České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

Katedra fyziky
Studijní program: Matematická fyzika

Zobecněné gravitační teorie a jejich kosmologické a astrofyzikální aplikace

Extended theories of gravity and their cosmological and astrophysical applications

MASTER THESIS

Vypracoval: Kamil Mudruňka
Vedoucí práce: doc. Ing. Mgr. Petr Jizba, Ph.D.
Rok: 2024

ZADÁNÍ DIPLOMOVÉ PRÁCE

ČVUT
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I. OSOBNÍ A STUDIJNÍ ÚDAJE

Příjmení: **Mudruňka** Jméno: **Kamil** Osobní číslo: **486396**

Fakulta/ústav: **Fakulta jaderná a fyzikálně inženýrská**

Zadávající katedra/ústav: **Katedra fyziky**

Studijní program: **Matematická fyzika**

II. ÚDAJE K DIPLOMOVÉ PRÁCI

Název diplomové práce:

Zobecněné gravitační teorie a jejich kosmologické a astrofyzikální aplikace

Název diplomové práce anglicky:

Extended theories of gravity and their cosmological and astrophysical applications

Pokyny pro vypracování:

1) Seznámení se s existujícími nestandardními řešeními vybraných gravitačních teorií. Speciální důraz bude kladen na Weylovu konformní gravitaci s Gauss–Bonneteho členem.
2) Vyšetřováni vybraných řešení poruchovým počtem. Důraz bude kladen jak na klasické tak i na kvantové fluktuace.
3) Formulace analogu Einstein-Rosenova mostu ve Weylově nekvantové gravitaci.
4) Diskuse získaných řešení.

Seznam doporučené literatury:

[1] S. Capozziello and V. Faraoni, Beyond Einstein Gravity; A Survey of Gravitational Theories for Cosmology and Astrophysics (Springer, New York, 2011).
[2] M. Hohmann, C. Pfeifer, M. Raidal and H. Veermäe, Wormholes in conformal gravity, Journal of Cosmology and Astroparticle Physics, 10 (2018) 003.
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[4] P.D. Mannheim, Imprint of galactic rotation curves and metric fluctuations on the recombination era anisotropy, Physics Letters B, 840 (2023) 137851
[5] S. Koh, S. Park and G. Tumurtushaa, Higgs Inflation with a Gauss-Bonnet term, arXiv:2308.00897 [gr-qc], https://doi.org/10.48550/arXiv.2308.00897

Jméno a pracoviště vedoucí(ho) diplomové práce:

doc. Ing. Mgr. Petr Jizba, Ph.D. katedra fyziky FJFI

Jméno a pracoviště druhé(ho) vedoucí(ho) nebo konzultanta(ky) diplomové práce:

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doc. Ing. Mgr. Petr Jizba, Ph.D.
podpis vedoucí(ho) práce

podpis vedoucí(ho) ústavu/katedry

doc. Ing. Václav Čuba, Ph.D.
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Diplomant bere na vědomí, že je povinen vypracovat diplomovou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací. Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v diplomové práci.

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Fakulta jaderná a fyzikálně inženýrská

Břehová 7
115 19 Praha 1

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ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

PROHLÁŠENÍ

Já, níže podepsaný

Jméno a příjmení studenta:	Kamil Mudruňka
Osobní číslo:	486396
Studijní program:	Matematická fyzika
Studijní obor:	-
Specializace:	-

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Zobecněné gravitační teorie a jejich kosmologické a astrofyzikální aplikace

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Poděkování

Děkuji vedoucímu práce doc. Ing. Mgr. Petru Jizbovi, Ph.D. za cenné diskuze, rady a poskytnutí tématu diplomové práce. Dále patří moje poděkování také rodině a přátelům, kteří mě při studiu podporovali.

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Kamil Mudruňka

Název práce:
Zobecněné gravitační teorie a jejich kosmologické a astrofyzikální aplikace

Autor: Kamil Mudruňka

Studijní program: Matematická fyzika
Obor: -
Druh práce: Master thesis

Vedoucí práce: doc. Ing. Mgr. Petr Jizba, Ph.D.
Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská ČVUT v Praze

Abstrakt: Tato práce se zabývá řešeními klasické Weylovy konformní gravitace a jejich poruchami. Nejdříve jsou diskutovány motivace a možnosti modifikace Obecné teorie relativity. Následně je prezentována Weylova konformní gravitace a její nejdůležitější známé výsledky. Pomocí Newman-Penroseova formalismu je odvozeno známé řešení popisující černou díru a nové řešení popisující červí díru. Následně jsou odvozeny rovnice popisující poruchy okolo černých děr v plné konformní teorii. Poslední část práce se zabývá výpočtem vlivu kvantových polí na černé díry v aproximaci kvantových polí na zakřiveném prostoročasu popsaném klasickou gravitací. Práce je zakončena krátkým úvodem do metody kvantování konformní gravitace představené P.D.Mannheimem v jeho článcích.
Klíčová slova: Weylova konformní gravitace, poruchová teorie, zobecněné černé díry, kvantová pole v zakřiveném prostoročase, kvantová gravitace

Title:
Extended theories of gravity and their cosmological and astrophysical applications
Author: Kamil Mudruňka

Abstract: The thesis focuses on Weyl conformal gravity solutions and their perturbations. First the motivation and possibilities of modifications of General theory of relativity are discussed. Weyl conformal gravity and its most important results are presented. The Newman-Penrose formalism is used to derive the known black hole solution of the theory and a new wormhole solution. The governing linearized equations for perturbations around black holes in the full conformal theory are derived. In the last part the effect of quantum fields on black holes in the approximation of quantum fields on curved spacetime described by classical gravity is computed. The thesis is ended with an introduction to the Weyl conformal gravity quantization scheme used by P.D. Mannheim in his articles.
Key words: Weyl conformal gravity, perturbation theory, generalized black holes, quantum fields in curved spacetime, quantum gravity

Contents

Used notation and conventions

Here we list the acronyms, notation and conventions used in this work.

Acronyms

- GR - General Relativity
- WCG - Weyl Conformal gravity
- ODE - Ordinary differential equation
- PDE - Partial differential equation
- NP formalism - Newman-Penrose formalism
- EFE - Einstein field equations
- MK solution - Mannheim-Kazanas solution
- SDS solution - Schwarzschild-De Sitter solution
- FLRW metric - Friedmann-Lemaître-Robertson-Walker metric
- QFT - Quantum field theory

Notation

- Greek indices range from 0 to 3, Latin indices from 1 to 3 (unless used in tetrads, where they also range from 0 to 3)
- Covariant derivative - ∇_μ, partial derivative - ∂_μ, Spin connection covariant derivative - $\mathcal{D}_\mu$
- Box operator - $\Box A = g^{\mu\nu}\nabla_\mu\nabla_\nu A$
- Metric of a unit sphere $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\varphi^2$
- Hilbert space operators are indicated by hats $\hat{H}$
- Functional integral Feynmann measure $\mathscr{D}\phi$

The mostly positive $(-,+,+,+)$ metric signature convention will be used, unless indicated otherwise in the text. The curvature tensors will follow the convention of M.W.T. [1], that is

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}{}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\gamma\mu}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta\mu}\Gamma^{\mu}{}_{\beta\gamma}. \tag{1}$$

Mannheim often uses the convention of Weinberg [2], which has reversed sign of the Riemann tensor and all quantities derived from it. In chapter 5, the mostly negative $(+,-,-,-)$ convention is going to be used for the explanation of energy-momentum tensor renormalization. The units $\hbar = c = 1$ will be used unless stated otherwise in text.

List of Figures

Introduction

Our current understanding of the physics of the Universe is formed by two theories: Einstein's General theory of relativity (GR), which describes gravity, and the Standard model, which describes matter and the other known forces of nature. While the latter is formulated in the framework of quantum field theory, Einstein's gravity remains a classical theory. In the quantum treatment it leads to a nonrenormalizable theory lacking predictive power at high energies. This unsatisfactory situation and the desire to have a unified theory of all forces of nature calls for questioning of the possible modifications to the theories.

GR, published in 1915, so far successfully withstood many experimental tests and predicted new astrophysical phenomena including the existence of black holes and gravitational waves, which were recently directly observed [3]. Despite its massive success, once galactic and cosmological scales are considered, GR has to be supported by copious amounts of dark matter and dark energy. With most of the galactic mass being invisible and yet not a single particle of dark matter being detected in any experiment, one is lead to question the validity of the theory. The same applies to the enormous discrepancy between the observed value of the cosmological constant and the particle physics prediction of the vacuum energy.

On the quantum side, even in the absence of a quantum theory of gravity, we can explore some of the interactions between quantum fields and gravity by coupling classical gravity to expectation values of quantum observables. Once curved spacetime is introduced, renormalization requires introduction of other curvature terms, namely R^2 and C^2 into the action, with quantum physics automatically creating a theory different from pure GR. Such action leads to fourth order field equations instead of the second order ones of GR.

With many modifications that can safely reproduce the Solar system scale behaviour of GR in existence, we should look for other principles to determine the correct one. In descriptions of the other 3 forces of nature symmetries of the theories play a crucial role. Perhaps the gravitational sector might also possess additional symmetries to the diffeomorphism symmetry of GR. In this work we will focus on Weyl conformal gravity (WCG), which replaces the Einstein-Hilbert Lagrangian by a conformally invariant C^2 term. The theory was pioneered by Philip D. Mannheim whose articles will be our main sources.

In chapter 3 the spherically symmetric black hole solutions of WCG will be explored using the Newman-Penrose formalism instead of the standard metric based approach. We will show that this approach can simplify the fourth order equations in certain cases and provide alternative derivation of the known black hole metrics. The Mannheim-Kazanas metric, which generalizes the Schwarzschild metric and provides a possible explanation for the shapes of galactic rotational curves without the introduction of dark matter will be discussed. Finally a new interesting wormhole solution will be presented.

In chapter 4 first order perturbation theory will be applied to the solutions. First we will introduce known results about fluctuations around flat spacetime, including gravitational waves, and around cosmological spacetimes. In the second part we will derive the governing equations for first order perturbations of a Schwarzschild black hole in the full fourth order theory. Finally we will explore the static case and derive the general solutions sourced by static spherically symmetric sources for both Schwarzschild and Mannheim-Kazanas black holes.

In chapter 5 the effect of vacuum energy of quantum fields on the black holes will be calculated. We will consider classical WCG field equations coupled to vacuum expectation values of massive quantum fields computed using DeWitt-Schwinger expansion of the one-loop effective action. Software developed to compute the complicated expressions appearing in the vacuum expectation value approximation will be attached to the work. The tiny first order corrections to the gravitational potential of Schwarzschild-De Sitter and Mannheim-Kazanas black holes will be presented.

Finally in chapter 6, a basic introduction to quantization of the gravitational field itself in WCG will be given. We will explain the idea of Mannheim based on ground state energy cancellation and PT symmetric quantization with a non-Hermitian Hamiltonian. Such approach is surely controversial as it modifies one of the fundamental principles of quantum physics, making it on the other hand possibly even more interesting and worth mentioning.

The appendices will contain an explanation of the tetrad formalism and the Newman-Penrose formalism that are used in some of the calculations presented in this work. Also some of the expressions and equations that were too long and impractical to be included in the main work will be listed. Finally, basic information and links to the software packages that were used for some of the calculations in this work will be provided.

Chapter 1

Beyond Einstein Gravity

In this chapter we present an overview of the physics and ideas reaching beyond the standard GR, both from classical and quantum physics perspective.

1.1 Why should we modify GR?

1.1.1 Dark matter and weak gravitational fields

While successfully withstanding all tests in the strong field regime, plain GR, that is Einstein-Hilbert action with ordinary matter Lagrangian, fails when it comes to very weak gravitational fields. The rotational velocity curves of galaxies are predicted to decay by GR and Newton's law, but according to our measurements they are flat or even slightly increasing. In order to explain this discrepancy either copious amounts of dark matter, whose nature is so far unknown to us, have to exist in the galaxies, or we have to abandon plain GR.

As dark matter is believed to comprise around 80%-90% of galactic mass, this is a very serious issue. The same problem manifests itself on the scale of galactic clusters as well, with around the same 80%-90% of the total cluster mass being invisible as measured through gravitational lensing. A review of dark matter and its nature can be found in [4].

While the existence of dark matter is indirectly suggested by many observations and it is an essential component of the standard cosmological ΛCDM model, no direct observation of dark matter (for example by particle detectors) has yet been made. One could thing of dark matter not as a physical substance, but rather as a mathematical correction to the incomplete Eisntein field equations (EFE) that makes them fit our reality better. However, as a purely mathematical correction, dark matter, its distribution and properties are a rather complicated one. Many of the observations could be explained by a much simpler fact - GR is invalid in the weak field regime. Instead it should be replaced by another theory, which would predict flat rotational curves.

There are many candidate theories in existence. An important example is Modified Newtonian dynamics (MOND). Its basic idea is that Newton's second law should be replaced by a different relation for accelerations lower than a chosen scale a_0. Instead of the standard $a = GM/r^2$ one would have $a^2/a_0 = GM/r^2$. A basic review of MOND by its author can be found in [5].

1.1.2 Cosmological constant problem

Dark matter is not the only problematic missing component in GR. The FLRW cosmological model predicts that the Universe is expanding. In order to achieve accelerated expansion which is observed, a nonzero cosmological constant has to be inserted into the field equations of GR. One could think of it as another mathematical correction, a new fundamental constant of nature. However, there is naturally a desire to give it a physical origin instead.

In the FLRW model, the cosmological constant serves as a homogeneous and isotropic source of gravity with and exotic equation of state

$$\omega = \frac{p}{\rho} = -1. \tag{1.1}$$

Such equation of state could be satisfied by vacuum energy of particle physics. The following explanation is inspired by [6]. Due to the uncertainty relations, the ground state of a quantum mechanical harmonic oscillator can not have a zero energy. Instead its energy is given by

$$E = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\hbar\omega. \tag{1.2}$$

If we take a scalar quantum field as an example, we can compute the energy of its ground state as

$$E = \langle 0|\hat{H}|0\rangle = \sum_i \frac{1}{2}\hbar\omega_i, \tag{1.3}$$

where i runs over all the modes of the field (with the sum being replaced by an integral in the continuous case). This quantity is clearly divergent. As long as we stay in the framework of special relativity only, this not a problem. Special relativity has no notion of absolute energy, any measured energy is simply the difference compared to a ground level, which we can set to zero. On the level of the Hamiltonian this can be done by prescribing a normal ordering of the creation and annihilation operators $:a^\dagger a := a^\dagger a$ and $: aa^\dagger := a^\dagger a$. This way the ground state energy of the Hamiltonian will be

$$E = \langle 0|{:}\hat{H}{:}|0\rangle = 0. \tag{1.4}$$

When we introduce gravity the situation changes. GR with its energy-momentum tensor as a source of gravity requires a notion of absolute energy. However, the apparent divergence is still not a physics breaking problem. In order to work with such expression in our calculations we can regularize it by imposing an upper cutoff at the mode frequencies. This is a logical thing to do, as we do not expect quantum field theory (QFT) to be valid on all scales. As a low energy approximation of a more fundamental underlying theory it will break at some very high energy cutoff scale, which terminates the sum and gives as a huge, but finite result. While the resulting energy-momentum tensor has the right properties, its magnitude is nowhere near the value of the cosmological constant needed in cosmology. If we take the cutoff scale to be the Planck scale, particle physics predicts a value 120 orders of magnitude higher that the observed one. Unless the vacuum energies of different fields cancel out to a very high degree of precision, such a large cosmological constant cannot be incorporated into the standard cosmology. Is there a way to explain this discrepancy? Or could it be that the QFT prediction is correct and instead GR or the ΛCDM model has to be replaced by a different theory able to incorporate such a huge cosmological constant?

1.1.3 High energy physics and inflation

Unlike our understanding of the other fundamental forces of nature, GR is a purely classical theory. It is reasonable to expect that it can be only a low energy effective description of a more fundamental quantum theory of gravity. At high energies such theory is likely to deviate significantly and introduce new terms into the Lagrangian. While unobservable in the low energy world surrounding us, these corrections would play a significant role in the very early evolution of the Universe.

One way to observe the high energy behavior of gravity is through the observation of the tiny anisotropies of the cosmic microwave background (CMB) that are predicted to originate from the fluctuations of the inflaton field and metric during the cosmological inflation epoch. Currently there are many models in existence that try to explain the observed power spectra of these fluctuations by postulating various Lagrangians for the inflaton field.

A detailed overview of inflation and the associated calculations can be found in [7]. Here we briefly summarize the key points. In order to analyze the fluctuations first order perturbation theory around the expanding homogeneous and isotropic[1] FLRW background is used. The perturbations are decomposed through a SVT (scalar, vector, tensor) scheme into[2]

$$ds^2 = -\left(1+2A\right)dt^2 + 2aB_i dx^i dt + a^2\left[\left(1-2\psi\right)\delta_{ij} + E_{ij}\right]dx^i dx^j, \tag{1.5}$$

with B and E being further decomposable into

$$B_i = \partial_i B + S_i, \quad \partial^i S_i = 0, \tag{1.6}$$

$$E_{ij} = \partial_i \partial_j E + 2\partial_{(i}F_{j)} + h_{ij}, \quad \partial^i F_i = 0, \quad h^i_i = \partial^i h_{ij} = 0. \tag{1.7}$$

The perturbations A, ψ, B and E are called scalar perturbations, S_i and F_i are called vector perturbations and h_{ij} is called a tensor perturbation. The reason for these names becomes clear when one looks at the transformation rules for the different perturbations under changes of the spatial coordinates. There are also redundant gauge degrees of freedom in this scheme, due to which there are in total two independent tensor degrees of freedom, two vector degrees of freedom and two scalar degrees of freedom. The inflaton field itself is another scalar degree of freedom.

We can treat the perturbations as quantum fields living on the FLRW background. The power spectrum $\mathcal{P}_\psi$ of the fluctuations of a field $\hat{\psi}$ is defined as

$$\langle 0|\hat{\psi}^\dagger \hat{\psi}|0\rangle = \int d\left(\log\left(k\right)\right)\mathcal{P}_\psi\left(k\right), \tag{1.8}$$

where $|0\rangle$ is usually taken to be the Bunch-Davies vacuum defined as the Minkowski vacuum of a comoving observer in the far past. For a quantum field expanded as

$$\hat{\psi} = \int \frac{d^3k}{(2\pi)^3}\left[v_k\left(\tau\right)\hat{a}_k e^{ik\cdot x} + v_k^*\left(\tau\right)\hat{a}_k^\dagger e^{-ik\cdot x}\right], \tag{1.9}$$

this results in the boundary condition

$$\lim_{\tau\to-\infty} v_k = \frac{e^{-i\omega\tau}}{\sqrt{2k}}, \tag{1.10}$$

[1] This means the background evolution is a function of the time variable only and thus simplifies to solving the Friedmann equation.

[2] Here we present the result for the flat case of the FLRW metric. The same decomposition applies to curved cases as well, but one has to replace all partial derivatives by covariant ones.

where τ is the conformal time defined as $a(t)d\tau = dt$. The two most important variables of interest are the scalar spectral index n_s and tensor-to-scalar ratio r defined as

$$n_s - 1 = \frac{d\log\left(\mathcal{P}_\psi(k)\right)}{d\log(k)}, \quad r = \frac{\mathcal{P}_T}{\mathcal{P}_\psi}, \tag{1.11}$$

where $\mathcal{P}_T$ is the power spectrum of the tensor perturbations. Their values are constrained to lie between $0.96 < n_s < 0.97, r < 0.05$ by the recent Planck satellite data[8].

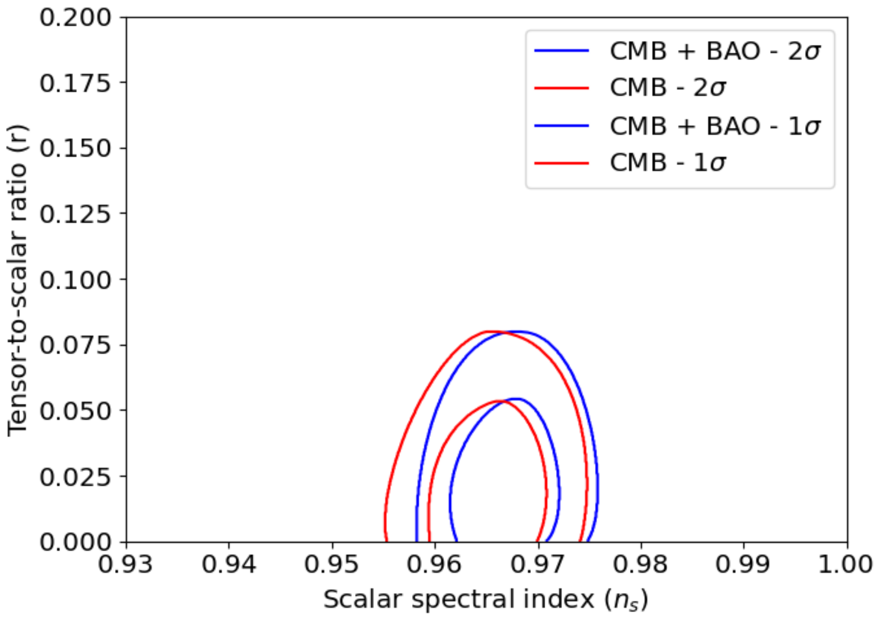

Figure 1.1: The constraints for n_s and r from the Planck satellite data. The 1σ and 2σ confidence levels are shown. Red boundaries represent data from CMB measurements alone. Blue boundaries represent data from combination of CMB and acoustic baryonic oscillation observations.

The typical Lagrangian that is considered is $\mathcal{L} = \frac{M_{pl}^2}{2}R + \mathcal{L}_\phi$, where $\mathcal{L}_\phi$ is the inflaton field Lagrangian. The simplest scalar inflaton model with

$$\mathcal{L}_\phi = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \tag{1.12}$$

is ruled out by the most recent CMB observations as it predicts too large tensor-to-scalar ratio [7]. However, the famous formulas for these predictions are valid only if plain GR is taken to be the underlying gravitational theory. If additional higher curvature corrections are considered, the predictions drastically change. For example the Gauss-Bonnet term coming from string theory alters the predictions of the ϕ^2 model in a favourable way [9]. Does this mean the inflaton potential has a very nontrivial shape unlike the simple polynomial interactions of the Standard model, or does it mean that GR is not the correct theory of gravity in the high energy regime?

1.1.4 Additional symmetries

GR on a given manifold M is a theory invariant under the group of diffeomorphisms $Diff(M)$. In the tetrad formulation[3] there is another Lorentz $SO(1,3)$ symmetry in the freedom of choice of the tetrads. The Lorentz group is also a symmetry group of the Standard model and its forces. Each of the forces then has additional symmetry group, together forming the $U(1) \times SU(2) \times SU(3)$ Standard model Lagrangian.

There might be additional symmetries to gravity which we might not be able to observe in the low energy regime where they would be broken. A promising candidate is the conformal symmetry. The primordial fluctuation power spectra display a near scale invariance with the scalar spectral index being close to one. This suggests that a scale symmetry might be a feature of gravity at high energies with additional small corrections present.

The local variant of global scale transformation, that is invariance under the change

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}, \tag{1.13}$$

where $\Omega(x)$ is an arbitrary everywhere nonzero smooth function on the spacetime manifold, is called conformal[4] transformation. There are two ways in which such symmetry could be implemented in a theory. One is to gauge it in the standard way as the other force symmetries. This would create a theory of gravity possessing an additional field apart from the metric. The other way would be to completely abandon the Einstein-Hilbert action and create a purely metric theory by a suitable choice of Lagrangian. One such theory will be researched in this work.

The presence of conformal symmetry would also restrict the possible couplings and interactions in the matter sector. It clearly prohibits any explicit dimensional couplings and mass terms. This agrees with the fact that the Standard model particles are also massless above the electroweak phase transition temperature and gain their masses dynamically through interaction with the Higgs field at low energies. The Maxwell equations of electromagnetism

$$\nabla_\mu F^{\mu\nu} = 0, \quad \nabla_{[\mu} F_{\alpha\beta]} = 0 \tag{1.14}$$

are invariant under conformal transformations provided that the electromagnetic field strength tensor is unchanged by the transformation[5].

$$F_{\mu\nu} \to F_{\mu\nu}. \tag{1.15}$$

This means the Lagrangian

$$L_1 = \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \tag{1.16}$$

is also invariant. For massless spin 1/2 field described by the Lagrangian

$$L_{1/2} = -\int d^4x \sqrt{-g} \bar{\psi} \gamma^\mu(x) \mathcal{D}_\mu \psi \tag{1.17}$$

[3]The tetrad formalism is explained in appendix A.

[4]Some literature uses the term Weyl transformation instead and gives a different meaning to the term conformal transformation.

[5]This holds for the covariant components. The contravariant components transform as $F^{\mu\nu} \to \Omega^{-4} F^{\mu\nu}$

the invariance can be forced by the the field transforming as[6]

$$\psi \to \Omega^{-3/2}\psi. \tag{1.18}$$

The fermions can gain mass through Yukawa coupling to a scalar field which transforms as $\phi \to \Omega^{-1}\phi$

$$L_{Yukawa} = -\int d^4x\sqrt{-g}\lambda\bar{\psi}\phi\psi. \tag{1.19}$$

Finally for a scalar field we can fix the free field Lagrangian by adding a coupling to the Ricci scalar with the coupling constant ξ equal to[7] $1/12$. The resulting Lagrangian is

$$L_0 = -\int d^4x\left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \xi R\phi^2\right), \tag{1.20}$$

and results in the equation of motion

$$\Box\phi - \frac{R}{6}\phi = 0. \tag{1.21}$$

The model can be further extended to also contain a quadratic self interaction

$$L_{int} = -\int d^4x\sqrt{-g}\lambda\phi^4. \tag{1.22}$$

We see that conformal symmetry does not only preserve the Standard model and its interactions, but also severely restricts possible gravitational Lagrangians.

1.1.5 Quantum fields in curved spacetime

The quantization of fields brings a problem not encountered in classical quantum mechanics: diverging quantities and the subsequent need for renormalization. The renormalization process might introduce new necessary counterterms into the theories. In flat spacetime it significantly limits the possible Lagrangians. However, once we transfer to curved spacetimes, another new phenomenon emerges. The renormalization of $\langle T_{\mu\nu}\rangle$ introduces counterterms involving curvature. More specifically, if we postulate a semiclassical theory of gravity by coupling the classical curvature terms to a vacuum expectation value of the energy-momentum tensor

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle \tag{1.23}$$

and renormalize $\langle T_{\mu\nu}\rangle$, the resulting field equations will be [10]

$$G_{\mu\nu} + \alpha A_{\mu\nu} + \beta B_{\mu\nu} + \gamma C_{\mu\nu} + \Lambda_{ren} g_{\mu\nu} = 8\pi G_{ren}\langle T_{\mu\nu}\rangle_{ren}, \tag{1.24}$$

where the coupling constants in the equation stand for their renormalized values which have to be fixed through experiment. The tensors $A_{\mu\nu}$, $B_{\mu\nu}$ and $C_{\mu\nu}$ come from variations of the terms

$$A_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}R^2, \tag{1.25}$$

$$B_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}R^{\alpha\beta}R_{\alpha\beta}, \tag{1.26}$$

$$C_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}. \tag{1.27}$$

[6] The key to this fact is to notice that there is another factor of Ω^{-1} buried in the transformation of the vierbein multiplying the flat spacetime gamma matrices $\gamma^\mu = e^\mu{}_a\gamma^a$.

[7] This is called a conformal coupling. Another interesting often discussed case is the minimal coupling, i.e. $\xi = 0$.

In $d = 4$ dimensions the result can be simplified using the fact that the Gauss-Bonnet term

$$\int d^4x \left(\sqrt{-g}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2\right) \tag{1.28}$$

is a topological invariant and thus its functional derivative with respect to the metric vanishes (which can be also checked by a direct computation, as we will do later in this chapter). As a result the tensors $A_{\mu\nu}$, $B_{\mu\nu}$ and $C_{\mu\nu}$ are not independent, but are related by

$$C_{\mu\nu} = -A_{\mu\nu} + 4B_{\mu\nu}. \tag{1.29}$$

In other words, the action can be rewritten to include only the R^2 and $R^{\alpha\beta}R_{\alpha\beta}$ terms. Also the square of the Weyl tensor (2.1)

$$C^2 = C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2 \tag{1.30}$$

can be put in instead of the square of the Riemann tensor $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$. The generated counterterms are therefore R^2 and C^2. We see that the existence of quantum field in curved spacetime requires the existence of a gravitational Lagrangian, which is not the standard Einstein-Hilbert one, but also includes addtional corrections. The existence of the R^2 term is the key idea behind Starobinsky inflation, which will be discussed later in this chapter.

1.1.6 Quantization of gravity

The Einstein-Hilbert action yields a nonrenormalizable quantum theory. This can be seen immediately through the dimension of the Newtonian gravity coupling constant. Renormalization of GR would lead to an infinite series of counterterms whose strength would have to be determined experimentally, leaving us with a theory with zero predictive power as all the counterterms would eventually play a significant role at high enough energies. GR also seems to be incomplete in the classical picture as singularities arise.

Many approaches to quantization of gravity exist. Perhaps the most straightforward way would be to treat the metric (or rather its difference from some fixed background) as a new quantum field and postulate the theory in the standard framework of QFT. This would require a serious modification of the gravitational Lagrangian as the nonrenormalizable Einstein-Hilbert term could not be present. One such theory will be presented at the end of this work.

Another promising idea is string theory, extending the point-like particles of QFT to one dimensional strings. Even though the theory started as a theory of strong nuclear force, it was discovered that it contains gravity among the other forces. The other famous approach, loop quantum gravity, instead aims at formulating an explicitly background independent quantum theory. The resulting spacetime itself has a discrete nature.

1.2 Uniqueness of the Einstein-Hilbert action

Having discussed physical motivations for modifications of GR, we will present an overview of its mathematical structure, which could serve as a guide for potential modifications. The Einstein-Hilbert action of GR given in four spacetime dimensions by

$$S_{EH} = \int d^4x\sqrt{-g}\frac{M_{Pl}^2 R}{2}, \tag{1.31}$$

where $M_{Pl} = 1/\sqrt{8\pi G}$ is the reduced Planck mass, seems to be the simplest possible nontrivial action involving curvature terms. While the precise meaning of the word simple might be subjective and different for each reader, there is indeed a sense in which this action is unique. It can be shown that in four dimensions the only rank-2 curvature tensor satisfying

$$T_{\mu\nu} = T_{\mu\nu}\left(g_{\alpha\beta}, \partial_\rho g_{\alpha\beta}, \partial_\sigma\partial_\rho g_{\alpha\beta}\right), \tag{1.32}$$

$$T_{\mu\nu} = T_{\nu\mu}, \tag{1.33}$$

$$\nabla^\mu T_{\mu\nu} = 0, \tag{1.34}$$

$$T_{\mu\nu} \text{ is linear in } \partial_\sigma\partial_\rho g_{\alpha\beta}. \tag{1.35}$$

is the Einstein tensor augmented by a cosmological constant [11]

$$AG_{\mu\nu} + Bg_{\mu\nu} = A\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + Bg_{\mu\nu}. \tag{1.36}$$

At the same time, as is stated by Vermeil's theorem [11], the only nontrivial curvature scalar that is linear in second derivatives of the metric is the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{1.37}$$

Thus if we demand our field equations to be linear in the second derivatives of the metric, we inevitably arrive at Einstein's field equations. Derived this way, they would only be a result of mathematical constraints coming from our (possibly wrong) intuition that laws of physics should be simple. The real world physics would enter through the choice of the coupling constant which have have to adjust to match Newton's law of gravity in the weak field limit.

1.2.1 $f(R)$ gravity and higher derivative theories

Looking at the Einstein-Hilbert action (1.31) we notice something suspicious. Compared to the other forces of nature, Einstein's gravity requires a dimensional coupling constant. This raises concern especially in discussion of quantum gravity and its renormalization as mentioned before. A simple way to fix this problem would be to find a different curvature scalar with the correct dimension, like for example R^2.

This modified action would no longer be linear in the second derivatives of metric, but since we are looking for the simplest action consistent with the other laws of physics we know of, rather than the really simplest one, there is no reason to strictly demand this linearity. For now let us explore how a different action built purely from R would behave. The most general action of this type would be

$$S_f = \int d^4x\sqrt{-g}\frac{f(R)}{2\kappa}, \tag{1.38}$$

where $f(R)$ is an arbitrary (well behaved) scalar function. For $f(R) = R^2$ the coupling constant $1/\kappa$ would be dimensionless. In order to compute the variation of

(1.38) let us recall the variations

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{1.39}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\rho\left(g^{\alpha\beta}\delta\Gamma^\rho{}_{\alpha\beta} - g^{\alpha\rho}\delta\Gamma^\beta{}_{\alpha\beta}\right). \tag{1.40}$$

The last term in (1.40) when combined with $\sqrt{-g}$ is a total divergence and therefore does not contribute to Einstein's field equations. This fails to be the case in $f(R)$ gravity because

$$\delta S_f = \frac{1}{2\kappa}\int d^4x\left(\sqrt{-g}\frac{df}{dR}\delta R + f(R)\delta\sqrt{-g}\right). \tag{1.41}$$

Now because the term $\nabla_\rho\left(g^{\alpha\beta}\delta\Gamma^\rho{}_{\alpha\beta} - g^{\alpha\rho}\Gamma^\beta{}_{\alpha\beta}\right)$ is multiplied by f' we have to rewrite it in terms of the metric and integrate by parts to get a term with $\delta g^{\mu\nu}$ instead of its derivatives. After rewriting

$$\nabla_\rho\left(g^{\alpha\beta}\delta\Gamma^\rho{}_{\alpha\beta} - g^{\alpha\rho}\delta\Gamma^\beta{}_{\alpha\beta}\right) = g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}, \tag{1.42}$$

we discover the vacuum field equations to be

$$f'(R)\,R_{\mu\nu} - \frac{1}{2}f(R)\,g_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f'(R) = 0. \tag{1.43}$$

If $f(R)$ is just a linear function we obtain Einstein's field equations with cosmological constant. On the other hand, in case $f'(R) \neq const.$ this yields equation that is of fourth order in the metric. For the quadratic action $f(R) = R^2$ we get

$$2RR_{\mu\nu} - \frac{1}{2}R^2 g_{\mu\nu} + 2\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)R = 0. \tag{1.44}$$

More about $f(R)$ gravity and its behaviour can be found in [12].

1.2.2 Ghosts and Ostrogradsky instability

Even though higher derivative theories might seem to be a promising solution, they are destined to suffer from a completely new type of problem - in general they are plagued by instabilities and ghosts. To illustrate this phenomenon we take the example from [13] and consider a classical Lagrangian of the form

$$L = L\left(\phi, \dot\phi, \ddot\phi\right). \tag{1.45}$$

This Lagrangian in the general case leads to fourth order equations of motion. If we introduce a new variable $q = \dot\phi$ the Lagrangian can be equivalently recast in the form

$$L_2 = L(\phi, q, \dot q) + \pi\left(\dot\phi - q\right). \tag{1.46}$$

The conjugate momenta to q and ϕ are

$$p = \frac{\partial L}{\partial \dot q}, \qquad \pi = \frac{\partial L}{\partial \dot\phi}. \tag{1.47}$$

If $p = p(\phi, q, \dot q)$ is solvable for $\dot q$ the Hamiltonian function can be written as

$$H_2 = \dot\phi\pi + \dot q p - L_2 = \dot q p - L + \pi q. \tag{1.48}$$

Because of the πq term the resulting Hamiltonian is unbounded from both below and above and the theory contains a propagating ghost mode. Such a system will be badly behaved in both classical and quantum physics. This phenomenon is called the Ostrogradsky instability.

However, not all higher derivative theories have this sort of instability. Possible ways to avoid the Ostrogradsky instability in higher derivative theories are discussed for example in [13]. One class of Lagrangians which avoid the Ostrogradsky instability are those for which

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^2}\right) = 0. \tag{1.49}$$

In this case it will not be possible to solve for $\dot{q}$ as a function of p, ϕ and q and the previous argument fails.

1.3 Quadratic actions

After investigating $f(R)$ gravity we see another reason why the Einstein-Hilbert action has a special place among all the possible gravitational theories. It leads to second order PDE's for the metric instead of fourth order as the more general actions do. Having second order equations on one hand seems very natural as the other laws of physics are also second order differential equations and avoids the potential Ostrogradsky instability, on the other hand it seems very unlikely that we would be lucky enough for a generic action to produce them.

If we want to have a dimensionless coupling constant we need action quadratic in curvature. So far we have considered only the Ricci scalar R, but nothing prevents us from constructing and trying out other curvature scalars. Let us show a few of them and compute their variations.

$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$

The term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ also known as the Kretschmann scalar is a quadratic curvature scalar that is nonvanishing in the Schwarzschild geometry[8]. Its variation is given by [10]

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} &= \frac{1}{2}g_{\mu\nu}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} - 4\Box R_{\mu\nu} \\ &- 2\nabla_\mu\nabla_\nu R + 4R_{\mu\alpha}R^{\alpha}{}_{\nu} - 4R^{\alpha\beta}R_{\alpha\mu\beta\nu}.\end{aligned} \tag{1.50}$$

$R_{\mu\nu}R^{\mu\nu}$

In contrast with the Kretschmann scalar, the square of the Ricci tensor is not as useful invariant in plain GR, because it is zero for all the vacuum solutions to EFE. Its variation is given by [10]

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}R^{\alpha\beta}R_{\alpha\beta} &= 2\nabla_\alpha\nabla_\nu R_{\mu}{}^{\alpha} - \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R - 2R_{\mu\alpha}R^{\alpha}{}_{\nu} \\ &+ \frac{1}{2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}.\end{aligned} \tag{1.51}$$

[8] For a Schwarzschild black hole with mass M the Kretschmann scalar equates to $\frac{48G^2M^2}{r^6}$ which proves that there is a real physical sigularity at $r = 0$.

The Gauss-Bonnet term

In $d = 4$ dimensions a particular linear combination of the terms R^2, $R^{\alpha\beta}R_{\alpha\beta}$ and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ has a vanishing variation. It is called the Gauss-Bonnet term (GB^2) and is given by

$$GB^2 = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2. \tag{1.52}$$

When integrated over the manifold it generates a topological invariant called the Euler number, defined by (1.28). It should be noted that it is only a topological invariant when not coupled to any other fields. When coupled, as in the theory described by the action

$$S = \int d^4x\sqrt{-g}\left[M_{pl}^2\frac{1}{2}R - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) - \frac{1}{16}\xi(\phi)R_{GB}^2\right], \tag{1.53}$$

it makes a nonzero contribution, rendering the field equations to be

$$\begin{aligned}&M_{pl}^2G_{\mu\nu} + \frac{1}{4}R\nabla_\mu\nabla_\nu\xi + \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\Box\xi - \left(\nabla_\rho\nabla_{(\mu}\xi\right)R_{\nu)}{}^\rho + \\ &\frac{1}{2}g_{\mu\nu}R^{\rho\sigma}\nabla_\rho\nabla_\sigma\xi - \frac{1}{2}R_\mu{}^\rho{}_\nu{}^\sigma\nabla_\rho\nabla_\sigma\xi = T_{\mu\nu}\,.\end{aligned} \tag{1.54}$$

Such a modification becomes trivial if the field rests at a constant value. Therefore if present in the high energy effective action of gravity and ϕ being the inflaton, which has decayed into other Standard model particles, it might be difficult if not impossible to directly measure the presence of the Gauss-Bonnet term at present age. However, its footprint can be detected in the primordial fluctuations and produced gravitational wave spectrum [9]. Inflation models with the Gauss-Bonnet term also display other interesting phenomena, such as alteration of the speed at which gravitational waves propagate during the inflationary epoch [14].

1.3.1 Lovelock's theorem

The question we would like to answer is having these other terms at hand, is it possible to combine them in such a way that the higher derivative terms in the variation of the action cancel out and we are left with second order equations only? The answer is given by the famous theorem by Lovelock[15]:

Theorem 1 (Lovelock's theorem)**.** The only tensor $T_{\mu\nu}$ which satisfies (1.32), (1.33) and (1.34) in an arbitrary number of spacetime dimensions is

$$T^\mu{}_\nu = \sum_{p=1}^{\infty} a_p\delta^{\mu,\mu1,\ldots,\mu2p}_{\nu,\nu1,\ldots,\nu2p}R_{\mu1\mu2}{}^{\nu1\nu2}R_{\mu3\mu4}{}^{\nu3\nu4}\ldots R_{\mu2p-1\mu2p}{}^{\nu2p-1\nu2p} + a\delta^\mu{}_\nu, \tag{1.55}$$

where a, a_p are arbitrary constants and the δ symbol is given by

$$\delta^{\mu1,\ldots,\mu n}_{\nu1,\ldots,\nu n} = \det\begin{pmatrix}\delta^{\mu1}{}_{\nu1} & \cdots & \delta^{\mu1}{}_{\nu n}\\ \vdots & \ddots & \vdots\\ \delta^{\mu n}{}_{\nu1} & \cdots & \delta^{\mu n}{}_{\nu n}\end{pmatrix}. \tag{1.56}$$

Theorem 2 (Lovelock gravity)**.** Lagrangian density whose Euler-Lagrange equation is

$$\sqrt{-g}T^\mu{}_\nu = 0, \tag{1.57}$$

where $T^{\mu}{}_{\nu}$ is given by (1.55) is

$$\mathcal{L} = \sqrt{-g}\sum_{p=1}^{\infty} 2a_p \delta^{\mu 1,\ldots,\mu 2p}_{\nu 1,\ldots,\nu 2p} R_{\mu 1 \mu 2}{}^{\nu 1 \nu 2} R_{\mu 3 \mu 4}{}^{\nu 3 \nu 4} \ldots R_{\mu 2p-1 \mu 2p}{}^{\nu 2p-1 \nu 2p} + 2a\sqrt{-g}. \quad (1.58)$$

It is clear that the sum contains only a finite number of terms as in d dimensions all the delta symbols with more than d pairs of indices vanish due to their antisymmetry. The first few terms the Lovelock Lagrangian (1.58) are

$$\mathcal{L} = \sqrt{-g}\left(a_0 + a_1 R + a_2 R^2_{GB} + \ \ldots\right). \quad (1.59)$$

In $d = 4$ spacetime dimensions these are all of the possible terms. The Gauss-Bonnet term is a topological invariant in $d = 4$ dimensions and therefore does not contribute to the field equations. Therefore we can conclude that the Einstein field equations with cosmological constant are the most general second order equations even if we release the requirement of linearity in second derivatives of the metric.

The general conclusion here is that at least on the classical level it is impossible to create a theory different from GR without diving into possibly problematic or speculative new physics. Unless we give up on the idea that gravity is sourced by a divergence free symmetric rank two tensor and/or change the number of spacetime dimensions, we either have to introduce new fields into gravity or work with higher derivatives.

1.4 Theories with additional fields

In this section a brief introduction to scalar-tensor theories will be presented. Detailed information about this class of theories can be found in [12], which is also the source of the examples shown here.

1.4.1 Scalar-tensor theories

Scalar-tensor theories, as their name suggests, extend GR by coupling the Ricci scalar to an additional scalar field which effectively serves as a variable gravitational constant. They are described by the action

$$S_{S-T} = \int d^4x\sqrt{-g}\left[\frac{M^2_{Pl}}{2}\left(\phi R + \frac{\omega(\phi)}{\phi} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) - V(\phi)\right] \quad (1.60)$$

The most famous example of a scalar-tensor theory is the Brans-Dicke theory described by the action

$$S_{B-D} = \int d^4x\sqrt{-g}\frac{1}{16\pi}\left[\phi R - \frac{\omega}{\phi} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + S_{Matter} \quad (1.61)$$

where the parameter ω is dimensionless. The field equations are obtained by variation with respect to $g^{\mu\nu}$ and ϕ. Explicitly they are given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} =& \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) \\ &- \frac{V}{2\phi}g_{\mu\nu}, \end{aligned} \quad (1.62)$$

$$\Box\phi = \frac{1}{3+2\omega}\left(8\pi T + \phi\frac{dV}{d\phi} - 2V\right), \tag{1.63}$$

where T is the trace of the energy-momentum tensor.

1.4.2 Jordan frame and Einstein frame

The action (1.60) corresponds to what is usually called the Jordan frame. In the Jordan frame the scalar field ϕ is coupled to the Ricci scalar and as such there is no pure Einstein-Hilbert term in the action. By performing a suitable conformal transformation, we could get rid of this coupling and free the Ricci scalar, effectively transforming the theory into Einstein-Hilbert action plus additional matter action for the scalar field. To do this we recall that the Ricci scalar in $d = 4$ spacetime dimensions transforms under a conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ as [16]

$$R \to \frac{1}{\Omega^2}\left(R + \frac{6}{\Omega}\Box\Omega\right). \tag{1.64}$$

If we take the scale factor $\Omega = \sqrt{\phi}$ we get rid of the ϕ term coupled to the Ricci scalar and transform the action into what is called the Einstein frame. In the Einstein frame we have the Einstein-Hilbert action plus additional action for the scalar field. After redefinition of the scalar field

$$\tilde{\phi} = M_{Pl}\sqrt{\frac{2\omega+3}{2}}\log\left(\frac{\phi}{M_{Pl}}\right), \tag{1.65}$$

the Brans-Dicke action transforms into

$$S_{BD} = \int d^4x \left[\frac{M_{Pl}^2 R}{2} - \frac{1}{2}\nabla^\lambda\nabla_\lambda\tilde{\phi} - V(\phi(\tilde{\phi}))\exp\left(-8\sqrt{\frac{\pi G}{2\omega+3}}\tilde{\phi}\right)\right]. \tag{1.66}$$

1.4.3 Equivalence with $f(R)$ gravity

$f(R)$ theories can be alternatively formulated as scalar-tensor theories by introduction of a field $\phi = R$. The action (1.38) can be then replaced by

$$S_f = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[f'(\phi)R - \phi f'(\phi) + f(\phi)\right]. \tag{1.67}$$

Variation with respect to ϕ yields

$$(R-\phi)\, f''(R) = 0, \tag{1.68}$$

which is equivalent to $R = \phi$ if $f''(R) \neq 0$. By considering $\psi = f'(\phi)$ we have a scalar-tensor type action (1.60) with a scalar field ψ and $\omega = 0$.

This idea can be applied to many theories outside the $f(R)$ family as well. It can help us to analyze the behaviour of higher order theories by reformulating them in a second order formalism. It can also be helpful when analyzing the Ostrogradsky instability and presence of ghosts, as the presence of these can be immediatelly spotted if the scalar-tensor formulation contatins fields with the wrong sign of the kinetic term for example.

1.4.4 Example: Starobinsky inflation

As discussed before, pure GR is not a viable theory if we decide to describe matter by means of QFT. The R^2 counterterm required to renormalize the vacuum expectation value of the energy-momentum tensor, even if its effect is very small at present age, drastically changes the standard FLRW cosmology. It replaces the initial singularity by an epoch of rapid expansion - the cosmological inflation. The model originated in [17], where Starobinsky studies EFE paired with the renormalized energy-momentum tensor of massless conformally coupled field on the FLRW background of the form

$$\begin{aligned}\langle T_{\mu\nu}\rangle =&\frac{k_2}{2880\pi^2}\left(R_{\mu\lambda}R^{\lambda}{}_{\nu}-\frac{2}{3}RR_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}+\frac{1}{4}g_{\mu\nu}R^2\right)\\&+\frac{1}{6}\frac{k_3}{2880\pi^2}\left(2\nabla_\mu\nabla_\nu R-g_{\mu\nu}\Box R-2RR_{\mu\nu}+\frac{1}{2}g_{\mu\nu}R^2\right),\end{aligned} \tag{1.69}$$

where k_2 and k_3 are constants dependent on the spin of the fields. The Starobinsky inflation model is described by the action

$$S_{Star}=\frac{1}{2}\int d^4x\sqrt{-g}\left(M_{Pl}^2R+\frac{R^2}{6M^2}\right), \tag{1.70}$$

where M is a constant. The transformation to scalar-tensor theory results in

$$S_{Star}=\int d^4x\sqrt{-g}\left(\frac{M_{Pl}^2}{2}R+\frac{R\psi}{M}-3\psi^2\right). \tag{1.71}$$

To get to the Einstein frame the conformal transformation with

$$\Omega^2=\exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right) \tag{1.72}$$

is used. The Einstein frame action is

$$S_{Star}=\int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R-\frac{1}{2}\partial^\lambda\phi\partial_\lambda\phi-\frac{3M_{Pl}^4}{4}M^2\left(1-e^{-\sqrt{\frac{2}{3}}\frac{\psi}{M_{Pl}}}\right)^2\right]. \tag{1.73}$$

We have reduced the originally fourth order field equations to GR plus a scalar inflaton with an untypical potential. This version is much easier to analyze. The power spectral variables evaluate to

$$n_s-1=-\frac{2}{N},\quad r=\frac{12}{N^2}, \tag{1.74}$$

where N is the number of e-folds the inflation era lasted for [18]. Assuming inflation lasted for 60 e-folds, the prediction is in a very good agreement with the Planck satellite data.

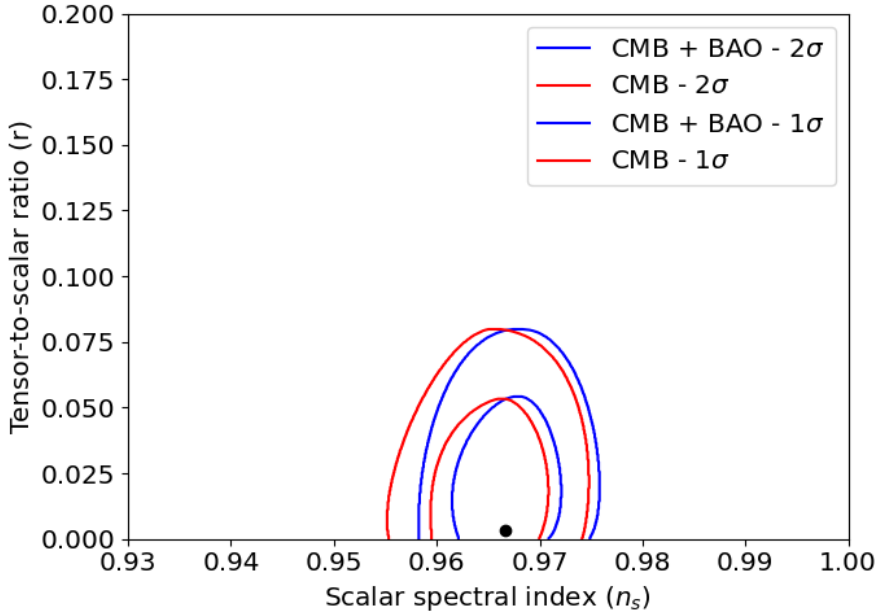

Figure 1.2: The n_s and r prediction of the Starobinsky inflation for the inflation epoch lasting 60 e-folds marked by a black dot. Red boundaries represent data from CMB measurements alone. Blue boundaries represent data from combination of CMB and acoustic baryonic oscillation observations.

Starobinsky inflation is also interesting in another way. It is a purely geometrical model that emerges naturally without the assumption of any inflaton fields. We see that many of the problems of classical gravity can be solved by quantization of matter alone.

Chapter 2

Weyl Conformal Gravity

Starting with GR, we have seen how the R^2 correction from QFT changes the early Universe cosmology. In this chapter the C^2 term will be added. In total there would be three curvature terms in the Lagrangian - R, R^2 and C^2, each with its respective renormalized coupling constant. These have to be fixed by experiments and in principle nothing prevents them from being zero. That includes the original Einstein-Hilbert term as well. If the C^2 Lagrangian could reproduce the known behaviour of GR on its own, we might end up with a new theory completely missing the original Einstein-Hilbert term.

2.1 Conformally invariant Lagrangian

The theory generated by C^2 Lagrangian is called Weyl Conformal gravity (WCG), often referred to just as Conformal gravity or Weyl gravity. In this chapter its basic behaviour will be introduced. Because the Weyl tensor of type $(1,3)$

$$C^{\mu}{}_{\nu\alpha\beta} = R^{\mu}{}_{\nu\alpha\beta} + \frac{2}{n-2}(\delta^{\mu}{}_{[\beta}R_{\alpha]\nu} + g_{\nu[\alpha}R_{\beta]}{}^{\mu}) + \frac{2}{(n-1)(n-2)}\delta^{\mu}{}_{[\alpha}g_{\beta]\nu}R \tag{2.1}$$

is invariant under conformal transformations $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, the C^2 action is conformally invariant.[1][2]. Also immediately see that the coupling constant G_W in a theory given by the action

$$S_W = G_W \int d^4x\sqrt{-g}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} \tag{2.2}$$

is dimensionless. This suggests that such a theory could be renormalizable in the power counting sense. The action (2.2) can be further simplified using the fact that the Gauss-Bonnet term is a total divergence in $d = 4$ dimensions to [16]

$$S = 2G_W \int d^4x\sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right), \tag{2.3}$$

which can be more useful in certain cases. Variation of (2.2) yields the Bach equation

$$B^{\mu\nu} = (2\nabla_{\rho}\nabla_{\sigma} - R_{\rho\sigma})C^{\rho\mu\sigma\nu} = 0, \tag{2.4}$$

[1] Note that this action is invariant only in $d = 4$ spacetime dimensions.

[2] Strictly speaking this is actually not the only Weyl invariant term we can think of in $d = 4$ spacetime dimensions. The Chern-Simons term $\sqrt{-g}{}^{*}C^{\mu}{}_{\nu}{}^{\alpha\beta}C^{\nu}{}_{\mu\alpha\beta} = \sqrt{-g}{}^{*}R^{\mu}{}_{\nu}{}^{\alpha\beta}R^{\nu}{}_{\mu\alpha\beta}$, ${}^{*}R^{\mu}{}_{\nu}{}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}R^{\mu}{}_{\nu\rho\sigma}$ is also conformally invariant, but it only contributes a boundary term to the action and does not influence the field equations. If we were to couple it to another field, a parity violating term would by introduced into the field equations. More on the effect of this term can be found in [19]

while variations of (2.3) yields a different, but equivalent form

$$\begin{aligned}\frac{1}{2}g^{\mu\nu}R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{6}g^{\mu\nu}R^2 - 2R^{\nu}{}_{\alpha}R^{\mu\alpha} + \nabla_\lambda\nabla^\nu R^{\lambda\mu} + \nabla_\lambda\nabla^\mu R^{\lambda\nu} - \Box R^{\mu\nu} + \\ + \frac{2}{3}R^{\mu\nu}R - \frac{2}{3}\nabla^\mu\nabla^\nu R + \frac{1}{6}g^{\mu\nu}\Box R = 0.\end{aligned} \tag{2.5}$$

The detailed computation of the variations can be found in [20].It is immediately clear that any vacuum solution of Einstein field equation is also a solution of Bach equation[3]. The converse does not hold as the Bach equation, being fourth order in derivatives of the metric, admits more general solutions.

The phenomenology of WCG is very rich, with most of the important areas including cosmology and galactic rotational curves being explored in detail by Mannheim and other authors. A good summary of the theory and it's predictions can be found in [21]. An overview of conformal transformations and symmetry can be found in [16].

2.2 Conformal transformations on solutions

From a purely mathematical point of view it might be better to interpret the Bach equation not as an equation for the metric itself, but rather an equation for equivalence classes of the relation "There exists a smooth function $\Omega(x)$ on the spacetime manifold such that $g_{\mu\nu} = \Omega^2(x)h_{\mu\nu}$". If we know one particular solution, we can generate an infinite class of solutions by applying conformal transformations to it. We can thing of the conformal factor as a gauge degree of freedom which further reduces the number of independent metric components by 1.

The conformal symmetry of the original theory is clearly broken in our Universe. This means that while the different conformally related solutions were physically indistinguishable in the unbroken symmetry phase, they describe physically completely different Universes at present age. The gauge is fixed by nature. The symmetry breaking at the classical level becomes evident at low temperatures when the Standard model particles gain masses through the interaction with the Higgs field. When the masses are explicitly written into the Lagrangian, the theory is no longer conformally invariant. Actually, as suggested in [22], the conformal symmetry, if present, points to the quantum nature of the Universe and gravity. A classical conformally invariant theory cannot have a nonflat solution, as any curvature would introduce a length scale, which would violate the conformal invariance. The only way to introduce scales into the solutions is some kind of spontaneous symmetry breaking, which is, however, a quantum phenomenon.

2.2.1 Conformally flat solutions

The trivial equivalence class is formed by solutions of the form

$$g_{\mu\nu} = \Omega(x)^2\eta_{\mu\nu}, \tag{2.6}$$

[3] In the version including the Gauss-Bonnet term his follows from $\nabla_\lambda C^{\lambda\mu\nu\alpha} = \nabla^\alpha R^{\mu\nu} - \nabla^\nu R^{\mu\alpha} + \frac{1}{2(n-1)}\left(\nabla^\nu R g^{\mu\alpha} - \nabla^\alpha R g^{\mu\nu}\right)$.

where $\eta_{\mu\nu}$ is the flat Minkowski metric expressed in an arbitrary set of coordinates[4]. Clearly $C^{\mu}{}_{\nu\alpha\beta} = 0$ for all these solutions. The converse also holds. In $d > 3$ dimensions any metric with $C^{\mu}{}_{\nu\alpha\beta} = 0$ is of the form (2.6). This class of solutions also contains a few of the commonly used metrics, most importantly the FLRW metrics

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \tag{2.7}$$

Another interesting example is the Schwarzschild interior solution. It is given by

$$ds^2 = -\frac{1}{4} A^2 dt^2 + B^{-1} dr^2 + r^2 d\Omega^2, \tag{2.8}$$

where

$$A = 3\sqrt{1 - \frac{r_s}{r_g}} - \sqrt{1 - \frac{r^2 r_s}{r_g^3}}, \quad B = 1 - \frac{r^2 r_s}{r_g^3} \tag{2.9}$$

and r_s and r_g are the Schwarzschild radius and the body radius respectively. At $r = r_g$ it can be matched with the exterior Schwarzschild solution to create a vacuum solution of (2.4) that behaves like the Schwarzschild solution outside a given radius, but does not contain any singularity at $r = 0$. In GR it is the most general conformally flat solution with the energy-momentum tensor of a non-expanding perfect fluid with positive denstity and pressure[23].

2.2.2 Wormholes

When thinking about modifications of GR, a topic that naturally comes to mind are various exotic solutions that are forbidden by GR such as wormholes or warp drives, that would theoretically allow travel at superluminal velocities. In GR these all require unphysical energy-momentum tensors that violate the energy conditions.

On the other hand, WCG not only allows wormholes that do not require presence of exotic matter near the throat[24][25], but also permits wormhole metrics that do not require any matter at all and are pure vacuum solutions. One such class of wormholes was presented in [26]. The construction begins with an arbitrary static spherically symmetric metric of the form

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2. \tag{2.10}$$

A suitable conformal transformation followed by a coordinate transformation brings in into the form

$$ds^2 = -B(l)dt^2 + \frac{dr^2}{B(l)} + (l^2 + d^2)d\Omega^2, \tag{2.11}$$

where l is the new radial coordinate, d is an arbitrary constant determining the throat radius and

$$B(l) = \left(1 + \frac{l^2}{d^2}\right) \eta(l)^2 a(d\eta(l)^{-1}), \tag{2.12}$$

where $\eta(l) = \eta_0 - \arctan(l/d)$ and η_0 is an arbitrary positive constant. l and r are related by

$$r(l) = \frac{d}{\eta(l)}. \tag{2.13}$$

In the next chapter a new vacuum wormhole solution with interesting properties will be presented.

[4]Apart from $Weyl(M)$ we still have the $Diff(M)$ symmetry.

2.3 Cosmology in Weyl Conformal Gravity

Unlike in GR, the possible energy-momentum tensors on the right hand side of the Bach equation are further constrained by the Bach tensor being traceless, a result directly related to the conformal invariance of the theory [16]. This forbids any explicit mass terms in the matter section of the Lagrangian. However, as mentioned before, the masses can be still generated by a symmetry breaking mechanism which introduces scales into the previously scaleless conformally invariant model. A detailed description of the cosmological model can be found in [27]. Here we briefly summarize it. The following matter action is proposed:

$$S_M = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu S\partial^\mu S + \frac{RS^2}{12} + \lambda S^4 + \bar{\psi}\gamma^\mu \mathcal{D}_\mu \psi + hS\bar{\psi}\psi\right). \tag{2.14}$$

(2.14) is invariant under conformal transformations with S transforming according to $S \to \Omega^{-1}S$ and the fermions as $\psi \to \Omega^{-3/2}\psi$. The scalar field S serves as a simplified model of what is expected to be a phase transition order parameter in an effective Landau-Ginzburg theory. The term λS^4 represents the negative potential energy corresponding to the minima of the ordered phase coming from the effective potential of the shape $aS^4 - 2a(T_V^2 - T^2)S^2$. Above the phase transition temperature T_V the conformal symmetry would be unbroken (and thus all standard model particles would have to be massless). The symmetry breaking phase transition would then generate an effective cosmological constant.

Because of the conformal symmetry of the model we can always work in a $S = S_0 = const.$ gauge. The FLRW metrics are conformally flat and therefore the C^2 term does not affect the cosmological evolution at all. The Friedmann equations are simply

$$T_{\mu\nu} = 0 \tag{2.15}$$

with

$$T_{\mu\nu} = T_{fermion} - \frac{S_0^2}{6}G_{\mu\nu} - g_{\mu\nu}\lambda S_0^4. \tag{2.16}$$

This is nothing but ordinary GR cosmology with a negative gravitational constant making everything, including matter, repulsive. This automatically ensures accelerating expansion of the Universe. It turns out that in order to have non-trivial solutions to $T_{\mu\nu} = 0$ in the high temperature phase with unbroken conformal symmetry, that is $S_0 = 0$, the spatial curvature of the Universe has to be negative [28]. When radiation domination is assumed the Friedmann equations can be solved analytically to yield

$$a(t)^2 = \frac{k(1-\beta)}{2\alpha} + \frac{k\beta\sinh^2(\sqrt{\alpha}t)}{\alpha}, \tag{2.17}$$

where

$$\alpha = -2\lambda S_0^2, \quad \beta = \sqrt{1 - \frac{16A\lambda}{k^2}}, \quad A = \rho(t)a(t)^4, \tag{2.18}$$

where $\rho(t) = \sigma T^4$ is the radiation energy density. For a sufficiently cold Universe the effective cosmological constant density parameter turns out to be

$$\Omega_\Lambda = \tanh^2\left(\sqrt{\alpha}t\right). \tag{2.19}$$

This result is completely independent of the actual value of the effective cosmological constant and thus WCG provides a possible solution to the cosmological constant problem.

Chapter 3

Spherically symmetric solutions in Newman-Penrose formalism

In this chapter we will derive and discuss the most important class of (electro) vacuum Petrov type D solutions in WCG, namely spherically symmetric metrics. Apart from the well known black hole solutions, a new exotic wormhole solution will be presented. The Newman-Penrose (NP) formalism will be used is this section. Explanation of the formalism can be found in appendix A, section B. The Bach equations in the NP formalism can be found in appendix A, section C.

In GR, the vacuum field equations written in the NP formalism simply state that all of the Ricci scalars are zero. Plugging these into the NP equations results in a coupled system of first order differential equations. In the same way, in WCG the NP formalism allows us to trade the fourth order Bach equations for a system of at most second order equations. As we will see, this allows us to solve the field equations more easily in certain cases.

3.1 Black hole solution

The most general static spherically symmetric static solution was obtained by Mannheim and Kazanas[29]. The derivation presented in the article results in a third order equation, which is then analytically solved. Here we present an alternative derivation through the NP formalism, where the problem becomes very simple as the two resulting second order equations completely decouple and can be solved separately. We start by noting that a general static spherically symmetric metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{3.1}$$

can be always brought to the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2 \tag{3.2}$$

by a suitable Weyl transformation followed by a coordinate transformation [29]. In the conformal phase this would correspond to our voluntary gauge choice. Based on our observations, the Schwarzschild solution is the correct metric at least on the scales of the Solar system. Therefore we can conclude that the "Schwarzschild gauge" where $A = 1/B$ is the preferred one in current broken symmetry era. We choose the null tetrad to be

$$l_\mu = \partial_\mu - \frac{1}{B(r)} r_\mu, \quad l^\mu = -\frac{1}{B(r)} t^\mu - r^\mu \tag{3.3}$$

$$n_\mu = \frac{B(r)}{2} t_\mu + \frac{1}{2} r_\mu, \quad n^\mu = -\frac{1}{2} t^\mu + \frac{B(r)}{2} r^\mu \tag{3.4}$$

$$m_\mu = -\frac{r}{\sqrt{2}} \theta_\mu - \frac{ir\sin(\theta)}{\sqrt{2}} \varphi_\mu, \quad m^\mu = -\frac{\sqrt{2}}{r} \theta^\mu - \frac{i}{\sqrt{2} r \sin(\theta)} \varphi^\mu. \tag{3.5}$$

The non-zero spin coefficients for a metric of the form (3.2) in this tetrad are

$$\alpha = -\beta = \frac{\sqrt{2}}{4r\tan(\theta)}, \tag{3.6}$$

$$\gamma = -\frac{1}{4} B'(r), \tag{3.7}$$

$$\rho = \frac{1}{r}, \tag{3.8}$$

$$\mu = \frac{B(r)}{2r}. \tag{3.9}$$

Non-zero Ricci scalars are

$$\Phi_{11} = \frac{r^2 B''(r) - 2B(r) + 2}{2r^2}, \tag{3.10}$$

$$\Lambda = -\frac{r^2 B''(r) + 4rB'(r) + 2B(r) - 2}{24r^2}. \tag{3.11}$$

and the only non-zero Weyl scalar is

$$\Psi_2 = \frac{r^2 B''(r) - 2rB'(r) + 2B(r) - 2}{12r^2}. \tag{3.12}$$

As a result, every static spherically symmetric metric is of Petrov type D. This result agrees well with the fact that gravitational radiation is not sourced by monopoles and dipoles, so it cannot be present in a perfectly spherically symmetric spacetime.

We notice that all the non-zero quantities are real, so every time a complex conjugate is encountered in the Bach equations we just multiply the variable by two instead. Also all the quantities depend only on r, so the angular derivatives δ and $\bar{\delta}$ drop out automatically. In this setting the Bach equations simplify substantially. The equation (A.90) simplifies to

$$DD\Psi_2 - 6\rho D\Psi_2 + 9\rho^2\Psi_2 - 3\Psi_2 D\rho = 0. \tag{3.13}$$

We have chosen the tetrad such that

$$D = -\partial_r, \tag{3.14}$$

so we obtain a single second order equation for Ψ_2

$$r^2\Psi_2'' + 6r\Psi_2' + 6\Psi_2 = 0. \tag{3.15}$$

If we were to solve to Bach equation in the covariant (metric) formalism, we would have arrived at a third order equation for $B(r)$. In the NP formalism with our choice of the null tetrad this equation instead splits into two second order equations which are easy to solve. The general solution of (3.15) is given by

$$\Psi_2 = \frac{c_1}{r^3} + \frac{c_2}{r^2}, \tag{3.16}$$

where c_1 and c_2 are arbitrary integration constants. Now we can proceed to obtain an ansatz for $B(r)$ from (3.12). We find the general solution to[1]

$$r^2 B''(r) - 2rB'(r) + 2B(r) = \frac{c_1}{r^3} + \frac{c_2}{r^2} \tag{3.17}$$

to be

$$B(r) = \frac{c_2}{2} + \frac{c_1}{6r} + c_3 r + c_4 r^2 = a + \frac{p}{r} + qr + \zeta r^2. \tag{3.18}$$

Now we plug this in back into the other components of the Bach equation to check if there are any further constraints on the four integration constants. The only nontrivial one is (A.95) which reduces to

$$-4\Phi_{11}\Psi_2 - 2D\Delta\Psi_2 - 4\mu D\Psi_2 + 4\rho\Delta\Psi_2 + 6\rho\mu\Psi_2 - 6\Psi_2 D\mu = 0, \tag{3.19}$$

which for the ansatz (3.18) yields the algebraic constraint

$$a^2 - 3pq - 1 = 0. \tag{3.20}$$

This equals the original result by Mannheim and Kazanas[29]. To be more concrete we can set q to be a arbitrary parameter and $p = -2M + \tilde{p}$ to match the Schwarzschild solution.

3.1.1 Charged black hole solution

The previous calculation can be trivially extended to include electromagnetic field and solve the Bach-Maxwell equations. Inspired by the classical Reisner-Nordstrom solution of GR we try to take the potential to be

$$A_\mu = \frac{Q}{r} r_\mu. \tag{3.21}$$

We compute the NP-Maxwell scalars to be

$$\phi_0 = \phi_2 = 0, \quad \phi_1 = \frac{Q}{4r^2}. \tag{3.22}$$

We check that this choice solves the Maxwell equations (A.89). In this case they reduce to

$$D\phi_1 = 2\rho\phi_1, \tag{3.23}$$
$$\Delta\phi_1 = -2\mu\phi_1. \tag{3.24}$$

If this indeed was a solution to the Maxwell equations, then the only modified component of the Bach equation would be (3.19) and therefore our ansatz (3.18) would not change. Sticking to it and by plugging in the values of μ and ρ we check that (3.22) really solves the Maxwell equations.

[1] Here we absorbed the factors 1/12 and 2 into the integration constants c_1 and c_2.

As mentioned above the only change to the Bach equation is now the addition of

$$\frac{\phi_1^2}{G_W} \tag{3.25}$$

to the right hand side of (3.19). This modifies the algebraic constraint (3.20) to

$$a^2 - 3pq - 1 = \frac{3Q^2}{8G_W}, \tag{3.26}$$

which is exactly the solution of Mannheim and Kazanas [30].

3.2 Properties of the Mannheim-Kazanas solution

Before moving to solve for a wormhole metric, the Mannheim-Kazanas (MK) solution will be discussed. As is the case for the Schwarzschild solution of GR, the MK solution describes the most general vacuum spherically symmetric solution of the Bach equations. We have derived a general constraint for its parameters, but apart from this their choice can be arbitrary. In literature the usually considered parametrization is

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2. \tag{3.27}$$

If $\beta\gamma \ll 1$ (as is suggested by observational data) then we have the Schwarzschild-De Sitter (SDS) solution with additional linear term. Note that the De Sitter term appears without the presence of any cosmological constant.

In his articles Mannheim proposes the linear term as a solution to the flat galactic rotational curves. With it WCG resembles a MOND type theory that add a minimal gravitational acceleration a body will always exert on other bodies. In [31][32][33][34] the rotational curves of 207 galaxies were successfully fitted under the assumption that the γ term is sourced by both the cosmological background and the visible matter in the galaxies as $\gamma = \gamma_0 + N^*\gamma^*$, where N^* is the number of solar masses contained within the galaxy. The velocities force in the Newtonian limit are then given by

$$\frac{v_W^2}{r} = c^2\left(\frac{N^*\beta^*}{r^2} + \frac{N^*\gamma^*}{2} + \frac{\gamma_0}{2} - \kappa r^2\right). \tag{3.28}$$

The data suggests values $\gamma^* = 5.42 \times 10^{-41} cm^{-1}$, $\gamma_0 = 3.06 \times 10^{-30} cm^{-1}$ and $\kappa = 9.54 \times 10^{-54} cm^{-2}$.

While we have 3 independent variables, the linear term is actually gauge dependent and can be removed by a suitable conformal transformation [35]. Because the conformal transformations preserve angles and null geodesics, the MK solution has the same causal structure as the SDS one. This means there are two horizons, one cosmological and one black hole horizon, beyond which the role of the radial and time coordinate switches. The maximal analytic extension of the MK solution can be created by applying a suitable conformal transformation on that of SDS spacetime.

For a long time it seemed that the linear term leads to problems with gravitational lensing. This has been resolved by realizing that the formula used in GR is not valid in globally curved backgrounds [36]. If the curvature of the background is taken into account, WCG produces correct results [37]. The predicted perihelion precession formula and the resulting constraints from the observational data also seem to be consistent with the value of γ obtained from galactic rotational curves [38].

3.2.1 Comparison with the Reissner-Nordstrom solution

The MK and Schwarzschild solutions might be impossible to tell apart on the scales of Solar system if γ and κ would be sufficiently small. However, the situation changes significantly once we introduce charge to the black hole. In GR, the charged black hole is described by the Reissner-Nordstrom solution

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2, \tag{3.29}$$

$$A(r) = 1 - \frac{2\beta}{r} + \frac{Q^2}{r^2}, \tag{3.30}$$

where Q^2 is the charge scaled by an appropriate constant. The charge introduces a $1/r^2$ term, which is very different from the $1/r$ it introduces in WCG. If we were able measure gravitational field of a sufficiently strongly charged object, we could distinguish between the two theories even on short distances.

3.2.2 Gauging away the singularity

While the choice of the Schwarzschild conformal gauge is clear in the black hole exterior region, the interior of black holes is inaccessible by experiments. This opens the possibility that the Schwarzschild like singularity at located at $r = 0$ could be just a conformal gauge artifact. In the Schwarzschild gauge the singularity is present as indicated by the curvature scalars

$$R = \frac{6\beta\gamma}{r^2} - \frac{6\gamma}{r} + 12\kappa, \tag{3.31}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{18\beta^2\gamma^2}{r^4} - \frac{24\beta\gamma^2}{r^3} + \frac{36\beta\gamma\kappa}{r^2} + \frac{10\gamma^2}{r^2} - \frac{36\gamma\kappa}{r} + 36\kappa^2, \tag{3.32}$$

$$\begin{aligned} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} =& \frac{4}{r^6}\Big(27\beta^4\gamma^2 - 18\beta^3\gamma^2 r - 36\beta^3\gamma + 9\beta^2\gamma^2 r^2 + 12\beta^2\gamma r + 12\beta^2 \\ & -6\beta\gamma^2 r^3 + 6\beta\gamma\kappa r^4 + 2\gamma^2 r^4 - 6\gamma\kappa r^5 + 6\kappa^2 r^6\Big) \end{aligned} \tag{3.33}$$

Due to the nontrivial transformation properties of the Riemann tensor under conformal transformations one may find $\Omega(x)^2$ such that these terms become nonsingular at $r = 0$. In the case of the Schwarzschild metric one such conformal factor is [39]

$$\Omega(r)^2 = \left(1 + \frac{l^2}{r^2}\right)^2, \tag{3.34}$$

where l^2 is a positive constant. This factor itself blows up at $r = 0$, but the resulting curvature scalar do not, indicating that a different set of coordinates might be found such that the metric as well as $\Omega(x)^2$ also become regular. The same conformal factor also works in the case of the full MK solution, with the resulting curvature scalars being presented in appendix C under (C.3), (C.4) and (C.5).

3.3 Wormhole solution

Finally we present a new wormhole solution to the theory. Let us consider the metric

$$ds^2 = -dt^2 + dr^2 + L^2(r)d\Omega^2 \tag{3.35}$$

where we allow the radial coordinate r to take values on the whole real line $(-\infty, +\infty)$. $L(r)$ will be the function determining the wormhole throat shape. For a throat to exist, we demand that $L(r) > 0$ for all values of r. The throat radius will simply be given by $L(0)$.

We choose the null tetrad to be

$$l_\mu = \frac{1}{\sqrt{2}}\left(t_\mu - r_\mu\right), \quad l^\mu = -\frac{1}{\sqrt{2}}\left(t^\mu + r^\mu\right) \tag{3.36}$$

$$n_\mu = \frac{1}{\sqrt{2}}\left(t_\mu + r_\mu\right), \quad n^\mu = \frac{1}{\sqrt{2}}\left(-t^\mu + r^\mu\right) \tag{3.37}$$

$$m_\mu = -\frac{L}{\sqrt{2}}\theta_\mu - \frac{iL\sin(\theta)}{\sqrt{2}}\varphi_\mu, \quad m^\mu = -\frac{\sqrt{2}}{L}\theta^\mu - \frac{i}{\sqrt{2}L\sin(\theta)}\varphi^\mu \tag{3.38}$$

The non-zero spin coefficients in this tetrad are

$$\alpha = -\beta = \frac{\sqrt{2}}{4L\tan(\theta)}, \quad \rho = \mu = \frac{1}{\sqrt{2}}\frac{L'}{L}. \tag{3.39}$$

Non-zero Ricci scalars are

$$\Phi_{00} = \Phi_{22} = -\frac{L''}{2L}, \quad \Phi_{11} = \frac{1 - L'^2}{4L^2}, \quad \Lambda = \frac{1 - 2LL'' + L'^2}{12L^2}, \tag{3.40}$$

indicating that the only vacuum solution to EFE with the form (3.35) is the flat spacetime. The only non-zero Weyl scalar is

$$\Psi_2 = \frac{-LL'' + L'^2 - 1}{6L^2}. \tag{3.41}$$

Under these conditions the Bach equations reduce to

$$\mathrm{D}\mathrm{D}\Psi_2 - 6\rho\mathrm{D}\Psi_2 + 9\rho^2\Psi_2 - 3\Psi_2\mathrm{D}\rho + \Phi_{00}\Psi_2 = 0, \tag{3.42}$$

$$-\mathrm{D}\Delta\Psi_2 - 2\mu\mathrm{D}\Psi_2 + 2\rho\Delta\Psi_2 + 3\mu\rho\Psi_2 - 3\Psi_2\mathrm{D}\mu - 2\Phi_{00}\Psi_2 = 0, \tag{3.43}$$

$$\Delta\Delta\Psi_2 + 6\mu\Delta\Psi_2 + 9\mu^2\Psi_2 + 3\Psi_2\Delta\mu + \Psi_{22}\Psi_2 = 0. \tag{3.44}$$

Plugging in the expressions for the derivatives and spin coefficients we immediately see that (3.42) and (3.44) are the same. There is a huge difference from the black hole case in the fact that expression for Ψ_2 and the Bach equations are not decoupled. The independent equations are thus

$$-L^2L'''' - 2LL'L''' + LL''^2 + 2L'^2L'' = 0, \tag{3.45}$$

$$-L^3L'''' + 2LL'^2L'' - L'^4 + 1 = 0. \tag{3.46}$$

These are fourth order ODEs whose general solution might be difficult to obtain. However, it is easy to notice that (3.45) always has three Ls in every term, just the derivatives are distributed differently among them. It is clear that any function satisfying $L'' = aL$ where a is a constant also satisfies (3.45). Based on this observation we can guess the ansatz[2]

$$L = pe^{hr} + qe^{-hr}. \tag{3.47}$$

The second equation (3.46) yields an algebraic constraint

$$16p^2q^2h^4 = 1. \tag{3.48}$$

By the choice $p = q$ the wormhole can be made symmetric on both sides of the throat. The corresponding metric is

$$ds^2 = -dt^2 + dr^2 + l_0^2 \cosh^2\left(\frac{r}{l_0}\right) d\Omega^2, \tag{3.49}$$

where l_0 is the wormhole throat diameter.

The resulting geometry resembles the hyperbolic case of the FLRW metric, but with the hyperbolic sine replaced by hyperbolic cosine. Both functions have the same asymptotic behavior. The curvature scalars demonstrate this well as they approach constants for large r.

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{1}{l_0^4}\left[12\tanh^4\left(\frac{r}{l_0}\right) + \frac{8}{\cosh^2\left(\frac{r}{l_0}\right)} + \frac{4}{\cosh^4\left(\frac{r}{l_0}\right)}\right] \tag{3.50}$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{4}{l_0^4}\left[1 + 2\tanh^4\left(\frac{r}{l_0}\right)\right] \tag{3.51}$$

$$R = -\frac{1}{l_0^2}\left[6 - \frac{4}{\cosh^2\left(\frac{r}{l_0}\right)}\right] \tag{3.52}$$

The geodesics for purely radial motion are

$$\frac{\partial^2 t}{\partial s^2} = 0, \quad \frac{\partial^2 r}{\partial s^2} = 0. \tag{3.53}$$

Unless affected by an external force, an observer standing still in this spacetime will remain doing so forever. Time flows at the same rate everywhere and observers can freely pass through the throat and back at any time. Through a conformal transformation and redefinition of the time coordinate an arbitrary scale factor can be introduce into the metric. One should however remember that in the presence of matter such scale factor (and the whole wormhole metric) will be constrained by $T_{\mu\nu} = 0$. Certainly the scale factor can not depend only on time if we do not want it to be constant, because unlike the FLRW model, the wormhole metric is not spatially homogeneous.

Finally let us note that in order to explore the properties of the wormhole, a different coordinate system, in which the hyperbolic functions are replaced by rational functions, might be more useful. The metric (3.49) can be alternatively expressed as

$$ds^2 = -dt^2 + \frac{l_0^2}{l_0^2 + r'^2}dr' + \left(l_0^2 + r'^2\right) d\Omega^2, \tag{3.54}$$

with the transformation between the two coordinate systems given by

$$r = l_0 \operatorname{argsinh}\left(\frac{r'}{l_0}\right). \tag{3.55}$$

[2] We should also not forget about the case $a = 0$. In that case the anzatz reduces to $L = pr + q$ and the flat spacetime appears.

Chapter 4

Perturbations in Weyl conformal gravity

In this chapter the linearized perturbation equations around selected solutions of WCG will be derived and discussed. Specifically we will discuss the flat Minkowski spacetime, De Sitter spacetime and FLRW spacetimes. These were already well studied in many articles and we will introduce and summarize the key results.

In the second half of the chapter we will derive the governing equations for first order perturbations around the Schwarzschild solution. Through a suitable conformal transformation these solutions can be converted to ones around a background containing the linear term of the MK solution as well. Finally we will derive the stationary perturbation equations for the horizon of the MK solution, which we will couple to the vacuum energy of quantum fields in the next chapter.

4.1 Perturbation theory for conformal gravity

Let us briefly summarize the ideas of perturbation theory. We start with a decomposition of the metric into the form

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{4.1}$$

where $g_{\mu\nu}$ is a fixed background metric and $h_{\mu\nu}$ is of very small magnitude compared to $g_{\mu\nu}$. We expand both sides of the field equations up to a desired order in $h_{\mu\nu}$

$$B^{(0)}_{\mu\nu} + B^{(1)}_{\mu\nu} + B^{(2)}_{\mu\nu} + ... = \frac{1}{4G_W}\left(T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + ...\right), \tag{4.2}$$

where $X^{(n)} = \mathcal{O}\left(h^n_{\mu\nu}\right)$. Now we demand that the equations hold at every order separately. Typically we are only interested in first order terms, as these are linear in the perturbed metric components and therefore significantly easier to solve both analytically and numerically.

The general perturbation equations are going to be extremely complicated even for simple background geometries as the number of different combinantions of coordinates in the derivatives up to fourth order is very high and likely all or most of them are going to be present. The perturbations are much more easily studied with the Bach equations expressed in the form (2.5). Nevertheless, we will also derive the results from the form (2.4) when feasible, as it provides a good consistency check.

Let us start by mentioning two useful identities. Because the Bach and Weyl tensors are traceless, it holds to first order that

$$B^{\mu\nu}h_{\mu\nu} = -B^{\mu\nu(1)}g_{\mu\nu}, \quad C^{\mu\alpha\nu\beta}h_{\mu\nu} = -C^{\mu\alpha\nu\beta(1)}g_{\mu\nu}. \tag{4.3}$$

This especially means that the Bach tensor perturbation will still be traceless when the expansion is done around a vacuum solution of WCG. The Bach tensor perturbation can be written as

$$B^{\mu\nu(1)} = \left(2\nabla_\alpha\nabla_\beta - R_{\alpha\beta}\right)C^{\mu\alpha\nu\beta(1)} + \left(2\nabla_\alpha\nabla_\beta - R_{\alpha\beta}\right)^{(1)}C^{\mu\alpha\nu\beta}. \tag{4.4}$$

While the fourth order equations will be much more difficult to analyze than the second order perturbed EFE, we can exploit the conformal symmetry to generate new solutions out of existing ones. If $h_{\mu\nu}$ solves the first order perturbed Bach equations around a background given by $g_{\mu\nu}$, then $\Omega(x)^2 h_{\mu\nu}$ will solve the first order perturbed equations around the background given by $\Omega(x)^2 g_{\mu\nu}$. The first order perturbation equations will of course change, as they are dependent on the background metric. As the application of conformal transformations also changes the background metric, we cannot freely use them to simplify the perturbations we consider. The gauge is fixed by our choice of the background metric.

4.1.1 Coordinate gauge freedom

What is meant by perturbation depends on our definition of the unperturbed background and how we identify points on the two manifolds. A good explanatory example comes from cosmology: the Friedmann equations are typically formulated in cosmological time, which we can view as the proper time of comoving observers. With such time coordinate the constant time slices of the Universe are not perfectly spatially homogeneous and isotropic - there are tiny density fluctuations present, which eventually collapse and evolve into the large scale structure of the Universe that we observe today. However, if we define the time coordinate through postulating that slices of constant time are those of constant density, then there suddenly are no density perturbations and the Universe is perfectly homogeneous. This of course does not mean that the Universe is truly unperturbed, but rather that the density perturbation is not gauge invariant. Under an infinitesimal coordinate transformation of the form

$$x^{\mu\prime} = x^\mu + \xi^\mu, \quad \xi^\mu \ll 1 \tag{4.5}$$

the metric transforms as

$$g'_{\mu\nu} = g_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu. \tag{4.6}$$

Therefore the perturbations also transform as

$$h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu. \tag{4.7}$$

This allows us to simplify many of our calculations by clever gauge choice that eliminates certain components of the perturbed metric. Also if we are able to find variables that are gauge invariant and compute their behaviour, we have a definite way of determining if real physical perturbations are present in a system.

4.2 Conformally flat backgrounds

In the case of conformally flat backgrounds (4.4) reduces to

$$\left(2\nabla_\alpha\nabla_\beta - R_{\alpha\beta}\right) C^{\mu\alpha\nu\beta(1)} = S^{\mu\nu}, \tag{4.8}$$

where $S^{\mu\nu}$ is a source of the perturbations. This is a second order equation for the perturbed components of the Weyl tensor. We could use this equation to study the asymptotic behaviour of perturbations for both the Schwarzschild and MK solutions as well as the FLRW wormhole one. The two most important cases are, however, the flat spacetime and the FLRW cosmological spacetimes. These have been studied extensively and the results will be summarized here.

4.2.1 Flat background

A special case of conformally flat background is flat Minkowski spacetime. There are two ways to obtain first order perturbation equations around it. Either by direct first order expansion of the full field equations or by a second order expansion of the action followed by computing its variation. As having our theory described by action is always beneficial, the second approach, which was done in [40], will be presented in this section We decompose the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The second order action around the flat Minkowski background is

$$\begin{aligned} S = 2G_W \int d^4x \left(\frac{1}{6}\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \frac{1}{2}\partial_\nu\partial_\alpha h^{\alpha\beta}\partial^\mu\partial^\nu h_{\mu\beta} + \frac{1}{4}\Box h_{\mu\nu}\Box h^{\mu\nu} + \right. \\ \left. + \frac{1}{6}\partial^\mu\partial^\nu h_{\mu\nu}\Box h - \frac{1}{12}\Box h\Box h\right), \end{aligned} \tag{4.9}$$

where $h = g_{\mu\nu}h^{\mu\nu}$ is the trace of the perturbation. Now we can use the SVT decomposition to find the separate equations of motion for the scalar, vector and tensor modes. The scalar mode Lagrangian evaluates to

$$\begin{aligned} \mathcal{L}_S = &\frac{4G_W}{3}\left(A\Delta^2 A + 2A\Delta^2\psi + \psi\Delta^2\psi - B\Delta^2\ddot{B} + \ddot{E}\Delta^2\ddot{E} - 2A\Delta^2\dot{B} + 2A\Delta^2\ddot{E}\right. \\ &\left. + 2\psi\Delta^2\ddot{E} - 2\psi\Delta^2\dot{B} - 2\ddot{E}\Delta^2\dot{B}\right). \end{aligned} \tag{4.10}$$

As this contains no time derivatives A and ψ, these two modes do not propagate. In fact if we plug in the algebraic constraint it generates

$$A = \dot{B} - \ddot{E} - \psi \tag{4.11}$$

back into the Lagrangian, it vanishes completely. There are no scalar modes propagating in WCG a thus no scalar particles emerge if we quantize the linearized theory. This is often discussed with connection to Birkhoff's theorem, which indeed holds in WCG. We will mention it in more detail in the section about black hole perturbations.

When analyzing the vector modes, it is easier to work with the variable $V_i = S_i - \dot{F}_i$ instead of the original S_i and F_i. Unlike the original ones, this new variable is gauge invariant and is thus the correct candidate for testing the presence of vector modes. The Lagrangian for the vector modes is

$$\mathcal{L}_V = G_W \Delta V_i \Box V_i \tag{4.12}$$

and the equation of motion is

$$\Box V_i = 0. \tag{4.13}$$

Therefore unlike GR, which has only tensor modes when linearized around flat background, WCG also contains two (as we are bound by $\partial^i V_i = 0$) propagating vector modes. These are well behaved as they obey the standard second order wave equation.

Finally the tensor mode Lagrangian is

$$\mathcal{L}_T = \frac{G_W}{2}\Box h_{ij}\Box h^{ij}. \tag{4.14}$$

This Lagrangian describes gravitational waves in WCG which will be discussed in the next section. As it leads to fourth order equation of motion, non-standard behaviour will arise. It turns out there are two normal tensor modes and two ghost modes.

Exactly the same results were obtained by first order expansion of the Bach equations around flat spacetime in [41]. The resulting equations of motion are

$$\Box\Box\bar{h}_{\mu\nu} + \partial_\mu W_\nu + \partial_\nu W_\mu - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda W^\lambda = 0, \tag{4.15}$$

where

$$W_\mu = \frac{1}{3}\nabla_\mu\nabla^\alpha\nabla^\beta\bar{h}_{\alpha\beta} - \Box\nabla^\alpha\bar{h}_{\alpha\mu} \tag{4.16}$$

and $\bar{h}_{\mu\nu}$ is the traceless part of $h_{\mu\nu}$ given by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}h. \tag{4.17}$$

We summarized this section by linearized WCG having 6 degrees of freedom, 2 massless tensor modes, two massless vector modes and two massless ghost tensor modes.

4.2.2 Gravitational waves

The governing equations for gravitational waves can be obtained in many ways. Either from the expansion around flat spacetime flat spacetime(4.15), by setting $\beta = 0$ in the Schwarzschild black hole analysis or from the Lagrangian (4.14). If we adopt the transverse-traceless gauge $\partial_\mu h^\mu{}_\nu = h = 0$ we obtain the biharmonic equation

$$\Box\Box h_{\mu\nu} = 0. \tag{4.18}$$

This is solvable by a Fourier transformation

$$h_{\mu\nu} = \int d^3x \; h^k_{\mu\nu} e^{-ik\cdot x}, \tag{4.19}$$

which leads to

$$\left(\frac{\partial^4}{\partial t^4} - 2k^2\frac{\partial^2}{\partial t^2} + k^4\right) h^k_{\mu\nu} = 0. \tag{4.20}$$

The characteristic polynomial of (4.20) has two double roots. Therefore the general solution also contains two modes growing linearly in time. The complete solution is

$$h^k_{\mu\nu} = \sum_{i\in\{+,\times\}} \int d^3k \; A^i_{\mu\nu}(k)e^{i\omega t} + tB^i_{\mu\nu}(k)e^{i\omega t} + C^i_{\mu\nu}(k)e^{-i\omega t} + tD^i_{\mu\nu}(k)e^{-i\omega t}, \tag{4.21}$$

where the sum goes ever the two GW polarizations and $\omega = k$. $X^i_{\mu\nu}$ are the tranverse-traceless GW polarization tensors. We see that the solution contains two modes which grow linearly in time. To see the effect of the additional t factors we can consider a explicit choice of the two polarizations

$$A^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^\times_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.22}$$

and calculate the resulting Ricci tensors. A plane wave of the form $t\sin(t-z)A_{\mu\nu}$ then generates Ricci tensor perturbation of the form

$$R^{(1)}_{\mu\nu} = \cos(t-z)A_{\mu\nu} \tag{4.23}$$

for both polarizations with an additional change of sign for $A^+_{\mu\nu}$. These are again plane waves that solve $\Box R^{(1)}_{\mu\nu} = 0$.

Even though the second order GR gravitational waves which solve $\Box h_{\mu\nu} = 0$ are solutions of linearized WCG, they actually do not carry any energy [42]. It can be checked that they are not only solutions to the linearized theory, but also to the full Bach equations (2.4).

4.2.3 Cosmological perturbations

Because of the conformal invariance of the Weyl squared action, the action (4.9) applies to any conformally flat spacetime including FLRW. This means that unlike GR, WCG propagates the same degrees of freedom in flat and FLRW spacetimes. Most importantly, there is no scalar degree of freedom. In order to obtain the fluctuations around expanding spacetime, the fluctuations around flat spacetime simply have to be multiplied by the scale factor a^2. In the case of De-Sitter spacetime in the conformal time given by

$$a^2 = \frac{1}{H^2\tau^2}, \tag{4.24}$$

GR and WCG produce the same result when supplemented by the Bunch-Davies vacuum boundary condition (1.10), which effectively requires as to only keep the positive frequency modes in the far past, and a Neumann boundary condition [43][40]

$$\frac{\partial}{\partial\tau} g_{\mu\nu}|_{\tau=0} = 0. \tag{4.25}$$

The resulting fluctuations are given by

$$h_{\mu\nu} = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{H^2\tau^2} A_{\mu\nu} e^{i(k\cdot x - k\tau)} + \frac{ik}{H^2\tau} A_{\mu\nu} e^{i(k\cdot x - k\tau)}. \tag{4.26}$$

In GR this result is achieved because the equations for the perturbations nontrivially depends on the scale factor, while in WCG it is a consequence of existence of the runaway modes that grown linearly in time. The condition (4.25) kills the vector modes. This can be seen in the $S_i = 0$ gauge, where it results in

$$\dot{F}_i|_{\tau=0} = 0. \tag{4.27}$$

However, this implies that also the gauge invariant V_i vanishes at $\tau \to -\infty$. Together with the Bunch-Davies condition this this means that V_i can not be excited at all.

The perturbations around the FLRW background can be also obtained from the flat spacetime ones by a suitable conformal and in the case of non-flat backgrounds coordinate transformations. The perturbations around a general conformal to flat spacetimes were studied in [44][45][46]. The resulting equations depend only on the traceless part of $h_{\mu\nu}$, a consequence of the conformal invariance. The relevant case in WCG is the negatively curved FLRW metric

$$ds^2 = \Omega(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right], \tag{4.28}$$

with the SVT decomposition of the perturbation

$$\begin{aligned} \tilde{ds}^2 = \Omega(\tau)^2 \Big[&-2\phi d\tau^2 + 2(\nabla_i B + S_i) d\tau dx^i + (-2\psi\gamma_{ij} + 2\nabla_i\nabla_j E + \nabla_i F_j \\ &+ \nabla_j F_i + 2h_{ij}) dx^i dx^j \Big], \end{aligned} \tag{4.29}$$

where γ_{ij} is the the spatial slice metric and $\partial^i S_i = \partial^i F_i = 0$, $h^i_i = \partial^i h_{ij} = 0$, has four gauge independent perturbations:

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \quad S_i - \dot{F}_i, \quad h_{ij}, \quad \gamma = -\dot{\Omega}\Omega\psi + B - \dot{E}. \tag{4.30}$$

When perturbing the cosmological model (2.14) and ignoring matter perturbations, as the effect of matter becomes completely negligible extremely quickly in the evolution of the Universe, we find out that γ is related to α by

$$\gamma = \frac{\Omega}{2\dot{\Omega}} \left[\frac{\eta}{3\Omega^2} \left(3\partial_\tau^2 - \nabla^i \nabla_i \right) \alpha - \alpha \right] \tag{4.31}$$

and the perturbations evolve as

$$\alpha = \frac{P^{1/2-K}_{-1/2+i\tau}(\cosh(\rho))}{\sqrt{\sinh(\rho)}} Z_S(x_i), \tag{4.32}$$

$$S_i - \dot{F}_i = A_i \sqrt{\sinh(\rho)} P^{1/2-K}_{-1/2+i\tau}(\cosh(\rho)) Z_V(x_i), \tag{4.33}$$

$$h_{ij} = B_{ij} \sqrt{\sinh^3(\rho)} P^{1/2-K}_{-1/2+i\tau}(\cosh(\rho)) Z_T(x_i), \tag{4.34}$$

where $\rho = \sqrt{-k}\tau$, $\eta = -24G_W/S_0^2$, $K = -1/2 + (1/4 - 1/\zeta)$, $\zeta = 48\lambda G_W$, A_i is a transverse polarization tensor, B_{ij} is a transverse traceless polarization tensor, P are the associated Legendre conical functions of the first kind and the $Z_X(x_i)$ are the functions describing the spatial dependence of the perturbations. The associated Legendre conical functions behave for large values of $\cosh(\rho)$ as

$$P^{1/2-K}_{-1/2+i\tau}(\cosh(\rho)) \approx \frac{1}{\sqrt{\cosh(\rho)}}. \tag{4.35}$$

We see that in WCG the perturbations grow much faster than in standard GR cosmology. Specifically, for fluctuation around the light cone, the scalar perturbations grow as

$$\frac{\alpha(t_1)}{\alpha(t_2)} = \frac{T^2(t_1)}{T^2(t_2)}. \tag{4.36}$$

This means that for example between nucleosynthesis and recombination, the perturbations grow by a factor of 10^{12}. With such rapid growth the fluctuations in the CMB only need to originate around the time of nucleosynthesis.

4.3 Black hole perturbations

In this section we shall present the basic results that we derived for black hole perturbations in WCG. In the framework of GR, two approaches to black hole perturbations are commonly used: either the metric is perturbed and equations governing the perturbations are derived from EFE expressed in the covariant formalism, or the NP formalism is used and a second order equation for the perturbations of Ψ_1 and Ψ_4, called the Teukolsky equation (A.84),(A.85), is derived. The original metric perturbations can then be reconstructed from the results. The second approach turns out to be extremely powerful as the equations decouple not only for the Schwarzschild solution, but also for the rotating Kerr black hole.

While we could apply both to the Bach equations, we quickly encounter serious obstacles with the Teukolsky equation approach. The simplicity of the original Teukolsky equation relies on the EFE, which in NP formalism for vacuum solutions are equal to vanishing of all of the Ricci scalars. On the other hand, in the case of WCG, the Ricci scalars depend on the Weyl scalars in a complicated way. Also all the derivative operators in the original Teukolsky act on the Weyl scalars with zero background value and therefore can be left unperturbed. In WCG the perturbed Ricci scalars would also contain terms such as $\Delta\Delta\Psi_2$ whose proper treatment would involve perturbing the derivative direction. That would mean that the metric perturbations directly enter the equations. For this reason we did not choose this approach, although it might be interesting to explore it as well. One possibility would be to express (2.5) in the NP formalism instead of (2.4). This way we would get rid of the Weyl scalars completely and obtain an equation with Ricci scalars only.

The case of the direct approach is feasible, but also poses new challenges unencountered in GR. When solving the Bach equations analytically, we are typically interested in finding the most general solution with given properties, e.g. spherical symmetry. In perturbation theory, we are interested in evolution of perturbations around a given background. Such a problem involves initial and boundary conditions. For fourth order equations, we need two additional conditions which might be hard to obtain. We are also exposed to the risk of the Ostrogradski instability and other unique consequences of having higher derivatives in the equations. However, even if present in the classical theory, we should remember that the ghosts and instabilities might be resolved on the quantum level, where, as we will show in the last chapter, the theory can be made unitary via a special quantization procedure.

We have seen that every vacuum solution of EFE is automatically also a vacuum solution of the Bach equations. Let us explicitly show that this also holds for the first order perturbations around such solutions. The linearized EFE are

$$G_{\mu\nu}{}^{(1)} = R_{\mu\nu}{}^{(1)} - \frac{1}{2}R^{(1)}g_{\mu\nu} - \frac{1}{2}Rh_{\mu\nu}. \tag{4.37}$$

For Ricci-flat background $R = 0$ and therefore also $R^{(1)} = 0$. The EFE reduce to $R_{\mu\nu}{}^{(1)} = 0$. The perturbation of the Ricci tensor can be expressed in terms of $h_{\mu\nu}$ as

$$R_{\mu\nu}{}^{(1)} = \frac{1}{2}\left(\nabla_\lambda\nabla_\mu h^\lambda{}_\nu + \nabla_\lambda\nabla_\nu h^\lambda{}_\mu - \Box h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right) = 0. \tag{4.38}$$

By looking at the form (2.5) the result is obvious as all the terms are proportional to the Ricci tensor or the Ricci scalar or their products. Let us therefore also verify it by examining the form (2.4). Plugging ${R_{\mu\nu}}^{(1)} = 0$ and $R_{\mu\nu} = 0$ into (4.4) we obtain

$$B^{\mu\nu(1)} = 2\nabla_\alpha\nabla_\beta C^{\mu\alpha\nu\beta(1)} + 2(\nabla_\alpha\nabla_\beta)^{(1)}C^{\mu\alpha\nu\beta}. \tag{4.39}$$

The first order perturbed second covariant derivative can be expanded as

$$(\nabla_\alpha\nabla_\beta)^{(1)} = \nabla_\alpha\nabla_\beta^{(1)} + \nabla_\alpha^{(1)}\nabla_\beta. \tag{4.40}$$

The covariant divergence of the Weyl tensor is given by the Cotton tensor

$$\nabla_\lambda C^{\lambda\mu\nu\alpha} = K^{\mu\nu\alpha}, \tag{4.41}$$

$$K^{\mu\nu\alpha} = \nabla^\alpha R^{\mu\nu} - \nabla^\nu R^{\mu\alpha} + \frac{1}{2(n-1)}\left(\nabla^\nu R g^{\mu\alpha} - \nabla^\alpha R g^{\mu\nu}\right), \tag{4.42}$$

which evaluates to zero for vacuum EFE solutions. This leaves us with

$$2(\nabla_\alpha\nabla_\beta)^{(1)}C^{\mu\alpha\nu\beta} = 2\nabla_\alpha\nabla_\beta^{(1)}C^{\mu\alpha\nu\beta}. \tag{4.43}$$

By perturbing (4.41) we get

$$\nabla_\alpha\nabla_\beta^{(1)}C^{\mu\alpha\nu\beta} + \nabla_\beta C^{\mu\alpha\nu\beta(1)} = K^{\mu\nu\alpha(1)}. \tag{4.44}$$

By looking at the definition of the Cotton tensor (4.42) we observe that

$$K^{\mu\nu\alpha(1)} = 0 \tag{4.45}$$

and therefore

$$\nabla_\alpha\nabla_\beta^{(1)}C^{\mu\alpha\nu\beta} = -\nabla_\alpha\nabla_\beta C^{\mu\alpha\nu\beta(1)}. \tag{4.46}$$

As a result $B^{\mu\nu(1)} = 0$ and we have just proven the following theorem:

Theorem 3 (First order perturbations around vacuum solutions of EFE)**.** Let $h_{\mu\nu}$ be the first order perturbation to a Ricci flat metric satisfying (4.38). Then $h_{\mu\nu}$ is also a solution to

$${B_{\mu\nu}}^{(1)} = 0. \tag{4.47}$$

This means that all the results about stability of Schwarzschild black hole solutions from GR will also apply to WCG, if the other, more general solutions admitted by WCG are not excited.

4.3.1 The perturbed field equations

Let us explore the actual perturbed Bach equations. Because of the spherical symmetry of the background, it is convenient to decompose the angular dependence of the perturbations using an orthogonal basis of functions on unit sphere such as the spherical harmonics. The radial behavior for each mode can then be examined. The dependence on φ can be easily ignored as the eigenvalues of the Laplace operator corresponding to the different spherical harmonics do not depend on the magnetic quantum number.

The perturbations can be divided into two categories: odd parity pertubations and even parity perturbations. The odd parity perturbation modes are described by

$$h_{\mu\nu}^{odd} = \begin{pmatrix} 0 & Q \\ Q^T & P \end{pmatrix}, \tag{4.48}$$

where

$$Q = \begin{pmatrix} -h_0(\partial/\partial \sin\theta\varphi)Y_n^m & h_0(\sin\theta\partial/\partial\theta)Y_n^m \\ -h_1(\partial/\partial \sin\theta\varphi)Y_n^m & h_1(\sin\theta\partial/\partial\theta)Y_n^m \end{pmatrix} \tag{4.49}$$

and

$$P = \begin{pmatrix} h_2(\sin\theta\partial^2/\partial\theta\partial\varphi - \cos\theta\partial/\partial\sin^2\theta\varphi)Y_n^m & Sym \\ \frac{1}{2}h_2(\sin\theta\partial^2/\partial\varphi\partial\varphi + \cos\theta\partial\partial\theta & -h_2(\sin\theta\partial^2/\partial\theta\partial\varphi \\ -\sin\theta\partial^2/\partial\theta\partial\theta)Y_n^m & -\cos\theta\partial/\partial\varphi)Y_n^m \end{pmatrix}, \tag{4.50}$$

while the even parity ones are given by

$$h_{\mu\nu}^{even} = \begin{pmatrix} T & Q \\ Q^T & P \end{pmatrix}. \tag{4.51}$$

where

$$T = \begin{pmatrix} (1-2\beta/r)H_0Y_n^m & H_1Y_n^m \\ H_1Y_n^m & (1-2\beta/r)^{-1}H_2Y_n^m \end{pmatrix}, \tag{4.52}$$

$$Q = \begin{pmatrix} h_0(\partial/\partial\theta)Y_n^m & h_0(\partial/\partial\varphi)Y_n^m \\ h_1(\partial/\partial\theta)Y_n^m & h_1(\partial/\partial\varphi)Y_n^m \end{pmatrix} \tag{4.53}$$

and

$$P = \begin{pmatrix} r^2\left[K + G(\partial^2/\partial\theta^2)\right]Y_n^m & Sym \\ r^2G(\partial^2/\partial\theta\partial\varphi - \cos\theta\partial/\sin\theta\partial\varphi)Y_n^m & r^2\left[K\sin^2\theta + G(\partial^2/\partial\varphi^2 + \right. \\ & \left. \sin\theta\cos\theta\partial/\partial\theta)\right]Y_n^m \end{pmatrix} \tag{4.54}$$

In these equations Sym means that the matrix should be filled in so that the result is symmetric. Y_n^m are the spherical harmonics and h_0, h_1, h_2, H_0, H_1, H_2, K and G are functions of the quantum numbers m, n and the radial and time coordinate. Q^T stands for the matrix transpose of Q.

The full perturbed Bach equations around a vacuum solution of EFE are

$$\nabla_\lambda\nabla^\nu R^{\lambda\mu(1)} + \nabla_\lambda\nabla^\mu R^{\lambda\nu(1)} - \Box R^{\mu\nu(1)} - \frac{2}{3}\nabla^\mu\nabla^\nu R^{(1)} + \frac{1}{6}g^{\mu\nu}\Box R^{(1)} = 0. \tag{4.55}$$

We see that these are second order equations for the components of the Ricci tensor perturbation. The trace of (4.55) is equal to

$$\left(2\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)R^{\mu\nu(1)} = 0, \tag{4.56}$$

which, however, does not contain any new information as it is trivially satisfied because of

$$0 = \nabla_\mu G^{\mu\nu(1)} = \nabla_\mu\left(R^{\mu\nu(1)} - \frac{1}{2}g^{\mu\nu}R^{(1)}\right). \tag{4.57}$$

This is consistent with our previous observation that the perturbed Bach tensor should be also traceless around vacuum solutions of WCG. The fact that the full perturbation equations for $h_{\mu\nu}$ split into two independent blocks enables us to write them explicitly in terms of the perturbations in a reasonable way.

In the following derivations we will assume vacuum equations with no source on the right hand side of the equations. If a source is present, the corresponding energy-momentum tensor components (multiplied by prefactors that we have thrown away in some of the equations) have to be included.

4.3.2 Odd parity perturbations

While the even parity perturbations are difficult to analyze, the odd parity sector turns out to be relatively simple even in the full fourth order theory. Due to residual gauge freedom, we do not have to consider all of them, but instead we can work in the Regge-Wheeler gauge, where only two perturbations are present[47]. The perturbation we consider is given by

$$h_{\mu\nu} = \sum_{n=0}^{\infty} h^n_{\mu\nu} \sin(\theta) \frac{\partial P_n(\cos(\theta))}{\partial\theta}, \tag{4.58}$$

where P_n is the n-th Legendre polynomial and

$$h^n_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h^n_0(t,r) \\ 0 & 0 & 0 & h^n_1(t,r) \\ 0 & 0 & 0 & 0 \\ h^n_0(t,r) & h^n_1(t,r) & 0 & 0 \end{pmatrix}. \tag{4.59}$$

By plugging this expansion into (4.38) we obtain the perturbed Ricci tensor of the form

$$R^{(1)}_{\mu\nu} = \sum_{n=0}^{\infty} R^{n(1)}_{\mu\nu}, \tag{4.60}$$

$$R^{n(1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & R^{n(1)}_0 \sin(\theta) P \\ 0 & 0 & 0 & R^{n(1)}_1 \sin(\theta) P \\ 0 & 0 & 0 & R^{n(1)}_2 Q \\ R^{n(1)}_0 \sin(\theta) P & R^{n(1)}_1 \sin(\theta) P & R^{n(1)}_2 Q & 0 \end{pmatrix}, \tag{4.61}$$

where

$$P = \frac{\partial P_n(\cos(\theta))}{\partial\theta} \tag{4.62}$$

and

$$Q = \frac{2cos(\theta)\frac{\partial P_n(cos(\theta))}{\partial\theta} - 2sin^2(\theta)\frac{\partial^2 P_n(cos(\theta))}{\partial\theta^2}}{4L}. \tag{4.63}$$

The key to this result and motivation for our choice of basis in (4.60) is the identity

$$-\frac{1}{\sin^2(\theta)}\frac{\partial P_n(\cos(\theta))}{\partial\theta} + \frac{\cos(\theta)}{\sin(\theta)}\frac{\partial^2 P_n(\cos(\theta))}{\partial\theta^2} + \frac{\partial^3 P_n(\cos(\theta))}{\partial\theta^3} = -n(n+1)\frac{\partial P_n(\cos(\theta))}{\partial\theta}, \tag{4.64}$$

which is obtained by differentiating the eigenvalue equation for spherical harmonics with $m = 0$

$$\frac{\cos(\theta)}{\sin(\theta)}\frac{\partial P_n(\cos(\theta))}{\partial\theta} + \frac{\partial^2 P_n(\cos(\theta))}{\partial\theta^2} = -n(n+1)P_n(\cos(\theta)). \tag{4.65}$$

The functions $R^{n(1)}_0$ and $R^{n(1)}_1$ are given by

$$R^{n(1)}_0 = \frac{L}{2}\left(\frac{\partial^2 h^n_1}{\partial t\partial r} - \frac{\partial^2 h^n_0}{\partial r^2}\right) - \frac{L'}{r}h^n_0 + \frac{L}{r}\frac{\partial h^n_1}{\partial t} + \frac{n(n+1)h^n_0}{2r^2}, \tag{4.66}$$

$$R^{n(1)}_1 = \frac{1}{2L}\left(\frac{\partial^2 h^n_1}{\partial t^2} - \frac{\partial^2 h^n_0}{\partial t\partial r} + \frac{1}{r}\frac{\partial h^n_0}{\partial t}\right) - \frac{L'}{r}h^n_1 - \frac{L}{r^2}h^n_1 + \frac{n(n+1)h^n_1}{2r^2}. \tag{4.67}$$

The function $R_2^{n(1)}$ is given by

$$R_2^{n(1)} = \frac{1}{4L}\left[L\frac{\partial}{\partial r}\left(Lh_1^n\right) + \frac{\partial}{\partial t}h_0^n\right], \tag{4.68}$$

where we have denoted the Schwarzschild metric term $-g_{tt} = 1 - \frac{2\beta}{r}$ by L. We will keep this notation from now on as it makes the resulting equations visually more compact. Its derivative with respect to the radial coordinate r will be denoted by $'$.

Let us think about what would happen if the perturbation $R_2^{n(1)}$ was equal to zero. That would be a general condition relating h_0^n and h_1^n independently of n, as it does not appear in the term. The perturbed Ricci tensor would have exactly the same form as the perturbed metric we began with. Because it has only nonzero off-diagonal components, the Ricci scalar pertubation would be zero

$$R^{(1)} = R_{\mu\nu}h^{\mu\nu} + R^{(1)}_{\mu\nu}g^{\mu\nu} = 0. \tag{4.69}$$

Then all the $R^{(1)}$ terms of (4.55) drop out and we would end up with

$$\nabla_\lambda\nabla^\nu R^{\lambda\mu(1)} + \nabla_\lambda\nabla^\mu R^{\lambda\nu(1)} - \Box R^{\mu\nu(1)} = 0, \tag{4.70}$$

which is exactly the same equation as the pertubed EFE, only with the metric perturbation $h_{\mu\nu}$ replaced by ${R_{\mu\nu}}^{(1)}$. Now we can return to the perturbed equations and also include $R_2^{(1)}$. Because of the linearity, we can compute its effect independently and add it to the original equations. For perturbation of the form

$${R_{\mu\nu}}^{n(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_2^{n(1)} \\ 0 & 0 & R_2^{n(1)} & 0 \end{pmatrix}\left(2\cos(\theta)\frac{\partial P_n(\cos(\theta))}{\partial\theta} - 2\sin(\theta)\frac{\partial^2 P_n(\cos(\theta))}{\partial\theta^2}\right), \tag{4.71}$$

we obtained the following terms:

$$B^{(1)}_{t\varphi} = \frac{2\left[n(n+1)-2\right]}{r^2}\frac{\partial R_2^{n(1)}}{\partial t}\sin(\theta)\frac{\partial P_n(\cos(\theta))}{\partial\theta}, \tag{4.72}$$

$$B^{(1)}_{r\varphi} = \left[n(n+1)-2\right]\frac{\partial}{\partial r}\left(\frac{R_2^{n(1)}}{r^2}\right)\sin(\theta)\frac{\partial P_n(\cos(\theta))}{\partial\theta}, \tag{4.73}$$

$$B^{(1)}_{\theta\varphi} = \frac{4}{r^2}\left[r^2\frac{\partial^2 R_2^{n(1)}}{\partial t^2} + r^2L^2\frac{\partial^2 R_2^{n(1)}}{\partial r^2} - \left(LL' + 2rL^2\right)\frac{\partial R_2^{n(1)}}{\partial r} + R_2^{n(1)}\left(2L - 4L^2\right)\right]Q \tag{4.74}$$

The resulting equations are

$$L\frac{\partial}{\partial r}\left(LR_1^{n(1)}\right)+\frac{\partial}{\partial t}R_0^{n(1)}+Z_n=0, \tag{4.75}$$

$$\frac{1}{2L}\left(\frac{\partial^2 R_1^{n(1)}}{\partial t^2}-\frac{\partial^2 R_0^{n(1)}}{\partial t\partial r}+\frac{1}{r}\frac{\partial R_0^{n(1)}}{\partial t}\right)-\frac{L'}{r}R_1^{n(1)}-\frac{L}{r^2}R_1^{n(1)}+\frac{n(n+1)R_1^{n(1)}}{2r^2}$$
$$+\left[n(n+1)-2\right]\frac{\partial}{\partial r}\left(\frac{R_2^{n(1)}}{r^2}\right)=0, \tag{4.76}$$

$$\frac{L}{2}\left(\frac{\partial^2 R_1^{n(1)}}{\partial t\partial r}-\frac{\partial^2 R_0^{n(1)}}{\partial r^2}\right)-\frac{L'}{r}R_0^{n(1)}+\frac{L}{r}\frac{\partial R_1^{n(1)}}{\partial t}+\frac{n(n+1)R_0^{n(1)}}{2r^2} \tag{4.77}$$

$$+\frac{2\left[n(n+1)-2\right]}{r^2}\frac{\partial R_2^{n(1)}}{\partial t}=0, \tag{4.78}$$

where

$$Z_n=\frac{4}{r^2}\left[r^2\frac{\partial^2 R_2^{n(1)}}{\partial t^2}+r^2L^2\frac{\partial^2 R_2^{n(1)}}{\partial r^2}-\left(LL'+2rL^2\right)\frac{\partial R_2^{n(1)}}{\partial r}+R_2^{n(1)}\left(2L-4L^2\right)\right]. \tag{4.79}$$

The equation (4.75) does not contain any n dependent term and constraints all the modes in the same way.

In total we have 3 equations for 3 unknown variables. However, the equations can not be independent. The true variables we are solving for are the original h_0^n and h_1^n. That means one of the equations must be redundant if the system is consistent.

4.3.3 Even parity perturbations

Even parity perturbations are not only more complicated because of the larger number of independent perturbations we have to consider, but also because of the additional terms that will appear in the perturbed Bach equation because of the Ricci scalar perturbations. Once again we use the Regge-Wheeler gauge, which this time reads [47]

$$h_{\mu\nu}=\sum_{n=0}^{\infty}h_{\mu\nu}^n P_n(\cos(\theta)), \tag{4.80}$$

where P_n is the n-th Legendre polynomial and

$$h_{\mu\nu}^n=\begin{pmatrix} L(r)h_0^n(t,r) & H^n(t,r) & 0 & 0\\ H^n(t,r) & h_1^n(t,r)/L(r) & 0 & 0\\ 0 & 0 & r^2K^n(t,r) & 0\\ 0 & 0 & 0 & r^2\sin^2(\theta)K^n(t,r)\end{pmatrix}. \tag{4.81}$$

By plugging this expansion into (4.38) we obtain the perturbed Ricci tensor of the form

$$R_{\mu\nu}^{(1)}=\sum_{n=0}^{\infty}R_{\mu\nu}^{n(1)}, \tag{4.82}$$

$$R_{\mu\nu}^{n(1)}=X_{\mu\nu}^{n(1)}P_n(\cos(\theta))+Y_{\mu\nu}^{n(1)}\frac{\partial}{\partial\theta}P_n(\cos(\theta)), \tag{4.83}$$

$$X^{n(1)}_{\mu\nu} = \begin{pmatrix} L(r)R^{n(1)}_{tt} & R^{n(1)}_{rt} & 0 & 0 \\ R^{n(1)}_{rt} & R^{n(1)}_{rr}/L(r) & 0 & 0 \\ 0 & 0 & r^2 R^{n(1)}_{\theta\theta} & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta)R^{n(1)}_{\varphi\varphi} \end{pmatrix}, \tag{4.84}$$

$$Y^{n(1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & R^{n(1)}_{t\theta} & 0 \\ 0 & 0 & R^{n(1)}_{r\theta} & 0 \\ R^{n(1)}_{t\theta} & R^{n(1)}_{r\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.85}$$

The $R^{n(1)}_{\mu\nu}$ are given by

$$R^{n(1)}_{tt} = -\frac{L\frac{\partial^2}{\partial r^2}h_0}{2} - \frac{h_0\frac{d^2}{dr^2}L}{2} - \frac{h_1\frac{d^2}{dr^2}L}{2} + \frac{\frac{\partial}{\partial r}KL'}{2} - \frac{3L'\frac{\partial}{\partial r}h_0}{4} - \frac{L'\frac{\partial}{\partial r}h_1}{4} + \frac{\partial^2}{\partial t\partial r}H$$
$$+ \frac{\frac{\partial}{\partial t}HL'}{2L} - \frac{\frac{\partial^2}{\partial t^2}K}{L} - \frac{\frac{\partial^2}{\partial t^2}h_1}{2L} - \frac{L\frac{\partial}{\partial r}h_0}{r} - \frac{h_0L'}{r} - \frac{h_1L'}{r} + \frac{2\frac{\partial}{\partial t}H}{r} + \frac{n(n+1)}{2r^2}h_0, \tag{4.86}$$

$$R^{n(1)}_{rr} = -L\frac{\partial^2}{\partial r^2}K + \frac{L\frac{\partial^2}{\partial r^2}h_0}{2} - \frac{\frac{\partial}{\partial r}KL'}{2} + \frac{3L'\frac{\partial}{\partial r}h_0}{4} + \frac{L'\frac{\partial}{\partial r}h_1}{4} - \frac{\partial^2}{\partial t\partial r}H - \frac{\frac{\partial}{\partial t}HL'}{2L}$$
$$+ \frac{\frac{\partial^2}{\partial t^2}h_1}{2L} - \frac{2L\frac{\partial}{\partial r}K}{r} + \frac{L\frac{\partial}{\partial r}h_1}{r} + \frac{n(n+1)}{2r^2}h_1, \tag{4.87}$$

$$R^{n(1)}_{\theta\theta} = -\frac{L\frac{\partial^2}{\partial r^2}K}{2} - \frac{\frac{\partial}{\partial r}KL'}{2} + \frac{\frac{\partial^2}{\partial t^2}K}{2L} - \frac{KL'}{r} - \frac{2L\frac{\partial}{\partial r}K}{r} + \frac{L\frac{\partial}{\partial r}h_0}{2r} + \frac{L\frac{\partial}{\partial r}h_1}{2r} + \frac{h_1L'}{r}$$
$$- \frac{\frac{\partial}{\partial t}H}{r} - \frac{KL}{r^2} + \frac{Lh_1}{r^2} + \frac{n(n+1)}{2r^2}K$$
$$+ \frac{1}{P_n(\cos(\theta))}\left[\frac{h_0\frac{d^2}{d\theta^2}P_n(\cos(\theta))}{2r^2} - \frac{h_1\frac{d^2}{d\theta^2}P_n(\cos(\theta))}{2r^2}\right], \tag{4.88}$$

$$R^{n(1)}_{\varphi\varphi} = -\frac{L\frac{\partial^2}{\partial r^2}K}{2} - \frac{\frac{\partial}{\partial r}KL'}{2} + \frac{\frac{\partial^2}{\partial t^2}K}{2L} - \frac{KL'}{r} - \frac{2L\frac{\partial}{\partial r}K}{r} + \frac{L\frac{\partial}{\partial r}h_0}{2r} + \frac{L\frac{\partial}{\partial r}h_1}{2r} + \frac{h_1L'}{r}$$
$$- \frac{\frac{\partial}{\partial t}H}{r} - \frac{KL}{r^2} + \frac{Lh_1}{r^2} + \frac{n(n+1)}{2r^2}K$$
$$+ \frac{1}{P_n(\cos(\theta))}\left[\frac{h_0\frac{d}{d\theta}P_n(\cos(\theta))}{2r^2\tan(\theta)} - \frac{h_1\frac{d}{d\theta}P_n(\cos(\theta))}{2r^2\tan(\theta)}\right], \tag{4.89}$$

$$R^{n(1)}_{rt} = -\frac{H\frac{d^2}{dr^2}L}{2} - \frac{\partial^2}{\partial t\partial r}K + \frac{\frac{\partial}{\partial t}KL'}{2L} - \frac{HL'}{r} - \frac{\frac{\partial}{\partial t}K}{r} + \frac{\frac{\partial}{\partial t}h_1}{r} + \frac{n(n+1)}{2r^2}H \tag{4.90}$$

$$R^{n(1)}_{t\theta} = \frac{1}{2}\left(HL' + L\frac{\partial}{\partial r}H - \frac{\partial}{\partial t}K - \frac{\partial}{\partial t}h_1\right), \tag{4.91}$$

$$R^{n(1)}_{r\theta} = \frac{1}{2L}\left(-L\frac{\partial}{\partial r}K + L\frac{\partial}{\partial r}h_1 + h_1L' - \frac{\partial}{\partial t}H\right). \tag{4.92}$$

There is no mixing between odd and even parity perturbations at the level of GR, each type perturbs a separate sector of the Ricci tensor. The same will also hold in WCG as the perturbed sectors of the Bach tensor are also separate.

Looking at the equation coming from $B^{(1)}_{t\theta}$

$$\left(L\frac{\partial}{\partial r}R^{n(1)}_{rt} - L\frac{\partial^2}{\partial r^2}R^{n(1)}_{t\theta} + L\frac{\partial^2}{\partial t\partial r}R^{n(1)}_{r\theta} + R^{n(1)}_{rt}L' - \frac{2\frac{\partial}{\partial t}R^{n(1)}_{\varphi\varphi}}{3} + \frac{\frac{\partial}{\partial t}R^{n(1)}_{\theta\theta}}{3} - \frac{2\frac{\partial}{\partial t}R^{n(1)}_{rr}}{3} - \frac{\frac{\partial}{\partial t}R^{n(1)}_{tt}}{3} + \frac{2L\frac{\partial}{\partial t}R^{n(1)}_{r\theta}}{r} - \frac{2R^{n(1)}_{t\theta}L'}{r}\right)\frac{d}{d\theta}P_n\left(\cos(\theta)\right) - \frac{P_n\left(\cos(\theta)\right)\frac{\partial}{\partial t}R^{n(1)}_{\phi\phi}}{\tan\left(\theta\right)} + \frac{P_n\left(\cos(\theta)\right)\frac{\partial}{\partial t}R^{n(1)}_{\theta\theta}}{\tan\left(\theta\right)} = 0, \tag{4.93}$$

we notice that there are two terms with a different angular dependence, having a factor of $\frac{P_n(\cos(\theta))}{\tan(\theta)}$ instead of $\frac{d}{d\theta}P_n\left(\cos(\theta)\right)$. If all of the equations have to be satisfied for all values of θ, then because the functions we are solving for depend only on r and t, these terms have to be set to zero separatelly from the rest of the equations. Therefore we have

$$\frac{\partial}{\partial t}R^{n(1)}_{\theta\theta} = \frac{\partial}{\partial t}R^{n(1)}_{\phi\phi}. \tag{4.94}$$

By the same argument from $B^{(1)}_{r\theta}$, which is equal to

$$\left(-\frac{2\frac{\partial}{\partial r}R^{(1)}_{\varphi\varphi}}{3} + \frac{\frac{\partial}{\partial r}R^{(1)}_{\theta\theta}}{3} + \frac{\frac{\partial}{\partial r}R^{(1)}_{rr}}{3} + \frac{2\frac{\partial}{\partial r}R^{(1)}_{tt}}{3} + \frac{R^{(1)}_{rr}L'}{2L} + \frac{R^{(1)}_{tt}L'}{2L} + \frac{\frac{\partial^2}{\partial t^2}R^{(1)}_{r\theta}}{L} - \frac{\frac{\partial}{\partial t}R^{(1)}_{rt}}{L} - \frac{\frac{\partial^2}{\partial t\partial r}R^{(1)}_{t\theta}}{L} - \frac{R^{(1)}_{\varphi\varphi}}{3r} - \frac{R^{(1)}_{\theta\theta}}{3r} - \frac{2R^{(1)}_{r\theta}L'}{r} + \frac{2R^{(1)}_{rr}}{3r} - \frac{2R^{(1)}_{tt}}{3r} + \frac{2\frac{\partial}{\partial t}R^{(1)}_{t\theta}}{rL} - \frac{2LR^{(1)}_{r\theta}}{r^2}\right)\frac{d}{d\theta}P_n\left(\cos(\theta)\right) - \frac{P_n\left(\cos(\theta)\right)\frac{\partial}{\partial r}R^{(1)}_{\phi\phi}(r,t)}{\tan\left(\theta\right)} + \frac{P_n\left(\cos(\theta)\right)\frac{\partial}{\partial r}R^{(1)}_{\theta\theta}(r,t)}{\tan\left(\theta\right)} = 0, \tag{4.95}$$

we obtain

$$\frac{\partial}{\partial r}R^{n(1)}_{\theta\theta} = \frac{\partial}{\partial r}R^{n(1)}_{\phi\phi}. \tag{4.96}$$

As a result, the two function must only differ by a constant. If the perturbations were to describe solution for a time dependent initial value problem, then they would be fixed by initial conditions. Demanding that the black hole starts unperturbed at $t = -\infty$, the constant should be zero. If we were to consider static perturbations sourced by a spherically symmetric source, which we will consider later in this work, the resulting spacetime again should be spherically symmetric and therefore the constant should again be zero[1]. Therefore we conclude that

$$R^{n(1)}_{\theta\theta} = R^{n(1)}_{\phi\phi}. \tag{4.97}$$

Importantly, by substracting the two terms, we get

$$0 = R^{n(1)}_{\theta\theta} - \frac{R^{n(1)}_{\phi\phi}}{\sin^2(\theta)} = \frac{1}{2}\left[h_0\frac{d^2}{d\theta^2}P_n\left(\cos(\theta)\right) - h_0\frac{\frac{d}{d\theta}P_n\left(\cos(\theta)\right)}{\tan(\theta)} - h_1\frac{d^2}{d\theta^2}P_n\left(\cos(\theta)\right) + h_1\frac{\frac{d}{d\theta}P_n\left(\cos(\theta)\right)}{\tan(\theta)}\right] \tag{4.98}$$

[1]This argument also applies in the case of nonzero right hand side. A spherically symmteric energy-momentum tensor will have $T_{\varphi\varphi} = \sin^2(\theta)T_{\theta\theta}$

resulting in $h_0 = h_1$, a constraint that also arises in GR. Now looking at $B^{(1)}_{\theta\theta}$ we see that there are again problematic terms present, namely

$$-\frac{2R^{(1)}_{tt}}{3\tan(\theta)}\frac{d}{d\theta}P_n\left(\cos(\theta)\right), \quad \frac{2R^{(1)}_{rr}}{3\tan(\theta)}\frac{d}{d\theta}P_n\left(\cos(\theta)\right), \quad -\frac{2R^{(1)}_{\theta\theta}}{3\tan(\theta)}\frac{d}{d\theta}P_n\left(\cos(\theta)\right)$$

We therefore have

$$R^{(1)}_{tt} = R^{(1)}_{rr} - R^{(1)}_{\theta\theta}. \tag{4.99}$$

From $B^{(1)}_{\theta\theta}\sin^2(\theta) - B^{(1)}_{\varphi\varphi}$ we get

$$\begin{aligned}
&L^2\sin(\theta)\frac{d^2}{d\theta^2}P_n\left(\cos(\theta)\right)\frac{\partial}{\partial r}R^{(1)}_{r\theta} - L^2\cos(\theta)\frac{d}{d\theta}P_n\left(\cos(\theta)\right)\frac{\partial}{\partial r}R^{(1)}_{r\theta}\\
&+ LR^{(1)}_{r\theta}\sin(\theta)L'\frac{d^2}{d\theta^2}P_n\left(\cos(\theta)\right) - LR^{(1)}_{r\theta}\cos(\theta)L'\frac{d}{d\theta}P_n\left(\cos(\theta)\right)\\
&- \sin(\theta)\frac{d^2}{d\theta^2}P_n\left(\cos(\theta)\right)\frac{\partial}{\partial t}R^{(1)}_{t\theta} + \cos(\theta)\frac{d}{d\theta}P_n\left(\cos(\theta)\right)\frac{\partial}{\partial t}R^{(1)}_{t\theta} = 0,
\end{aligned} \tag{4.100}$$

which tells us that

$$L\frac{\partial}{\partial r}\left(LR^{(1)}_{r\theta}\right) = \frac{\partial}{\partial t}R^{(1)}_{t\theta}. \tag{4.101}$$

The final equations are quite long and will be presented in appendix B.

4.3.4 Birkhoff's theorem in WCG

Because of the complexity of the fourth order equations, we might want simplify the problem by considering only spherically symmetric perturbations, dropping the angular dependence. However, the propagation of spherically symmetric perturbation in WCG is prohibited by Birkhoff's theorem. For WCG it was proven in [48] and states

Theorem 4 (Birkhoff's theorem in WCG)**.** The most general spherically symmetric solution to the Bach-Maxwell equations

$$B_{\mu\nu} = \frac{1}{4G_W}\left(F_{\mu\lambda}F^{\lambda}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\mu\nu}F^{\mu\nu}\right), \quad \nabla_\mu F^{\mu\nu} = 0 \tag{4.102}$$

can be brought by coordinate transformations, conformal transformations and gauge transformations of the electromagnetic field into the form

$$ds^2 - \left(ar^2 + br + c + \frac{d}{r}\right)dt^2 + \left(ar^2 + br + c + \frac{d}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{4.103}$$

$$A = \frac{Q}{r}dt, \tag{4.104}$$

where a, b, c, d and Q are constants bound by

$$3bd - c^2 + 1 - \frac{3Q^2}{2G_W} = 0. \tag{4.105}$$

In regions where $ar^2 + br + c + \frac{d}{r} > 0$ the solutions are static.

4.4 Static perturbations

In the next chapter we are going to use the perturbation equations to study the effect of the energy of vacuum state fluctuations of quantum fields on the WCG solutions. In order to do that, we compute the governing equations for static perturbations. In this chapter we will first discuss them in the framework of classical physics.

As discussed in chapter 2, because the FLRW metrics are conformally flat, WCG requires the total traceless energy-momentum to be approximately zero on large scales. In real world the conformal symmetry would be broken by quantum effects and a trace anomaly will develop. However, its amplitude will be very small compared to the energy-momentum tensors induced by macroscopic large structures in the Universe. Therefore we can consider only a traceless energy-momentum tensor when dealing with the effects of classical macroscopic perturbations. The perturbation could be sourced for example by a conformally coupled scalar field or electromagnetic field.

4.4.1 Schwarzschild black hole

In the case of black holes, we are mostly interested in the shift of the horizon or presence of new horizons caused by distribution of matter around the black hole. Because we are starting with a metric of the form

$$ds^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{4.106}$$

we will consider the perturbed metric to be

$$ds^2 = -\left[A(r) + h(r)\right] dt^2 + \left[A(r) + h(r)\right]^{-1} dr^2 + r^2 d\Omega^2. \tag{4.107}$$

The angular part of the metric is fixed by our choice of coordinates. The perturbation K will not be considered here. If we were to solve the full nonlinear equations, we would have started with $g_{\theta\theta} = r^2$ by our choice of coordinates anyway. The off diagonal perturbations are set to zero by requiring that the resulting spacetime is static and spherically symmetric. Finally, because we have derived that in the even parity sector h_0 is equal to h_1, we have $h_{rr} = -h_{tt}/A^2$, which is, after expanding to first order

$$\frac{1}{A+h} = \frac{1}{A} - \frac{h}{A^2}, \tag{4.108}$$

exactly our choice.

For the metric of the form (4.107) the first order perturbation of the Bach tensor around a Schwarzschild black hole background is given by

$$B^{(1)}_{tt} = \frac{1}{6r^5}\Big[4\beta^2 r^3 h'''' + 10\beta^2 r^2 h''' - 12\beta^2 r h'' + 12\beta^2 h' - 4\beta r^4 h'''' - 11\beta r^3 h''' + 8\beta r^2 h'' - 10\beta r h' + 4\beta h + r^5 h''' + 3r^4 h''' - r^3 h'' + 2r^2 h' - 2rh\Big] \tag{4.109}$$

$$B^{(1)}_{rr} = \frac{1}{6r^3\left(2\beta - r\right)}\left[3\beta r^2 h''' + 6\beta r h'' - 6\beta h' - r^3 h''' - r^2 h'' + 2rh' - 2h\right] \tag{4.110}$$

$$B^{(1)}_{\theta\theta} = \frac{1}{12r^2}\Big[-2\beta r^3 h'''' - 2\beta r^2 h''' + 12\beta r h'' - 12\beta h' + r^4 h'''' + 2r^3 h''' - 2r^2 h'' + 4rh' - 4h\Big] \tag{4.111}$$

$$B^{(1)}_{\varphi\varphi} = \sin^2\left(\theta\right) B^{(1)}_{\theta\theta} \tag{4.112}$$

4.4.2 Homogeneous solution

First we explore the solution of the homogeneous case $T_{\mu\nu} = 0$. Because the equation is linear, if we are able to find the fundamental system, we can use it to find particular solutions for any right hand side through variation of constants. All the equations are not independent. The radial component equation (4.110) is only a third order ODE and as pointed out in [29], in a conformally invariant theory all the information is contained in this equation only. Even though the equation is of third order, the three linearly independent solutions can be easily guessed by looking at the structure of the MK solution. The full Bach equations extend Schwarzschild black hole by a linear and a quadratic term. We can therefore expect that these will also be present in the perturbed Schwarzschild black hole. One can easily check that indeed the fundamental system is formed by the set

$$h_1 = \frac{1}{r}, \quad h_2 = r^2, \quad h_3 = r - 3\beta. \tag{4.113}$$

The general homogeneous solution is then a linear combination of these three. h_1 corresponds to changing the black hole mass while h_2 and h_3 introduce the two additional integration constants of the MK solution. The term $3\beta^2\gamma/r$ is not present at the linear level. We also checked that these solve the other equations.

4.4.3 Schwarzschild-De Sitter black hole

In order to find the fundamental system for the full MK spacetime perturbed equations, it is convenient to first solve the SDS case. Because the two solutions are related by a conformal transformation, we can apply the same transformation to the fundamental system and convert it to the MK one. The perturbation equations for

$$A(r) = 1 - \frac{2\beta}{r} - \kappa r^2 \tag{4.114}$$

are

$$\begin{aligned} B_{tt}^{(1)} = &\frac{1}{6r^5}\Big[4\beta^2r^3h'''' + 10\beta^2r^2h''' - 12\beta^2rh'' + 12\beta^2h' + 4\beta\kappa r^6h'''' \\ &+13\beta\kappa r^5h''' - 6\beta\kappa r^4h'' + 6\beta\kappa r^3h' - 4\beta r^4h'''' - 11\beta r^3h''' + 8\beta r^2h'' \\ &-10\beta rh' + 4\beta h + \kappa^2r^9h'''' + 4\kappa^2r^8h''' - 2\kappa r^7h'''' - 7\kappa r^6h''' - \kappa r^5h'' \\ &-2\kappa r^4h' + 2\kappa r^3h + r^5h'''' + 3r^4h''' - r^3h'' + 2r^2h' - 2rh\Big] \end{aligned} \tag{4.115}$$

$$B_{rr}^{(1)} = \frac{1}{6r^3\left(-2\beta + \kappa r^3 + r\right)}\Big[-3\beta r^2h''' - 6\beta rh'' + 6\beta h' + r^3h''' + r^2h'' - 2rh' + 2h\Big] \tag{4.116}$$

$$\begin{aligned} B_{\theta\theta}^{(1)} = &\frac{1}{12r^2}\Big[-2\beta r^3h'''' - 2\beta r^2h''' + 12\beta rh'' - 12\beta h' - \kappa r^6h'''' - 4\kappa r^5h''' + r^4h'''' \\ &+2r^3h''' - 2r^2h'' + 4rh' - 4h\Big] \end{aligned} \tag{4.117}$$

$$B_{\varphi\varphi}^{(1)} = \sin^2(\theta)\, B_{\theta\theta}^{(1)} \tag{4.118}$$

The radial component equation (4.116) is apart from an additional κr^3 in the prefactor exactly the same as (4.110). The fundamental system is therefore again

$$h_1 = \frac{1}{r}, \quad h_2 = r^2, \quad h_3 = r - 3\beta. \tag{4.119}$$

We have also checked that it solves the other equations as well.

4.4.4 MK black hole

The full perturbed equations around the MK solutions are very long and will not be written here (they are presented in appendix (C.6), (C.7), (C.8)). In order to find their fundamental system we use the relation between the SDS and MK metrics that was found in [35]. Starting with a SDS metric (4.114) we perform a conformal transformation of the form

$$\Omega(r) = \frac{1}{1+\alpha r} \tag{4.120}$$

followed by a coordinate transformation

$$\tilde{r} = \frac{r}{1-\alpha r}. \tag{4.121}$$

This brings the metric into a form with an additional linear term

$$A(r) = 2\alpha^3\beta r^2 - 6\alpha^2\beta r + \alpha^2r^2 + 6\alpha\beta - 2\alpha r - \frac{2\beta}{r} - \kappa r^2 + 1. \tag{4.122}$$

Performing this transformation on the fundamental system of Schwazrschild-De Sitter solution we obtain

$$\frac{1}{r} \to -\frac{(\alpha r-1)^3}{r}, \tag{4.123}$$

$$r^2 \to r^2, \tag{4.124}$$

$$r - 3\beta \to -\left(\alpha r - 1\right)\left(3\beta\left(\alpha r - 1\right) + r\right). \tag{4.125}$$

Plugging these back into the perturbed MK equations we discover that in order to have the correct transformation to a solution of (C.6), (C.7), (C.8), we need to choose

$$\alpha_1 = \frac{\gamma}{3\beta\gamma - 2}, \quad \alpha_3 = \frac{\gamma}{6\beta\gamma - 4}, \tag{4.126}$$

where the α_1 factor has to be applied to the original $\frac{1}{r}$ solution and α_3 to $r - 3\beta$. We arrive at the fundamental system for the MK solution[2]

$$h_1 = \frac{(-3\beta\gamma + \gamma r + 2)^3}{r}, \tag{4.127}$$

$$h_2 = r^2, \tag{4.128}$$

$$h_3 = (3\beta(-6\beta\gamma + \gamma r + 4) + 2r(3\beta\gamma - 2))(-6\beta\gamma + \gamma r + 4). \tag{4.129}$$

The solutions were checked to solve the other equations as well.

4.4.5 Nonhomogeneous solutions

Having computed the fundamental system, the solutions to nonhomogeneous problems can be straightforwardly found using the method of variation of constants. First we divide the equation by the coefficient in front of h'''. Then after computing the Wronskian

$$W = det \begin{pmatrix} h_1 & h_2 & h_3 \\ h_1' & h_2' & h_3' \\ h_1'' & h_2'' & h_3'' \end{pmatrix}, \tag{4.130}$$

the general solution with the right hand side $S(r)$ can be expressed as

$$h = \sum_{i=1}^{3} h_i \int \frac{W_i}{W} dr \tag{4.131}$$

where W_i results from the Cramer rule prescription for solving systems of linear equations and is obtained by replacing the i-th column of the Wronski matrix by $0, 0, S$. The explicit forms of the Wronskians are

$$W_S = W_{S-DS} = \frac{6(3\beta - r)}{r^2}, \tag{4.132}$$

$$W_{MK} = \frac{24}{r^2}\Big(729\beta^6\gamma^5 - 486\beta^5\gamma^5 r - 2430\beta^5\gamma^4 + 81\beta^4\gamma^5 r^2 + 1458\beta^4\gamma^4 r + 3240\beta^4\gamma^3 \\ -216\beta^3\gamma^4 r^2 - 1728\beta^3\gamma^3 r - 2160\beta^3\gamma^2 + 216\beta^2\gamma^3 r^2 + 1008\beta^2\gamma^2 r + 720\beta^2\gamma \\ -96\beta\gamma^2 r^2 - 288\beta\gamma r - 96\beta + 16\gamma r^2 + 32r\Big). \tag{4.133}$$

4.4.6 Fourth order Poisson equation

For the case of a spherically symmetric stationary energy-momentum tensor the equation

$$\Delta^2 B = B'''' + \frac{4}{r}B''' = \frac{3G_W\left(T^t_{\ t} - T^r_{\ r}\right)}{B} \tag{4.134}$$

[2]Constant prefactors were removed from the functions as these do not matter for linear equations.

can be derived from the Bach equations [49]. This general case is nonlinear because of the B on the right hand side (unless we have a very specific energy-momentum tensor). If the energy-momentum tensor described a weak source distributed around a very massive black hole or star, we could use the Schwarschild or MK geometry as an approximate solution and linearize the equation around it. This approach could be useful for example to describe the gravitational field generated by a gas cloud surrounding a star provided that the gas mass is small compared to that of the star. Also, because the Universe is homogeneous and isotropic to a high degree of precision on large enough scales, the equation also describes the effect of cosmological background perturbations on black holes.

The expansion around the Schwarzschild metric for $B\Delta^2 B$ yields

$$\left(B\Delta^2 B\right)^{(1)} = -\frac{2\beta h''''}{r} - \frac{8\beta h'''}{r^2} + h'''' + \frac{4h'''}{r}, \tag{4.135}$$

$$\left(B\Delta^2 B\right)^{(2)} = \frac{2\left(rh'''' + 4h'''\right)h}{r}, \tag{4.136}$$

$$\left(B\Delta^2 B\right)^{(n)} = 0 \quad n > 2. \tag{4.137}$$

Because (4.134) comes from the Bach equations, all the solutions to (4.110) will also be solutions of (4.135). The fourth linearly independent one is a simple constant function, that is $h_4 = 1$. The linear combination

$$h = \sum_{i=1}^{4} a_i h_i \tag{4.138}$$

is also a solution of $(B\Delta^2 B)^{(2)} = 0$. Hence we see that (4.134) is a consequence of the Bach equations, but is not equivalent to it as there must be exist another independent equation that would constraint the four integration constants to obtain the MK solution.

Considering the right hand side to be the source of the first order perturbation and dividing by $1 - \frac{2\beta}{r}$, we obtain

$$h'''' + \frac{4}{r}h''' = \frac{3G_W\left(T^t_{\ t} - T^r_{\ r}\right)}{1 - \frac{2\beta}{r}} = S(r). \tag{4.139}$$

The Wronskian of the fundamental system $\{1, 1/r, r, r^2\}$ is

$$W = det \begin{pmatrix} 1/r & r^2 & r & 1 \\ -1/r^2 & 2r & 1 & 0 \\ 2/r^3 & 2 & 0 & 0 \\ -6/r^4 & 0 & 0 & 0 \end{pmatrix} = -\frac{12}{r^4} \tag{4.140}$$

which also confirms the linear independence. Now the completely general solution can be expressed as

$$h = \sum_{i=1}^{4} h_i \int \frac{W_i}{W} dr. \tag{4.141}$$

The explicit evaluation of h can be effectively done using the Laplace expansion of the determinant and yields:

$$h = -\frac{1}{r}\int \frac{Sr^4}{6}dr + r^2 \int \frac{Sr}{6}dr - r\int \frac{Sr^2}{2}dr + \int \frac{Sr^3}{2}dr + \frac{a_1}{r} + a_2 r^2 + a_3 r + a4. \tag{4.142}$$

This is the most general perturbation to the black that can be caused by inhomogeneities in the cosmological background. The generalization to the full MK metric is done trivially by replacing the $1 - \frac{2\beta}{r}$ denominator in (4.139) by (3.27). Our calculation agrees with the result given in [21].

Chapter 5

Quantum field corrections to black holes

In this chapter we compute the tiny corrections to WCG black holes caused by the energy-momentum tensors generated by vacuum fluctuations of quantum fields. Specifically we are going to focus on massive fields with spin 0, 1/2 and 1. First the computation and renormalization of the energy-momentum tensor of quantum fields in curved spacetime will be summarized. Then the DeWitt-Schwinger expansion of the propagator will be used to calculate an approximation of the renormalized expectation value of energy-momentum tensor of massive quantum fields in the Schwarzschild, SDS and MK spacetimes. Finally the first order corrections to the corresponding metrics will be derived.

5.1 Energy-momentum tensor renormalization

We could always add a conformally coupled scalar field to our solutions and set it to zero everywhere in order not to generate a contribution to the traceless energy-momentum tensor of WCG. While this works in classical physics, upon quantization the scalar field will inevitably generate a nonzero energy-momentum tensor because of the nonzero energy of the vacuum state and associated fluctuations. In this section we explain the method that we used for the calculation of the resulting energy-momentum tensor for massive quantum fields. The explanation in the next three sections comes from [10].

In order to avoid confusion, in this explanation the **mostly negative** metric signature $(+,-,-,-)$ will be used, as it is more often used in QFT and the conversion of quantities appearing in QFT can be quite complicated. At the end, we will switch back to $(-,+,+,+)$ when analyzing the resulting effective action. The masses of the quantum fields will be denoted by M. We will use dimensional regularization to identify the divergent parts of the action, so the integrals will be in general performed in d dimensions with the limit $d \to 4$ taken at the end.

Let us start with the field equations of WCG. On the classical level we have the Bach equations with the energy-momentum tensor $T_{\mu\nu}$ serving as a source on the right hand side. If we were to quantize $T_{\mu\nu}$ and replace it with $\hat{T}_{\mu\nu}$, we would have to quantize qravity as well. Even though WCG can be successfully quantized and is renormalizable, this task is not within the scope of this work. A basic introduction to quantization of WCG will be given in the last chapter. We are instead going to work with the vacuum expectation value $\langle\hat{T}_{\mu\nu}\rangle = \langle 0|\hat{T}_{\mu\nu}|0\rangle$ where $|0\rangle$ stands for our choice of vacuum. The field equations we are going to solve will be

$$\alpha B_{\mu\nu} = \langle\hat{T}_{\mu\nu}\rangle, \tag{5.1}$$

where α is a dimensionless coupling constant. As one would expect of QFT, the naively calculated value of $\langle\hat{T}_{\mu\nu}\rangle$ diverges because of the products of two fields taken at the same spacetime points. Therefore it is necessary to introduce a regularization and renormalization scheme that will tame these divergences.

In this work we are going to approach the problem of calculation of $\langle\hat{T}_{\mu\nu}\rangle$ through the adiabatic expansion of the one-loop effective action written in the DeWitt-Schwinger representation of the Feynmann propagator. A more general method providing exact result based on point splitting exists [50]. The core idea is isolating the divergent terms in the limit

$$\langle\hat{T}_{\mu\nu}(x)\rangle = \lim_{x\to x'}\langle\hat{T}_{\mu\nu}(x,x')\rangle, \tag{5.2}$$

where $\langle\hat{T}_{\mu\nu}(x,x')\rangle$ is created by taking the value of one of the fields in the products at x while taking the other one at x'. However, it requires computationally expensive numerical solutions and will not be used here[1].

5.2 One-loop effective action

In order to compute $\langle\hat{T}_{\mu\nu}\rangle$ we are going to look for an effective action W such that

$$\frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g^{\mu\nu}} = \langle\hat{T}_{\mu\nu}\rangle. \tag{5.3}$$

By variation of the generating functional

$$Z[J] = \int \mathscr{D}\phi e^{iS_m[\phi]+i\int d^4xJ\phi} \tag{5.4}$$

we obtain

$$\delta Z[0] = i\int \mathscr{D}\phi\delta S_m[\phi]e^{iS_m[\phi]} = i\langle 0,\mathrm{out}|\delta S_m[\phi]|0,\mathrm{in}\rangle. \tag{5.5}$$

After dividing both sides by $Z[0] = \langle 0,\mathrm{in}|0,\mathrm{out}\rangle$ we see that the effective action W is nothing but

$$W = -i\log\left(Z[0]\right). \tag{5.6}$$

Since $Z[0]$ is proportional to the Green function[2] $Z[0] \propto \sqrt{-\det(G_F)}$, W can be expressed in terms of G_F as

$$W = -\frac{i}{2}\,\mathrm{tr}\left[\log\left(-G_F\right)\right]. \tag{5.7}$$

[1]As the authors of [50] mention, their result can also be used as an purely analytic approximation. However, in most cases, including the ones of our interest, it is badly behaved near the horizons, making it unsuitable for us.

[2]The infinite normalization factor of the functional integral will not affect physics when added to the action, therefore we can forget it.

This is usually called the one-loop effective action. In order to compute the trace we should treat the Green function as an operator on the space spanned by vectors $|x\rangle$, whose scalar product normalizes the vectors to delta functions

$$\langle x|x'\rangle = \frac{\delta(x-x')}{\sqrt{-g(x)}}, \quad G_F(x,x') = \langle x|G_F|x'\rangle. \tag{5.8}$$

The trace is then given by

$$\mathrm{tr}\, G_F = \int d^d x \sqrt{-g}\langle x|G_F|x\rangle. \tag{5.9}$$

5.2.1 DeWitt-Schwinger representation of the Green function

The metric around a point x can be expressed as the expansion around another point x' in the Riemann normal coordinates as

$$\begin{aligned} g_{\mu\nu}(x) &= \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta - \frac{1}{6}\nabla_\gamma R_{\mu\alpha\nu\beta}y^\alpha y^\beta y^\gamma \\ &+ \frac{1}{20}\nabla_\delta\nabla_\gamma R_{\mu\alpha\nu\beta}y^\alpha y^\beta y^\gamma y^\delta + \frac{2}{45}R_{\alpha\mu\beta\lambda}R^\lambda{}_{\gamma\nu\delta}y^\alpha y^\beta y^\gamma y^\delta + ... \end{aligned} \tag{5.10}$$

where y^μ is the separation between the two points and the curvature tensors are evaluated at x'. Plugging in this expansion into the definition of Green function

$$\left(\Box + M^2 + \xi R\right) G_F(x,x') = -\frac{\delta(x-x')}{\sqrt{-g(x)}} \tag{5.11}$$

and converting to Fourier space allows us to find asymptotic expansion

$$\sqrt{-g(x)}G_F(x,x') \approx \int \frac{d^d k}{(2\pi)^n}e^{-iky}\sum_{j=0}^{\infty}\left[a_j(x,x')\left(-\frac{\partial}{\partial m^2}\right)^j \frac{1}{k^2-M^2}\right], \tag{5.12}$$

where the coefficients a_j are geometric terms depending on the separation between x and x'. The first three are

$$a_0(x,x') = 1, \tag{5.13}$$

$$a_1(x,x') = \left(\frac{1}{6}-\xi\right)R - \frac{1}{2}\left(\frac{1}{6}-\xi\right)\nabla_\alpha R y^\alpha - \frac{1}{3}a_{\alpha\beta}y^\alpha y^\beta, \tag{5.14}$$

$$a_2(x,x') = \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R^2 + \frac{1}{3}a^\alpha{}_\alpha, \tag{5.15}$$

where

$$\begin{aligned} a_{\alpha\beta} =& \frac{1}{2}\left(\xi - \frac{1}{6} + \frac{1}{60}\right)\nabla_\alpha\nabla_\beta R - \frac{1}{40}\Box R_{\alpha\beta} + \frac{1}{30}R_\alpha{}^\lambda R_{\lambda\beta} \\ &- \frac{1}{60}R_\alpha{}^\lambda{}_\beta{}^\rho R_{\lambda\rho} - \frac{1}{60}R^{\lambda\rho\sigma}{}_\alpha R_{\lambda\rho\sigma\beta}. \end{aligned} \tag{5.16}$$

The result can by re-expressed using the identity

$$\frac{1}{k^2-M^2+i\epsilon} = -i\int_0^\infty ds e^{is(k^2-M^2+i\epsilon)} \tag{5.17}$$

as

$$G_F(x,x') = -i\Delta^{1/2}(x,x')(4\pi)^{-d/2}\int_0^\infty ids(is)^{-d/2}e^{-iM^2s+(\sigma/2is)}\sum_{j=0}^\infty a_j(x,x')(is)^j, \tag{5.18}$$

where $\sigma = \frac{1}{2}y_\mu y^\mu$ and $\Delta(x,x')$ is given by

$$-\frac{det(\partial_\mu\partial_\nu\sigma(x,x'))}{\sqrt{g(x)\ g(x')}}, \tag{5.19}$$

which reduces to $1/\sqrt{-g}$ in the Riemann normal coordinates around x'. We have also dropped the $i\epsilon$ term in the final expression as we are not going to explicitly refer to it anymore.

5.2.2 Effective action renormalization

Having expanded $G_F(x,x')$ we can use this representation to expand the one-loop effective potential. Comparing (5.18) with

$$G_F = -\left(\Box + M^2 + \xi R\right)^{-1}, \tag{5.20}$$

which can be expressed as

$$\left(\Box + M^2 + \xi R\right)^{-1} = i\int_0^\infty dse^{-is\left(\Box+M^2+\xi R\right)}, \tag{5.21}$$

we can directly read the matrix elements $\langle x| - is\,(\Box + M^2 + \xi R)\,|x'\rangle$ to be

$$i\Delta^{1/2}(x,x')(4\pi)^{-d/2}(is)^{d/2}e^{-iM^2s+(\sigma/2is)}\sum_{j=0}^\infty a_j(x,x')(is)^j. \tag{5.22}$$

Now if add a small imaginary part to the exponent and use the approximation

$$\int_a^\infty \frac{e^{-is\left(\Box+M^2+\xi R\right)}}{is}ids = -\gamma - \log\left(ia\left(\Box + M^2 + \xi R\right)\right) - \mathcal{O}\left(ia\left(\Box + M^2 + \xi R\right)\right), \tag{5.23}$$

we have

$$\log\left(-G_F\right) = \int_0^\infty \frac{e^{-is\left(\Box+M^2+\xi R\right)}}{is}ids + N_{log}, \tag{5.24}$$

where N_{log} as an infinite constant coming from taking $a \to 0$. Since it is a constant, it will not affect the resulting physics when included in the action and we can safely ignore it. Finally putting this into the trace formula (5.9) an expression of the effective Lagrangian of the form

$$L_{eff} = \lim_{x\to x'}\frac{\Delta^{1/2}(x,x')}{2(4\pi)^{d/2}}\sum_{j=0}^\infty a_j(x,x')\int_0^\infty ids(is)^{j-1-d/2}e^{-i(M^2s-\sigma/2s)} \tag{5.25}$$

emerges. Now taking $d \to 4$ and the coincidence limit $x \to x'$ and $\sigma \to 0$ brings it into the form

$$L_{eff} = \frac{1}{32\pi^2}\sum_{j=0}^\infty \frac{a_j}{M^{2(i-j)}}\Gamma(j-2), \tag{5.26}$$

where

$$a_i = \lim_{x \to x'} a_i(x, x') \tag{5.27}$$

and Γ is the gamma function. From here we can isolate the divergent terms from the finite part. In the next section, we are going to present the resulting effective action that will be used in our calculation. For that, we will switch back to the **mostly positive** metric signature $(-, +, +, +)$ from now on.

5.3 Lowest order finite term

The first three terms of (5.26) are divergent because of the poles of the gamma function and have to be supplemented by counterterms in order to obtain the renormalized values. Among these are the sources of the C^2 and R^2 terms in (1.24). The first finite coefficient is a_3, which generates the resulting renormalized effective action [51]

$$\begin{aligned} W_{\text{finite}} =& \frac{1}{192\pi^2 M^2} \int d^4x \sqrt{-g} \Big(c_1 R \Box R + c_2 R_{pq} \Box R^{pq} + c_3 R^3 + c_4 R R_{pq} R^{pq} \\ &+ c_5 R_{pq} R^p{}_r R^{qr} + c_6 R_{pq} R_{rs} R^{prqs} + c_7 R R_{pqrs} R^{pqrs} + c_8 R_{pq} R^p{}_{rst} R^{qrst} \\ &+ c_9 R_{pqrs} R^{pquv} R^{rs}{}_{uv} + c_{10} R_{prqs} R^p{}_u{}^q{}_v R^{rusv} \Big) . \end{aligned} \tag{5.28}$$

This result is valid for fields of any spin and the coefficients c_i depend on the spin of the field and are summarized in the following table

Coefficient c_i list			
Field spin	$s = 0$	$s = 1/2$	$s = 1$
c_1	$(1/2)\xi^2 - (1/5)\xi + 1/56$	$-3/280$	$-27/280$
c_2	$1/140$	$1/28$	$9/28$
c_3	$-(\xi - 1/6)^3$	$1/864$	$-5/72$
c_4	$(1/30)(\xi - 1/6)$	$-1/180$	$31/60$
c_5	$-8/945$	$-25/756$	$-52/63$
c_6	$2/315$	$47/1260$	$-19/105$
c_7	$-(1/30)(\xi - 1/6)$	$-7/1440$	$-1/10$
c_8	$1/1260$	$19/1260$	$61/140$
c_9	$17/7560$	$29/7560$	$-67/2520$
c_{10}	$-1/270$	$-1/108$	$1/18$

Computing the variation of (5.28) is an extremely tedious task and has been done in a many articles, see for example [52][51]. We are going to use the result from [52] as it is presented in a nicely organized way that makes it easy to copy and implement it in any computer algebra system or programming language.

Finally we should note that the resulting approximation is only valid for fields with a large enough mass as it is an expansion in powers of $1/M^2$. We will see that the resulting expressions are divergent in the $M \to 0$ limit. Also we should notice that the resulting expression is a purely geometric term totally independent on the quantum state of the field. We have not specified any boundary conditions on the Green function. Because the low order expansion is a good approximation in a small neighbourhood of any point, the energy-momentum tensor constructed this way probes well the behavior of short wavelength modes of the field, and not the nonlocal contributions of the low frequency modes sensitive to the vacuum state.

5.4 Trace anomaly and compensation

Having computed the renormalized energy-momentum tensor we proceed to use it a source term in a suitable set of field equations describing the resulting curvature. As we focus on WCG, a logical choice would be

$$aB_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle \tag{5.29}$$

where a a coupling constant. This equation choice is, however, inconsistent due to the trace of $\langle \hat{T}_{\mu\nu} \rangle$. Because we are working with massive fields that have the mass explicitly built in into their Lagrangians, unlike in the case of dynamical mass generation used by Mannheim, the resulting energy-momentum tensor has a nonzero trace.

In fact there is no way around this problem on the level of $\langle \hat{T}_{\mu\nu} \rangle$, because of a curved spacetime QFT phenomenon called trace anomaly. Even if we start with a massless and conformally invariant field action, the vacuum expectation value of the energy-momentum tensor develops a nonzero trace through the renormalization procedure. This makes sense as the introduction of renormalization scale breaks the conformal symmetry.

If we want to have the curvature in our equations described by a pure Bach tensor term, we need a traceless energy-momentum. As demonstrated in [21], traceless energy-momentum tensors for matter can be obtained if the masses are generated by symmetry breaking in originally conformally invariant theory.

Exploring the behaviour of renormalized energy-momentum tensors of pairs of interacting fields is beyond the scope of this work. Instead we compensate the trace part of $T_{\mu\nu}$ by postulating the presence of small curvature corrections to WCG. The field equations consider are

$$aB_{\mu\nu} + bD_{\mu\nu} + cG_{\mu\nu} + \Lambda g_{\mu\nu} + \text{o.c.} = \langle \hat{T}_{\mu\nu} \rangle, \tag{5.30}$$

where a, b, c are renormalized coupling constants that have to be fixed by experiment and

$$D_{\mu\nu} = 2RR_{\mu\nu} - \frac{1}{2}R^2 g_{\mu\nu} + 2\left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right) R \tag{5.31}$$

is the left hand side of the R^2 gravity field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \tag{5.32}$$

is the Einstein tensor, Λ is the renormalized cosmological constant and *o.c.* stands for other corrections. These extra corrections will be responsible for matching the traces and we will be left with

$$\left(aB_{\mu\nu} + bD_{\mu\nu} + cG_{\mu\nu} + \Lambda g_{\mu\nu} + \text{o.c.}\right)_{traceless} = \langle \hat{T}_{\mu\nu} \rangle - \frac{1}{4} g_{\mu\nu} \langle T \rangle. \tag{5.33}$$

If we require the theory to be dominated by the Bach tensor term, we can neglect the other traceless term on the left hand side leaving us with

$$aB_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle - \frac{1}{4} g_{\mu\nu} \langle T \rangle \tag{5.34}$$

The background value of $B_{\mu\nu}$ is zero, so we replace it by its first order perturbation

$$aB^{(1)}_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle - \frac{1}{4} g_{\mu\nu} \langle T \rangle. \tag{5.35}$$

We should also perturb the right hand side of (5.35). However, not only would this have a negligible effect for small perturbations as $\langle \hat{T}_{\mu\nu} \rangle$ is already microscopic in magnitude, but also the resulting expression would contain higher derivatives of the perturbation as the $1/M^2$ order of the DeWitt-Schwinger expansion is already made of up to sixth derivatives of the metric. We therefore do not perturb it. Should any significant new terms in the metric appear, we can always recalculate $\langle \hat{T}_{\mu\nu} \rangle$ to reflect the correction. We will set the coupling constant a equal to one in our calculations.

5.5 Black hole horizon corrections

5.5.1 Evaluation of the energy-momentum tensor

Evaluation of the variation of (5.28) by hand is (realistically speaking) impossible, unless our metric has a special structure and most of the terms drop out (Ricci flat metrics for example). Because our favorite curvature tensor manipulation software is the GraviPy library for Python, using it we have created a script that computes all the necessary terms for us. The results of the calculation were checked against the results from [52][53] for the cases of Schwarzschild and Reissner-Nordstrom metric. Another consistency check was made by checking that the resulting expressions satififsy the conservation law $\nabla_\mu \langle \hat{T}^{\mu\nu} \rangle = 0$. This was successfully achieved for spin 1/2 and 1 fields, but failed for a scalar field on the MK spacetime. For this reason, we are not going to consider it in the full MK solution analysis. We were not able locate the origin of the small nonzero $\nabla_\mu \langle \hat{T}^{\mu\nu} \rangle$ that is generated in this case. As we will see, scalar fields behave in a similar way as spin 1/2 fields, so we do not miss on any interesting solutions by not considering them. A more detailed description of the program is given in appendix F.

5.5.2 Schwarzschild solution

The renormalized energy-momentum tensor components for spin 0, 1/2 and 1 for Schwarzschild spacetime are presented in A. They are completely regular at the horizon $r = 2\beta$ a rapidly fall towards zero as $r \to \infty$. Their rather simple structure allows for explicit integration of the Cramer rule Wronskians with a closed form expression for the resulting perturbation.

The results also represent the tiny correction to Newton's law of gravity resulting from the vacuum energy of quantum fields. In the Newtonian limit we have

$$g_{00} = -1 - 2\phi, \tag{5.36}$$

where ϕ is the gravitational potential. The radial force is then equal to $F_r = -\partial_r \phi$. To understand how tiny this additional force is, we have to remember that we set $\hbar$ and the WCG coupling constant equal to 1 in our calculations.

Dirac field correction

For a spin 1/2 field the resulting correction is

$$
\begin{aligned}
h = & -\frac{41\beta^2}{20160M^2\pi^2r^4} + \frac{29\beta}{100800M^2\pi^2r^3} + \frac{29}{60480M^2\pi^2r^2} - \frac{29\log(r)}{45360M^2\beta\pi^2r} \\
& + \frac{29\log(|-3\beta+r|)}{45360M^2\beta\pi^2r} - \frac{319}{272160M^2\beta\pi^2r} + \frac{29\log(r)}{45360M^2\beta^2\pi^2} - \frac{29\log(|-3\beta+r|)}{45360M^2\beta^2\pi^2} \\
& + \frac{29}{54432M^2\beta^2\pi^2} - \frac{29r\log(r)}{136080M^2\beta^3\pi^2} + \frac{29r\log(|-3\beta+r|)}{136080M^2\beta^3\pi^2} - \frac{29r}{408240M^2\beta^3\pi^2} \\
& + \frac{29r^2\log(r)}{1224720M^2\beta^4\pi^2} - \frac{29r^2\log(|-3\beta+r|)}{1224720M^2\beta^4\pi^2}
\end{aligned}
\tag{5.37}
$$

Plotting the potential for $\beta = 1$ gives us

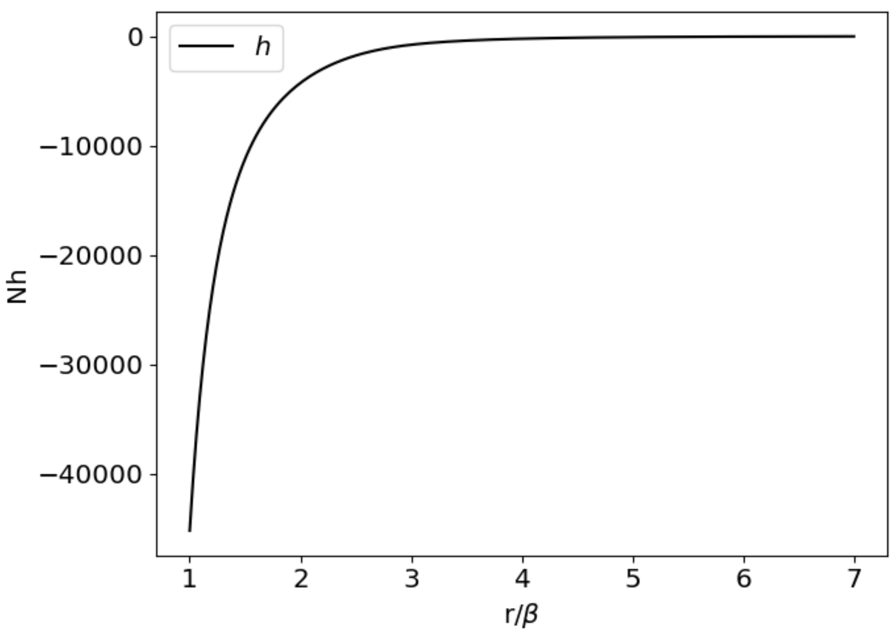

Figure 5.1: The gravitational potential correction for a Schwarzschild black hole with $\beta = 1$ generated by presence of a spin 1/2 quantum field. $N = 24494400M^2\pi^2\beta^2$.

where we multiplied h by a normalization factor $N = 24494400M^2\pi^2\beta^2$. The limits $r \to \infty$ and $\beta \to 0$ are both equal to zero. Despite the singularity in the Wronskian at $r = 3\beta$ h is completely regular and well behaved everywhere.

Proca field correction

A spin 1 field generates the correction

$$
\begin{aligned}
h =& \frac{51\beta^2}{4480M^2\pi^2r^4} - \frac{113\beta}{67200M^2\pi^2r^3} - \frac{113}{40320M^2\pi^2r^2} + \frac{113\log\left(|r|\right)}{30240M^2\beta\pi^2r} \\
&- \frac{113\log\left(|3\beta - r|\right)}{30240M^2\beta\pi^2r} + \frac{1243}{181440M^2\beta\pi^2r} - \frac{113\log\left(|r|\right)}{30240M^2\beta^2\pi^2} + \frac{113\log\left(|3\beta - r|\right)}{30240M^2\beta^2\pi^2} \\
&- \frac{113}{36288M^2\beta^2\pi^2} + \frac{113r\log\left(|r|\right)}{90720M^2\beta^3\pi^2} - \frac{113r\log\left(|3\beta - r|\right)}{90720M^2\beta^3\pi^2} + \frac{113r}{272160M^2\beta^3\pi^2} \\
&- \frac{113r^2\log\left(|r|\right)}{816480M^2\beta^4\pi^2} + \frac{113r^2\log\left(|3\beta - r|\right)}{816480M^2\beta^4\pi^2}
\end{aligned}
\tag{5.38}
$$

Plotting the potential for $\beta = 1$ gives us

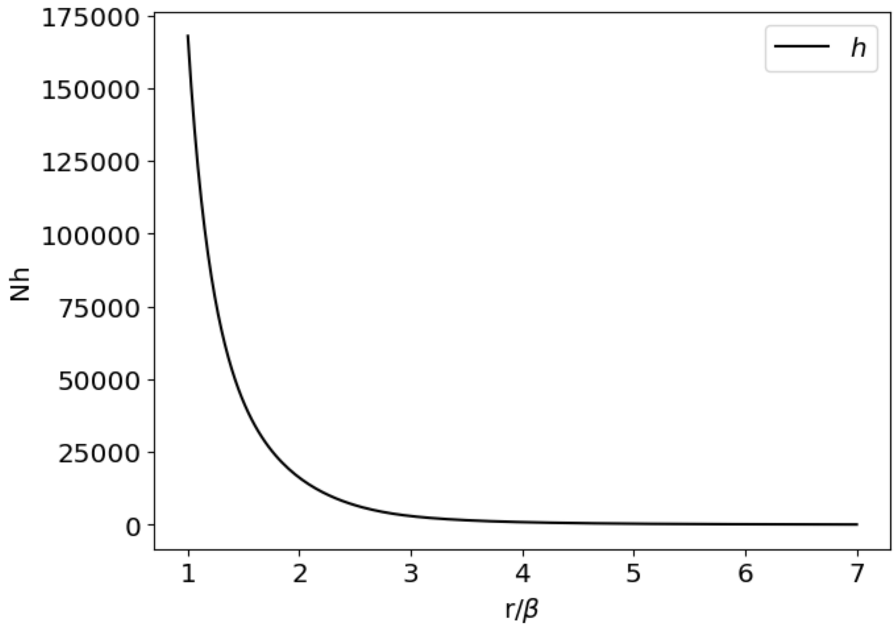

Figure 5.2: The gravitational potential correction for a Schwarzschild black hole with $\beta = 1$ generated by presence of a spin 1 quantum field. $N = 16329600M^2\pi^2\beta^2$.

where we multiplied h by a normalization factor $N = 16329600M^2\pi^2\beta^2$. The limits $r \to \infty$ and $\beta \to 0$ are again both equal to zero as expected. The huge difference when compared to the Dirac field result is the opposite sign of the potential. Proca fields source a repulsive gravitational potential. This can be also seen from the opposite sign of the correspoding energy-momentum tensor for low values of r/β. Near the $r = 0$ singularity the limit

$$\lim_{r\to 0} h(r) = +\infty \tag{5.39}$$

overpowers the original $1/r$ term and allows for geodesics that avoid the singularity. This is, however, a purely mathematical thought, because the presence of an actual observer (especially a macroscopic one) would heavily disturb the structure of the inner horizons created by the quantum fields. One should also remember that the quantization of spin 1 gauge fields introduces the Faddeev-Popov ghosts into the model, whose contribution to the renormalized energy-momentum tensor also should be included. Their effect could potentially change the behaviour of the correction.

Conformally coupled scalar field

To complete the solutions we also present the result for a conformally coupled scalar field. The correction is

$$\begin{aligned} h = &- \frac{71\beta^2}{40320M^2\pi^2r^4} + \frac{11\beta}{40320M^2\pi^2r^3} + \frac{11}{24192M^2\pi^2r^2} - \frac{11\log(|r|)}{18144M^2\beta\pi^2r} \\ &+ \frac{11\log(|3\beta - r|)}{18144M^2\beta\pi^2r} - \frac{121}{108864M^2\beta\pi^2r} + \frac{11\log(|r|)}{18144M^2\beta^2\pi^2} - \frac{11\log(|3\beta - r|)}{18144M^2\beta^2\pi^2} \\ &+ \frac{55}{108864M^2\beta^2\pi^2} - \frac{11r\log(|r|)}{54432M^2\beta^3\pi^2} + \frac{11r\log(|3\beta - r|)}{54432M^2\beta^3\pi^2} - \frac{11r}{163296M^2\beta^3\pi^2} + \\ &\frac{11r^2\log(|r|)}{489888M^2\beta^4\pi^2} - \frac{11r^2\log(|3\beta - r|)}{489888M^2\beta^4\pi^2}. \end{aligned} \tag{5.40}$$

When plotted for $\beta = 1$ we see similar behaviour to the spin 1/2 case:

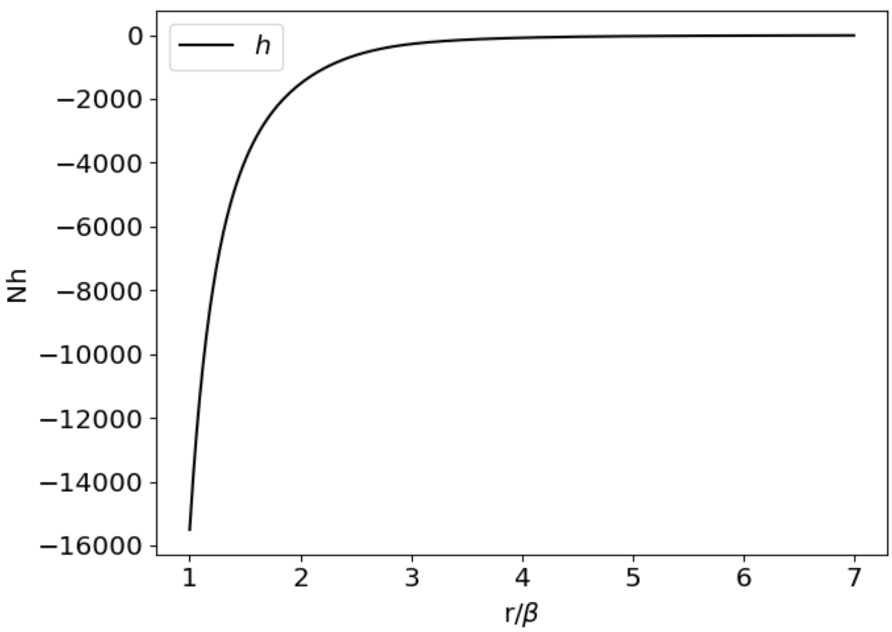

Figure 5.3: The gravitational potential correction for a Schwarzschild black hole with $\beta = 1$ generated by presence of a conformally coupled spin 0 quantum field. $N = 9797760M^2\pi^2\beta^2$.

The normalization factor N is equal to $9797760M^2\pi^2\beta^2$ this time. Again both limits $r \to \infty$ and $\beta \to 0$ are zero.

5.5.3 Structure of the corrections

The corrections are composed of $1/r^4$, $1/r^3$, $1/r^2$ and $1/r$ terms and an additional expression containing logarithms. The inverse quartic and linear term have an opposite sign compared to the others. The inverse quartic one determines the behaviour near the $r = 0$ singularity.

The term proportional to $1/r$ represents a correction to the black hole mass. However, it can be set to zero because $1/r$ is also one of the homogeneous solutions. The other two homogeneous solution contributions are to be set by boundary conditions. The fact that no corrections of the form r or r^2 are present agrees with Mannheim's observation that these terms in the MK solution should be sourced mainly by the effect of the homogeneous cosmological background and its inhomogeneties respectively [34].

To explore the residual logarithmic term we will take (5.40) as an example. Let us denote the expression remaining after subtracting the $1/r^4$, $1/r^3$, $1/r^2$ and $1/r$ terms by $\rho(r)$. Near the horizon, we have

$$\lim_{r\to 0} \rho(r) = +\infty. \tag{5.41}$$

When plotted, we observe that it generates a repulsive potential all the way to the $r = 0$ singularity:

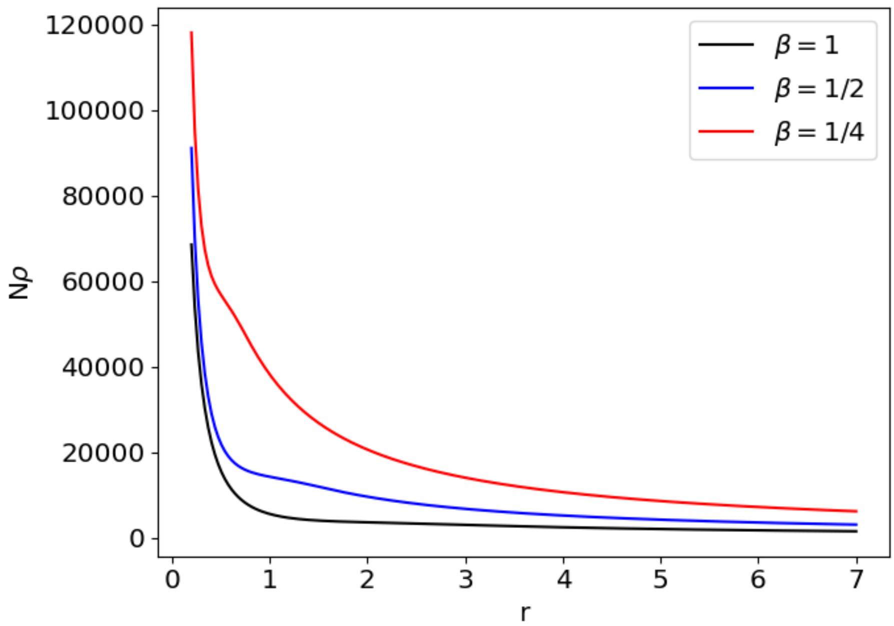

Figure 5.4: The residual gravitational potential correction for a Schwarzschild black hole generated by presence of a conformally coupled spin 0 quantum field after subtracting the terms proportional to $1/r^4$, $1/r^3$, $1/r^2$ and $1/r$. $N = 9797760M^2\pi^2$.

In this plot the normalization factor N is the same as in the conformally coupled scalar field case.

5.5.4 Schwarzschild-De Sitter solution

The Schwarzschild solution energy-momentum tensors as well as the resulting correction strenght decays rapidly with increasing distance from the horizon.

When we add the r^2 term into the solution the large distance limits of the energy-momentum tensors will no longer be zero, but instead they will tend towards a constant.

The closed form expression for the correction generated by a spin 1/2 field is very long and therefore will not be presented here. We present it in the appendix (E.1). Despite the $3\beta - r$ term in the denominator, the expression is regular at $r = 3\beta$ with the limit being

$$\lim_{r\to 3\beta} h(r) = \frac{18900\beta^2\kappa - 263}{8164800M^2\beta^2\pi^2}. \tag{5.42}$$

Perhaps surprisingly the limit $r \to \infty$ of the correction is zero. There are no terms in (E.1) that contain only κ, so although κ is present in the expression, the existence of the correction relies on β. Once we remove the black hole, there is no effect on the spacetime. The limit $\beta \to 0$ is again zero, confirming our result is well behaved.

Why does the cosmological constant have no effect can be seen by looking at the right hand side of (4.139). It evaluates to zero. This means that WCG ignores any cosmological constant type perturbations. The effective cosmological constant in Mannheim's model only drives the expansion of the Universe through the sign reversed Einstein-Hilbert term coming from the conformal coupling of S, but is completely decoupled from the local gravity, which is governed by the C^2 term instead. Because (4.139) is valid in any spherically symmetric spacetime, the same result also applies to the full MK solution.

The correction from spin 1 field was computed to be (E.2). Again it generates a repulsive potential that drops to zero rapidly with increasing r. The cosmological constant term it generates is equal to

$$\Lambda_1 = \frac{25\kappa^3}{3360M^2\pi^2}, \tag{5.43}$$

which has the same sign as the one from a spin 1/2 field given by

$$\Lambda_{1/2} = \frac{31\kappa^3}{40320M^2\pi^2}. \tag{5.44}$$

5.5.5 MK solution and the linear term

Finally we compute the massive quantum field corrections to the full MK metric. Due to the Wronskians and the renormalized energy-momentum tensor components being rational functions, a closed form expression for the result can be extracted with the help of a powerful enough computer algebra system. However, due to the possibly irreducible quadratic terms in the denominators coming from the fundamental system (4.127),(4.128),(4.129), the result is an extremely complicated, long and unpractical expression. We were not able to extract the full formula using Sympy in reasonable time.

Looking at the renormalized energy-momentum tensor components for the MK solution presented in appendix D (D.23), we see that the addition of the linear term means a significant change to the equations, compared to the addition of an r^2 term to the Schwarzschild solution. Most importantly, letting $\beta \to 0$ does not lead to a cosmological constant kind of energy-momentum tensor, but other γ dependent terms are present. Because γ is expected to be very small and the terms have high powers of r in them, we might expect them to be important only far away from the black hole horizon, possibly affecting the strength of the additional linear and quadratic terms in the MK solution. In order to examine the effect of the linear term, we will set both κ and β in the equations to zero.

Dirac field correction

In the case of a spin 1/2 field the correction is

$$h = -\frac{3\gamma^4 r^2 \log(|r|)}{8960M^2\pi^2} + \frac{3\gamma^4 r^2 \log(|\gamma r + 2|)}{8960M^2\pi^2} - \frac{9\gamma^3 r \log(|r|)}{4480M^2\pi^2} + \frac{9\gamma^3 r \log(|\gamma r + 2|)}{4480M^2\pi^2} - \frac{3\gamma^3 r}{4480M^2\pi^2} + \frac{9\gamma^2 \log(|\gamma r + 2|)}{2240M^2\pi^2} + \frac{3\gamma^2}{1120M^2\pi^2} + \frac{3\gamma \log(|\gamma r + 2|)}{1120M^2\pi^2 r} + \frac{\gamma}{2016M^2\pi^2 r}. \tag{5.45}$$

The limit as $\gamma \to 0$ is zero as expected. What is different from all the previous corrections is the $r \to \infty$ limit. Here we have

$$\lim_{r\to\infty} \frac{h}{r^2} = \frac{3\gamma^4 \log(|\gamma|)}{8960M^2\pi^2}. \tag{5.46}$$

This means that the additional terms of the MK solution are indeed affected. Terms up to r^2 are generated, unless $\gamma = \pm 1$. In that case, the fastest growing term is given by

$$\lim_{r\to\infty} \frac{h}{\log(r)} = \frac{9}{2240M^2\pi^2}. \tag{5.47}$$

The correction changes its sign in the asymptotic region depending on the sign of $\gamma - 1$. Figure 5.5 shows the corrections for three different values of γ. The normalization factor is $N = 80640M^2\pi^2$.

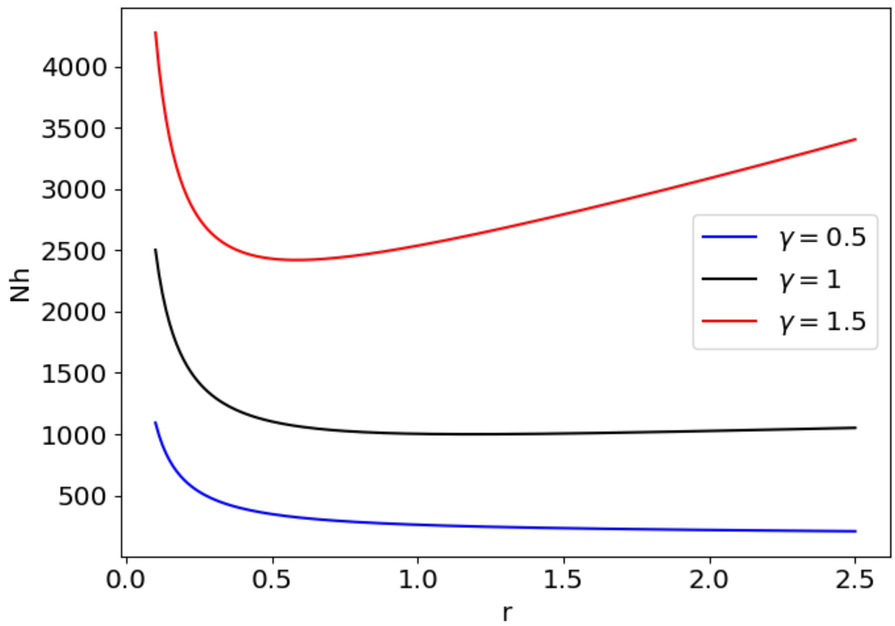

Figure 5.5: The corrections coming from a spin 1/2 field for a MK metric with $\beta = \kappa = 0$. $N = 80640M^2\pi^2$.

Proca field correction

A spin 1 field generates the correction

$$h = -\frac{89\gamma^4 r^2 \log(|r|)}{17920M^2\pi^2} + \frac{89\gamma^4 r^2 \log(|\gamma r+2|)}{17920M^2\pi^2} - \frac{267\gamma^3 r \log(|r|)}{8960M^2\pi^2} + \frac{267\gamma^3 r \log(|\gamma r+2|)}{8960M^2\pi^2} - \frac{89\gamma^3 r}{8960M^2\pi^2} + \frac{267\gamma^2 \log(|\gamma r+2|)}{4480M^2\pi^2} + \frac{89\gamma^2}{2240M^2\pi^2} + \frac{89\gamma \log(|\gamma r+2|)}{2240M^2\pi^2 r} + \frac{397\gamma}{40320M^2\pi^2 r}. \tag{5.48}$$

This again vanishes in the $\gamma \to 0$ limit. The large r behaviour is the same as in the case of spin 1/2 field

$$\lim_{r\to\infty}\frac{h}{r^2} = \frac{89\gamma^4 \log(|\gamma|)}{17920M^2\pi^2}. \tag{5.49}$$

When plotted it again shows a similar behaviour. The normalization factor is $N = 161280M^2\pi^2$.

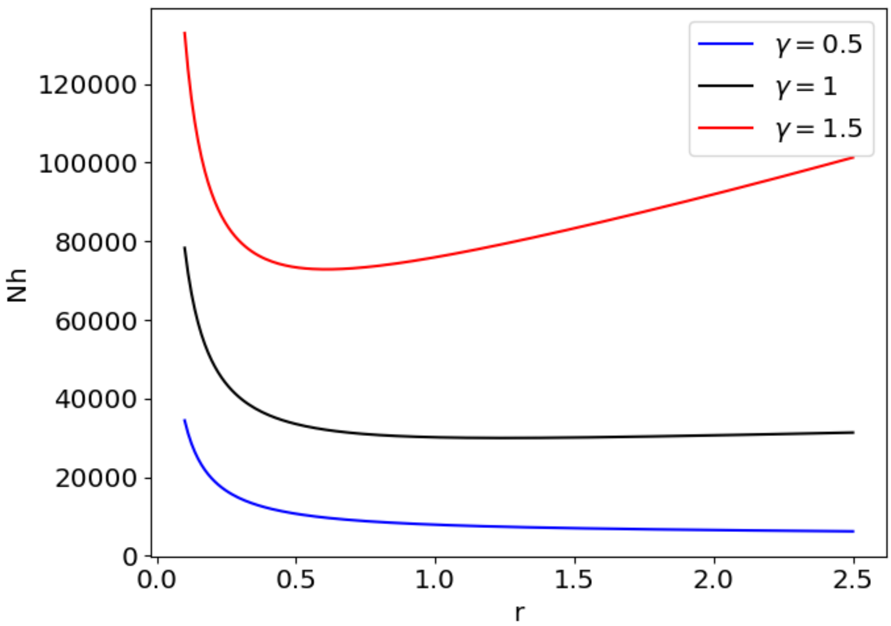

Figure 5.6: The corrections coming from a spin 1 field for a MK metric with $\beta = \kappa = 0$. $N = 161280M^2\pi^2$.

5.5.6 Validity of the results

When interpreting the results, we should remember that they were computed through a perturbative expansion which assumed that the corrections a very small compared to the background potential. This assumption is clearly valid in the case of Schwarzschild and SDS metrics, where the corrections rapidly decay to zero and overpower the $1/r$ potential of the original solution only extremely close to the singularity at $r = 0$. As the classical theory is most likely invalid is this region anyway, it poses no problem.

On the other hand, the corrections for the linear term are unbounded. Unless $|\gamma| = 1$, they grow faster than the background linear term. Their validity thus should be questioned and a solution to the full nonlinear Bach equations should be used instead. If we introduce the quadratic $-\kappa r^2$ term into the background, the calculations start to break down. In the example of a spin 1/2 field, the limit $\gamma \to 0$ is infinite. The limit $(\gamma, \kappa) \to 0$ therefore does not exist as it depends on the order, in which the partial limits are taken. If we integrate the fourth order Poisson equation (4.139) instead, the limit exists and is zero. Furthermore a term proportional to $r^2 \log(r)$ is generated, which again grows faster than the background potential.

5.5.7 Comment on cancellation of the energy-momentum trace

In our calculations we have assumed that the trace anomaly of the energy-momentum tensor is canceled out by additional terms in the field equations whose other effects we completely ignored. In the next chapter we will show that quantization of WCG as done by Mannheim in his articles results in total cancellation of the vacuum state energies of all the other fields by that of gravity.

An attempt was made at solving the system (5.30) with explicitly including the R^2 term variation as well

$$aB_{\mu\nu} + bD_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle. \tag{5.50}$$

The resulting fundamental system for the equation coming from $\langle \hat{T}_{rr} \rangle$ is difficult to obtain as one of the functions seems not to be analytic, because a Laurent series expansion plugged into the equation provides only two nonzero solutions. In the case of nonzero right hand side, the solution to the equation can be represented by Laurent series. For the case of Schwarzschild metric and a spin 1/2 field we get

$$h(x) = \sum_{n=-\infty}^{\infty} a_n x^n, \tag{5.51}$$

$$a_{-2} = 0, \tag{5.52}$$

$$a_{-3} = -\frac{\beta^2}{240M^2\pi^2(5a-48b)}, \tag{5.53}$$

$$a_{-4} = \frac{\beta^3(-19a-12b)}{10080M^2\pi^2(a-12b)(5a-48b)}, \tag{5.54}$$

$$a_{-5} = -\frac{\beta^4\cdot(5a-252b)(19a+12b)}{33600M^2\pi^2(a-12b)(5a-48b)(7a-96b)}, \tag{5.55}$$

$$a_{-6} = -\frac{\beta^5(a-8b)(5a-252b)(19a+12b)}{20160M^2\pi^2(a-15b)(a-12b)(5a-48b)(7a-96b)}, \tag{5.56}$$

$$a_{-7} = -\frac{\beta^6(a-8b)(5a-252b)(7a-60b)(19a+12b)}{22400M^2\pi^2(a-15b)(a-12b)(5a-56b)(5a-48b)(7a-96b)}, \tag{5.57}$$

$$a_{-8} = -\frac{3\beta^7(a-9b)(a-8b)(5a-252b)(7a-60b)(19a+12b)}{1600M^2\pi^2(a-15b)(a-12b)(5a-56b)(5a-48b)(7a-96b)(22a-369b)}, \tag{5.58}$$

$$a_{-9} = ..., \tag{5.59}$$

a_{-1} and a_2 being arbitrary and all the other $a_i, i \geq 0$ being zero. If we assume $b \ll a$, then $h \approx \langle \hat{T}_{\mu\nu} \rangle / a$. However, when balancing the trace, to which only the $bD_{\mu\nu}$ term contributes, we discover that $bh = \langle \hat{T} \rangle$. That means that very roughly $b/a \approx 1$ and both terms would play a non-negligible role in the theory. Therefore more terms would be needed to cancel the trace if the Bach tensor should heavily dominate the curvature side of the equations by multiple orders of magnitude.

Chapter 6

Quantization of conformal gravity

In the last chapter we provide a brief introduction to quantization of WCG. We will present the approach proposed by Mannheim[22][54], which is based on PT-symmetric quantization and cancellation of vacuum state energy. For a more general introduction to quantization of conformal gravity see for example [55].

In order to motivate the quantization we repeat the argument already mentioned when talking about the conformal symmetry breaking in chapter 2. In this work we have explored some phenomenology of classical conformal theory and its solutions without realizing an important question: How is it possible that the solutions like for example (3.27) contain dimensional parameters when they should be forbidden by the conformal symmetry. The answer is of course given by spontaneous breaking of the conformal symmetry. But spontaneous symmetry breaking is quantum effect. In other words, a purely classical conformal gravity would never be able to generate any curvature as it would require the existence of a length scale describing it. Therefore conformal gravity curvature must be generated by a quantum theory.

6.1 Vacuum state energy

In the Lagrangian formulation of physics, two quantities derived from variation of the action are usually studied. The first one are the field equations of motion obtained by variation with respect to the fields. The other one is the energy-momentum tensor, which is obtained by variation with respect to the metric. In the case of matter fields or gauge fields of the forces of nature, these variations are completely independent. Gravity is, on the other hand, different, as the metric itself now becomes the field, and the energy-momentum tensor and the left hand side of the equations of motion coincide. The equations of motion for the gravitational field are then simply

$$\frac{\delta S_{total}}{\delta g^{\mu\nu}} = 0, \tag{6.1}$$

where S_{total} is the action of the whole system including the matter sector.

Due to the the conformal invariance of the C^2 Lagrangian, the Bach tensor must be, as we already mentioned many times, traceless. In the matter section renormalization, we have shown that the matter energy-momentum tensor has both a finite and an infinite part, coming from taking the products of two fields taken at the same spacetime point. Cancelation of the infinite part requires the introduction of (at the level of bare coupling constants also infinite) curvature counterterms.

If we were to keep only the C^2 counterterm, the resulting theory (5.1) would be inconsistent. Because the finite part of matter energy-momentum tensor has a non-zero trace, the gravitational equation of motion (6.1) can not be satisfied.

The solution is to quantize gravity as a well. If the energy of the $\hat{B}_{\mu\nu}$ exactly cancelled that of the matter sector, the problem would be gone, as would also the need to renormalize the matter sector, because the infinite contributions of both sides would have to cancel out identically. If such quantization would be possible, it would have an interesting consequence. In order to generate infinite terms we need products of at least two quantum fields. This can be seen from

$$\langle 0|\hat{a}|0\rangle = 0, \quad \langle 0|\hat{a}^\dagger|0\rangle = 0, \quad \langle 0|\hat{a}\hat{a}^\dagger|0\rangle = 1. \tag{6.2}$$

Therefore when expanding around flat spacetime, the term that cancels the matter field $T_{\mu\nu}$ is not $B^{(1)}_{\mu\nu}$, but $B^{(2)}_{\mu\nu}$. When expanded in powers of $\hbar$, the vacuum energy of matter $T_{\mu\nu}$ is of order $\hbar$, therefore we need $h_{\mu\nu}$ to be of order $\hbar^{1/2}$. We also have to cancel out the whole energy-momentum tensor, including the pressure, not just the energy density component. We also see that it is only necessary to discuss perturbations around flat spacetime, because if $h_{\mu\nu} \approx \hbar^{1/2}$, or in other words there is no classical gravity at all, the higher order contributions to matter energy-momentum tensor will be curvature dependent.

6.2 Quantization of the linearized theory

The general solution for the tensorial part of the linearized WCG equations around flat spacetime (4.21) can be used to construct a quantum theory by insertion of creation and annihilation operators in front of the Fourier modes to get

$$\begin{aligned} h_{\mu\nu} =& \frac{1}{2\sqrt{G_W}} \sum_{i\in\{+,\times\}} \int \frac{d^3k}{(2\pi)^{3/2}\omega_k^{3/2}} \Big[\hat{A}^i(k)A^i_{\mu\nu}e^{-i\omega_k t+ik\cdot x} + \hat{B}^i(k)i\omega_k t B^i_{\mu\nu}e^{-i\omega_k t+ik\cdot x} \\ &+\hat{A}^i_c(k)A^i_{\mu\nu}e^{-i\omega_k t-ik\cdot x} - \hat{B}^i_c(k)i\omega_k t B^i_{\mu\nu}e^{-i\omega_k t-ik\cdot x}\Big]. \end{aligned} \tag{6.3}$$

For reasons that will be seen later we so far do not say that the operators denoted by lower index c are hermitian conjugates of their unindexed counterparts. Now we need to postulate commutation relations between $\hat{A}^i$, $\hat{B}^i$, $\hat{A}^i_c$ and $\hat{B}^i_c$ in such a way, that the resulting $B^{(2)}_{\mu\nu}$ will cancel out the flat spacetime matter $T_{\mu\nu}$. We recall that $B^{(2)}_{\mu\nu}$ around flat spacetime is given by the variation of (4.14). This cancellation cannot be achieved by simply postulating

$$\left[\hat{A}^i(k), \hat{A}^j_c(\tilde{k})\right] = \delta^{ij}\delta(k-\tilde{k}), \quad \left[\hat{A}^i(k), \hat{A}^j(\tilde{k})\right] = 0, \quad \left[\hat{A}^i(k)_c, \hat{A}^j_c(\tilde{k})\right] = 0 \tag{6.4}$$

and the same for $\hat{B}$ and $\hat{B}_c$ operators, as that would lead to time dependence. Instead, the commutators $\left[\hat{B}^i(k), \hat{B}^j_c(\tilde{k})\right]$ have to vanish. The consistent commutation relations are

$$\left[\hat{B}^i(k), \hat{B}^j_c(\tilde{k})\right] = \left[\hat{A}^i(k), \hat{A}^j_c(\tilde{k})\right] = 0 \tag{6.5}$$

$$\left[\hat{A}^i(k), \hat{B}^j(\tilde{k})\right] = \left[\hat{A}^i_c(k), \hat{B}^j_c(\tilde{k})\right] = 0 \tag{6.6}$$

$$\left[\hat{A}^i(k), \hat{B}^j_c(\tilde{k})\right] = \left[\hat{B}^i(k), \hat{A}^j_c(\tilde{k})\right] = Z\delta_{ij}\delta(k-\tilde{k}), \tag{6.7}$$

where Z is to be fixed by the matter sector so that the vacuum state energy cancels out. A gauge boson contributes a positive +1 factor into the divergent zero point energy while each fermion a negative -1. So for a theory with M boson fields and N fermion fields one has to fix $Z = (N - M)/2$ (the factor 1/2 appears because there are two graviton polarizations). The Standard model has $N = 16$ and $M = 12$, so there it leads to a reasonable positive value of Z. This relation puts constraints on the contents of any extensions of the Standard model, as negative Z would lead to unwanted states with negative norms. The resulting Hamiltonian is, after isolating the divergent part denoted as *div*,

$$\hat{H} = \sum_{i\in\{+,\times\}} \int d^3k\omega_k \left[\hat{A}^i_c(k)\hat{B}^i(k) + \hat{B}^i_c(k)\hat{A}^i(k) + 2\hat{B}^i_c(k)\hat{B}^i(k) + \text{div.}\right]. \qquad (6.8)$$

6.2.1 One-particle states

Quantized this way, WCG displays a significant difference compared to GR. The one particle state corresponding to the normal second order theory graviton $\hat{A}^i_c(k)|0\rangle$ is not an eigenstate. This is not a coincidence. As we have already mentioned in the discussion of gravitational waves in WCG, the harmonic waves that solve the second order wave equation are conformally flat and do not carry any energy. These states therefore do not propagate and the single gravitons can not be detected. The fourth order theory gravitons $\hat{B}^i_c(k)|0\rangle$ are eigenstates, but due to the modification of the Hilbert state norm required to keep the theory unitary, which we explain in the next section, they are actually zero norm states and therefore are not physically detectable either.

6.3 Problem of unitarity

The previous choice of commutation relations has another fundamental reason behind it. It makes the theory into a consistent unitary one without any negative norm states present. To see where the negative norm states might come from in a fourth order theory we notice that the action (4.14) does not mix the components of $h_{\mu\nu}$ in any way and therefore we can simplify the analysis by considering a scalar field action instead. Our action we will be

$$S_S = \frac{1}{2}\int d^4x \left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + \left(M_1^2 + M_2^2\right)\partial_\mu\phi\partial^\mu\phi + M_1^2 M_2^2\phi^2\right], \qquad (6.9)$$

where M_1 and M_2 are constants. The action (6.9) leads to the equations of motion

$$\left(\Box - M_1^2\right)\left(\Box - M_2^2\right)\phi = 0, \qquad (6.10)$$

which in the limit $M_1, M_2 \to 0$ reduce to the fourth order wave equation for gravitational waves in WCG (4.18). The resulting propagator is

$$G(x, x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}}{M_2^2 - M_1^2}\left(\frac{1}{k^2 + M_1^2} - \frac{1}{k^2 + M_2^2}\right). \qquad (6.11)$$

As this is a difference of two normal second order theory propagators, we can expect that a set of negative norm states will have to appear as the completeness relations will have to be of the form

$$\sum_n |n\rangle\langle n| - \sum_m |m\rangle\langle m| = 1. \qquad (6.12)$$

The problem lies in the assumption, that the Hamiltonian is a Hermitian operator. While in standard quantum mechanics this is a fundamental postulate of the theory, in Mannheim's quantization of WCG it has to be dropped. The issue of negative norms can be resolved if the operators $\hat{A}_c^i$ and $\hat{B}_c^i$ of (6.3) are not Hermitian conjugates of the unindexed ones, but rather a different operators that annihilate the $H^\dagger$ ground eigenstate $\langle 0_L|$, which itself is not the bra $|0\rangle$. We could generalize the standard Dirac norm $|\psi|^2 = \langle\psi|\psi\rangle$ by thinking of it as $|\psi|^2 = \langle\psi_L|\psi_R\rangle$ where the vectors ψ_R and ψ_L are related by $\hat{H}|\psi_R\rangle = E|\psi_R\rangle$ if and only if $\langle\psi_L|\hat{H}^\dagger = E\langle\psi_L|$, or in other words we have to distinguish between left and right eigenvalues of the Hamiltonian. If the Hamiltonian is Hermitian, then $\langle\psi_L| = |\psi_R\rangle^\dagger$ and we are back at the classical quantum theory.

To relate the action (6.9) to the original (4.14), we notice that when taking the limit $M_1, M_2 \to 0$, the four linearly independent solutions to (6.10) collapse to only two, because the ω_k from the dispersion relations coincide for $M_1 = M_2 = 0$. The additional time dependent solution does not suddenly appear magically, but instead was hidden in the difference of the pairs of the original solutions. The linear combination

$$\frac{2i\omega_k}{M_1^2 - M_2^2}\left(\psi_1 - \psi_2\right), \tag{6.13}$$

where ψ_1 and ψ_2 are the solutions to (6.10) that incorporate M_1 and M_2 respectively into their dispersion relations, gives the correct limit. However, as it is not an eigenstate of $i\partial_t$, it will not be a solution of the time-independent Schrodinger equation. The Hamiltonian (6.8) therefore does not have enough eigenstates to be diagonalizable and so cannot be Hermitian either. However, the theory is still unitary[56].

6.3.1 PT symmetry and real eigenvalues

With a non-Hermitian Hamiltonian, one may doubt the reality of its eigenvalues. For the case discussed here, it can be shown that there exists a similarity transformation S such that $\langle\psi_L| = \langle\psi_R|S$ and for $S = e^{-\hat{Q}}$, $e^{-\hat{Q}/2}\hat{H}e^{\hat{Q}/2}$ is Hermitian [57]. This also shows that $\hat{H}$ has a real spectrum, as one would expect from a reasonable physical theory. The fact that the eigenvalues are real is not a coincidence, but a consequence of PT symmetry. P is the parity operator and T is the time inversion operator, which is anti-Hermitian. The basics of PT symmetric Hamiltonian theory can be found in [58]. Most importantly, PT invariance is both a sufficient and necessary condition for the eigenvalue equation $\hat{H}|\psi\rangle - \lambda|\psi\rangle = 0$ to be real. Because all the poles of the propagator (6.11) lie on the real axis, the WCG Hamiltonian must be PT symmetric.

Conclusion

In this work we have explored Weyl conformal gravity and its implications for black hole physics and cosmology. WCG stands out among the possible alternatives to GR by being a purely metric theory possessing additional local conformal symmetry. The conformal symmetry is well motivated by the almost scale invariant primordial fluctuations that occurred during the inflation epoch of the Universe. At the same time, the theory is also motivated by its Lagrangian being one of the counterterms necessary to renormalize the energy-momentum tensors of quantum fields in curved spacetime. While the presence of up to fourth derivatives in the field equations and the associated risk of instabilities might look like a serious drawback, the Lovelock theorem shows that unless we add additional fields that represent gravity or modify the spacetime dimension, it can not be avoided.

As all the vacuum solutions of GR are also solutions of WCG, the theory leads to viable phenomenology. The most general black hole solution augments the Schwarzschild black hole with two additional terms, a quadratic De Sitter-like background one and a linear one, which might serve as an explanation of the galactic rotational curves without the introduction of any dark matter. The known analytic solutions to the WCG field equations also comprise a rich family of exotic metrics, including wormholes. We have found an interesting wormhole solution that asymptotically approaches the hyperbolic case of the FLRW metric.

Apart from dark matter, WCG also provides a possible solution to the dark energy problem. Because of the conformal flatness of the FLRW metrics, the Lagrangian of WCG is not sensitive to them and only describes the local, attractive gravity. Cosmology is instead driven the Einstein-Hilbert term coming from the spontaneous symmetry breaking. The resulting model are ordinary Friedmann equations, but with opposite sign of the gravitational constant, rendering every component repulsive. The cosmological constant term is the order parameter of the broken symmetry phase representing the negative energy difference between the two phases. The resulting evolution ensures that $0 < \Omega_\Lambda < 1$ at present time, independent of the actual magnitude of the cosmological constant.

Solving the fourth order field equations of WCG poses a significant mathematical challenge. In this work we tried solving them by using the Newman-Penrose formalism. In this approach the fourth order equations were traded for a system a system of coupled second order ones. In the case of a static spherically symmetric metric we found a suitable choice of the NP tetrad that decoupled the equations and allowed us to obtain the Mannheim-Kazanas solution by solving a series of two easily solvable second order differential equations.

The linearized version of the theory around a flat spacetime contains two vector modes, two regular tensor modes and two ghost tensor modes that grow linearly in time. There is no propagating scalar degree of freedom. This structure is preserved by conformal transformations and thus also holds in De Sitter space. There, when supplied by a suitable Neumann boundary condition and the Bunch-Davies vacuum, WCG produces the same result as GR. In the FLRW spacetimes the cosmological perturbations grown faster than in GR.

The first order perturbation theory around the Schwarzschild solution was established and the perturbation equations in the Regge-Wheeler gauge were obtained. They consist of a set of second order PDEs for the Ricci tensor perturbations supported by the perturbation equations of GR relating them to the original metric perturbations. While all of the solutions from GR are also solutions in WCG, more general solutions exist. Their study would be an interesting topic of further research. Also studying the perturbations of the wormhole solutions might lead to new interesting results. The stationary case of the equations describes the effect of perturbations on the cosmological background on the black hole, showing the origin of the additional quadratic term in the Mannheim-Kazanas solution.

The renormalized energy-momentum tensors of massive quantum fields in the black hole spacetimes of WCG were computed. We used the DeWitt-Schwinger representation of the Green function to expand to one-loop effective action up to the order of $1/M^2$, where M is the mass of the field. In the large mass limit the resulting approximation is expected to be very accurate. In the case of Schwarzschild black hole, the resulting expressions rapidly approach zero with increasing distance from the black hole horizon, while in the SDS and full MK cases they approach an effective cosmological constant term.

Corrections to the horizons of the Schwarzschild, SDS and MK black holes were computed, with the results showing additional corrections to the gravitational potential proportional to $1/r$, $1/r^2$, $1/r^3$, $1/r^4$ and an additional term of a more complicated nature, all of which rapidly decay with increasing distance from the horizon. In the cases of spin 0 and spin 1/2 fields these corrections generate an attractive force, for spin 1 field the resulting force is repulsive. The asymptotic contribution to the energy-momentum tensors resembling a cosmological constant does not affect the corrections in any way. This result well agrees with the fact that the cosmological evolution and local gravity are two separate, decoupled phenomena in WCG. In the case of the MK solution, a simplified variant leaving only the linear γ term was examined, with the resulting corrections growing faster than the original potential, indicating a possible breakdown of the linear perturbation theory in the asymptotically nonflat regime.

The canonical quantization based on vacuum state energy cancellation and PT symmetric quantization with a non-Hermitian Hamiltonian was presented. It provides an interesting solution to the potential instabilities and non-unitarity of the fourth order theory. Together with power counting renormalizability, this provides the way to creating a fully consistent quantum theory of gravity, whose classical behavior is well examined and not only matches GR on the Solar system scales, but also provides a possible solutions to the problems of both dark matter and dark energy.

Overall WCG is a rich, well motivated and promising theory of gravity. A lot of work has been done by Mannheim and other authors and huge part of the phenomenology of the theory has been explored. The results provide a possible answers to the cosmological constant problems and can fit the rotational curves of a large set of galaxies without any dark matter required. Also, because of its quadratic action, it is renormalizable in the power counting sense and therefore could serve as a candidate theory of quantum gravity. As such, the theory is definitely worth further research and exploration.

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Appendix

Appendix A

Tetrad and Newman-Penrose formalism

In this appendix the mathematical formulation of gravitational theories will be discussed. Alternative description to the commonly used covariant formalism where the metric tensor is the basic variable will be explained. Such formulation might be not only more practical and useful in solving certain problems, but also sometimes necessary to expand and modify GR.

A Tetrad formalism

As postulated by the equivalence principle, for every observer in spacetime a local inertial frame exists, in which the laws of physics reduce to those of special relativity. In this frame an arbitrary coordinate chart $\hat{x}^\mu$ can be chosen. An observer would naturally perform all measurements in these local coordinates rather than some global coordinates that are typically being used in GR. In terms of $\hat{x}^\mu$ the metric and Christoffel symbols can be expressed as

$$g_{\mu\nu} = \frac{\partial x^\mu}{\partial \hat{x^\mu}} \frac{\partial x^\nu}{\partial \hat{x^\nu}} g_{\hat{\mu}\hat{\nu}}, \qquad \Gamma^\mu{}_{\lambda\alpha} = \frac{\partial \hat{x}^\mu}{\partial x^\lambda} \frac{\partial^2 x^\mu}{\partial \hat{x}^\mu \partial x^\alpha}. \tag{A.1}$$

Clearly these expressions do not depend on the choice of the local coordinates because all the hat indices are contracted in every expression. Because the Christoffel symbols can be expressed in terms of $g_{\mu\nu}$ alone without any reference to the local inertial frame, the Lorentz symmetry and degrees of freedom associated with the choice of local inertial frame completely disappear from the theory when the metric is used as the basic variable from which the curvature terms are constructed. Or in other words, the covariant derivative of a tensor transforms the right way under the local Lorentz transformations, keeping the equations the same in all local inertial frames.

This description in terms of $g_{\mu\nu}$ from an outer observer's viewpoint is definitely useful and practical for exploring a wide variety of gravitational phenomena and applications like for example cosmology. However, it is unsuitable once objects with non-tensorial transformation properties are introduced. The most important class of such objects are fermion fields, i.e. spinors. In order to incorporate spinors into a geometrical theory of gravity, we need to replace the standard covariant approach with what is called the tetrad formalism.

Consider the viewpoint of a free falling observer located somewhere in the spacetime. In order to do physics in his local inertial frame he will choose a set of coordinates in the frame. A logical choice will be orthonormal cartesian coordinates where the time coordinate will always point to his future and the remaining three will span the space around him. The indices corresponding to these coordinates will be denoted by Latin letters instead of the Greek ones which will be kept for the global coordinates.

The local coordinates are spanned by four vectors $\{x_0^\mu, x_1^\mu, x_2^\mu, x_3^\mu\}$. The tranformations between the local and global coordinates are given by projections onto the local basis

$$T^{ab...}{}_{gh...} = \frac{\partial x^a}{\partial x^\mu}\frac{\partial x^b}{\partial x^\nu}...\frac{\partial x^\alpha}{\partial x^g}\frac{\partial x^\beta}{\partial x^h}...T^{\mu\nu...}{}_{\alpha\beta...}\,. \tag{A.2}$$

We introduce new objects called tetrads defined as

$$e^a{}_\mu = \frac{\partial x^a}{\partial x^\mu}, \quad e^\mu{}_a = \frac{\partial x^\mu}{\partial x^a}. \tag{A.3}$$

Obviously $e^\nu{}_a e^a{}_\mu = \delta^\nu{}_\mu$ and $e^a{}_\mu e^\mu{}_b = \delta^a{}_b$. The metric in the local coordinates will always be by definition the flat Minkowsi metric

$$\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1). \tag{A.4}$$

The Greek indices can be raised and lowered by $g_{\mu\nu}$ while the Latin indices are operated by η_{ab}. The knowledge of $e^a{}_\mu$ also implies the knowledge of $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$. The tetrads in a sense serve as the square root of the metric. Making them the fundamental variable instead of the metric, we discover that there is not a one to one correspondence between the two, but there is an infinite amount of tetrads giving the same metric. They can be arbitrarily rotated and boosted. For a Lorentz transformation $e^a{}_\mu = \Lambda^a{}_b e^b{}_\mu$ the metric remains unchanged as

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} = \Lambda^a{}_c \Lambda^b{}_d e^c{}_\mu e^d{}_\nu \eta_{ab} = e^c{}_\mu e^d{}_\nu \eta_{cd}. \tag{A.5}$$

Formulation of our theory in this local basis has the advantage of the additional degree of freedom in the choice of the tetrads. In a sense it is a more general and fundamental description than the metric formalism. This is because not every tetrad choice corresponds a to a set of global coordinates. A set of vectors $\{x_0, x_1, x_2, x_3\}$ spans a Lie algebra with the commutator

$$[x_i, x_j] = f_{ij}{}^k x_k. \tag{A.6}$$

If the tetrad results from a set of coordinates, then $x_i = \partial_i$ and all $f_{ij}{}^k = 0$. On the other hand since $f_{ij}{}^k = x^k\left([x_i, x_j]\right)$, where x^k is the dual (one-form) corresponding to x_k, $f_{ij}{}^k = 0$ implies that $dx^k = 0$. By the Poincaré lemma this closed one form is exact on a sufficiently small enough neighbourhood of the point in consideration. In other words, x_k come from a set of coordinates. A basis is non-coordinate if and only if at least one of $f_{ij}{}^k \neq 0$.

A.1 An example: Dirac equation

We mentioned the the tetrad formalism provides us with a way to incorporate spinors into curved spacetime. As we will be dealing only with the Dirac equation here and not gravity, we will use the **mostly negative** $(+,-,-,-)$ metric signature in this section. The flat spacetime Dirac equation

$$(i\gamma^a\partial_a - m)\,\psi = 0 \tag{A.7}$$

is Lorentz invariant if the spinors ψ transform as

$$\psi \to S\psi, \quad S = \exp\left(-\frac{i}{4}\omega_{ab}\sigma^{ab}\right), \tag{A.8}$$

where $\sigma^{ab} = \frac{i}{2}\left[\gamma^a, \gamma^b\right]$ and γ^i are the constant flat spacetime Dirac matrices forming the Clifford algebra given by $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$.

In order to generalize a theory described by vectors in flat spacetime to curved spacetime, the affine connection is introduced. This ensures that the equations have the same form in every reference frame. In order to do the same thing with spinors, we need a connection that would also transform correctly under the spinorial representation of the local Lorentz transformations. This is where tetrads enter the equation. We define the connection one-form as

$$\omega^a{}_{b\mu} = e^a{}_\lambda \nabla_\mu e^\lambda{}_b. \tag{A.9}$$

The covariant derivative with the correct transformation properties can be expressed as

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \frac{1}{2}\omega_{ab\mu}\sigma^{ab}\psi. \tag{A.10}$$

If we started with a transformation of the form

$$S = \exp\left(-\frac{i}{4}\omega_{ab}M^{ab}\right), \tag{A.11}$$

where M^{ab} are the generators of the fundamental representation of $SO(1,3)$, we would end up with the standard covariant derivative of vector fields. Generally, this method allows us to construct covariant derivatives for objects transforming under any representation of the Lorentz group. Finally in order to plug in the flat spacetime Dirac matrices, we have to introduce one more tetrad and either project the covariant derivative index onto the local basis, or equivalently, define the curved spacetime Dirac matrices as

$$\gamma^\mu = e^\mu{}_a\gamma^a. \tag{A.12}$$

The final result is the Dirac equation in curved spacetime:

$$(i\gamma^\mu\mathcal{D}_\mu - m)\,\psi = 0. \tag{A.13}$$

B Newman-Penrose formalism

In chapter 3 of this work a specific form of the tetrad formalism is used to construct alternative derivation of black hole solutions in conformal gravity. More precisely null tetrads ($x_a x^a = 0$) are employed. The choice of null tetrad is the core idea of what is called the Newman-Penrose (NP) formalism or spin-coefficient formalism as some literature calls it. Four complex null $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ vectors are chosen with the following normalization

$$l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0, \tag{A.14}$$
$$l^\mu n_\mu = -1 \qquad m^\mu \bar{m}_\mu = 1, \tag{A.15}$$
$$l^\mu m_\mu = l^\mu \bar{m}_\mu = n^\mu m_\mu = n^\mu \bar{m}_\mu = 0. \tag{A.16}$$

In other words the metric components in the local basis $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ are set to

$$g_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad g^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.17}$$

The covariant derivatives in the direction of the tetrad vectors are denoted by $l^\mu \nabla_\mu = \mathrm{D}$, $n^\mu \nabla_\mu = \Delta$, $m^\mu \nabla_\mu = \delta$ and $\bar{m}^\mu \nabla_\mu = \bar{\delta}$. The fundamental variables in the Newman-Penrose formalism are the spin coefficients

$$\kappa = -m^\mu \mathrm{D} l_\mu, \quad \tau = -m^\mu \Delta l_\mu, \quad \sigma = -m^\mu \delta l_\mu, \quad \rho = -m^\mu \bar{\delta} l_\mu, \tag{A.18}$$
$$\pi = \bar{m}^\mu \mathrm{D} n_\mu, \quad \nu = \bar{m}^\mu \Delta n_\mu, \quad \mu = \bar{m}^\mu \delta n_\mu, \quad \lambda = \bar{m}^\mu \bar{\delta} n_\mu, \tag{A.19}$$
$$\alpha = -\frac{1}{2}\left(n^\mu \bar{\delta} l_\mu - \bar{m}^\mu \bar{\delta} m_\mu\right), \quad \beta = -\frac{1}{2}\left(n^\mu \delta l_\mu - \bar{m}^\mu \delta m_\mu\right), \tag{A.20}$$
$$\gamma = -\frac{1}{2}\left(n^\mu \Delta l_\mu - \bar{m}^\mu \Delta m_\mu\right), \quad \epsilon = -\frac{1}{2}\left(n^\mu \mathrm{D} l_\mu - \bar{m}^\mu \mathrm{D} m_\mu\right). \tag{A.21}$$

The curvature terms are encoded in five complex Weyl scalars,

$$\Psi_0 = l^\mu m^\nu l^\alpha m^\beta C_{\mu\nu\alpha\beta}, \quad \Psi_1 = l^\mu n^\nu l^\alpha m^\beta C_{\mu\nu\alpha\beta}, \quad \Psi_2 = l^\mu m^\nu \bar{m}^\alpha n^\beta C_{\mu\nu\alpha\beta}, \tag{A.22}$$
$$\Psi_3 = l^\mu n^\nu \bar{m}^\alpha n^\beta C_{\mu\nu\alpha\beta}, \quad \Psi_4 = n^\mu \bar{m}^\nu n^\alpha \bar{m}^\beta C_{\mu\nu\alpha\beta}, \tag{A.23}$$

3 complex Ricci scalars

$$\Phi_{01} = \frac{1}{2} R_{\mu\nu} l^\mu m^\nu, \quad \Phi_{02} = \frac{1}{2} R_{\mu\nu} m^\mu m^\nu, \quad \Phi_{12} = \frac{1}{2} R_{\mu\nu} n^\mu m^\nu \tag{A.24}$$

and 4 real Ricci scalars

$$\Phi_{00} = \frac{1}{2} R_{\mu\nu} l^\mu l^\nu, \quad \Phi_{22} = \frac{1}{2} R_{\mu\nu} n^\mu n^\nu, \tag{A.25}$$
$$\Phi_{11} = \frac{1}{4}\left(R_{\mu\nu} l^\mu n^\nu + R_{\mu\nu} m^\mu \bar{m}^\nu\right), \quad \Lambda = \frac{R}{24}. \tag{A.26}$$

We shall note that the signs of certain quantities may differ between various sources as multiple conventions are used throughout the literature. The convention we use here is that of [59]. Weyl and Ricci scalars can be computed from the spin coefficients through the following set of equations. These come from known identities in differential geometry and are therefore valid in all theories.

$$
\begin{aligned}
D\sigma - \delta\kappa &= \sigma\left(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}\right) + \kappa\left(\bar{\pi} - \tau - 3\beta - \alpha\right) + \Psi_0, && \text{(A.27)}\\
D\rho - \bar{\delta}\kappa &= \left(\rho^2 + \sigma\bar{\sigma}\right) + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau + \kappa(\pi - 3\alpha - \bar{\beta}) + \Phi_{00}, && \text{(A.28)}\\
D\tau - \Delta\kappa &= \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \tau(\epsilon - \bar{\epsilon}) - \kappa(3\gamma + \bar{\gamma}) + \Psi_1 + \Phi_{01}, && \text{(A.29)}\\
D\alpha - \bar{\delta}\epsilon &= \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho) + \Phi_{10}, && \text{(A.30)}\\
D\beta - \delta\epsilon &= \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) + \epsilon(\bar{\pi} - \bar{\alpha}) + \Psi_1, && \text{(A.31)}\\
D\lambda - \bar{\delta}\pi &= (\rho\lambda + \bar{\sigma}\mu) + \pi(\pi + \alpha - \beta) - \nu\bar{\kappa} + \lambda(\bar{\epsilon} - 3\epsilon) + \Phi_{20}, && \text{(A.32)}\\
D\nu - \Delta\pi &= \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\epsilon + \bar{\epsilon}) + \Psi_3 + \Phi_{21}, && \text{(A.33)}\\
\Delta\alpha - \bar{\delta}\gamma &= \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3, && \text{(A.34)}\\
\Delta\lambda - \bar{\delta}\nu &= \lambda(\bar{\gamma} - 3\gamma - \mu - \bar{\mu}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4, && \text{(A.35)}\\
\delta\rho - \bar{\delta}\sigma &= \rho(\bar{\alpha} + \beta) + \sigma(\bar{\beta} - 3\alpha) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01}, && \text{(A.36)}\\
\delta\lambda - \bar{\delta}\mu &= \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21}, && \text{(A.37)}\\
\delta\nu - \Delta\mu &= \left(\mu^2 + \lambda\bar{\lambda}\right) + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22}, && \text{(A.38)}\\
\delta\gamma - \Delta\beta &= \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} + \beta(\mu - \gamma + \bar{\gamma}) + \alpha\bar{\lambda} + \Phi_{12}, && \text{(A.39)}\\
\delta\tau - \Delta\sigma &= (\mu\sigma + \bar{\lambda}\rho) + \tau(\tau + \beta - \bar{\alpha}) + \sigma(\bar{\gamma} - 3\gamma) - \kappa\bar{\nu} + \Phi_{02}, && \text{(A.40)}\\
D\mu - \delta\pi &= (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \nu\kappa + \Psi_2 + 2\Lambda, && \text{(A.41)}\\
\Delta\rho - \bar{\delta}\tau &= -(\rho\bar{\mu} + \sigma\lambda) + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \nu\kappa - \Psi_2 - 2\Lambda, && \text{(A.42)}\\
D\gamma - \Delta\epsilon &= \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda, && \text{(A.43)}\\
\delta\alpha - \bar{\delta}\beta &= (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda. && \text{(A.44)}
\end{aligned}
$$

We can use these identities to easily evaluate the Weyl scalars without having to compute the components of the Weyl tensor. Four of the scalars are readily expressed through equations (A.27), (A.31), (A.34), (A.35):

$$
\begin{aligned}
\Psi_0 &= D\sigma - \delta\kappa - \sigma\left(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}\right) - \kappa\left(\bar{\pi} - \tau - 3\beta - \alpha\right), && \text{(A.45)}\\
\Psi_1 &= D\beta - \delta\epsilon - \sigma(\alpha + \pi) - \beta(\bar{\rho} - \bar{\epsilon}) + \kappa(\mu + \gamma) - \epsilon(\bar{\pi} - \bar{\alpha}), && \text{(A.46)}\\
\Psi_3 &= \bar{\delta}\gamma - \Delta\alpha + \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}), && \text{(A.47)}\\
\Psi_4 &= \bar{\delta}\nu - \Delta\lambda + \lambda(\bar{\gamma} - 3\gamma - \mu - \bar{\mu}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}). && \text{(A.48)}
\end{aligned}
$$

In order to obtain the expression for Ψ_2 we subtract (A.44) from (A.43) and add (A.42). The resulting expression is

$$
\begin{aligned}
\Psi_2 = \frac{1}{3}\Big[&(\bar{\delta} - 2\alpha + \bar{\beta} - \pi - \bar{\tau})\beta - (\delta - \bar{\alpha} + \bar{\pi} + \tau)\alpha + (D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})\gamma\\
&- (\Delta - \bar{\gamma} - \gamma + \bar{\mu} - \mu)\epsilon + (\bar{\delta} - \alpha + \bar{\beta} - \bar{\tau} - \pi)\tau - (\Delta - \bar{\gamma} - \gamma + \bar{\mu} - \mu)\rho\\
&+2(\nu\kappa - \lambda\sigma)\Big].
\end{aligned}
\tag{A.49}
$$

The readily obtainable Ricci scalars are

$$\Phi_{00} = \mathrm{D}\rho - \bar{\delta}\kappa - \left(\rho^2 + \sigma\bar{\sigma}\right) - \rho(\epsilon + \bar{\epsilon}) + \bar{\kappa}\tau - \kappa(\pi - 3\alpha - \bar{\beta}), \tag{A.50}$$

$$\Phi_{10} = \mathrm{D}\alpha - \bar{\delta}\epsilon - \alpha(\rho + \bar{\epsilon} - 2\epsilon) - \beta\bar{\sigma} + \bar{\beta}\epsilon + \kappa\lambda + \bar{\kappa}\gamma - \pi(\epsilon + \rho), \tag{A.51}$$

$$\Phi_{20} = \mathrm{D}\lambda - \bar{\delta}\pi - (\rho\lambda + \bar{\sigma}\mu) - \pi(\pi + \alpha - \beta) + \nu\bar{\kappa} - \lambda(\bar{\epsilon} - 3\epsilon), \tag{A.52}$$

$$\Phi_{22} = \delta\nu - \Delta\mu - \left(\mu^2 + \lambda\bar{\lambda}\right) - \mu(\gamma + \bar{\gamma}) + \bar{\nu}\pi - \nu(\tau - 3\beta - \bar{\alpha}), \tag{A.53}$$

$$\Phi_{12} = \delta\gamma - \Delta\beta - \gamma(\tau - \bar{\alpha} - \beta) - \mu\tau + \sigma\nu + \epsilon\bar{\nu} - \beta(\mu - \gamma + \bar{\gamma}) - \alpha\bar{\lambda}. \tag{A.54}$$

The remaining two independent ones are

$$\Lambda = \frac{1}{2}\left[\mathrm{D}\mu - \delta\pi - (\bar{\rho}\mu + \sigma\lambda) - \pi(\bar{\pi} - \bar{\alpha} + \beta) + \mu(\epsilon + \bar{\epsilon}) + \nu\kappa - \Psi_2\right], \tag{A.55}$$

$$\begin{aligned}\Phi_{11} = \frac{1}{2}&\Big[\mathrm{D}\gamma - \Delta\epsilon + \delta\alpha - \bar{\delta}\beta - \alpha(\tau + \bar{\pi}) - \beta(\bar{\tau} + \pi) + \gamma(\epsilon + \bar{\epsilon}) + \epsilon(\gamma + \bar{\gamma}) \\ &-\tau\pi + \nu\kappa\Big] - \frac{1}{2}\Big[(\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu})\Big].\end{aligned} \tag{A.56}$$

B.1 Petrov classification

The NP formalism and its equations provide a completely different formulation of geometric theories of gravity that is completely equivalent to the original covariant one. While it might look complicated and the NP equations are merely mathematical identities and lack any physical content, the NP quantities indeed have a direct physical interpretation. The Weyl scalars can tell us a lot about the asymptotic behaviour of the gravitational field, that would be hard to read from the covariant form of the field equations. They also provide a shortcut to determining the algebraic symmetries of the Weyl tensor, which are the basis of what is called the Petrov classification. An explanation can be found for example in [60], from where the one provided here is taken.

While the metric, Christoffel symbols and curvature tensor components expressed in global coordinates are independent of the tetrad choice, the NP formalism quantities are not. The possible Lorentz transformations of the NP null tetrad can be split into three categories:

1. Class I rotations - $l \to l$, $n \to n + \bar{a}m + a\bar{m} + a\bar{a}l$, $m \to m + al$, $\bar{m} \to +\bar{a}l$
2. Class II rotations - $l \to l + \bar{b}m + b\bar{m} + b\bar{b}n$, $n \to n$, $m \to m + bn$, $\bar{m} \to \bar{m} + \bar{b}n$
3. Class III rotations - $l \to A^{-1}l$, $n \to An$, $m \to e^{i\theta}m$, $\bar{m} \to e^{-i\theta}\bar{m}$

In these equations a and b are complex and θ is real. The Weyl scalars transform under these as

1. Class I:

$$\Psi_0 \to \Psi_0 \tag{A.57}$$

$$\Psi_1 \to \Psi_1 + \bar{a}\Psi_0 \tag{A.58}$$

$$\Psi_2 \to \Psi_2 + 2\bar{a}\Psi_1 + \bar{a}^2\Psi_0 \tag{A.59}$$

$$\Psi_3 \to \Psi_3 + 3\bar{a}\Psi_2 + 3\bar{a}^2\Psi_1 + \bar{a}^3\Psi_0 \tag{A.60}$$

$$\Psi_4 \to \Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_4 \tag{A.61}$$

2. Class II:

$$\Psi_0 \to \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 \quad \text{(A.62)}$$
$$\Psi_1 \to \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4 \quad \text{(A.63)}$$
$$\Psi_2 \to \Psi_2 + 2b\Psi_3 + b^2\Psi_4 \quad \text{(A.64)}$$
$$\Psi_3 \to \Psi_3 + b\Psi_4 \quad \text{(A.65)}$$
$$\Psi_4 \to \Psi_4 \quad \text{(A.66)}$$

3. Class III:

$$\Psi_0 \to A^{-2}e^{2i\theta}\Psi_0 \quad \text{(A.67)}$$
$$\Psi_1 \to A^{-1}e^{i\theta}\Psi_1 \quad \text{(A.68)}$$
$$\Psi_2 \to \Psi_2 \quad \text{(A.69)}$$
$$\Psi_3 \to Ae^{-i\theta}\Psi_3 \quad \text{(A.70)}$$
$$\Psi_4 \to A^2e^{-2i\theta}\Psi_4 \quad \text{(A.71)}$$

Provided the spacetime is not conformally flat, Ψ_0 can be always made to vanish by a suitable class II rotation. This corresponds to solving the quartic equation[1]

$$\Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0 \quad \text{(A.72)}$$

As b is a complex number, there will always be four solutions. Depending on the multiplicity of the roots, the following classification of Weyl tensors exists:

1. 4 distinct roots - Petrov type I

   Ψ_0 vanishes as result of the class II rotation and Ψ_4 can be made to vanish by a class I rotation. Ψ_1, Ψ_2, Ψ_3 can not be made nonzero by a further class III rotation. Petrov type I spacetimes are those, where Ψ_0 can be made to vanish and at least one of the remaining Weyl scalars is simultaneously nonzero.

2. 1 double root - Petrov type II

   Ψ_0 vanishes, but due to the multiplicity of the root, the derivative with respect to b of (A.72) also holds. Therefore the b that makes Ψ_0 vanish also solves

   $$\Psi_4 b^3 + 3\Psi_3 b^2 + 3\Psi_2 b + \Psi_1 = 0 \quad \text{(A.73)}$$

   and Ψ_1 will also simultaneously vanish. Ψ_4 can be set to zero by a class I rotation and Ψ_2 and Ψ_3 can not be made zero by a further class III rotation. A spacetime is of Petrov type II if Ψ_0 and Ψ_1 can be simultaneously set to zero and there is at the same time at least one nonzero Weyl scalar.

3. 2 double roots - Petrov type D

[1] Ψ_4 can always be made nonzero by a suitable class I rotation, which does not affect Ψ_0.

Ψ_0 turns into $(b-b_1)^2(b-b_2)^2\Psi_4$. By gradually differentiating (A.72) the expressions for the other Weyl scalar transformations turn out to be

$$\Psi_1 \to \frac{\Psi_4}{2}(b-b_1)(b-b_2)(2b-b_1-b_2) \tag{A.74}$$

$$\Psi_2 \to \frac{\Psi_4}{3}\left[(b-b_1)(b-b_2)+\frac{1}{2}(2b-b_1-b_2)^2\right] \tag{A.75}$$

$$\Psi_3 \to \frac{\Psi_4}{2}(2b-b_1-b_2) \tag{A.76}$$

$$\Psi_4 \to \Psi_4 \tag{A.77}$$

Ψ_0 and Ψ_1 is made to vanish by choosing one of the roots, here for example b_1. Then $\Psi_2 \to \frac{\Psi_4}{6}(b_1-b_2)^2$ and $\Psi_3 \to \frac{\Psi_4}{2}(b_1-b_2)$. A subsequent class I rotation with $\bar{a} = \frac{1}{b_2-b_1}$ not only makes Ψ_4 vanish, but also Ψ_3 without affecting Ψ_0 and Ψ_1. A spacetime is of Petrov type D if all Weyl scalars except Ψ_2 can be made to vanish.

4. Triple root - Petrov type III

   As a result of satisfying both first and second derivatives of (A.72) Ψ_0, Ψ_1 and Ψ_2 can all be simultaneously set to zero by a class II rotation and Ψ_4 can be zeroed by a further class I rotation without affecting Ψ_0, Ψ_1 and Ψ_2. As Ψ_3 can not be affected by class I rotation if all Ψ_0, Ψ_1 and Ψ_2 are zero, it will always remain nonzero. A class III transformation will not change it either. A spacetime is of Petrov type III if Ψ_0, Ψ_1 and Ψ_2 can be made to vanish simultaneously and $\Psi_3 \neq 0$ at the same time.

5. Quadruple root - Petrov type N In this case even the third derivative of (A.72) is satisfied and thus a NP tetrad can be chosen such that $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$. If all of these are zero, a class I rotation will not be able to make Ψ_4 to vanish. A spacetime is of Petrov type N if all Ψ_0, Ψ_1, Ψ_2, Ψ_3 can be made to vanish and $\Psi_4 \neq 0$ at the same time.

6. Conformally flat - Petrov type O Finally one should not forget that all of the previous cases assumed at least one of the Weyl scalars to be nonzero. In the opposite case the spacetime is of Petrov type O.

B.2 Physical interpretation and peeling theorem

The asymptotic behaviour of the Weyl scalars can be analytically determined. For asymptotically flat spacetimes[2] the NP equations were integrated [61], with the resulting behaviour being

$$\Psi_0 = \Psi_0^0 r^{-5} + \mathcal{O}(r^{-6}), \tag{A.78}$$

$$\Psi_1 = \Psi_1^0 r^{-4} + \mathcal{O}(r^{-5}), \tag{A.79}$$

$$\Psi_2 = \Psi_2^0 r^{-3} + \mathcal{O}(r^{-4}), \tag{A.80}$$

$$\Psi_3 = \Psi_3^0 r^{-2} + \mathcal{O}(r^{-3}), \tag{A.81}$$

$$\Psi_4 = \Psi_4^0 r^{-1} + \mathcal{O}(r^{-2}). \tag{A.82}$$

(A.83)

[2] In [61] the condition $\Psi_0 \to \mathcal{O}(r^{-5})$ is used to guarantee the asymptotic flatness.

This can be generalized to asymptotically conformally flat spacetimes in the Peeling theorem. As one moves to asymptotic infinity, various components of the gravitational fields decay at different rates and the spacetime gradually transfers through the different Petrov types. Far away the only remaining gravitational field is of Petrov type N, which corresponds to transverse gravitational waves.

We see that Ψ_2 falls of as $\mathcal{O}\left(r^{-3}\right)$. The Weyl tensor contains the information about tidal forces produced by the gravitational field which do not change the volume of an infinitesimal cloud of test particles, only its shape. In pure GR it coincides with the Riemann tensor in vacuum. The r^{-3} falloff nicely agrees with the inverse square Newtonian gravity law. Indeed the Petrov type D spacetimes correspond to gravitational fields of isolated massive objects. Far enough they look like point sources.

Analysis of Petrov type D spacetimes is very convenient in the NP formalism because of the following theorem [60]: If the spacetime is of type D, then $\kappa = \sigma = \nu = \lambda = 0$, $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$. The converse also holds. Based on this it can easily be checked that the Schwarzschild and Kerr spacetimes are of type D. We will use this theorem to show that the solutions to conformal gravity discussed in this work are of Petrov type D.

B.3 Perturbation theory in NP formalism

The NP formalism turn out to be extremely useful for perturbation theory around Petrov type D spacetimes. A small perturbation in the metric translates to perturbation of the tetrad, which in turn results in perturbed spin coefficients and curvature scalars. For the tetrad choice satisfying $\kappa = \sigma = \nu = \lambda = 0$ and $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ Teukolsky derived a master equation for first order perturbations of Ψ_0 and Ψ_4 in GR of the form [62]

$$\begin{aligned}&\left[\left(\mathrm{D} - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho}\right)\left(\Delta - 4\gamma + \mu\right) - \right.\\&\left.\left(\delta + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau\right)\left(\bar{\delta} + \pi - 4\alpha\right) - 3\Psi_2\,\right]\Psi_0 = 4\pi T_0,\end{aligned} \tag{A.84}$$

$$\begin{aligned}&\left[\left(\Delta - 3\gamma + \bar{\gamma} - 4\mu - \bar{\mu}\right)\left(\mathcal{D} - 4\epsilon + \rho\right) - \right.\\&\left.\left(\bar{\delta} + \bar{\tau} - \bar{\beta} - 3\alpha - 4\pi\right)\left(\delta + \tau - 4\beta\right) - 3\Psi_2\,\right]\Psi_4 = 4\pi T_4,\end{aligned} \tag{A.85}$$

where the source terms are given by

$$\begin{aligned}T_0 = &\left(\delta + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau\right)\left[\left(D - 2\epsilon - 2\bar{\rho}\right)T_{lm} - \left(\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta\right)T_{ll}\right]\\&+ \left(D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho}\right)\left[\left(\delta + 2\bar{\pi} - 2\beta\right)T_{lm} - \left(D - 2\epsilon + 2\bar{\epsilon} - \bar{\rho}\right)T_{mm}\right],\end{aligned} \tag{A.86}$$

$$\begin{aligned}T_4 = &\left(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu}\right)\left[\left(\bar{\delta} - 2\bar{\tau} + 2\alpha\right)T_{n\bar{m}} - \left(\Delta + 2\gamma - 2\bar{\gamma} + \bar{\mu}\right)T_{\bar{m}\bar{m}}\right]\\&+ \left(\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi\right)\left[\left(\Delta + 2\gamma + 2\bar{\mu}\right)T_{n\bar{m}} - \left(\bar{\delta} - \bar{\tau} + 2\bar{\beta} + 2\alpha\right)T_{nn}\right].\end{aligned} \tag{A.87}$$

In these equations Ψ_2 is the unperturbed background value and the projections of T represent the projections of the energy-momentum tensor perturbations. Ψ_0 and Ψ_4 are their perturbations around the zero background value. The two equations are completely decoupled, which makes them a very powerful tool for investigating black hole perturbations in GR.

B.4 Maxwell equations in the NP formalism

The Maxwell equations can be efficiently expressed in the NP formalism through the quantities

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu, \quad \phi_1 = \frac{1}{2} F_{\mu\nu} \left(l^\mu n^\nu + \bar{m}^\mu m^\nu \right), \quad \phi_2 = F_{\mu\nu} \bar{m}^\mu n^\nu. \tag{A.88}$$

Under the assumptions $\kappa = \sigma = \nu = \lambda = 0$, $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$, the Maxwell equations with a source 4-current J_μ are [62]

$$\begin{aligned} (D - 2\rho)\phi_1 - \left(\bar{\delta} + \pi - 2\alpha\right) \phi_0 &= 2\pi J_l, \\ (\delta - 2\tau)\phi_1 - (\Delta + \mu - 2\gamma)\phi_0 &= 2\pi J_m, \\ (D - \rho + 2\epsilon)\phi_2 - \left(\bar{\delta} + 2\pi\right) \phi_1 &= 2\pi J_{\bar{m}}, \\ (\delta - \tau + 2\beta)\phi_2 - (\Delta + 2\mu)\phi_1 &= 2\pi J_n. \end{aligned} \tag{A.89}$$

C Bach equations in the Newman-Penrose formalism

The expression for the Bach tensor in the NP formalism was found in [59]. Here we summarize the equations from the article that we used in the derivation of the MK and wormhole solutions. Despite their complicated nature, they simplify drastically under the assumptions on spherical symmetry. In the following formulas *c.c.* stands for complex conjugate of the previous expressions. The fourth derivative part $B^Z_{\mu\nu} = \nabla^\alpha \nabla^\beta C_{\mu\alpha\nu\beta}$ is given by

$$\begin{aligned} B^Z_{ll} &= \bar{\delta}\Psi_0 - \mathrm{D}\bar{\delta}\Psi_1 - \bar{\delta}\mathrm{D}\Psi_1 + \mathrm{DD}\Psi_2 + \lambda\mathrm{D}\Psi_0 + \bar{\sigma}\Delta\Psi_0 + (2\pi - 7\alpha - \bar{\beta})\bar{\delta}\Psi_0 \\ &+ (5\alpha + \bar{\beta} - 3\pi)\mathrm{D}\Psi_1 - \bar{\kappa}\Delta\Psi_1 - \bar{\sigma}\delta\Psi_1 + (3\epsilon + \bar{\epsilon} + 7\rho)\bar{\delta}\Psi_1 \\ &- (\epsilon + \bar{\epsilon} + 6\rho)\mathrm{D}\Psi_2 + \bar{\kappa}\delta\Psi_2 - 5\kappa\bar{\delta}\Psi_2 + 4\kappa\mathrm{D}\Psi_3 \\ &+ \Psi_0[\bar{\kappa}\nu + 4\alpha(3\alpha + \bar{\beta}) - (\epsilon + \bar{\epsilon} + 3\rho)\lambda + \pi(\pi - 7\alpha - \bar{\beta}) + \bar{\sigma}(\mu - 4\gamma) \\ &+ \mathrm{D}\lambda - 4\bar{\delta}\alpha + \bar{\delta}\pi] \\ &+ 2\Psi_1[2\kappa\lambda + \bar{\kappa}(\gamma - \mu) + \rho(5\pi - 9\alpha - 2\bar{\beta}) + \bar{\sigma}(\beta + 2\tau) + \epsilon(2\pi - 4\alpha - \bar{\beta}) \\ &+ \bar{\epsilon}(\pi - \alpha) + \mathrm{D}\alpha - \mathrm{D}\pi + \bar{\delta}\epsilon + 2\bar{\delta}\rho] \\ &+ 3\Psi_2[\kappa(3\alpha + \bar{\beta} - 3\pi) - \bar{\kappa}\tau + \rho(\epsilon + \bar{\epsilon} + 3\rho) - \sigma\bar{\sigma} - \mathrm{D}\rho - \bar{\delta}\kappa] \\ &+ 2\Psi_3[\kappa(\epsilon - \bar{\epsilon} - 5\rho) + \bar{\kappa}\sigma + \mathrm{D}\kappa] + 2\Psi_4\kappa^2 + c.c. \end{aligned} \tag{A.90}$$

$$
\begin{aligned}
B^Z_{ln} &= \bar{\delta}\Delta\Psi_1 - \mathrm{D}\Delta\Psi_2 - \bar{\delta}\delta\Psi_2 + \mathrm{D}\delta\Psi_3 - \lambda\Delta\Psi_0 - \nu\bar{\delta}\Psi_0 \\
&+ 2\nu\mathrm{D}\Psi_1 + (2\pi - \alpha + \bar{\beta})\Delta\Psi_1 + \lambda\delta\Psi_1 + (2\mu - \bar{\mu} - 2\gamma)\bar{\delta}\Psi_1 \\
&+ (\bar{\mu} - 3\mu)\mathrm{D}\Psi_2 + (2\rho - \epsilon - \bar{\epsilon})\Delta\Psi_2 + (\alpha - \bar{\beta} - 2\pi)\delta\Psi_2 + (\bar{\pi} + 3\tau)\bar{\delta}\Psi_2 \\
&+ (2\beta - \bar{\pi} - 2\tau)\mathrm{D}\Psi_3 - \kappa\Delta\Psi_3 + (\epsilon + \bar{\epsilon} - 2\rho)\delta\Psi_3 - 2\sigma\bar{\delta}\Psi_3 + \sigma\mathrm{D}\Psi_4 + \kappa\delta\Psi_4 \\
&+ \Psi_0[\lambda(4\gamma - \mu + \bar{\mu}) + \nu(\alpha - \bar{\beta} - 2\pi) - \bar{\delta}\nu] \\
&+ 2\Psi_1[\gamma(\alpha - \bar{\beta} - 2\pi) - \lambda(\beta + \bar{\pi} + 2\tau) + \mu(\bar{\beta} - \alpha + 2\pi) + \bar{\mu}(\alpha - \pi) \\
&+ \nu(\epsilon + \bar{\epsilon} - 2\rho) + \mathrm{D}\nu - \bar{\delta}\gamma + \bar{\delta}\mu] \\
&+ 3\Psi_2[\kappa\nu + \mu(2\rho - \epsilon - \bar{\epsilon}) - \bar{\mu}\rho + \pi\bar{\pi} + \lambda\sigma + \tau(2\pi - \alpha + \bar{\beta}) - \mathrm{D}\mu + \bar{\delta}\tau] \\
&+ 2\Psi_3[\kappa(\bar{\mu} - 2\mu - \gamma) + \epsilon(\beta - \tau - \bar{\pi}) + \bar{\epsilon}(\beta - \tau) + \rho(\bar{\pi} - 2\beta + 2\tau) \\
&+ \sigma(\alpha - \bar{\beta} - 2\pi) + \mathrm{D}\beta - \mathrm{D}\tau - \bar{\delta}\sigma] \\
&+ \Psi_4[\kappa(4\beta - \bar{\pi} - \tau) + \sigma(\epsilon + \bar{\epsilon} - 2\rho) + \mathrm{D}\sigma] + c.c.
\end{aligned}
\tag{A.91}
$$

$$
\begin{aligned}
B^Z_{lm} &= \bar{\delta}\Delta\Psi_0 - \mathrm{D}\Delta\Psi_1 - \bar{\delta}\delta\Psi_1 + \mathrm{D}\delta\Psi_2 \\
&+ \nu\mathrm{D}\Psi_0 + (\pi - 3\alpha + \bar{\beta})\Delta\Psi_0 + (\mu - \bar{\mu} - 4\gamma)\bar{\delta}\Psi_0 \\
&+ (2\gamma - 2\mu + \bar{\mu})\mathrm{D}\Psi_1 + (\epsilon - \bar{\epsilon} + 3\rho)\Delta\Psi_1 + (3\alpha - \bar{\beta} - \pi)\delta\Psi_1 \\
&+ (2\beta + \bar{\pi} + 4\tau)\bar{\delta}\Psi_1 \\
&- (\bar{\pi} + 3\tau)\mathrm{D}\Psi_2 - 2\kappa\Delta\Psi_2 - (\epsilon - \bar{\epsilon} + 3\rho)\delta\Psi_2 - 3\sigma\bar{\delta}\Psi_2 + 2\sigma\mathrm{D}\Psi_3 + 2\kappa\delta\Psi_3 \\
&+ \Psi_0[(4\gamma - \mu)(3\alpha - \bar{\beta} - \pi) + \bar{\mu}(4\alpha - \pi) + \nu(\bar{\epsilon} - \epsilon - 3\rho) - \lambda\bar{\pi} \\
&+ \mathrm{D}\nu - 4\bar{\delta}\gamma + \bar{\delta}\mu] \\
&+ 2\Psi_1[2\kappa\nu + (\mu - \gamma)(\epsilon - \bar{\epsilon} + 3\rho) - \bar{\mu}(2\rho + \epsilon) + (\beta + 2\tau)(\pi - 3\alpha + \bar{\beta}) \\
&+ \bar{\pi}(\pi - \alpha) + \mathrm{D}\gamma - \mathrm{D}\mu + \bar{\delta}\beta + 2\bar{\delta}\tau] \\
&+ 3\Psi_2[\kappa(\bar{\mu} - 2\mu) + \bar{\pi}\rho + \sigma(3\alpha - \bar{\beta} - \pi) + \tau(\epsilon - \bar{\epsilon} + 3\rho) - \mathrm{D}\tau - \bar{\delta}\sigma] \\
&+ 2\Psi_3[\kappa(2\beta - \bar{\pi} - 2\tau) + \sigma(\bar{\epsilon} - \epsilon - 3\rho) + \mathrm{D}\sigma] + 2\Psi_4\kappa\sigma \\
&+ \delta\delta\bar{\Psi}_1 - \delta\mathrm{D}\bar{\Psi}_2 - \mathrm{D}\delta\bar{\Psi}_2 + \mathrm{D}\mathrm{D}\bar{\Psi}_3 \\
&- 2\bar{\lambda}\delta\bar{\Psi}_0 + 3\bar{\lambda}\mathrm{D}\bar{\Psi}_1 + \sigma\Delta\bar{\Psi}_1 + (4\bar{\pi} - 3\bar{\alpha} - \beta)\delta\bar{\Psi}_1 \\
&+ (\bar{\alpha} + \beta - 5\bar{\pi})\mathrm{D}\bar{\Psi}_2 - \kappa\Delta\bar{\Psi}_2 + (\epsilon - \bar{\epsilon} + 5\bar{\rho})\delta\bar{\Psi}_2 - \sigma\bar{\delta}\bar{\Psi}_2 \\
&+ (3\bar{\epsilon} - \epsilon - 4\bar{\rho})\mathrm{D}\bar{\Psi}_3 - 3\bar{\kappa}\delta\bar{\Psi}_3 + \kappa\bar{\delta}\bar{\Psi}_3 + 2\bar{\kappa}\mathrm{D}\bar{\Psi}_4 \\
&+ \bar{\Psi}_0[\bar{\lambda}(5\bar{\alpha} + \beta - 3\bar{\pi}) - \bar{\nu}\sigma - \delta\bar{\lambda}] \\
&+ 2\bar{\Psi}_1[\kappa\bar{\nu} + \bar{\alpha}(\bar{\alpha} + \beta) + \bar{\pi}(2\bar{\pi} - 3\bar{\alpha} - \beta) - \bar{\lambda}(4\bar{\rho} + \epsilon) + \sigma(\bar{\mu} - \bar{\gamma}) \\
&+ \mathrm{D}\bar{\lambda} - \delta\bar{\alpha} + \delta\bar{\pi}] \\
&+ 3\bar{\Psi}_2[2\bar{\kappa}\bar{\lambda} - \kappa\bar{\mu} + \bar{\pi}(\epsilon - \bar{\epsilon}) + \bar{\rho}(4\bar{\pi} - \bar{\alpha} - \beta) + \sigma\bar{\tau} - \mathrm{D}\bar{\pi} + \delta\bar{\rho}] \\
&+ 2\bar{\Psi}_3(\kappa(\bar{\beta} - \bar{\tau}) + \bar{\kappa}(\beta - 4\bar{\pi}) - \sigma\bar{\sigma} + (\bar{\rho} - \bar{\epsilon})(\epsilon - \bar{\epsilon} + 2\bar{\rho}) + \mathrm{D}\bar{\epsilon} - \mathrm{D}\bar{\rho} - \delta\bar{\kappa}) \\
&+ \bar{\Psi}_4[\bar{\kappa}(5\bar{\epsilon} - \epsilon - 3\bar{\rho}) + \kappa\bar{\sigma} + \mathrm{D}\bar{\kappa}],
\end{aligned}
\tag{A.92}
$$

$$
\begin{aligned}
B^{Z}_{nm} =& \Delta\Delta\Psi_1 - \Delta\delta\Psi_2 - \delta\Delta\Psi_2 + \delta\delta\Psi_3 \\
& - 2\nu\Delta\Psi_0 + (4\mu - 3\gamma + \bar{\gamma})\Delta\Psi_1 + 3\nu\delta\Psi_1 - \bar{\nu}\bar{\delta}\Psi_1 \\
& + \bar{\nu}\mathrm{D}\Psi_2 + (5\tau - \bar{\alpha} - \beta)\Delta\Psi_2 + (\gamma - \bar{\gamma} - 5\mu)\delta\Psi_2 + \bar{\lambda}\bar{\delta}\Psi_2 \\
& - \bar{\lambda}\mathrm{D}\Psi_3 - 3\sigma\Delta\Psi_3 + (\bar{\alpha} + 3\beta - 4\tau)\delta\Psi_3 + 2\sigma\delta\Psi_4 \\
& + \Psi_0[\nu(5\gamma - \bar{\gamma} - 3\mu) + \lambda\bar{\nu} - \Delta\nu] \\
& + 2\Psi_1[\nu(\bar{\alpha} - 4\tau) + \bar{\nu}(\alpha - \pi) - \lambda\bar{\lambda} + (\gamma - \mu)(\gamma - \bar{\gamma} - 2\mu) \\
& - \Delta\gamma + \Delta\mu + \delta\nu] \\
& + 3\Psi_2[\mu(4\tau - \bar{\alpha} - \beta) + \bar{\lambda}\pi - \bar{\nu}\rho + 2\nu\sigma + \tau(\bar{\gamma} - \gamma) + \Delta\tau - \delta\mu] \\
& + 2\Psi_3[\kappa\bar{\nu} - \sigma(\bar{\gamma} + 4\mu) + \tau(2\tau - \bar{\alpha} - 3\beta) + \beta(\bar{\alpha} + \beta) \\
& + \bar{\lambda}(\rho - \epsilon) - \Delta\sigma + \delta\beta - \delta\tau] \\
& + \Psi_4[-\kappa\bar{\lambda} + \sigma(\bar{\alpha} + 5\beta - 3\tau) + \delta\sigma] \\
& - \Delta\mathrm{D}\bar{\Psi}_3 + \Delta\delta\bar{\Psi}_2 + \bar{\delta}\mathrm{D}\bar{\Psi}_4 - \bar{\delta}\delta\bar{\Psi}_3 \\
& - 2\bar{\lambda}\Delta\bar{\Psi}_1 - 2\bar{\nu}\delta\bar{\Psi}_1 + 2\bar{\nu}\mathrm{D}\bar{\Psi}_2 + (3\bar{\pi} + \tau)\Delta\bar{\Psi}_2 + (\bar{\gamma} - \gamma + 3\bar{\mu})\delta\bar{\Psi}_2 + 3\bar{\lambda}\bar{\delta}\bar{\Psi}_2 \\
& + (\gamma - \bar{\gamma} - 3\bar{\mu})\mathrm{D}\bar{\Psi}_3 + (2\bar{\rho} - \rho - 2\bar{\epsilon})\Delta\bar{\Psi}_3 + (\alpha - 3\bar{\beta} + \bar{\tau})\delta\bar{\Psi}_3 \\
& - (2\bar{\alpha} + 4\bar{\pi} + \tau)\bar{\delta}\bar{\Psi}_3 + (3\bar{\beta} - \alpha - \bar{\tau})\mathrm{D}\bar{\Psi}_4 - \bar{\kappa}\Delta\bar{\Psi}_4 + (4\bar{\epsilon} + \rho - \bar{\rho})\bar{\delta}\bar{\Psi}_4 \\
& + 2\bar{\Psi}_0\bar{\lambda}\bar{\nu} + 2\bar{\Psi}_1[\bar{\lambda}(\gamma - \bar{\gamma} - 3\bar{\mu}) + \bar{\nu}(2\bar{\alpha} - 2\bar{\pi} - \tau) - \Delta\bar{\lambda}] \\
& + 3\bar{\Psi}_2[\bar{\lambda}(3\bar{\beta} - \bar{\tau} - \alpha) + \bar{\pi}(3\bar{\mu} - \gamma + \bar{\gamma}) + \bar{\nu}(\rho - 2\bar{\rho}) + \bar{\mu}\tau + \Delta\bar{\pi} + \bar{\delta}\bar{\lambda}] \\
& + 2\bar{\Psi}_3[2\bar{\kappa}\bar{\nu} + (\bar{\epsilon} - \bar{\rho})(\gamma - \bar{\gamma} - 3\bar{\mu}) - \rho(\bar{\gamma} + 2\bar{\mu}) + \tau(\bar{\tau} - \bar{\beta}) + (\bar{\alpha} \\
& + 2\bar{\pi})(\alpha - 3\bar{\beta} + \bar{\tau}) - \Delta\bar{\epsilon} + \Delta\bar{\rho} - \bar{\delta}\bar{\alpha} - 2\bar{\delta}\bar{\pi}] \\
& + \bar{\Psi}_4[\bar{\kappa}(\gamma - \bar{\gamma} - 3\bar{\mu}) + \rho(4\bar{\beta} - \bar{\tau}) + \bar{\rho}(\alpha - 3\bar{\beta} + \bar{\tau}) + 4\bar{\epsilon}(3\bar{\beta} - \bar{\tau} - \alpha) - \bar{\sigma}\tau \\
& - \Delta\bar{\kappa} + 4\bar{\delta}\bar{\epsilon} - \bar{\delta}\bar{\rho}]
\end{aligned}
\tag{A.93}
$$

$$
\begin{aligned}
B^Z_{mm} =&\Delta\Delta\Psi_0 - \Delta\delta\Psi_1 - \delta\Delta\Psi_1 + \delta\delta\Psi_2 \\
&+ (2\mu - 7\gamma + \bar{\gamma})\Delta\Psi_0 + \nu\delta\Psi_0 - \bar{\nu}\bar{\delta}\Psi_0 \\
&+ \bar{\nu}\Psi_1 + (7\tau - \bar{\alpha} + 3\beta)\Delta\Psi_1 + (5\gamma - \bar{\gamma} - 3\mu)\delta\Psi_1 + \bar{\lambda}\bar{\delta}\Psi_1 \\
&- \bar{\lambda}\mathrm{D}\Psi_2 - 5\sigma\Delta\Psi_2 + (\bar{\alpha} - \beta - 6\tau)\delta\Psi_2 + 4\sigma\delta\Psi_3 \\
&+ \Psi_0[\mu(\mu - 7\gamma + \bar{\gamma}) + \nu(\bar{\alpha} - \beta - 3\tau) + \bar{\nu}(4\alpha - \pi) + 4\gamma(3\gamma - \bar{\gamma}) \\
&- \lambda\bar{\lambda} - 4\Delta\gamma + \Delta\mu + \delta\nu] \\
&+ 2\Psi_1[2\nu\sigma - \bar{\nu}(\epsilon + 2\rho) + \bar{\lambda}(\pi - \alpha) + (\bar{\gamma} - 2\gamma)(\beta + 2\tau) \\
&+ (\mu - \gamma)(5\tau - \bar{\alpha} + 2\beta) + \Delta\beta + 2\Delta\tau + \delta\gamma - \delta\mu] \\
&+ 3\Psi_2[\kappa\bar{\nu} + \bar{\lambda}\rho + \sigma(3\gamma - \bar{\gamma} - 3\mu) + \tau(3\tau - \bar{\alpha} + \beta) - \Delta\sigma - \delta\tau] \\
&+ 2\Psi_3[-\kappa\bar{\lambda} + \sigma(\bar{\alpha} + \beta - 5\tau) + \delta\sigma] + 2\Psi_4\sigma^2 \\
&+ \mathrm{D}\mathrm{D}\bar{\Psi}_4 - \mathrm{D}\delta\bar{\Psi}_3 - \delta\mathrm{D}\bar{\Psi}_3 + \delta\delta\bar{\Psi}_2 \\
&- 4\bar{\lambda}\delta\bar{\Psi}_1 + 5\bar{\lambda}\mathrm{D}\bar{\Psi}_2 + \sigma\Delta\bar{\Psi}_2 + (\bar{\alpha} - \beta + 6\bar{\pi})\delta\bar{\Psi}_2 \\
&+ (\beta - 3\bar{\alpha} - 7\bar{\pi})\mathrm{D}\bar{\Psi}_3 - \kappa\Delta\bar{\Psi}_3 + (\epsilon - 5\bar{\epsilon} + 3\bar{\rho})\delta\bar{\Psi}_3 - \sigma\bar{\delta}\bar{\Psi}_3 \\
&+ (7\bar{\epsilon} - \epsilon - 2\bar{\rho})\mathrm{D}\bar{\Psi}_4 - \bar{\kappa}\delta\bar{\Psi}_4 + \kappa\bar{\delta}\bar{\Psi}_4 \\
&+ 2\bar{\Psi}_0\bar{\lambda}^2 + 2\bar{\Psi}_1[\bar{\lambda}(\bar{\alpha} + \beta - 5\bar{\pi}) - \bar{\nu}\sigma - \delta\bar{\lambda}] \\
&+ 3\bar{\Psi}_2[\kappa\bar{\nu} + \bar{\lambda}(3\bar{\epsilon} - \epsilon - 3\bar{\rho}) + \bar{\mu}\sigma + \bar{\pi}(\bar{\alpha} - \beta + 3\bar{\pi}) + \mathrm{D}\bar{\lambda} + \delta\bar{\pi}] \\
&+ 2\bar{\Psi}_3[2\bar{\kappa}\bar{\lambda} - \kappa(2\bar{\mu} + \bar{\gamma}) + \sigma(\bar{\tau} - \bar{\beta}) + (\bar{\rho} - \bar{\epsilon})(2\bar{\alpha} - \beta + 5\bar{\pi}) \\
&+ (\epsilon - 2\bar{\epsilon})(2\bar{\pi} + \bar{\alpha}) - \mathrm{D}\bar{\alpha} - 2\mathrm{D}\bar{\pi} - \delta\bar{\epsilon} + \delta\bar{\rho}] \\
&+ \bar{\Psi}_4[\kappa(4\bar{\beta} - \bar{\tau}) + \bar{\kappa}(\beta - \bar{\alpha} - 3\bar{\pi}) + (\bar{\rho} - 4\bar{\epsilon})(\epsilon - 3\bar{\epsilon} + \bar{\rho}) - \sigma\bar{\sigma} \\
&+ 4\mathrm{D}\bar{\epsilon} - \mathrm{D}\bar{\rho} - \delta\bar{\kappa}],
\end{aligned}
\tag{A.94}
$$

$$
\begin{aligned}
B^Z_{nn} =&\Delta\Delta\Psi_2 - \Delta\delta\Psi_3 - \delta\Delta\Psi_3 + \delta\delta\Psi_4 \\
&- 4\nu\Delta\Psi_1 + (\gamma + \bar{\gamma} + 6\mu)\Delta\Psi_2 + 5\nu\delta\Psi_2 - \bar{\nu}\bar{\delta}\Psi_2 \\
&+ \bar{\nu}\mathrm{D}\Psi_3 + (3\tau - \bar{\alpha} - 5\beta)\Delta\Psi_3 - (3\gamma + \bar{\gamma} + 7\mu)\delta\Psi_3 + \bar{\lambda}\bar{\delta}\Psi_3 \\
&- \bar{\lambda}\mathrm{D}\Psi_4 - \sigma\Delta\Psi_4 + (\bar{\alpha} + 7\beta - 2\tau)\delta\Psi_4 \\
&+ 2\Psi_0\nu^2 + 2\Psi_1[\nu(\gamma - \bar{\gamma} - 5\mu) + \lambda\bar{\nu} - \Delta\nu] \\
&+ 3\Psi_2[\mu(\gamma + \bar{\gamma} + 3\mu) + \nu(\bar{\alpha} + 3\beta - 3\tau) - \lambda\bar{\lambda} - \bar{\nu}\pi + \Delta\mu + \delta\nu] \\
&+ 2\Psi_3[\bar{\nu}(\epsilon - \rho) + \bar{\lambda}(\alpha + 2\pi) + \gamma(2\tau - \bar{\alpha} - 4\beta) + \bar{\gamma}(\tau - \beta) \\
&+ \mu(5\tau - 2\bar{\alpha} - 9\beta) + 2\nu\sigma - \Delta\beta + \Delta\tau - \delta\gamma - 2\delta\mu] \\
&+ \Psi_4[\kappa\bar{\nu} + \bar{\lambda}(\rho - 4\epsilon) - \sigma(\gamma + \bar{\gamma} + 3\mu) + 4\beta(3\beta + \bar{\alpha}) + \tau(\tau - \bar{\alpha} - 7\beta) \\
&- \Delta\sigma + 4\delta\beta - \delta\tau] + c.c.
\end{aligned}
\tag{A.95}
$$

The second Ricci tensor part $B^R_{\mu\nu} = \frac{1}{2}R_{\alpha\beta}C_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}$ is given by

$$B^R_{ll} =\Phi_{20}\Psi_0 + \Phi_{02}\bar{\Psi}_0 - 2\Phi_{01}\Psi_1 - 2\Phi_{10}\bar{\Psi}_1 + \Phi_{00}(\Psi_2 + \bar{\Psi}_2), \tag{A.96}$$

$$B^R_{ln} =\Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 + \Phi_{01}\Psi_3 + 2\Phi_{10}\bar{\Psi}_3 - 2\Phi_{11}(\Psi_2 + \bar{\Psi}_2), \tag{A.97}$$

$$B^R_{lm} =\Phi_{21}\Psi_0 - 2\Phi_{11}\Psi_1 + \Phi_{01}(\Psi_2 - 2\bar{\Psi}_2) + \Phi_{02}\bar{\Psi}_1 + \Phi_{00}\bar{\Psi}_3, \tag{A.98}$$

$$B^R_{nm} =\Phi_{22}\Psi_1 + \Phi_{12}(\bar{\Psi}_2 - 2\Psi_2) + \Phi_{02}\Psi_3 - 2\Phi_{11}\bar{\Psi}_3 + \Phi_{01}\bar{\Psi}_4, \tag{A.99}$$

$$B^R_{mm} =\Phi_{22}\Psi_0 - 2\Phi_{12}\Psi_1 + \Phi_{02}(\Psi_2 + \bar{\Psi}_2) - \Phi_{01}\bar{\Psi}_3 + \Phi_{00}\bar{\Psi}_4, \tag{A.100}$$

$$B^R_{nn} =\Phi_{22}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{12}(\Psi_3 + \bar{\Psi}_3) + \Phi_{02}\Psi_4 + \Phi_{20}\bar{\Psi}_4 \tag{A.101}$$

The rest of the equations can be obtained by complex conjugation. Also the tracelessness of B implies that $B_{ln}=B_{m\bar{m}}$.

Appendix B

Even parity perturbations around Schwarzschild solution

In this appendix we present the full equations for the Ricci tensor perturbations around a Schwarzschild black hole of the form The derivative of L with respect to

$$ds^2 = -L(r)dt^2 + \frac{dr^2}{L(r)} + r^2 d\Omega^2, \quad L(r) = 1 - \frac{2\beta}{r}. \tag{B.1}$$

the radial coordinate r is denoted by L'. With only three independent perturbations h_0, H and K, we expect that only three of the equations will be independent.

$$r\left(L\frac{\partial}{\partial r}R^{n(1)}_{rt}-L\frac{\partial^2}{\partial r^2}R^{n(1)}_{t\theta}+L\frac{\partial^2}{\partial t\partial r}R^{n(1)}_{r\theta}+R^{n(1)}_{rt}L'-\frac{\partial}{\partial t}R^{n(1)}_{rr}\right)+2L\frac{\partial}{\partial t}R^{n(1)}_{r\theta} \\ -2R^{n(1)}_{t\theta}L'=0, \tag{B.2}$$

$$2rL\left(-L\frac{\partial^2}{\partial r^2}R^{n(1)}_{r\theta}-R^{n(1)}_{r\theta}\frac{d^2}{dr^2}L-3L'\frac{\partial}{\partial r}R^{n(1)}_{r\theta}-\frac{\partial}{\partial r}R^{n(1)}_{\theta\theta}+\frac{\partial}{\partial r}R^{n(1)}_{rr}\right) \\ +r\left(-R^{n(1)}_{\theta\theta}L'-2R^{n(1)}_{r\theta}\left(L'\right)^2+2R^{n(1)}_{rr}L'+2\frac{\partial^2}{\partial t^2}R^{n(1)}_{r\theta}-2\frac{\partial}{\partial t}R^{n(1)}_{rt}\right) \\ +4L^2\frac{\partial}{\partial r}R^{n(1)}_{r\theta}-4L^2R^{n(1)}_{r\theta}=0, \tag{B.3}$$

$$-2r^2LR^{n(1)}_{rt}\frac{d^2}{dr^2}L-2r^2L\frac{\partial^2}{\partial t\partial r}R^{n(1)}_{\theta\theta}+r^2L'\frac{\partial}{\partial t}R^{n(1)}_{\theta\theta}-4rLR^{n(1)}_{rt}L'-4rL\frac{\partial}{\partial t}R^{n(1)}_{\theta\theta} \\ +4rL\frac{\partial}{\partial t}R^{n(1)}_{rr}+2Ln(n+1)R^{n(1)}_{rt}-2Ln(n+1)\frac{\partial}{\partial t}R^{n(1)}_{r\theta}-2Ln(n+1)\frac{\partial}{\partial r}R^{n(1)}_{t\theta} \\ +2n(n+1)L'R^{n(1)}_{t\theta}=0, \tag{B.4}$$

$$r^2\left(L^2\frac{\partial^2}{\partial r^2}R^{n(1)}_{\theta\theta}-2L^2\frac{\partial^2}{\partial r^2}R^{n(1)}_{rr}+2LR^{n(1)}_{\theta\theta}\frac{d^2}{dr^2}L-4LR^{n(1)}_{rr}\frac{d^2}{dr^2}L+3LL'\frac{\partial}{\partial r}R^{n(1)}_{\theta\theta}\right. \\ \left.-4LL'\frac{\partial}{\partial r}R^{n(1)}_{rr}+4L\frac{\partial^2}{\partial t\partial r}R^{n(1)}_{rt}+2L'\frac{\partial}{\partial t}R^{n(1)}_{rt}-\frac{\partial^2}{\partial t^2}R^{n(1)}_{\theta\theta}-2\frac{\partial^2}{\partial t^2}R^{n(1)}_{rr}\right) \\ +2rL\left(L\frac{\partial}{\partial r}R^{n(1)}_{\theta\theta}-2L\frac{\partial}{\partial r}R^{n(1)}_{rr}+2R^{n(1)}_{\theta\theta}L'-4R^{n(1)}_{rr}L'+4\frac{\partial}{\partial t}R^{n(1)}_{rt}\right) \\ -Ln(n+1)\left(4L\frac{\partial}{\partial r}R^{n(1)}_{r\theta}+R^{n(1)}_{\theta\theta}+2R^{n(1)}_{r\theta}L'-2R^{n(1)}_{rr}\right)=0, \tag{B.5}$$

$$r^2\left(-2L'\frac{\partial}{\partial t}R^{n(1)}_{rt}-\frac{\partial^2}{\partial t^2}R^{n(1)}_{\theta\theta}+2\frac{\partial^2}{\partial t^2}R^{n(1)}_{rr}\right)-n(n+1)L\left(4L\frac{\partial}{\partial r}R^{n(1)}_{r\theta}+R^{n(1)}_{\theta\theta}\right. \\ \left.+2R^{n(1)}_{r\theta}L'-2R^{n(1)}_{rr}\right)+L\left(-3r^2L\frac{\partial^2}{\partial r^2}R^{n(1)}_{\theta\theta}+2r^2L\frac{\partial^2}{\partial r^2}R^{n(1)}_{rr}-3r^2L'\frac{\partial}{\partial r}R^{n(1)}_{\theta\theta}\right. \\ \left.+4r^2L'\frac{\partial}{\partial r}R^{n(1)}_{rr}-4r^2\frac{\partial^2}{\partial t\partial r}R^{n(1)}_{rt}-6rL\frac{\partial}{\partial r}R^{n(1)}_{\theta\theta}+4rL\frac{\partial}{\partial r}R^{n(1)}_{rr}\right)=0, \tag{B.6}$$

$$-r^3L^2\frac{\partial^2}{\partial r^2}R^{n(1)}_{\theta\theta}-r^3LL'\frac{\partial}{\partial r}R^{n(1)}_{\theta\theta}+r^3\frac{\partial^2}{\partial t^2}R^{n(1)}_{\theta\theta}-6r^2L^2\frac{\partial}{\partial r}R^{n(1)}_{\theta\theta}+4rL^2R^{n(1)}_{rr} \\ +4r^2L^2\frac{\partial}{\partial r}R^{n(1)}_{rr}-4r^2LR^{n(1)}_{\theta\theta}L'+4r^2LR^{n(1)}_{rr}L'-4r^2L\frac{\partial}{\partial t}R^{n(1)}_{rt}-4rL^2R^{n(1)}_{\theta\theta} \\ +rLn(n+1)R^{n(1)}_{\theta\theta}-4L^2n(n+1)(r)R^{n(1)}_{r\theta}=0. \tag{B.7}$$

Appendix C

Mannheim-Kazanas solution

In this section the formulas related to the Mannheim-Kazanas solution that too long to be displayed in the main text are presented. We assume the MK metric metric of the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \tag{C.1}$$

$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2. \tag{C.2}$$

A Curvature scalars in the nonsingular gauge

The curvature scalars in the nonsingular gauge are given by the following expressions. Importantly, they are all equal to zero at the coordinate origin $r = 0$.

$$R = \frac{6r\left(-12\beta^2\gamma l^2 + 7\beta\gamma l^2 r + \beta\gamma r^3 + 8\beta l^2 - \gamma l^2 r^2 - \gamma r^4 + 2\kappa r^5 - 2l^2 r\right)}{\left(l^2+r^2\right)^3}, \tag{C.3}$$

$$\begin{aligned}
R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = {} & \frac{2r^2}{\left(l^2+r^2\right)^{10}}\Big[\left(l^2+r^2\right)^2\Big(3\beta^2\gamma l^4 + 24\beta^2\gamma l^2 r^2 - 3\beta^2\gamma r^4 \\
& -24\beta\gamma l^2 r^3 - 2\beta l^4 - 16\beta l^2 r^2 + 2\beta r^4 - \gamma l^4 r^2 + 8\gamma l^2 r^4 + \gamma r^6 + 2\kappa l^4 r^3 \\
& -8\kappa l^2 r^5 - 2\kappa r^7 + 8l^2 r^3\Big)^2 + 2\left(l^2+r^2\right)^2\Big(15\beta^2\gamma l^4 + 30\beta^2\gamma l^2 r^2 + 3\beta^2\gamma r^4 \\
& -6\beta\gamma l^4 r - 18\beta\gamma l^2 r^3 - 10\beta l^4 - 20\beta l^2 r^2 - 2\beta r^4 + 4\gamma l^2 r^4 + \kappa l^4 r^3 - 4\kappa l^2 r^5 \\
& -\kappa r^7 + 2l^4 r + 6l^2 r^3\Big)^2 + 2\left(l^2+r^2\right)^2\Big(-3\beta^2\gamma l^4 + 6\beta^2\gamma l^2 r^2 - 3\beta^2\gamma r^4 + 3\beta\gamma l^4 r \\
& -6\beta\gamma l^2 r^3 + 3\beta\gamma r^5 + 2\beta l^4 - 4\beta l^2 r^2 + 2\beta r^4 - \gamma l^4 r^2 + 2\gamma l^2 r^4 - \gamma r^6 + \kappa l^4 r^3 \\
& -2\kappa l^2 r^5 + \kappa r^7 + 4l^2 r^3\Big)^2 + \left(l^4 - r^4\right)^2\Big(15\beta^2\gamma l^2 + 3\beta^2\gamma r^2 - 12\beta\gamma l^2 r - 10\beta l^2 \\
& - 2\beta r^2 + 3\gamma l^2 r^2 - \gamma r^4 - 2\kappa l^2 r^3 + 2\kappa r^5 + 4l^2 r\Big)^2\Big],
\end{aligned} \tag{C.4}$$

$$
\begin{aligned}
R_{\mu\nu}R^{\mu\nu} = \frac{2r^2}{\left(l^2+r^2\right)^8} \Big(& 756\beta^4\gamma^2 l^8 + 1512\beta^4\gamma^2 l^6 r^2 + 1620\beta^4\gamma^2 l^4 r^4 - 864\beta^3\gamma^2 l^8 r \\
& -1584\beta^3\gamma^2 l^6 r^3 - 2448\beta^3\gamma^2 l^4 r^5 - 1008\beta^3\gamma l^8 - 2016\beta^3\gamma l^6 r^2 - 2160\beta^3\gamma l^4 r^4 \\
& +381\beta^2\gamma^2 l^8 r^2 + 576\beta^2\gamma^2 l^6 r^4 + 1698\beta^2\gamma^2 l^4 r^6 + 72\beta^2\gamma^2 l^2 r^8 + 9\beta^2\gamma^2 r^{10} \\
& -72\beta^2\gamma\kappa l^6 r^5 - 864\beta^2\gamma\kappa l^4 r^7 - 216\beta^2\gamma\kappa l^2 r^9 + 840\beta^2\gamma l^8 r + 1536\beta^2\gamma l^6 r^3 \\
& +2424\beta^2\gamma l^4 r^5 + 336\beta^2 l^8 + 672\beta^2 l^6 r^2 + 720\beta^2 l^4 r^4 - 84\beta\gamma^2 l^8 r^3 - 48\beta\gamma^2 l^6 r^5 \\
& -576\beta\gamma^2 l^4 r^7 - 48\beta\gamma^2 l^2 r^9 - 12\beta\gamma^2 r^{11} + 18\beta\gamma\kappa l^8 r^4 - 48\beta\gamma\kappa l^6 r^6 + 636\beta\gamma\kappa l^4 r^8 \\
& +144\beta\gamma\kappa l^2 r^{10} + 18\beta\gamma\kappa r^{12} - 236\beta\gamma l^8 r^2 - 348\beta\gamma l^6 r^4 - 1108\beta\gamma l^4 r^6 - 36\beta\gamma l^2 r^8 \\
& +48\beta\kappa l^6 r^5 + 576\beta\kappa l^4 r^7 + 144\beta\kappa l^2 r^9 - 176\beta l^8 r - 320\beta l^6 r^3 - 528\beta l^4 r^5 + 9\gamma^2 l^8 r^4 \\
& -12\gamma^2 l^6 r^6 + 82\gamma^2 l^4 r^8 + 12\gamma^2 l^2 r^{10} + 5\gamma^2 r^{12} - 10\gamma\kappa l^8 r^5 + 48\gamma\kappa l^6 r^7 - 164\gamma\kappa l^4 r^9 \\
& -48\gamma\kappa l^2 r^{11} - 18\gamma\kappa r^{13} + 24\gamma l^8 r^3 + 8\gamma l^6 r^5 + 184\gamma l^4 r^7 + 8\gamma l^2 r^9 + 6\kappa^2 l^8 r^6 \\
& +72\kappa^2 l^4 r^{10} + 36\kappa^2 l^2 r^{12} + 18\kappa^2 r^{14} - 4\kappa l^8 r^4 + 20\kappa l^6 r^6 - 204\kappa l^4 r^8 - 36\kappa l^2 r^{10} \\
& +32 l^6 r^4 + 104 l^4 r^6 - 36\kappa^2 l^6 r^8 + 24 l^8 r^2 \Big) .
\end{aligned}
\tag{C.5}
$$

B Horizon perturbation equations for MK metric

The equations governing the perturbation of the horizon of full MK parametrized by (3.27) metric are

$$\begin{aligned}B_{tt}^{(1)} = \frac{1}{12r^5}\Big[&18\beta^4\gamma^2r^3h'''' + 45\beta^4\gamma^2r^2h''' - 54\beta^4\gamma^2rh'' + 54\beta^4\gamma^2h' \\ &-99\beta^3\gamma^2r^3h''' + 72\beta^3\gamma^2r^2h'' - 90\beta^3\gamma^2rh' + 36\beta^3\gamma^2h - 24\beta^3\gamma r^3h'''' \\ &-60\beta^3\gamma r^2h''' + 72\beta^3\gamma rh'' - 72\beta^3\gamma h' + 30\beta^2\gamma^2r^5h'''' + 90\beta^2\gamma^2r^4h''' \\ &-36\beta^2\gamma^2r^3h'' + 54\beta^2\gamma^2r^2h' - 36\beta^2\gamma^2rh - 12\beta^2\gamma\kappa r^6h'''' - 39\beta^2\gamma\kappa r^5h''' \\ &+18\beta^2\gamma\kappa r^4h'' - 18\beta^2\gamma\kappa r^3h' + 36\beta^2\gamma r^4h'''' + 99\beta^2\gamma r^3h''' - 72\beta^2\gamma r^2h'' \\ &+90\beta^2\gamma rh' - 36\beta^2\gamma h + 8\beta^2r^3h'''' + 20\beta^2r^2h''' - 24\beta^2rh'' + 24\beta^2h' \\ &-12\beta\gamma^2r^6h'''' - 39\beta\gamma^2r^5h''' + 6\beta\gamma^2r^4h'' - 12\beta\gamma^2r^3h' + 12\beta\gamma^2r^2h \\ &+12\beta\gamma\kappa r^7h'''' + 42\beta\gamma\kappa r^6h''' - 6\beta\gamma\kappa r^5h'' + 12\beta\gamma\kappa r^4h' - 12\beta\gamma\kappa r^3h \\ &-20\beta\gamma r^5h'''' - 60\beta\gamma r^4h''' + 24\beta\gamma r^3h'' - 36\beta\gamma r^2h' + 24\beta\gamma rh \\ &+26\beta\kappa r^5h''' - 12\beta\kappa r^4h'' + 12\beta\kappa r^3h' - 8\beta r^4h'''' - 22\beta r^3h''' + 16\beta r^2h'' \\ &-20\beta rh' + 8\beta h + 2\gamma^2r^7h'''' + 7\gamma^2r^6h''' - 4\gamma\kappa r^8h'''' - 15\gamma\kappa r^7h''' \\ &+13\gamma r^5h''' - 2\gamma r^4h'' + 4\gamma r^3h' - 4\gamma r^2h + 2\kappa^2r^9h'''' + 8\kappa^2r^8h''' \\ &-14\kappa r^6h''' + 2\kappa r^5h'' - 4\kappa r^4h' + 4\kappa r^3h + 2r^5h'''' + 6r^4h''' - 2r^3h'' \\ &-4rh - 36\beta^3\gamma^2r^4h'''' + 4\gamma r^6h'''' + 4r^2h' - 4\kappa r^7h'''' + 8\beta\kappa r^6h''''\Big],\end{aligned} \tag{C.6}$$

$$\begin{aligned}B_{rr}^{(1)} = \frac{1}{12r^3\left(3\beta^2\gamma - 3\beta\gamma r - 2\beta + \gamma r^2 - \kappa r^3 + r\right)}\Big[&9\beta^2\gamma r^2h''' + 18\beta^2\gamma rh'' \\ &-18\beta^2\gamma h' - 6\beta\gamma r^3h''' - 6\beta\gamma r^2h'' + 12\beta\gamma rh' - 12\beta\gamma h - 6\beta r^2h''' \\ &+12\beta h' + \gamma r^4h''' + 2r^3h''' + 2r^2h'' - 4rh' + 4h - 12\beta rh''\Big].\end{aligned} \tag{C.7}$$

$$\begin{aligned}B_{\theta\theta}^{(1)} = \frac{1}{12r^2}\Big[&3\beta^2\gamma r^3h'''' + 3\beta^2\gamma r^2h''' - 18\beta^2\gamma rh'' + 18\beta^2\gamma h' - 3\beta\gamma r^4h'''' \\ &-6\beta\gamma r^3h''' + 6\beta\gamma r^2h'' - 12\beta\gamma rh' + 12\beta\gamma h - 2\beta r^3h'''' - 2\beta r^2h''' \\ &-12\beta h' + \gamma r^5h'''' + 3\gamma r^4h''' - \kappa r^6h'''' - 4\kappa r^5h''' + r^4h'''' + 2r^3h''' \\ &-2r^2h'' + 4rh' - 4h + 12\beta rh''\Big]\end{aligned} \tag{C.8}$$

Appendix D

Renormalized energy-momentum tensors

In this appendix we present the approximations for the vacuum expectation values of the renormalized energy-momentum of spin 0, spin 1/2 and spin 1 massive fields in the geometries discussed in this work. All the results displayed here were checked to satisfy the covariant conservation law $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$. The scalar field calculation failed to do so in certain cases. Those results will not be presented here.

A Schwarzschild metric

For the Schwarzschild metric of the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \quad B(r) = 1 - \frac{2\beta}{r} \tag{D.1}$$

we have computed the following values:

A.1 Scalar field

$$\langle T^t{}_t \rangle = \frac{\beta^2 \left(-11088\beta\xi + 2474\beta + 5040\xi r - 1125r\right)}{10080 M^2 \pi^2 r^9} \tag{D.2}$$

$$\langle T^r{}_r \rangle = \frac{\beta^2 \left(432\beta\xi - 94\beta - 288\xi r + 63r\right)}{1440 M^2 \pi^2 r^9} \tag{D.3}$$

$$\langle T^\theta{}_\theta \rangle = \frac{\beta^2 \left(-14112\beta\xi + 3086\beta + 6048\xi r - 1323r\right)}{10080 M^2 \pi^2 r^9} \tag{D.4}$$

$$\langle T \rangle = \frac{\beta^2 \left(-18144\beta\xi + 3994\beta + 7560\xi r - 1665r\right)}{5040 M^2 \pi^2 r^9} \tag{D.5}$$

For the conformal coupling $\xi = \frac{1}{6}$ these evaluate to

$$\langle T^t{}_t \rangle = \frac{\beta^2 (626\beta - 285r)}{10080 M^2 \pi^2 r^9} \tag{D.6}$$

$$\langle T^r{}_r \rangle = \frac{\beta^2 (-22\beta + 15r)}{1440 M^2 \pi^2 r^9} \tag{D.7}$$

$$\langle T^\theta{}_\theta \rangle = \frac{\beta^2 (734\beta - 315r)}{10080 M^2 \pi^2 r^9} \tag{D.8}$$

$$\langle T \rangle = \frac{\beta^2 (194\beta - 81r)}{1008 M^2 \pi^2 r^9} \tag{D.9}$$

For the minimal coupling $\xi = 0$ these evaluate to

$$\langle T^t{}_t \rangle = \frac{\beta^2 (2474\beta - 1125r)}{10080 M^2 \pi^2 r^9} \tag{D.10}$$

$$\langle T^r{}_r \rangle = \frac{\beta^2 (-94\beta + 63r)}{1440 M^2 \pi^2 r^9} \tag{D.11}$$

$$\langle T^\theta{}_\theta \rangle = \frac{\beta^2 (3086\beta - 1323r)}{10080 M^2 \pi^2 r^9} \tag{D.12}$$

$$\langle T \rangle = \frac{\beta^2 (3994\beta - 1665r)}{5040 M^2 \pi^2 r^9} \tag{D.13}$$

A.2 Dirac field

$$\langle T^t{}_t \rangle = \frac{\beta^2 (298\beta - 135r)}{5040 M^2 \pi^2 r^9} \tag{D.14}$$

$$\langle T^r{}_r \rangle = \frac{\beta^2 (-14\beta + 9r)}{720 M^2 \pi^2 r^9} \tag{D.15}$$

$$\langle T^\theta{}_\theta \rangle = \frac{\beta^2 (442\beta - 189r)}{5040 M^2 \pi^2 r^9} \tag{D.16}$$

$$\langle T \rangle = \frac{\beta^2 (542\beta - 225r)}{2520 M^2 \pi^2 r^9} \tag{D.17}$$

A.3 Proca field

$$\langle T^t{}_t \rangle = \frac{\beta^2 (-1222\beta + 555r)}{3360 M^2 \pi^2 r^9} \tag{D.18}$$

$$\langle T^r{}_r \rangle = \frac{\beta^2 (50\beta - 33r)}{480 M^2 \pi^2 r^9} \tag{D.19}$$

$$\langle T^\theta{}_\theta \rangle = \frac{\beta^2 (-1618\beta + 693r)}{3360 M^2 \pi^2 r^9} \tag{D.20}$$

$$\langle T \rangle = \frac{\beta^2 (-2054\beta + 855r)}{1680 M^2 \pi^2 r^9} \tag{D.21}$$

B MK black hole

For the full MK metric of the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \quad B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2 \tag{D.22}$$

we have the following values:

B.1 Dirac field

$$\langle T^t{}_t \rangle = \frac{1}{40320M^2\pi^2r^9}\Big[-8046\beta^6\gamma^3 + 2376\beta^5\gamma^3 r + 16092\beta^5\gamma^2 + 4302\beta^4\gamma^3r^2 + 1674\beta^4\gamma^2\kappa r^3 - 5598\beta^4\gamma^2 r - 10728\beta^4\gamma - 3078\beta^3\gamma^3r^3 - 4224\beta^3\gamma^2r^2 - 2232\beta^3\gamma\kappa r^3 + 4296\beta^3\gamma r + 2384\beta^3 + 492\beta^2\gamma^3r^4 - 204\beta^2\gamma^2\kappa r^5 + 2637\beta^2\gamma^2r^3 - 672\beta^2\gamma\kappa r^4 + 904\beta^2\gamma r^2 + 744\beta^2\kappa r^3 - 1080\beta^2 r + 3\beta\gamma^3r^5 - 328\beta\gamma^2r^4 + 21\beta\gamma\kappa^2r^7 + 136\beta\gamma\kappa r^5 - 360\beta\gamma r^3 + 1008\beta^3\gamma^2\kappa r^4 - 6\gamma^3r^6 + \gamma^2\kappa r^7 - \gamma^2r^5 - 14\gamma\kappa^2r^8 + 31\kappa^3r^9\Big] \tag{D.23}$$

$$\langle T^r{}_r \rangle = \frac{1}{40320M^2\pi^2r^9}\Big[2646\beta^6\gamma^3 - 3780\beta^5\gamma^3 r - 5292\beta^5\gamma^2 + 954\beta^4\gamma^3r^2 - 1890\beta^4\gamma^2\kappa r^3 + 6174\beta^4\gamma^2 r + 3528\beta^4\gamma + 810\beta^3\gamma^3r^3 - 504\beta^3\gamma^2\kappa r^4 - 1272\beta^3\gamma^2r^2 + 2520\beta^3\gamma\kappa r^3 - 3192\beta^3\gamma r - 784\beta^3 - 192\beta^2\gamma^3r^4 - 747\beta^2\gamma^2r^3 + 336\beta^2\gamma\kappa r^4 + 424\beta^2\gamma r^2 - 840\beta^2\kappa r^3 + 504\beta^2 r + 128\beta\gamma^2r^4 + 21\beta\gamma\kappa^2r^7 - 392\beta\gamma\kappa r^5 + 168\beta\gamma r^3 + 30\gamma^3r^6 - 35\gamma^2\kappa r^7 - 105\beta\gamma^3r^5 + 588\beta^2\gamma^2\kappa r^5 + 35\gamma^2r^5 - 14\gamma\kappa^2r^8 + 31\kappa^3r^9\Big] \tag{D.24}$$

$$\langle T^\theta{}_\theta \rangle = \frac{1}{40320M^2\pi^2r^9}\Big[-11934\beta^6\gamma^3 + 10206\beta^5\gamma^3 r + 23868\beta^5\gamma^2 - 900\beta^4\gamma^3r^2 + 1998\beta^4\gamma^2\kappa r^3 - 17010\beta^4\gamma^2 r - 15912\beta^4\gamma - 1242\beta^3\gamma^3r^3 + 1200\beta^3\gamma^2r^2 - 2664\beta^3\gamma\kappa r^3 + 9072\beta^3\gamma r + 3536\beta^3 + 63\beta^2\gamma^3r^4 - 192\beta^2\gamma^2\kappa r^5 + 1242\beta^2\gamma^2r^3 - 400\beta^2\gamma r^2 + 888\beta^2\kappa r^3 - 1512\beta^2 r + 105\beta\gamma^3r^5 - 42\beta\gamma^2r^4 + 128\beta\gamma\kappa r^5 - 336\beta\gamma r^3 - 6\gamma^3r^6 - 18\gamma^2\kappa r^7 - 35\gamma^2r^5 - 7\gamma\kappa^2r^8 + 31\kappa^3r^9\Big] \tag{D.25}$$

$$\langle T \rangle = \frac{1}{20160M^2\pi^2r^9}\Big[-14634\beta^6\gamma^3 + 9504\beta^5\gamma^3 r + 29268\beta^5\gamma^2 + 1728\beta^4\gamma^3r^2 + 1890\beta^4\gamma^2\kappa r^3 - 16722\beta^4\gamma^2 r - 19512\beta^4\gamma - 2376\beta^3\gamma^3r^3 + 252\beta^3\gamma^2\kappa r^4 - 1548\beta^3\gamma^2r^2 - 2520\beta^3\gamma\kappa r^3 + 9624\beta^3\gamma r + 4336\beta^3 + 213\beta^2\gamma^3r^4 + 2187\beta^2\gamma^2r^3 - 168\beta^2\gamma\kappa r^4 + 264\beta^2\gamma r^2 + 840\beta^2\kappa r^3 - 1800\beta^2 r + 54\beta\gamma^3r^5 - 142\beta\gamma^2r^4 + 21\beta\gamma\kappa^2r^7 - 432\beta\gamma r^3 + 6\gamma^3r^6 - 35\gamma^2\kappa r^7 - 18\gamma^2r^5 - 21\gamma\kappa^2r^8 + 62\kappa^3r^9\Big] \tag{D.26}$$

B.2 Proca field

$$\langle T^{t}{}_{t}\rangle = \frac{1}{40320M^2\pi^2r^9}\Big[49491\beta^6\gamma^3 - 88803\beta^5\gamma^3 r - 98982\beta^5\gamma^2 + 63099\beta^4\gamma^3 r^2 -10449\beta^4\gamma^2\kappa r^3 + 133389\beta^4\gamma^2 r + 65988\beta^4\gamma - 21150\beta^3\gamma^3 r^3 + 5796\beta^3\gamma^2\kappa r^4 -71028\beta^3\gamma^2 r^2 + 13932\beta^3\gamma\kappa r^3 - 59448\beta^3\gamma r - 14664\beta^3 + 2811\beta^2\gamma^3 r^4 -198\beta^2\gamma^2\kappa r^5 + 17538\beta^2\gamma^2 r^3 - 3864\beta^2\gamma\kappa r^4 + 19308\beta^2\gamma r^2 - 4644\beta^2\kappa r^3 +6660\beta^2 r + 195\beta\gamma^3 r^5 - 420\beta\gamma^2\kappa r^6 - 1874\beta\gamma^2 r^4 + 672\beta\gamma\kappa^2 r^7 + 552\beta\gamma\kappa r^5 -2148\beta\gamma r^3 - 89\gamma^3 r^6 + 65\gamma^2\kappa r^7 - 65\gamma^2 r^5 - 168\gamma\kappa^2 r^8 + 300\kappa^3 r^9\Big] \tag{D.27}$$

$$\langle T^{r}{}_{r}\rangle = \frac{1}{40320M^2\pi^2r^9}\Big[-14175\beta^6\gamma^3 + 44415\beta^5\gamma^3 r + 28350\beta^5\gamma^2 - 45783\beta^4\gamma^3 r^2 +10773\beta^4\gamma^2\kappa r^3 - 65457\beta^4\gamma^2 r - 18900\beta^4\gamma + 17622\beta^3\gamma^3 r^3 - 14868\beta^3\gamma^2\kappa r^4 +50964\beta^3\gamma^2 r^2 - 14364\beta^3\gamma\kappa r^3 + 28056\beta^3\gamma r + 4200\beta^3 + 897\beta^2\gamma^3 r^4 +2478\beta^2\gamma^2\kappa r^5 - 14346\beta^2\gamma^2 r^3 + 9912\beta^2\gamma\kappa r^4 - 13628\beta^2\gamma r^2 + 4788\beta^2\kappa r^3 -2772\beta^2 r - 2247\beta\gamma^3 r^5 + 2100\beta\gamma^2\kappa r^6 + 242\beta\gamma^2 r^4 - 168\beta\gamma\kappa^2 r^7 -2072\beta\gamma\kappa r^5 + 1876\beta\gamma r^3 + 445\gamma^3 r^6 - 749\gamma^2\kappa r^7 + 469\gamma^2 r^5 + 112\gamma\kappa^2 r^8 -280\gamma\kappa r^6 + 300\kappa^3 r^9\Big] \tag{D.28}$$

$$\langle T^{\theta}{}_{\theta}\rangle = \frac{1}{40320M^2\pi^2r^9}\Big[65529\beta^6\gamma^3 - 150633\beta^5\gamma^3 r - 131058\beta^5\gamma^2 + 113679\beta^4\gamma^3 r^2 -10935\beta^4\gamma^2\kappa r^3 + 219555\beta^4\gamma^2 r + 87372\beta^4\gamma - 28818\beta^3\gamma^3 r^3 + 10080\beta^3\gamma^2\kappa r^4 -126372\beta^3\gamma^2 r^2 + 14580\beta^3\gamma\kappa r^3 - 91896\beta^3\gamma r - 19416\beta^3 - 3255\beta^2\gamma^3 r^4 +750\beta^2\gamma^2\kappa r^5 + 24408\beta^2\gamma^2 r^3 - 6720\beta^2\gamma\kappa r^4 + 33724\beta^2\gamma r^2 - 4860\beta^2\kappa r^3 +8316\beta^2 r + 2037\beta\gamma^3 r^5 - 420\beta\gamma^2\kappa r^6 + 910\beta\gamma^2 r^4 - 420\beta\gamma\kappa^2 r^7 - 80\beta\gamma\kappa r^5 -3752\beta\gamma r^3 - 89\gamma^3 r^6 - 337\gamma^2\kappa r^7 - 469\gamma^2 r^5 + 196\gamma\kappa^2 r^8 + 140\gamma\kappa r^6 + 300\kappa^3 r^9\Big] \tag{D.29}$$

$$\langle T\rangle = \frac{1}{20160M^2\pi^2r^9}\Big[83187\beta^6\gamma^3 - 172827\beta^5\gamma^3 r - 166374\beta^5\gamma^2 + 122337\beta^4\gamma^3 r^2 -10773\beta^4\gamma^2\kappa r^3 + 253521\beta^4\gamma^2 r + 110916\beta^4\gamma - 30582\beta^3\gamma^3 r^3 + 5544\beta^3\gamma^2\kappa r^4 -136404\beta^3\gamma^2 r^2 + 14364\beta^3\gamma\kappa r^3 - 107592\beta^3\gamma r - 24648\beta^3 - 1401\beta^2\gamma^3 r^4 +1890\beta^2\gamma^2\kappa r^5 + 26004\beta^2\gamma^2 r^3 - 3696\beta^2\gamma\kappa r^4 + 36564\beta^2\gamma r^2 - 4788\beta^2\kappa r^3 +10260\beta^2 r + 1011\beta\gamma^3 r^5 + 420\beta\gamma^2\kappa r^6 + 94\beta\gamma^2 r^4 - 168\beta\gamma\kappa^2 r^7 - 840\beta\gamma\kappa r^5 -3888\beta\gamma r^3 + 89\gamma^3 r^6 - 679\gamma^2\kappa r^7 - 267\gamma^2 r^5 + 168\gamma\kappa^2 r^8 + 600\kappa^3 r^9\Big] \tag{D.30}$$

Appendix E

Schwarzschild-De Sitter black hole horizon corrections

The black hole horizon correction coming from the renormalized energy-momentum tensor of quantum fields for the SDS black hole is presented here. For spin 1/2 field the correction we obtained is

$$\begin{aligned}
h =&\frac{1}{24494400M^2\beta^4\pi^2r^4\cdot(3\beta-r)}\Big[-149445\beta^7+1530900\beta^6\kappa r^3\log\left(|r|\right)\\
&-1530900\beta^6\kappa r^3\log\left(|3\beta-r|\right)+2806650\beta^6\kappa r^3+70956\beta^6 r\\
&+2041200\beta^5\kappa r^4\log\left(|3\beta-r|\right)-2211300\beta^5\kappa r^4+28188\beta^5 r^2\\
&-1020600\beta^4\kappa r^5\log\left(|3\beta-r|\right)+595350\beta^4\kappa r^5-46980\beta^4 r^3\log\left(|r|\right)\\
&+46980\beta^4 r^3\log\left(|3\beta-r|\right)-97875\beta^4 r^3-226800\beta^3\kappa r^6\log\left(|r|\right)\\
&+226800\beta^3\kappa r^6\log\left(|3\beta-r|\right)-56700\beta^3\kappa r^6+62640\beta^3 r^4\log\left(|r|\right)\\
&-62640\beta^3 r^4\log\left(|3\beta-r|\right)+67860\beta^3 r^4+18900\beta^2\kappa r^7\log\left(|r|\right)\\
&-18900\beta^2\kappa r^7\log\left(|3\beta-r|\right)-31320\beta^2 r^5\log\left(|r|\right)+31320\beta^2 r^5\log\left(|3\beta-r|\right)\\
&-18270\beta^2 r^5+6960\beta r^6\log\left(|r|\right)-6960\beta r^6\log\left(|3\beta-r|\right)+1740\beta r^6\\
&-580r^7\log\left(|r|\right)+580r^7\log\left(|3\beta-r|\right)-2041200\beta^5\kappa r^4\log\left(|r|\right)\\
&+1020600\beta^4\kappa r^5\log\left(|r|\right)\Big]\,.
\end{aligned}\tag{E.1}$$

For spin 1 field the correction is

$$\begin{aligned}
h =&\frac{1}{16329600M^2\beta^4\pi^2r^4\cdot(3\beta-r)}\Big[557685\beta^7-5817420\beta^6\kappa r^3\log\left(|r|\right)\\
&+5817420\beta^6\kappa r^3\log\left(|3\beta-r|\right)-10665270\beta^6\kappa r^3-268272\beta^6 r\\
&+7756560\beta^5\kappa r^4\log\left(|r|\right)-7756560\beta^5\kappa r^4\log\left(|3\beta-r|\right)+8402940\beta^5\kappa r^4\\
&-109836\beta^5 r^2-3878280\beta^4\kappa r^5\log\left(|r|\right)+3878280\beta^4\kappa r^5\log\left(|3\beta-r|\right)\\
&-2262330\beta^4\kappa r^5+183060\beta^4 r^3\log\left(|r|\right)-183060\beta^4 r^3\log\left(|3\beta-r|\right)\\
&+861840\beta^3\kappa r^6\log\left(|r|\right)-861840\beta^3\kappa r^6\log\left(|3\beta-r|\right)+215460\beta^3\kappa r^6\\
&-244080\beta^3 r^4\log\left(|r|\right)+244080\beta^3 r^4\log\left(|3\beta-r|\right)-264420\beta^3 r^4\\
&-71820\beta^2\kappa r^7\log\left(|r|\right)+71820\beta^2\kappa r^7\log\left(|3\beta-r|\right)+122040\beta^2 r^5\log\left(|r|\right)\\
&-122040\beta^2 r^5\log\left(|3\beta-r|\right)+71190\beta^2 r^5-27120\beta r^6\log\left(|r|\right)+381375\beta^4 r^3\\
&+27120\beta r^6\log\left(|3\beta-r|\right)-6780\beta r^6+2260r^7\log\left(|r|\right)-2260r^7\log\left(|3\beta-r|\right)\Big]\,.
\end{aligned}\tag{E.2}$$

Appendix F

Software package

In order to evaluate the complicated formulas for the vacuum expectation value of the renormalized energy-momentum tensors a Python script was developed. It is based on the freely available Sympy and GraviPy libraries. Sympy was also used to check various expressions and solutions in this work. The program together with description and example scripts in the form of Jupyter notebooks will be attached to the work.

The program runs at a reasonable speed with the calculation of all $\langle \hat{T}_{\mu\nu} \rangle$ components for the full MK solution taking less than one hour on an AMD Ryzen 7 6800H CPU[1].

Overall Sympy seems to be working very well when evaluating the components of the curvature tensors. Because it supports creation of custom tensors, the attached scripts also include an implementation of Weyl and Bach tensors. The (so far) downside of Sympy seemed to be its lack of ability to recognize the Laplace equation in spherical coordinates and treat the spherical harmonics as its eigenvalues and instead differentiating the explicitly. While this made the package unsuitable for working with equations where the spherical harmonics appear as eigenvalues, it might be improved in the future.

Sympy can be found at https://www.sympy.org/en/index.html.

GraviPy can be found at https://github.com/wojciechczaja/GraviPy.

The scripts are available at https://github.com/KM1011G/GraviPy-REMT-WCG.

[1]This could be probably further improved by calling the expression simplification procedure less/more often.