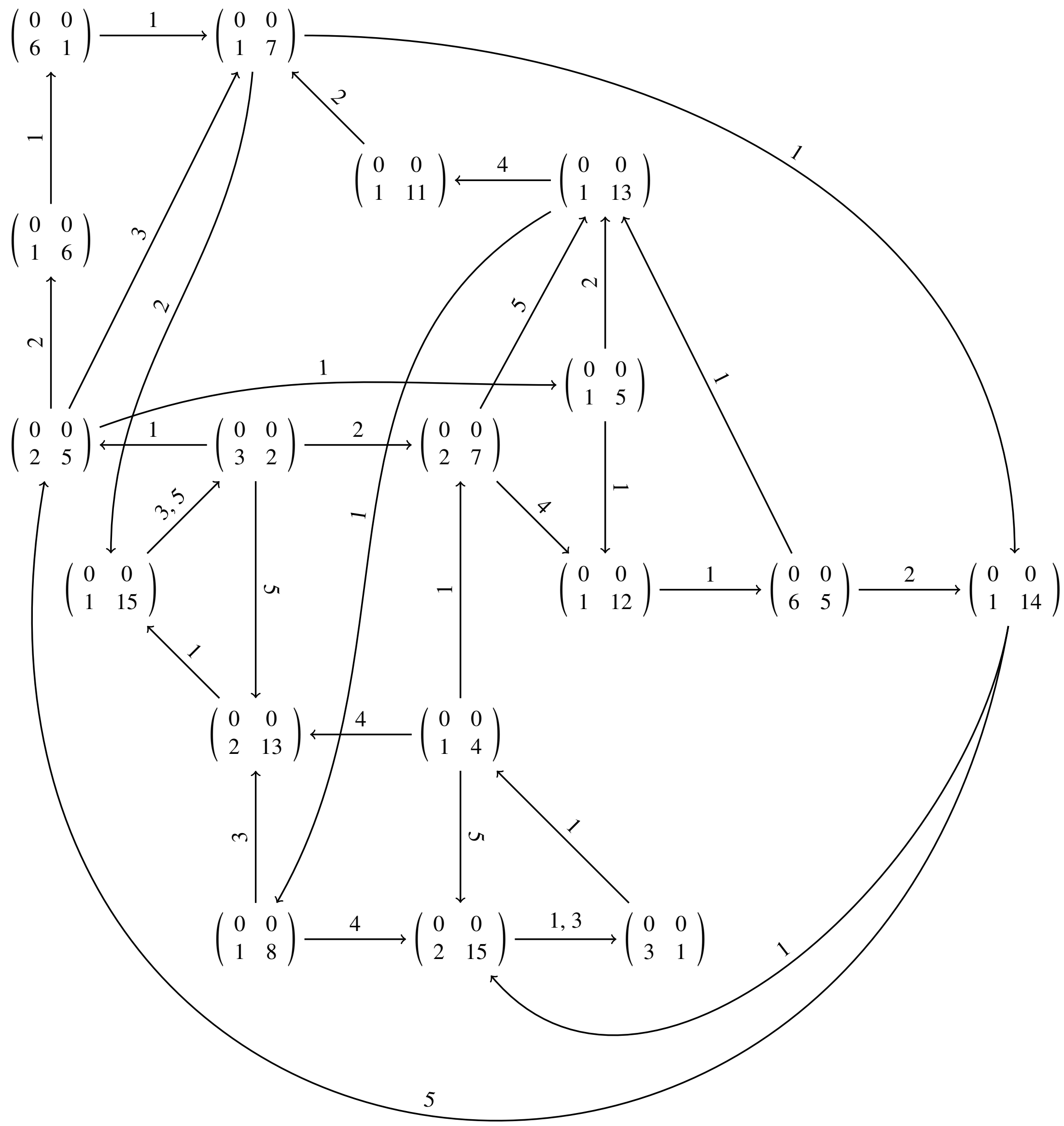

Figure 6.7: The graph of admissible tails with modified edge labels after all reductions for $P = 6$, $P' = 108$ and $\alpha = 135.186$.

Part II

Critical exponent of morphic sequences

7 Critical exponent of morphic images of a sequence

In our work, we often encountered the problem of computing the critical exponent of a morphic image of a sequence. This problem occurred, for example, in [8, 14]. In these papers, we used a morphism to construct a sequence with the least number of palindromes or complemented words, while having a small critical exponent. To complete the proofs, it was necessary to show what the critical exponent of the morphic images is.

In this part, we will first show the results of [8, 14] to motivate our choice of morphisms in Section 7.1. We will then present the main ideas for calculating the critical exponent of a morphic image of an arbitrary sequence $\mathbf{u}$ when we know the form of bispecial factors in the sequence $\mathbf{u}$ in Section 7.2. In Chapter 8, we will describe bispecial factors and their return words in the sequence $\mathbf{p}$ defined in the same section. Finally, we will demonstrate how this method works on four given morphic images of $\mathbf{p}$ in Chapters 9, 10, 11, and 12.

7.1 Motivation

Since the definition of repetitions in sequence, people have asked what the lower bound for repetitions in a sequence is. The question has been answered in general, and also for certain classes of sequences, such as sequences over an alphabet of given cardinality, Sturmian sequences, balanced sequences etc. Naturally, if we restrict our class of sequences, the repetition threshold (i.e. the infimum of critical exponents of sequences in the class) will be at least the bound for the larger class.

7.1.1 Complement avoidance

Let us recall that the complement to a word u over an alphabet $\{0, 1\}$ is a word $\bar{u}$ obtained by switching $0 \to 1$ and $1 \to 0$. We call u and $\bar{u}$ complemented factors. In [8], the authors studied the repetition threshold for the classes of sequences

$$\begin{aligned} \mathrm{CAL}_\ell &= \{\mathbf{u} \text{ binary sequence} : (\forall u \text{ factor of } \mathbf{u})(\bar{u} \text{ factor of } \mathbf{u} \Rightarrow |u| < \ell)\} \\ \mathrm{CAN}_n &= \{\mathbf{u} \text{ binary sequence} : \#\{u \text{ factor of } \mathbf{u} : \bar{u} \text{ factor of } \mathbf{u}\} \le n\}, \end{aligned}$$

i.e. the sequences that avoid complemented factors of length larger than ℓ or have only a limited number of complemented factors. It is clear that $\mathrm{CAL}_1 = \mathrm{CAN}_0 = \mathrm{CAN}_1 = \{(0)^\omega, (1)^\omega\}$ and the repetition threshold for these classes is infinite.

Theorem 7.1 ([8])**.** *The repetition threshold for class CAL_ℓ is β, for the following pairs (ℓ,β). Moreover, this list of pairs is optimal and the bound is attained.*

(*) $(1, +\infty)$,

(a) $(3, 2 + \alpha)$, *where* $\alpha = (1 + \sqrt{5})/2$,

(b) $(5, 3)$,

(c) $(7, \frac{8}{3})$,

(d) $(8, \frac{5}{2})$,

(e) $(11, \gamma')$, *where* $\gamma' \doteq 2.48086$ *is the critical exponent of* $\mathbf{p}$ *(see Chapter 8),*

(f) $(13, \frac{7}{3})$.

Optimality means that if ℓ is between any two values $\ell_1 < \ell_2$ in the list, then the critical exponent is the same as for ℓ_1. For example, the repetition threshold for $\ell = 6$ is 3. The proof in the article always gave us the sequence attaining the bound and showed that any binary word having the same critical exponent while avoiding shorter complemented factors must be finite. To finish the proof, it was necessary to calculate the critical exponent of the given sequences. We calculated the critical exponent for Item (d) and Item (e) since it used the sequences $\xi(\mathbf{p})$ and $\psi(\mathbf{p})$. The calculations can be found in Chapters 12 and 11.

The results can then be extended to the class CAN_n.

Theorem 7.2 ([8]). *The repetition threshold for class* CAN_n *is* β, *for the following pairs* (n, β). *Moreover, this list of pairs is optimal and the bound is attained.*

(*) $(0, +\infty)$,

(a) $(4, 2 + \alpha)$, *where* $\alpha = (1 + \sqrt{5})/2$,

(b) $(8, 3)$,

(c) $(24, \frac{8}{3})$,

(d) $(36, \frac{5}{2})$,

(e) $(64, \gamma')$, *where* $\gamma' \doteq 2.4808627161472369$ *is the critical exponent of* $\mathbf{p}$,

(f) $(90, \frac{7}{3})$.

7.1.2 Palindrome avoidance

In [14], the authors found the repetition threshold for the classes of binary sequences containing at most n distinct factors that are palindromes.

Theorem 7.3 ([14]). *The repetition threshold for binary sequences containing only* p *distinct palindromes is* β *for the following pairs* (p, β). *Moreover, this list of pairs is optimal, and the bound is attained.*

(a) $(11, \frac{10}{3})$,

(b) $(12, \frac{23}{7})$,

(c) $(13, 3)$,

(d) $(15, \frac{8}{3})$,

(e) $(18, \frac{28}{11})$,

(f) $(20, \frac{5}{2})$,

(g) $(25, \frac{7}{3})$.

The optimality again means that if $p_1 < p_2 < p_3$, where p_1 and p_3 correspond to an item in the theorem, then the repetition threshold for sequences containing at most p_2 palindromes is the same as the repetition threshold for the class of sequences containing at most p_1 palindromes (i.e. allowing more palindromes does not lower the repetition threshold). The proofs, analogously to complement avoidance, present sequences reaching the threshold and then show that any word having the same critical exponent and containing fewer palindromes must be finite. To complete the proofs, it was necessary to calculate the critical exponents of the given sequences. The sequences for Item (e) and (f) correspond with $\mu(\mathbf{p})$ and $\nu(\mathbf{p})$ and the calculation is part of this thesis, Chapters 10 and 9.

7.2 General approach to computation of critical exponent

Definition 7.4. *Let* $\mathbf{u}$ *be a sequence over an alphabet* $\mathcal{A}$ *and let* $\Psi : \mathcal{A}^* \to \mathcal{B}^*$ *be a morphism. Consider a factor* w *of* $\Psi(\mathbf{u})$*. We say that* (w_1, w_2) *is a* synchronization point *of* w *if* $w = w_1 w_2$ *and for all factors* $p, s \in \mathcal{L}(\Psi(\mathbf{u}))$ *and* $v \in \mathcal{L}(\mathbf{u})$ *such that* $\Psi(v) = pws$ *there exists a factorization* $v = v_1 v_2$ *of* v *with* $\Psi(v_1) = pw_1$ *and* $\Psi(v_2) = w_2 s$*. We denote the synchronization point by* $w_1 \bullet w_2$*.*

Example 7.5. Let us consider morphism ν, where

$$\nu(0) = 011, \qquad \nu(1) = 0, \qquad \nu(2) = 01.$$

First, we can see that the letter 0 appears only as a prefix of letter images. Therefore, any appearance of 0 symbolizes start of a letter image and there will be synchronization point in front of 0 regardless of the preceding letter.

The end of letter image might be harder to determine since $\nu(1)$ is a prefix of $\nu(2)$ which is a prefix of $\nu(0)$. But from the form of the morphism, we can see that the factor 11 can be obtained only from $\nu(0)$, therefore, after 11, there will be a synchronization point and it will be always followed by 0.

Lastly, we can see that any word of length at least two must contain either 0 or 11 and therefore, it has a synchronization point.

Let $\mathbf{u}$ be a fixed point of the morphism χ, let w be a bispecial factor in $\mathbf{u}$ such that there exist letters $a, b, c, d \in \mathcal{A}$, $a \neq b$ and $c \neq d$ such that $aw, bw, wc, wd \in \mathcal{L}(\mathbf{u})$.

Let Ψ be a morphism $\Psi : \mathcal{A}^* \to \mathcal{B}^*$ and $\mathbf{v} = \Psi(\mathbf{u})$. Since $aw, bw, wc, wd \in \mathcal{L}(\mathbf{u})$, we have $\Psi(aw), \Psi(bw), \Psi(wc), \Psi(wd) \in \mathcal{L}(\Psi(\mathbf{u})) = \mathcal{L}(\mathbf{v})$.

Now, depending on the form of Ψ, we might be able to identify that for long enough w (so long that $\Psi(w)$ has a synchonization point) $x\Psi(w)y$ is a bispecial factor in $\mathbf{v}$, where we need to find x, y depending on the letter images a, b, c, d under the morphism Ψ.

Let us show how to find y. There are two possibilities, either $\Psi(c), \Psi(d)$ differ at some position. In this case y is their longest common prefix such that $|y| < \min\{|\Psi(c)|, |\Psi(d)|\}$. Or, without loss of generality, $\Psi(c)$ is a prefix of $\Psi(d)$. In this case, we have to prolong the factor wc, resp. wd to wu, resp. wv using our knowledge of the form of χ (often some factors are forbidden in $\mathbf{u}$). Since the morphism Ψ is injective and since for long enough extension wu the factors $\Psi(w)$ and $\Psi(u)$ contain a synchronization point, clearly wu is not a prefix of wv. Similarly, wv is not a prefix of wu. The word y will then be the longest common prefix of $\Psi(u)$ and $\Psi(v)$. We can determine the word x analogously, we just need to consider suffixes instead of prefixes.

This procedure will allow us to find all bispecial factors in $\mathbf{v}$ that are "long enough" and the next step is to find the short bispecial factors, which is often done by generating a prefix of the sequence $\mathbf{v}$.

Similarly to finding the bispecial factors, we can determine the form of the return words in $\mathbf{v}$.

Lemma 7.6. *Let* $\mathbf{u}$ *be a sequence over an alphabet* $\mathcal{A}$*. Let* $\Psi : \mathcal{A}^* \to \mathcal{B}^*$ *be an injective morphism and let* $L \in \mathbb{N}$ *be such that every factor* v *of* $\Psi(\mathbf{u})$*,* $|v| \geq L$*, has a synchronization point. Then any factor* $w \in \mathcal{L}(\Psi(\mathbf{u}))$ *such that* $|w| \geq 2L + 1$ *has two synchronization points and can be factorized as* $w = s \bullet v \bullet t$*, where* $|s| \leq L, |t| \leq L$ *are minimal and there exists a unique* $x \in \mathcal{L}(\mathbf{u}) \setminus \{\varepsilon\}$ *such that* $v = \Psi(x)$*.*

Moreover, if w *is a bispecial factor in* $\Psi(\mathbf{u})$*, then* x *is a bispecial factor in* $\mathbf{u}$*.*

Proof. For $w \in \mathcal{L}(\Psi(\mathbf{u}))$, $|w| \geq 2L + 1$, we can write $w = hyz$, where $|h| = L$, $|z| = L$ and $|y| \geq 1$. Furthermore, the factors h and z each have a synchronization point $h = h_1 \bullet h_2$, $z = z_1 \bullet z_2$. Therefore, there exists a factorization $w = s \bullet v \bullet t$, where $|v| \geq |h_2 y z_1| \geq 1$, $|s| \leq |h_1| \leq |h| \leq L$, and $|t| \leq |z_2| \leq |z| \leq L$. Let us consider a factorization such that $|s|$ and $|t|$ are minimal. From the definition of synchronization point, we can see that $v = \Psi(x)$ for some $x \in \mathcal{L}(\mathbf{u})$ and by the injectivity of Ψ, the factor x is unique.

Now, let w be a bispecial factor of $\Psi(\mathbf{u})$, $|w| \geq 2L + 1$ and let $w = s \bullet v \bullet t$ from the previous point, $\Psi(x) = v$. Let us now assume that x is not a bispecial factor of $\mathbf{u}$. Without loss of generality, let us assume that x is not left special (for right special the proof is analogous, only we use prefixes instead of suffixes). Then any occurrence of x in $\mathbf{u}$ is preceded by a unique letter $a \in \mathcal{A}$. Therefore, any occurrence of v as a factor of w in $\Psi(\mathbf{u})$ is preceded by $\Psi(a)$. If $|s| \geq |\Psi(a)|$, then $\Psi(a)$ must be a suffix of s since it is a unique left extension of v in $\Psi(\mathbf{u})$. But this is a contradiction since we chose $|s|$ minimal and this would give us a synchronization point $w = s' \bullet \Psi(a)v \bullet t$, $|s'| < |s|$. Therefore, $|s| < |\Psi(a)|$ and s is a proper suffix of $\Psi(a)$. Then any occurrence of $w = svt$ has left extension dictated by the letter in $\Psi(a)$ preceeding s which gives us a contradiction since w is a left special factor in $\Psi(\mathbf{u})$. □

Theorem 7.7 ([14])**.** *Let* $\mathbf{u}$ *be an infinite word over an alphabet* $\mathcal{A}$ *such that the letter densities in* $\mathbf{u}$ *exist. Let* $\Psi : \mathcal{A}^* \to \mathcal{B}^*$ *be an injective morphism and let* $L \in \mathbb{N}$ *be such that every factor* v *of* $\Psi(\mathbf{u})$, $|v| \geq L$, *has a synchronization point. Then* $\mathrm{E}^*(\mathbf{u}) = \mathrm{E}^*(\Psi(\mathbf{u}))$.

Proof. The inequality $E^*(\psi(\mathbf{u})) \geq E^*(\mathbf{u})$ is proven in [27] for any non-erasing morphism under the assumption of existence of letter densities in $\mathbf{u}$. Let us prove the opposite inequality. According to the definition of $E^*(\psi(\mathbf{u}))$, there exist sequences (w_n) and (v_n) such that

1. $\lim_{n\to\infty} |v_n| = +\infty$;
2. w_n is a factor of $\psi(\mathbf{u})$ for each $n \in \mathbb{N}$;
3. w_n is a prefix of the periodic word $(v_n)^\omega$ for each $n \in \mathbb{N}$;
4. $E^*(\psi(\mathbf{u})) = \lim_{n\to\infty} \frac{|w_n|}{|v_n|}$.

If $E^*(\psi(\mathbf{u})) = 1$, then, clearly, $E^*(\psi(\mathbf{u})) \leq E^*(\mathbf{u})$. Assume in the sequel that $E^*(\psi(\mathbf{u})) > 1$, then we have for large enough n that $|w_n| > |v_n|$ and moreover, by the first item, $|v_n| \geq L$. By assumption, both v_n and w_n have synchronization points and since v_n is a prefix of w_n for large enough n, we may write

$$w_n = x_n \bullet \psi(w_n') \bullet y_n \quad \text{and} \quad v_n = x_n \bullet \psi(v_n') \bullet z_n\,,$$

where we highlighted the first and the last synchronization point (not necessarily distinct) in w_n and v_n and where w_n' and v_n' are uniquely given factors of $\mathbf{u}$ and the lengths of x_n, y_n, z_n are smaller than L.

By the third item, we have

$$w_n = v_n^k u_n = (x_n\psi(v_n')z_n)^k u_n\,,$$

where u_n is a proper prefix of v_n and $k \in \mathbb{N}, k \geq 1$.

There are two possible cases for (u_n).

(a) Either $(|u_n|)$ is bounded, but as $E^*(\psi(\mathbf{u})) > 1$, it follows that $k \geq 2$ for large enough n.

(b) Or there is a subsequence (u_{j_n}) of (u_n) such that for all $n \in \mathbb{N}$ we have $|u_{j_n}| \geq L$. Then by assumption, u_{j_n} has a synchronization point and we may write $u_{j_n} = x_{j_n} \bullet \psi(u'_{j_n}) \bullet y_{j_n}$, where we highlighted the first and the last synchronization point in u_{j_n} and u'_{j_n} is a prefix of v'_{j_n} by injectivity of ψ.

(a) In the first case, since $k \geq 2$ for large enough n, the factor w_n starts with $(x_n\psi(v_n')z_n)^2$. By the definition of synchronization points and injectivity of ψ, there exists a unique factor t_n of $\mathbf{u}$ such that $\psi(v_n')z_nx_n = \psi(t_n)$. Consequently, $w_n = (x_n\psi(v_n')z_n)^k u_n = x_n\psi(t_n^{k-1}v_n')z_nu_n$. Therefore, $t_n^{k-1}v_n'$ is a factor of $\mathbf{u}$ and it is a prefix of $(t_n)^\omega$ and

$$E^*(\psi(\mathbf{u})) = \lim_{n\to\infty} \frac{|w_n|}{|v_n|} = \lim_{n\to\infty} \frac{|x_n\psi({t_n}^{k-1}v_n')z_nu_n|}{|\psi(t_n)|} = \lim_{n\to\infty} \frac{|\psi({t_n}^{k-1}v_n')|}{|\psi(t_n)|}\,,$$

where the last equality holds thanks to boundedness of $(|x_n|)$, $(|z_n|)$ and $(|u_n|)$.

(b) In the second case, $w_{j_n} = (x_{j_n}\psi(v'_{j_n})z_{j_n})^k x_{j_n}\psi(u'_{j_n})y_{j_n}$, where $k \geq 1$. By definition of synchronization points and injectivity of ψ, there exists a unique factor t_{j_n} of $\mathbf{u}$ such that $\psi(v'_{j_n})z_{j_n}x_{j_n} = \psi(t_{j_n})$. Consequently, $(t_{j_n})^k u'_{j_n}$ is a factor of $\mathbf{u}$ and it is a prefix of $(t_{j_n})^\omega$ and

$$E^*(\psi(\mathbf{u})) = \lim_{n\to\infty} \frac{|w_n|}{|v_n|} = \lim_{n\to\infty} \frac{|x_{j_n}\psi((t_{j_n})^k u'_{j_n})y_{j_n}|}{|\psi(t_{j_n})|} = \lim_{n\to\infty} \frac{|\psi((t_{j_n})^k u'_{j_n})|}{|\psi(t_{j_n})|},$$

where the last equality holds thanks to boundedness of $(|x_n|)$ and $(|y_n|)$.

Combining two simple facts:

- $\frac{|\psi(u)|}{|u|} = \vec{1}^T M_\psi \frac{\vec{u}}{|u|}$ for each word u over $\mathcal{A}$, where $\vec{1}$ is a vector with all coordinates equal to one and M_ψ is a matrix of size $\#\mathcal{A} \times \#\mathcal{A}$, where $(M_\psi)_{i,j} = |\psi(j)|_i$ for $i, j \in \mathcal{A}$ characterizes how the morphism ψ changes Parikh vectors;
- for each sequence (s_n) of factors of $\mathbf{u}$ with $\lim_{n\to\infty} |s_n| = \infty$ we have, by existence of letter densities in $\mathbf{u}$, $\lim\limits_{n\to\infty} \frac{\vec{s_n}}{|s_n|} = \vec{f}$, where $\vec{f}$ is the vector of letter densities in $\mathbf{u}$,

we obtain

$$\lim_{n\to\infty} \frac{|\psi(s_n)|}{|s_n|} = \vec{1}^T M_\psi \vec{f}. \tag{7.1}$$

Consequently,

(a) in the first case, since $\lim_{n\to\infty} |t_n| = \infty$, we obtain using (7.1)

$$\begin{aligned} E^*(\psi(\mathbf{u})) &= \lim_{n\to\infty} \frac{|\psi(t_n{}^{k-1}v'_n)|}{|\psi(t_n)|} \\ &= \lim_{n\to\infty} \frac{|\psi(t_n{}^{k-1}v'_n)|}{|t_n{}^{k-1}v'_n|} \frac{|t_n|}{|\psi(t_n)|} \frac{|t_n{}^{k-1}v'_n|}{|t_n|} \\ &= \lim_{n\to\infty} \frac{|t_n{}^{k-1}v'_n|}{|t_n|} \leq E^*(\mathbf{u}), \end{aligned}$$

where the last inequality follows from the fact that $(t_n)^{k-1}v'_n \in \mathcal{L}(\mathbf{u})$ and $(t_n)^{k-1}v'_n$ is a power of t_n;

(b) in the second case, since $\lim |t_{j_n}| = \infty$, we obtain using (7.1)

$$\begin{aligned} E^*(\psi(\mathbf{u})) &= \lim_{n\to\infty} \frac{|\psi\big((t_{j_n})^k u'_{j_n}\big)|}{|\psi(t_{j_n})|} \\ &= \lim_{n\to\infty} \frac{|\psi\big((t_{j_n})^k u'_{j_n}\big)|}{|(t_{j_n})^k u'_{j_n}|} \frac{|t_{j_n}|}{|\psi(t_{j_n})|} \frac{|(t_{j_n})^k u'_{j_n}|}{|t_{j_n}|} \\ &= \lim_{n\to\infty} \frac{|(t_{j_n})^k u'_{j_n}|}{|t_{j_n}|} \leq E^*(\mathbf{u}), \end{aligned}$$

where the last inequality follows from the fact that $(t_{j_n})^k u'_{j_n} \in \mathcal{L}(\mathbf{u})$ and $(t_{j_n})^k u'_{j_n}$ is a power of t_{j_n}.

□

Next, we will be working with return words, and the following lemma will help us prolong the return words simultaneously with prolongation of the factors.

Lemma 7.8. *Let w be a factor of a recurrent sequence $\mathbf{u}$ and let v be its return word. If w has a unique right extension a, then v is a return word to wa, too. If w has a unique left extension b, then bvb^{-1} is a return word to bw, where b^{-1} means erasing the letter b from the end of the word. In particular, the Parikh vectors of the corresponding return words are the same.*

Proof. It follows immediately from the fact that the left or right extension is unique; therefore, since w is a prefix and suffix of vw, then wa is a prefix and suffix of vwa and each factor of the form vw has a as a right extension. Also, since w does not appear in vw in the middle, wa cannot appear in the middle of vwa. The proof for the left extension follows analogously. □

8 Sequence p

The sequence $\mathbf{p}$ is defined as the fixed point of morphism φ in the form

$$\varphi : 0 \to 01, \quad 1 \to 21, \quad 2 \to 0.$$

Therefore, the prefix of $\mathbf{p}$ is

$$\mathbf{p} = 0121021010210121010210121021010121010\cdots$$

The following characteristics of $\mathbf{p}$ are known [9]:

- It is sequence A287072 in the On-Line Encyclopedia of Integer Sequences (OEIS) and can be found in `https://oeis.org/A287072`.
- The factor complexity of $\mathbf{p}$ is $C(n) = 2n + 1$.
- The word $\mathbf{p}$ is not closed under reversal; $02 \in \mathcal{L}(\mathbf{p})$, but $20 \notin \mathcal{L}(\mathbf{p})$.
- The word $\mathbf{p}$ is uniformly recurrent and $\mathbf{p}$ has letter densities because φ is primitive (see [13]).
- The morphism φ is injective. Moreover, each non-empty factor w of $\mathbf{p}$ has a synchronization point.

8.1 Bispecial factors in p

To find bispecial factors, we need to find factors that are both left and right special. So, first, we examine the form of left special factors. Using the form of φ, we observe

- 0 has only one left extension: 1,
- 1 has two left extensions: 0 and 2,
- 2 has two left extensions: 0 and 1.

Therefore, every left special factor either starts with 1 and has left extensions $\{0, 2\}$, or starts with 2 with left extensions $\{0, 1\}$.

Lemma 8.1. *Let $w \neq \varepsilon$, $w \in \mathcal{L}(\mathbf{p})$.*

- *If w is a left special factor such that $0w, 1w \in \mathcal{L}(\mathbf{p})$, then $1\varphi(w)$ is a left special factor such that $01\varphi(w), 21\varphi(w) \in \mathcal{L}(\mathbf{p})$.*
- *If w is a left special factor such that $0w, 2w \in \mathcal{L}(\mathbf{p})$, then $\varphi(w)$ is a left special factor such that $0\varphi(w), 1\varphi(w) \in \mathcal{L}(\mathbf{p})$.*

Proof. It follows from the form of φ and the fact that $\mathbf{p}$ is the fixed point of the morphism φ, i.e., if $u \in \mathcal{L}(\mathbf{p})$, then $\varphi(u) \in \mathcal{L}(\mathbf{p})$. □

We can find the form of right special factors analogously. We observe

- 0 has two right extensions: 1 and 2,
- 1 has two right extensions: 0 and 2,
- 2 has only one right extension: 1.

Therefore, every right special factor either ends with 0 with right extensions $\{1, 2\}$, or ends in 1 with right extensions $\{0, 2\}$.

Lemma 8.2. *Let $w \neq \varepsilon$, $w \in \mathcal{L}(\mathbf{p})$.*

- *If w is a right special factor such that $w0, w2 \in \mathcal{L}(\mathbf{p})$, then $\varphi(w)0$ is a right special factor such that $\varphi(w)01, \varphi(w)02 \in \mathcal{L}(\mathbf{p})$.*
- *If w is a right special factor such that $w1, w2 \in \mathcal{L}(\mathbf{p})$, then $\varphi(w)$ is a right special factor such that $\varphi(w)2, \varphi(w)0 \in \mathcal{L}(\mathbf{p})$.*

Combining these two lemmata, we can see that we have at most 4 possible kinds of non-empty bispecial factors in $\mathbf{p}$.

Theorem 8.3. *Let v be a non-empty bispecial factor in $\mathbf{p}$.*

1. *$0v, 2v, v0, v2 \in \mathcal{L}(\mathbf{p})$ if and only if there exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 1\varphi(w)$ and $0w, 1w, w1, w2$ are factors of $\mathbf{p}$.*
2. *$0v, 1v, v1, v2 \in \mathcal{L}(\mathbf{p})$ if and only if there exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \varphi(w)0$ and $0w, 2w, w0, w2$ are factors of $\mathbf{p}$.*
3. *$0v, 2v, v1, v2 \in \mathcal{L}(\mathbf{p})$ if and only if there exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 1\varphi(w)0$ and $0w, 1w, w0, w2$ are factors of $\mathbf{p}$.*
4. *$0v, 1v, v0, v2 \in \mathcal{L}(\mathbf{p})$ if and only if there exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \varphi(w)$ and $0w, 2w, w1, w2$ are factors of $\mathbf{p}$.*

Proof. The implication ($\Leftarrow$) follows from Lemmata 8.1 and 8.2. We will prove the opposite implication for Item 1, the other cases may be proven analogously. If v is a non-empty factor of $\mathbf{p}$ such that $0v, 2v, v0, v2 \in \mathcal{L}(\mathbf{p})$, then v necessarily starts and ends with the letter 1. By the form of φ, we have the following synchronization points $v = 1 \bullet \hat{v} \bullet$ ($\hat{v}$ may be empty). Hence, by injectivity of φ, there exists a unique w in $\mathbf{p}$ such that $v = 1\varphi(w)$. Thus, using again the form of φ and the knowledge of possible right extensions, the factor w is bispecial and $0w, 1w, w1, w2 \in \mathcal{L}(\mathbf{p})$. $\square$

We can see that the only bispecial factor of length one is 1, it has left extensions $0, 2$ and right extensions $0, 2$. Applying Theorem 8.3 Item 2, we obtain that $\varphi(1)0$ is bispecial with left extensions $0, 1$ and right extensions $1, 2$. Theorem 8.3 Item 1 gives us that $1\varphi^2(1)\varphi(0)$ is bispecial with left extensions $0, 2$ and right extensions $0, 2$. This process can be iterated providing us with infinitely many bispecial factors:

$$\begin{aligned} &1 \to \varphi(1)0 \to 1\varphi^2(1)\varphi(0) \to \varphi(1)\varphi^3(1)\varphi^2(0)0 \to \\ &\to 1\varphi^2(1)\varphi^4(1)\varphi^3(0)\varphi(0) \to \varphi(1)\varphi^3(1)\varphi^5(1)\varphi^4(0)\varphi^2(0)0 \quad \dots \end{aligned} \tag{8.1}$$

The only bispecial factor of length two is 10, it has left extensions $0, 2$ and right extensions $1, 2$. Applying Theorem 8.3 Item 4, we obtain that $\varphi(1)\varphi(0)$ is bispecial factor with left extensions $0, 1$ and right extensions $0, 2$. Theorem 8.3 Item 3 gives us that $1\varphi^2(1)\varphi^2(0)0$ is bispecial factor with left extensions $0, 2$ and right extensions $1, 2$. This process can be iterated providing us again with infinitely many bispecial factors:

$$\begin{aligned} &10 \to \varphi(1)\varphi(0) \to 1\varphi^2(1)\varphi^2(0)0 \to \varphi(1)\varphi^3(1)\varphi^3(0)\varphi(0) \to \\ &\to 1\varphi^2(1)\varphi^4(1)\varphi^4(0)\varphi^2(0)0 \to \varphi(1)\varphi^3(1)\varphi^5(1)\varphi^5(0)\varphi^3(0)\varphi(0) \quad \dots \end{aligned} \tag{8.2}$$

Each bispecial factor v of length greater than two has at least two synchronization points and the corresponding bispecial factor w from Theorem 8.3 is non-empty. In other words, the bispecial factor v makes part of one of the sequences (8.1) and (8.2) of bispecial factors.

As a consequence of Theorem 8.3 and the above arguments, we get a complete description of bispecial factors in $\mathbf{p}$.

Theorem 8.4. *Let w be a non-empty bispecial factor in $\mathbf{p}$. Then it has one of the following forms:*

($\mathcal{A}$)
$$w_A^{(n)} = 1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0)$$

for $n \geq 1$. If $n = 0$, then we set $w_A^{(0)} = 1$.

The Parikh vector of $w_A^{(n)}$ is the same as of the word $1\varphi(012)\varphi^3(012)\ldots\varphi^{2n-1}(012)$.

($\mathcal{B}$)
$$w_B^{(n)} = \varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0$$

for $n \geq 0$.

The Parikh vector of $w_B^{(n)}$ is the same as of the word $012\varphi^2(012)\varphi^4(012)\ldots\varphi^{2n}(012)$.

(C)
$$w_C^{(n)} = 1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0$$

for $n \geq 0$.

The Parikh vector of $w_C^{(n)}$ is the same as of the word $01\varphi^2(01)\varphi^4(01)\ldots\varphi^{2n}(01)$.

($\mathcal{D}$)
$$w_D^{(n)} = \varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n+1}(0)\varphi^{2n-1}(0)\cdots\varphi(0)$$

for $n \geq 0$.

The Parikh vector of $w_D^{(n)}$ is the same as of the word $\varphi(01)\varphi^3(01)\ldots\varphi^{2n+1}(01)$.

8.2 The shortest return words to bispecial factors in p

To find the shortest return word to a bispecial factor in $\mathbf{p}$, it is convenient to know, how many return words the factor has. The next theorem will help us with that.

Theorem 8.5 (Theorem 5.7 in [5])**.** *Let $\mathbf{u}$ be a uniformly recurrent infinite word. Then each factor of $\mathbf{u}$ has exactly 3 return words if and only if $C(n) = 2n + 1$ and $\mathbf{u}$ has no weak bispecial factors.*

Next, to use the theorem above, we will show that each bispecial factor is ordinary, and therefore each factor of $\mathbf{p}$ has 3 return words.

Lemma 8.6. *All bispecial factors in $\mathbf{p}$ are ordinary.*

Proof. The empty word is ordinary because all factors of length two are $10, 01, 02, 12, 21$. Thus $b(\varepsilon) = 5 - 3 - 3 + 1 = 0$. It is easy to verify that each non-empty bispecial factor w has three extensions. In particular,

- extensions of $w = w_A^{(n)}$ are: $0w2, 2w0, 0w0$,
- extensions of $w = w_B^{(n)}$ are: $2w2, 2w1, 0w2$,
- extensions of $w = w_C^{(n)}$ are: $0w0, 0w2, 1w0$,

- extensions of $w = w_D^{(n)}$ are: $1w2, 0w1, 1w1$.

Consequently, $b(w) = 3 - 2 - 2 + 1 = 0$. □

As we have seen in the case of generating bispecial factors, it is convenient to examine the return words to short bispecial factors. The return words can be found using the prefix of **p** generated at the beginning of this section and the knowledge that there are exactly three for each bispecial factor.

- The return words to ε are $0, 1, 2$.
- The return words to 1 are $12, 102, 10$.
- The return words to $\varphi(1)0$ are $210 = \varphi(1)0$, 21010, 2101. The shortest one is 210 and it is a prefix of all of them.
- The return words to 10 are 10, 102, 1012. The shortest one is 10 and it is a prefix of all of them.

Lemma 8.7. *If w is a non-empty bispecial factor of* **p** *and v is its return word, then $\varphi(v)$ is a return word to $\varphi(w)$.*

Proof. On one hand, since vw contains w as a prefix and as a suffix, $\varphi(v)\varphi(w)$ contains $\varphi(w)$ as a prefix and as a suffix, too. On the other hand, w starts in 1 or 2 and ends in 0 or 1, thus $\varphi(w)$ starts in 0 or 2 and ends in 1, therefore it has the following synchronization points $\bullet\varphi(w)\bullet$. Consequently, $\varphi(v)\varphi(w)$ cannot contain $\varphi(w)$ somewhere in the middle because in such a case, by injectivity of φ, vw would contain w also somewhere in the middle. □

Example 8.8. Consider the bispecial factor 10 with the shortest return word 10 (being a prefix of the other two return words), then by Lemma 8.7 the bispecial factor $\varphi(1)\varphi(0)$ has the shortest return word equal to $\varphi(10)$. By Lemma 8.7, the factor $\varphi^2(1)\varphi^2(0)$ has the shortest return word equal to $\varphi^2(10)$ and by Lemma 7.8, the shortest return word to the bispecial factor $1\varphi^2(1)\varphi^2(0)0$ has the same Parikh vector as $\varphi^2(10)$.

Using Lemmata 8.7 and 7.8, and the knowledge of how the bispecial factors are constructed, as illustrated in Example 8.8, we obtain the following statement about the shortest return words to bispecial factors in **p**.

Theorem 8.9. *The shortest return words to bispecial factors in* **p** *have the following properties.*

($\mathcal{A}$) *The shortest return words to $w_A^{(n)}$ are*

  (i) 12 *and* 10 *for $n = 0$,*

  (ii) $r_A^{(n)}$ *with the same Parikh vector as $\varphi^{2n-1}(012)$ for $n \geq 1$.*

($\mathcal{B}$) *The shortest return word $r_B^{(n)}$ to $w_B^{(n)}$ has the same Parikh vector as $\varphi^{2n}(012)$.*

($\mathcal{C}$) *The shortest return word $r_C^{(n)}$ to $w_C^{(n)}$ has the same Parikh vector as $\varphi^{2n}(01)$.*

($\mathcal{D}$) *The shortest return word $r_D^{(n)}$ to $w_D^{(n)}$ has the same Parikh vector as $\varphi^{2n+1}(01)$.*

Proof. We will prove case ($\mathcal{A}$). The other cases are similar. The shortest return words to $w_A^{(0)} = 1$ are 12 and 10 as can be seen in the prefix. Let us proceed by induction on n. Consider the bispecial factor $w_A^{(1)} = 1\varphi^2(1)\varphi(0) = 1\varphi(\varphi(1)0)$. By description of the shortest return words to short bispecial factors, we know that 210 is the shortest return word (moreover prefix of all other return words) to the bispecial factor $\varphi(1)0$. Using Lemma 8.7, $\varphi(210)$ is the shortest return word to the factor $\varphi^2(1)\varphi(0)$. By Lemma 7.8, the

Parikh vector of the shortest return word to $w_A^{(1)} = 1\varphi^2(1)\varphi(0)$ is equal to the Parikh vector of $\varphi(210)$, hence also to the Parikh vector of $\varphi(012)$.

Assume for a fixed $n \geq 1$, the bispecial factor

$$w_A^{(n)} = 1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0)$$

has the shortest return word with the Parikh vector $\varphi^{2n-1}(012)$ and this return word is a prefix of all other return words. By definition, $w_A^{(n+1)} = 1\varphi^2(w_A^{(n)})\varphi(0)$. By Lemma 8.7 and by induction assumption, the shortest return word to the factor $\varphi^2(w_A^{(n)})$ has the same Parikh vector as $\varphi^{2n+1}(012)$. Using Lemma 7.8, we obtain that the shortest return word to $w_A^{(n+1)}$ has the same Parikh vector as the factor $\varphi^{2n+1}(012)$, too. □

8.3 The asymptotic critical exponent of p

With the knowledge of the form of bispecial factors and their shortest return words, we can now use Theorem 2.2 and we get $\mathrm{E}^*(\mathbf{p}) = 1 + \max\{A', B', C', D'\}$, where

$$A' = \limsup_{n\to\infty} \frac{|w_A^{(n)}|}{|r_A^{(n)}|} = \limsup_{n\to\infty} \frac{|1\varphi(012)\varphi^3(012)\ldots\varphi^{2n-1}(012))|}{|\varphi^{2n-1}(012)|};$$

$$B' = \limsup_{n\to\infty} \frac{|w_B^{(n)}|}{|r_B^{(n)}|} = \limsup_{n\to\infty} \frac{|012\varphi^2(012)\varphi^4(012)\ldots\varphi^{2n}(012))|}{|\varphi^{2n}(012)|};$$

$$C' = \limsup_{n\to\infty} \frac{|w_C^{(n)}|}{|r_C^{(n)}|} = \limsup_{n\to\infty} \frac{|01\varphi^2(01)\varphi^4(01)\ldots\varphi^{2n}(01)|}{|\varphi^{2n}(01)|};$$

$$D' = \limsup_{n\to\infty} \frac{|w_D^{(n)}|}{|r_D^{(n)}|} = \limsup_{n\to\infty} \frac{|\varphi(01)\varphi^3(01)\ldots\varphi^{2n+1}(01)|}{|\varphi^{2n+1}(01)|}.$$

Let us denote

$$M_\varphi = \begin{pmatrix} |\varphi(0)|_0 & |\varphi(1)|_0 & |\varphi(2)|_0 \\ |\varphi(0)|_1 & |\varphi(1)|_1 & |\varphi(2)|_1 \\ |\varphi(0)|_2 & |\varphi(1)|_2 & |\varphi(2)|_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{8.3}$$

Remark. By the Hamilton-Cayley theorem, we have $M_\varphi^3 - 2M_\varphi^2 + M_\varphi - I = 0$.

The following lemma shows us a nice way to compute the lengths of the bispecial factors and their shortest return words in **p** and later also in the morphic images of **p**.

Lemma 8.10. *Let $\vec{v} \in \mathbb{N}_0^3$, M_φ be the matrix defined in* (8.3) *and $c_0, c_1, c_2 \in \mathbb{N}_0$ and define $\ell_n = (c_0, c_1, c_2)\left(M_\varphi\right)^n \vec{v}$. Then ℓ_n satisfies the recurrence relation*

$$\ell_{n+3} - 2\ell_{n+2} + \ell_{n+1} - \ell_n = 0 \tag{8.4}$$

with the initial conditions

$$\begin{aligned} \ell_0 &= (c_0, c_1, c_2)\vec{v}, \\ \ell_1 &= (c_0, c_1, c_2)M_\varphi\vec{v}, \\ \ell_2 &= (c_0, c_1, c_2)(M_\varphi)^2\vec{v}. \end{aligned}$$

Let $\beta, \lambda_1, \lambda_2$ be the real and two complex roots of the polynomial $t^3 - 2t^2 + t - 1$, i.e.

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}; \qquad |\lambda_1| \doteq 0.75488 < 1. \tag{8.5}$$

Then ℓ_n can be computed as $\ell_n = A_1\beta^n + B_1\lambda_1^n + C_1\lambda_2^n$, where

$$\begin{aligned}
A_1 &= \frac{\ell_0|\lambda_1|^2 - 2\ell_1 \operatorname{Re}(\lambda_1) + \ell_2}{|\beta - \lambda_1|^2},\\
B_1 &= \frac{\ell_0\beta\lambda_2 - \ell_1(\beta + \lambda_2) + \ell_2}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)},\\
C_1 &= \overline{B_1}.
\end{aligned}$$

Proof. By the remark above, the sequence $(M_\varphi)^n$ follows the recurrence relation $(M_\varphi)^{n+3} - 2(M_\varphi)^{n+2} + M_\varphi^{n+1} - M_\varphi^n = 0$. Multiplying the equation from left by (c_0, c_1, c_2) and from right by $\vec{v}$ gives us the relation for ℓ_n.

The coefficients A_1, B_1 and C_1 are then the solution of the system

$$\begin{pmatrix} 1 & 1 & 1\\ \beta & \lambda_1 & \lambda_2\\ \beta^2 & \lambda_1^2 & \lambda_2^2 \end{pmatrix}\begin{pmatrix} A_1\\ B_1\\ C_1 \end{pmatrix} = \begin{pmatrix} \ell_0\\ \ell_1\\ \ell_2 \end{pmatrix},$$

which can be solved and gives us the closed formulas in the lemma. $\square$

For our case, we can denote $f_n = |\varphi^n(w)| = (1,1,1)(M_\varphi)^n\vec{V}(w)$ and use Lemma 8.10, i.e. $f_n = A_1\beta^n + B_1\lambda_1^n + C_1\lambda_2^n$ for some coefficients A_1, B_1, C_1, where $A_1 \neq 0$ for $w \in \{01, 012\}$.

Since β is strictly larger than the modulus of the other roots of the characteristic polynomial, we obtain:

$$\begin{aligned}
A' &= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \beta^{2k-1}}{\beta^{2n-1}} &= \tfrac{\beta^2}{\beta^2-1};\\
B' = C' &= \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \beta^{2k}}{\beta^{2n}} &= \tfrac{\beta^2}{\beta^2-1};\\
D' &= \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \beta^{2k+1}}{\beta^{2n+1}} &= \tfrac{\beta^2}{\beta^2-1}.
\end{aligned}$$

Consequently, $\mathrm{E}^*(\mathbf{p}) = 1 + \frac{\beta^2}{\beta^2-1} \doteq 2.48$.

9 The infinite word $\nu(\mathbf{p})$

The morphism ν has the form:

$$\nu : 0 \to 011, \quad 1 \to 0, \quad 2 \to 01.$$

Therefore,

$$\nu(\mathbf{p}) = 01100100110100110011010011001 0011\cdots$$

and ν is injective.

Lemma 9.1. *Any factor v of $\nu(\mathbf{p})$ of length at least two has a synchronization point.*

Proof. It is enough to show that each factor of $\nu(\mathbf{p})$ of length 2 has a synchronization point since v contains at least one of them as a prefix. The factors of length two are

$$\bullet 0 \bullet 0, \qquad \bullet 01, \qquad 1 \bullet 0, \qquad 11\bullet,$$

where we highlighted the synchronization points as we discussed them in Example 7.5. ◻

Using Lemma 9.1 and Theorem 7.7, we deduce that

$$\mathrm{E}^*(\nu(\mathbf{p})) = \mathrm{E}^*(\mathbf{p}).$$

9.1 Bispecial factors in $\nu(\mathbf{p})$

Lemma 9.2. *Let $v \in \mathcal{L}(\nu(\mathbf{p}))$ be a bispecial factor of length at least five. Then one of the items holds.*

1. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 1\nu(w)01$ and $0w, 2w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*
2. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \nu(w)0$ and $0w, 1w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
3. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 1\nu(w)0$ and $0w, 2w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
4. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \nu(w)01$ and $0w, 1w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*

Proof. The statement follows from Lemma 9.1, Lemma 7.6, from the form of the morphism ν and from the possible left and right extensions of factors in $\mathbf{p}$. ◻

Combining Lemma 9.2 and Theorem 8.4, we get a complete description of bispecial factors of length at least five in $\nu(\mathbf{p})$.

Theorem 9.3. *Let v be a non-empty bispecial factor in $\nu(\mathbf{p})$ of length at least five. Then v has one of the following forms:*

($\mathcal{A}$)

$$v_A^{(n)} = 1\nu(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0))01$$

for $n \geq 1$.

The Parikh vector of $v_A^{(n)}$, $n \geq 1$ is the same as of the word $011\nu(1\varphi(012)\varphi^3(012)\ldots\varphi^{2n-1}(012))$.

($\mathcal{B}$)
$$v_B^{(n)} = \nu(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0)0$$

for $n \geq 0$.

The Parikh vector of $v_B^{(n)}$ *is the same as of the word* $0\nu(012\varphi^2(012)\varphi^4(012)\ldots\varphi^{2n}(012))$.

($\mathcal{C}$)
$$v_C^{(n)} = 1\nu(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0)0$$

for $n \geq 0$.

The Parikh vector of $v_C^{(n)}$ *is the same as of the word* $01\nu(01\varphi^2(01)\varphi^4(01)\ldots\varphi^{2n}(01))$.

($\mathcal{D}$)
$$v_D^{(n)} = \nu(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n+1}(0)\varphi^{2n-1}(0)\cdots\varphi(0))01$$

for $n \geq 0$.

The Parikh vector of $v_D^{(n)}$ *is the same as of the word* $01\nu(\varphi(01)\varphi^3(01)\ldots\varphi^{2n+1}(01))$.

9.2 The shortest return words to bispecial factors in $\nu(\mathbf{p})$

Lemma 9.4. *If w is a non-empty bispecial factor in* $\mathbf{p}$ *and v is its return word, then $\nu(v)$ is a return word to $\nu(w)0$.*

Proof. On one hand, consider any occurrence of vw and denote a the following letter, then $\nu(v)\nu(w)0$ is a prefix of $\nu(vwa)$. Since vw contains w as a prefix and as a suffix, then $\nu(v)\nu(w)0$ contains $\nu(w)0$ as a prefix and as a suffix, too. On the other hand, w starts in 1 or 2, thus $\nu(w)0$ starts in 0 and ends in 0, therefore $\nu(w)0$ has the following synchronization points $\bullet\nu(w)\bullet 0$. Consequently, $\nu(v)\nu(w)0$ cannot contain $\nu(w)0$ somewhere in the middle because in such a case, by injectivity of ν, vw would contain w also somewhere in the middle. □

Applying Lemmata 9.4, 7.8 and Theorem 8.9, we have the following description of the shortest return words to bispecial factors.

Theorem 9.5. *The shortest return words to bispecial factors of length at least five in $\nu(\mathbf{p})$ have the following properties.*

($\mathcal{A}$) *The shortest return word $\hat{r}_A^{(n)}$ to $v_A^{(n)}$ has the same Parikh vector as $\nu(\varphi^{2n-1}(012))$ for $n \geq 1$.*

($\mathcal{B}$) *The shortest return word $\hat{r}_B^{(n)}$ to $v_B^{(n)}$ has the same Parikh vector as $\nu(\varphi^{2n}(012))$.*

($\mathcal{C}$) *The shortest return word $\hat{r}_C^{(n)}$ to $v_C^{(n)}$ has the same Parikh vector as $\nu(\varphi^{2n}(01))$.*

($\mathcal{D}$) *The shortest return word $\hat{r}_D^{(n)}$ to $v_D^{(n)}$ has the same Parikh vector as $\nu(\varphi^{2n+1}(01))$.*

Proof. We will prove case ($\mathcal{A}$). The other cases are similar.

Let us consider $n \geq 1$ and the bispecial factor

$$v_A^{(n)} = 1\nu(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0))01 = 1\nu(w_A^{(n)})01.$$

Using Theorem 8.9, we know that the shortest return word to $w_A^{(n)}$ has the same Parikh vector as $\varphi^{2n-1}(012)$, moreover the shortest return word is a prefix of all other return words.

Using Lemma 9.4, we obtain that the shortest return word to $\nu(w_A^{(n)})0$ has the same Parikh vector as $\nu(\varphi^{2n-1}(012))$. Using Lemma 7.8 twice, we obtain that the shortest return word to $1\nu(w_A^{(n)})01$ has the same Parikh vector as $\nu(\varphi^{2n-1}(012))$, since adding 1 at the beginning and erasing 1 at the end does not change the Parikh vector. □

9.3 The critical exponent of $\nu(\mathbf{p})$

Using Theorem 2.2 and the description of bispecial factors from Theorem 9.3 and of their shortest return words from Theorem 9.5, we obtain the following formula for the critical exponent of $\nu(\mathbf{p})$.

$$\mathrm{E}(\nu(\mathbf{p})) = 1 + \max\{A, B, C, D, F\} ,$$

where

$$\begin{aligned}
A &= \sup\left\{\frac{|v_A^{(n)}|}{|\hat{r}_A^{(n)}|} : n \geq 1\right\} = \sup\left\{\frac{|011\nu(1\varphi(012)\varphi^3(012)\dots\varphi^{2n-1}(012))|}{|\nu(\varphi^{2n-1}(012))|} : n \geq 1\right\}\\
B &= \sup\left\{\frac{|v_B^{(n)}|}{|\hat{r}_B^{(n)}|} : n \geq 0\right\} = \sup\left\{\frac{|0\nu(012\varphi^2(012)\varphi^4(012)\dots\varphi^{2n}(012))|}{|\nu(\varphi^{2n}(012))|} : n \geq 0\right\} ;\\
C &= \sup\left\{\frac{|v_C^{(n)}|}{|\hat{r}_C^{(n)}|} : n \geq 0\right\} = \sup\left\{\frac{|01\nu(01\varphi^2(01)\varphi^4(01)\dots\varphi^{2n}(01))|}{|\nu(\varphi^{2n}(01))|} : n \geq 0\right\};\\
D &= \sup\left\{\frac{|v_D^{(n)}|}{|\hat{r}_D^{(n)}|} : n \geq 0\right\} = \sup\left\{\frac{|01\nu(\varphi(01)\varphi^3(01)\dots\varphi^{2n+1}(01))|}{|\nu(\varphi^{2n+1}(01))|} : n \geq 0\right\} ;\\
F &= \max\left\{\frac{|w|}{|r|} : w \text{ bispecial factor in } \nu(\mathbf{p}) \text{ of length at most four and } r \text{ its shortest return word}\right\} .
\end{aligned}$$

Theorem 9.6. *The critical exponent of* $\nu(\mathbf{p})$ *equals*

$$\mathrm{E}(\nu(\mathbf{p})) = \frac{5}{2}.$$

Proof. To determine the critical exponent, we need to examine the values of A, B, C, D and F.

1. First, we calculate the value of F. The bispecial factors of length at most four in $\nu(\mathbf{p})$ with their shortest return words are

   - 0 with the shortest return word 0.
   - 1 with the shortest return word 1.
   - 01 with the shortest return words 010, 011.
   - 10 with the shortest return word 10.
   - 1001 with the shortest return word 100.

   Therefore, we get $F = \max\left\{1, \frac{2}{3}, \frac{4}{3}\right\} = \frac{4}{3}$.

2. Computation of A and B.

   Let us denote $a_n := |\nu(\varphi^n(012))|$, then from the form of ν, we get

   $$a_n = (3, 1, 2)(M_\varphi)^n(1, 1, 1)^T.$$

   Using Lemma 8.10 with $a_0 = 6, a_1 = 10$ and $a_2 = 17$, we get the explicit solution

   $$a_n = A_1\beta^n + B_1\lambda_1^n + C_1\lambda_2^n,$$

   where

   $$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

and

$$\begin{aligned}
A_1 &= \frac{6|\lambda_1|^2 - 20\,\mathrm{Re}(\lambda_1) + 17}{|\beta - \lambda_1|^2} \doteq 5.58131\,;\\
B_1 &= \frac{6\beta\lambda_2 - 10(\beta + \lambda_2) + 17}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq 0.20935 - 0.10348i\,;\\
C_1 &= \overline{B_1}\,.
\end{aligned}$$

Let us show that $A \le \frac{3}{2}$.

We need to show for all $n \ge 1$ that

$$\begin{aligned}
\frac{4 + A_1 \sum_{k=1}^{n} \beta^{2k-1} + B_1 \sum_{k=1}^{n} \lambda_1^{2k-1} + C_1 \sum_{k=1}^{n} \lambda_2^{2k-1}}{A_1\beta^{2n-1} + B_1\lambda_1^{2n-1} + C_1\lambda_2^{2n-1}} &\le^? \frac{3}{2},\\
8 + 2A_1 \sum_{k=1}^{n} \beta^{2k-1} + 4\,\mathrm{Re}\left(B_1 \sum_{k=1}^{n} \lambda_1^{2k-1}\right) &\le^? 3A_1\beta^{2n-1} + 6\,\mathrm{Re}(B_1\lambda_1^{2n-1}),\\
8 + 2A_1 \sum_{k=1}^{n-1} \beta^{2k-1} + 4\,\mathrm{Re}\left(B_1 \sum_{k=1}^{n-1} \lambda_1^{2k-1}\right) &\le^? A_1\beta^{2n-1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}),\\
8 + 2A_1\left(\frac{\beta^{2n-1}}{\beta^2 - 1} - \frac{\beta}{\beta^2 - 1}\right) + 4\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\le^? A_1\beta^{2n-1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}).
\end{aligned}$$

Since

$$\frac{2}{\beta^2 - 1} \le 1,$$

we need to prove the inequality in the form

$$8 + 4\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) \le^? 2A_1 \frac{\beta}{\beta^2 - 1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}).$$

For the left side, we can write for $n \ge 1$

$$\begin{aligned}
8 + 4\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\le 8 + 4|B_1||\lambda_1| \frac{|\lambda_1|^{2n-2} + 1}{|\lambda_1^2 - 1|}\\
&\le 8 + 4|B_1||\lambda_1| \frac{2}{|\lambda_1^2 - 1|}.
\end{aligned}$$

For the right side, we can write for $n \ge 1$

$$\begin{aligned}
2A_1 \frac{\beta}{\beta^2 - 1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}) &\ge 2A_1 \frac{\beta}{\beta^2 - 1} - 2|B_1||\lambda_1|^{2n-1}\\
&\ge 2A_1 \frac{\beta}{\beta^2 - 1} - 2|B_1||\lambda_1|.
\end{aligned}$$

Since the inequality

$$8 + 4|B_1||\lambda_1| \frac{2}{|\lambda_1^2 - 1|} \le 2A_1 \frac{\beta}{\beta^2 - 1} - 2|B_1||\lambda_1|$$

holds true for given values, we obtain $A \leq \frac{3}{2}$.

Next, we will show that $B \leq \frac{3}{2}$.

For all $n \geq 0$ we have to show that

$$\begin{aligned}
\frac{1 + A_1 \sum_{k=0}^{n} \beta^{2k} + B_1 \sum_{k=0}^{n} \lambda_1^{2k} + C_1 \sum_{k=0}^{n} \lambda_2^{2k}}{A_1 \beta^{2n} + B_1 \lambda_1^{2n} + C_1 \lambda_2^{2n}} &\leq^? \frac{3}{2}, \\
2 + 2A_1 \sum_{k=0}^{n-1} \beta^{2k} + 4\,\mathrm{Re}\left(B_1 \sum_{k=0}^{n-1} \lambda_1^{2k}\right) &\leq^? A_1 \beta^{2n} + 2\,\mathrm{Re}\left(B_1 \lambda_1^{2n}\right), \\
2 + 2A_1 \frac{\beta^{2n} - 1}{\beta^2 - 1} + 4\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right) &\leq^? A_1 \beta^{2n} + 2\,\mathrm{Re}\left(B_1 \lambda_1^{2n}\right).
\end{aligned}$$

Since

$$\frac{2}{\beta^2 - 1} \leq 1,$$

we need to prove the inequality in the form

$$2 + 4\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right) \leq^? A_1 \frac{2}{\beta^2 - 1} + 2\,\mathrm{Re}\left(B_1 \lambda_1^{2n}\right).$$

For the left side, we can write for $n \geq 0$

$$\begin{aligned}
2 + 4\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right) &\leq 2 + 4|B_1| \frac{|\lambda_1|^{2n} + 1}{|\lambda_1^2 - 1|} \\
&\leq 2 + 4|B_1| \frac{2}{|\lambda_1^2 - 1|}.
\end{aligned}$$

For the right side, we can write for $n \geq 0$

$$\begin{aligned}
A_1 \frac{2}{\beta^2 - 1} + 2\,\mathrm{Re}\left(B_1 \lambda_1^{2n}\right) &\geq A_1 \frac{2}{\beta^2 - 1} - 2|B_1||\lambda_1|^{2n} \\
&\geq A_1 \frac{2}{\beta^2 - 1} - 2|B_1|.
\end{aligned}$$

Since the inequality

$$2 + 4|B_1| \frac{2}{|\lambda_1^2 - 1|} \leq A_1 \frac{2}{\beta^2 - 1} - 2|B_1|$$

holds true for given values, we obtain $B \leq \frac{3}{2}$.

3. Computation of C and D.

   Denote $b_n := |\nu(\varphi^n(01))| = (3, 1, 2)(M_\varphi)^n (1, 1, 0)^T$. Then, using Lemma 8.4 with initial conditions $b_0 = 4, b_1 = 7$ and $b_2 = 13$, we obtain the explicit solution

$$b_n = A_2 \beta^n + B_2 \lambda_1^n + C_2 \lambda_2^n,$$

   where

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

and

$$\begin{aligned}
A_2 &= \frac{4|\lambda_1|^2 - 14\,\mathrm{Re}(\lambda_1) + 13}{|\beta - \lambda_1|^2} \doteq 4.21321\,;\\
B_2 &= \frac{4\beta\lambda_2 - 7(\beta + \lambda_2) + 13}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq -0.10660 + 0.24672i\,;\\
C_2 &= \overline{B_2}\,.
\end{aligned}$$

First, we will show that $C = \frac{3}{2}$.

For $n = 0$, we get $\frac{2+b_0}{b_0} = \frac{3}{2}$ and for $n = 1$, we obtain $\frac{2+b_0+b_2}{b_2} = \frac{19}{13} < \frac{3}{2}$. Therefore, we have $C \geq \frac{3}{2}$ and it suffices to show for all $n \geq 2$ that

$$\begin{aligned}
\frac{2 + A_2\sum_{k=0}^{n}\beta^{2k} + B_2\sum_{k=0}^{n}\lambda_1^{2k} + C_2\sum_{k=0}^{n}\lambda_2^{2k}}{A_2\beta^{2n} + B_2\lambda_1^{2n} + C_2\lambda_2^{2n}} &\overset{?}{\leq} \frac{3}{2},\\
4 + 2A_2\sum_{k=0}^{n-1}\beta^{2k} + 4\,\mathrm{Re}\left(B_2\sum_{k=0}^{n-1}\lambda_1^{2k}\right) &\overset{?}{\leq} A_2\beta^{2n} + 2\,\mathrm{Re}\left(B_2\lambda_1^{2n}\right),\\
4 + 2A_2\frac{\beta^{2n}-1}{\beta^2-1} + 4\,\mathrm{Re}\left(B_2\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right) &\overset{?}{\leq} A_2\beta^{2n} + 2\,\mathrm{Re}\left(B_2\lambda_1^{2n}\right).
\end{aligned}$$

Since

$$\frac{2}{\beta^2 - 1} \leq 1,$$

we need to prove the inequality in the form

$$4 + 4\,\mathrm{Re}\left(B_2\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right) \overset{?}{\leq} A_2\frac{2}{\beta^2-1} + 2\,\mathrm{Re}\left(B_2\lambda_1^{2n}\right).$$

Now, we need to be more careful with the approximations. For the left side, we can write for $n \geq 2$

$$\begin{aligned}
4 + 4\,\mathrm{Re}\left(B_2\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right) &\leq 4 + 4\,\mathrm{Re}\left(\frac{B_2}{1-\lambda_1^2}\right) + 4|B_2|\frac{|\lambda_1|^{2n}}{|1-\lambda_1^2|}\\
&\leq 4 + 4\,\mathrm{Re}\left(\frac{B_2}{1-\lambda_1^2}\right) + 4|B_2|\frac{|\lambda_1|^{4}}{|1-\lambda_1^2|}.
\end{aligned}$$

For the right side, we can write for $n \geq 2$

$$\begin{aligned}
A_2\frac{2}{\beta^2-1} + 2\,\mathrm{Re}\left(B_2\lambda_1^{2n}\right) &\geq A_2\frac{2}{\beta^2-1} - 2|B_2||\lambda_1|^{2n}\\
&\geq A_2\frac{2}{\beta^2-1} - 2|B_2||\lambda_1|^{4}.
\end{aligned}$$

Since the inequality

$$4 + 4\,\mathrm{Re}\left(\frac{B_2}{1-\lambda_1^2}\right) + 4|B_2|\frac{|\lambda_1|^{4}}{|1-\lambda_1^2|} \leq A_2\frac{2}{\beta^2-1} - 2|B_2||\lambda_1|^{4}$$

holds for given values and the cases $n \in \{0, 1\}$ were already examined, we have proven that $C = \frac{3}{2}$.

It remains to prove $D \leq \frac{3}{2}$, however, the steps are the same as in the proof of inequality $A \leq \frac{3}{2}$, only the values A_1, resp. B_1 will be substituted for A_2, resp. B_2 and the offset value 4 will be replaced by 2. Therefore, it is enough to check the inequality

$$4 + 4|B_2||\lambda_1|\frac{2}{|\lambda_1^2 - 1|} \quad \leq \quad 2A_2\frac{\beta}{\beta^2 - 1} - 2|B_2||\lambda_1|,$$

which is true for given values.

We have shown that $\max\{A, B, C, D\} = \frac{3}{2}$, and $F = \frac{4}{3}$. Consequently,

$$\mathrm{E}(\nu(\mathbf{p})) = 1 + \max\{A, B, C, D, F\} = \frac{5}{2}.$$

$\square$

10 The infinite word $\mu(\mathbf{p})$

Since the examination of the word $\mu(\mathbf{p})$ proceeds analogously as for $\nu(\mathbf{p})$ and we would quickly run out of possible notation, therefore, in this chapter, we will use the same notation for bispecial factors, their shortest return words and other variables as in the previous section. We will not use any variables from Chapter 9; the calculation of the critical exponent of $\mu(\mathbf{p})$ is independent.

The morphism μ has the form:

$$\begin{aligned} \mu : 0 &\to 011001, \\ 1 &\to 1001, \\ 2 &\to 0. \end{aligned}$$

Therefore, μ is injective and

$$\mu(\mathbf{p}) = 0110011001010010110010100101100110010110 0101\cdots$$

Lemma 10.1. *Any factor v of $\mu(\mathbf{p})$ of length at least six has a synchronization point.*

Proof. Similarly to Lemma 9.1, it is enough to show that any factor of length six has a synchronization point. The factors are

$011001\bullet$	$11001\bullet 0$	$1\bullet 0\bullet 1001\bullet$	$1\bullet 01100$
$11001\bullet 1$	$1001\bullet 01$	$\bullet 0\bullet 1001\bullet 0$	
$1001\bullet 10$	$001\bullet 0\bullet 10$	$001\bullet 011$	
$001\bullet 100$	$01\bullet 0\bullet 100$	$01\bullet 0110$	

where we highlighted the synchronization points. □

Using Lemma 10.1 and Theorem 7.7, we deduce that

$$E^*(\mu(\mathbf{p})) = E^*(\mathbf{p}).$$

10.1 Bispecial factors in $\mu(\mathbf{p})$

Using Lemma 10.1, Lemma 7.6, and the form of μ with the knowledge of possible left and right extension in $\mathbf{p}$, we obtain the following lemma.

Lemma 10.2. *Let $v \in \mathcal{L}(\mu(\mathbf{p}))$ be a bispecial factor of length at least 13. Then one of the items holds.*

1. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \mu(w)01$ and $0w, 2w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*
2. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 011001\mu(w)$ and $0w, 1w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
3. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \mu(w)$ and $0w, 2w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
4. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 011001\mu(w)01$ and $0w, 1w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*

Theorem 10.3. *Let v be a bispecial factor in $\mu(\mathbf{p})$ of length at least 13. Then v has one of the following forms:*

($\mathcal{A}$)
$$v_A^{(n)} = \mu(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0))01$$
for $n \geq 1$.
The Parikh vector of $v_A^{(n)}$ *is the same as of the word* $01\mu(1\varphi(012)\varphi^3(012)\ldots\varphi^{2n-1}(012))$.

($\mathcal{B}$)
$$v_B^{(n)} = 011001\mu(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0)$$
for $n \geq 0$.
The Parikh vector of $v_B^{(n)}$ *is the same as of the word* $000111\mu(012\varphi^2(012)\varphi^4(012)\ldots\varphi^{2n}(012))$.

($\mathcal{C}$)
$$v_C^{(n)} = \mu(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0)$$
for $n \geq 1$.
The Parikh vector of $v_C^{(n)}$ *is the same as of the word* $\mu(01\varphi^2(01)\varphi^4(01)\ldots\varphi^{2n}(01))$.

($\mathcal{D}$)
$$v_D^{(n)} = 011001\mu(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n+1}(0)\varphi^{2n-1}(0)\cdots\varphi(0))01$$
for $n \geq 0$.
The Parikh vector of $v_D^{(n)}$ *is the same as of the word* $00001111\mu(\varphi(01)\varphi^3(01)\ldots\varphi^{2n+1}(01))$.

10.2 The shortest return words to bispecial factors in $\mu(\mathbf{p})$

Lemma 10.4. *If* w *is a bispecial factor of* $\mathbf{p}$, $|w| \geq 2$, *and* v *is its return word, then* $\mu(v)$ *is a return word to* $\mu(w)$.

Proof. On one hand, since vw contains w as a prefix and as a suffix, then $\mu(v)\mu(w)$ contains $\mu(w)$ as a prefix and as a suffix, too. On the other hand, w starts in 10 or 21 and ends in 0 or 1, therefore $\mu(w)$ has the following synchronization points $\bullet\mu(w)\bullet$. Consequently, $\mu(v)\mu(w)$ cannot contain $\mu(w)$ somewhere in the middle because in such a case, by injectivity of μ, vw would contain w also somewhere in the middle. □

Applying Lemma 10.4, Lemma 7.8 and Theorem 8.9, we have the following description of the shortest return words to bispecial factors.

Theorem 10.5. *The shortest return words to bispecial factors of length at least 13 in* $\mu(\mathbf{p})$ *have the following properties.*

($\mathcal{A}$) *The shortest return word* $\hat{r}_A^{(n)}$ *to* $v_A^{(n)}$ *has the same Parikh vector as* $\mu(\varphi^{2n-1}(012))$ *for* $n \geq 1$.

($\mathcal{B}$) *The shortest return word* $\hat{r}_B^{(n)}$ *to* $v_B^{(n)}$ *has the same Parikh vector as* $\mu(\varphi^{2n}(012))$.

($\mathcal{C}$) *The shortest return word* $\hat{r}_C^{(n)}$ *to* $v_C^{(n)}$ *has the same Parikh vector as* $\mu(\varphi^{2n}(01))$ *for* $n \geq 1$.

($\mathcal{D}$) *The shortest return word* $\hat{r}_D^{(n)}$ *to* $v_D^{(n)}$ *has the same Parikh vector as* $\mu(\varphi^{2n+1}(01))$.

Proof. We will prove case ($\mathcal{A}$). The other cases are similar. Let us consider $n \geq 1$ and the bispecial factor
$$v_A^{(n)} = \mu(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0))01 = \mu(w_A^{(n)})01.$$
Using Theorem 8.9, we know that the shortest return word to $w_A^{(n)}$ has the same Parikh vector as $\varphi^{2n-1}(012)$, moreover the shortest return word is a prefix of all of the return words.

Using Lemma 10.4, we obtain that the shortest return word to $\mu(w_A^{(n)})$ has the same Parikh vector as $\mu(\varphi^{2n-1}(012))$. Using Lemma 7.8 twice, we obtain that the shortest return word to $\mu(w_A^{(n)})01$ has the same Parikh vector as $\mu(\varphi^{2n-1}(012))$. □

10.3 The critical exponent of $\mu(\mathbf{p})$

Using Theorem 2.2 and the description of bispecial factors from Theorem 10.3 and of their shortest return words from Theorem 10.5, we obtain the following formula for the critical exponent of $\mu(\mathbf{p})$.

$$\mathrm{E}(\mu(\mathbf{p})) = 1 + \max\{A, B, C, D, F\} ,$$

where

$$A = \sup\left\{\frac{|v_A^{(n)}|}{|\hat{r}_A^{(n)}|} : n \geq 1\right\} = \sup\left\{\frac{|01\mu(1\varphi(012)\varphi^3(012)\ldots\varphi^{2n-1}(012))|}{|\mu(\varphi^{2n-1}(012))|} : n \geq 1\right\} ;$$

$$B = \sup\left\{\frac{|v_B^{(n)}|}{|\hat{r}_B^{(n)}|} : n \geq 0\right\} = \sup\left\{\frac{|000111\mu(012\varphi^2(012)\varphi^4(012)\ldots\varphi^{2n}(012))|}{|\mu(\varphi^{2n}(012))|} : n \geq 0\right\} ;$$

$$C = \sup\left\{\frac{|v_C^{(n)}|}{|\hat{r}_C^{(n)}|} : n \geq 1\right\} = \sup\left\{\frac{|\mu(01\varphi^2(01)\varphi^4(01)\ldots\varphi^{2n}(01))|}{|\mu(\varphi^{2n}(01))|} : n \geq 1\right\} ;$$

$$D = \sup\left\{\frac{|v_D^{(n)}|}{|\hat{r}_D^{(n)}|} : n \geq 0\right\} = \sup\left\{\frac{|00001111\mu(\varphi(01)\varphi^3(01)\ldots\varphi^{2n+1}(01))|}{|\mu(\varphi^{2n+1}(01))|} : n \geq 0\right\} ;$$

$$F = \max\left\{\frac{|w|}{|r|} : w \text{ bispecial factor in } \mu(\mathbf{p}) \text{ of length at most 12 and } r \text{ its shortest return word}\right\} .$$

Theorem 10.6. *The critical exponent of $\mu(\mathbf{p})$ is equal to*

$$\mathrm{E}(\mu(\mathbf{p})) = \frac{28}{11}.$$

Proof. To calculate the critical exponent of $\mu(\mathbf{p})$, we need to examine the values of A, B, C, D and F.

1. To determine the value of F, we need to find all bispecial factors of length at most 12 and their shortest return words. The bispecial factors are

   - 1 with the shortest return word 1.
   - 0 with the shortest return word 0.
   - 10 with the shortest return word 10.
   - 01 with the shortest return word 01.
   - 010 with the shortest return word 01.
   - 1001 with the shortest return word 1001.
   - 011001 with the shortest return word 0110.
   - 100101 with the shortest return word 10010.
   - 01100101 with the shortest return word 011001.
   - 1001011001 with the shortest return word 1001011001.

   Therefore, $F = \max\left\{1, \frac{3}{2}, \frac{6}{5}, \frac{8}{6}\right\} = \frac{3}{2} < \frac{17}{11}$.

2. Computation of A and B.

   Denote $a_n := |\mu(\varphi^n(012))| = (6, 4, 1)(M_\varphi)^n(1, 1, 1)^T$. Then using Lemma 8.10 with initial conditions $a_0 = 11, a_1 = 21$ and $a_2 = 36$, we obtain the explicit solution

$$a_n = A_1\beta^n + B_1\lambda_1^n + C_1\lambda_2^n,$$

where

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

and

$$\begin{aligned}
A_1 &= \frac{11|\lambda_1|^2 - 42\,\mathrm{Re}(\lambda_1) + 36}{|\beta - \lambda_1|^2} \doteq 11.53075\,; \\
B_1 &= \frac{11\beta\lambda_2 - 21(\beta + \lambda_2) + 36}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq -0.26538 - 0.55714i\,; \\
C_1 &= \overline{B_1}\,.
\end{aligned}$$

Let us show that $A \leq \frac{17}{11}$. We have to show for all $n \geq 1$ that

$$\begin{aligned}
\frac{6 + A_1 \sum_{k=1}^{n} \beta^{2k-1} + B_1 \sum_{k=1}^{n} \lambda_1^{2k-1} + C_1 \sum_{k=1}^{n} \lambda_2^{2k-1}}{A_1\beta^{2n-1} + B_1\lambda_1^{2n-1} + C_1\lambda_2^{2n-1}} &\leq^? \frac{17}{11}, \\
66 + 11A_1 \sum_{k=1}^{n} \beta^{2k-1} + 22\,\mathrm{Re}\left(B_1 \sum_{k=1}^{n} \lambda_1^{2k-1}\right) &\leq^? 17A_1\beta^{2n-1} + 34\,\mathrm{Re}(B_1\lambda_1^{2n-1}), \\
66 + 11A_1 \sum_{k=1}^{n-1} \beta^{2k-1} + 22\,\mathrm{Re}\left(B_1 \sum_{k=1}^{n-1} \lambda_1^{2k-1}\right) &\leq^? 6A_1\beta^{2n-1} + 12\,\mathrm{Re}(B_1\lambda_1^{2n-1}), \\
66 + 11A_1\left(\frac{\beta^{2n-1}}{\beta^2 - 1} - \frac{\beta}{\beta^2 - 1}\right) + 22\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\leq^? 6A_1\beta^{2n-1} + 12\,\mathrm{Re}(B_1\lambda_1^{2n-1}).
\end{aligned}$$

Since

$$\frac{11}{\beta^2 - 1} \leq 6,$$

we need to prove the inequality in the form

$$66 + 22\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) \leq^? 11A_1 \frac{\beta}{\beta^2 - 1} + 12\,\mathrm{Re}(B_1\lambda_1^{2n-1}).$$

For the left side, we can write for $n \geq 1$

$$\begin{aligned}
66 + 22\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\leq 66 + 22|B_1||\lambda_1| \frac{|\lambda_1|^{2n-2} + 1}{|\lambda_1^2 - 1|} \\
&\leq 66 + 22|B_1||\lambda_1| \frac{2}{|\lambda_1^2 - 1|}.
\end{aligned}$$

For the right side, we can write for $n \geq 1$

$$\begin{aligned}
11A_1 \frac{\beta}{\beta^2 - 1} + 12\,\mathrm{Re}(B_1\lambda_1^{2n-1}) &\geq 11A_1 \frac{\beta}{\beta^2 - 1} - 12|B_1||\lambda_1|^{2n-1} \\
&\geq 11A_1 \frac{\beta}{\beta^2 - 1} - 12|B_1||\lambda_1|.
\end{aligned}$$

Since the inequality

$$66 + 22|B_1||\lambda_1| \frac{2}{|\lambda_1^2 - 1|} \leq 11A_1 \frac{\beta}{\beta^2 - 1} - 12|B_1||\lambda_1|$$

holds true for given values, we obtain $A \leq \frac{17}{11}$.

Next, we will show that $B = \frac{17}{11}$.

Since for $n = 0$ we have $\frac{|v_B^{(0)}|}{|\hat{r}_B^{(0)}|} = \frac{6+11}{11} = \frac{17}{11}$, it remains to show that for all $n \geq 1$

$$
\begin{aligned}
\frac{6 + A_1 \sum_{k=0}^{n} \beta^{2k} + B_1 \sum_{k=0}^{n} \lambda_1^{2k} + C_1 \sum_{k=0}^{n} \lambda_2^{2k}}{A_1\beta^{2n} + B_1\lambda_1^{2n} + C_1\lambda_2^{2n}} &\leq^? \frac{17}{11}, \\
66 + 11A_1 \sum_{k=0}^{n-1} \beta^{2k} + 22\,\mathrm{Re}\left(B_1 \sum_{k=0}^{n-1} \lambda_1^{2k}\right) &\leq^? 6A_1\beta^{2n} + 12\,\mathrm{Re}\left(B_1\lambda_1^{2n}\right), \\
66 + 11A_1 \frac{\beta^{2n} - 1}{\beta^2 - 1} + 22\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right) &\leq^? 6A_1\beta^{2n} + 12\,\mathrm{Re}\left(B_1\lambda_1^{2n}\right).
\end{aligned}
$$

Now, we need to be more careful with the approximations, we will therefore prove the inequality in the form

$$
66 + 22\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right) \leq^? 11A_1 \frac{1}{\beta^2 - 1} + A_1\beta^{2n}\left(6 - \frac{11}{\beta^2 - 1}\right) + 12\,\mathrm{Re}\left(B_1\lambda_1^{2n}\right).
$$

For the left side, we can write for $n \geq 1$

$$
\begin{aligned}
66 + 22\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right) &\leq 66 + 22|B_1| \frac{|\lambda_1|^{2n} + 1}{|\lambda_1^2 - 1|} \\
&\leq 66 + 22|B_1| \frac{1 + |\lambda_1|^2}{|\lambda_1^2 - 1|}.
\end{aligned}
$$

For the right side, we can write for $n \geq 1$

$$
\begin{aligned}
A_1 \frac{11}{\beta^2 - 1} + A_1\beta^{2n}\left(6 - \frac{11}{\beta^2 - 1}\right) + 12\,\mathrm{Re}\left(B_1\lambda_1^{2n}\right) &\geq A_1 \frac{11}{\beta^2 - 1} + A_1\beta^2\left(6 - \frac{11}{\beta^2 - 1}\right) - 12|B_1||\lambda_1|^{2n} \\
&\geq A_1 \frac{11}{\beta^2 - 1} + A_1\beta^2\left(6 - \frac{11}{\beta^2 - 1}\right) - 12|B_1||\lambda_1|^2.
\end{aligned}
$$

Since the inequality

$$
66 + 22|B_1| \frac{1 + |\lambda_1|^2}{|\lambda_1^2 - 1|} \leq A_1 \frac{11}{\beta^2 - 1} + A_1\beta^2\left(6 - \frac{11}{\beta^2 - 1}\right) - 12|B_1||\lambda_1|^2
$$

holds true for given values and the bound is attained for $n = 0$, we conclude $B = \frac{17}{11}$.

3. Computation of C and D.

   Denote $b_n := |\mu(\varphi^n(01))| = (6, 4, 1)(M_\varphi)^n(1, 1, 0)^T$. Then using Lemma 8.4 with initial conditions $b_0 = 10, b_1 = 15$ and $b_2 = 26$, we get the explicit solution in the form

$$
b_n = A_2\beta^n + B_2\lambda_1^n + C_2\lambda_2^n,
$$

   where

$$
\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}
$$

and

$$
\begin{aligned}
A_2 &= \frac{10|\lambda_1|^2 - 30\,\mathrm{Re}(\lambda_1) + 26}{|\beta - \lambda_1|^2} \doteq 8.70431\,;\\
B_2 &= \frac{10\beta\lambda_2 - 15(\beta + \lambda_2) + 26}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq 0.64785 + 0.29119i\,;\\
C_2 &= \overline{B_2}\,.
\end{aligned}
$$

The computation for $C \leq \frac{17}{11}$ is the same as for B, only we substitute A_1, B_1 and C_1 by A_2, B_2 and C_2 and the offset 6 by 0. Since the case for $n = 0$ falls under the computation of F, it is enough to check the inequality

$$
0 + 22|B_2|\frac{1 + |\lambda_1|^2}{|\lambda_1^2 - 1|} \quad \leq \quad A_2\frac{11}{\beta^2 - 1} + A_2\beta^2\left(6 - \frac{11}{\beta^2 - 1}\right) - 12|B_2||\lambda_1|^2,
$$

which holds for given values.

Let us show that $D \leq \frac{17}{11}$**.**

We have to show for $n \geq 1$ that

$$
\begin{aligned}
\frac{8 + A_2\sum_{k=1}^{n}\beta^{2k-1} + B_2\sum_{k=1}^{n}\lambda_1^{2k-1} + C_2\sum_{k=1}^{n}\lambda_2^{2k-1}}{A_2\beta^{2n-1} + B_2\lambda_1^{2n-1} + C_2\lambda_2^{2n-1}} &\leq^? \quad \frac{17}{11},\\
88 + 11A_2\left(\frac{\beta^{2n-1}}{\beta^2 - 1} - \frac{\beta}{\beta^2 - 1}\right) + 22\,\mathrm{Re}\left(B_2\lambda_1\frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\leq^? \quad 6A_2\beta^{2n-1} + 12\,\mathrm{Re}(B_2\lambda_1^{2n-1}).
\end{aligned}
$$

Since

$$
\frac{11}{\beta^2 - 1} \leq 6,
$$

we need to prove the inequality in the form

$$
88 + 22\,\mathrm{Re}\left(B_2\lambda_1\frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) \quad \leq^? \quad 11A_2\frac{\beta}{\beta^2 - 1} + 12\,\mathrm{Re}(B_2\lambda_1^{2n-1}).
$$

For the left side, we now need to be more careful with the approximations. We can write for $n \geq 1$

$$
\begin{aligned}
88 + 22\,\mathrm{Re}\left(B_2\lambda_1\frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) \quad &\leq \quad 88 + 22\,\mathrm{Re}\left(\frac{B_2\lambda_1}{1 - \lambda_1^2}\right) + 22|B_2|\frac{|\lambda_1|^{2n-1}}{|1 - \lambda_1^2|}\\
&\leq \quad 88 + 22\,\mathrm{Re}\left(\frac{B_2\lambda_1}{1 - \lambda_1^2}\right) + 22|B_2|\frac{|\lambda_1|}{|1 - \lambda_1^2|}.
\end{aligned}
$$

For the right side, we can write for $n \geq 1$

$$
\begin{aligned}
11A_2\frac{\beta}{\beta^2 - 1} + 12\,\mathrm{Re}(B_2\lambda_1^{2n-1}) \quad &\geq \quad 11A_2\frac{\beta}{\beta^2 - 1} - 12|B_2||\lambda_1|^{2n-1}\\
&\geq \quad 11A_2\frac{\beta}{\beta^2 - 1} - 12|B_2||\lambda_1|.
\end{aligned}
$$

Since the inequality

$$
88 + 22\,\mathrm{Re}\left(\frac{B_2\lambda_1}{1 - \lambda_1^2}\right) + 22|B_2|\frac{|\lambda_1|}{|1 - \lambda_1^2|} \quad \leq \quad 11A_2\frac{\beta}{\beta^2 - 1} - 12|B_2||\lambda_1|
$$

holds true for the given values, we obtain $D \leq \frac{17}{11}$.

We have shown that $\max\{A, B, C, D\} = B = \frac{17}{11}$, and $F < \frac{17}{11}$. Consequently, $\mathrm{E}(\mu(\mathbf{p})) = 1 + \max\{A, B, C, D, F\} = \frac{28}{11}$. □

11 The infinite word $\psi(\mathbf{p})$

Since the examination of the word $\psi(\mathbf{p})$ proceeds analogously as for $\nu(\mathbf{p})$ and $\mu(\mathbf{p})$, we will use the same notation for bispecial factors, their shortest return words and other variables as in the previous chapter. We will not use any variables from Chapters 9 and 10.

The morphism ψ has the form:

$$\begin{aligned}\psi : 0 &\to 011001,\\ 1 &\to 0,\\ 2 &\to 01101.\end{aligned}$$

Therefore, ψ is injective and

$$\psi(\mathbf{p}) = 0110010011010011001011010011001001100101 1010\cdots$$

Lemma 11.1. *Any factor v of $\psi(\mathbf{p})$ of length at least five has a synchronization point.*

Proof. Similarly to Lemma 9.1, it is enough to show that any factor of length five has a synchronization point. The factors are:

$\bullet 01100$	$01 \bullet 0 \bullet 01$	$1101 \bullet 0$	$1 \bullet 0110$
$11001\bullet$	$1 \bullet 0 \bullet 011$	$101 \bullet 0 \bullet 0$	
$1001 \bullet 0$	$0 \bullet 0110$	$001 \bullet 01$	
$001 \bullet 0 \bullet 0$	$\bullet 01101\bullet$	$01 \bullet 011$	

where we highlighted the synchronization points. □

Using Lemma 11.1 and Theorem 7.7, we deduce that

$$\mathrm{E}^*(\psi(\mathbf{p})) = \mathrm{E}^*(\mathbf{p}).$$

11.1 Bispecial factors in $\psi(\mathbf{p})$

Using Lemma 11.1, Lemma 7.6, and the form of ψ with the knowledge of possible left and right extension in $\mathbf{p}$, we obtain the following Lemma.

Lemma 11.2. *Let $v \in \mathcal{L}(\psi(\mathbf{p}))$ be a bispecial factor in $\psi(\mathbf{p})$ of length at least 11. Then one of the following items is true.*

1. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 01\psi(w)0110$ and $0w, 2w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*
2. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \psi(w)0$ and $0w, 1w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
3. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 01\psi(w)0$ and $0w, 2w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
4. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \psi(w)0110$ and $0w, 1w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*

Theorem 11.3. *Let $v \in \mathcal{L}(\psi(\mathbf{p}))$. If v is a bispecial factor of length at least 11, then v has one of the following forms:*

$\mathcal{A}$)
$$v_A^{(n)} = 01\psi(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0))0110 \qquad \text{for } n \geq 1.$$
The Parikh vector of $v_A^{(n)}$ *is the same as that of the word* $000111\psi(1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012))$.

$\mathcal{B}$)
$$v_B^{(n)} = \psi(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0)0 \qquad \text{for } n \geq 0.$$
The Parikh vector of $v_B^{(n)}$ *is the same as that of the word* $0\psi(012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012))$.

$\mathcal{C}$)
$$v_C^{(n)} = 01\psi(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0)0 \qquad \text{for } n \geq 1.$$
The Parikh vector of $v_C^{(n)}$ *is the same as of the word* $001\psi(01\varphi^2(01)\varphi^4(01)\cdots\varphi^{2n}(01))$.

$\mathcal{D}$)
$$v_D^{(n)} = \psi(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n+1}(0)\varphi^{2n-1}(0)\cdots\varphi(0))0110 \qquad \text{for } n \geq 0.$$
The Parikh vector of $v_D^{(n)}$ *is the same as of the word* $0011\psi(\varphi(01)\varphi^3(01)\cdots\varphi^{2n+1}(01))$.

11.2 The shortest return words to bispecial factors in $\psi(\mathbf{p})$

Lemma 11.4. *Let w be a bispecial factor in* $\mathbf{p}$, $|w| \geq 2$, *and v its return word.*

- *If w has right extensions* $\{1, 2\}$, *then $\psi(v)$ is a return word to $\psi(w)$.*
- *If w has right extensions* $\{0, 2\}$, *then $\psi(v)$ is a return word to $\psi(w)0110$*

Proof. Any bispecial factor in $\mathbf{p}$ of length at least two either starts in 10 or in 2. Therefore, $\psi(w)$ starts in $\bullet 0011001$ or $\bullet 01101$.

- If w has right extensions $\{1, 2\}$, then it ends in 0. Therefore $\psi(w)$ ends in $011001\bullet$ and has following synchronization points $\bullet\psi(w)\bullet$. Since vw has w as a prefix, certainly $\psi(v)\psi(w)$ contains $\psi(w)$ as a prefix. It cannot contain $\psi(w)$ in the middle, because then, from injectivity of ψ, the factor vw would contain w in the middle.
- If w has right extension $\{0, 2\}$, then it ends in $\varphi(0) = 01$. Therefore, $\psi(w)$ ends in 0 and is always followed by 0110. Combining this, $\psi(w)0110$ has following synchronization points $\bullet\psi(w) \bullet 0110$.

  Now, consider any occurrence of vw and denote $a \in \{0, 2\}$ the following letter, then $\psi(v)\psi(w)0110$ is a prefix of $\psi(vwa)$. Since vw contains w as a prefix and as a suffix, then $\psi(v)\psi(w)0110$ contains $\psi(w)0110$ as a prefix and as a suffix, too.

  From the position of synchronization points, $\psi(v)\psi(w)0110$ cannot contain $\psi(w)0110$ somewhere in the middle because in such a case, by injectivity of ψ, vw would contain w also somewhere in the middle. □

Applying Lemma 11.4, Lemma 7.8 and Theorem 8.9, we have the following description of the shortest return words to bispecial factors.

Theorem 11.5. *The shortest return words to bispecial factors of length at least 11 in* $\psi(\mathbf{p})$ *have the following properties.*

$\mathcal{A}$) *The shortest return word* $r_A^{(n)}$ *to* $v_A^{(n)}$, $n \geq 1$, *has the same Parikh vector as the factor* $\psi(\varphi^{2n-1}(012))$.

$\mathcal{B}$) *The shortest return word* $r_B^{(n)}$ *to* $v_B^{(n)}$, $n \geq 0$, *has the same Parikh vector as the factor* $\psi(\varphi^{2n}(012))$.

C) The shortest return word $r_C^{(n)}$ to $v_C^{(n)}$, $n \geq 1$, has the same Parikh vector as the factor $\psi(\varphi^{2n}(01))$.

$\mathcal{D}$) The shortest return word $r_D^{(n)}$ to $v_D^{(n)}$, $n \geq 0$, has the same Parikh vector as the factor $\psi(\varphi^{2n+1}(01))$.

Proof. We will prove case ($\mathcal{A}$). The other cases are similar. Let us consider $n \geq 1$ and the bispecial factor

$$v_A^{(n)} = 01\psi(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0))0110 = 01\psi(w_A^{(n)})0110,$$

where the bispecial factor $w_A^{(n)}$ has right extensions $0, 2$. Using Theorem 8.9, we know that the shortest return word to $w_A^{(n)}$ has the same Parikh vector as $\varphi^{2n-1}(012)$, moreover the shortest return word is a prefix of all of the return words.

Using Lemma 11.4, the second item, we obtain that the shortest return word to $\psi(w_A^{(n)})0110$ has the same Parikh vector as $\psi(\varphi^{2n-1}(012))$. Using Lemma 7.8 twice, we obtain that the shortest return word to $01\psi(w_A^{(n)})0110$ has the same Parikh vector as $\psi(\varphi^{2n-1}(012))$.

□

11.3 The critical exponent of $\psi(\mathbf{p})$

Using Theorem 2.2 and the description of bispecial factors from Theorem 11.3 and of their shortest return words from Theorem 11.5, we obtain the following formula for the critical exponent of $\psi(\mathbf{p})$.

$$\mathrm{E}(\psi(\mathbf{p})) = 1 + \max\{A, B, C, D, F\},$$

where

$$A = \sup\left\{\frac{|v_A^{(n)}|}{|r_A^{(n)}|} : n \geq 1\right\} = \sup\left\{\frac{|000111\psi(1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012))|}{|\psi(\varphi^{2n-1}(012))|} : n \geq 1\right\};$$

$$B = \sup\left\{\frac{|v_B^{(n)}|}{|r_B^{(n)}|} : n \geq 0\right\} = \sup\left\{\frac{|0\psi(012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012))|}{|\psi(\varphi^{2n}(012))|} : n \geq 0\right\};$$

$$C = \sup\left\{\frac{|v_C^{(n)}|}{|r_C^{(n)}|} : n \geq 1\right\} = \sup\left\{\frac{|001\psi(01\varphi^2(01)\varphi^4(01)\cdots\varphi^{2n}(01))|}{|\psi(\varphi^{2n}(01))|} : n \geq 1\right\};$$

$$D = \sup\left\{\frac{|v_D^{(n)}|}{|r_D^{(n)}|} : n \geq 0\right\} = \sup\left\{\frac{|0011\psi(\varphi(01)\varphi^3(01)\cdots\varphi^{2n+1}(01))|}{|\psi(\varphi^{2n+1}(01))|} : n \geq 0\right\};$$

$$F = \left\{\frac{|w|}{|r|} : w \text{ bispecial factor in } \psi(\mathbf{p}) \text{ of length at most ten, } r \text{ the shortest return word to } w\right\}.$$

Theorem 11.6. *The critical exponent of $\psi(\mathbf{p})$ is equal to*

$$\mathrm{E}(\psi(\mathbf{p})) = \mathrm{E}^*(\mathbf{p}) = 1 + \frac{\beta^2}{\beta^2 - 1},$$

where β is the real root of the polynomial $t^3 - 2t^2 + t - 1$, i.e. $\beta \doteq 1.75488$.

Proof. To calculate the critical exponent of $\psi(\mathbf{p})$, we need to examine the values of A, B, C, D and F. First, let us point out that the inequality $\mathrm{E}(\psi(\mathbf{p})) \geq \mathrm{E}^*(\psi(\mathbf{p})) = \mathrm{E}^*(\mathbf{p})$ follows from the definition. Therefore, it is sufficient to show $\max\{A, B, C, D, F\} \leq \frac{\beta^2}{\beta^2-1}$.

1. To determine the value of F, we need to find all bispecial factors of length at most ten and their shortest return words. The bispecial factors are:

- 1 with the shortest return word 1.
- 0 with the shortest return word 0.
- 10 with the shortest return word 10.
- 01 with the shortest return word 01.
- 010 with the shortest return word 01011.
- 101 with the shortest return word 101.
- 1001 with the shortest return word 100.
- 0110 with the shortest return words 011001, 011010.
- 0100110 with the shortest return word 010011.
- 0100110010 with the shortest return word 0100110.

Therefore, $F = \max\left\{1, \frac{3}{5}, \frac{4}{3}, \frac{2}{3}, \frac{7}{6}, \frac{10}{7}\right\} = \frac{10}{7} \doteq 1.43 < \frac{\beta^2}{\beta^2-1} \doteq 1.48$.

2. Computation of A and B.

Denote $a_n := |\psi(\varphi^n(012))| = (6,1,5)(M_\varphi)^n(1,1,1)^T$. Then using Lemma 8.10 with initial conditions $a_0 = 12, a_1 = 19$ and $a_2 = 32$, we obtain the explicit solution

$$a_n = A_1\beta^n + B_1\lambda_1^n + C_1\lambda_2^n,$$

where

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

and

$$\begin{aligned} A_1 &= \frac{12|\lambda_1|^2 - 38\,\mathrm{Re}(\lambda_1) + 32}{|\beta - \lambda_1|^2} \doteq 10.61753\,; \\ B_1 &= \frac{12\beta\lambda_2 - 19(\beta + \lambda_2) + 32}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq 0.69124 - 0.13298i\,; \\ C_1 &= \overline{B_1}\,. \end{aligned}$$

Let us show that $A \le \frac{\beta^2}{\beta^2-1}$.

Since for $n = 1$, we have $\frac{7+a_1}{a_1} = 1 + \frac{7}{19} < \frac{\beta^2}{\beta^2-1}$, we have to show for all $n \ge 2$ that

$$\begin{aligned} \frac{7 + A_1\sum_{k=1}^n \beta^{2k-1} + B_1\sum_{k=1}^n \lambda_1^{2k-1} + C_1\sum_{k=1}^n \lambda_2^{2k-1}}{A_1\beta^{2n-1} + B_1\lambda_1^{2n-1} + C_1\lambda_2^{2n-1}} &\le^? \quad \frac{\beta^2}{\beta^2-1} \\ (\beta^2-1)\left(7 + 2\,\mathrm{Re}\left(B_1\lambda_1\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right)\right) + A_1\beta^{2n+1} - A_1\beta &\le^? \quad 2\beta^2\,\mathrm{Re}(B_1\lambda_1^{2n-1}) + A_1\beta^{2n+1} \\ (\beta^2-1)\left(7 + 2\,\mathrm{Re}\left(B_1\lambda_1\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right)\right) &\le^? \quad 2\beta^2\,\mathrm{Re}(B_1\lambda_1^{2n-1}) + A_1\beta\,. \end{aligned}$$

Now, on the one hand, for $n \ge 2$, we have

$$\begin{aligned} (\beta^2-1)\left(7 + 2\,\mathrm{Re}\left(B_1\lambda_1\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right)\right) &\le (\beta^2-1)\left(7 + 2|B_1||\lambda_1|\frac{|\lambda_1|^{2n}+1}{|\lambda_1^2-1|}\right) \\ &\le (\beta^2-1)\left(7 + 2|B_1||\lambda_1|\frac{|\lambda_1|^4+1}{|\lambda_1^2-1|}\right). \end{aligned}$$

On the other hand for $n \geq 2$,

$$\begin{aligned} 2\beta^2 \operatorname{Re}(B_1\lambda_1^{2n-1}) + A_1\beta &\geq A_1\beta - 2\beta^2|B_1||\lambda_1|^{2n-1} \\ &\geq A_1\beta - 2\beta^2|B_1||\lambda_1|^3. \end{aligned}$$

And if we substitute the values, we get

$$(\beta^2-1)\left(7 + 2|B_1||\lambda_1|\frac{|\lambda_1|^4+1}{|\lambda_1^2-1|}\right) \leq A_1\beta - 2\beta^2|B_1||\lambda_1|^3.$$

Therefore $A \leq \frac{\beta^2}{\beta^2-1}$.

Next, we will show that $B \leq \frac{\beta^2}{\beta^2-1}$.

Since for $n = 0$ we get $\frac{1+a_0}{a_0} = \frac{1+12}{12} < \frac{\beta^2}{\beta^2-1}$ and for $n = 1$, we get $\frac{1+a_0+a_2}{a_2} = 1 + \frac{1+12}{32} < \frac{\beta^2}{\beta^2-1}$, it remains to show that for all $n \geq 2$

$$\begin{aligned} \frac{1 + A_1\sum_{k=0}^{n}\beta^{2k} + B_1\sum_{k=0}^{n}\lambda_1^{2k} + C_1\sum_{k=0}^{n}\lambda_2^{2k}}{A_1\beta^{2n} + B_1\lambda_1^{2n} + C_1\lambda_2^{2n}} &\leq^? \quad \frac{\beta^2}{\beta^2-1} \\ (\beta^2-1)\left(1 + 2\operatorname{Re}\left(B_1\frac{\lambda_1^{2n+2}-1}{\lambda_1^2-1}\right)\right) + A_1\beta^{2n+2} - A_1 &\leq^? \quad 2\beta^2\operatorname{Re}(B_1\lambda_1^{2n}) + A_1\beta^{2n+2} \\ (\beta^2-1)\left(1 + 2\operatorname{Re}\left(B_1\frac{\lambda_1^{2n+2}-1}{\lambda_1^2-1}\right)\right) &\leq^? \quad 2\beta^2\operatorname{Re}(B_1\lambda_1^{2n}) + A_1\,. \end{aligned}$$

Now, on the one hand, for all $n \geq 2$, we have

$$\begin{aligned} (\beta^2-1)\left(1 + 2\operatorname{Re}\left(B_1\frac{\lambda_1^{2n+2}-1}{\lambda_1^2-1}\right)\right) &\leq (\beta^2-1)\left(1 + 2\operatorname{Re}\left(\frac{B_1}{1-\lambda_1^2}\right) - 2\operatorname{Re}\left(B_1\frac{\lambda_1^{2n+2}}{1-\lambda_1^2}\right)\right) \\ &\leq (\beta^2-1)\left(1 + 2\operatorname{Re}\left(\frac{B_1}{1-\lambda_1^2}\right) + 2|B_1|\frac{|\lambda_1^{2n+2}|}{|1-\lambda_1^2|}\right) \\ &\leq (\beta^2-1)\left(1 + 2\operatorname{Re}\left(\frac{B_1}{1-\lambda_1^2}\right) + 2|B_1|\frac{|\lambda_1|^6}{|1-\lambda_1^2|}\right). \end{aligned}$$

On the other hand,

$$2\beta^2\operatorname{Re}(B_1\lambda_1^{2n}) + A_1 \geq A_1 - 2\beta^2|B_1||\lambda_1|^4.$$

Substituting the values proves that

$$(\beta^2-1)\left(1 + 2\operatorname{Re}\left(\frac{B_1}{1-\lambda_1^2}\right) + 2|B_1|\frac{|\lambda_1|^6}{|1-\lambda_1^2|}\right) < A_1 - 2\beta^2|B_1||\lambda_1|^4.$$

Therefore, $B \leq \frac{\beta^2}{\beta^2-1}$.

3. Computation of C and D.

   Denote $b_n := |\psi(\varphi^n(01))| = (6,1,5)(M_\varphi)^n(1,1,0)^T$. Then using Lemma 8.4 with initial conditions $b_0 = 7, b_1 = 13$ and $b_2 = 25$, we get the explicit solution in the form

$$b_n = A_2\beta^n + B_2\lambda_1^n + C_2\lambda_2^n,$$

where
$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$
and
$$\begin{aligned} A_2 &= \frac{7|\lambda_1|^2 - 13\,\mathrm{Re}(\lambda_1) + 25}{|\beta - \lambda_1|^2} \doteq 8.01494\,; \\ B_2 &= \frac{10\beta\lambda_2 - 15(\beta + \lambda_2) + 26}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq -0.50747 + 0.63155i\,; \\ C_2 &= \overline{B_2}\,. \end{aligned}$$

The computation for $C \le \frac{\beta^2}{\beta^2-1}$

It proceeds in the same way as for $B \le \frac{\beta^2}{\beta^2-1}$. So at first, we need to check the values for $n \in \{0, 1\}$: The case $n = 0$ falls under the computation of F and for $n = 1$, we get $\frac{3+c_0+c_2}{c_2} = 1 + \frac{3+7}{25} < \frac{\beta^2}{\beta^2-1}$. The rest is the same as for B, only we substitute A_1, B_1, C_1 by A_2, B_2, C_2 and the offset 1 by 3. Therefore, it remains to check the inequality
$$(\beta^2 - 1)\left(3 + 2\,\mathrm{Re}\left(\frac{B_2}{1 - \lambda_1^2}\right) + 2|B_2|\frac{|\lambda_1|^6}{|1 - \lambda_1^2|}\right) < A_2 - 2\beta^2|B_2||\lambda_1|^4,$$
which holds for given values and, therefore, $C \le \frac{\beta^2}{\beta^2-1}$.

To finish the proof, we need to show that $D \le \frac{\beta^2}{\beta^2-1}$. The computation for D is the same as for A, only we substitute A_1, B_1 and C_1 by A_2, B_2 and C_2 and the offset 7 by 4. Since for $n = 1$, we get $\frac{4+b_1}{b_1} = 1 + \frac{4}{13} < \frac{\beta^2}{\beta^2-1}$, it is enough to check the inequality
$$(\beta^2 - 1)\left(4 + 2|B_2||\lambda_1|\frac{|\lambda_1|^4 + 1}{|\lambda_1^2 - 1|}\right) \le A_2\beta - 2\beta^2|B_2||\lambda_1|^3,$$
which holds for given values. Therefore, $D \le \frac{\beta^2}{\beta^2-1}$.

We have shown that $1 + \max\{A, B, C, D, F\} \le \mathrm{E}^*(\psi(\mathbf{p}))$, and since $\mathrm{E}(\psi(\mathbf{p}) \ge \mathrm{E}^*(\psi(\mathbf{p}))$, we have $\mathrm{E}(\psi(\mathbf{p}) = \mathrm{E}^*(\psi(\mathbf{p})) = \mathrm{E}^*(\mathbf{p}) = 1 + \frac{\beta^2}{\beta^2-1}$. □

12 The infinite word $\xi(\mathbf{p})$

Since the examination of the word $\xi(\mathbf{p})$ proceeds analogously as for $\nu(\mathbf{p})$, $\mu(\mathbf{p})$ and $\psi(\mathbf{p})$, we will use the same notation for bispecial factors, their shortest return words and other variables as in the previous chapter. We will not use any values from Chapters 9, 10 or 11.

The morphism ξ has the form:

$$\begin{aligned} \xi : 0 &\to 01, \\ 1 &\to 0110, \\ 2 &\to 1. \end{aligned}$$

Therefore, ξ is injective and

$$\xi(\mathbf{p}) = 01011010110011011001011001101 10\cdots$$

Lemma 12.1. *Any factor v of $\xi(\mathbf{p})$ of length at least five has a synchronization point.*

Proof. Similarly to Lemma 9.1, it is enough to show that any factor of length five has a synchronization point. Since the factors $\bullet 011$, $01 \bullet 0$ and $0 \bullet 0$ each have a synchronization point, it is sufficient to check that any factor of $\xi(\mathbf{p})$ of length at least five must contain one of these factors. Going through all possible words in $\{0, 1\}^5$, we find that it contains either 111, which cannot be a factor of $\xi(\mathbf{p})$ or it contains one of the factors listed. $\square$

Remark. We need to work with factors of length at least five because, for example, the factor 1101 does not have a synchronization point.

Using Lemma 11.1 and Theorem 7.7, we deduce that

$$\mathrm{E}^*(\xi(\mathbf{p})) = \mathrm{E}^*(\mathbf{p}).$$

12.1 Bispecial factors in $\xi(\mathbf{p})$

Now, it is harder to determine how the left extensions $\{0, 2\}$ in $\mathbf{p}$ will project into $\xi(\mathbf{p})$, we will therefore at first prove the following lemma.

Lemma 12.2. *Let $w \neq \varepsilon$, $w \in \mathcal{L}(\mathbf{p})$ be a bispecial factor such that $0w, 2w \in \mathcal{L}(\mathbf{p})$. Then either $w = 1$ and both $1\xi(1)$ and $01\xi(1)$ are left special factors in $\xi(\mathbf{p})$, or $w = 10$ and both $1\xi(10)$ and $01\xi(10)$ are left special factors in $\xi(\mathbf{p})$, or $|w| \geq 3$ and $01\xi(w)$ is a left special factor in $\xi(\mathbf{p})$, while $1\xi(w)$ is not a left special factor in $\xi(\mathbf{p})$.*

Proof. From the form of bispecial factors in $\mathbf{p}$ with left extensions $0, 2$, we can see that either $w = 1$, $w = 10$ or w starts in $1\varphi^2(1) = 1021$.

If $w = 1$, then from $101, 021, 121 \in \mathcal{L}(\mathbf{p})$, we have that $001\xi(1), 011\xi(1), 101\xi(1) \in \mathcal{L}(\xi(\mathbf{p}))$ and both $1\xi(1)$ and $01\xi(1)$ are left special factors.

If $w = 10$, then from $1010, 0210, 1210 \in \mathcal{L}(\mathbf{p})$, we have that $001\xi(10), 011\xi(10), 101\xi(10) \in \mathcal{L}(\xi(\mathbf{p}))$ and both $1\xi(10)$ and $01\xi(10)$ are left special factors.

If w starts in 1021, then we can see that $02w \notin \mathcal{L}(\mathbf{p})$ since this word has $021021 = \varphi(2121) = \varphi^2(11)$ as a prefix and $11 \notin \mathcal{L}(\mathbf{p})$. Therefore, the only possible left extensions of w are $12w$, $10w$ and, therefore, $01\xi(w)$ is a left special factor in $\xi(\mathbf{p})$, while $1\xi(w)$ has only one possible left extension. $\square$

Remark. The cases $w = 1$ and $w = 10$ in the previous theorem generate bispecial factors in $\xi(\mathbf{p})$ of length at most $|01\xi(10)| = 8$, they will therefore be examined as short bispecial factors.

Lemma 12.3. *Let $v \in \mathcal{L}(\xi(\mathbf{p}))$ be a bispecial factor in $\xi(\mathbf{p})$ of length at least 11. Then one of the following items is true.*

1. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 01\xi(w)$ and $0w, 2w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*
2. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \xi(w)$ and $0w, 1w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
3. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 01\xi(w)$ and $0w, 2w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
4. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \xi(w)$ and $0w, 1w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*

Theorem 12.4. *Let $v \in \mathcal{L}(\xi(\mathbf{p}))$. If v is a bispecial factor of length at least 11, then v has one of the following forms:*

$\mathcal{A}$)

$$v_A^{(n)} = 01\xi(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0)) \qquad \textit{for } n \geq 1.$$

The Parikh vector of $v_A^{(n)}$ is the same as that of the word $01\xi(1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012))$.

$\mathcal{B}$)

$$v_B^{(n)} = \xi(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0) \qquad \textit{for } n \geq 0.$$

The Parikh vector of $v_B^{(n)}$ is the same as that of the word $\xi(012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012))$.

$\mathcal{C}$)

$$v_C^{(n)} = 01\xi(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0) \qquad \textit{for } n \geq 1.$$

The Parikh vector of $v_C^{(n)}$ is the same as of the word $01\xi(01\varphi^2(01)\varphi^4(01)\cdots\varphi^{2n}(01))$.

$\mathcal{D}$)

$$v_D^{(n)} = \xi(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n+1}(0)\varphi^{2n-1}(0)\cdots\varphi(0)) \qquad \textit{for } n \geq 0.$$

The Parikh vector of $v_D^{(n)}$ is the same as of the word $\xi(\varphi(01)\varphi^3(01)\cdots\varphi^{2n+1}(01))$.

12.2 The shortest return words to bispecial factors in $\xi(\mathbf{p})$

Lemma 12.5. *If w is a bispecial factor of* $\mathbf{p}$, $|w| \geq 4$, *and v is its return word, then $\xi(v)$ is a return word to $\xi(w)$.*

Proof. Since vw has w as a prefix, then $\xi(v)\xi(w)$ must have $\xi(w)$ as a prefix.

If w is a bispecial factor in $\mathbf{p}$, then w starts either in 1 or 2101 and ends in 10 or 01. Therefore, $\xi(w)$ starts in $\bullet 0110$ or $1 \bullet 0110$ and ends in $010110\bullet$ or $011001\bullet$.

- If $\xi(w)$ starts in $\bullet 011$ and ends in $010110\bullet$ or $011001\bullet$, then $\xi(w)$ has the following synchronization points $\bullet\xi(w)\bullet$. Therefore, by injectivity of ξ, $\xi(v)\xi(w)$ cannot have $\xi(w)$ somewhere in the middle since then vw would have w somewhere in the middle.
- If w starts in 2101, then we can see that the factor $w' = 2^{-1}w$ is not left special since the only left special factor with the first letter 1 in $\mathbf{p}$ starts in 102. This however means that $\xi(w)$ has the synchronization points $\bullet\xi(w)\bullet$ in this case, too. Applying the same reasoning as above, we observe that $\xi(v)\xi(w)$ cannot have $\xi(w)$ somewhere in the middle since then vw would have w somewhere in the middle. $\square$

Applying Lemma 12.5, Lemma 7.8 and Theorem 8.9, we have the following description of the shortest return words to bispecial factors.

Theorem 12.6. *The shortest return words to bispecial factors of length at least 11 in $\xi(\mathbf{p})$ have the following properties.*

$\mathcal{A}$) The shortest return word $r_A^{(n)}$ to $v_A^{(n)}$, $n \geq 1$, has the same Parikh vector as $\xi(\varphi^{2n-1}(012))$.

$\mathcal{B}$) The shortest return word $r_B^{(n)}$ to $v_B^{(n)}$, $n \geq 0$, has the same Parikh vector as $\xi(\varphi^{2n}(012))$.

C) The shortest return word $r_C^{(n)}$ to $v_C^{(n)}$, $n \geq 1$, has the same Parikh vector as $\xi(\varphi^{2n}(01))$.

$\mathcal{D}$) The shortest return word $r_D^{(n)}$ to $v_D^{(n)}$, $n \geq 0$, has the same Parikh vector as $\xi(\varphi^{2n+1}(01))$.

Proof. We will prove case ($\mathcal{A}$). The other cases are similar. Let us consider $n \geq 1$ and the bispecial factor

$$v_A^{(n)} = 01\xi(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0)) = 01\xi(w_A^{(n)}).$$

Using Theorem 8.9, we know that the shortest return word to $w_A^{(n)}$ has the same Parikh vector as $\varphi^{2n-1}(012)$, moreover the shortest return word is a prefix of all of the return words.

Using Lemma 12.5, we obtain that the shortest return word to $\xi(w_A^{(n)})$ has the same Parikh vector as $\xi(\varphi^{2n-1}(012))$. Using Lemma 7.8 twice, we obtain that the shortest return word to $01\xi(w_A^{(n)})$ has the same Parikh vector as $\xi(\varphi^{2n-1}(012))$. □

12.3 The critical exponent of $\xi(\mathbf{p})$

Using Theorem 2.2 and the description of bispecial factors from Theorem 12.4 and of their shortest return words from Theorem 12.6, we obtain the following formula for the critical exponent of $\xi(\mathbf{p})$.

$$\mathrm{E}(\xi(\mathbf{p})) = 1 + \max\{A, B, C, D, F\}\ ,$$

where

$$A = \sup\left\{\frac{|01\xi(1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012))|}{|\xi(\varphi^{2n-1}(012))|} : n \geq 1\right\}\ ;$$
$$B = \sup\left\{\frac{|\xi(012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012))|}{|\xi(\varphi^{2n}(012))|} : n \geq 0\right\}\ ;$$
$$C = \sup\left\{\frac{|01\xi(01\varphi^2(01)\varphi^4(01)\cdots\varphi^{2n}(01))|}{|\xi(\varphi^{2n}(01))|} : n \geq 1\right\}\ ;$$
$$D = \sup\left\{\frac{|\xi(\varphi(01)\varphi^3(01)\cdots\varphi^{2n+1}(01))|}{|\xi(\varphi^{2n+1}(01))|} : n \geq 0\right\}\ ;$$
$$F = \max\left\{\frac{|v|}{|r|} : v \text{ bispecial factor in } \xi(\mathbf{p}) \text{ of length at most ten and } r \text{ the shortest return word to } v\right\}.$$

Theorem 12.7. *The critical exponent of $\xi(\mathbf{p})$ is equal to*

$$\mathrm{E}(\xi(\mathbf{p})) = \frac{5}{2}.$$

Proof. To calculate the critical exponent of $\xi(\mathbf{p})$, we need to examine the values of A, B, C, D and F.

1. To determine the value of F, we need to find all bispecial factors of length at most ten and their shortest return words. The bispecial factors are

- 1 with the shortest return word 1.
- 0 with the shortest return word 0.
- 10 with the shortest return word 10
- 01 with the shortest return word 01.
- 101 with the shortest return word 10.
- 0110 with the shortest return word 011.
- 10110 with the shortest return word 10110
- 01101 with the shortest return words 011010110, 011011001
- 010110 with the shortest return word 01011
- 1011001 with the shortest return word 101100
- 01011001 with the shortest return word 010110

Therefore, $F = \max\left\{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{9}, \frac{6}{5}, \frac{7}{6}\right\} = \frac{3}{2}$.

The computation of B, C, D will proceed in the same way as for $\nu(\mathbf{p})$ in Chapter 9, Theorem 9.6, only the values of $A_1, A_2, B_1, B_2, C_1, C_2$ and the offsets will differ. We will therefore omit rewriting the approximations and write only the first and last steps, where the steps in-between are analogous as in Chapter 9.

2. Computation of A and B.

Denote $a_n := |\xi(\varphi^n(012))| = (2,4,1)(M_\varphi)^n(1,1,1)^T$. Then using Lemma 8.10 with initial conditions $a_0 = 7, a_1 = 13$ and $a_2 = 24$, we obtain the explicit solution

$$a_n = A_1\beta^n + B_1\lambda_1^n + C_1\lambda_2^n,$$

where

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

and

$$\begin{aligned} A_1 &= \frac{7|\lambda_1|^2 - 26\,\mathrm{Re}(\lambda_1) + 24}{|\beta - \lambda_1|^2} \doteq 7.70431\,; \\ B_1 &= \frac{7\beta\lambda_2 - 13(\beta + \lambda_2) + 24}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq -0.35215 + 0.29119i\,; \\ C_1 &= \overline{B_1}\,. \end{aligned}$$

Let us show that $A \le \frac{3}{2}$.

For $n = 1$, we obtain $\frac{6+a_1}{a_1} = \frac{19}{13} < \frac{3}{2}$.

Therefore, it is enough to show for all $n \ge 2$ that

$$\begin{aligned} \frac{6 + A_1\sum_{k=1}^n \beta^{2k-1} + B_1\sum_{k=1}^n \lambda_1^{2k-1} + C_1\sum_{k=1}^n \lambda_2^{2k-1}}{A_1\beta^{2n-1} + B_1\lambda_1^{2n-1} + C_1\lambda_2^{2n-1}} &\overset{?}{\le} \frac{3}{2}, \\ 12 + 2A_1\sum_{k=1}^{n}\beta^{2k-1} + 4\,\mathrm{Re}\left(B_1\sum_{k=1}^{n}\lambda_1^{2k-1}\right) &\overset{?}{\le} 3A_1\beta^{2n-1} + 6\,\mathrm{Re}(B_1\lambda_1^{2n-1}), \\ 12 + 2A_1\sum_{k=1}^{n-1}\beta^{2k-1} + 4\,\mathrm{Re}\left(B_1\sum_{k=1}^{n-1}\lambda_1^{2k-1}\right) &\overset{?}{\le} A_1\beta^{2n-1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}), \\ 12 + 2A_1\left(\frac{\beta^{2n-1}}{\beta^2 - 1} - \frac{\beta}{\beta^2 - 1}\right) + 4\,\mathrm{Re}\left(B_1\lambda_1\frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\overset{?}{\le} A_1\beta^{2n-1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}). \end{aligned}$$

$$12+4\,\mathrm{Re}\left(B_1\lambda_1\frac{1-\lambda_1^{2n-2}}{1-\lambda_1^2}\right) \quad \leq^? \; 2A_1\frac{\beta}{\beta^2-1}+A_1\beta^{2n-1}\left(1-\frac{2}{\beta^2-1}\right)+2\,\mathrm{Re}(B_1\lambda_1^{2n-1}).$$

For the left side, we can write for $n \geq 2$

$$\begin{aligned} 12+4\,\mathrm{Re}\left(B_1\lambda_1\frac{1-\lambda_1^{2n-2}}{1-\lambda_1^2}\right) &\leq 12+4|B_1||\lambda_1|\frac{|\lambda_1|^{2n-2}+1}{|\lambda_1^2-1|} \\ &\leq 12+4|B_1||\lambda_1|\frac{|\lambda_1|^2+1}{|\lambda_1^2-1|}. \end{aligned}$$

For the right side, we can write for $n \geq 2$

$$\begin{aligned} 2A_1\frac{\beta}{\beta^2-1}+A_1\beta^{2n-1}\left(1-\frac{2}{\beta^2-1}\right)+2\,\mathrm{Re}(B_1\lambda_1^{2n-1}) \quad &\geq \\ &\geq 2A_1\frac{\beta}{\beta^2-1}+A_1\beta^{2n-1}\left(1-\frac{2}{\beta^2-1}\right)-2|B_1||\lambda_1|^{2n-1} \\ &\geq 2A_1\frac{\beta}{\beta^2-1}+A_1\beta^{3}\left(1-\frac{2}{\beta^2-1}\right)-2|B_1||\lambda_1|^{3}. \end{aligned}$$

Since the inequality

$$12+4|B_1||\lambda_1|\frac{|\lambda_1|^2+1}{|\lambda_1^2-1|} \quad \leq \quad 2A_1\frac{\beta}{\beta^2-1}+A_1\beta^{3}\left(1-\frac{2}{\beta^2-1}\right)-2|B_1||\lambda_1|^{3}.$$

holds true for the given values, we obtain $A \leq \frac{3}{2}$.

Therefore, $A \leq \frac{3}{2}$.

Next, we will show that $B \leq \frac{3}{2}$.

For all $n \geq 0$ we have to show that

$$\frac{1+A_1\sum_{k=0}^n\beta^{2k}+B_1\sum_{k=0}^n\lambda_1^{2k}+C_1\sum_{k=0}^n\lambda_2^{2k}}{A_1\beta^{2n}+B_1\lambda_1^{2n}+C_1\lambda_2^{2n}} \quad \leq^? \quad \frac{3}{2}.$$

Using the same approximations as in Theorem 9.6, it is enough to check that the inequality

$$2+4|B_1|\frac{2}{|\lambda_1^2-1|} \quad \leq \quad A_1\frac{2}{\beta^2-1}-2|B_1|$$

holds true for given values. Therefore, $B \leq \frac{3}{2}$.

3. Computation of C and D.

   Denote $b_n := |\xi(\varphi^n(01))| = (2,4,1)(M_\varphi)^n(1,1,0)^T$. Then, using Lemma 8.4 with initial conditions $b_0 = 6, b_1 = 11$ and $b_2 = 18$, we obtain the explicit solution

$$b_n = A_2\beta^n + B_2\lambda_1^n + C_2\lambda_2^n,$$

   where

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

and

$$
\begin{aligned}
A_2 &= \frac{6|\lambda_1|^2 - 22\,\mathrm{Re}(\lambda_1) + 18}{|\beta - \lambda_1|^2} \doteq 5.81581\,; \\
B_2 &= \frac{6\beta\lambda_2 - 11(\beta + \lambda_2) + 18}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq 0.09210 - 0.51781i\,; \\
C_2 &= \overline{B_2}\,.
\end{aligned}
$$

First, we will show that $C \le \frac{3}{2}$.

The case $n = 0$ falls to the computation of F, and for $n = 1$, we obtain $\frac{2+b_0+b_2}{b_2} = \frac{13}{9} \doteq 1.44 < \frac{3}{2}$. Therefore, it suffices to show for all $n \ge 2$ that

$$
\frac{2 + A_2 \sum_{k=0}^{n} \beta^{2k} + B_2 \sum_{k=0}^{n} \lambda_1^{2k} + C_2 \sum_{k=0}^{n} \lambda_2^{2k}}{A_2\beta^{2n} + B_2\lambda_1^{2n} + C_2\lambda_2^{2n}} \quad \overset{?}{\le} \quad \frac{3}{2},
$$

Using the same approximations as in Theorem 9.6, it is enough to check that the inequality

$$
4 + 4\,\mathrm{Re}\left(\frac{B_2}{1 - \lambda_1^2}\right) + 4|B_2|\frac{|\lambda_1|^4}{|1 - \lambda_1^2|} \quad \le \quad A_2\frac{2}{\beta^2 - 1} - 2|B_2||\lambda_1|^4
$$

holds for given values. Therefore, we have proven that $C \le \frac{3}{2}$.

It remains to prove $D \le \frac{3}{2}$,

We need to check the inequality

$$
\frac{0 + A_2 \sum_{k=1}^{n} \beta^{2k-1} + B_2 \sum_{k=1}^{n} \lambda_1^{2k-1} + C_2 \sum_{k=1}^{n} \lambda_2^{2k-1}}{A_2\beta^{2n-1} + B_2\lambda_1^{2n-1} + C_2\lambda_2^{2n-1}} \quad \overset{?}{\le} \quad \frac{3}{2}.
$$

Using the same approximations as in Theorem 9.6, it is enough to check that the inequality

$$
0 + 4|B_2||\lambda_1|\frac{2}{|\lambda_1^2 - 1|} \quad \le \quad 2A_2\frac{\beta}{\beta^2 - 1} - 2|B_2||\lambda_1|,
$$

which is true for given values.

Therefore, $D \le \frac{3}{2}$.

We have shown that $\max\{A, B, C, D, F\} = F = \frac{3}{2}$. Therefore, $\mathrm{E}(\xi(\mathbf{p})) = 1 + F = \frac{5}{2}$. □

Conclusion

Conclusions

The first part of this thesis follows up on the research conducted in my bachelor thesis [31]. In the bachelor thesis, we focused on the minimal critical exponent of balanced sequences over a given alphabet. At first, we described how to compute the (asymptotic) critical exponent of a given balanced sequence using the knowledge of the length of bispecial factors and their return words. Then the computation was implemented to obtain two computer programs that calculate the (asymptotic) critical exponent for a given balanced sequence.

In this thesis, we expand the theory from [12] so that instead of computing the asymptotic critical exponent of a given sequence, we use the inequalities to find a balanced sequence that has low asymptotic critical exponent. To do this, we introduce a new tool – a graph of admissible tails that allows us to find all balanced sequences over a given alphabet that have an asymptotic critical exponent lower than a given bound.

We described the theory in our recently published paper [16]; therefore, we omit the proofs in this thesis. Using the theory, I designed and implemented the algorithms for generating the graph of admissible tails, including the reductions and searching the cycles in the graph. The implementation was done in C++ using object-oriented programming.

We used it in combination with the program from [31] to find the minimal asymptotic critical exponent for balanced sequences over a given alphabet of up to eleven letters. The results were quite interesting. For example, we learned that the minimum is attained and the period of θ is determined exactly and is only one possible (up to cyclic permutations). We have shown that for $d \in \{3, 4, 5\}$, the minimal asymptotic critical exponent coincides with the minimal critical exponent, while for $d \geq 6$, the minimal asymptotic critical exponent is strictly smaller than the minimal critical exponent.

Increasing the alphabet size proved difficult. Not only does the number of combinations of P and P' increase with alphabet size but also the minimal value of the asymptotic critical exponent decreases, which forces α to increase. All of this causes the graph of admissible tails to contain more vertices and more edges, which slows down the program. The main problem is similar to what we have seen for a 10-letter alphabet. As there are many vertices and edges, it happens that the graph of admissible tails cannot be reduced to obtain only one cycle but contains two or more interlocking cycles. Therefore, in our next work, we will need a different approach to solve this problem.

In the second part of the thesis, we at first examined the bispecials and return words in the sequence $\mathbf{p}$, which is the fixed point of the morphism φ defined in Chapter 8. Next, we calculated the critical exponent of some morphic images of the sequence $\mathbf{p}$, which have some nice properties, mainly they contain only a limited number of complemented factors or a limited number of palindromes while having a small critical exponent. To do so, we had to determine the form of bispecial factors and their return words in the sequence $\mathbf{p}$ and also find the relations between the bispecial factors in $\mathbf{p}$ and its morphic image under four different morphisms. After this, the problem was reduced to finding the maximal exponent of a small number of short factors and using estimates to prove inequalities for longer factors. These computations completed the proofs in [8, 14].

Although we managed to expand our knowledge of both the asymptotic and critical exponents of balanced sequences in our work, there are still some open questions.

- In [17], we conjectured that the infimum of the critical exponents of balanced sequences over d-letter alphabet is $\frac{d-1}{d-2}$ for $d \geq 11$. In the same paper, we managed to show that it cannot be lower for $d \geq 11$ and we found sequences over d-lettered alphabets, where $d \in \{11\} \cup \{2\delta : \delta \in \mathbb{N}, \delta \geq 6\}$ that attain the bound. Therefore, it remains to prove or refute our conjecture for d odd larger than 11.

- In [16] and this thesis, we found the infimum of asymptotic critical exponents of balanced sequences over d-letter alphabet with $d \in \{3, 4, \ldots, 10\}$. We also found sequences that attain the minimum and showed that the slope of the sequence attaining minimum is quadratic irrational and its period is unique up to cyclic permutations. It remains to find the minimal asymptotic critical exponent of balanced sequences over d-letter alphabets with d larger than 10.

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A Program

In this chapter, we will describe the algorithm for creating and processing the graph of admissible tails. The implementation was completed in C++ using object-oriented programming. Most of the program was written from scratch, including working with the linked list etc. In the thesis, we use pseudocode to describe the main idea behind the main functions.

The input parameters are P, P' and α, where P, resp. P', is the length of the period of a constant gap sequence $\mathbf{y}$, resp. $\mathbf{y}'$, and $\alpha > 0$. The output is a graph of $\left(1 + \frac{1}{\alpha}\right)$-admissible tails after reductions. We choose to work with $\alpha = \frac{1}{\beta}$ because for bigger alphabets, the minimal asymptotic critical exponent approaches 1 and therefore β soon becomes a very small number and rounding errors in the program might be significant. Also, for example, $\frac{1}{k} > \beta$ becomes $k < \alpha$, which is better for evaluation. The construction can be divided into six parts, each part will be discussed separately.

1. At first, two lists of rows are generated. Both lists consist of the representative of each class of equivalence $\sim_f$, where $f \in \{Y, Y'\}$, and the list of their equivalent rows as defined below in Definition A.1.
2. Using the prepared lists, we generate all classes of equivalence $[A]_\equiv$ with the rows of matrix A being the representatives of their class for $\sim_Y$, resp. $\sim_{Y'}$.
3. For each class $[A]_\equiv$, the interval $\mathcal{D}(\beta, A)$ is generated.
4. The graph is constructed from the list of classes and their intervals.
5. The graph is reduced to the strongly connected components.
6. If necessary, forward and backward reductions are used in combination with the reduction to strongly connected components.

A.1 List of rows

If we look closely at the definition of equivalence $\equiv$, we can see that it defines two equivalences on rows of the matrix A.

Definition A.1. *We say that the row $(a, b) \in \mathbb{Z}^{1\times 2}$ is equivalent to the row $(a', b') \in \mathbb{Z}^{1\times 2}$ with respect to an integer Y if there exists $c \in \mathbb{Z}$ coprime with Y such that*

$$ca \equiv a' \mod Y \quad \textit{and} \quad cb \equiv b' \mod Y.$$

We write $(a, b) \sim_Y (a', b')$.

Using this definition, the matrices $A, B \in \mathbb{Z}^{2\times 2}$ are equivalent according to Definition 5.6 if and only if $(a_{11}, a_{12}) \sim_Y (b_{11}, b_{12})$ and $(a_{21}, a_{22}) \sim_{Y'} (b_{21}, b_{22})$.

Since we will need to search in the list of rows, we have additional requirements on the representative of each group of equivalence. We want the first element of the row to be a divisor of Y. And since the rows are rows of the unimodular matrix, we allow only the rows (a, b) where $a \cdot b = 0$ or $\gcd\{a, b\} = 1$.

A.1.1 Generating list of rows

The input parameter is $f \in \mathbb{N}$ (in our case it is either Y or Y'). If $f = 1$, we have only one possible row (0,0). If $f \geq 2$, we use the pseudocode described in Algorithm 1 to generate the list L.

Algorithm 1: Generating RowList L with given $f \geq 2$

```
L := ∅;
add (0,1) to L;
for b = 0, 1, ..., f − 1 do
    add (1, b) to L;
for a = 2, 3, ..., f − 1 do
    if f = 0  mod a then
        Generate a field F of length f and set all values to zero;
        for b = 1, 2, ..., f − 1 do
            if gcd(a, b) > 1 then
                F[b] ← 1; /* This row is not allowed.                    */
            else
                add (a, b) to L;
                F[b] ← 1;
                for g ∈ {kf/a + 1 : k ∈ ℕ} ∩ (1, f − 1) do
                    /* Find equivalent rows                              */
                    b0 ← b · g  mod f;
                    if F[b0] = 0 then
                        F[b0] ← 1;
                        add (a, b0) as an equivalent row to (a, b); /* The row is equivalent
                           because gcd{g, f} = 1 and g(a, b) ~f (kf + a, b0) ~f (a, b0).  */
```

A.1.2 Searching for equivalent rows

Now we want to find the chosen representative of a class $[(a,b)]_{\sim_f}$, where $a, b \in \{0, 1, \ldots, f-1\}$ and either $a \cdot b = 0$ or $\gcd\{a, b\} = 1$. The pseudocode used to do so can be found in Algortihm 2.

For $c \neq 0$ coprime with f, we will denote c^{-1_f} the modular multiplicative inverse of c modulo f.

A.2 List of matrices

Given the numbers P and P', we first compute Y and Y'. Then we generate the lists of rows L1 with the parameter Y and L2 with Y' as described in Algorithm 1.

The procedure of generating all the matrices that were unimodular before applying modulo is in detail described in [31] in Chapter 7, more precisely in Section 7.1.2.1. In the program, we use the algorithm described by pseudocode in Algorithm 3.

This process will insert representatives of all classes of equivalence $\equiv$ exactly once in the list.

Algorithm 2: Searching in the RowList for representative of the class

if $f = 1$ **then**
  return $(0, 0)$; /* Only one possible row. */
/* Now, we will consider $f > 1$, so a and b cannot be zero simultaneously. */
if $a = 0$ **then**
  /* So $b \neq 0$ must be coprime with f, because otherwise the matrix could not have been unimodular */
  return $(0, 1)$;
  /* Because $(0, 1) \sim_f b^{-1_f}(0, b)$ and b^{-1_f} is also coprime with f */
else if $a = 1$ **then**
  return $(1, b)$;
else if $0 \neq f \mod a$ **then**
  $d \leftarrow \gcd\{a, f\}$;
  $a_0 \leftarrow \frac{a}{d}$; /* Then a_0 is coprime with f. */
  $b' \leftarrow b \cdot a_0^{-1_f} \mod f$; /* Now $a' = d$ is a divisor of f and $(a, b) \sim_f (a', b')$. */
  Find (d, b') in the list and return the representative to which it was added as an equivalent row.;
  return the representative.
else
  Find (a, b) in the list and return the representative to which it was added as an equivalent row.;
  return the representative.

Algorithm 3: Generating list M of modulated unimodular matrices

$M = \emptyset$;

$$E \leftarrow \begin{pmatrix} 1 \mod Y & 0 \\ 0 & 1 \mod Y' \end{pmatrix};$$

add E to M;
Set the pointer to the first matrix in the list;
while *pointer is not NULL* **do**
  Take the matrix $A \leftarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ given by the pointer;
  Generate two matrices B and C such that:

$$B = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix} \qquad C = \begin{pmatrix} a_{11} + a_{12} & a_{12} \\ a_{21} + a_{22} & a_{22} \end{pmatrix}$$

  for *B and C* **do**
    Find the representative of class $[B]_\equiv$, resp. $[C]_\equiv$, by searching the first row of B, respectively C, in L1 and the second row in L2.;
    Look if the representative matrix is already in the list. If not, add it at the end of the list.
  Move the pointer to the next matrix in the list

A.3 Intervals $\mathcal{D}(\beta, A)$

The input of the function for the generation of the interval $\mathcal{D}(\beta, A)$ is

- matrix A (or two rows that define the matrix),
- Y and Y',
- $H = \gcd\{P, P'\}$,
- $\alpha := \frac{1}{\beta}$
- The left and right points of the interval specified by the backward reduction in Section 5.5.2: $C, D \in [0, 1]$, where $C < D$. If not specified, $C := 0$ and $D = 1$.

The pseudocode can be found in Algorithm 4.

A.4 Graph

We have a list of classes of equivalence, know its length, denoted by M, and for each class, we have generated the set $\mathcal{D}(\beta, A)$.

For easier searching, we will copy the matrices into the field and index them accordingly. Next, we will generate the field of adjacency lists adj and fill it by the procedure described in Algorithm 5.

At the end of the procedure, we will have the graph Γ_β represented as a field of vertices and a field of adjacent lists for each vertex.

The labels of edges are not important for finding the paths, but we will use them for the forward and backward reductions.

A.5 Reduction to strongly connected components

This reduction is quite simple. We walk through the graph, and if we find a vertex without an edge going into it, we delete all edges going from it and vice versa. The pseudocode can be found in Algorithm 6.

Algorithm 4: Computation of the set $\mathcal{D}(\beta, A)$

$\mathrm{Max} \leftarrow \lceil L(1+\beta) \rceil - 2$; /* Calculated from Y, Y', H as in Lemma 5.11 */
$I \leftarrow (1, \mathrm{Max})$;
for $m = 0, 1, \ldots, Max$ **do**
  $A_m \leftarrow A \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$;
  $k \leftarrow 0$;
  while $k \leq (2+m) \cdot \alpha$ **do**
    $\ell \leftarrow 0$;
    while $\ell \leq (2+m) \cdot \alpha$ ***and*** $k + \ell \leq Y \cdot Y' \cdot H^2$ **do**
      if $I = \emptyset$ **then**
        $\mathcal{D}(\beta, A) \leftarrow \emptyset$;
        return;
      else if $\begin{cases} k + \ell = 0, \\ or\ k = \ell \wedge k \geq \alpha, \\ or\ k > \ell \wedge \frac{k + \ell(m+C)}{(1+m+C)} > \alpha, \\ or\ k < \ell \wedge \frac{k+\ell(m+D)}{(1+m+D)} > \alpha, \\ or\ A_m \begin{pmatrix} \frac{\ell}{H} \\ \frac{k}{H} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} Y \\ Y' \end{pmatrix}, \end{cases}$ **then**
        increase ℓ by H;
        /* It means that $\binom{\ell}{k}$ does not comply with $\mathfrak{P}3$ or $\mathfrak{P}2$, resp. the inequality in (5.9). */
      else
        $J := \emptyset$;
        if $\ell > 1$ **then**
          $J \leftarrow \left(\frac{k+1+m(\ell-1)}{\ell-1}, \mathrm{Max}\right)$;
        if $k > 1$ **then**
          $J \leftarrow J \cup \left(\left(1, \frac{k-1+m(\ell+1)}{\ell+1}\right) \cap (1, m+1)\right)$;
        if $k = 0$ **then**
          $J \leftarrow J \cup (1, m+1)$;
        $I \leftarrow I \cap J$;
        $\ell \leftarrow \ell + H$;
    $k \leftarrow k + H$;
$\mathcal{D}(\beta, A) \leftarrow I$;
return $\mathcal{D}(\beta, A)$.

Algorithm 5: Filling the adjacent list adj

```
for $i = 0, 1, \ldots, M-1$ do
    Take the $i$-th matrix and denote it by $A_i$;
    Set the pointer to the first interval in $\mathcal{D}(\beta, A_i)$;
    Take the interval $(a, b)$ from the pointer;
    while The pointer is not NULL do
        $k \leftarrow \lfloor a \rfloor$;
        while $k < b$ do
            Find the representative of class $[A_i \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}]_{\equiv}$ and find it in the list of matrices. Denote its index as $j$;
            To the adj[i], add the edge to $j$ with the label $(k, k+1) \cap (a, b)$;
            $k \leftarrow k + 1$;
        Move the pointer to the next interval;
```

Algorithm 6: Reduction to strongly connected components

```
deleted $\leftarrow$ true;
while $deleted$ = true do
    /* While we delete some edges */
    deleted $\leftarrow$ false;
    for $i$ = 0,1,..., M-1 do
        if adj[i] is empty then
            /* If the vertex has no outgoing edges */
            $j \leftarrow 0$;
            while $j \leq M - 1$ do
                if adj[j] has an edge to i then
                    /* delete all edges going into it */
                    Delete the edge;
                    deleted $\leftarrow$ true;
                $j \leftarrow j + 1$;
        else
            isOk $\leftarrow$ false;
            for $j = 0, 1, \ldots, N-1$ do
                /* Check if there is an edge into the vertex */
                if adj[j] contains an edge to i then
                    isOk $\leftarrow$ true;
            if isOk = false then
                /* If there is no edge going into the vertex */
                adj[i] $\leftarrow \emptyset$;
                /* delete all edges going from it. */
                deleted $\leftarrow$ true;
```

A.6 Forward reduction

As discussed in Section 5.5, the forward reduction has two parts. Since the edge modification is pretty straightforward, we will discuss only the edge deletion in more detail.

A.6.1 Delete edges

At first, we will show how to determine whether an edge has a suitable prolongation to a path of length at least MinLength.

Let $[A_N]_\equiv$ be a class of equivalence $\equiv$, let (a, b) be an interval such that $\delta_{N+1} \in (a, b)$ and let k be the depth of recursion (at the beginning, it is set to 0). This procedure is described in Algorithm 7.

Algorithm 7: The pseudocode for the function hasSuitableEdge

```
input: Index of the matrix in the list i
       Boundary points of interval (a, b)
       The depth of recursion k
hasSuitableEdge(i,a,b,k):
isOk ← false;
Set pointer to the first interval in adj[i];
while The pointer is not NULL do
    if k = MinLength then
        return true;
    (e, f) ← edge label from the pointer;
    B ← vertex from the pointer;
    ℓ ← the index of matrix B in the field;
    if (e, f) ∩ (a, b) ≠ ∅ then
        if a, b ∈ ℕ then
            /* We have a suitable edge and no restrictions on the next one given by our
               edge.                                                                  */
            return true;
        /* Otherwise we compute new interval and recursively check if there exists a
           prolongation                                                               */
        (g, h) ← new interval from (e, f) ∩ (a, b) calculated using (5.6);
        j ← ⌊g⌋;
        while max{j, h} = h do
            (o, p) ← (g, h) ∩ (j, j + 1);
            isOk ← isOk or hasSuitableEdge(ℓ, o, p, k + 1); /* Using recursion      */
            j ← j + 1;
    Move the pointer to the next edge;
return isOk
```

Now we can use the function hasSuitableEdge to remove edges that do not have a suitable prolongation. We will repeat the procedure described in Algorithm 8 for all classes and all edges, where $[A]_\equiv$ is a class of equivalence and (a, b) an edge to class $[B]_\equiv$.

Algorithm 8: Edge deletion by forward reduction

if $a, b \in \mathbb{N}$ **then**
    /* We have no conditions on δ and we do nothing */
    return;
$(c, d) \leftarrow$ interval for δ_{N+1} computed from (a, b) using (5.6);
hasPath $\leftarrow$ false;
$j \leftarrow \lfloor c \rfloor$;
while $\max\{j, d\} = d$ ***and*** *hasPath* = *false* **do**
    $(e, f) \leftarrow (c, d) \cap (j, j+1)$;
    $\ell \leftarrow$ index of the vertex $[B]_{\equiv}$;
    hasPath $\leftarrow$ `hasSuitableEdge`$(\ell, e, f, 0)$; /* using function described in Algorithm 7 */
    $j \leftarrow j + 1$
if *hasPath* = *false* **then**
    /* We have not found a suitable path */
    remove the edge (a, b) from the graph;