CZECH TECHNICAL UNIVERSITY
IN PRAGUE

Faculty of Nuclear Sciences and Physical Engineering

# HABILITATION THESIS

## Lie algebras: their structure and applications

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# Preface

We present in this thesis a selection of our nine recent research papers. Although their topics are somewhat varied, they share one common feature: they involve Lie algebras and Lie groups either as a main subject of investigation or as an essential tool.

The papers contained in the thesis are divided into three thematic chapters preceded by an introductory review of our notation and essential background. In the first group consisting of four papers in Chapter 2 we study the structure of certain classes of solvable Lie algebras, establish their basic properties and construct their generalized Casimir invariants. We also investigate the structure of Lie algebras with nontrivial Levi decomposition, i.e. of algebras which are neither semisimple nor solvable. The notation and methods used in these papers are introduced in Sections 1.1,1.2.

In the second group of two papers in Chapter 3 we compute the Lie superalgebra of point (super)symmetries of certain partial differential equations defined on superspace. Next, we use it in construction of particular solutions of these equations. This is a generalization of Sophus Lie's approach to symmetries of differential equations which is reviewed in Section 1.3. Both similarities and differences between the ordinary, i.e. commuting, case and the superspace case are also discussed in the papers.

In the last group consisting of three papers in Chapter 4 we present several results concerning the Poisson–Lie T–duality/plurality of sigma models. In particular, we show that Poisson–Lie T–plurality can be interpreted as a canonical transformation in Section 4.1. Next, we discuss the transformation of boundary conditions for open strings under T–plurality. Some introductory remarks to these papers are contained in Section 1.4.

Most parts of the introduction in Chapter 1 review rather well–known material and consequently are quite concise – their main purpose is to establish the notation and also to refresh reader's knowledge of the theory by means of simple examples. The reader is referred to the literature for proofs of any theorems stated there. The only less standard and therefore more detailed part of the Introduction is Section 1.2 where Casimir operators and generalized Casimir invariants are introduced.

I would like to thank all coauthors of the presented papers for our successful collaboration. These involve my senior colleagues, professors Alfred Michel Grundland (Université du Québec à Trois–Rivières and Université de Montréal), Ladislav Hlavatý (Czech Technical University in Prague) and Pavel Winternitz (Université de Montréal); Cecilia Albertsson and Alexander Hariton who were postdoctoral fellows at Yukawa Institute and Université de

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# Chapter 1

# Introduction

Our intention is to review in the present chapter certain basic notions and results in the theory of Lie groups and Lie algebras, symmetries of differential equations, Poisson–Lie groups and sigma models built on them. For the Lie group part, we assume that the reader has basic familiarity with differential geometry. We omit proofs which can be found in numerous textbooks [1, 2, 3, 4, 5, 6, 7].

## 1.1 Lie algebras

### 1.1.1 Definition and basic properties of Lie algebras

A *Lie algebra* $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ equipped with a multiplication (also called a bracket), i.e. a bilinear map $[\ ,\ ]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, such that

$$[y,x] = -[x,y] \qquad \text{(antisymmetry)} \tag{1.1}$$

$$0 = \big[x,[y,z]\big] + \big[y,[z,x]\big] + \big[z,[x,y]\big] \qquad \text{(Jacobi identity)} \tag{1.2}$$

for all elements $x,y,z\in\mathfrak{g}$. In what follows we shall consider the fields $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ and finite–dimensional Lie algebras only.

The structure of the Lie algebra $\mathfrak{g}$ can be represented in any chosen basis $(e_j)_{j=1}^{\dim\mathfrak{g}}$ by the corresponding *structure constants* ${c_{jk}}^l$ in the basis $(e_j)_{j=1}^{\dim\mathfrak{g}}$

$$[e_j,e_k] = \sum_{l=1}^{\dim\mathfrak{g}} {c_{jk}}^l e_l. \tag{1.3}$$

When we write down Lie brackets specifying the structure of some Lie algebra, we usually omit vanishing brackets and suppose that antisymmetry holds.

For any pair of vector subspaces $V, W$ of $\mathfrak{g}$ we define their bracket as

$$[V, W] = \text{span}\{[x, y] | x \in V, y \in W\}. \tag{1.4}$$

We notice that the linear span is necessary in order for $[V, W]$ to be a vector subspace of $\mathfrak{g}$.

A *subalgebra* $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ which is closed under the bracket,

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}. \tag{1.5}$$

An *ideal* $I$ of the Lie algebra $\mathfrak{g}$ is a subalgebra such that

$$[I, \mathfrak{g}] \subseteq I. \tag{1.6}$$

The Lie algebra $\mathfrak{g}$ itself and $\{0\}$ are *trivial* ideals. A Lie algebra which does not possess any nontrivial ideal is called *simple.*

Three different series of ideals can be associated with any given Lie algebra. The dimensions of the ideals in each of these series are important characteristics of the given Lie algebra.

The *derived series* $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ is defined recursively

$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \quad \mathfrak{g}^{(0)} = \mathfrak{g}. \tag{1.7}$$

The second term in the series, namely $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, is called the *derived algebra* of $\mathfrak{g}$ and may also be denoted by $D(\mathfrak{g})$ or $\mathfrak{g}^2$. If the derived series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is called *solvable.*

The *lower central series* $\mathfrak{g} = \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ is again defined recursively

$$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \quad \mathfrak{g}^1 = \mathfrak{g}. \tag{1.8}$$

If the lower central series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ is called *nilpotent.* The highest value of $k$ for which we have $\mathfrak{g}^k \neq 0$ is the *degree of nilpotency* of the nilpotent Lie algebra $\mathfrak{g}$.

Obviously, a nilpotent Lie algebra is also solvable. An Abelian Lie algebra is nilpotent of degree 1.

The *upper central series* is $\mathfrak{z}_1(\mathfrak{g}) \subseteq \ldots \subseteq \mathfrak{z}_k(\mathfrak{g}) \subseteq \ldots \subseteq \mathfrak{g}$. In this series $\mathfrak{z}_1$ is the *center* of $\mathfrak{g}$

$$\mathfrak{z}_1(\mathfrak{g}) = C(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0, \ \forall y \in \mathfrak{g}\}. \tag{1.9}$$

Now let us consider the factor algebra $\mathfrak{f}_1 \simeq \mathfrak{g}/\mathfrak{z}_1(\mathfrak{g})$. Its center is $C(\mathfrak{f}_1) = C(\mathfrak{g}/\mathfrak{z}_1(\mathfrak{g}))$. We define the *second center* $\mathfrak{z}_2(\mathfrak{g})$ of $\mathfrak{g}$ to be the unique ideal in $\mathfrak{g}$ such that

$$\mathfrak{z}_2(\mathfrak{g})/\mathfrak{z}_1(\mathfrak{g}) = C(\mathfrak{g}/\mathfrak{z}_1(\mathfrak{g})). \tag{1.10}$$

Recursively we define the $k^{th}$*–center* $\mathfrak{z}_k(\mathfrak{g})$ as the unique ideal in $\mathfrak{g}$ such that its image under factorization by $\mathfrak{z}_{k-1}(\mathfrak{g})$ is the center of $\mathfrak{g}/\mathfrak{z}_{k-1}(\mathfrak{g})$, i.e.

$$\mathfrak{z}_k(\mathfrak{g})/\mathfrak{z}_{k-1}(\mathfrak{g}) = C(\mathfrak{g}/\mathfrak{z}_{k-1}(\mathfrak{g})). \tag{1.11}$$

The union of the upper central series is the *hypercenter*

$$\mathfrak{z}_\infty(\mathfrak{g}) = \bigcup_{i=1}^{\infty} \mathfrak{z}_i(\mathfrak{g}). \tag{1.12}$$

Since the higher centers $\mathfrak{z}_i(\mathfrak{g})$ are ordered by inclusion, the hypercenter can be also viewed as the largest set in the sequence. The upper central series terminates, i.e. the hypercenter is equal to the whole algebra $\mathfrak{g}$, if and only if $\mathfrak{g}$ is nilpotent.

We shall call these three series the *characteristic series* of the algebra $\mathfrak{g}$.

**Example 1.1** *Let us consider the Heisenberg algebra in one dimension* $\mathfrak{h}(1)$*, spanned by three vectors* $e_1, e_2, e_3$ *with the only nonvanishing Lie bracket*

$$[e_2, e_3] = e_1.$$

*Its characteristic series are*

$$\begin{aligned} (\mathfrak{h}(1))^{(1)} &= \mathrm{span}\{e_1\}, & (\mathfrak{h}(1))^{(2)} &= 0, \\ (\mathfrak{h}(1))^2 = (\mathfrak{h}(1))^{(1)} &= \mathrm{span}\{e_1\}, & (\mathfrak{h}(1))^3 &= 0, \\ \mathfrak{z}_1(\mathfrak{h}(1)) &= (\mathfrak{h}(1))^2, & \mathfrak{z}_2(\mathfrak{h}(1)) &= \mathfrak{h}(1). \end{aligned}$$

*The algebra* $\mathfrak{h}(1)$ *is nilpotent because* $(\mathfrak{h}(1))^3 = 0$ *or, equivalently, because* $\mathfrak{z}_2(\mathfrak{h}(1)) = \mathfrak{h}(1)$.

In any given Lie algebra we have two distinguished ideals, the radical and the nilradical. The *radical* is the maximal solvable ideal of $\mathfrak{g}$ denoted by $\mathrm{R}(\mathfrak{g})$. It is unique because it is a sum of all solvable ideals. The *nilradical* $\mathrm{NR}(\mathfrak{g})$ is the maximal nilpotent ideal of $\mathfrak{g}$, also unique. The existence of these ideals is a consequence of the fact that a sum of two solvable (nilpotent) ideals of $\mathfrak{g}$ is again a solvable (nilpotent) ideal, respectively.

The following relation between the radical and the nilradical holds

$$(\mathrm{R}(\mathfrak{g}))^2 \subseteq \mathrm{NR}(\mathfrak{g}) \subseteq \mathrm{R}(\mathfrak{g}).$$

When the radical of the algebra $\mathfrak{g}$ vanishes, i.e. $\mathfrak{g}$ has no nonvanishing solvable ideals, the algebra is called *semisimple*.

When the radical of $\mathfrak{g}$ is nonvanishing, also the nilradical is nonvanishing – either $(\mathrm{R}(\mathfrak{g}))^2 = 0$, i.e. both $\mathrm{NR}(\mathfrak{g})$ and $\mathrm{R}(\mathfrak{g})$ are Abelian and therefore coincide, or $0 \neq (\mathrm{R}(\mathfrak{g}))^2 \subseteq \mathrm{NR}(\mathfrak{g})$).

The hypercenter $\mathfrak{z}_\infty(\mathfrak{g})$ is always a nilpotent ideal of $\mathfrak{g}$ and therefore is contained in the nilradical.

**Example 1.2** *Let us consider the algebra $\mathfrak{e}(2)$ of infinitesimal Euclidean motions in 2–dimensional Euclidean space $\mathbb{R}^2$. It is spanned by three vectors $e_1, e_2, e_3$ with the nonvanishing Lie brackets*

$$[e_1, e_3] = -e_2, \qquad [e_2, e_3] = e_1.$$

*Its characteristic series are*

$$\begin{aligned} &(\mathfrak{e}(2))^{(1)} = \operatorname{span}\{e_1, e_2\}, && (\mathfrak{e}(2))^{(2)} = 0, \\ &(\mathfrak{e}(2))^2 = (\mathfrak{e}(2))^{(1)} = \operatorname{span}\{e_1, e_2\}, && (\mathfrak{e}(2))^3 = (\mathfrak{e}(2))^2, \\ &\mathfrak{z}_1(\mathfrak{e}(2)) = 0. \end{aligned}$$

*The algebra $\mathfrak{e}(2)$ is solvable because $(\mathfrak{e}(2))^{(2)} = 0$, but it is not nilpotent because $(\mathfrak{e}(2))^2 = (\mathfrak{e}(2))^3 = \ldots = (\mathfrak{e}(2))^k = \ldots \neq 0$.*

*The radical of $\mathfrak{e}(2)$ by definition coincides with $\mathfrak{e}(2)$. The nilradical of $\mathfrak{e}(2)$ must contain $(\mathfrak{e}(2))^2$ and therefore it coincides with the derived algebra*

$$\mathrm{NR}(\mathfrak{e}(2)) = (\mathfrak{e}(2))^2$$

*on dimensional grounds.*

The *centralizer* $\operatorname{cent}_{\mathfrak{g}}(\mathfrak{h})$ of a given subspace $\mathfrak{h} \subseteq \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\operatorname{cent}_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, y] = 0, \ \forall y \in \mathfrak{h}\}. \tag{1.13}$$

The *normalizer* $\operatorname{norm}_{\mathfrak{g}}(\mathfrak{h})$ of a given subspace $\mathfrak{h} \subseteq \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements $x$ in $\mathfrak{g}$ such that $[x, h]$ is in the subspace $\mathfrak{h}$ for any $h \in \mathfrak{h}$, i.e.

$$\operatorname{norm}_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, y] \in \mathfrak{h}, \ \forall y \in \mathfrak{h}\}. \tag{1.14}$$

When $\mathfrak{h}$ is a subalgebra then necessarily $\mathfrak{h} \subseteq \operatorname{norm}_{\mathfrak{g}}(\mathfrak{h})$. The normalizer of an ideal in $\mathfrak{g}$ is the whole algebra $\mathfrak{g}$.

A *representation* $\rho$ of a given Lie algebra $\mathfrak{g}$ on a vector space $V$ is a linear map of $\mathfrak{g}$ into the space $\mathfrak{gl}(V)$ of linear operators acting on $V$

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V) \colon x \to \rho(x)$$

such that for any pair $x$, $y$ of elements of $\mathfrak{g}$

$$\rho([x,y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) \tag{1.15}$$

holds. The field over which the vector space is defined must contain $\mathbb{F}$ in order to have the representation well–defined, i.e. we may have representations of real algebras on complex vector spaces but not vice versa. *Dimension of the representation* $\rho$ is understood to be the same as the dimension of the vector space $V$.

A subspace $W$ of $V$ is called *invariant* if

$$\rho(\mathfrak{g})W = \{\rho(x)w |\, x \in \mathfrak{g}, w \in W\} \subseteq W.$$

A representation $\rho$ of $\mathfrak{g}$ on $V$ is *reducible* if a proper nonvanishing invariant subspace $W$ of $V$ exists.

A representation $\rho$ of $\mathfrak{g}$ on $V$ is *irreducible* if no nontrivial invariant subspace of $V$ exists.

A representation $\rho$ of $\mathfrak{g}$ on $V$ is *fully reducible* when every invariant subspace $W$ of $V$ has an invariant complement $\tilde{W}$, i.e.

$$V = W \oplus \tilde{W}, \qquad \rho(\mathfrak{g})\tilde{W} \subseteq \tilde{W}. \tag{1.16}$$

In particular, any irreducible representation is also fully reducible.

An important criterion for irreducibility of a given representations is

**Theorem 1.1 (Schur's Lemma)** *Let $\mathfrak{g}$ be a complex Lie algebra and $\rho$ its representation on a finite–dimensional vector space $V$.*

1. *Let $\rho$ be irreducible. Then any operator $A$ on $V$ which commutes with all $\rho(x)$,*
$$[A, \rho(x)] = 0, \qquad \forall x \in \mathfrak{g},$$
*has the form $A = \lambda \mathbf{1}$ for some complex number $\lambda$.*

2. *Let $\rho$ be fully reducible and such that every operator $A$ on $V$ which commutes with all $\rho(x)$ has the form $A = \lambda \mathbf{1}$ for some complex number $\lambda$. Then $\rho$ is irreducible.*

The *adjoint representation* of a given Lie algebra $\mathfrak{g}$ is a linear map of $\mathfrak{g}$ into the space $\mathfrak{gl}(\mathfrak{g})$ of linear operators acting on $\mathfrak{g}$

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})\colon x \to \mathrm{ad}(x)$$

defined for any pair $x$, $y$ of elements of $\mathfrak{g}$ via

$$\mathrm{ad}(x)\, y = [x, y]. \tag{1.17}$$

When convenient we may also use an alternative notation $\mathrm{ad}_x = \mathrm{ad}(x)$. The image of ad is denoted by $\mathrm{ad}(\mathfrak{g})$.

A linear map $\Phi$ between two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ is a *homomorphism* if

$$\Phi([x,y]_1) = [\Phi(x), \Phi(y)]_2, \qquad \forall x, y \in \mathfrak{g}_1.$$

When a homomorphism is a bijection we call it an *isomorphism*.

An *automorphism* $\Phi$ of a given Lie algebra $\mathfrak{g}$ is a bijective linear map

$$\Phi : \ \mathfrak{g} \to \mathfrak{g}$$

such that

$$\Phi([x,y]) = [\Phi(x), \Phi(y)] \tag{1.18}$$

for any pair $x, y$ of elements of $\mathfrak{g}$.

A *derivation* $D$ of a given Lie algebra $\mathfrak{g}$ is a linear map

$$D\colon \mathfrak{g} \to \mathfrak{g}$$

such that for any pair $x$, $y$ of elements of $\mathfrak{g}$

$$D([x,y]) = [D(x), y] + [x, D(y)]. \tag{1.19}$$

If an element $z \in \mathfrak{g}$ exists, such that

$$D = \mathrm{ad}(z), \quad \text{i.e. } D(x) = [z, x], \ \forall x \in \mathfrak{g}$$

the derivation is called *inner*, any other one is *outer*.

**Example 1.3** *Let us consider the algebra of Euclidean motions in two dimensions* $\mathfrak{e}(2)$ *of Example 1.2. We have*

$$\mathrm{ad}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

*The algebra of derivations of* $\mathfrak{e}(2)$ *is spanned by inner derivations* $\mathrm{ad}(e_1)$, $\mathrm{ad}(e_2)$, $\mathrm{ad}(e_3)$ *together with one outer derivation*

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The set of all automorphisms of $\mathfrak{g}$ with the composition as the group law forms the Lie group $\mathrm{Aut}(\mathfrak{g})$ and its Lie algebra coincides with the algebra $\mathfrak{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$. Inner derivations $\mathfrak{Inn}(\mathfrak{g})$ form an ideal in $\mathfrak{Der}(\mathfrak{g})$.

Any ideal is invariant with respect to inner derivations. Ideals in the algebra $\mathfrak{g}$ which are invariant with respect to all derivations and automorphisms of $\mathfrak{g}$ are called *characteristic*. In particular, ideals in characteristic series are characteristic.

Several important theorems describe properties of representations of various classes of Lie algebras.

**Theorem 1.2 (Theorem of Engel)** *A real or complex Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(x)$ is nilpotent operator for all $x \in \mathfrak{g}$.*

*The representation space $V$ of a finite–dimensional representation $\rho$ of a nilpotent complex Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum of invariant subspaces*

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$$

*such that in a suitable basis the operators $\rho(x)|_{V_j}$ on the subspaces $V_j$ simultaneously take an upper triangular form with a multiple of unit matrix on the diagonal, i.e.*

$$\rho(x)|_{V_j} = \begin{pmatrix} \lambda(x) & ? & \ldots & ? \\ 0 & \lambda(x) & \ldots & ? \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda(x) \end{pmatrix}, \qquad x \in \mathfrak{g} \tag{1.20}$$

*for some linear functional $\lambda$ on $\mathfrak{g}$.*

**Example 1.4** *Let us consider the Heisenberg algebra $\mathfrak{h}(1)$ of Example 1.1. We have*

$$\mathrm{ad}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.21}$$

*Obviously, all operators $\mathrm{ad}(e_i)$ are nilpotent,*

$$\mathrm{ad}(e_1) = \mathrm{ad}(e_2)^2 = \mathrm{ad}(e_3)^2 = 0,$$

*in agreement with Engel's theorem. For the adjoint representation we have the functional $\lambda \equiv 0$.*

*The adjoint representation is reducible because any subspace of* $\mathfrak{h}(1)$ *containing* $e_1$ *is invariant. There is no invariant complementary subspace to the invariant subspace* $\mathrm{span}\{e_1\}$*; therefore, the adjoint representation is not fully reducible.*

*An arbitrary derivation of the Heisenberg algebra* $\mathfrak{h}(1)$ *is given by the matrix*

$$D = \begin{pmatrix} a+d & e & f \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \tag{1.22}$$

*where* $a, b, c, d, e, f$ *are arbitrary parameters. We have a 2–dimensional ideal* $\mathfrak{Inn}(\mathfrak{h}(1))$ *of inner derivations in* $\mathfrak{Der}(\mathfrak{h}(1))$*. The ideal* $\mathfrak{Inn}(\mathfrak{h}(1))$ *is spanned by* $\mathrm{ad}(e_2), \mathrm{ad}(e_3)$*, i.e. consists of the derivations of the form*

$$\begin{pmatrix} 0 & e & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Theorem 1.3 (Theorem of Lie)** *Any representation* $\rho$ *of a solvable Lie algebra* $\mathfrak{g}$ *on a complex finite–dimensional vector space* $V$ *contains a common eigenvector* $v \in V$*,* $v \neq 0$*, i.e.*

$$\rho(x)v = \lambda(x) \cdot v, \qquad x \in \mathfrak{g} \tag{1.23}$$

*for some linear functional* $\lambda$ *on* $\mathfrak{g}$*.*

*For any complex solvable Lie algebra* $\mathfrak{g}$ *there exists a filtration by codimension 1 ad–invariant subspaces, i.e.*

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_{\dim \mathfrak{g}} = \mathfrak{g}, \quad \dim V_k / \dim V_{k-1} = 1, \quad [\mathfrak{g}, V_k] \subseteq V_k. \tag{1.24}$$

Lie's theorem implies that any complex solvable Lie algebra $\mathfrak{g}$ has only one–dimensional irreducible representations and that the adjoint representation of any complex non–Abelian solvable Lie algebra $\mathfrak{g}$ is not fully reducible.

**Example 1.5** *Let us consider the algebra of Euclidean motions in two dimensions* $\mathfrak{e}(2)$ *of Examples 1.2 and 1.3. Common eigenvectors of all operators in the adjoint represention* $\mathrm{ad}(e_i)$ *exist only when* $\mathbb{F} = \mathbb{C}$*. In that case we have two eigenvectors*

$$e_\pm = e_1 \pm \mathrm{i}e_2, \qquad \mathrm{ad}(e_{1,2})\, e_\pm = 0, \quad \mathrm{ad}(e_3)\, e_\pm = \mp \mathrm{i}e_\pm.$$

*Denoting* $V_\pm = \mathrm{span}\{e_\pm\}$ *we have two ad–invariant filtrations*

$$0 \subsetneq V_\pm \subsetneq V_- \oplus V_+ \subsetneq \mathfrak{e}(2)$$

*where* $V_- \oplus V_+ = \mathrm{span}\{e_1, e_2\} = (\mathfrak{e}(2))^2$*.*

We have just seen that Lie's theorem indeed does not hold over the field of real numbers.

Last but not least, for semisimple Lie algebras we have

**Theorem 1.4 (Theorem of Weyl)** *A complex Lie algebra* $\mathfrak{g}$ *is semisimple if and only if all its finite–dimensional representations are fully reducible.*

Next, let us consider bilinear forms on Lie algebras. A symmetric bilinear form $B$ on a given Lie algebra $\mathfrak{g}$ such that

$$B(\mathrm{ad}(x)y, z) + B(y, \mathrm{ad}(x)z) = 0 \tag{1.25}$$

for every triplet $x, y, z \in \mathfrak{g}$ is called *ad–invariant* or *invariant.* A symmetric bilinear form $B$ is *invariant with respect to automorphisms* if

$$B(\Phi(x), \Phi(y)) = B(x, y) \tag{1.26}$$

holds for every automorphism $\Phi$ of $\mathfrak{g}$ and any pair $x, y \in \mathfrak{g}$.

We recall that any form $B$ invariant with respect to automorphisms is also ad–invariant. This statement follows from the fact that (1.26) implies upon differentiation

$$B(D(x), y) + B(x, D(y)) = 0 \tag{1.27}$$

for any derivation $D$, in particular for all inner derivations. The converse is not true in general because outer derivations of $\mathfrak{g}$ may exist or there may be discrete transformations in $\mathrm{Aut}(\mathfrak{g})$ which are not obtained by exponentiation of elements in $\mathrm{Der}(\mathfrak{g})$.

The *Killing form* $K$ of a given Lie algebra $\mathfrak{g}$ is a symmetric bilinear form on $\mathfrak{g}$ defined by

$$K(x, y) = \mathrm{tr}(\mathrm{ad}(x) \cdot \mathrm{ad}(y)). \tag{1.28}$$

The Killing form is invariant with respect to automorphisms. In the particular case of complex simple Lie algebras, any invariant symmetric bilinear form is a multiple of the Killing form.

The Killing form provides important criteria for semisimplicity and solvability known as Cartan criteria.

**Theorem 1.5 (1st Cartan criterion)** *A Lie algebra* $\mathfrak{g}$ *is solvable if and only if the restriction of its Killing form to the derived algebra vanishes.*

**Theorem 1.6 (2nd Cartan criterion)** *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if its Killing form is nondegenerate.*

All nilpotent Lie algebras have vanishing Killing form but, contrary to some claims in the literature (see e.g. [8], p. 669 or [9], p. 82), not every algebra with vanishing Killing form is nilpotent. Consider e.g. a solvable nonnilpotent algebra

$$\mathfrak{g} = \mathrm{span}\{e_1, e_2, e_3\}, \qquad [e_1, e_3] = e_1, \qquad [e_2, e_3] = \mathrm{i}e_2$$

whose Killing form vanishes identically.

**Example 1.6** *The Killing form of the algebra of Euclidean motions in 2 dimensions* $\mathfrak{e}(2)$*, Example 1.2, is given by the matrix*

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

*Recalling that* $(\mathfrak{e}(2))^2 = \mathrm{span}\{e_1, e_2\}$ *we see that the 1st Cartan criterion correctly predicts that* $\mathfrak{e}(2)$ *is solvable.*

The Killing form also allows us to consider the *orthogonal complement*

$$V^\perp = \{x \in \mathfrak{g} | K(x, y) = 0, \forall y \in V\}. \tag{1.29}$$

of a given subspace $V \subset \mathfrak{g}$ with respect to the Killing form (whether or not is $K$ nondegenerate).

An orthogonal complement $I^\perp$ of an ideal $I \subset \mathfrak{g}$ is again an ideal in $\mathfrak{g}$. This property implies that any semisimple Lie algebra is just a direct sum of its simple components.

The radical of $\mathfrak{g}$ can be constructed very efficiently once the Killing form of $\mathfrak{g}$ is computed. We have a simple formula

$$\mathrm{R}(\mathfrak{g}) = \left(\mathfrak{g}^2\right)^\perp. \tag{1.30}$$

A fundamental theorem due to E. E. Levi [10, 1, 11] provides a general scheme for the structure of Lie algebras.

**Theorem 1.7 (Theorem of Levi)** *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra over a field* $\mathbb{F}$ *and* $\mathfrak{r} = R(\mathfrak{g})$ *be its radical. Then there exists a semisimple subalgebra* $\mathfrak{p}$ *of* $\mathfrak{g}$ *such that*

$$\mathfrak{g} = \mathfrak{p} \dotplus \mathfrak{r}. \tag{1.31}$$

*The subalgebra* $\mathfrak{p}$ *is isomorphic to the factor algebra* $\mathfrak{g}/\mathfrak{r}$ *and is unique up to automorphisms of* $\mathfrak{g}$*.*

Because $\mathfrak{r}$ is a solvable ideal and $\mathfrak{p}$ a semisimple subalgebra we have

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{r}] \subseteq \mathfrak{r}, \qquad [\mathfrak{r}, \mathfrak{r}] \subsetneq \mathfrak{r}.$$

Levi's theorem implies that in order to classify Lie algebras up to any given dimension $n$, one needs to classify

1. all simple algebras and consequently all semisimple algebras up to dimension $n$,

2. all solvable algebras up to dimension $n$,

3. and construct all semidirect sums of the form (1.31) – that task reduces to consideration of representations of $\mathfrak{p}$ valued in $\mathfrak{Der}(\mathfrak{r})$.

The first task was accomplished already by W. Killing and É. Cartan over the fields of both complex [12, 13] and real numbers [14, 15]. Below in Section 1.1.2 we shall review that classification over the field of complex numbers.

The second task presently appears to be unsolvable in its full generality. It was completed only for very low dimension $n \leq 6$. We shall explain why it is not possible to classify solvable algebras using the same ideas as for semisimple algebras and we motivate a different approach: construction of solvable Lie algebras with a given structure of their nilradicals. Several such classifications are then presented in papers included in Section 2.

The third task was studied using several approaches in [16, 17, 18, 19]. We shall deal with it in more detail in [20] which is included in Section 2.4.

### 1.1.2 Classification of complex simple Lie algebras

Let $\mathfrak{g}$ be a complex Lie algebra. Any nilpotent subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ coinciding with its normalizer $\mathrm{norm}_{\mathfrak{g}}(\mathfrak{g}_0)$ is called a *Cartan subalgebra*. It can be constructed in the following way.

Let $x \in \mathfrak{g}$. Consider the linear operator $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ and find its generalized nullspace

$$\mathfrak{g}_0(x) = \lim_{k \to \infty} \ker \left(\mathrm{ad}(x)\right)^k .$$

When $\dim \mathfrak{g}_0(x)$ is minimal, i.e.

$$\dim \mathfrak{g}_0(x) = \min_{y \in \mathfrak{g}} \dim \mathfrak{g}_0(y),$$

we call the element $x \in \mathfrak{g}$ *regular*.

**Proposition 1.8** *Let $x \in \mathfrak{g}$ be a regular element of the complex Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}_0(x)$ is a Cartan subalgebra of $\mathfrak{g}$. Any other Cartan subalgebra of $\mathfrak{g}$ is related to $\mathfrak{g}_0(x)$ by an automorphism of $\mathfrak{g}$.*

Consequently, the dimension of the Cartan subalgebra $\mathfrak{g}_0(x)$ is independent of the choice of the regular element $x$ and is called the *rank* of the Lie algebra $\mathfrak{g}$. We point out that the proposition holds whether or not $\mathfrak{g}$ is semisimple, i.e. any complex Lie algebra has a Cartan subalgebra unique up to automorphisms (the uniqueness is lost for real algebras). On the other hand, Cartan subalgebras of semisimple algebras have special properties.

Cartan subalgebra $\mathfrak{g}_0$ of a semisimple Lie algebra is Abelian and $\mathrm{ad}(h) \in \mathfrak{gl}(\mathfrak{g})$ is diagonalizable for every $h \in \mathfrak{g}_0$. Therefore, there exist common eigenspaces $\mathfrak{g}_\lambda \subset \mathfrak{g}$ of all operators $\mathrm{ad}(h)$, $h \in \mathfrak{g}_0$ and nonvanishing functionals $\lambda \in \mathfrak{g}_0^*$ such that

$$\mathrm{ad}(h)e_\lambda = \lambda(h) \cdot e_\lambda, \qquad h \in \mathfrak{g}_0,\ e_\lambda \in \mathfrak{g}_\lambda.$$

These functionals $\lambda$ are called *roots* of the semisimple Lie algebra $\mathfrak{g}$. The collection of all roots is called the *root system* of the algebra $\mathfrak{g}$ and denoted by $\Delta$. The diagonalizability of $\mathrm{ad}(h)$ implies that

$$\mathfrak{g} = \mathfrak{g}_0 \dotplus (\dotplus\{\mathfrak{g}_\lambda | \, \lambda \in \Delta\}) \, .$$

It is always possible to introduce an ordering among the roots via a choice of $h_0 \in \mathfrak{g}_0$ such that $\lambda(h_0) \neq 0$ and $\lambda(h_0) \in \mathbb{R}$ for all roots $\lambda$. This ordering is not unique but different choices give results equivalent up to automorphism of $\mathfrak{g}$. For any pair of roots $\lambda, \kappa$ one writes $\lambda > \kappa$ if and only if $\lambda(h_0) > \kappa(h_0)$. Similarly one defines *positive roots* $\lambda > 0$, i.e. $\lambda(h_0) > 0$ and *negative roots* $\lambda < 0$, i.e. $\lambda(h_0) < 0$. The set of all positive roots is denoted $\Delta^+$, the set of negative roots $\Delta^-$. We have $\Delta = \Delta^+ \cup \Delta^-$. *Simple roots* are positive roots which cannot be written as a sum of two positive roots. We denote the set of all simple roots by $\Delta^S$.

We list the most important properties of the root system $\Delta$ and *root subspaces* $\mathfrak{g}_\lambda$ of a semisimple complex Lie algebra $\mathfrak{g}$:

1. the Killing form $K$ of $\mathfrak{g}$ when restricted to $\mathfrak{g}_0 \times \mathfrak{g}_0$ is nondegenerate;

2. if $\lambda$ is a root then so is $-\lambda$ and no other multiple of $\lambda$ is a root;

3. all root subspaces $\mathfrak{g}_\lambda$ are 1–dimensional;

4. $[\mathfrak{g}_\lambda, \mathfrak{g}_\kappa] = \mathfrak{g}_{\lambda+\kappa}$ whenever $\lambda, \kappa$ and $\lambda + \kappa$ are roots;

5. when $\lambda + \kappa$ is neither 0 nor a root we have $[\mathfrak{g}_\lambda, \mathfrak{g}_\kappa] = 0$;

6. $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \subset \mathfrak{g}_0$;

7. there is a basis of $\mathfrak{g}$ consisting of elements of the Cartan subalgebra $\mathfrak{g}_0$ and of root subspaces $\mathfrak{g}_\lambda$ such that the structure constants of $\mathfrak{g}$ in this basis are integers; such a basis is called the *Weyl–Chevalley basis* of $\mathfrak{g}$ and the real form of the Lie algebra $\mathfrak{g}$ corresponding to this choice of basis is called the *split real form* of $\mathfrak{g}$;

8. to any functional $\lambda \in \mathfrak{g}_0^*$ we can associate a unique element $h_\lambda \in \mathfrak{g}_0$ such that
$$\lambda(h) = K(h_\lambda, h), \qquad \forall h \in \mathfrak{g}_0$$
and we can define a nondegenerate bilinear symmetric form $\langle , \rangle$ on $\mathfrak{g}_0^*$ so that
$$\langle \lambda, \kappa \rangle = K(h_\lambda, h_\kappa), \qquad \forall \lambda, \kappa \in \mathfrak{g}_0^*;$$

9. simple roots are linearly independent;

10. any positive root is a linear combination of simple roots with nonnegative integer coefficients; therefore, the root system $\Delta$ is contained in the real subspace of $\mathfrak{g}_0^*$ spanned by the simple roots, we denote this subspace by $\mathfrak{h}^*$;

11. the whole Lie algebra $\mathfrak{g}$ is obtained by multiple Lie brackets of root vectors $e_\alpha$ where $\alpha \in \Delta^S$ or $-\alpha \in \Delta^S$;

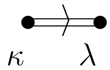
12. $\langle , \rangle$ defines a real scalar product on $\mathfrak{h}^*$;

13. the root system $\Delta$ is invariant under all reflections $S_\lambda$ of the form
$$S_\lambda(\alpha) = \alpha - 2\frac{\langle \alpha, \lambda \rangle}{\langle \lambda, \lambda \rangle}\lambda, \qquad \lambda, \alpha \in \Delta;$$
all such reflections generate a finite group called the *Weyl group* of the root system $\Delta$;

14. any root is an image of some simple root under the action of some element of the Weyl group; in particular, it has the same length.

It turns out that the structure of a semisimple complex Lie algebra is fully determined up to isomorphisms by angles and relative lengths of its simple roots in the Euclidean space $\mathfrak{h}^*$. This information is usually encoded either in the *Cartan matrix* $A = (a_{\kappa\lambda})$
$$a_{\kappa\lambda} = 2\frac{\langle \kappa, \lambda \rangle}{\langle \lambda, \lambda \rangle}, \qquad \kappa, \lambda \in \Delta^S$$

or equivalently in Dynkin diagrams. The Cartan matrix has only integer entries: 2 on the diagonal, $0, -1, -2, -3$ off the diagonal. Its associated *Dynkin diagram* is a graph with vertices corresponding to the simple roots and $a_{\kappa\lambda}$ edges connecting the vertices labelled by $\kappa$ and $\lambda$. Further, one distinguishes graphically between shorter and longer roots either by different symbols for vertices or different types of arrows connecting vertices. We shall use a convention that the arrow goes from the longer root to the shorter one, e.g. a subdiagram of the form




implies the following values of the Cartan matrix elements

$$a_{\kappa\lambda} = 2\frac{\langle \kappa, \lambda \rangle}{\langle \lambda, \lambda \rangle} = -2, \qquad a_{\lambda\kappa} = 2\frac{\langle \lambda, \kappa \rangle}{\langle \kappa, \kappa \rangle} = -1.$$

The structure of any root system can be shown to be such that

1. simple components $\mathfrak{g}_k$ of a semisimple Lie algebra $\mathfrak{g}$ correspond to connected subdiagrams of the Dynkin diagram of $\mathfrak{g}$;

2. there are no closed loops in Dynkin diagrams;

3. a connected Dynkin diagram is either *simply laced* meaning that it contains only simple edges and consequently all roots are of the same length, or the corresponding root system contains roots of precisely two different lengths.

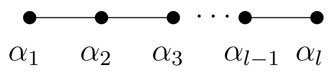
The fundamental classification result is due to W. Killing [12] and É. Cartan [13] whose computations were later significantly simplified by E. Dynkin [21, 22]. It states that a finite–dimensional complex simple Lie algebra $\mathfrak{g}$ either takes one of the following classical forms

- $\mathfrak{sl}(l+1, \mathbb{C})$, the algebra of traceless $(l+1) \times (l+1)$ matrices, also denoted $A_l$, of rank $l \geq 1$,

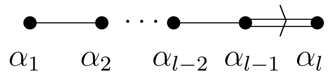
- $\mathfrak{so}(2l+1, \mathbb{C})$, the algebra of skew–symmetric $(2l+1) \times (2l+1)$ matrices, also denoted $B_l$, of rank $l \geq 2$,

- $\mathfrak{sp}(2l, \mathbb{C})$, the algebra of $2l \times 2l$ matrices skew–symmetric with respect to a nondegenerate antisymmetric form on $\mathbb{C}^{2l}$, also denoted $C_l$, of rank $l \geq 3$,

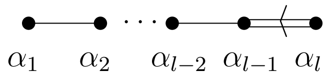
$A_l$



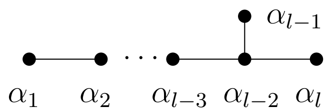

$B_l$



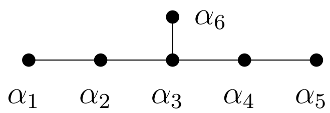

$C_l$



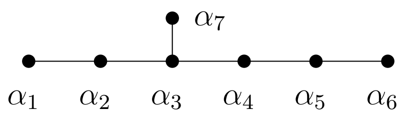

$D_l$



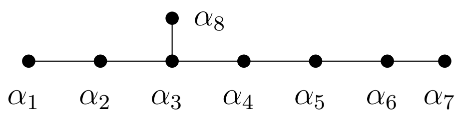

$E_6$



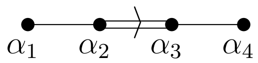

$E_7$




$E_8$




$F_4$




$G_2$

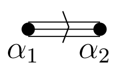



Table 1.1: Dynkin diagrams of simple Lie algebras.

- $\mathfrak{so}(2l, \mathbb{C})$, the algebra of skew–symmetric $2l \times 2l$ matrices, also denoted $D_l$, of rank $l \geq 4$,

or belongs among the five so–called exceptional algebras, denoted by $E_6$, $E_7$, $E_8$, $F_4$, $G_2$. Out of these algebras, the algebras $A_l$, $D_l$, $E_6$, $E_7$, $E_8$ are simply laced. The corresponding Dynkin diagrams are listed in Table 1.1.

### 1.1.3 Classification of solvable Lie algebras

Let us now turn our attention to solvable and nilpotent Lie algebras.

First of all, let us consider Cartan subalgebras of solvable and nilpotent Lie algebras. If $\mathfrak{g}$ is nilpotent then it is equal to its Cartan subalgebra by definition. Consequently, the notion of Cartan subalgebra is trivial in this case and of no help in any classification.

If $\mathfrak{g}$ is solvable and nonnilpotent the Cartan subalgebra is no longer trivial. Let us consider the example of the algebra $\mathfrak{e}(2)$. It has regular elements

$$e_{a_1,a_2} = e_3 + a_1 e_1 + a_2 e_2, \qquad a_1, a_2 \in \mathbb{F}$$

and the corresponding Cartan subalgebras are one–dimensional

$$\mathfrak{g}_0(e_{a_1,a_2}) = \mathrm{span}\{e_{a_1,a_2}\}.$$

It means that in this particular case we have

$$\mathfrak{e}(2) = (\mathfrak{e}(2))^2 \dotplus \mathfrak{g}_0(e_{a_1,a_2}).$$

Had such property held generally, the Cartan subalgebra would have been of great use in the classification of solvable Lie algebras. Unfortunately, that is not the case.

**Example 1.7** *Let us consider the 7–dimensional solvable matrix algebra $\mathfrak{g}$ consisting of all matrices of the form*

$$A(a_1, \ldots, a_7) = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ 0 & a_1 & a_3 & a_6 \\ 0 & 0 & a_1 & a_7 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a_i \in \mathbb{C}. \tag{1.32}$$

*Its nilradical has codimension one in $\mathfrak{g}$ and is spanned by the matrices of the form*

$$A(0, a_2, \ldots, a_7) = \begin{pmatrix} 0 & a_2 & a_4 & a_5 \\ 0 & 0 & a_3 & a_6 \\ 0 & 0 & 0 & a_7 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a_i \in \mathbb{C}.$$

*The derived algebra $\mathfrak{g}^2$ is spanned by*

$$A(0,0,0,a_4,\dots,a_7) = \begin{pmatrix} 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & a_7 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a_i \in \mathbb{C}.$$

*Regular elements of $\mathfrak{g}$ are matrices $A(a_1,\dots,a_7)$ of* (1.32) *with $a_1 \neq 0$. Let us for simplicity choose*

$$A_0 = A(1,0,\dots,0).$$

*The (generalized) nullspace $\mathfrak{g}_0$ of $\mathrm{ad}_{A_0}$ is 4–dimensional. It consists of matrices which commute with $\mathrm{ad}_{A_0}$, i.e.*

$$\begin{pmatrix} a_1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a_i \in \mathbb{C}.$$

*The structure of the Cartan subalgebra $\mathfrak{g}_0$ is the same as of a direct sum of an Abelian one–dimensional algebra spanned by $A_0$ and Heisenberg algebra $\mathfrak{h}(1)$ spanned by $A(0,a_2,a_3,a_4,0,0,0)$. We see that in this case the Cartan subalgebra is no longer Abelian and that it has a nontrivial intersection with the derived algebra $\mathfrak{g}^2$.*

*The root subspace is in this case defined as a common generalized eigenspace, i.e.*

$$\mathfrak{g}_\lambda = \bigcap_{x \in \mathfrak{g}_0} \lim_{k\to\infty} \ker(\mathrm{ad}(x) - \lambda(x)\mathbf{1})^k.$$

*There is only one root $\lambda :\ \lambda(A(a_1,a_2,a_3,a_4,0,0,0)) = a_1$ with its root subspace*

$$\mathfrak{g}_\lambda = \{A(0,0,0,0,a_5,a_6,a_7) | a_4,a_5,a_6 \in \mathbb{C}\}.$$

As we have just seen, a number of crucial properties of root systems that made the classification of semisimple complex algebra feasible, are lost for solvable algebras. Among these are the commutativity of the Cartan subalgebra, one–dimensionality of the root subspaces, the existence of an opposite root etc.

Therefore, we conclude that the notion of Cartan subalgebra is not of particular importance for the structure theory of nonsemisimple algebras and that a different approach is needed in this case.

In the papers presented in Sections 2.1, 2.2, 2.3 we have followed another route towards a partial classification of solvable Lie algebras, exploiting the fact that every solvable Lie algebra $\mathfrak{s}$ contains a unique maximal nilpotent ideal, i.e. its nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{s})$.

For a solvable Lie algebra $\mathfrak{s}$ the dimension of its nilradical $\mathfrak{n}$ satisfies estimates (see Section 2.4)

$$\dim \mathfrak{n} \geq \dim \mathfrak{s}^2, \qquad \dim \mathfrak{n} \geq \frac{1}{2}(\dim \mathfrak{s} + \dim \mathfrak{s}^{(2)}). \tag{1.33}$$

From now on we shall assume that $\mathfrak{s}$ is *indecomposable*, i.e. cannot be decomposed into a direct sum of two or more ideals. We also assume that $\mathfrak{s}$ is not nilpotent, $\mathfrak{s} \neq \mathfrak{n}$.

**General procedure for classifying all solvable Lie algebras with a given nilradical**

Let us first introduce some notions which we shall use later in this section. An element $x$ of $\mathfrak{s}$ will be called *nilpotent* if it satisfies

$$[x, \dots, [x, [x, y]] \dots] = 0, \qquad \forall y \in \mathfrak{s}$$

when the commutator is taken sufficiently many times. A set of elements $\{f_1, \dots, f_k\}$ of $\mathfrak{s}$ is called *linearly nilindependent* if no nontrivial linear combination of them is nilpotent.

We will often use the adjoint representation defined in equation (1.17) and its restriction to the nilradical $\mathfrak{n}$ of $\mathfrak{s}$ denoted by $\mathrm{ad}|_{\mathfrak{n}}$. This restriction is realized by matrices $A \in \mathbb{F}^{n\times n}$. If $x$ is a nilpotent element of $\mathfrak{s}$, it will be represented by a nilpotent matrix in the adjoint representation and consequently also $\mathrm{ad}|_{\mathfrak{n}}(x)$ will be nilpotent. A set of matrices in $\mathbb{F}^{n\times n}$ will be called *linearly nilindependent* if no nontrivial combination of them is a nilpotent matrix.

Let us consider a given nilpotent Lie algebra $\mathfrak{n}$ of dimension $n$ as a nilradical of a solvable Lie algebra $\mathfrak{s}$ of dimension $s$. We wish to find all such indecomposable solvable Lie algebras. Let us choose a basis $(e_1, \dots, e_n)$ of $\mathfrak{n}$ and extend it to a basis of $\mathfrak{s}$

$$(e_1, \dots, e_n, f_1, \dots, f_p), \qquad \text{where } n + p = s. \tag{1.34}$$

The derived algebra $\mathfrak{s}^2 = [\mathfrak{s}, \mathfrak{s}]$ of a solvable Lie algebra $\mathfrak{s}$ is contained in the nilradical

$$\mathfrak{s}^2 \subseteq \mathfrak{n}.$$

It follows that the commutation relations of the solvable Lie algebra $\mathfrak{s}$ in the basis (1.34) can be written as

$$[e_i, e_j] = \sum_{k=1}^{n} N_{ij}{}^k e_k, \tag{1.35}$$

$$[f_\alpha, e_i] = \sum_{j=1}^{n} (A_\alpha)_i^j e_j, \tag{1.36}$$

$$[f_\alpha, f_\beta] = \sum_{i=1}^{n} \gamma_{\alpha\beta}{}^i e_i \tag{1.37}$$

where Latin subscripts and superscripts $i, j = 1, \ldots, n$ refer to the basis of the nilradical $\mathfrak{n}$, Greek ones $\alpha, \beta = 1, \ldots, p$ to the basis of the complementary space $\mathfrak{f}$ spanned by $f_1, \ldots, f_p$. The structure constants $N_{ij}{}^k$ of the nilradical are assumed to be known (in the chosen basis $(e_1, \ldots, e_n)$).

The Jacobi identities for $\{e_i, e_j, e_k\}$ are assumed to be satisfied. The remaining Jacobi identities must be imposed. These identities for triplets $\{f_\alpha, e_i, e_j\}$, $\{f_\alpha, f_\beta, e_i\}$, $\{f_\alpha, f_\beta, f_\delta\}$ imply

$$\sum_{k=1}^{n} N_{ij}{}^k (A_\alpha)_k^l + N_{jk}{}^l (A_\alpha)_i^k - N_{ik}{}^l (A_\alpha)_j^k = 0, \tag{1.38}$$

$$([A_\alpha, A_\beta])_i^j = \sum_{k=1}^{n} \gamma_{\alpha\beta}{}^k N_{ik}{}^j, \tag{1.39}$$

$$\sum_{i=1}^{n} \gamma_{\beta\delta}{}^i (A_\alpha)_i^k + \gamma_{\alpha\beta}{}^i (A_\delta)_i^k + \gamma_{\delta\alpha}{}^i (A_\beta)_i^k = 0, \tag{1.40}$$

respectively.

Equation (1.38) is a system of $n^2(n-1)/2$ linear homogeneous equations for $n^2$ unknowns, i.e. the matrix elements $(A_\alpha)_i^j$, for each value of $\alpha \in \{1, \ldots, p\}$ separately (since $N_{ij}{}^k$ are known). It follows from (1.38) that the matrix $A_\alpha$ represents a derivation $D_\alpha$ of the nilradical. In the adjoint representation of the Lie algebra $\mathfrak{s}$ restricted to the nilradical $\mathfrak{n}$ we have

$$(D_1 = \mathrm{ad}|_{\mathfrak{n}}(f_1), \ldots, D_p = \mathrm{ad}|_{\mathfrak{n}}(f_p)) \simeq (A_1, \ldots, A_p) \tag{1.41}$$

where $D_\alpha(e_i) = \sum_{j=1}^{n} (A_\alpha)_i^j e_j$. Finding all sets of matrices satisfying equation (1.38) is equivalent to finding all sets of nilindependent derivations of the nilradical. They must be nilindependent since otherwise the nilradical would be larger (it would contain one or more elements of $\mathfrak{f}$). This also means that the derivations are outer ones: inner derivations of $\mathfrak{n}$ are always represented by nilpotent matrices.

Relations (1.39) exist for $p \geq 2$ and determine the properties of the set of matrices $(A_1, \ldots, A_p)$. In particular, if the nilradical is Abelian we have ${N_{ij}}^k = 0$ and hence the matrices commute

$$[A_\alpha, A_\beta] = 0. \tag{1.42}$$

In general relation (1.42) does not hold. However, the space $\mathfrak{f}$ is not uniquely defined by the solvable algebra $\mathfrak{s}$. Below we will discuss "allowed transformations" of the basis of $\mathfrak{s}$ that will be used to classify the Lie algebras $\mathfrak{s}$ into equivalence classes. In all cases considered so far it turned out that $\mathfrak{f}$ can be chosen so that matrices $A_\alpha$ commute.

Relations (1.40) represent a set of linear algebraic equations for the structure constants ${\gamma_{\alpha\beta}}^i$ once the matrices $A_\alpha$ are known. They exist for $p \geq 3$.

A classification of solvable Lie algebras $\mathfrak{s}$ with the given nilradical $\mathfrak{n}$ amounts to a classification of all matrices $A_\alpha$ and constants ${\gamma_{\alpha\beta}}^i$ satisfying equations (1.38), (1.39) and (1.40) under the following "allowed transformations":

1. Redefinition of the space $\mathfrak{f}$

$$\tilde{f}_\alpha = f_\alpha + r_\alpha^j e_j, \qquad r_\alpha^j \in \mathbb{F}. \tag{1.43}$$

   The equivalence defined by equation (1.43) implies that $D_\alpha$ should be viewed as equivalence classes, rather than outer derivations themselves. In other words, we combine inner derivations with the outer derivations $D_\alpha$ to modify the matrices $A_\alpha$.

2. Change of basis in the nilradical $\mathfrak{n}$

$$\tilde{e}_i = S_i^j e_j, \qquad S \in \mathrm{Aut}(\mathfrak{n}) \subseteq GL(n, \mathbb{F}). \tag{1.44}$$

   Thus the matrices $S$ form the group of automorphisms of the nilradical $\mathfrak{n}$, expressed in the chosen basis $(e_1, \ldots, e_n)$. By definition, they leave the set of commutation relations (1.35) invariant and consequently respect all basis independent properties of the nilradical (in particular all ideals in the derived and lower and upper central series).

3. Change of basis in $\mathfrak{f}$

$$\tilde{f}_i = G_i^j f_j, \qquad S \in GL(p, \mathbb{F}). \tag{1.45}$$

Such classification has been performed for the following classes of nilpotent Lie algebras: Heisenberg algebras $\mathfrak{h}(N)$ (where $\dim \mathfrak{h}(N) = 2N+1, N \geq 1$) [23], Abelian Lie algebras $\mathfrak{a}_n$, $n \geq 1$ [24, 25], "triangular" Lie algebras $\mathfrak{t}(N)$, ($\dim \mathfrak{t}(N) = N(N-1)/2, N \geq 2$) [26, 27], naturally graded and $\mathbb{Z}$–graded nilradicals of maximal degree of nilpotency [28, 29, 30, 31] and some other special types of nilradicals [32, 33, 34]. Out of these, the papers [28, 30, 33] are presented in this thesis.

Moreover, all solvable algebras in dimensions up to 6 have been found in this way [35, 36, 37].

We shall now review several low–dimensional examples in order to demonstrate the procedure.

First, let us consider the case of 3–dimensional indecomposable solvable Lie algebras. In this case the condition (1.33) shows that the dimension of the nilradical $\dim \mathrm{NR}(\mathfrak{s})$ is 2 or 3. When $\dim \mathrm{NR}(\mathfrak{s})$ is 3, the algebra is equal to its nilradical, i.e. nilpotent. It can be shown to be the Heisenberg algebra $\mathfrak{h}(1)$. When $\dim \mathrm{NR}(\mathfrak{s}) = 2$ we have an Abelian nilradical since no other nilpotent 2–dimensional Lie algebra exists. The solvable algebra $\mathfrak{s}$ is determined once the action of one nonnilpotent element $f_1$ on the nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{s}) = \mathrm{span}\{e_1, e_2\}$ is specified. Any change of basis in the nilradical is allowed because any regular linear map is an automorphism of $\mathfrak{n}$ and consequently the task is reduced to the classification of $2 \times 2$ nonnilpotent matrices with respect to conjugation and overall rescaling by nonzero number. We find the following canonical forms for the matrix $A_1$:

- Over the field of complex numbers the matrix $A_1$ has one of the following two forms
$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
where the parameter $a$ satisfies $0 < |a| \leq 1$, if $|a| = 1$ then $\arg(a) \leq \pi$.

- Over the field of real numbers the matrix $A_1$ has one of the following three forms
$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \qquad \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
where the parameters $a, \alpha$ satisfy $-1 \leq a \leq 1$, $a \neq 0$, $\alpha \geq 0$.

The condition $a \neq 0$ arises from the restriction to indecomposable algebras. The matrix $\begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}$ is present only over the field of real numbers because over the field of complex numbers it is upon rescaling conjugated to $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ with the choice $a = \frac{\alpha+\mathrm{i}}{\alpha-\mathrm{i}}$.

The corresponding solvable algebras are

$\mathfrak{s}_{3,a}$

| | $e_1$ | $e_2$ | $f_1$ |
|---|---|---|---|
| $e_1$ | 0 | 0 | $-e_1$ |
| $e_2$ | 0 | 0 | $-ae_2$ |
| $f_1$ | $e_1$ | $ae_2$ | 0 |

,

$\mathfrak{s}_{3,\mathbb{R},\alpha}$

| | $e_1$ | $e_2$ | $f_1$ |
|---|---|---|---|
| $e_1$ | 0 | 0 | $-\alpha e_1 + e_2$ |
| $e_2$ | 0 | 0 | $-e_1 - \alpha e_2$ |
| $f_1$ | $\alpha e_1 - e_2$ | $e_1 + \alpha e_2$ | 0 |

which is isomorphic to $\mathfrak{s}_{3,a}$ over the field of complex numbers, and

$\mathfrak{s}_{3,\mathrm{Jordan}}$

| | $e_1$ | $e_2$ | $f_1$ |
|---|---|---|---|
| $e_1$ | 0 | 0 | $-e_1$ |
| $e_2$ | 0 | 0 | $-e_1 - e_2$ |
| $f_1$ | $e_1$ | $e_1 + e_2$ | 0 |

.

A similar investigation can be performed in any dimension when the nilradical is Abelian and has codimension one in $\mathfrak{s}$. When the codimension of the Abelian nilradical is greater than one the situation becomes more involved – one has to classify all Abelian subalgebras of the matrix algebra $\mathfrak{gl}(\mathfrak{n})$ and then use this classification in the construction of non–isomorphic solvable algebras with the Abelian nilradical $\mathfrak{n}$.

When the nilradical $\mathfrak{n}$ is not Abelian its Lie brackets put restrictions on automorphisms of $\mathfrak{n}$, i.e. they are no longer arbitrary regular linear maps. Consequently, not all changes of basis in the nilradical are allowed. Similarly, the space of derivations of $\mathfrak{n}$ is restricted.

As an example let us consider solvable Lie algebras with the Heisenberg nilradical of Example 1.1, i.e. $\mathfrak{n} = \mathfrak{h}(1)$

| | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|
| $e_1$ | 0 | 0 | 0 |
| $e_2$ | | 0 | $e_1$ |

.

Now any derivation $D$ and any automorphism $\phi$ must preserve the ideal $\mathfrak{n}^2 = \mathrm{span}\{e_1\}$ and in addition $D(e_1)$ or $\phi(e_1)$ is determined by the action of $D$, $\phi$, respectively, on $e_2, e_3$. We find that an arbitrary derivation of $\mathfrak{n}$ has the matrix form

$$D = \begin{pmatrix} \mathrm{tr}X & \vec{b} \\ 0 & X \end{pmatrix}, \quad X \in \mathbb{F}^{2,2}, \quad \vec{b} \in \mathbb{F}^2 \qquad (1.46)$$

and any automorphism takes the form

$$\phi = \begin{pmatrix} \det C & \vec{d} \\ 0 & C \end{pmatrix}, \quad C \in \mathbb{F}^{2,2}, \det C \neq 0, \quad \vec{d} \in \mathbb{F}^2. \tag{1.47}$$

Let $\mathfrak{s}$ be a solvable Lie algebra with the nilradical $\mathfrak{h}(1)$. Using the criterion proven in Section 2.4 we find that

$$\dim \mathfrak{s} \leq \dim \mathfrak{h}(1) + 2 = 5.$$

The inner derivations of $\mathfrak{h}(1)$ have the form (1.21). Therefore, the derivations $D_j|_{\mathfrak{n}} = \mathrm{ad}(f_j)$ can be brought to the form

$$D_j = \begin{pmatrix} \mathrm{tr} X_j & 0 \\ 0 & X_j \end{pmatrix}, \quad X_j \in \mathbb{F}^{2,2} \tag{1.48}$$

by the transformation (1.43). If $\dim \mathfrak{s} = 4$ we have to classify all $2 \times 2$ nonnilpotent matrices $X_j$. This is the same task as accomplished above for the Abelian two–dimensional nilradical (the matrices $A$); the only difference is in the values of parameters which lead to decomposable algebras and are therefore excluded.

The corresponding algebras are described by the following Lie brackets:

| | $e_1$ | $e_2$ | $e_3$ | $f_1$ |
|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $-(1+a)e_1$ |
| $e_2$ | $0$ | $0$ | $e_1$ | $-e_2$ |
| $e_3$ | $0$ | $-e_1$ | $0$ | $-ae_3$ |
| $f_1$ | $(1+a)e_1$ | $e_2$ | $ae_3$ | $0$ |

$0 \leq |a| \leq 1, \text{ if } |a| = 1 \text{ then } \arg(a) \leq \pi$

| | $e_1$ | $e_2$ | $e_3$ | $f_1$ |
|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $-2e_1$ |
| $e_2$ | $0$ | $0$ | $e_1$ | $-e_2$ |
| $e_3$ | $0$ | $-e_1$ | $0$ | $-e_2 - e_3$ |
| $f_1$ | $2e_1$ | $e_2$ | $e_2 + e_3$ | $0$ |

over the field of complex numbers, and an additional real form

| | $e_1$ | $e_2$ | $e_3$ | $f_1$ |
|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $-2\alpha e_1$ |
| $e_2$ | $0$ | $0$ | $e_1$ | $-e_2 + e_3$ |
| $e_3$ | $0$ | $-e_1$ | $0$ | $-e_2 - \alpha e_3$ |
| $f_1$ | $2\alpha e_1$ | $e_2 - e_3$ | $e_2 + \alpha e_3$ | $0$ |

, $\quad \alpha \geq 0.$

When $\dim \mathfrak{s} = 5$ we have two derivations $D_1, D_2$. Their commutator must be an inner derivation but that must vanish due to their form (1.48). Therefore $D_1, D_2$ commute. We have at our disposal the conjugation (1.44) by the automorphism (1.47) and the change of basis (1.45). The task at hand is reduced to classification of pairs of commuting nonnilpotent $2 \times 2$ matrices $X_1, X_2$ up to conjugation and linear combinations. It turns out that we can bring any pair of commuting derivations (1.48) to a unique canonical form

$$D_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

when the field is complex. Over the field of real numbers we have one more possibility, namely

$$D_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The corresponding Lie brackets defining the solvable codimension 2 extensions of $\mathfrak{h}(1)$ are

| | $e_1$ | $e_2$ | $e_3$ | $f_1$ | $f_2$ |
|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $-2e_1$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_1$ | $-e_2$ | $-e_2$ |
| $e_3$ | $0$ | $-e_1$ | $0$ | $-e_3$ | $e_3$ |
| $f_1$ | $2e_1$ | $e_2$ | $e_3$ | $0$ | $0$ |
| $f_2$ | $0$ | $e_2$ | $-e_3$ | $0$ | $0$ |

and

| | $e_1$ | $e_2$ | $e_3$ | $f_1$ | $f_2$ |
|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $-2e_1$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_1$ | $-e_2$ | $-e_3$ |
| $e_3$ | $0$ | $-e_1$ | $0$ | $-e_3$ | $e_2$ |
| $f_1$ | $2e_1$ | $e_2$ | $e_3$ | $0$ | $0$ |
| $f_2$ | $0$ | $e_3$ | $-e_2$ | $0$ | $0$ |

,

respectively.

## 1.2 Casimir invariants

### 1.2.1 Universal enveloping algebras and Casimir operators

Universal enveloping algebra is an important object in the representation theory of Lie algebras. It is defined as a certain factoralgebra of the tensor algebra of a given Lie algebra $\mathfrak{g}$.

The *tensor algebra* (or *free algebra*) of the vector space $V$ over the field $\mathbb{F}$ is the vector space

$$\mathcal{T}(V) = \oplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{F} \oplus V \oplus V \otimes V \oplus \ldots \oplus V^{\otimes k} \oplus \ldots$$

equipped with the associative multiplication generated by the multiplication of decomposable elements

$$(v_1 \otimes v_2 \otimes \ldots \otimes v_k) \cdot (w_1 \otimes \ldots \otimes w_l) = v_1 \otimes v_2 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_l.$$

When the vector space $V$ is in addition a Lie algebra $V = \mathfrak{g}$, one may consider a two–sided ideal $\mathcal{J}$ in the associative algebra $\mathcal{T}(\mathfrak{g})$ generated by the elements of the form $x \otimes y - y \otimes x - [x, y]$, i.e.

$$\mathcal{J} = \operatorname{span}\left\{ A \otimes (x \otimes y - y \otimes x - [x, y]) \otimes B \,|\, x, y \in \mathfrak{g},\ A, B \in \mathcal{T}(\mathfrak{g}) \right\}.$$

The factoralgebra

$$\mathfrak{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{J} \tag{1.49}$$

is called the *universal enveloping algebra* of the Lie algebra $\mathfrak{g}$. It is obvious that universal enveloping algebras are associative algebras, i.e. the notion of a universal enveloping algebra allows us to construct an infinite dimensional associative algebra out of any Lie algebra in a canonical way.

The main reason why universal enveloping algebras are useful is the following observation: any representation $\rho$ of a Lie algebra $\mathfrak{g}$ on a (finite–dimensional, for simplicity) vector space $V$ gives rise to a representation $\tilde{\rho}$ of the tensor algebra $\mathcal{T}(\mathfrak{g})$ defined by

$$\tilde{\rho}(x_1 \otimes x_2 \otimes \ldots \otimes x_k) = \rho(x_1) \cdot \rho(x_2) \ldots \rho(x_k).$$

The definition of a representation $\rho$, equation (1.15), implies that $\tilde{\rho}(\mathcal{J}) = 0$. Consequently, $\tilde{\rho}$ defines also a representation $\hat{\rho}$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on the vector space $V$

$$\hat{\rho}(a) = \tilde{\rho}(A), \qquad a = A \bmod \mathcal{J} \in \mathfrak{U}(\mathfrak{g}), \quad A \in \mathcal{T}(\mathfrak{g}).$$

*Casimir operators* are elements of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ [38, 39, 40], i.e. such $c \in \mathfrak{U}(\mathfrak{g})$ that

$$c \cdot a = a \cdot c$$

holds for all $a \in \mathfrak{U}(\mathfrak{g})$. A necessary and sufficient condition for $c$ to be a Casimir operator is

$$c \cdot x = x \cdot c, \qquad \forall x \in \mathfrak{g} \simeq \mathfrak{g}^{\otimes 1}/\mathcal{J}.$$

We shall consider nontrivial Casimir operators only, i.e. those different from elements of $\mathbb{F}/\mathcal{J} \simeq \mathbb{F}$. In order to avoid writing mod $\mathcal{J}$ at all times we adopt a convention that Casimir operators shall be written as totally symmetric expressions in the elements of $\mathfrak{g}$. This can be always accomplished using the identity

$$x \otimes y \bmod \mathcal{J} = \frac{1}{2}(x \otimes y + y \otimes x) + \frac{1}{2}[x, y] \bmod \mathcal{J}$$

as many times as needed, starting from the highest order terms and proceeding order by order. Such a procedure also implies the uniqueness of such totally symmetric representative of the equivalence class mod $\mathcal{J}$. We shall occasionally suppress the tensor product sign, i.e. $xy \equiv x \otimes y$.

The importance of Casimir operators for the representation theory of complex Lie algebras comes from the Schur's lemma, Theorem 1.1. In any representation $\rho$ we have

$$[\hat{\rho}(c), \rho(x)] = 0, \qquad \forall x \in \mathfrak{g}.$$

Consequently, if the representation $\rho$ is irreducible, $\hat{\rho}(c)$ must be a multiple of the identity operator, $\lambda \mathbf{1}$. The number $\lambda$ depends on the choice of the representation $\rho$ and the Casimir operator $c$. If two irreducible representations $\rho_1$ on $V_1$ and $\rho_2$ on $V_2$ are *equivalent*, i.e. if a linear transformation $T : V_1 \to V_2$ exists such that

$$\rho_2(x) = T \circ \rho_1(x) \circ T^{-1}, \qquad \forall x \in \mathfrak{g},$$

then necessarily we have $\lambda_1 = \lambda_2$ for the given Casimir invariant $c$. That means that the eigenvalues of $\hat{\rho}(c)$ can be used to distinguish inequivalent irreducible representations.

If $\rho$ is fully reducible but not irreducible then we may use the knowledge of Casimir operators of $\mathfrak{g}$ in the decomposition of $\rho$ into irreducible components. In particular, we construct common eigenspaces of all known

Casimir operators and we know that each of them is an invariant subspace (not necessarily irreducible for general $\mathfrak{g}$).

The existence of nontrivial Casimir operators was established for certain classes of Lie algebras only, e.g. for semisimple ones. Also Lie algebras with nonvanishing center, including all nilpotent ones, do possess nontrivial Casimir operators; namely, the elements of the center themselves. On the other hand some Lie algebras are known to have no nontrivial Casimir invariants.

Let us consider a semisimple complex Lie algebra $\mathfrak{g}$ and its Killing form $K$. Let us take any basis $(e_1, \ldots, e_{\dim \mathfrak{g}})$ of $\mathfrak{g}$ and find the dual basis $(\tilde{e}^1, \ldots, \tilde{e}^{\dim \mathfrak{g}})$ such that

$$K(e_k, \tilde{e}^j) = \delta_k^j.$$

Let us assume that ${c_{ij}}^k$ are the structure constants (1.3) of the Lie algebra $\mathfrak{g}$ in the basis $(e_1, \ldots, e_k)$. By the invariance of the Killing form $K$ we have

$$K(e_k, [e_a, \tilde{e}^j]) = -K([e_a, e_k], \tilde{e}^j) = -{c_{ak}}^j = -K(e_k, \sum_{m=1}^{\dim \mathfrak{g}} {c_{am}}^j \tilde{e}^m)$$

which by nondegeneracy of $K$ implies that

$$[e_a, \tilde{e}^j] = \sum_{m=1}^{\dim \mathfrak{g}} {c_{ma}}^j \tilde{e}^m.$$

Let us construct an element of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the form

$$C = \sum_{k=1}^{\dim \mathfrak{g}} \tilde{e}^k \otimes e_k = \sum_{k=1}^{\dim \mathfrak{g}} e_k \otimes \tilde{e}^k \tag{1.50}$$

(its symmetry comes from the fact that the Killing form is symmetric).

Suppressing the tensor product signs and computing $\bmod \mathcal{J}$, we have for the commutator between $e_a \in \mathfrak{g}$ and $C \in \mathfrak{U}(\mathfrak{g})$

$$\begin{aligned}
[e_a, C] &= \sum_{k=1}^{\dim \mathfrak{g}} \left(e_a\, \tilde{e}^k\, e_k - \tilde{e}^k\, e_k\, e_a\right) = \\
&= \sum_{k=1}^{\dim \mathfrak{g}} \left((e_a\, \tilde{e}^k - \tilde{e}^k\, e_a)\, e_k + \tilde{e}^k\, (e_a\, e_k - e_k\, e_a)\right) = \\
&= \sum_{k=1}^{\dim \mathfrak{g}} ([e_a, \tilde{e}^k]\, e_k + \tilde{e}^k\, [e_a, e_k]) = \sum_{k,l=1}^{\dim \mathfrak{g}} ({c_{la}}^k \tilde{e}^l\, e_k + {c_{ak}}^l \tilde{e}^k\, e_l) = 0.
\end{aligned}$$

We conclude that $C$ is a Casimir operator of $\mathfrak{g}$. It is called the *quadratic Casimir operator* [38]. For its application in the proof of Weyl's theorem, see [39].

We remark that the quadratic Casimir operator does not exhaust all independent Casimir operators of the semisimple Lie algebra $\mathfrak{g}$ when we have rank $\mathfrak{g} > 1$. It is known that any semisimple Lie algebra of rank $l$ has $l$ independent Casimir operators which generate the whole center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ through their products and linear combinations. Their explicit form depends on the details of the structure of the considered algebra $\mathfrak{g}$.

Casimir invariants are of primordial importance in physics. They often represent such important quantities as angular momentum, elementary particle mass and spin, Hamiltonians of various physical systems etc.

**Example 1.8** *Let us consider the angular momentum algebra*

$$\mathfrak{so}(3) = \mathrm{span}\{L_1, L_2, L_3\}$$

*with*

$$[L_j, L_k] = \sum_{l=1}^{3} \epsilon_{jkl} L_l. \tag{1.51}$$

*The quadratic Casimir operator* (1.50) *is*

$$C = -\frac{1}{2} \sum_{l=1}^{3} L_l^2, \tag{1.52}$$

*i.e. it coincides up to a numerical factor* $1/2$ *with the square of angular momentum, familiar from the construction of irreducible representations of the angular momentum algebra in quantum mechanics.*

Notice that the sign of the Casimir operator (1.52) is in fact the same as used in physics: in quantum mechanics the operators of angular momentum $\hat{L}_j$ (measured in multiples of $\hbar$) satisfy the commutation relations

$$[\hat{L}_j, \hat{L}_k] = \sum_{l=1}^{3} \mathrm{i}\epsilon_{jkl} \hat{L}_l$$

which differ from the ones in equation (1.51) by an extra imaginary unit. This extra i factor can be traced to the requirement that observables are described by Hermitean operators; the generators of unitary representations of Lie groups are, on the contrary, anti–Hermitean. An obvious remedy is to formally introduce a "physical" basis of a given real Lie algebra

$$\hat{e}_j = \mathrm{i}e_j \tag{1.53}$$

in which the original real structure constants

$$[e_j, e_k] = \sum_l f_{jk}{}^l e_l$$

become explicitly purely imaginary

$$[\hat{e}_j, \hat{e}_k] = \sum_l \mathrm{i}f_{jk}{}^l \hat{e}_l.$$

**Example 1.9** *Let us consider the Poincaré algebra* $\mathfrak{iso}(1,3)$ *(a.k.a. inhomogeneous Lorentz algebra) spanned by* $M^{\mu\nu}, P^\mu$, $\mu, \nu = 0, \ldots, 3$ *with the nonvanishing commutation relations*

$$\begin{aligned} [M^{\mu\nu}, P^\rho] &= \eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu, \\ [M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}, \end{aligned} \tag{1.54}$$

*where* $\eta$ *is the Minkowski metric* $\eta^{\mu\nu} = \eta_{\mu\nu} = diag(1, -1, -1, -1)$*. We shall use the metric* $\eta$ *to move indices up and down, as is common in the theory of relativity, and denote by* $\epsilon_{\mu\nu\rho\sigma}$ *the covariant totally antisymmetric tensor.*

*The Poincaré algebra has a nontrivial Levi decomposition* (1.31)

$$\mathfrak{iso}(1,3) = \mathfrak{so}(1,3) \dotplus \mathfrak{r}$$

*with its semisimple factor being the Lorentz algebra*

$$\mathfrak{so}(1,3) = \mathrm{span}\{M^{\mu\nu}\}_{\mu,\nu=0,1,2,3}$$

*and an Abelian radical*

$$\mathfrak{r} = \mathrm{span}\{P^\mu\}_{\mu=0,1,2,3}.$$

*There are two independent Casimir operators of this Lie algebra, which are usually expressed as*

$$P^2 = \sum_{\mu=0}^{3} \eta_{\mu\nu}P^\mu P^\nu \quad \text{and} \quad W^2 = \sum_{\mu=0}^{3} \eta_{\mu\nu}W^\mu W^\nu$$

*where the quadruplet of quadratic elements of* $\mathfrak{U}(\mathfrak{g})$

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma$$

*is called the Pauli–Lubanski vector. That means that in this case one of the Casimir operators is of second order in generators whereas the other is of fourth order.*

*These two Casimir operators are essential in the construction of irreducible representations of the Poincaré algebra in relativistic quantum field theory. Notice that in this case one constructs infinite–dimensional unitary representations.*

### Energy spectrum of hydrogen atom in quantum mechanics

In order to further demonstrate the relevance of Casimir operators to physics, let us review another application, namely an algebraic determination of the hydrogen spectrum in quantum mechanics. This computation is originally due to Wolfgang Pauli [41].

The Hamiltonian of an electron in hydrogen atom is

$$\hat{H} = \frac{1}{2M} \sum_j \hat{P}_j \hat{P}_j - \frac{Q}{r}, \tag{1.55}$$

where $\hat{P}_j = -\mathrm{i}\hbar \frac{\partial}{\partial x_j}$ are operators of linear momenta in $\mathbb{R}^3$ with the coordinates $x_1, x_2, x_3$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $M$ is the mass of the electron and $Q = \frac{e^2}{4\pi\epsilon_0}$ in SI units.

The Hamiltonian (1.55) has three obvious integrals of motion, namely the angular momenta

$$\hat{L}_j = \frac{1}{\hbar} \sum_{k,l} \epsilon_{jkl} \hat{X}_k \hat{P}_l,$$

(chosen dimensionless for convenience) and three less obvious integrals of motion, namely the components of the Laplace–Runge–Lenz vector

$$\hat{K}_i = \frac{1}{2MQ} \sum_k \sum_j \epsilon_{ikj} (\hat{P}_k \hat{L}_j + \hat{L}_j \hat{P}_k) - \frac{1}{\hbar} \frac{x_i}{r}. \tag{1.56}$$

The expression $\frac{x_i}{r}$ should be interpreted as the operator of multiplication by the given function of coordinates. For future reference, let us denote

$$\hat{L}^2 = \sum_{j=1}^{3} \hat{L}_j \hat{L}_j, \qquad \hat{K}^2 = \sum_{j=1}^{3} \hat{K}_j \hat{K}_j.$$

As it turns out, the knowledge of these integrals of motions and their algebraic structure is enough to determine the spectrum of bound states in the hydrogen atom.

The crucial ingredients are the commutators between various components $\hat{L}_j$ and $\hat{K}_j$. By a somewhat lengthy but straightforward calculation we find

$$
\begin{aligned}
[\hat{L}_j, \hat{L}_k] &= \mathrm{i}\sum_{l=1}^{3} \epsilon_{jkl}\hat{L}_l, &(1.57)\\
[\hat{L}_j, \hat{K}_k] &= \mathrm{i}\sum_{l=1}^{3} \epsilon_{jkl}\hat{K}_l, &(1.58)\\
[\hat{K}_j, \hat{K}_k] &= -\frac{2\mathrm{i}}{MQ^2}\sum_{l=1}^{3} \epsilon_{jkl}\hat{L}_l\hat{H}. &(1.59)
\end{aligned}
$$

Another important observation is the operator identity

$$\sum_{j=1}^{3} \hat{K}_j\hat{L}_j = 0. \tag{1.60}$$

The commutator (1.59) prevents the operators $\hat{L}_j, \hat{K}_j$ from forming a Lie algebra. Nevertheless, this bothersome property can be circumvented if we consider a given energy level, i.e. a subspace $\mathcal{H}_E$ of the Hilbert space $\mathcal{H}$ consisting of all eigenvectors of $\hat{H}$ with the given energy $E$. Operators $\hat{L}_j, \hat{K}_j$ can be all restricted to $\mathcal{H}_E$ because they commute with $\hat{H}$. When such restriction is understood, the $\hat{H}$ in equation (1.59) can be replaced by a numerical factor $E$ and the algebra of $\hat{L}_j, \hat{K}_j$ closes. In particular, when $E < 0$ it is isomorphic to the Lie algebra $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. When $E > 0$ the difference in sign leads to a different real form of the same complex Lie algebra, namely to $\mathfrak{so}(1,3)$. We shall be interested in bound states here, i.e. we assume $E < 0$.

Once the energy is fixed we may introduce the operators

$$\hat{L}_{(1)j} = \frac{1}{2}\left(\hat{L}_j + \sqrt{-\frac{MQ^2}{2E}}\hat{K}_j\right)$$

and

$$\hat{L}_{(2)j} = \frac{1}{2}\left(\hat{L}_j - \sqrt{-\frac{MQ^2}{2E}}\hat{K}_j\right)$$

(notice that $-\frac{MQ^2}{2E}$ is by assumption a positive number). The commutators of $\hat{L}_{(1)j}$ and $\hat{L}_{(2)j}$ now become

$$
\begin{aligned}
[\hat{L}_{(1)j}, \hat{L}_{(1)k}] &= \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \hat{L}_{(1)l}, \\
[\hat{L}_{(2)j}, \hat{L}_{(2)k}] &= \mathrm{i} \sum_{l=1}^{3} \epsilon_{jkl} \hat{L}_{(2)l}, \\
[\hat{L}_{(1)j}, \hat{L}_{(2)k}] &= 0.
\end{aligned}
$$

That means that we have an explicit decomposition of our realization of $\mathfrak{so}(4)$ into the direct sum $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and that the two independent Casimir operators of $\mathfrak{so}(4)$ can be expressed as

$$
C_1 = \sum_{j=1}^{3} \hat{L}^2_{(1)j}, \qquad C_2 = \sum_{j=1}^{3} \hat{L}^2_{(2)j},
$$

or equivalently as

$$
C_1 = \frac{1}{4} \sum_{j=1}^{3} \left( \hat{L}_j + \sqrt{-\frac{MQ^2}{2E}} \hat{K}_j \right)^2, \qquad C_2 = \frac{1}{4} \sum_{j=1}^{3} \left( \hat{L}_j - \sqrt{-\frac{MQ^2}{2E}} \hat{K}_j \right)^2. \tag{1.61}
$$

The sum of these two Casimir operators, i.e. $C_1 + C_2$, gives the quadratic Casimir operator (1.50) of $\mathfrak{so}(4)$.

From the theory of angular momentum, i.e. of representations of the Lie algebra $\mathfrak{so}(3)$, we know that in any irreducible representation of $\mathfrak{so}(4)$ we have

$$
C_1 = p(p+1)\mathbf{1}, \qquad C_2 = q(q+1)\mathbf{1}
$$

for some nonnegative integer or half–integer values of $p$ and $q$. The irreducible representation of $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ determined by these values of the Casimir operators has dimension equal to $(2p+1) \times (2q+1)$.

When we expand the expressions for the Casimir operators (1.61) and subtract them, we find that

$$
C_1 - C_2 = \sqrt{-\frac{MQ^2}{2E}} \sum_{j=1}^{3} \hat{L}_j \hat{K}_j
$$

which vanishes in our representation, as we already know (cf. (1.60)). Therefore, only irreducible representations of $\mathfrak{so}(4)$ with $p = q$ arise in our problem.

Let us now consider such a representation of $\mathfrak{so}(4)$ with the given values of $E$ and $p$. The angular momentum $\hat{L}_j$ can be expressed as

$$\hat{L}_j = \hat{L}_{(1)j} + \hat{L}_{(2)j},$$

i.e. we can employ the standard result concerning the composition of two independent angular momenta and conclude that $\hat{L}^2$ takes all integer values between $|p - p| = 0$ and $p + p = 2p$. In particular, the s–state, i.e. the state with $\hat{L}^2 = 0$, exists in our representation and is of interest to us. Let $\psi$ be any s–state, i.e. a vector $\psi \in \mathcal{H}$ such that $\hat{L}_j \psi = 0$. Obviously, $\psi$ is a function of the radial coordinate $r$ only. We have

$$\hat{L}^2 \psi = 0$$

and

$$\hat{K}^2 \psi = \frac{2}{MQ^2} \hat{H} \psi + \frac{1}{\hbar^2} \psi$$

by inspection of both sides of the equation when expanded in terms of $\hat{X}_j, \hat{P}_j$ etc.

When $\psi$ in addition belongs to our representation of $\mathfrak{so}(4)$ determined by the values of $E$ and $p$, we have the following value for the quadratic Casimir operator (1.50) of $\mathfrak{so}(4)$

$$\begin{aligned} (C_1 + C_2)\psi = 2p(p+1)\psi &= \frac{1}{2} \sum_j \left( \hat{L}_j^2 \psi - \frac{MQ^2}{2E} \hat{K}_j^2 \psi \right) \\ &= -\frac{MQ^2}{4E} \left( \frac{2E}{MQ^2} + \frac{1}{\hbar^2} \right) \psi. \end{aligned} \tag{1.62}$$

Thus we have arrived at the condition

$$8p(p+1) = -2 - \frac{MQ^2}{\hbar^2 E}$$

which is just a different formulation of the celebrated Rydberg formula

$$E = -\frac{MQ^2}{2\hbar^2} \frac{1}{(2p+1)^2} \tag{1.63}$$

where the potentially half–integer valued parameter $p$ is traditionally replaced by the integer $n = 2p + 1 > 0$. Once we have established that $E$ is determined by the value of $p$ by equation (1.63) we also see that $\mathcal{H}_E$ coincides with the representation space of the $\mathfrak{so}(4)$ irreducible representation labelled by $p$ and $q = p$. On $\mathcal{H}_E$ we may also write equation (1.62) in the form

$$C_1 + C_2 = -\left( \frac{MQ^2}{4\hbar^2 \hat{H}} + \frac{1}{2} \right) \tag{1.64}$$

since both $C_1 + C_2$ and $\hat{H}$ take a constant value on $\mathcal{H}_E$. While it may be tempting to consider this to be an operator identity valid on the whole Hilbert space $\mathcal{H}$, we don't consider such interpretation legitimate. In particular, on scattering states ($E > 0$) we even have a different Lie algebra. Therefore, equation (1.64) should be considered at most on the bound state sector of our Hilbert space $\mathcal{H}$.

To sum up, we have seen that the spectrum of hydrogen atom can be derived using the theory of Lie algebras, without explicit construction of eigenfunctions. More precisely, we have derived a necessary condition (1.63) that any energy eigenvalue must satisfy. That this formula is physically relevant for all values of $p \geq 0$ such that $2p \in \mathbb{Z}$ is not a consequence of the computation just shown and shall be established by other means (e.g. by an explicit construction of s–states introduced above). Once the existence of at least one state with the energy $E_n = -\frac{MQ^2}{2\hbar^2}\frac{1}{n^2}$ is shown, the degeneracy $n^2$ of the energy level $E_n$ also follows directly from algebraic considerations.

### 1.2.2 Generalized Casimir invariants

As was shown by Kirillov in [42] and will be explained below, Casimir operators are in one–to–one correspondence with polynomial invariants characterizing orbits of the coadjoint representation of $\mathfrak{g}$. The search for invariants of the coadjoint representation is algorithmic and amounts to solving a system of linear first order partial differential equations [43, 44, 45, 46, 47, 48, 23, 25, 27]. Alternatively, global properties of the coadjoint representation can be used [47, 49, 50, 51]. In general, solutions are not necessarily polynomials and we shall call the nonpolynomial solutions *generalized Casimir invariants.*

For certain classes of Lie algebras, including semisimple Lie algebras, perfect Lie algebras, nilpotent Lie algebras, and more generally algebraic Lie algebras, all invariants of the coadjoint representation are functions of polynomial ones [43, 44].

On the other hand, in the representation theory of solvable Lie algebras their invariants are not necessarily polynomials, i.e. they can be genuinely generalized Casimir invariants. In addition to their importance in representation theory, they may occur in physics. Indeed, Hamiltonians and integrals of motion of classical integrable Hamiltonian systems are not necessarily polynomials in the momenta [52, 53], though typically they are invariants of some group action.

In order to calculate the (generalized) Casimir invariants we consider some basis $(e_1, \ldots, e_n)$ of $\mathfrak{g}$, in which the structure constants are ${c_{ij}}^k$. The coadjoint representation $\mathrm{ad}^*$ of $\mathfrak{g}$ is the representation on $\mathfrak{g}^*$ obtained via transposition of the operators in the adjoint representation

$$\langle \mathrm{ad}^*(x)\phi, y\rangle = -\langle \phi, \mathrm{ad}(x)y\rangle, \qquad \forall x, y \in \mathfrak{g}, \phi \in \mathfrak{g}^*.$$

A basis for the coadjoint representation is given by the first order differential operators acting on functions on $\mathfrak{g}^*$, i.e. vector fields,

$$\widehat{E}_k = \sum_{a,b=1}^{n} e_b c_{ka}{}^{b} \frac{\partial}{\partial e_a}, \qquad 1 \le k \le n. \tag{1.65}$$

In equation (1.65) the quantities $e_a$ are commuting independent variables – the coordinates in the basis of the space $\mathfrak{g}^*$, dual to the algebra $\mathfrak{g}$. Using the relation $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ one can identify them with the basis vectors of $\mathfrak{g}$.

The invariants of the coadjoint representation, i.e. the *generalized Casimir invariants*, are solutions of the following system of partial differential equations

$$\widehat{E}_k I(e_1, \ldots, e_n) = 0, \;\; k = 1, \ldots, n. \tag{1.66}$$

The relation to Casimir operators, i.e. the 1–1 correspondence between polynomial solutions of equation (1.66) and the elements of the center of the enveloping algebra comes from the following observations.

Firstly, it is obvious that both the operation on $\mathfrak{U}(\mathfrak{g})$ of taking the commutator with a fixed element $e_k \in \mathfrak{g}$ and the application of the first order differential operator $\widehat{E}_k$ satisfy Leibniz rule

$$[e_k, a_1 a_2] = [e_k, a_1]a_2 + a_1[e_k, a_2], \qquad a_1, a_2 \in \mathfrak{U}(\mathfrak{g}),$$

$$\widehat{E}_k(F_1 F_2) = \widehat{E}_k(F_1)F_2 + F_1\widehat{E}_k(F_2), \qquad F_1, F_2 \in C^\infty(\mathfrak{g}^*).$$

Further ingredient of the proof is the fact that $[e_k, \cdot]$ and $\widehat{E}_k$ give the same answer when applied to $e_l$, namely

$$[e_k, e_l] = \sum_{m=1}^{n} c_{kl}{}^{m} e_m, \qquad \widehat{E}_k(e_l) = \sum_{m=1}^{n} c_{kl}{}^{m} e_m, \tag{1.67}$$

where it is understood that $e_l \in \mathfrak{g} \subset \mathfrak{U}(\mathfrak{g})$ in the first equality and $e_l \in (\mathfrak{g}^*)^*$ in the second.

Now, let us consider a polynomial function $F$ on $\mathfrak{g}^*$. We express it as a completely symmetric expression in the basis functionals $e_l \in (\mathfrak{g}^*)^*$ – since as functions they commute that does not in fact change anything. Next, we associate to it an element $\tilde{F}$ of the universal enveloping algebra by simply changing the interpretation of the generators $e_k \in (\mathfrak{g}^*)^* \to e_k \in \mathfrak{g} \subset \mathfrak{U}(\mathfrak{g})$. Recalling that the totally symmetric representative of a given element $A \in$

$\mathfrak{U}(\mathfrak{g})$ is unique and observing that $[e_k, \tilde{F}]$ is by construction again a totally symmetric expression in the generators $e_l$, we find that

$$[e_k, \tilde{F}] = 0 \Leftrightarrow \widehat{E}_k(F) = 0$$

by Leibniz rule and equation (1.67). Thus, polynomial invariants of the coadjoint representation can indeed be identified with Casimir operators in a bijective way.

Let us first determine the number of functionally independent solutions of the system (1.66). We can rewrite this system as

$$C \cdot \nabla I = 0 \tag{1.68}$$

where $C$ is the antisymmetric matrix

$$C = \begin{pmatrix} 0 & {c_{12}}^b e_b & \dots & {c_{1n}}^b e_b \\ -{c_{12}}^b e_b & 0 & \dots & {c_{2n}}^b e_b \\ \vdots & & & \vdots \\ -{c_{1,n-1}}^b e_b & \dots & 0 & {c_{n-1,n}}^b e_b \\ -{c_{1n}}^b e_b & \dots & -{c_{n-1,n}}^b e_b & 0 \end{pmatrix} \tag{1.69}$$

in which summation over the repeated index $b$ is to be understood in each term and $\nabla$ is the gradient operator $\nabla = (\partial_{e_1}, \dots, \partial_{e_n})^t$ (where $t$ stands for transposition). The number of independent equations in the system (1.66) is $r(C)$, the generic rank of the matrix $C$. The number of functionally independent solutions of the system (1.66) is hence

$$n_I = n - r(C). \tag{1.70}$$

Since $C$ is antisymmetric, its rank is even. Hence $n_I$ has the same parity as $n$. Equation (1.70) gives the number of functionally independent generalized Casimir invariants.

The individual equations in the system of partial differential equations (PDEs) (1.66) can be solved by the method of characteristics, or, equivalently by integration of the vector fields (1.65).

### 1.2.3 Method of characteristics

The method of characteristics is applicable to linear homogeneous first order PDEs

$$\sum_{j=0}^{n} f^a(e_1, \dots, e_n) \frac{\partial}{\partial e_a} u(e_1, \dots, e_n) = 0 \tag{1.71}$$

for an unknown function $u$. (It may be generalized to inhomogeneous PDEs but we shall not need that here.) Instead of attempting to solve equation (1.71) directly, we can consider an associated system of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}\tilde{e}_k(t)}{\mathrm{d}t} = f^k(\tilde{e}_1(t), \ldots, \tilde{e}_n(t)), \qquad 1 \le k \le n, \tag{1.72}$$

and find its solution satisfying a generic initial condition

$$\tilde{e}_k(0) = e_k, \qquad 1 \le k \le n. \tag{1.73}$$

In the language of differential geometry this means that we are constructing the flow, i.e. the collection of all integral curves, of the vector field

$$\widehat{F} = \sum_{j=0}^{n} f^a(e_1, \ldots, e_n) \frac{\partial}{\partial e_a}. \tag{1.74}$$

Once integral curves, i.e. solutions of (1.72), are known, we construct functionally independent functions which are constant along integral curves in the following way. We choose a hypersurface in $\mathbb{R}^n$ such that it is transversal to all integral curves (this is often done only locally). We associate to every integral curve its intersection with the chosen hypersurface. The coordinates of that point of intersection are invariants of the vector field $\widehat{F}$, i.e. solutions of equation (1.71), because they are by construction the same for any pair of points connected by an integral curve of $\widehat{F}$ and consequently are annihilated by the vector field.

For the sake of the argument let us assume that the hypersurface is expressed in our coordinates as the hyperplane $e_1 = 1$. Let us take $(e_1, \ldots, e_n)$ as the initial condition (1.73). We determine the value of the curve parameter $t(e_1, \ldots, e_n)$ such that $\tilde{e}(t(e_1, \ldots, e_n))$ lies on the hyperplane $e_1 = 1$, i.e. $\tilde{e}_1(t(e_1, \ldots, e_n)) = 1$. The remaining $n-1$ coordinates $\tilde{e}_k(t(e_1, \ldots, e_n))$, $2 \le k \le n$ of the intersection of the integral curve with the hyperplane $e_1 = 1$ are invariants of the vector field (1.74)

$$I_k(e_1, \ldots, e_n) = \tilde{e}_{k+1}(t(e_1, \ldots, e_n)),\ 1 \le k \le n-1.$$

They are by construction functionally independent.

We remark that any invariant of the vector field $\widehat{F}$ is obviously also an invariant of the vector field $\widehat{G} = f\widehat{F}$ for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$. On the other hand, the integral curves of $\widehat{G}$ differ from those of $\widehat{F}$ by a reparametrization, i.e. the differential equations (1.72) are different for $\widehat{G}$

and for $\widehat{F}$. Consequently, the solution of the system of ODEs (1.72) can be often significantly simplified through a suitable choice of the function $f$. This independence of the invariants on the reparametrization of integral curves is symbolically depicted by rewriting of the system (1.72) in the form

$$\frac{\mathrm{d}e_1}{f^1(e_1,\ldots,e_n)} = \frac{\mathrm{d}e_2}{f^2(e_1,\ldots,e_n)} = \ldots = \frac{\mathrm{d}e_n}{f^n(e_1,\ldots,e_n)}. \tag{1.75}$$

The method of characteristics relies on our ability to find the integral curves of the vector field $\widehat{F}$, i.e. to solve the system of ODEs (1.72). This can be done explicitly only for certain classes of functions $f^1,\ldots,f^n : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{R}^n \to \mathbb{C}$). One particular case when we can integrate the system (1.72) is when all functions $f^k$ are linear in the coordinates $e_j$. This is the case for the vector fields $\widehat{E}_a$ encountered in equation (1.65). Therefore we shall now study this case in some detail.

Let the vector field $\widehat{F}$ take the form

$$\widehat{F} = \sum_{k=1}^{n}\sum_{j=1}^{n} \alpha_k^j e_j \frac{\partial}{\partial e_k}. \tag{1.76}$$

Let $e$ denote the column vector of the coordinates $(e_1,\ldots,e_n)^t$. To the vector field (1.76) we associate its flow determined by the equations

$$\frac{\mathrm{d}\tilde{e}_k(t)}{\mathrm{d}t} = \sum_{j=1}^{n} \alpha_k^j \tilde{e}_j, \qquad 1 \le k \le n,$$

or, in vector notation,

$$\frac{\mathrm{d}\tilde{e}(t)}{\mathrm{d}t} = F\cdot\tilde{e}(t), \qquad\qquad F = \begin{pmatrix} \alpha_1^1 & \alpha_1^2 & \ldots & \alpha_1^n \\ \alpha_2^1 & \alpha_2^2 & \ldots & \alpha_2^n \\ \vdots & \vdots & & \vdots \\ \alpha_n^1 & \alpha_n^2 & \ldots & \alpha_n^n \end{pmatrix} \tag{1.77}$$

subject to the initial conditions $\tilde{e}(0) = e$.

We can perform a linear change of coordinates $(e_1,\ldots,e_n)$ to simplify the ODE system (1.77). Putting

$$y = S\cdot e \tag{1.78}$$

where $S = (s_k^j)$ is a constant invertible matrix, we transform equation (1.77) to

$$\frac{\mathrm{d}\tilde{y}(t)}{\mathrm{d}t} = \tilde{F}\cdot\tilde{y}(t), \qquad \tilde{F} = S\cdot F\cdot S^{-1} \tag{1.79}$$

with the initial condition $\tilde{y}(0) = y$.

If $F$ is diagonalizable we can choose $S^{-1}$ to be a matrix of eigenvectors of $F$ and completely decouple the system (1.77). The solution of system (1.79) is

$$\tilde{y}_k(t) = y_k \mathrm{e}^{\alpha_k t}, \qquad k = 1, \dots, n \tag{1.80}$$

where $\alpha_k$ are eigenvalues of the matrix $F$ ordered so that we have $\alpha_1 \neq 0$. We choose our hypersurface as $y_1 = 1$ and compute $t(y_1, \dots, y_n)$ such that $\tilde{y}_1(t(y_1, \dots, y_n)) = 1$. The result is

$$t = -\frac{1}{\alpha_1} \ln(y_1) .$$

where $t = t(y_1, \dots, y_n)$. The remaining coordinates $\tilde{y}_k(t)$ of the intersection of the integral curve and the hyperplane are then the invariants of the vector field $\widehat{F}$. We obtain $n-1$ invariants

$$I_{k-1} = \frac{y_k}{y_1^{\frac{\alpha_k}{\alpha_1}}}, \qquad k = 2, \dots, n. \tag{1.81}$$

In terms of the original coordinates $(e_1, \dots, e_n)$ our invariants read

$$I_{k-1} = \frac{\sum_{j=1}^n s_k^j e_j}{\left(\sum_{j=1}^n s_1^j e_j\right)^{\frac{\alpha_k}{\alpha_1}}}, \qquad k = 2, \dots, n. \tag{1.82}$$

The situation becomes more complicated when the matrix $F$ is not diagonalizable. If that is the case we reduce it to its Jordan canonical form $\widetilde{F}$ (over the field of complex numbers) as in equation (1.79). In order to see the general pattern we have to consider two additional cases.

Consider the vector field $\widehat{F}_0$ specified by the matrix

$$F_0 = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

Now $\widehat{F}_0$ takes the simple form

$$\widehat{F}_0 = e_1 \, \partial_{e_2} + \dots + e_{n-1} \, \partial_{e_n} . \tag{1.83}$$

Equation (1.77) becomes

$$\begin{aligned} \frac{\mathrm{d}\tilde{e}_1(t)}{\mathrm{d}t} &= 0, \\ \frac{\mathrm{d}\tilde{e}_k(t)}{\mathrm{d}t} &= \tilde{e}_{k-1}(t). \end{aligned}$$

The integral curves are given by the formula

$$\tilde{e}_1(t) = e_1, \qquad \tilde{e}_k(t) = \sum_{j=0}^{k-1} \frac{t^j}{j!}\, e_{k-j}. \tag{1.84}$$

In this case we obviously cannot choose our hyperplane as $e_1 = 1$ because $e_1$ is constant along the integral curves (1.84). A convenient choice of the hyperplane is

$$e_2 = 0$$

which implies

$$t = -\frac{e_2}{e_1}. \tag{1.85}$$

Substituting equation (1.85) into (1.84) we obtain the invariants

$$I_1 = e_1, \qquad \tilde{I}_k = \sum_{j=0}^{k-1} \frac{e_{k-j}}{j!}(-1)^j \left(\frac{e_2}{e_1}\right)^j, \qquad k = 3, \ldots, n.$$

Multiplying $\tilde{I}_k$ by the invariant $(e_1)^{k-1}$ and shifting the label $k$ we obtain $n-1$ invariants of the vector field $\widehat{F_0}$ which are all homogeneous polynomials, namely

$$\begin{aligned} I_1 &= e_1, \\ I_k &= \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} e_1^{k-1-j}\, e_2^j\, e_{k+1-j}, \qquad\qquad 2 \leq k \leq n-1. \end{aligned} \tag{1.86}$$

Next, consider the vector field $\widehat{F_1}$ defined by the matrix

$$F_1 = \begin{pmatrix} \alpha & & & & \\ 1 & \alpha & & & \\ & & \ddots & & \\ & & \ddots & \alpha & \\ & & & 1 & \alpha \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

The operator $\widehat{F_1}$ differs from $\widehat{F_0}$ of equation (1.83) by a dilation operator

$$\widehat{D} = \alpha \sum_{j=1}^{n} e_j\, \partial_{e_j}$$

which acts on any homogeneous polynomial by multiplication by its degree together with an overall multiplication by $\alpha$. Therefore, any ratio of two polynomials of the same degree such that both are invariant with respect to $\widehat{F}_0$ is an invariant of the vector field $\widehat{F}_1$. This gives us $n-2$ functionally independent invariants

$$J_k = \frac{I_k}{(e_1)^k}, \quad k = 2, \dots, n-1, \tag{1.87}$$

where $I_k$ were defined in equation (1.86). The last invariant is obtained from the first two coordinates of the integral curves of $\widehat{F}_1$

$$\tilde{e}_k(t) = \sum_{j=1}^{k} \frac{e_{k-j+1}}{(j-1)!} t^{j-1} \mathrm{e}^{\alpha t}$$

via the choice of hyperplane $e_1 = 1$, i.e. $t = -\frac{1}{\alpha} \ln e_1$, and consequently we have an invariant

$$K_1 = \alpha \frac{e_2}{e_1} - \ln(e_1), \qquad \alpha \neq 0. \tag{1.88}$$

Notice that proceeding as before, i.e. determining all functionally independent invariants as the coordinates of the intersection of the integral curve with the chosen hyperplane, we can immediately obtain a different but equivalent complete set of invariants

$$I_{k-1} = \sum_{j=1}^{k} \frac{e_{k-j+1}}{(j-1)! e_1} (-\frac{1}{\alpha} \ln e_1)^{j-1}.$$

This set is less convenient than the one chosen above due to the presence of powers of logarithms.

Combining these results, one may construct invariants of any vector field of the form (1.76) provided one considers holomorphic functions of complex variables $e_1, \dots, e_n$. Over the field of real numbers the situation is further complicated by the existence of matrices which are not diagonalizable over reals but are diagonalizable over complex numbers. This leads to invariants involving trigonometric functions. Consideration of such cases is beyond the scope of the present introduction.

### 1.2.4 Computation of generalized Casimir invariants

At least two conceptually different options exist for solving the system of linear first order PDEs (1.66).

One is to solve one of the equations using the method of characteristics and thus find $n-1$ invariants $I_k$ of the first vector field. Next, we transform all remaining vector fields to a new set of coordinates

$$(e_1, \ldots, e_n) \rightarrow (I_1, \ldots, I_{n-1}, s), \tag{1.89}$$

where $s$ is an arbitrarily chosen function of $e_1, \ldots, e_n$ functionally independent of the invariants $I_1, \ldots, I_{n-1}$. We obtain

$$\begin{aligned} E_1 &= \frac{\partial}{\partial s}, \\ \hat{E}_k &= \sum_{c=1}^{n-1} \phi_k^c(I_1, \ldots, I_{n-1}, s) \frac{\partial}{\partial I_c} + \phi_k^s(I_1, \ldots, I_{n-1}, s) \frac{\partial}{\partial s}, \\ \phi_k^c &= \sum_{a,b=1}^{n} e_b c_{ka}{}^b \frac{\partial I_c}{\partial e_a}, \qquad \phi_k^s = \sum_{a,b=1}^{n} e_b c_{ka}{}^b \frac{\partial s}{\partial e_a}, \qquad 2 \leq k \leq n. \end{aligned} \tag{1.90}$$

Any function $J$ of $I_1, \ldots, I_{n-1}$ is an invariant of the vector field $E_1$. For $J$ to be an invariant of the entire Lie algebra it must be a solution of the system of equations

$$\sum_{c=1}^{n-1} \phi_k^c(I_1, \ldots, I_{n-1}, s) \frac{\partial J}{\partial I_c} = 0, \qquad 2 \leq k \leq n \tag{1.91}$$

for all values of the noninvariant parameter $s$. Since the vector fields $E_k$, $1 \leq k \leq n$ span a Lie algebra, that is an integrable distribution in the sense of the Frobenius theorem, the system (1.91) is compatible. It will have precisely $n_I$ functionally independent solutions, as stated in (1.70). We can continue by solving another chosen equation of the system (1.91) using the method of characteristics. In this way we may be able to fully solve the system (1.66) equation by equation. However after the first step, the substitution of invariants of the first vector field $\hat{E}_1$ into the system, the vector fields no longer have linear coefficients. Consequently, it may be difficult or indeed impossible to find the solution in closed form.

Another method of computation of the generalized Casimir invariants is called the method of moving frames. It goes back to Cartan [54, 55, 56], its modern formulation is due to M. Fels and P. Olver [49, 50, 51] (and a related method was developed and applied in [47]). Boyko et al. adapted the method of moving frames to the case of coadjoint representations. They presented an algebraic procedure for calculating (generalized) Casimir operators and applied it to a large number of solvable Lie algebras [57, 58, 59, 60, 61].

Here we outline the method of moving frames from the practical viewpoint, i.e. an algorithm. For a more detailed description of the method with all the necessary proofs and technical assumptions see [49, 50, 51].

The method of moving frames can be roughly divided into the following steps.

1. Integration of the coadjoint action of the Lie algebra $\mathfrak{g}$ on its dual $\mathfrak{g}^*$ as given by the vector fields (1.65) to the (local) action of the group $G$.

   This is usually realized by choosing a convenient (local) parametrization of $G$ in terms of one–parameter subgroups, e.g.

   $$g(\vec{\alpha}) = \exp(\alpha_N e_N) \cdot \ldots \cdot \exp(\alpha_2 e_2) \cdot \exp(\alpha_1 e_1) \in G, \quad \vec{\alpha} = (\alpha_1, \ldots, \alpha_N) \tag{1.92}$$

   and correspondingly composing the flows $\Psi^{\alpha_k}_{\hat{E}_k}$ of the vector fields $\hat{E}_k$ defined in equation (1.65)

   $$\frac{\mathrm{d}\Psi^{\alpha_k}_{\hat{E}_k}(p)}{\mathrm{d}\alpha_\mathrm{k}} = \hat{E}_k(\Psi^{\alpha_k}_{\hat{E}_k}(p)), \qquad p \in \mathfrak{g}^*, \tag{1.93}$$

   i.e. we have

   $$\Psi(g(\vec{\alpha})) = \Psi^{\alpha_N}_{\hat{E}_N} \circ \ldots \circ \Psi^{\alpha_2}_{\hat{E}_2} \circ \Psi^{\alpha_1}_{\hat{E}_1}. \tag{1.94}$$

   For a given point $p \in \mathfrak{g}^*$ with coordinates $e_k = e_k(p),\ e = (e_1, \ldots, e_N)$ we denote the coordinates of the transformed point $\Psi(g(\vec{\alpha}))p$ by $\tilde{e}_k$

   $$\tilde{e}_k \equiv \Psi_k(\vec{\alpha})e = e_k\left(\Psi(g(\vec{\alpha}))p\right). \tag{1.95}$$

   We consider $\tilde{e}_k$ to be a function of both the group parameters $\vec{\alpha}$ and the coordinates $e$ of the original point $p$.

2. Choice of a section cutting through the orbits of the action $\Psi$.

   We need to choose in a smooth way a single point on each of the (generic) orbits of the action of the group $G$. Typically this is done as follows: we find a subset of $r$ coordinates, say $(e_{\pi(i)})_{i=1}^r$, on which the group $G$ acts transitively, at least locally in an open neighborhood of chosen values $(e^0_{\pi(i)})_{i=1}^r$. Here $\pi$ denotes a suitable injection $\pi : \{1, \ldots, r\} \to \{1, \ldots, N\}$ and $r$ is the rank of the matrix $C$ in equation (1.69). Points whose coordinates satisfy

   $$e_{\pi(i)} = e^0_{\pi(i)},\ 1 \le i \le r \tag{1.96}$$

   form our section $\Sigma$, intersecting each generic orbit once.

3. Construction of invariants.

   For a given point $p \in \mathfrak{g}^*$ we find group elements transforming $p$ into $\tilde{p} \in \Sigma$ by the action $\Psi$. We express as many of their parameters as possible (i.e. $r$ of them) in terms of the original coordinates $e$ and substitute them back into equation (1.95). This gives us $\tilde{e}_k$ as functions of $e$ only. Out of them, $\tilde{e}_{\pi(i)},\ i = 1, \ldots, r$ have the prescribed fixed values $e^0_{\pi(i)}$. The remaining $N - r$ functions $\tilde{e}_k$ are by construction invariant under the coadjoint action of $G$, i.e. define the invariants of the coadjoint representation.

Technically, it may not be necessary to evaluate all the functions $\tilde{e}_k$ so that a suitable choice of the basis in $\mathfrak{g}$ can substantially simplify the whole procedure. This happens when only a smaller subset of say $r_0$ group parameters $\alpha_k$ enters into the computation of $N - r + r_0$ functions $\tilde{e}_k,\ k = 1, \ldots, N - r + r_0$ (possibly after a rearrangement of the coordinates $e_k$). In this case the remaining parameters can be ignored throughout the computation. They are specified by the remaining equations

$$\tilde{e}_i = e^0_i,\ N - r + r_0 + 1 \leq i \leq N \tag{1.97}$$

but do not enter into the expressions for $\tilde{e}_k,\ 1 \leq k \leq N - r + r_0$ which define the invariants.

The method of moving frames exploits the fact that the flows (1.93) can be computed as in Section 1.2.3, provided one is able to find the respective eigenvalues and Jordan canonical form. That is due to the linear dependence on the coordinates in the coefficients of the vector fields (1.65). Consequently, the problem is reduced to a suitable choice of the section and the elimination of group parameters, i.e. to a system of algebraic equations. Unfortunately, the resulting equations may be difficult or impossible to solve explicitly. In addition, the complexity of the computation strongly depends on arbitrary choices involved in the selection of the section.

To sum up, both methods of solving the system (1.66) have their own advantages and disadvantages and it is hard to predict which of the two methods will lead to more efficient computation in any individual case. Obviously the two methods give equivalent results.

We mention that invariants found using either of the methods may not be in the most convenient form. That can be remedied once we find them. For example, as we already mentioned, generalized Casimir invariants of a nilpotent Lie algebra can be always chosen as polynomials, i.e. proper Casimir invariants. The method of moving frames may naturally give us nonpolynomial ones. Nevertheless, it is usually quite easy to construct polynomials out of them.

**Example 1.10** *Let us consider the Lie algebra with the nonvanishing Lie brackets*

$$[e_2, e_3] = e_1, \qquad [e_2, e_4] = e_3, \qquad [e_3, e_4] = -e_2. \tag{1.98}$$

*This algebra has its nilradical spanned by $e_1, e_2, e_3$, isomorphic to the Heisenberg algebra $\mathfrak{h}(1)$ of Example 1.1.*

*The vector fields* (1.65) *are*

$$\begin{aligned} \widehat{E}_1 &= 0, & \widehat{E}_2 &= e_1\partial_{e_3} + e_3\partial_{e_4}, \\ \widehat{E}_3 &= -e_1\partial_{e_2} - e_2\partial_{e_4}, & \widehat{E}_4 &= -e_3\partial_{e_2} + e_2\partial_{e_3}. \end{aligned} \tag{1.99}$$

*Consequently, the matrix $C$ takes the form*

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_3 \\ 0 & -e_1 & 0 & -e_2 \\ 0 & -e_3 & e_2 & 0 \end{pmatrix}. \tag{1.100}$$

*The generic rank of $C$ is* 2 *and the number* (1.70) *of functionally independent Casimir invariants is*

$$n_I = 4 - 2 = 2.$$

*Since the first column of $C$ consists of zeros, $e_1$ is a solution. We take $\widehat{E}_2$ as the first vector field to which we apply the method of characteristics. We have*

$$\frac{\mathrm{d}e_3}{e_1} = \frac{\mathrm{d}e_4}{e_3}$$

*and the invariants of $\widehat{E}_2$ are $e_1, e_2$ and $\xi = e_3^2 - 2e_1e_4$. Therefore, any Casimir invariant of the algebra* (1.98) *must be of the from $J = J(e_1, e_2, \xi)$. When we apply $\widehat{E}_3$ to such $J$ we get*

$$\widehat{E}_3 J = e_1 \left( 2e_2 \frac{\partial J}{\partial \xi} - \frac{\partial J}{\partial e_2} \right) \tag{1.101}$$

*and we obtain a solution of $\widehat{E}_3 J = 0$ in the form $\eta = e_2^2 + e_3^2 - 2e_1e_4$. Both $e_1$ and $\eta$ are also annihilated by $\widehat{E}_4$. Altogether, we have found that our algebra* (1.98) *has two Casimir invariants*

$$I_1 = e_1, \qquad I_2 = e_2^2 + e_3^2 - 2e_1e_4. \tag{1.102}$$

*Let us redo the same calculation using the method of moving frames. The flows of the vector fields $\widehat{E}_1, \ldots, \widehat{E}_4$ are*

$$\begin{aligned}
&\Psi^{\alpha_1}_{\widehat{E}_1}(\vec{e}) = (e_1, e_2, e_3, e_4),\\
&\Psi^{\alpha_2}_{\widehat{E}_2}(\vec{e}) = (e_1, e_2, \alpha_2 e_1 + e_3, \frac{\alpha_2^2}{2} e_1 + \alpha_2 e_3 + e_4),\\
&\Psi^{\alpha_3}_{\widehat{E}_3}(\vec{e}) = (e_1, -\alpha_3 e_1 + e_2, e_3, \frac{\alpha_3^2}{2} e_1 - \alpha_3 e_2 + e_4),\\
&\Psi^{\alpha_4}_{\widehat{E}_4}(\vec{e}) = (e_1, e_2 \cos\alpha_4 - e_3 \sin\alpha_4, e_2 \sin\alpha_4 + e_3 \cos\alpha_4, e_4)
\end{aligned}$$

*where $\vec{e} = (e_1, e_2, e_3, e_4)$. We compose the flows as in equation (1.94) and obtain*

$$\begin{aligned}
\Psi(g(\vec{\alpha})) &= \Psi^{\alpha_4}_{\hat{E}_4} \circ \Psi^{\alpha_3}_{\hat{E}_3} \circ \Psi^{\alpha_2}_{\hat{E}_2} \circ \Psi^{\alpha_1}_{\hat{E}_1},\\
\Psi(g(\vec{\alpha}))(\vec{e}) &= (e_1, \cos\alpha_4(-\alpha_3 e_1 + e_2) - \sin\alpha_4(\alpha_2 e_1 + e_3),\\
&\quad \sin\alpha_4(-\alpha_3 e_1 + e_2) + \cos\alpha_4(\alpha_2 e_1 + e_3),\\
&\quad \left.\frac{\alpha_2^2 + \alpha_3^2}{2} e_1 - \alpha_3 e_2 + \alpha_2 e_3 + e_4\right)
\end{aligned} \tag{1.103}$$

*where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. We choose a section $\Sigma$ given by the equations*

$$e_2 = 0, \qquad e_3 = 1. \tag{1.104}$$

*The intersection of our section $\Sigma$ with the orbit $\Psi(g(\vec{\alpha}))(\vec{e})$ starting from the point $\vec{e} = (e_1, e_2, e_3, e_4)$ has the following values of $\alpha_2, \alpha_3$*

$$\alpha_2 = \frac{\cos\alpha_4 - e_3}{e_1}, \qquad \alpha_3 = \frac{e_2 - \sin\alpha_4}{e_1} \tag{1.105}$$

*(generically, i.e. when $e_1 \neq 0$). The coordinates of the intersection*

$$\left(e_1, 0, 1, \frac{2e_1 e_4 - e_2^2 - e_3^2 + 1}{2e_1}\right). \tag{1.106}$$

*are independent of the remaining two parameters $\alpha_1, \alpha_4$. That means that we have found using the method of moving frames that two functionally independent functions $e_1$ and $\frac{2e_1e_4 - e_2^2 - e_3^2 + 1}{2e_1}$ are generalized Casimir invariants. Equivalently, $e_1$ and $e_2^2 + e_3^2 - 2e_1 e_4$ are Casimir invariants of our algebra.*

In Sections 2.1, 2.2 and 2.3 we compute Casimir operators of nilpotent algebras considered there and generalized Casimir invariants of all solvable Lie algebras constructed there. In most cases these are nonpolynomial, i.e. genuine generalized Casimir invariants.

## 1.3 Lie Groups

In this section we shall review several basic notions in the theory of Lie groups and their actions. Next, we discuss symmetry groups of algebraic and differential equations. For more details we refer the reader to [2, 3, 62] and [5, 6], respectively.

### 1.3.1 Definition of Lie group and its Lie algebra

Let us consider a real smooth manifold $G$ (of finite dimension). If the manifold $G$ is also a group, i.e. equipped with an associative product such that a multiplicative unit $e$ and an inverse $g^{-1}$ exist, we may contemplate the compatibility of these two structures on $G$. When both the product[1]

$$\cdot\ : G \times G \to G$$

and the inverse

$$(\ )^{-1}\ : G \to G$$

are smooth (i.e. differentiable) maps, we call $G$ a *Lie group*. One may also consider *complex Lie groups* which are complex manifolds such that the group operations are holomorphic but we shall not use them here.

Lie groups form a class of manifolds with rather special properties. Let us define two particular sets of diffeomorphisms of $G$, the left and right translations

$$L_g\ : G \to G, \quad L_g(h) = gh$$

and

$$R_g\ : G \to G, \quad R_g(h) = hg$$

defined for any chosen $g \in G$. Since these maps are diffeomorphisms their tangent maps $(L_g)_*, (R_g)_*$ define isomorphisms of the infinite–dimensional Lie algebra $\mathfrak{X}(G)$ of vector fields on $G$. A vector field $X \in \mathfrak{X}(G)$ is called *left–invariant* if

$$(L_g)_* X = X$$

for all $g \in G$. (Similarly for right–invariant fields.) The definition of a left–invariant vector field can be phrased also in a different way. Let us view both $X \in \mathfrak{X}(G)$ and the pullback $(L_g)^*$ as endomorphisms of the vector space $\mathcal{F}(G)$ of all smooth functions on $G$. Then $X$ is left–invariant if and only if

$$X \circ (L_g)^* = (L_g)^* \circ X, \qquad \forall g \in G. \tag{1.107}$$

[1] often written without an explicit product sign $\cdot$

The formulation (1.107) makes evident a crucial property of left–invariant vector fields: they form not only a subspace but a subalgebra of $\mathfrak{X}(G)$ because

$$[X, Y] \circ (L_g)^* = X \circ Y \circ (L_g)^* - Y \circ X \circ (L_g)^*$$
$$= (L_g)^* \circ X \circ Y - (L_g)^* \circ Y \circ X = (L_g)^* \circ [X, Y]$$

for any left–invariant vector fields $X, Y$. The algebra of left–invariant vector fields is called the *Lie algebra of the Lie group* $G$ and denoted by $\mathfrak{g}$.

Elements of $\mathfrak{g}$ are uniquely specified by their value at any chosen point $g \in G$. Conventionally, this identification is performed at the group unit, i.e. we identify

$$\mathfrak{g} \simeq T_e G.$$

Therefore, the dimension of $\mathfrak{g}$ is the same as dimension of the Lie group $G$.

One of the properties of left–invariant vector fields is that they are complete, i.e. any integral curve $\gamma(t)$

$$\dot{\gamma}(t) = X(\gamma(t))$$

of $X \in \mathfrak{g}$ can be extended to all real values of the curve parameter $t \in \mathbb{R}$. This property allows us to define the *exponential map* from the Lie algebra to the Lie group

$$\exp : \mathfrak{g} \to G : \quad X \to \gamma_X(1) \qquad \text{where } \dot{\gamma}_X(t) = X(\gamma_X(t)), \quad \gamma_X(0) = e. \tag{1.108}$$

The exponential map is a local diffeomorphism of $\mathfrak{g}$ into $G$, i.e. is smooth and is a diffeomorphism of some open neighborhood $U$ of $0 \in \mathfrak{g}$ onto the open neighborhood $\exp(U)$ of $e \in G$.

Using the exponential map one may relate properties of Lie groups and their Lie algebras. In essence any local property of Lie groups has its counterpart in the properties of Lie algebras. Therefore, one may say that locally, i.e. up to topological issues, a Lie group and its Lie algebra encode the same information. Because Lie algebras are vector spaces, most computations in the theory of Lie algebras reduce to problems of linear algebra and consequently are much easier to handle than the corresponding computation in Lie groups. Therefore, using the local diffeomorphism exp (1.108) one may solve many problems on Lie groups which would be intractable on a general smooth manifold (or on a general, e.g. discrete, group).

### 1.3.2 Left–invariant forms on Lie groups

Let us now turn our attention to the space of 1–forms dual to the Lie algebra of left–invariant fields $\mathfrak{g}$ and review its essential properties.

We may define a canonical map $\nu_g^L$ between the tangent space $T_gG$ of the Lie group $G$ at an arbitrary point $g \in G$ and the corresponding Lie algebra $\mathfrak{g}$. The map $\nu_g^L$ is for any $u \in T_gG$ defined by the prescription

$$\nu_g^L(u) = X_u, \tag{1.109}$$

where $X_u$ is the unique left–invariant field on $G$ such that $X_u(g) = u$. Introducing such maps at all points of the group $G$ we obtain a differential 1–form $\nu^L$ on $G$ valued in the Lie algebra $\mathfrak{g}$, i.e. $\nu^L \in \Omega^1(G) \otimes \mathfrak{g}$. The 1–form $\nu^L$ is called the (left) *Maurer–Cartan 1–form* on $G$.

When we choose a basis $(X_i)$ of the Lie algebra $\mathfrak{g}$, i.e. linearly independent left–invariant vector fields $X_1, \ldots, X_{\dim G}$, we can express the Maurer–Cartan 1–form in components

$$\nu^L = \sum_{i=1}^{\dim G} \sigma^i \otimes X_i$$

where $\sigma^i$ are some differential 1–forms on $G$. They have a particular property, namely they are *left–invariant* in the sense that $\sigma^i(X)$ is a constant function on $G$ for every left–invariant vector field $X$. The 1–forms $\sigma^i$ form a basis of the vector space $\Omega_L^1(G)$ of all left–invariant 1–forms on $G$, dual to the basis $(X_i)$ of the Lie algebra $\mathfrak{g}$.

Similarly, one defines a *left–invariant p–form* on $G$ as any differential $p$–form on $G$ such that it gives a constant when evaluated on any $p$–tuple of left–invariant vector fields. As it turns out, the vector space $\Omega_L^\bullet(G)$ of all left–invariant forms (i.e. of any degree $p$) on $G$ is closed under exterior product and exterior derivative, i.e. $\Omega_L^\bullet(G)$ is a differential subalgebra of the exterior differential algebra $\Omega^\bullet(G)$ of all differential forms on $G$. In particular, the identity

$$\mathrm{d}\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \sigma([X, Y])$$

valid for any differential 1–form $\omega$ and any pair of vector fields $X, Y$ implies the formula

$$\mathrm{d}\sigma(X, Y) = -\sigma([X, Y]) \tag{1.110}$$

for any left–invariant 1–form $\sigma$. Equation (1.110) in turn implies the *Maurer–Cartan structure equations*

$$\mathrm{d}\sigma^i = -\sum_{j,k=1}^{\dim G} c_{jk}{}^i \sigma^j \otimes \sigma^k = -\frac{1}{2} \sum_{j,k=1}^{\dim G} c_{jk}{}^i \sigma^j \wedge \sigma^k \tag{1.111}$$

for the basis $(\sigma^i)$ of left–invariant 1–forms dual to the basis $(X_i)$ of the Lie algebra $\mathfrak{g}$. The Jacobi identity now becomes formally identical to the condition that the 2–forms $\mathrm{d}\sigma^i$ are closed, i.e.

$$\mathrm{d}^2\sigma^i = 0.$$

We mention that this observation is often used while checking whether a prescribed set of structure constants ${c_{ij}}^k$, antisymmetric in indices $i, j$, defines a Lie algebra. One simply defines formal 2–forms

$$\tau^i = -\frac{1}{2}\sum_{j,k=1}^{\dim G} {c_{jk}}^i \sigma^j \wedge \sigma^k$$

and computes their exterior derivatives using the rule for the derivative of an exterior product of 1–forms

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\,\omega_1 \wedge \omega_2 - \omega_1 \wedge \mathrm{d}\,\omega_2$$

and equation (1.111). If the resulting expression for $\mathrm{d}\tau^i$ vanishes for every index $i$, the constants ${c_{jk}}^i$ define a bracket satisfying the Jacobi identity, i.e. a Lie algebra.

While we used the left–invariant formalism in the definition of the Maurer–Cartan 1–form and its component left–invariant 1–forms, as is the usual convention in the literature, we may similarly introduce also right Maurer–Cartan 1–form and right–invariant forms. E.g. the right Maurer–Cartan 1–form assigns to a vector field $X$ evaluated at the point $g$ the right–invariant vector field $X_R$ such that $X_R(g) = X(g)$. The only difference arises in signs in several formulae. This is due to the following fact: let $(X_i^L)$ and $(X_i^R)$ be bases of spaces of left– and right–invariant vector fields, respectively, such that they coincide in the group unit, $X_i^L(e) = X_i^R(e)$. Let ${c_{ij}}^k$ be the structure constants of the Lie algebra $\mathfrak{g}$ in the basis $(X_i^L)$, i.e.

$$[X_i^L, X_j^L] = \sum_{k=1}^{\dim G} {c_{ij}}^k X_k^L.$$

Then the right–invariant fields $X_i^R$ satisfy the following commutation relation

$$[X_i^R, X_j^R] = -\sum_{k=1}^{\dim G} {c_{ij}}^k X_k^R.$$

We shall use the notion of right–invariant 1–forms on the Lie group $G$ in Section 1.4.3 and in Chapter 4.

### 1.3.3 Actions of Lie groups

For applications in both mathematics and physics we need a formalism allowing us to view Lie groups as sets of certain transformations of some objects. This leads us to the notion of an action of the group.

A (left) *action* of the Lie group $G$ on a manifold $M$ is a smooth map

$$\triangleright : \ G \times M \to M : \ (g, m) \to g \triangleright m$$

such that $g_1 \triangleright (g_2 \triangleright m) = (g_1 g_2) \triangleright m$ and $e \triangleright m = m$ for all $g_1, g_2 \in G$, $m \in M$.

Similarly one may consider also *right actions* $\triangleleft : M \times G \to G$ which satisfy $(m \triangleleft g_1) \triangleleft g_2 = m \triangleleft (g_1 g_2)$ and $m \triangleleft e = m$. Any left action $\triangleright$ defines a right action $\triangleleft$ through $m \triangleleft g = g^{-1} \triangleright m$ and vice versa.

An action $\triangleright$ of $G$ on $M$ is called *effective* if for every $g \in G$ different from the group unit $e$ an element $m \in M$ exists such that $g \triangleright m \neq m$. Consequently, we can reconstruct the group multiplication on the group $G$ from the knowledge of its effective action.

Examples of left actions of the group $G$ on itself are

$$g \triangleright h = gh, \qquad g \triangleright h = h \cdot g^{-1}$$

and the *adjoint action*

$$Ad : \ G \times G \to G : \ Ad_g(h) \equiv Ad(g, h) = g \cdot h \cdot g^{-1}.$$

When the manifold $M$ is a vector space and the action of $G$ on $M$ is *linear*

$$g \triangleright (av + w) = a(g \triangleright v) + g \triangleright w, \qquad \forall g \in G, v, w \in M, a \in \mathbb{R}$$

it is equivalent to a *representation* of the group $G$ on the vector space $M$. A representation of the Lie group $G$ on a vector space $V$ is any (smooth) map

$$\rho : G \to \mathrm{End}(V)$$

which satisfies

$$\rho(e) = \mathbf{1}, \qquad \rho(g_1 g_2) = \rho(g_1)\rho(g_2), \quad \forall g_1, g_2 \in G.$$

A representation can be associated to any linear action by the prescription

$$\rho : G \to \mathrm{End}(M) : \ \rho(g)v = g \triangleright v.$$

Whether we speak about a linear action or a representation is just a matter of convenience in the problem at hand.

A particular representation of the Lie group $G$ on its algebra $\mathfrak{g}$ is defined by the derivation of the adjoint action

$$\mathrm{Ad}: \; G \rightarrow \mathfrak{gl}(\mathfrak{g}): \; \mathrm{Ad}(g) = (Ad_g)_*.$$

This representation is called the *adjoint* representation of $G$.

Further differentiating we get the already known adjoint representation of the Lie algebra $\mathfrak{g}$ on itself (1.17)

$$\mathrm{ad}: \; \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}): \; \mathrm{ad} = \mathrm{Ad}_*.$$

Sometimes we may encounter actions which are not well–defined for all pairs $(g, m)$. Formally, one defines a *local (left) action* of a Lie group $G$ on a manifold $M$ to be a smooth map $\triangleright: U \rightarrow M$ where $U$ is some open neighborhood in $G \times M$ which contains the whole subset $\{e\} \times M$ and satisfies the properties

$$e \triangleright m = m, \qquad \forall m \in M$$

and

$$g_1 \triangleright (g_2 \triangleright m) = (g_1 g_2) \triangleright m$$

whenever $(g_2, m)$ and $(g_1, g_2 \triangleright m) \in U$.

When we consider an abstract Lie group $G$ together with its prescribed (local) effective action on some manifold $M$ we often speak about a (local) *group of transformations* or *group of motions* of $M$. In fact, this notion was what Sophus Lie had in mind in his pioneering works [63, 64, 65, 66] on what we now call Lie groups and Lie algebras.

An *infinitesimal action* of the Lie algebra $\mathfrak{g}$ on $M$ is a homomorphism $\mu: \mathfrak{g} \rightarrow \mathfrak{X}(M)$. We often write the image of $x \in \mathfrak{g}$ in capital letters, $\mu(x) \equiv X$. A Lie algebra equipped with an injective infinitesimal action on some manifold $M$ is called an *algebra of infinitesimal transformations*.

Any local action of $G$ on $M$ gives rise to an infinitesimal action of the Lie algebra $\mathfrak{g}$ on $M$ through the prescription

$$(\mu(x)f)(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\exp(tx) \triangleright m), \qquad \forall f \in \mathcal{F}(M), m \in M. \tag{1.112}$$

### 1.3.4 Symmetries of algebraic equations

Now we shall introduce the notion of a symmetry of a given equation. Next, we apply it in particular to differential equations. Again we present only the essential notions and ideas. For proofs see [5, 6].

Let

$$f(x) = 0, \qquad f: \mathrm{Dom}(f) \subset \mathbb{F}^N \rightarrow \mathbb{F}^{\tilde{N}} \tag{1.113}$$

be a system of algebraic equations (or just one equation when $\tilde{N} = 1$) and $\mathcal{S}_f$ be its *solution set*

$$\mathcal{S}_f = \{x \in \mathrm{Dom}(f) \mid f(x) = 0\}.$$

A *symmetry* of the equation (1.113) is any transformation

$$T : \mathrm{Dom}(f) \to \mathrm{Dom}(f)$$

such that it preserves the solution set

$$T(\mathcal{S}_f) = \mathcal{S}_f. \tag{1.114}$$

Usually, we restrict our attention to transformations $T$ which are diffeomorphisms, $T \in \mathrm{Diff}(\mathrm{Dom}(f))$.

It follows from the definition of a symmetry that symmetries of a given equation form a group, i.e. a subgroup of $\mathrm{Diff}(\mathrm{Dom}(f))$. Let us denote this *group of symmetries* of the equation (1.113) by $Sym(f = 0)$.

The group of all diffeomorphisms $\mathrm{Diff}(\mathrm{Dom}(f))$ is infinite–dimensional. While the use of the theory of Lie algebras as introduced above is not completely rigorous in this case, we may in a certain sense view the algebra $\mathfrak{X}(\mathrm{Dom}(f))$ of vector fields on $\mathrm{Dom}(f)$ as a Lie algebra of $\mathrm{Diff}(\mathrm{Dom}(f))$. When $Sym(f = 0)$ happens to be a a Lie group (more precisely, a Lie group of transformations), the corresponding algebra $\mathfrak{sym}(f = 0)$ of infinitesimal transformations defines a subalgebra of $\mathfrak{X}(\mathrm{Dom}(f))$. Its relation to the function $f$ is derived using the notion of a 1–parameter subgroup.

A *1–parameter subgroup* $\sigma$ of a group $G$ is a homomorphism of the additive group $(\mathbb{R}, +)$ into the group $G$. While $G$ may not necessarily be a Lie group (cf. $\mathrm{Diff}(\mathrm{Dom}(f))$), the image $\sigma(\mathbb{R})$ has a natural structure of a 1–dimensional Lie group (or 0–dimensional if $\sigma(t) = e$ for all $t \in \mathbb{R}$). Consequently, one may consider its Lie algebra. When $G$ is a group of transformations of $M$ and $\sigma$ its 1–parameter subgroup we have a 1–dimensional algebra of infinitesimal transformations spanned by its generator $X_\sigma \in \mathfrak{X}(M)$:

$$X_\sigma j(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j\left(\sigma(t) \triangleright m\right), \qquad \forall j \in \mathcal{F}(M).$$

Let $Sym(f = 0)$ be the group of symmetries of the equation (1.113). We shall call the vector subspace of $\mathfrak{X}(\mathrm{Dom}(f))$ spanned by all generators $X_\sigma$ of 1–parametric subgroups of the group $Sym(f = 0)$ the *algebra of infinitesimal symmetries* of the equation $f = 0$ and denote it by $\mathfrak{sym}(f = 0)$. It turns out that $\mathfrak{sym}(f = 0)$ is a subalgebra of $\mathfrak{X}(\mathrm{Dom}(f))$. The algebra $\mathfrak{sym}(f = 0)$

coincides with the algebra of infinitesimal transformations arising from the group of transformation $Sym(f=0)$ when $Sym(f=0)$ is a Lie group.

Let us take $m \in \mathcal{S}_f$ and $X_\sigma \in \mathfrak{sym}(f=0)$. Because $\sigma(t)$ lies in the symmetry group $Sym(f=0)$ for all $t \in \mathbb{R}$ we have $f(\sigma(t) \triangleright m) = 0$ and consequently

$$X_\sigma f(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\sigma(t) \triangleright m\right) = 0.$$

That means that the vector fields $X$ in the algebra of infinitesimal symmetries $\mathfrak{sym}(f=0)$ of the equation $f=0$ satisfy

$$\left.Xf\right|_{f=0} = 0, \qquad \text{i.e. } Xf(m) = 0, \quad \forall m \in \mathcal{S}_f. \tag{1.115}$$

Let us consider the converse problem. We recall that the *flow of the vector field* $X$ is the map

$$\Phi_X : U \to M : \Phi_X(0,m) = m, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_X(t,m) = X(\Phi_X(t,m)), \quad \forall (t,m) \in U, \tag{1.116}$$

where $U$ is some open neighborhood $U \subset \mathbb{R} \times M$ such that $(0, M) \subset U$.

Let $X \in \mathfrak{X}(\mathrm{Dom}(f))$ satisfy the condition (1.115). Is it true that the flow $\Phi_X$ defines a 1–parameter group of symmetries of the equation (1.113)?

In general, the answer is negative for two reasons.

Firstly, the flow may not be defined on the whole $\mathbb{R} \times \mathrm{Dom}(f)$, i.e. the vector field may not be *complete*. That is why we introduced the notion of a local action of a group: the flow of a vector field defines in general a local action of a 1–parameter group.

Secondly, even locally the flow may not define symmetries of the given equation (1.113).

**Example 1.11** *Let us consider a system of equations*

$$x_1 - x_2^2 = 0, \qquad x_1 = 0. \tag{1.117}$$

*Its set of solutions is* $\mathcal{S} = \{(0,0)\}$*. On the other hand, the condition* (1.115) *is satisfied by the vector field*

$$X = \partial_{x_2}$$

*whose flow is*

$$\Phi_X : \mathbb{R} \times (\mathbb{R} \times \mathbb{R}) \to (\mathbb{R} \times \mathbb{R}) : \Phi_X(t, x_1, x_2) = (x_1, x_2 + t).$$

*Now the action of the group element* $t \neq 0$*,* $\Phi_X(t, \cdot)$*, takes the solution* $(0,0)$ *to a point* $(0,t)$ *which is not a solution of the equation* (1.117).

It turns out that the condition on the function $f$ which prevents such pathological behaviour is the maximality of the rank of the Jacobian, rank $\frac{\partial f_j}{\partial x_k}\big|_{\mathcal{S}_f} = \tilde{N}$. These results are the content of

**Theorem 1.9 (On infinitesimal generators of symmetries)** *Let* $f : \mathrm{Dom}(f) \subset \mathbb{R}^N \to \mathbb{R}^{\tilde{N}}$ *define a system of equations*

$$f(x) = 0 \tag{1.118}$$

*such that*

$$\operatorname{rank} \frac{\partial f_j}{\partial x_k}(x) = \tilde{N}, \qquad \forall x \in \mathcal{S}_f. \tag{1.119}$$

*Then a vector field* $X \in \mathfrak{X}(\mathrm{Dom}(f))$ *generates a local 1–parameter group of symmetries of the equation* (1.118) *if and only if*

$$(Xf)(m) = 0, \quad \forall m \in \mathcal{S}_f. \tag{1.120}$$

We see that under the assumption of regularity of the function $f$ (1.119) we can determine the algebra of infinitesimal symmetries $\mathfrak{sym}(f = 0)$ of the given equation $f = 0$ through solution of a linear system of equations (1.115) for the coefficient functions $X^i \in \mathcal{F}(\mathrm{Dom}(f))$ of the vector field

$$X : \; X(x) = \sum_{i=1}^{N} X^i(x) \frac{\partial}{\partial x^i}\bigg|_x .$$

Infinitesimal symmetries can be converted into actual symmetries through computation of the corresponding flows; composing the flows one may construct a local group of symmetries of the given equation $f = 0$. In this way, the description of infinitesimal symmetries in terms of the condition (1.115) significantly simplifies the search for symmetries of the given equation.

Detection of symmetries which cannot be connected to identity transformation by flows of infinitesimal symmetries, e.g. belonging to different connected components of the symmetry group, is a much harder problem and we shall not discuss it here.

### 1.3.5 Symmetries of differential equations

Let us now shift our attention to differential equations.

Let us for simplicity start with one ordinary differential equation

$$F(x, u(x), u'(x), \ldots, u^{(p)}(x)) = 0 \tag{1.121}$$

on some domain $M \subset \mathbb{R}$.

The concept of symmetry remains the same: symmetries are transformations leaving the set of solutions invariant. The question is what kind of transformations do we admit?

In principle, we may allow any transformation on the infinite–dimensional space of all functions on $M$ differentiable up to order $p$. Such a broad definition would, however, entail numerous computational difficulties. Therefore, one a priori restricts the class of allowed transformations.

The most restrictive and most often used class of allowed transformations is the following one: we allow any invertible transformation of the space of dependent and independent variables, i.e. $u$ and $x$,

$$\hat{x} = g_1(x, u), \qquad \hat{u} = g_2(x, u). \tag{1.122}$$

Such transformations are called *point transformation.* The effect of such a transformation on any function $f : M \to \mathbb{R}$ is defined using the transformation of the graph of the function $f(x)$.

Let $f$ be a function on the domain $M \subset \mathbb{R}$. Its *graph* is the following subset of $M \times \mathbb{R}$

$$\Gamma_f = \{(x, f(x)) \mid x \in M\}. \tag{1.123}$$

$\Gamma \subset M \times \mathbb{R}$ defines a function $f$ on some subset of $M$ such that $\Gamma = \Gamma_f$ if and only if for every pair of points $(x_1, u_1), (x_2, u_2) \in \Gamma$ the relation $x_1 = x_2$ implies $u_1 = u_2$.

When $f$ is at least $k$–times differentiable we define also the $k^{th}$*–prolonged graph* of the function $f$

$$\Gamma_f^{(k)} = \{\big(x, f(x), f'(x), \ldots, f^{(k)}(x)\big) \mid x \in M\} \subset M \times \mathbb{R}^{1+k}. \tag{1.124}$$

We denote the coordinates on $M \times \mathbb{R}^{1+k}$ by $x, u, u', \ldots, u^{(k)}$ for obvious reasons.

Let us assume that a (local) group $G$ of transformations of the form (1.122) is given. We define the action of $g \in G$ on the graph $\Gamma_f$ in a natural way

$$g \triangleright \Gamma_f = \{g \triangleright (x, f(x)) \mid x \in M\}.$$

In this way we obtain a new subset $g \triangleright \Gamma_f$ of $M \times \mathbb{R}$. When $g \triangleright \Gamma_f$ is a graph of some function $\hat{f}$

$$g \triangleright \Gamma_f = \Gamma_{\hat{f}}$$

we call $\hat{f} \equiv g \triangleright f$ the transformation of the function $f$ under the point transformation $g$ of $M \times \mathbb{R}$.

Such construction of the transformation $f \to \hat{f}$ introduces another source of locality into our transformation groups. In particular, even if the action of $G$ on the space of dependent and independent coordinates $M \times \mathbb{R}$ is globally defined, its induced action on functions is not: $g \triangleright \Gamma_f$ may fail to define a graph of a new function; there may be two different points $(x, u_1)$ and $(x, u_2)$ in $g \triangleright \Gamma_f$. Therefore, the induced action of $G$ on the space of functions $\mathcal{F}(M)$ is only a local action.

A local 1–parameter group of point transformations

$$(\hat{x}, \hat{u}) = t \triangleright (x, u) : \qquad \hat{x} = g_1(x, u; t), \quad \hat{y} = g_2(x, u; t) \tag{1.125}$$

of $M \times \mathbb{R}$ is a *1–parameter symmetry group* of the differential equation (1.121) if for every solution $u : M \to \mathbb{R}$ of equation (1.121) and every $t \in \mathbb{R}$ such that $\hat{u} = t \triangleright u$ is defined we have

$$F(x, \hat{u}(x), \hat{u}'(x), \ldots, \hat{u}^{(n)}(x)) = 0.$$

In order to establish a symmetry criterion in terms of a vector field generating the 1–parameter group of transformations we have to analyze how do the derivatives transform. Let us assume that a function $u = f(x)$ is given. We have its graph $\Gamma_f$ and its prolonged graph $\Gamma_f^{(1)}$. We transform $\Gamma_f$ by a 1–parameter group of point transformations $\phi : \mathbb{R} \to \mathrm{Diff}(M \times \mathbb{R})$ and consequently we also obtain $\hat{f}_t = t \triangleright f$ whenever it is defined. What is the relation between the derivatives of the function $f$ and of the functions $\hat{f}_t$? In other words, how are the prolonged graphs of these functions related?

The points of the graph $\Gamma_f$ transform under the action (1.125) into the points of the graph $\Gamma_{\hat{f}}$ as

$$\hat{x} = g_1(x, f(x); t), \qquad \hat{f}(\hat{x}) = g_2(x, f(x); t).$$

We obtain by differentiation and use of the chain rule an expression for the derivative of $\hat{f}$,

$$\hat{f}'(\hat{x}) \equiv \frac{\mathrm{d}\hat{f}}{\mathrm{d}\hat{x}}(\hat{x}) = \frac{\frac{\mathrm{d}}{\mathrm{d}x} g_2(x, f(x); t)}{\frac{\mathrm{d}}{\mathrm{d}x} g_1(x, f(x); t)} = \left. \frac{\frac{\partial g_2}{\partial x} + f'(x)\frac{\partial g_2}{\partial u}}{\frac{\partial g_1}{\partial x} + f'(x)\frac{\partial g_1}{\partial y}} \right|_{(x, f(x); t)}.$$

We see that the transformation (1.125) induces a unique point transformation of $U \times \mathbb{R}^2$

$$\hat{x} = g_1(x, u; t), \quad \hat{u} = g_2(x, u; t), \quad \hat{u}' = \frac{\frac{\partial g_2}{\partial x}(x, u; t) + u' \frac{\partial g_2}{\partial u}(x, u; t)}{\frac{\partial g_1}{\partial x}(x, u; t) + u' \frac{\partial g_1}{\partial u}(x, u; t)} \tag{1.126}$$

such that the prolonged graph $\Gamma_f^{(1)}$ of any function $f : M \to \mathbb{R}$ is transformed by the transformation (1.126) into the prolonged graph $\Gamma_{\hat{f}_t}^{(1)}$ of the transformed function $\hat{f}_t = t \triangleright f$ whenever $\hat{f}_t$ exists. By induction, this concept can be readily generalized to $k^{th}$–prolonged graphs.

Let us now convert these ideas to the infinitesimal language. Let us assume that the 1–parameter group of transformations (1.125) is generated by the vector field $X$ on $M \times \mathbb{R}$,

$$X \in \mathfrak{X}(M \times \mathbb{R}), \qquad X = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}. \tag{1.127}$$

What is the corresponding vector field $\tilde{X} \in \mathfrak{X}(M \times \mathbb{R} \times \mathbb{R})$ generating the action on the prolonged graphs?

We differentiate equation (1.126) with respect to $t$ and set $t = 0$. We notice that by definition of the generator $X$ of the 1–parameter group (1.125) we have

$$g_1(x,u;0) = x,\ g_2(x,u;0) = u, \quad \frac{\partial g_1}{\partial t}(x,u;0) = \xi(x,u),\ \frac{\partial g_2}{\partial t}(x,u;0) = \eta(x,u).$$

Altogether, we find that

$$\tilde{X} = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \left(\mathcal{D}_x\eta(x,u,u') - u'\mathcal{D}_x\xi(x,u,u')\right)\frac{\partial}{\partial u'} \tag{1.128}$$

where $\mathcal{D}_x = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u}$ is called the *operator of total derivative* on $\mathcal{F}(M \times \mathbb{R})$. We call the vector field (1.128) the first *prolongation* of the vector field $X$ and denote it by $\mathrm{pr}^{(1)}X$. Repeating the same procedure for higher derivatives we find that the action of the 1–parameter group (1.125) on $k^{th}$–prolonged graphs is generated by the vector field $\mathrm{pr}^{(k)}X \in \mathfrak{X}(M \times \mathbb{R}^{1+k})$

$$\mathrm{pr}^{(k)}X = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \sum_{j=1}^{k} \eta^{(j)}(x,u,u',\dots,u^{(j)})\frac{\partial}{\partial u^{(j)}} \tag{1.129}$$

where the components $\eta^{(j)}(x,u,u',\dots,u^{(j)})$ are constructed recursively

$$\eta^{(j)}(x,u,u',\dots,u^{(j)}) = \mathcal{D}_x\eta^{(j-1)} - u^{(j)}\mathcal{D}_x\xi \tag{1.130}$$

using the operator of total derivative

$$\mathcal{D}_x = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + \sum_{j=1}^{k-1} u^{(j+1)}\frac{\partial}{\partial u^{(j)}}.$$

That means that the vector field (1.129) encodes in itself the fact that the derivatives $u'(x), \ldots, u^{(n)}(x)$ in the differential equation (1.121) transform in a unique way once the point transformation (1.122) is chosen. Provided that we work only with generators of the form (1.129), we may now for our purposes view the differential equation (1.121) as an algebraic equation for a set of unknowns $x, u, u', \ldots, u^{(p)}$. This determines certain solution hypersurface $\Sigma$ in $M \times \mathbb{R}^{1+p}$,

$$\Sigma = \{(x, u, u', \ldots, u^{(p)}) \in M \times \mathbb{R}^{1+p} | F(x, u, u', \ldots, u^{(p)}) = 0\}.$$

Any $p$–times differentiable function $f : M \to \mathbb{R}$ whose $p^{th}$–prolonged graph $\Gamma_f^{(p)}$ lies in the hypersurface $\Sigma$ is a solution of the differential equation (1.121).

Combining the results on symmetries of algebraic equations and the prolongation of vector fields, we can formulate a criterion on generators of point symmetries of differential equations.

**Theorem 1.10 (On generators of symmetries of ODEs)** *Let $M \subset \mathbb{R}$ and let $F : M \times \mathbb{R}^{1+p} \to \mathbb{R}$ define a differential equation*

$$F(x, u(x), u'(x), \ldots, u^{(p)}(x)) = 0. \tag{1.131}$$

*Let*

$$\Sigma_F = \{(x, u, u', \ldots, u^{(p)}) \in M \times \mathbb{R}^{1+p} | F(x, u, u', \ldots, u^{(p)}) = 0\}$$

*and*

$$\mathrm{d}F(v) \neq 0, \qquad \forall v \in \Sigma_F. \tag{1.132}$$

*Then a vector field $X \in \mathfrak{X}(M \times \mathbb{R})$ generates a local 1–parameter group of point symmetries of the differential equation* (1.131) *if and only if*

$$\mathrm{pr}^{(p)} F(v) = 0, \quad \forall v \in \Sigma_F. \tag{1.133}$$

We notice that the regularity condition (1.132) is satisfied e.g. for any differential equation solved with respect to the highest derivative.

Let us mention that point transformations are not the only class of transformations one may consider in the context of symmetry analysis of differential equations. Another, less restrictive choice is defined by transformations on $\mathbb{R}^3$ (with coordinates $x, u, u'$) of the form

$$\hat{x} = g_1(x, u, u'), \qquad \hat{u} = g_2(x, u, u'), \qquad \hat{u}' = g_3(x, u, u') \tag{1.134}$$

subject to a consistency condition

$$\frac{\partial g_2(x, u, u')}{\partial u'} = g_3(x, u, u') \frac{\partial g_1(x, u, u')}{\partial u'}. \tag{1.135}$$

This condition comes from the requirement that first derivatives of the function $u = f(x)$ should transform independently of second and higher derivatives of $f(x)$.

Transformations (1.134) are called *contact transformations*. While for certain differential equations the group of contact symmetries is larger than the group of point symmetries, in most cases both groups are isomorphic.

We shall restrict ourselves to point transformations in the following.

Theorem 1.10 can be readily generalized to systems of ordinary differential equations and also to partial differential equations.

Let us consider a system of $q$ partial differential equations of order at most $p$

$$F_\nu(x^i, u^\alpha, \dots, u^\alpha_J) = 0, \qquad \nu = 1, \dots, q, \ |J| \leq p. \tag{1.136}$$

where $(x^i)_{i=1}^m$ are independent variables, $(u^\alpha)_{\alpha=1}^n$ are dependent variables. (Collectively, we denote them $x$ and $u$, respectively.) We define the multi–index $J = (j_1, \dots, j_m)$, where $j_i \in \mathbb{N} \cup \{0\}$, $|J| = j_1 + \ldots + j_m$ and

$$u^\alpha_J = \frac{\partial^{|J|} u^\alpha}{\partial^{j_1} x_1 \partial^{j_2} x_2 \ldots \partial^{j_m} x_m}.$$

We suppose that solutions $u(x)$ of PDE (1.136) are defined on a domain $M \subset \mathbb{R}^m$ and take values in $N \subset \mathbb{R}^n$ where $M$ and $N$ are some open subsets.

As before, the coordinates $x^i, u^\alpha$ on $M \times N$ are formally extended to the so–called $k^{th}$ jet bundle

$$\mathcal{J}_k = \{(x^i, u^\alpha, u^\alpha_J) \,|\, |J| \leq k\} \tag{1.137}$$

which includes both coordinates on $M \times N$ and all derivatives of the dependent variables $u^\alpha$ of order less or equal to $k$ (we identify $\mathcal{J}_0 \equiv M \times N$). On the jet bundle, we define the total derivatives

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} + \sum_{\alpha, J} u^\alpha_{J_i} \frac{\partial}{\partial u^\alpha_J}, \tag{1.138}$$

where

$$J_i = (j_1, \dots, j_{i-1}, j_i + 1, j_{i+1}, \dots, j_m).$$

More generally, for $J = (j_1, j_2, \dots, j_m)$, we define

$$\mathcal{D}_J = \underbrace{\mathcal{D}_1 \mathcal{D}_1 \cdots \mathcal{D}_1}_{j_1} \cdots \ \underbrace{\mathcal{D}_n \mathcal{D}_n \cdots \mathcal{D}_n}_{j_m}. \tag{1.139}$$

The prolongation of a 1–parameter group action to the jet bundle $\mathcal{J}_k$ as before induces a prolongation of the generating vector field. For the vector field $X$ given by

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{1.140}$$

the $k^{th}$ order prolongation of $X$ is

$$\mathrm{pr}^{(k)}(X) = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \sum_{\alpha,|J|\neq 0} \eta^\alpha_J(x,u,\ldots,u_{(|J|)})\frac{\partial}{\partial u^\alpha_J}, \tag{1.141}$$

where $\eta^\alpha_J(x,u,\ldots,u_{(|J|)})$ are functions on the $|J|$–th jet bundle and are given by the recursive formula

$$\eta^\alpha_{J_j} = \mathcal{D}_j\eta^\alpha_J - \sum_i(\mathcal{D}_j\xi^i)u^\alpha_{J_i} \tag{1.142}$$

or, equivalently, by the formula

$$\eta^\alpha_J = \mathcal{D}_J\left(\eta^\alpha - \xi^i\frac{\partial u^\alpha}{\partial x^i}\right) + \xi^i u^\alpha_{J_i}. \tag{1.143}$$

An analogue of the symmetry criterion 1.10 can now be stated as follows

**Theorem 1.11 (On generators of symmetries of PDEs)** *Let*

$$F_\nu(x^i, u^\alpha, \ldots, u^\alpha_J) = 0, \qquad \nu = 1,\ldots,q,\ |J| \leq p.$$

*be a non–degenerate system of partial differential equations (meaning that the system is locally solvable with respect to highest derivatives and is of maximal rank at every point $p \in \mathcal{J}_k$ such that $F_\nu(p) = 0$, $\nu = 1,\ldots,q$) and $G$ be a connected Lie group (locally) acting on $\mathcal{J}_0 = M \times N$ through the transformations*

$$\tilde{x}_i = A^i(x,u,g), \qquad \tilde{u}^\alpha = B^\alpha(x,u,g).$$

*Let the Lie algebra $\mathfrak{g}$ of the Lie group $G$ together with its induced infinitesimal action* (1.112) *be the corresponding algebra of infinitesimal transformations. Then $G$ is a group of point symmetries of the PDE system $F = 0$ if and only if*

$$\left[\mathrm{pr}^{(p)}(X)\right](F_\nu) = 0, \quad \nu = 1,\ldots,q, \qquad \textit{whenever } F = 0 \tag{1.144}$$

*for every infinitesimal generator $X$ representing the infinitesimal action of $x \in \mathfrak{g}$.*

Theorem 1.11 applies also to ODEs and systems of ODEs (when $m = 1$).

A practical determination of the symmetry algebra of a given $p^{th}$ order system (1.136) of differential equations

$$F_\nu(x^i, u^\alpha, \dots, u^\alpha_J) = 0, \qquad \nu = 1, \dots, q$$

involves several steps:

1. we have to compute $p^{th}$ prolongation of an arbitrary vector field $X$ (1.140) on $\mathcal{J}_0$,

2. evaluate $\mathrm{pr}^{(p)}(X)F_\nu$,

3. substitute into it all equations $F_\nu = 0$ and their differential consequences (if necessary); preferably, we eliminate the highest order derivatives using $F_\nu = 0$.

   These three steps can be rather lengthy and tedious, but are algorithmic and can be efficiently and reliably performed using computer algebra systems.

4. Now that $F = 0$ was imposed, the resulting equations

   $$\mathrm{pr}^{(p)}(X)F_\nu|_{F=0} = 0$$

   are to be viewed as equations for the unknown components $\xi^i, \eta^\alpha$ of the vector field $X$ which must hold for any values of the remaining jet space coordinates $u^\alpha_J, |J| \geq 1$. After we separate independent terms in $u^\alpha_J$, we obtain a highly overdetermined[2] system of linear partial differential equations for the functions $\xi^i(x,u), \eta^\alpha(x,u)$. Its solution provides us with all generators $X$ which satisfy equation (1.144) of Theorem 1.11.

   Although this step is often also entrusted to computers, it does sometimes happen that computer programs miss some of the solutions and the resulting symmetry algebra is incomplete.

After the symmetry generators are found, it is sensible to check their consistency by verifying that the symmetry algebra is closed under commutators. Next, one may integrate the generators to 1–parameter subgroups and compose them to obtain the connected component of the symmetry group.

Other possible components of the symmetry group cannot be deduced directly from the infinitesimal approach. Although some methods for their determination exist (see e.g. [5, 6]) we shall not consider them here.

[2] in almost all cases

Let us now iluminate the presented abstract concepts by a concrete example. Because its full derivation and intermediate calculations are rather long, we shall only review and interpret the results. We use an abbreviated notation, $\partial_a \equiv \frac{\partial}{\partial a}$.

**Example 1.12** *The heat equation*

$$\partial_t u - \partial_{xx} u = 0 \tag{1.145}$$

*has an infinite dimensional algebra of infinitesimal point symmetries. It consists of the six vector fields*

$$\begin{aligned}
X_1 &= 4xt\partial_x + 4t^2\partial_t - (2t + x^2)u\partial_u, \\
X_2 &= 2x\partial_x + 4t\partial_t + u\partial_u, \\
X_3 &= \partial_t, \\
X_4 &= -2t\partial_x + xu\partial_u, \\
X_5 &= u\partial_u, \\
X_6 &= \partial_x
\end{aligned}$$

*together with an infinite set of generators*

$$X_V = V(x,t)\partial_u$$

*where $V(x,t)$ is an arbitrary solution of the heat equation (1.145).*

*It is instructive to interpret these vector fields in terms of the corresponding finite transformations. The vector fields $X_3, X_6$ generate translations in $t$ and $x$. These symmetries are obvious from the onset – they just represent the fact that the heat equation* (1.145) *is autonomous, i.e. does not involve $t$ and $x$ explicitly.*

*The vector fields $X_2, X_5$ represent invariance of the heat equation under two independent scalings $u \to \lambda u$ and $x \to \lambda x, t \to \lambda^2 t$.*

*The vector field $X_4$ indicates invariance under the Galilei transformation $x \to x - \lambda t$ accompanied by a suitable redefinition of $u(x,t)$.*

*Finally, $X_V$ generates the invariance under the transformation $u \to u + \lambda V$ where $V$ is another arbitrary solution of the heat equation* (1.145)*, i.e. represents its linearity.*

*Altogether, all the symmetry generators $X_2, \ldots, X_6, X_V$ can be guessed without any calculations. They close into a subalgebra of the full symmetry algebra $\mathfrak{sym}(\partial_t u - \partial_{xx} u = 0) = \mathrm{span}\{X_1, \ldots, X_6, X_V\}_{\partial_t V - \partial_{xx} V = 0}$. Without explicit computation of the symmetry algebra one would probably miss the generator $X_1$ which does not possess any obvious physical interpretation.*

*As far as the algebraic structure of the Lie algebra $\mathfrak{sym}(\partial_t u - \partial_{xx} u = 0)$ is considered, we notice that it splits into a direct sum,*

$$\mathfrak{sym}(\partial_t u - \partial_{xx} u = 0) = \mathrm{span}\{X_V\}_{\partial_t V - \partial_{xx} V = 0} \oplus \mathrm{span}\{X_1, \dots, X_6\}$$

*where* $\mathrm{span}\{X_V\}_{\partial_t V - \partial_{xx} V = 0}$ *is an infinite–dimensional Abelian Lie algebra and* $\mathrm{span}\{X_1, \dots, X_6\}$ *is a finite dimensional Levi decomposable algebra. It has a simple factor* $\mathrm{span}\{X_1, X_2, X_3\}$ *isomorphic to* $\mathfrak{sl}(2)$ *and a nilpotent radical* $\mathrm{span}\{X_4, X_5, X_6\}$ *isomorphic to the Heisenberg algebra* $\mathfrak{h}(1)$ *of Example 1.1.*

*We observe that the infinite dimensional algebra* $\mathrm{span}\{X_V\}_{\partial_t V - \partial_{xx} V = 0}$ *is often truncated to a finite dimensional subalgebra when the symmetries are computed using algorithms implemented in computer algebra systems (e.g. procedure* Infinitesimals *in Maple 13).*

We have noticed in this example that often most, if not all, infinitesimal symmetries of the given differential equation can be found by inspection, without any computation. Unfortunately, there is no easy way of establishing the completeness of the symmetry algebra guessed in this way, e.g. there is no method of independent determination of dimension of the symmetry algebra. The only reliable method is to perform the full computation of symmetries and check whether anything unexpected arises.

In Chapter 3 we determine the Lie superalgebra of infinitesimal symmetries of several supersymmetric field equations, i.e. partial differential equations involving not only ordinary commuting variables but also anticommuting (sometimes called fermionic) variables. The method used there [67, 68] is a rather straightforward generalization of the procedure outlined here – one only has to pay close attention to ordering of anticommuting terms and possible sign changes. As was hinted in the previous paragraph, in most cases nothing unexpected happens and only the symmetries build into our models from the beginning (i.e. Lorentz invariance and supersymmetry) are recovered. Only in one case an additional scaling is found, also rather easy to guess.

Once the symmetry algebra of the given equation(s) is determined, one can use it in several different ways such as:

1. Exponentiate infinitesimal symmetries to 1–parameter subgroups and use the resulting transformations to generate new solutions from the known ones.

2. Use the symmetry algebra as a necessary criterion for equivalence of two differential equations. If any pair of differential equations can be transformed one into the other by a point transformation then necessarily

their symmetry algebras must be isomorphic. Thus we have a necessary (though far from sufficient) condition for equivalence. In addition, when an explicit transformation between two equations is sought, it is often convenient to construct point transformations taking one symmetry algebra into the other and only then look for transformations taking one equation into the other inside this class.

In particular, when a given PDE has an infinite dimensional Abelian subalgebra of infinitesimal symmetries involving an arbitrary solution of some linear PDE we may interpret it as a strong indication that our prescribed equation may be linearizable by some point transformation.

3. Reduce the order of an ODE. This method is based on a simple observation that an ODE
$$F(x, y, \ldots, y^{(p)}) = 0$$
which possesses an infinitesimal symmetry $\partial_y$ must be independent of the dependent variable $y$, i.e. in the form
$$F(x, y', \ldots, y^{(p)}) = 0. \tag{1.146}$$
Obviously, we may lower its order by one through the substitution $z = y'$, then attempt to solve the new ODE
$$F(x, z, \ldots, z^{(p-1)}) = 0$$
and once its solution $z(x)$ is known, we may write the solution of the original equation (1.146) in quadrature
$$y(x) = \int z(x)\mathrm{d}x.$$
Hence, the substance of the method is the following: starting from an arbitrary nonvanishing infinitesimal symmetry $X = \xi\partial_x + \eta\partial_y$ we look for a point transformation, i.e. a change of coordinates on $M \times N$, such that in the new coordinates $\tilde{x}, \tilde{y}$ our vector field $X$ takes the form $X = \partial_{\tilde{y}}$. According to the rules for transformation of the components of a vector field these new coordinates must satisfy equations
$$X(\tilde{x}) = 0, \qquad X(\tilde{y}) = 1.$$
These equations are solved using the method of characteristics, see Section 1.2.3. Their solution is in general not unique, but any particular solution with nonconstant $\tilde{x}$ can be used.

Once $\tilde{x}, \tilde{y}$ are found, we lower the order of our equation in the new coordinates, solve it (if possible), and at the end transform the solution to the original coordinates.

This approach generalizes many particular methods used in solution of ODEs.

**Example 1.13** *Let*

$$F(y, \ldots, y^{(p)}) = 0$$

*be an autonomous ODE, i.e. not depending explicitly on* $x$. *It is invariant under translations in the independent variable* $x$, *generated by* $X = \partial_x$. *Therefore, if we interchange the roles of independent and dependent variable* $\hat{x} = y$, $\hat{y} = x$, *the vector field becomes* $X = \partial_{\tilde{y}}$ *and we may lower the order of the differential equation for the inverse function* $x(y)$ *by one.*

**Example 1.14** *Let*

$$F(x, y, \ldots, y^{(p)}) = 0 \tag{1.147}$$

*be invariant under the scaling* $x \to \lambda x$, $y \to \lambda^\alpha y$. *Such scaling is obtained as the 1–parameter group of transformations generated by the vector field*

$$X = x\partial_x + \alpha y \partial_y.$$

*The new coordinates* $\tilde{x}, \tilde{y}$ *can be chosen as*

$$\tilde{x} = \frac{y}{x^\alpha}, \qquad \tilde{y} = \ln x.$$

*Once we rewrite the original ODE* (1.147) *in these coordinates we may again lower its order by one.*

We remark that the reduced equation may have a group of symmetries rather distinct from the original one. In particular, other symmetries of the original equation may not survive the reduction. Only the symmetries generated by such vector fields $Y \in \mathfrak{X}(M \times N)$ that a constant $\alpha \in \mathbb{F}$ exists satisfying

$$[Y, X] = \alpha X$$

are guaranteed to survive the reduction.

By induction, a $k$–dimensional algebra of infinitesimal symmetries of a given ODE with a complete flag of ideals as in Lie's theorem (Theorem 1.3) allows us to reduce the order by $k$ provided we can find suitable coordinates in each step, of course. That was the original motivation for the definition of a solvable algebra – although, as we have seen in Lie's theorem, it is in the current terminology well justified only if we consider complex Lie algebras and complex (holomorphic) ODEs.

This reduction can be immediately generalized to systems of ODEs but not to PDEs. For PDEs, another method is available.

4. Construction of group–invariant solutions of PDEs. As already mentioned, the method described above does not work for PDEs since the fact that a PDE does not involve the dependent variable explicitly does not in general provide any help in its solution. Nevertheless, we may employ the symmetries in construction of particular solutions of a given PDE.

   The essential observation is as simple as above. Let us suppose that a given PDE
   $$F(x^i, u^\alpha, \dots, u^\alpha_J) = 0$$
   has a symmetry generator
   $$X = \partial_{x^1}. \tag{1.148}$$
   That means that $F$ is invariant with respect to translations in $x^1$, i.e. does not depend on it explicitly. Consequently, we may suppose that our solution $u^\alpha$ depends only on the remaining independent variables $x^i$, $i = 2, \dots, m$ and in this way we obtain a well–defined PDE with one less independent variables. Any solution of this PDE is also a solution of the original equation which in addition is invariant with respect to the 1–parameter group of symmetries generated by the vector field $X$; hence its name *group–invariant solution.*

   Similarly as before, the method boils down to the construction of suitable coordinates $\tilde{x}^i, \tilde{u}^\alpha$ on $M \times N$ in which a given symmetry generator $X$ takes the form (1.148). Again, the method of characteristics is used. In fact, it turns out that we need to compute only the invariant coordinates
   $$\tilde{x}^i : X(\tilde{x}^i) = 0,\ i = 2, \dots, m, \qquad \tilde{u}^\alpha : X(\tilde{u}^\alpha) = 0,\ \alpha = 1, \dots, n$$
   in the process, as the following example will demonstrate.

**Example 1.15** *Let us consider the heat equation of Example 1.12 and the vector field*
$$X_4 = -2t\partial_x + xu\partial_u.$$

*This vector field has the following invariants*

$$\tau = t, \qquad I = u e^{\frac{x^2}{4t}}.$$

*Therefore, we substitute* $u(x,t) = I(t)e^{-\frac{x^2}{4t}}$ *into the heat equation* (1.145) *and obtain a reduced equation for* $I(t)$

$$2tI'(t) + I(t) = 0.$$

*Its general solution is* $I(t) = \frac{C}{\sqrt{t}}$. *Altogether, we have recovered the fundamental solution (when* $C = \frac{1}{\sqrt{4\pi}}$*) of the heat equation*

$$u(x,t) = \frac{C}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

*as the solution invariant with respect to Galilei transformations generated by the vector field* $X_4$.

As before, the reduced equation may have symmetries which are of no direct relation to the original ones. If we want to be able to further reduce the number of independent variables we again need a solvable symmetry algebra and an appropriate choice of generators of 1–parameter subgroups (i.e. a basis respecting the flag of codimension 1 ideals, starting from the smallest one).

We notice that solutions invariant with respect to vector fields $X$ and $\tilde{X} = Ad_g X$ are related: we may obtain a solution $\tilde{u}(x)$ invariant with respect to $\tilde{X}$ from $u(x)$ simply by setting $\tilde{u}(x) = g \triangleright u(x)$. Therefore, one shall first classify 1–dimensional subalgebras of the symmetry algebra under conjugation by $g \in G$ (or higher–dimensional subalgebras if reduction with respect to more independent variables is intended) and only then perform the reduction with respect to nonequivalent generators.

In Chapter 3 we shall encounter a generalization of this procedure to PDEs involving anticommuting variables. There, 1–dimensional subalgebras of the Lie superalgebras of infinitesimal symmetries are divided into conjugacy classes and the symmetry reduction to ODE is performed for nonequivalent symmetry generators. As we will see, the procedure sometimes fails because of nilpotency of some of the generators.

## 1.4 Poisson–Lie T–dual sigma models

One of the many application of Lie groups and algebras in modern physics are the so–called Poisson–Lie T–dual sigma models which we have spend considerable time investigating during the last decade [69, 70, 71, 72, 73, 74, 75, 76]. In Chapter 4 we present three recent papers dealing with particular properties of such models.

### 1.4.1 Sigma models

A sigma model of the simplest variant is a field theoretical model whose dynamical fields are components of a map $\Phi$ between two (pseudo)Riemannian manifolds $(\Sigma, \gamma)$ and $(M, g)$. Dynamics of the map $\Phi$ is determined from the action

$$S = \int_\Sigma \langle \gamma, \Phi^* g \rangle \mathrm{d}vol_\gamma. \tag{1.149}$$

We call the $D$–dimensional manifold $M$ the *target manifold* and its metric $g$ the *background metric*. The terminology for $\Sigma$ depends on its dimension. When $\dim \Sigma = 2$ which shall be the case of interest to us here we call it the *worldsheet* and $\gamma$ the *worldsheet metric*.

In local coordinates $x^\mu$ on $\Sigma$ and $y^a$ on $M$ the action (1.149) reads

$$S = \int_\Sigma \gamma^{\mu\nu}(x) g_{ab}\left(\phi(x)\right) \frac{\partial \phi^a(x)}{\partial x^\mu} \frac{\partial \phi^b(x)}{\partial x^\nu} \sqrt{\pm|\gamma(x)|}\, \mathrm{d}x^1 \ldots \mathrm{d}x^{\dim \Sigma} \tag{1.150}$$

where $\mu, \nu = 1, \ldots, \dim \Sigma$, $a, b = 1, \ldots, D = \dim M$, $\gamma^{\mu\nu}\gamma_{\nu\kappa} = \delta^\mu_\kappa$ and summation over repeated indices is assumed here and in the rest of this Section. The sign of the determinant $|\gamma| = \det \gamma_{\mu\nu}$ under the square root is chosen so that the square root is real–valued, i.e. the sign depends on the signature of the metric $\gamma$.

The equations of motion are

$$\frac{1}{\sqrt{\pm|\gamma|}} \partial_\mu (\sqrt{\pm|\gamma|} \partial^\mu \phi^a) + \Gamma^a_{bc} \partial_\mu \phi^b \partial^\mu \phi^c = 0 \tag{1.151}$$

where $\Gamma^a_{bc}$ are components of the Levi–Civita connection on the target manifold $M$, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\partial^\mu = \gamma^{\mu\nu}\partial_\nu$.

We mention that solutions of a sigma model are called *harmonic maps* in mathematics.

Sigma model action (1.149) can be viewed as the simplest action we can postulate for a map between two Riemannian manifolds; in particular, it is quadratic in derivatives of our field and does not involve any potential or mass terms. That is why it is of interest in field theory as the simplest action defined on nonlinear targets $M$, i.e. such that they don't possess the structure of a vector space.

Sigma models are of particular interest in string theory because they are cousins of the Polyakov action for bosonic string. (Also supersymmetric sigma models which are analogues of superstring actions exist). Let us recall that the Polyakov action looks formally very much like the two–dimensional sigma model action (1.150) expressed in coordinates

$$S_P = \frac{1}{2}\int_\Sigma \gamma^{\mu\nu}(x) g_{ab}\left(\phi(x)\right)\frac{\partial\phi^a(x)}{\partial x^\mu}\frac{\partial\phi^b(x)}{\partial x^\nu}\sqrt{-|\gamma(x)|}\mathrm{d}x^0\ldots\mathrm{d}x^1 \tag{1.152}$$

where $\mu,\nu = 0,1$ and we have assumed that $\Sigma$ is pseudo–Riemannian, i.e. the metric $\gamma$ is indefinite. The difference is that in Polyakov action (1.152) not only the field $\Phi$ but also the worldsheet metric $\gamma$ are dynamic. Consequently, in addition to equations (1.151) we have also equations coming from the variation of the metric $\gamma$. They are neatly expressed in the statement that the worldsheet stress tensor $T_{\mu\nu}$ must vanish

$$T_{\mu\nu} \equiv \frac{\delta S_P}{\delta\gamma^{\mu\nu}} = 0. \tag{1.153}$$

However, the worldsheet metric $\gamma$ is in a sense a gauge degree of freedom. Any metric on a two–dimensional manifold is conformally flat. Therefore, by a change of coordinates on the worldsheet $\Sigma$ we can always locally bring the metric $\gamma$ to our preferred form $\gamma = \mathrm{e}^\omega\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ where $\omega$ is some function on $\Sigma$. Another interesting observation is that $\omega$ drops out completely from the action (1.152). Consequently, up to possible topological obstructions we may bring the metric $\gamma$ in the Polyakov action (1.152) to any fixed metric. In this way we recover the sigma model (1.150) from the Polyakov action.

One shall not miss one important point – although the worldsheet metric dynamics was "gauged away", its equations of motion (1.153) remain and have to be imposed on the solutions $\phi$ of the sigma model (1.150). Therefore, the Polyakov action (1.152) is equivalent to the corresponding sigma model complemented by the vanishing stress tensor constraint (1.153).

For further considerations we shall need a slightly more general definition of the sigma model than we have introduced in equation (1.149). Firstly, we consider an antisymmetric 2–form $B$ on the target manifold and add a term $\int_\Sigma \phi^*(B)$ to the action of our 2–dimensional sigma model. The two–form $B$ is referred to as the *torsion potential* or simply the *B–field*. Its effect on the equations of motion (1.151) can be viewed as a modification of the Levi–Civita connection by a torsion term $H = \mathrm{d}B$.

The second modification makes sense only when the worldsheet has a topology of an open strip infinitely extended in timelike direction, e.g. $\Sigma \simeq [0,\pi][\sigma] \times \mathbb{R}[\tau]$. Then we may couple a one–form $A$ defined on the target manifold $M$ to the field $\phi$ at the two string endpoints $\sigma = 0, \pi$

$$S = \int_\Sigma \langle \gamma, \Phi^* g \rangle \mathrm{d}vol_\gamma + \int_\Sigma \phi^*(B) + q_1 \int \phi^* A|_{\sigma=0} + q_2 \int \phi^*(A)|_{\sigma=\pi} \quad (1.154)$$

where $q_1, q_2$ are two charges associated with the respective endpoints of the open string.

### 1.4.2 T–duality of sigma models

Dualities play an important role in string theory. They can relate strongly and weakly coupled string theories or provide a connection between models on geometrically different targets. In the context of superstrings they even relate different types of string theories (i.e. different periodicity/boundary conditions in the fermionic sector).

One class of such transformations is T–duality, or the target space duality. In its simplest, Abelian, version it applies to sigma models (1.154) (with $A = 0$ for simplicity) possessing an isometry, i.e. a Killing vector field $X$ on $M$ such that the Lie derivatives with respect to $X = X^a \frac{\partial}{\partial y^a}$ of both the background metric $g$ and the B–field $B$ vanish

$$\mathscr{L}_X g = 0, \qquad \mathscr{L}_X B = 0.$$

Then the action is invariant under an infinitesimal transformation of the form

$$\delta \phi^a = \epsilon X^a.$$

We may choose our coordinates $(y^a)_{a=0}^{D-1}$ on $M$ so that $X = \frac{\partial}{\partial y^0}$. Then our original sigma model and the one written in terms of transformed background fields

$$\begin{aligned}
\tilde{G}_{00} &= \frac{1}{G_{00}}, \quad \tilde{G}_{0i} = \frac{1}{G_{00}} B_{0i}, \quad \tilde{B}_{0i} = \frac{1}{G_{00}} G_{0i}, \\
\tilde{G}_{ij} &= G_{ij} - \frac{1}{G_{00}} (G_{0i} G_{0j} + B_{i0} B_{0j}), \\
\tilde{B}_{ij} &= B_{ij} + \frac{1}{G_{00}} (G_{0i} B_{0j} + B_{i0} G_{0j})
\end{aligned} \quad (1.155)$$

($i = 1, \dots, D-1$) are equivalent in the sense that both can be obtained by two different reductions from a single action involving more fields [77, 78]. Also the solutions of the two models are directly related. The formulae (1.155) are called Buscher's formulae in the literature.

When one applies Buscher's formulae (1.155) twice one obtains back the original model – that is why the transformation is called duality. If we have several independent commuting isometries $X_i$ such that

$$\mathscr{L}_{X_i} g = 0, \qquad \mathscr{L}_{X_i} B = 0,$$

i.e. there is an Abelian algebra of isometries, then there are more mutually dual models. The dualization may proceed in several steps, using one vector field at each step for the dualization as prescribed by Buscher's formulae.

A question naturally arised what can be done for models with non–Abelian algebra $\mathfrak{g}$ of isometries. The first attempt at such generalization was performed in [79] but suffered from a substantial deficiency: the transformation worked only one way, the "dual" sigma model might have no isometries at all.

Attempting to resolve this problem, C. Klimčík and P. Ševera in [80] constructed a transformation which they called Poisson–Lie T–duality of sigma models.

### 1.4.3 Poisson–Lie T–dual sigma models

Let us assume that the target manifold $M$ has two additional structures – namely, that it is both a Lie group $M = G$ and a Poisson manifold, i.e. equipped by Poisson bracket $\{\,,\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that it is bilinear, antisymmetric and satisfies both Leibniz rule and Jacobi identity (see e.g. [7, 81]). Furthermore, we require compatibility between the two structures, i.e. the group multiplication should be a Poisson map. Such structure $(G, \cdot, \{,\})$ is called a *Poisson–Lie group*. Locally it is encoded in its Lie bialgebra.

A *Lie coalgebra* is an algebraic structure which is naturally obtained on the dual of a Lie algebra. In particular, let $\mathfrak{g}^*$ be the dual space of the Lie algebra $\mathfrak{g}$. Then we can define a linear map $\delta : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ such that

$$\delta(\alpha)(x \otimes y) = \alpha([x, y])$$

for all $\alpha \in \mathfrak{g}^*, x, y \in \mathfrak{g}$. If we denote by $\tau_{12} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ the flip map $\tau_{12}(\alpha_1 \otimes \alpha_2) = \alpha_2 \otimes \alpha_1$ (and similarly when more factors are present in the tensor product) we find that $\delta$ has the following properties

$$\begin{aligned} \tau_{12} \circ \delta &= -\delta, & (1.156) \\ (id \otimes \delta) \circ \delta &= (\delta \otimes id) \circ \delta - \tau_{23} \circ (\delta \otimes id) \circ \delta. & (1.157) \end{aligned}$$

The properties (1.156),(1.157) are taken as the defining properties of a *Lie cobracket* $\delta : V \to V \otimes V$ on any vector space $V$ (implicitly assuming that $\delta$ is linear).

A *Lie bialgebra* $(\mathfrak{g}, [,], \delta)$ is an algebraic structure which combines both a Lie algebra and a Lie coalgebra in a compatible way. That means that $[,]$ is a Lie bracket on $\mathfrak{g}$, $\delta$ is a Lie cobracket on $\mathfrak{g}$ and

$$\delta([x,y]) = (\mathrm{ad}_x \otimes id + id \otimes \mathrm{ad}_x)\delta(y) - (\mathrm{ad}_y \otimes id + id \otimes \mathrm{ad}_y)\delta(x) \quad (1.158)$$

Equation (1.158) is the so–called 1–cocycle condition because it means that $\delta$ can be viewed as 1–cocycle in Chevalley cohomology with values in $\mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$. In a basis $(X_i)$ of the Lie bialgebra $\mathfrak{g}$ such that

$$[X_i, X_j] = f_{ij}{}^k X_k, \qquad \delta(X_i) = \tilde{f^{jk}}_i X_j \otimes X_k$$

the axioms (1.156),(1.157) of the Lie coalgebra are expressed as

$$\tilde{f^{ij}}_k = -\tilde{f^{ji}}_k$$

and

$$\tilde{f^{kl}}_m \tilde{f^{ij}}_l + \tilde{f^{il}}_m \tilde{f^{jk}}_l + \tilde{f^{jl}}_m \tilde{f^{ki}}_l = 0$$

whereas the bialgebra axiom (1.158) becomes

$$\tilde{f^{jk}}_l f_{mi}{}^l + \tilde{f^{kl}}_m f_{li}{}^j + \tilde{f^{jl}}_i f_{lm}{}^k + \tilde{f^{jl}}_m f_{il}{}^k + \tilde{f^{kl}}_i f_{lm}{}^j = 0.$$

It is interesting to notice that the dual space $\mathfrak{g}^*$ of a Lie bialgebra $\mathfrak{g}$ is also a Lie bialgebra with the canonical definition

$$\delta_*(\alpha)(x \otimes y) = \alpha([x,y]), \qquad [\alpha, \beta]_*(x) = \alpha \otimes \beta(\delta(x)),$$

i.e. the roles of the Lie bracket and cobracket get interchanged.

The *Drinfeld double* $D$ is defined as a connected Lie group such that its Lie algebra $\mathfrak{d}$ equipped with a symmetric ad–invariant nondegenerate bilinear form $\langle \,.\,,\,.\, \rangle$ can be decomposed into a pair of subalgebras $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ maximally isotropic with respect to $\langle \,.\,,\,.\, \rangle$. The dimensions of the subalgebras $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ are necessarily equal – otherwise $\langle \,.\,,\,.\, \rangle$ would be degenerate. Any such decomposition is called a *Manin triple* and denoted $(\mathfrak{g}|\tilde{\mathfrak{g}})$. For a given Drinfeld double several Manin triples may exist, i.e.

$$(\mathfrak{g}|\tilde{\mathfrak{g}}) \cong (\tilde{\mathfrak{g}}|\mathfrak{g}) \cong (\mathfrak{g}'|\tilde{\mathfrak{g}}') \cong \ldots$$

In particular, a given Lie bialgebra $\mathfrak{g}$ gives rise to a unique Drinfeld double such that

$$\mathfrak{d} = \mathfrak{g} \dot{+} \mathfrak{g}^*.$$

Any Drinfeld double is by itself a Poisson–Lie group. Its Lie cobracket is obtained by the natural identification of $\mathfrak{d}$ and $\mathfrak{d}^*$ using the bilinear symmetric form $\langle\,.\,,.\,\rangle$.

The bases $(X_i), (\tilde{X}^i)$ in the subalgebras can be chosen so that

$$\langle X_i, X_j \rangle = 0,\ \langle X_i, \tilde{X}^j \rangle = \langle \tilde{X}^j, X_i \rangle = \delta_i^j,\ \langle \tilde{X}^i, \tilde{X}^j \rangle = 0. \tag{1.159}$$

We shall assume that any considered basis of any Manin triple satisfies (1.159).

Due to the ad-invariance of $\langle\,.\,,.\,\rangle$ the structure constants of $\mathfrak{d}$ are fully determined by the structure of its maximally isotropic subalgebras $\mathfrak{g}, \tilde{\mathfrak{g}}$, i.e. if in bases $(X_i), (\tilde{X}^i)$ the Lie brackets are given by

$$[X_i, X_j] = f_{ij}{}^k X_k,\ [\tilde{X}^i, \tilde{X}^j] = \tilde{f^{ij}}_k \tilde{X}^k$$

then

$$[X_i, \tilde{X}^j] = f_{ki}{}^j \tilde{X}^k + \tilde{f^{jk}}_i X_k. \tag{1.160}$$

Let $G$ ($\tilde{G}$) be the subgroup of $D$ whose Lie algebra is $\mathfrak{g}$ ($\tilde{\mathfrak{g}}$). We mention that all constructions below are in general permissible only locally, in a vicinity of the group unit.

The Lagrangian of the dualizable sigma models introduced by C. Klimčík and P. Ševera has the generic sigma model structure

$$L = F_{ij}(y)\partial_- y^i \partial_+ y^j\,, \qquad i, j = 1, \ldots, n = \dim \mathfrak{g} \tag{1.161}$$

where the background field $F$ on the Lie group $G$ has a particular, very restricted form. It is more easily expressed in terms of right–invariant fields, i.e. components of right–invariant 1–forms of Section 1.3.2. We have

$$L = E_{ab}(g)(\partial_- g g^{-1})^a (\partial_+ g g^{-1})^b,\ g \in G \tag{1.162}$$

where $(\partial_\pm g g^{-1})^a$ in physicist's notation denote the components of the pull-back of the right Maurer–Cartan 1–form $\nu^R$ on $G$ by the map $g : \Sigma \to G$,

$$\sum_{\epsilon=\pm} (\partial_\epsilon g g^{-1})^a \mathrm{d}x^\epsilon X_a^R = g^*(\nu^R)$$

and

$$E(g) = (E_0^{-1} + \Pi(g))^{-1}\,, \qquad \Pi(g) = b(g)a(g)^{-1} = -\Pi(g)^t \tag{1.163}$$

where $t$ denotes transposition and $a(g), b(g), d(g)$ are $n \times n$ submatrices of the adjoint representation of the group $G$ on $\mathfrak{d}$ in the basis $(X_i, \tilde{X}^j)$.

$$Ad(g)^t = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}, \tag{1.164}$$

$$a(g)^{-1} = d(g)^t, \qquad b(g)^t a(g) = -a(g)^t b(g). \tag{1.165}$$

The tensor field $\Pi$ defined on the Poisson–Lie group $G$ by equation (1.163) in fact describes the Poisson structure. It can be shown that the Poisson bracket becomes

$$\{f_1, f_2\}(g) = \Pi^{ab}(g) \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(\exp(tX_a)g) \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(\exp(tX_b)g).$$

Altogether, it means that the background field $F$ in equation (1.161) is

$$F_{ij}(y) = e_i^a(g(y)) E_{ab}(g(y)) e_j^b(g(y)) \tag{1.166}$$

where $e_i^a$ are components of right–invariant basis 1–forms (also called vielbeins) $\nu_R^a = e_i^a \mathrm{d}y^i$.

The covariant tensor field $F$ on $G$ is thus determined by the decomposition $\mathfrak{d} = (\mathfrak{g}|\tilde{\mathfrak{g}})$ and by the matrix $E_0$. It can be understood as a sum of the metric and the torsion potential defining the geometric properties of the manifold $G$. Necessary condition for invertibility of the metric of sigma models is

$$\det(E_0 + E_0^t) \neq 0\,. \tag{1.167}$$

It turns out that usually this condition is sufficient only in the vicinity of the group unit.

The possibility to decompose some Drinfeld doubles into more than two Manin triples[3] enables us to construct more than two equivalent sigma models and this property was called Poisson-Lie T-plurality [82]. Let $X_j, \tilde{X}^k$ where $j, k = 1, \ldots, n$ be basis vectors of the subalgebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ of the Manin triple associated with the Lagrangian (1.162) and $U_j, \tilde{U}^k$ be basis vectors of some other Manin triple $(\mathfrak{g}_u, \tilde{\mathfrak{g}}_u)$ in the same Drinfeld double related by a $2n \times 2n$ transformation matrix:

$$\begin{pmatrix} \vec{X} \\ \vec{\tilde{X}} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \cdot \begin{pmatrix} \vec{U} \\ \vec{\tilde{U}} \end{pmatrix}, \tag{1.168}$$

where

$$\vec{X} = (X_1, \ldots, X_n)^t, \ldots, \ \vec{\tilde{U}} = (\tilde{U}^1, \ldots, \tilde{U}^n)^t\,.$$

[3]Two decompositions always exist, namely $(\mathfrak{g}|\tilde{\mathfrak{g}})$ and $(\tilde{\mathfrak{g}}|\mathfrak{g})$.

The transformed model is then given by the Lagrangian of the same form as in equation (1.162) but with $E(g)$ replaced by

$$E_u(g_u) = M(N + \Pi_u\, M)^{-1} = (E_{0u}^{-1} + \Pi_u)^{-1} \tag{1.169}$$

where

$$M = S^t E_0 - Q^t\,, \qquad N = P^t - R^t E_0\,, \qquad E_{0u} = MN^{-1} \tag{1.170}$$

and $\Pi_u$ is calculated as in equation (1.163) from the adjoint representation of the group $G_u$ on $\mathfrak{d}$ expressed in the dual bases $(U_j), (\tilde{U}^k)$. The transformation of $E_0$ corresponds to the invariance of the vector subspace $\mathcal{E}^+ = \mathrm{span}\{X_j + (E_0)_{jk}\tilde{X}^k\} = \mathrm{span}\{U_j + (E_{0u})_{jk}\tilde{U}^k\}$. Notice that for $P = S = 0$, $Q = R = \mathbf{1}$ we get the dual model with $E_{0u} = E_0^{-1}$, corresponding to the interchange $\mathfrak{g} \leftrightarrow \tilde{\mathfrak{g}}$ so that the duality transformation is a special case of the plurality transformation (1.168)–(1.170).

### 1.4.4 Further developments

Various properties of Poisson–Lie T–dual sigma models were gradually studied by several authors. C. Klimčík and P. Ševera have extended their analysis to open strings in [83, 84, 85]. The global properties of closed strings, i.e. the problem of zero modes, were studied in [86] and in [87] where it was shown that a Poisson–Lie T–dual to a closed string is a particular, so–called monodromic, open string.

An interpretation of Poisson–Lie T–duality as a canonical transformation was presented by K. Sfetsos in [88].

The question of (one–loop) renormalizability of Poisson–Lie T–dual sigma models was originally investigated by K. Sfetsos in [89] and C. Klimčík and G. Valent in [90] and finally affirmatively answered by G. Valent, C. Klimčík and R. Squellari in [91]. The equivalence of the renormalization group flow for the original and dual model, i.e. true one–loop quantum equivalence of the two models, was established by K. Sfetsos and K. Siampos in [92].

Our own contributions to this field were often related to exploitation of different Drinfeld double decompositions. Such a possibility was known from the beginning [80] but no explicit examples were known at the time and most papers dealt with Poisson–Lie T–duality as a transformation between the dual algebras $\mathfrak{g}$ ad $\mathfrak{g}^*$ only. We first presented an example of models which correspond to different Manin triples in [69] in two dimensions. In [70] we presented a classification of all six–dimensional Drinfeld doubles together with their decompositions into Manin triples. These two papers were included in my doctoral thesis defended in 2002.

The paper [70] then served as a source of examples for several follow–up papers. It first found an application in R. von Unge's paper [82] where he constructed several conformally invariant backgrounds related by what he called *T–plurality*, i.e. corresponding to different Manin triples, and studied transformation of dilaton field under such transformations. We have further extended his analysis in [72, 73].

Other direction of research using [70] was explicit solution of sigma models using T–plurality transformations by L. Hlavatý and his students in [93, 94, 95]. In [96] they discussed the formalism when spectator fields are present.

In this thesis we present three recent papers [74, 75, 76] in Section 4. In these papers we deal with questions of canonical equivalence of models connected by Poisson–Lie T–plurality transformations in [74] and of transformation of boundary conditions in [75, 76].

# Chapter 2

# Structure of certain solvable and Levi decomposable algebras

In this chapter we present four papers

[**a**] L. Šnobl and P. Winternitz, A class of solvable Lie algebras and their Casimir Invariants, J. Phys. A: Math. Gen. 38 (2005) 2687-2700,

[**b**] L. Šnobl and P. Winternitz, All solvable extensions of a class of nilpotent Lie algebras of dimension $n$ and degree of nilpotency $n-1$, J. Phys. A: Math. Theor. 42 (2009) 105201,

[**c**] L. Šnobl and D. Karásek, Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras, Linear Algebra and its Applications 432 (2010) 1836-1850,

[**d**] L. Šnobl, On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras, J. Phys. A: Math. Theor. 43 (2010) 505202,

in which we study several aspects of Lie algebras.

In the first three papers we construct and classify all solvable Lie algebras with the three given series of nilradicals of high nilindex (maximal in [a,b] and almost maximal in [c]). The method employed in these papers is the one of Section 1.1.3 with certain modifications, in particular taking into account the results of [a] in [c]. Also their generalized Casimir invariants are computed, using the methods reviewed in Section 1.2.

In the last paper [d] we provide an improvement of the estimate on dimension of any solvable Lie algebra $\mathfrak{s}$ with its nilradical isomorphic to the given nilpotent Lie algebra $\mathfrak{n}$ due to G. M. Mubarakzyanov [97]. Next, we study the structure of algebras with nontrivial Levi decomposition, in particular the compatibility between the structure of the radical and Levi factor. From this perspective, we revisit the classification results of P. Turkowski [16, 17] and results of other authors [98, 99].

The papers [a,b] have already inspired several similar classifications by other authors, namely [29, 32, 31].

**N.B.** The convention for labelling of ideals in the lower central series used in [a] differs by a shift from the one in the rest of the papers and in Section 1.1.1.

2.1

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# A class of solvable Lie algebras and their Casimir invariants

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**Abstract**
A nilpotent Lie algebra $\mathfrak{n}_{n,1}$ with an $(n-1)$-dimensional Abelian ideal is studied. All indecomposable solvable Lie algebras with $\mathfrak{n}_{n,1}$ as their nilradical are obtained. Their dimension is at most $n+2$. The generalized Casimir invariants of $\mathfrak{n}_{n,1}$ and of its solvable extensions are calculated. For $n=4$ these algebras figure in the Petrov classification of Einstein spaces. For larger values of $n$ they can be used in a more general classification of Riemannian manifolds.

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## 1. Introduction

The purpose of this paper is to classify a certain type of finite-dimensional solvable Lie algebras, existing for any dimension $n$ with $n \geqslant 4$. These Lie algebras will be described below. Here we shall first present our motivation for performing this investigation.

Lie groups and Lie algebras appear in physics in many different guises. They may be *a priori* parts of the physical theory, like Lorentz or Galilei invariance of most theories, or the (semi-) simple Lie groups of the standard model in particle theory.

Alternatively, specific Lie groups may appear as consequences of specific dynamics. Consider any physical system with dynamics described by a system of ordinary or partial differential equations. This system of equations will be invariant under some local Lie group of local point transformations, taking solutions into solutions. This symmetry group $G$ and its Lie algebra $\mathfrak{g}$ can be determined in an algorithmic manner [1]. The Lie algebra $\mathfrak{g}$ is obtained as an algebra of vector fields, usually in some nonstandard basis, depending on the way in which the algorithm is applied.

 

An immediate task is to identify the algebra found as being isomorphic to some known abstract Lie algebra. To do this we must transform it to a canonical basis in which all basis-independent properties are manifest. Thus, if $\mathfrak{g}$ is decomposable into a direct sum, it should be explicitly decomposed into components that are further indecomposable

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k. \tag{1}$$

Each indecomposable component must be further identified. Let $\mathfrak{g}$ now denote such an indecomposable Lie algebra. A fundamental theorem due to Levi [2, 3] tells us that any finite-dimensional Lie algebra can be represented as the semidirect sum

$$\mathfrak{g} = \mathfrak{l} \niplus \mathfrak{r}, \qquad [\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}, \qquad [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}, \qquad [\mathfrak{l}, \mathfrak{r}] \subseteq \mathfrak{r}, \tag{2}$$

where $\mathfrak{l}$ is semisimple and $\mathfrak{r}$ is the radical of $\mathfrak{g}$, i.e. its maximal solvable ideal. If $\mathfrak{g}$ is simple, we have $\mathfrak{r} = 0$. If $\mathfrak{g}$ is solvable, we have $\mathfrak{l} = 0$.

Semisimple Lie algebras over the field of complex numbers $\mathbb{C}$ have been completely classified by Cartan [4], over the field of real numbers $\mathbb{R}$ by Gantmacher [5] (see, e.g., [6]).

Algorithms realizing decompositions (1), (2) exist [7]. The 'weak' link in the classification of Lie algebras is that not all solvable Lie algebras are known.

There are two ways of proceeding in the classification of Lie algebras, in particular solvable ones: by dimension, or by structure.

The dimensional approach for real Lie algebras was started by Bianchi [8] who classified all real Lie algebras of dimension 2 and 3. Those of dimension 4 were classified by Kruchkovich [9]. Further work in this direction is due to Morozov (nilpotent Lie algebras up to dimension 6) [10], Mubarakzyanov [11–14], Patera *et al* [15] and Turkowski [16, 17]. The classification of low-dimensional Lie algebras over $\mathbb{C}$ was started earlier by Lie himself [18].

The most interesting physical application of the classification of low-dimensional Lie algebras is in general relativity. Indeed, the classification of Einstein spaces according to their isometry groups [19] is based on the work of Bianchi and his successors [8, 9]. The Petrov classification concerns Einstein spaces of dimension 4 and hence involves isometry groups of relatively low dimensions [19, 20].

String theory [21, 22], brane cosmology [23] and some other elementary particle theories going beyond the standard model require the use of higher-dimensional spaces. Any attempt at a Lie group classification of such spaces will require knowledge of higher-dimensional Lie groups, including solvable ones.

It seems to be neither feasible nor fruitful to proceed by dimension in the classification of Lie algebras $\mathfrak{g}$ beyond $\dim \mathfrak{g} = 6$. It is however possible to proceed by structure.

Any solvable Lie algebra $\mathfrak{g}$ has a uniquely defined nilradical $\mathrm{NR}(\mathfrak{g})$, i.e. maximal nilpotent ideal, satisfying

$$\dim \mathrm{NR}(\mathfrak{g}) \geqslant \tfrac{1}{2} \dim \mathfrak{g}. \tag{3}$$

Hence we can consider a given nilpotent algebra of dimension $n$ as a nilradical and then find all of its extensions to solvable Lie algebras. In previous articles this has been performed for the following nilpotent Lie algebras: Heisenberg algebras $\mathfrak{h}(N)$ (where $\dim \mathfrak{h}(N) = 2N + 1, N \geqslant 1$) [24], Abelian Lie algebras $\mathfrak{a}_n, n \geqslant 1$ [25, 26], 'triangular' Lie algebras $\mathfrak{t}(N)$, ($\dim \mathfrak{t}(N) = \frac{N(N-1)}{2}, N \geqslant 2$) [27, 28].

Here we shall consider a class of nilpotent algebras that, for want of a better notation, we shall call $\mathfrak{n}_{n,1}$, where the subscript denotes the dimension of $\mathfrak{n}_{n,1}$, $n = 3, 4, \ldots$. This algebra has an $(n-1)$-dimensional Abelian ideal with the basis $(e_1, \ldots, e_{n-1})$. The Lie brackets are given by

$$\begin{aligned} &[e_j, e_k] = 0, \qquad 1 \leqslant j, k \leqslant n-1, \\ &[e_1, e_n] = 0, \\ &[e_k, e_n] = e_{k-1}, \qquad 2 \leqslant k \leqslant n-1. \end{aligned} \tag{4}$$

Thus the action of the element $e_n$ on the Abelian ideal is given by an indecomposable nilpotent Jordan matrix

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \in F^{(n-1)\times(n-1)}. \tag{5}$$

We shall consider this algebra over the field $F$, where we have $F = \mathbb{R}$, or $F = \mathbb{C}$.

We mention that for $n = 3$ we have $\mathfrak{n}_{3,1} \simeq \mathfrak{h}(1) \simeq \mathfrak{t}(3)$. The algebra $\mathfrak{n}_{4,1}$ is the only four-dimensional indecomposable nilpotent Lie algebra. The algebra $\mathfrak{n}_{n,1}$ exists for any integer $n$ satisfying $n \geqslant 3$ and for $n \geqslant 4$ it is no longer isomorphic to $\mathfrak{h}(N)$ nor $\mathfrak{t}(N)$.

## 2. Mathematical preliminaries

### *2.1. Basic concepts*

Three different series of subalgebras can be associated with any given Lie algebra. The dimensions of the subalgebras in each of these series are important characteristics of the given Lie algebra.

The *derived series* $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \dots \supseteq \mathfrak{g}^{(k)} \supseteq \dots$ is defined recursively

$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \qquad \mathfrak{g}^{(0)} = \mathfrak{g}.$$

If the derived series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is called a *solvable Lie algebra*.

The *lower central series* $\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \dots \supseteq \mathfrak{g}^k \supseteq \dots$ is again defined recursively

$$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \qquad \mathfrak{g}^0 = \mathfrak{g}.$$

If the lower central series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ is called a *nilpotent Lie algebra*. The lowest value of $k$ for which we have $\mathfrak{g}^k = 0$ is the degree of nilpotency of a nilpotent Lie algebra.

Obviously, a nilpotent Lie algebra is also solvable. An Abelian Lie algebra is nilpotent of degree 1.

The *upper central series* is $\mathfrak{z}_1 \subseteq \dots \subseteq \mathfrak{z}_k \subseteq \dots \subseteq \mathfrak{g}$. In this series $\mathfrak{z}_1$ is the *centre* of $\mathfrak{g}$

$$\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0, \forall y \in \mathfrak{g}\}.$$

Now let us consider the factor algebra $\mathfrak{f}_1 \simeq \mathfrak{g}/\mathfrak{z}_1$. Its centre is $C(\mathfrak{f}_1) = C(\mathfrak{g}/\mathfrak{z}_1)$. We define the *second centre* of $\mathfrak{g}$ to be

$$\mathfrak{z}_2 = \mathfrak{z}_1 \oplus C(\mathfrak{g}/\mathfrak{z}_1). \tag{6}$$

Recursively we define higher centres as

$$\mathfrak{z}_{k+1} = \mathfrak{z}_k \oplus C(\mathfrak{g}/\mathfrak{z}_k). \tag{7}$$

For nilpotent Lie algebras the upper central series terminates, i.e. there exists $l$ such that $\mathfrak{z}_l = \mathfrak{g}$. We shall call these three series the *characteristic series* of the algebra $\mathfrak{g}$. We shall use the notations $DS$, $CS$ and $US$ for sets of integers denoting the dimensions of subalgebras in the derived, lower central and upper central series, respectively.

The *centralizer* $\mathfrak{g}_{\mathfrak{h}}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_{\mathfrak{h}} = \{x \in \mathfrak{g} | [x, y] = 0, \forall y \in \mathfrak{h}\}. \tag{8}$$

A *derivation* $D$ of a given Lie algebra $\mathfrak{g}$ is a linear map

$$D : \mathfrak{g} \to \mathfrak{g}$$

such that for any pair $x$, $y$ of elements of $\mathfrak{g}$

$$D([x, y]) = [D(x), y] + [x, D(y)]. \tag{9}$$

If an element $z \in \mathfrak{g}$ exists, such that

$$D = \mathrm{ad}_z, \quad \text{i.e.} \quad D(x) = [z, x], \quad \forall x \in G,$$

the derivation is called an *inner derivation*, any other one is an *outer derivation*.

### 2.2. Solvable Lie algebras with a given nilradical

Any solvable Lie algebra $\mathfrak{s}$ contains a unique maximal nilpotent ideal $\mathfrak{n} = NR(\mathfrak{s})$, the nilradical $\mathfrak{n}$. The dimension of the nilradical satisfies (3) [3]. We will assume that $\mathfrak{n}$ is known. That is, in some basis $(e_1, \ldots, e_n)$ we know the Lie brackets

$$[e_a, e_b] = N_{ab}^c e_c. \tag{10}$$

We wish to extend the nilpotent algebra $\mathfrak{n}$ to all possible indecomposable solvable Lie algebras $\mathfrak{s}$ having $\mathfrak{n}$ as their nilradical. Thus, we add further elements $f_1, \ldots, f_p$ to the basis $(e_1, \ldots, e_n)$ which together form a basis of $\mathfrak{s}$. The derived algebra of a solvable Lie algebra is contained in the nilradical [3], i.e.

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{n}. \tag{11}$$

It follows that the Lie brackets on $\mathfrak{s}$ satisfy

$$[f_i, e_a] = (A_i)_a^b e_b, \qquad 1 \leqslant i \leqslant p, \qquad 1 \leqslant a \leqslant n, \tag{12}$$

$$[f_i, f_j] = \gamma_{ij}^a e_a, \qquad 1 \leqslant i, j \leqslant p. \tag{13}$$

The matrix elements of the matrices $A_i$ must satisfy certain linear relations following from the Jacobi relations between the elements $(f_i, e_a, e_b)$. The Jacobi identities between the triples $(f_i, f_j, e_a)$ will provide linear expressions for the structure constants $\gamma_{ij}^a$ in terms of the matrix elements of the commutators of the matrices $A_i$ and $A_j$.

Since $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{s}$, the matrices $A_i$ must satisfy another condition; no nontrivial linear combination of them is a nilpotent matrix, i.e. they are *linearly nil-independent*.

Let us now consider the adjoint representation of $\mathfrak{s}$, restrict it to the nilradical $\mathfrak{n}$ and find $\mathrm{ad}|_{\mathfrak{n}}(f_k)$. It follows from the Jacobi identities that $\mathrm{ad}|_{\mathfrak{n}}(f_k)$ is a derivation of $\mathfrak{n}$. In other words, finding all sets of matrices $A_i$ in (12) satisfying the Jacobi identities is equivalent to finding all sets of outer nil-independent derivations of $\mathfrak{n}$

$$D^1 = \mathrm{ad}|_{\mathfrak{n}}(f_1), \ldots, D^p = \mathrm{ad}|_{\mathfrak{n}}(f_p). \tag{14}$$

Furthermore, in view of (11), the commutators $[D^j, D^k]$ must be inner derivations of $\mathfrak{n}$. This requirement determines the structure constants $\gamma_{ij}^a$, i.e. the Lie brackets (13), up to elements in the centre $C(\mathfrak{n})$ of $\mathfrak{n}$.

Different sets of derivations may correspond to isomorphic Lie algebras, so redundancies must be eliminated. In terms of the Lie brackets (12) and (13) it means that the matrices $A_i$ and constants $\gamma_{ij}^a$ must be classified into equivalence classes and a representative of each class must be chosen. Equivalence is considered under the following transformations:

$$f_i \to \tilde{f}_i = \rho_{ij} f_j + \sigma_{ia} e_a, \qquad e_a \to \tilde{e}_a = R_{ab} e_b \tag{15}$$

where $\rho$ is an invertible $p \times p$ matrix, $\sigma$ is a $p \times n$ matrix and the invertible $n \times n$ matrix $R$ must be chosen so that the Lie brackets (10) are preserved.

### *2.3. A type of indecomposable nilpotent Lie algebras*

Any nilpotent Lie algebra $\mathfrak{n}$ will contain a maximal Abelian subalgebra $\mathfrak{a}$, not necessarily unique. We have [10]

$$\tfrac{1}{2}(\sqrt{8n+1}-1) \leqslant \dim \mathfrak{a} \leqslant \dim \mathfrak{n} = n. \tag{16}$$

If $\dim \mathfrak{a} = \dim \mathfrak{n}$, then $\mathfrak{n} = \mathfrak{a}$ is Abelian. The case that we are interested in is the next closest to Abelian, namely $\dim \mathfrak{n} = n$, $\dim \mathfrak{a} = n-1$. Let us choose a basis $(e_1, \ldots, e_{n-1}, e_n)$ of $\mathfrak{n}$, where $(e_1, \ldots, e_{n-1})$ is a basis of $\mathfrak{a}$. The Lie brackets for $\mathfrak{n}$ are

$$[e_j, e_k] = 0, \quad 1 \leqslant j, k \leqslant n-1, \qquad [e_k, e_n] = \sum_{l=1}^{n-1} N_{kl} e_l, \qquad 1 \leqslant k \leqslant n-1. \tag{17}$$

The matrix $N \in \mathrm{F}^{(n-1)\times(n-1)}$ must be a nilpotent matrix, otherwise the algebra $\mathfrak{n}$ will not be nilpotent. Elements of the centre $C(\mathfrak{n})$ will correspond to the kernel $\mathrm{Ker} N$ of the matrix $N$. Elements of the derived algebra $\mathfrak{n}^{(1)}$ will correspond to the image $\mathrm{Im}\, N$ of the matrix $N$. In order for the algebra $\mathfrak{n}$ to be indecomposable, we must have

$$\mathrm{Ker}\, N \subseteq \mathrm{Im}\, N. \tag{18}$$

Performing a change of basis within the Abelian algebra $\mathfrak{a}$, we can transform the matrix $N$ to its Jordan canonical form. This can be one indecomposable nilpotent block, or several blocks. Condition (18) forbids the presence of one-dimensional blocks. There are as many mutually nonisomorphic algebras as there are nonequivalent partitions of $n-1$ into sums of positive integers satisfying

$$n-1 = n_1 + n_2 + \cdots + n_l, \qquad n_i \geqslant n_{i-1}, \qquad n_i \neq 1. \tag{19}$$

We shall denote the corresponding Lie algebras $\mathfrak{n}_{n,k}$ where $n = \dim \mathfrak{n}_{n,k}$ and $k$ enumerates the different isomorphy classes for given $n$, i.e. the number of allowed partitions of $n-1$.

The rest of this paper will be devoted to the algebras $\mathfrak{n}_{n,1}$ with the matrix $N_{n,1}$ given by one indecomposable Jordan block. We shall find all extensions of these algebras to solvable Lie algebras with nilradical $\mathfrak{n}_{n,1}$. The Lie brackets for $\mathfrak{n}_{n,1}$ were already given in equation (4).

## 3. Classification of solvable Lie algebras with the nilradical $\mathfrak{n}_{\mathbf{n,1}}$

### *3.1. Nilpotent algebra $\mathfrak{n}_{n,1}$*

The Lie algebra $\mathfrak{n}_{n,1}$ is defined by the Lie brackets (4) of the introduction. We shall consider $n \geqslant 4$. The dimensions of the subalgebras in the characteristic series are

$$DS = [n, n-2, 0], \qquad CS = [n, n-2, n-3, \ldots, 1, 0], \qquad US = [1, 2, \ldots, n-2, n]. \tag{20}$$

Its maximal Abelian ideal $\mathfrak{a}$ can be identified with the centralizer of the highest centre $\mathfrak{z}_{n-2} = \mathrm{span}\{e_1, \ldots, e_{n-2}\}$, i.e. $\mathfrak{a} = \mathrm{span}\{e_1, \ldots, e_{n-1}\}$. Hence $\mathfrak{a}$ is unique.

In order to find all non-nilpotent derivations of $\mathfrak{n}$ we assign to $D$ its matrix

$$D(e_a) = D_{ab} e_b$$

and evaluate condition (9) for basis elements $x = e_i$, $y = e_j$. We find

- $i = 1 < j < n : D_{1n} e_{j-1} = 0$,
- $1 < i < j < n : D_{in} e_{j-1} - D_{jn} e_{i-1} = 0$,
- $i = 1, j = n : \sum_{k=1}^{n-1} D_{1,k+1} e_k = 0$,
- $1 < i < j = n : (D_{i-1,i-1} - D_{ii} - D_{nn}) e_{i-1} + \sum_{k=1, k\neq i-1}^{n-1} (D_{i-1,k} - D_{i,k+1}) e_k = 0$.

From the first and second equations we immediately get

$$D_{in} = 0, 1 \leqslant i < n,$$

from the third

$$D_{1k} = 0, \quad 1 < k.$$

In the last one the coefficients of linearly independent elements in the sum must be zero, therefore considering

$$(D_{i-1,k} - D_{i,k+1}) = 0, \qquad k \neq i - 1$$

we obtain by induction

$$D_{ij} = 0, \qquad D_{ji} = D_{j-1,i-1} = D_{j-i+1,1}, \qquad i < j. \tag{21}$$

The remaining recursion relations

$$D_{i-1,i-1} - D_{ii} - D_{nn} = 0$$

can be most easily solved from the 'lower right corner', denoting

$$D_{nn} = \alpha, \qquad D_{n-1,n-1} = \beta$$

we find

$$D_{ii} = (n - i - 1)\alpha + \beta, \qquad 1 \leqslant i \leqslant n - 1. \tag{22}$$

Thus we have solved the derivation property (9) for all basis elements of $\mathfrak{n}$. Finding the inner derivations is elementary and we may write

**Lemma 1.** *The algebra of derivations of the nilpotent algebra $\mathfrak{n}_{n,1}$ is expressible in the standard basis $(e_1, \ldots, e_n)$ of $\mathfrak{n}_{n,1}$ as the algebra of lower triangular matrices $D$ whose elements satisfy*

$$D_{ii} = (n - i + 1) D_{nn} + D_{n-1,n-1}, \qquad 1 \leqslant i \leqslant n - 1,$$
$$D_{ji} = D_{j-i+1,1}, \qquad 1 \leqslant i < j \leqslant n - 1.$$

*Its subalgebra of inner derivations is*

$$\mathrm{span}\{\mathrm{ad}_{e_2}, \ldots, \mathrm{ad}_{e_n}\},$$

*where*

$$(\mathrm{ad}_{e_j})_{ab} = \delta_{an} \delta_{b,j-1}, \qquad \forall j \leqslant n - 1, \qquad (\mathrm{ad}_{e_n})_{ab} = -\delta_{a,b+1}.$$

### *3.2. Construction of solvable Lie algebras with nilradical $\mathfrak{n}_{n,1}$*

As was explained in section 2.2, to find all solvable Lie algebras with nilradical $\mathfrak{n}_{n,1}$ we must find all nonequivalent nil-independent sets $\{D^1, \ldots, D^p\}$ of derivations $\mathfrak{n}_{n,1}$. The equivalence is generated by the following transformations:

(i) We may add any inner derivation to $D^k$.

(ii) We may perform a change of basis in $\mathfrak{n}$ such that the Lie brackets are not changed. Since any such change must inter alia preserve the lower central series and the subalgebra $\mathfrak{a}$, the matrix of such transformations must be lower triagonal. The preservation of Lie brackets then imposes certain further relations. We may decompose any such transformation into a composition of scaling

$$e_n \to \tilde{e}_n = \omega e_n, \qquad e_k \to \tilde{e}_k = \tau\omega^{n-k-1} e_k, \qquad 1 \leqslant k \leqslant n-1 \tag{23}$$

and the transformation

$$\begin{aligned} e_k \to \tilde{e}_k &= e_k + \sum_{j=1}^{k-1} u_{k-j} e_j, \qquad 1 \leqslant k \leqslant n-1, \qquad u_1, \ldots, u_{n-2} \in \mathrm{F}, \\ e_n \to \tilde{e}_n &= e_n + \sum_{j=1}^{n} v_j e_j, \qquad v_1, \ldots, v_{n-1} \in \mathrm{F}. \end{aligned} \tag{24}$$

(To prove that (24) gives a general form of such a transformation it is sufficient to consider $[\tilde{e}_k, e_n] = [\tilde{e}_k, \tilde{e}_n] = \tilde{e}_{k-1}$ and to use induction on $k$.) The scaling (23) acts on $D^k$ as

$$D^k \to S D^k S^{-1}$$

where $S = \mathrm{diag}(\tau\omega^{n-2}, \tau\omega^{n-3}, \ldots, \tau, \omega)$, transformation (24) as

$$D^k \to U D^k U^{-1}$$

where

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ u_1 & 1 & 0 & 0 & \ldots & 0 \\ u_2 & u_1 & 1 & 0 & \ldots & 0 \\ & \ddots & \ddots & \ddots & & \\ u_{n-2} & \ldots & u_2 & u_1 & 1 & 0 \\ v_1 & \ldots & v_{n-3} & v_{n-2} & v_{n-1} & 1 \end{pmatrix}.$$

(iii) We can change the basis in the space $\mathrm{span}\{D^1, \ldots, D^p\}$.

By adding inner derivations we can transform all $D^k$ into the form

$$D^k = \begin{pmatrix} d_1^k & 0 & 0 & 0 & \ldots & 0 \\ 0 & d_2^k & 0 & 0 & \ldots & 0 \\ a_3^k & 0 & d_3^k & 0 & \ldots & 0 \\ & \ddots & \ddots & \ddots & & \\ a_{n-1}^k & \ldots & a_3^k & 0 & \beta^k & 0 \\ 0 & \ldots & 0 & 0 & a_n^k & \alpha^k \end{pmatrix}, \tag{25}$$

where

$$d_1^k = (n-2)\alpha^k + \beta^k, \ldots, \qquad d_j^k = (n-1-j)\alpha^k + \beta^k, \ldots, \qquad d_{n-2}^k = \alpha^k + \beta^k.$$

We shall assume that $D^k$ are always brought to this form.

We see that the number of nil-independent elements $p$ can be at most two since a set of three or more derivations of the form (25) cannot be linearly nil-independent.

*Case 1.* $p = 1$ The entire structure of the associated solvable Lie algebra is encoded in the matrix $D$. The Lie brackets of the non-nilpotent element $f$ with nilpotent elements are given by

$$[f, e_k] = D(e_k) = D_{kl} e_l.$$

We shall divide our investigation into subcases determined by values of the parameters $\alpha, \beta$, at least one of which must be nonzero.

(i) $\alpha \neq 0$. We rescale $D$ to put $\alpha = 1$. Then by a change of basis (24) in $\mathfrak{n}$

$$\tilde{e}_k = e_k - \frac{1}{l-1} a_l e_{k-l+1}, \qquad l \leqslant k \leqslant n-1,$$

$$\tilde{e}_k = e_k, \qquad 1 \leqslant k \leqslant l-1$$

we put to zero first $a_3$, then $a_4$ etc up to $a_{n-1}$. From now on we assume that $a_k = 0$, $k \leqslant n-1$. If $\beta \neq 1$ (=$\alpha$) then a further change of basis

$$\tilde{e}_n = e_n - \frac{a_n}{\beta - 1} e_{n-1} \tag{26}$$

turns $a_n$ into zero and the matrix $D$ is diagonal

$$D = \mathrm{diag}(n-2+\beta, n-3+\beta, \ldots, \beta, 1). \tag{27}$$

If $\beta = 1$ then $a_n$ cannot be removed. The only remaining transformation is scaling (23) which allows us to scale any nonzero $a_n$ to 1. Therefore we find in addition to (27) with $\beta = 1$ another possibility, namely

$$D = \begin{pmatrix} n-1 & 0 & \ldots & 0 & 0 \\ 0 & n-2 & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 1 & 1 \end{pmatrix}. \tag{28}$$

(ii) $\alpha = 0$ We rescale $D$ to put $\beta = 1$. We use (24) to change $e_n$

$$\tilde{e}_n = e_n - a_n e_{n-1}$$

and transform $a_n$ into $a_n = 0$. If $D$ is diagonal, it cannot be further simplified. Let us assume that $D$ is not diagonal. For $\alpha = 0$ the matrix $D$ is invariant with respect to transformations (24) preserving $a_n = 0$, i.e. the parameters $a_k$ cannot be removed. The only transformation we still have at our disposal is scaling (23) which allows us to scale one chosen nonzero $a_k$ to 1 over the field $\mathbb{C}$. Over $\mathbb{R}$ one value $a_k$ can be scaled to 1 if $k$ is even, or to $\pm 1$ if $k$ is odd.

*Case 2.* $p = 2$. By taking linear combinations of $D^1, D^2$ we obtain $\alpha^1 = 1, \beta^1 = 0, \alpha^2 = 0, \beta^2 = 1$. Further by a change of basis in $\mathfrak{n}$ (24) we take $D^1$ to its canonical form

$$D^1 = \mathrm{diag}(n-2, n-3, \ldots, 2, 1, 0, 1)$$

found for $p = 1$. In order to define a solvable Lie algebra $\mathfrak{g}$ with nilradical $\mathfrak{n}$, the two derivations $D^1, D^2$ must commute to an inner derivation

$$[D^1, D^2] \in \mathrm{span}\{\mathrm{ad}_{e_2}, \ldots, \mathrm{ad}_{e_n}\}.$$

Computing the commutator for the above given forms of $D^1$, $D^2$ (note that $D^1$ is diagonal) we immediately find that

$$a_j^2 = 0, \qquad 3 \leqslant j \leqslant n$$

must hold.

Therefore there is a single canonical form of $D^1$, $D^2$

$$D^1 = \mathrm{diag}(n-2, n-3, \ldots, 2, 1, 0, 1), \qquad D^2 = \mathrm{diag}(1, 1, \ldots, 1, 1, 0). \tag{29}$$

The corresponding solvable Lie algebra is now almost specified, the Lie brackets of non-nilpotent elements $f_1$, $f_2$ being

$$[f_1, e_k] = (n-1-k)e_k, \qquad 1 \leqslant k < n, \qquad [f_1, e_n] = e_n,$$

$$[f_2, e_k] = e_k, \qquad 1 \leqslant k < n, \qquad [f_2, e_n] = 0.$$

It remains to fix the Lie bracket between $f_1$, $f_2$. Because $D^1$ and $D^2$ are commuting matrices representing $f_1$, $f_2$ in the adjoint representation of $\mathfrak{g}$ restricted to the ideal $\mathfrak{n}$, the Lie bracket of $f_1$, $f_2$ must be in the kernel of the representation map, i.e. in the centre of $\mathfrak{n}$

$$[f_1, f_2] = \gamma e_1. \tag{30}$$

The transformation

$$f_1 \to \tilde{f}_1 = f_1 + \gamma e_1$$

takes $\gamma$ in equation (30) into $\gamma = 0$ while leaving all other Lie brackets invariant. We conclude that in the case $p = 2$ the solvable Lie algebra with nilradical $\mathfrak{n}_{n,1}$ is unique.

### *3.3. Standard forms of solvable Lie algebras with nilradical* $\mathfrak{n}_{n,1}$

The results obtained above can be summed up as theorems. We give them for algebras over the field $F = \mathbb{R}$ or $\mathbb{C}$. We specify $F = \mathbb{R}$, or $\mathbb{C}$ only when the two cases differ. In all cases we give the dimensions of the subalgebras in the characteristic series. These dimensions are basis independent and are very useful for identifying the Lie algebras. The nilradical in all cases is $\mathfrak{n}_{n,1}$ with the Lie brackets (4). We shall specify the action of the non-nilpotent elements $f$ or $f_1$ and $f_2$ on the nilradical (see equation (12)).

In the theorems, 'solvable' will always mean solvable, indecomposable, non-nilpotent.

**Theorem 1.** *Any solvable Lie algebra* $\mathfrak{s}$ *with nilradical* $\mathfrak{n}_{n,1}$ *will have dimension* $\dim \mathfrak{s} = n+1$*, or* $\dim \mathfrak{s} = n+2$.

**Theorem 2.** *Three types of solvable Lie algebras of dimension* $\dim \mathfrak{s} = n+1$ *exist for any* $n \geqslant 4$. *They are represented by the following:*

*(i)* $A = A_1$ *in equation (12) diagonal*

$$[f, e_k] = ((n-k-1)\alpha + \beta)e_k, \qquad 1 \leqslant k \leqslant n-1, \qquad [f, e_n] = \alpha e_n. \tag{31}$$

*The mutually nonisomorphic algebras of this type are*

$$\mathfrak{s}_{n+1,1}(\beta): \quad \alpha = 1, \qquad \beta \in \mathrm{F}\backslash\{0, n-2\}, \tag{32}$$

$$DS = [n+1, n, n-2, 0], \qquad CS = [n+1, n, n, \ldots], \qquad US = [0],$$

$$\mathfrak{s}_{n+1,2}: \quad \alpha = 1, \qquad \beta = 0, \tag{33}$$

$$DS = [n+1, n-1, n-3, 0], \qquad CS = [n+1, n-1, n-1, \ldots], \qquad US = [0],$$

$$\mathfrak{s}_{n+1,3}: \quad \alpha = 1, \qquad \beta = 2-n, \tag{34}$$

$$DS = [n+1, n, n-2, 0], \qquad CS = [n+1, n, n, \ldots], \qquad US = [1, 1, \ldots],$$

$$\mathfrak{s}_{n+1,4}: \quad \alpha = 0, \qquad \beta = 1, \tag{35}$$

$$DS = [n+1, n-1, 0], \qquad CS = [n+1, n-1, n-1, \ldots], \qquad US = [0].$$

*(ii)* $A = A_1$ *in equation (12) nondiagonal, its diagonal determined by* $\alpha = \beta = 1$. *We have*

$$\begin{aligned} &\mathfrak{s}_{n+1,5}: \quad [f, e_k] = (n-k)e_k, \qquad 1 \leqslant k \leqslant n-1, \qquad [f, e_n] = e_n + e_{n-1}, \\ &DS = [n+1, n, n-2, 0], \qquad CS = [n+1, n, n, \ldots], \qquad US = [0]. \end{aligned} \tag{36}$$

*(iii)* $A = A_1$ *in equation (12) nondiagonal, its diagonal determined by* $\alpha = 0, \beta = 1$.

$$\begin{aligned} &\mathfrak{s}_{n+1,6}(a_3, \ldots, a_{n-1}) : [f, e_k] = e_k + \sum_{l=1}^{k-2} a_{k-l+1} e_l, \qquad 1 \leqslant k \leqslant n-1, \\ &[f, e_n] = 0, \end{aligned} \tag{37}$$

$a_j \in \mathrm{F}$, *at least one* $a_j$ *satisfies* $a_j \neq 0$.
*Over* $\mathbb{C}$*: the first nonzero* $a_j$ *satisfies* $a_j = 1$.
*Over* $\mathbb{R}$*: the first nonzero* $a_j$ *for even* $j$ *satisfies* $a_j = 1$. *If all* $a_j = 0$ *for* $j$ *even, then the first nonzero* $a_j$ *(*$j$ *odd) satisfies* $a_j = \pm 1$. *We have*

$$DS = [n+1, n-1, 0], \qquad CS = [n+1, n-1, n-1, \ldots], \qquad US = [0].$$

**Theorem 3.** *Precisely one class of solvable Lie algebras* $\mathfrak{s}_{n+2}$ *of* $\dim \mathfrak{s} = n+2$ *with nilradical* $\mathfrak{n}_{n,1}$ *exists. It is represented by a basis* $(e_1, \ldots, e_n, f_1, f_2)$ *and the Lie brackets involving* $f_1$ *and* $f_2$ *are*

$$\begin{aligned} &[f_1, e_k] = (n-1-k)e_k, \qquad 1 \leqslant k \leqslant n-1, \qquad [f_1, e_n] = e_n, \\ &[f_2, e_k] = e_k, \qquad 1 \leqslant k \leqslant n-1, \qquad [f_2, e_n] = 0, \qquad [f_1, f_2] = 0. \end{aligned} \tag{38}$$

*For this algebra*

$$DS = [n+2, n, n-2, 0], \qquad CS = [n+2, n, n, \ldots], \qquad US = [0]. \tag{39}$$

## 4. Generalized Casimir invariants

### *4.1. General method*

The term *Casimir operator*, or *Casimir invariant*, is usually reserved for elements of the centre of the enveloping algebra of a Lie algebra $\mathfrak{g}$ [29, 30]. These operators are in one-to-one correspondence with polynomial invariants characterizing orbits of the coadjoint representation of $\mathfrak{g}$ [31]. The search for invariants of the coadjoint representation is algorithmic and amounts to solving a system of linear first-order partial differential equations [32–36, 15, 24, 26, 28]. Alternatively, global properties of the coadjoint representation can be used [36]. In general, solutions are not necessarily polynomials and we shall call the nonpolynomial solutions *generalized Casimir invariants*. For certain classes of Lie algebras, including semisimple Lie algebras, perfect Lie algebras, nilpotent Lie algebras and more generally algebraic Lie algebras, all invariants of the coadjoint representation are functions of polynomial ones [32, 33].

Casimir invariants are of primordial importance in physics. They represent such important quantities as angular momentum, elementary particle's mass and spin, Hamiltonians of various physical systems etc.

In the representation theory of solvable Lie algebras the invariants are not necessarily polynomials, i.e. they can be genuinely generalized Casimir invariants. In addition to their importance in representation theory, they may occur in physics. Indeed, Hamiltonians and integrals of motion for classical integrable Hamiltonian systems are not necessarily polynomials in the momenta [37, 38], though typically they are invariants of some group action.

In order to calculate the (generalized) Casimir invariants we consider some basis $(g_1, \ldots, g_n)$ of $\mathfrak{g}$, in which the structure constants are $c_{ij}^k$. A basis for the coadjoint representation is given by the first-order differential operators

$$\hat{G}_k = g_b c_{ka}^b \frac{\partial}{\partial g_a}. \tag{40}$$

In equation (40) the quantities $g_a$ are commuting independent variables which can be identified with coordinates in the dual basis of the space $\mathfrak{g}^*$, dual to the algebra $\mathfrak{g}$.

The invariants of the coadjoint representation, i.e. the generalized Casimir invariants, are solutions of the following system of partial differential equations:

$$\hat{G}_k I(g_1, \ldots, g_n) = 0, \qquad k = 1, \ldots, n. \tag{41}$$

The number of functionally independent solutions of system (41) is

$$n_I = n - r(C) \tag{42}$$

where $C$ is the antisymmetric matrix

$$C = \begin{pmatrix} 0 & c_{12}^b g_b & \ldots & c_{1n}^b g_b \\ -c_{12}^b g_b & 0 & \ldots & c_{2n}^b g_b \\ \vdots & \vdots & & \\ -c_{1,n-1}^b g_b & \ldots & 0 & c_{n-1,n}^b g_b \\ -c_{1n}^b g_b & \ldots & -c_{n-1,n}^b g_b & 0 \end{pmatrix} \tag{43}$$

and $r(C)$ is the generic rank of $C$. Since $C$ is antisymmetric, its rank is even. Hence $n_I$ has the same parity as $n$.

Since the method of computation is generally known, we shall not present details and just give the results in the form of theorems. In all cases proofs consist of a direct calculation, i.e. solving equations (41).

### 4.2. *The generalized Casimir invariants*

The differential operators corresponding to the basis elements of $\mathfrak{n}_{n,1}$ are

$$\hat{E}_1 = 0, \qquad \hat{E}_k = e_{k-1}\frac{\partial}{\partial e_n}, \qquad 1 < k < n, \qquad \hat{E}_n = -\sum_{k=2}^{n-1} e_{k-1}\frac{\partial}{\partial e_k}. \tag{44}$$

The form of $\hat{E}_k$, $(1 < k < n)$ implies that the invariants do not depend on $e_n$. Solving the equation $\hat{E}_n I(e_1, e_2, \ldots, e_{n-1}) = 0$ by the method of characteristics, we obtain the following result

**Theorem 4.** *The nilpotent Lie algebra $\mathfrak{n}_{n,1}$ has $n-2$ functionally independent invariants. They can be chosen to be the following polynomials:*

$$\begin{aligned} \xi_0 &= e_1, \\ \xi_k &= \frac{(-1)^k k}{(k+1)!} e_2^{k+1} + \sum_{j=0}^{k-1} (-1)^j \frac{e_2^j e_{k+2-j} e_1^{k-j}}{j!}, \qquad 1 \leqslant k \leqslant n-3. \end{aligned} \tag{45}$$

Let us now consider the $(n+1)$-dimensional solvable Lie algebras of theorem 2. The operators $\hat{E}_i$ representing $\mathfrak{n}_{n,1}$ will each contain an additional term involving a derivative with respect to $f$. However, from the form of these operators we see that the invariants cannot depend on $f$. Moreover, they can only depend on the invariants (45) of $\mathfrak{n}_{n,1}$. To find the invariants of the algebras $\mathfrak{s}_{n+1,k}$ we must represent the element $f \in \mathfrak{s}_{n+1,k}$ by the appropriate 'truncated' differential operator $\hat{F}_T$ (by 'truncated' we mean that we keep only the part acting on $e_1, \ldots, e_{n-1}$). We must then solve the equation

$$\hat{F}_T I(\xi_0, \ldots, \xi_{n-3}) = 0. \tag{46}$$

For the algebras $\mathfrak{s}_{n+1,1}, \ldots, \mathfrak{s}_{n+1,5}$ of theorem 2, we have

$$\hat{F}_T = \sum_{k=1}^{n-1} ((n-1-k)\alpha + \beta) e_k \frac{\partial}{\partial e_k} \tag{47}$$

with $\alpha$ and $\beta$ as in theorem 2. For $\mathfrak{s}_{n+1,6}(a_3, \ldots, a_{n-1})$ we have

$$\hat{F}_T = e_1 \frac{\partial}{\partial e_1} + e_2 \frac{\partial}{\partial e_2} + \sum_{l=1}^{n-3} \left( e_{l+2} + \sum_{j=1}^{l} a_{l+3-j} e_j \right) \frac{\partial}{\partial e_{l+2}}. \tag{48}$$

Solving equation (46) in each case we obtain the following result.

**Theorem 5.** *The algebras $\mathfrak{s}_{n+1,1}(\beta), \ldots, \mathfrak{s}_{n+1,5}$ have $n-3$ invariants each. Their form is*

*(i)* $\mathfrak{s}_{n+1,1}(\beta), \mathfrak{s}_{n+1,2}$ *and* $\mathfrak{s}_{n+1,5}$

$$\chi_k = \frac{\xi_k}{\xi_0^{(k+1)\frac{n-3+\beta}{n-2+\beta}}}, \qquad 1 \leqslant k \leqslant n-3. \tag{49}$$

*For $\mathfrak{s}_{n+1,2}$ and $\mathfrak{s}_{n+1,5}$ we have $\beta = 0$ and $\beta = 1$, respectively, in equation (49).*

*(ii)* $\mathfrak{s}_{n+1,3}$

$$\chi_1 = \xi_0, \qquad \chi_k = \frac{\xi_k^2}{\xi_1^{k+1}}, \qquad 2 \leqslant k \leqslant n-3. \tag{50}$$

*(iii)* $\mathfrak{s}_{n+1,4}$

$$\chi_k = \frac{\xi_k}{\xi_0^{k+1}}, \qquad 1 \leqslant k \leqslant n-3. \tag{51}$$

*(iv)* $\mathfrak{s}_{n+1,6}(a_3, \ldots, a_{n-1})$

$$\chi_k = \sum_{m=0}^{[\frac{k+1}{2}]} (-1)^m \frac{(\ln \xi_0)^m}{m!} \left( \sum_{i_1+\ldots+i_m=k-2m+1} a_{i_1+3} a_{i_2+3} \ldots a_{i_m+3} \right. \tag{52}$$
$$\left. + \sum_{j+i_1+\ldots+i_m=k-2m-1} \frac{\xi_{j+1}}{\xi_0^{j+2}} a_{i_1+3} a_{i_2+3} \ldots a_{i_m+3} \right), \qquad 1 \leqslant k \leqslant n-3.$$

*The summation indices take the values $0 \leqslant j, i_1, \ldots, i_m \leqslant k+1$.*

Finally, let us consider the $(n+2)$-dimensional algebra $\mathfrak{s}_{n+2}$. The invariants can again depend only on $\xi_0, \ldots, \xi_{n-3}$. We have two additional truncated differential operators, namely

$$\hat{F}_{1T} = \sum_{k=1}^{n-1} (n-1-k) e_k \frac{\partial}{\partial e_k}, \qquad \hat{F}_{2T} = \sum_{k=1}^{n-1} e_k \frac{\partial}{\partial e_k}. \tag{53}$$

Imposing two equations of form (46) we obtain the following result.

**Table 1.** Number of linearly nil-independent elements that can be added to the nilradical.

| Nilradical $\mathfrak{n}$ | dim $\mathfrak{n}$ | | $p_{\max}$ |
|---|---|---|---|
| $\mathfrak{n}_{n,1}$ | $n \geqslant 4$ | | 2 |
| $\mathfrak{a}_n$ | $n \geqslant 3$ | | $n-1$ |
| $\mathfrak{a}_1$ | $n = 1$ | | 1 |
| $\mathfrak{a}_2$ | $n = 2$ | | $1\,(\mathbb{C}), 2\,(\mathbb{R})$ |
| $\mathfrak{h}(N)$ | $n = 2N+1,$ | $N \geqslant 1$ | $N+1$ |
| $\mathfrak{t}(N)$ | $n = \frac{(N-1)N}{2},$ | $N \geqslant 2$ | $N-1$ |

**Theorem 6.** *The Lie algebra $\mathfrak{s}_{n+2}$ of theorem 3 has $n-4$ functionally independent invariants that can be chosen to be*

$$\chi_k = \frac{\xi_{k+1}}{\xi_1^{\frac{k+2}{2}}}, \qquad 1 \leqslant k \leqslant n-4. \tag{54}$$

We see that for the algebra $\mathfrak{s}_{n+1,6}(a_3, \ldots, a_{n-1})$ the invariants involve powers of the logarithm $\ln \xi_0$. In all other cases we obtain sets of ratios of powers of $\xi_k$.

A specific class of solvable Lie algebras, namely rigid ones, was considered by Campoamor-Stursberg [35], who calculated their generalized Casimir invariants for dimensions up to $N = 8$ inclusively. Our algebras $\mathfrak{s}_{n+2}$ fall into this category (with $N = n+2$). Our results for $n \leqslant 6$ agree with those of [35].

## 5. Conclusions

The main results of this paper are summed up in theorems 1, 2 and 3 of section 3 and theorems 4, 5 and 6 of section 4.

The results on the structure of indecomposable solvable Lie algebras with the nilradical $\mathfrak{n}_{n,1}$ are quite simple and it is interesting to compare them with results for other nilradicals. This comparison is performed in table 1. There $\mathfrak{a}_n$ denotes an $n$-dimensional Abelian Lie algebra, $\mathfrak{h}(N)$ a Heisenberg algebra in an $N$-dimensional space and $\mathfrak{t}(N)$ is the subalgebra of strictly upper triangular matrices of $\mathfrak{sl}(N, \mathrm{F})$. In the third column $p_{\max}$ is the maximal number of non-nilpotent elements we can add in order to obtain an indecomposable solvable Lie algebra. Note that $p_{\max}$ is independent of the dimension of the nilradical only for $\mathfrak{n}_{n,1}$.

The results on generalized Casimir invariants are also quite simple. For the nilradical $\mathfrak{n}_{n,1}$ the number of invariants is $n_I = n-2-p$, where $p$ is the number of non-nilpotent elements (i.e. $p = 1$ or $p = 2$). In comparison, for the Abelian nilradical $\mathfrak{a}_n$, the number of invariants is $n_I = n - p$. In both cases the invariants depend only on elements of the nilradical and can be polynomials, ratios of powers of polynomials, or may involve logarithms.

Finally, a few words about applications of the Lie algebras obtained above. The algebras $\mathfrak{s}_{n+1,j}$ of theorem 2, for $n = 4$, appear in Petrov's classification [19] of gravitational fields admitting groups of motion (isometry groups) of dimension 5. The fact that we have a complete list of all such Lie algebras for arbitrary $n$ would enable us to construct the corresponding invariant Riemann, or pseudo-Riemann metrics in spaces of arbitrary dimension. An invariant metric then makes it possible to write invariant classical, or quantum integrable systems in such spaces and to investigate the separation of variables in Hamilton–Jacobi and Schrödinger equations.

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# 2.2

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# All solvable extensions of a class of nilpotent Lie algebras of dimension $n$ and degree of nilpotency $n-1$

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**Abstract**
We construct all solvable Lie algebras with a specific $n$-dimensional nilradical $\mathfrak{n}_{n,2}$ (of degree of nilpotency $n-1$ and with an $(n-2)$-dimensional maximal Abelian ideal). We find that for given $n$ such a solvable algebra is unique up to isomorphisms. Using the method of moving frames we construct a basis for the Casimir invariants of the nilradical $\mathfrak{n}_{n,2}$. We also construct a basis for the generalized Casimir invariants of its solvable extension $\mathfrak{s}_{n+1}$ consisting entirely of rational functions of the chosen invariants of the nilradical.

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## 1. Introduction

The purpose of this paper is to construct all solvable Lie algebras with a specific nilradical that in an appropriate basis $(e_1, \ldots, e_n)$ has the Lie brackets

$$\begin{aligned}
&[e_j, e_k] = 0, && 1 \leqslant j, k \leqslant n-2, \\
&[e_1, e_{n-1}] = [e_2, e_{n-1}] = 0, && \\
&[e_k, e_{n-1}] = e_{k-2}, && 3 \leqslant k \leqslant n-2, \\
&[e_1, e_n] = 0, && \\
&[e_k, e_n] = e_{k-1}, && 2 \leqslant k \leqslant n-1.
\end{aligned} \tag{1}$$

The nilpotent Lie algebra (1) of dimension $n$ exists for all $n \geqslant 5$, has degree of nilpotency $n-1$ and has a uniquely defined maximal Abelian ideal $\mathfrak{a}$ of dimension $n-2$, equal to its derived algebra.

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This paper is part of a research program devoted to the classification of Lie algebras over the fields of complex and real numbers. Levi's theorem [27, 30] tells us that any finite-dimensional Lie algebra is isomorphic to a semidirect sum of a semisimple Lie algebra and a solvable one. The semisimple Lie algebras have been classified [17, 23] (for a more recent reference, see [25]) and Levi's theorem reduces the classification of all Lie algebras to the classification of solvable ones and some representation theory of the semisimple ones.

Solvable Lie algebras cannot be completely classified. Mubarakzyanov has provided a classification of real and complex Lie algebras of dimension $n \leqslant 5$ [33, 34] (equivalent in dimension 3 to previous classifications by Bianchi [6] and Lie [31]) and a partial classification for $n = 6$ [35]. The classification for $n = 6$ was continued by Turkowski [53] who also considered the classification of semidirect sums of semisimple and solvable Lie algebras [52, 54].

The method of classifying and constructing solvable Lie algebras used by Mubarakzyanov and Turkowski was based on the fact that every solvable Lie algebra has a uniquely defined nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{s})$ [27]. Its dimension satisfies [33]

$$\dim \mathfrak{n} \geqslant \tfrac{1}{2} \dim \mathfrak{s}. \tag{2}$$

Hence we can consider a given nilpotent Lie algebra $\mathfrak{n}$ of dimension $n$ and classify all of its extensions to solvable Lie algebras. This is an open-ended task since infinitely many different series of nilpotent Lie algebras exist and they themselves have not been classified.

Nilpotent Lie algebras of dimension $n = 6$ over complex numbers were classified by Umlauf [55], over an arbitrary field of characteristic zero by Morozov [32] who also gives a list of all lower dimensional ones. Those of dimensions $n = 7$ and $n = 8$ were classified by Safiulina [46] and Tsagas [50], respectively, and some results for $n = 9$ are known [51]. We mention that the number of nonequivalent nilpotent algebras increases very rapidly with their dimension $n$ and for $n \geqslant 7$ becomes infinite, i.e. classes of nonisomorphic nilpotent algebras depending on parameters arise. For reviews of this field of research with extensive bibliographies, see e.g. [24, 28]. For classifications of specific type of nilpotent Lie algebras with $n \leqslant 9$ see e.g. [3, 4, 24, 28].

A more manageable task is to start from series of nilpotent Lie algebras that already exist in low dimensions. Thus, for $n = 1, 2$ a nilpotent Lie algebra must be Abelian. For $n = 2k + 1, k \geqslant 1$ another series exists, namely the Heisenberg algebras. A further series exists for all $n \geqslant 4$. For lack of better name it was called $\mathfrak{n}_{n,1}$ in our earlier article [47]. Six inequivalent indecomposable nilpotent Lie algebras exist for $n = 5$, among them a Heisenberg algebra, the algebra $\mathfrak{n}_{5,1}$ and also the algebra which we shall denote $\mathfrak{n}_{5,2}$ (and more generally $\mathfrak{n}_{n,2}, n \geqslant 5$) with the commutation relations as in equation (1) above.

Earlier articles were devoted to solvable extensions of Heisenberg algebras [45], Abelian Lie algebras [36, 37], 'triangular' Lie algebras [48, 49] and the algebras $\mathfrak{n}_{n,1}$ [47].

The motivation for providing a classification of Lie algebras was discussed in our previous article [47]. Let us mention that string theory and other elementary particle theories require the use of higher dimensional spaces. A classification of such spaces, analogous to the Petrov classification of Einstein spaces [42] is based on the classification of higher dimensional Lie groups, in particular the solvable ones.

A more general reason why a classification of Lie algebras (and by extension Lie groups) is needed in physics is that Lie groups typically occur as groups of transformations of the solution space of some equations. These equations may be of differential, difference, integral or some other type. Different systems may have isomorphic symmetry groups and in this case the results obtained for one theory can be transferred to another one.

In the representation theory of Lie algebras and Lie groups an important role is played by Casimir operators or generalized Casimir operators, i.e. polynomial and nonpolynomial invariants of the coadjoint representation. Casimir invariants, corresponding to elements of the center of the enveloping algebra of a Lie algebra, are of primordial importance in physics. They represent such important quantities as angular momentum, elementary particle mass and spin, Hamiltonians of various physical systems, etc. Also the generalized Casimir invariants occur in physics. Indeed, Hamiltonians and integrals of motion for classical integrable Hamiltonian systems are not necessarily polynomials in the momenta [26, 44], though typically they are invariants of some group action.

Part of our classification program [36, 37, 45, 47–49] is to construct a basis, i.e. a maximal set of functionally independent generalized Casimir operators of each algebra obtained in the classification. We do this for the algebras $\mathfrak{n}_{n,2}$ below in section 4.

The present paper is organized as follows. Section 2 is devoted to some mathematical preliminaries. In section 3 we present the complete classification of solvable Lie algebras with the nilradical $\mathfrak{n}_{n,2}$ and show that precisely one non-nilpotent element can be added. This element has a diagonal action on the nilradical. In section 4 we first calculate the Casimir invariants of the nilpotent algebras $\mathfrak{n}_{n,2}$, i.e. the polynomial invariants of the coadjoint representation of $\mathfrak{n}_{n,2}$. There exist $n-4$ functionally independent Casimir invariants $\xi_0 = e_1, \xi_1, \ldots, \xi_{n-5}$. We then show that the solvable Lie algebra $\mathfrak{s}_{n+1}$ has exactly $n-5$ functionally independent generalized Casimir operators which can be chosen in the form

$$\chi_j = \frac{\xi_j^n}{\xi_0^{(n-1)(j+2)}}, \qquad 1 \leqslant j \leqslant n-5. \tag{3}$$

## 2. Mathematical preliminaries

### *2.1. Basic concepts*

Three different series of subalgebras can be associated with any given Lie algebra. The dimensions of the subalgebras in each of these series are important characteristics of the given Lie algebra.

The *derived series* $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \cdots \supseteq \mathfrak{g}^{(k)} \supseteq \cdots$ is defined recursively

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \qquad \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \qquad k \geqslant 1. \tag{4}$$

If the derived series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is called a *solvable Lie algebra*.

The *lower central series* $\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots \supseteq \mathfrak{g}^k \supseteq \cdots$ is again defined recursively

$$\mathfrak{g}^1 = \mathfrak{g}, \qquad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \qquad k \geqslant 2. \tag{5}$$

If the lower central series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ is called a *nilpotent Lie algebra*. The highest value of $k$ for which we have $\mathfrak{g}^k \neq 0$ is the degree of nilpotency of a nilpotent Lie algebra.

Obviously, a nilpotent Lie algebra is also solvable. An Abelian Lie algebra is nilpotent of degree 1.

The *upper central series* is $\mathfrak{z}_1 \subseteq \cdots \subseteq \mathfrak{z}_k \subseteq \cdots \subseteq \mathfrak{g}$. In this series $\mathfrak{z}_1$ is the *center* of $\mathfrak{g}$

$$\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [x, y] = 0, \forall y \in \mathfrak{g}\}. \tag{6}$$

Further we define recursively $\mathfrak{z}_k$ as the unique ideal in $\mathfrak{g}$ such that $\mathfrak{z}_k/\mathfrak{z}_{k-1}$ is the center of $\mathfrak{g}/\mathfrak{z}_{k-1}$. The upper central series terminates, i.e. a number $k$ exists such that $\mathfrak{z}_k = \mathfrak{g}$, if and only if $\mathfrak{g}$ is nilpotent [27].

We shall call these three series the *characteristic series* of the algebra $\mathfrak{g}$. We shall use the notations DS, CS and US for (ordered) lists of integers denoting the dimensions of subalgebras in the derived, lower central and upper central series, respectively. We list the last (then repeated) entry only once (e.g., we write CS $= [n, n-1]$ rather than CS $= [n, n-1, n-1, n-1, \ldots]$).

The *centralizer* $\mathfrak{g}_\mathfrak{h}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_\mathfrak{h} = \{x \in \mathfrak{g} \,|\, [x, y] = 0, \forall y \in \mathfrak{h}\}. \tag{7}$$

An *automorphism* $\Phi$ of a given Lie algebra $\mathfrak{g}$ is a bijective linear map

$$\Phi : \mathfrak{g} \to \mathfrak{g}$$

such that for any pair $x, y$ of elements of $\mathfrak{g}$

$$\Phi([x, y]) = [\Phi(x), \Phi(y)]. \tag{8}$$

We recall that all automorphisms of $\mathfrak{g}$ form a Lie group Aut($\mathfrak{g}$). Its Lie algebra is then the algebra of *derivations* of $\mathfrak{g}$, i.e. of linear maps

$$D : \mathfrak{g} \to \mathfrak{g}$$

such that for any pair $x, y$ of elements of $\mathfrak{g}$

$$D([x, y]) = [D(x), y] + [x, D(y)]. \tag{9}$$

If an element $z \in \mathfrak{g}$ exists, such that

$$D = \mathrm{ad}(z), \qquad \text{i.e.} \quad D(x) = [z, x], \quad \forall x \in G,$$

the derivation is called an *inner derivation*, any other one is an *outer derivation*.

### 2.2. Solvable Lie algebras with a given nilradical

Any solvable Lie algebra $\mathfrak{s}$ contains a unique maximal nilpotent ideal $\mathfrak{n} = \mathrm{NR}(\mathfrak{s})$, the *nilradical* $\mathfrak{n}$. We will assume that $\mathfrak{n}$ is known. That is, in some basis $(e_1, \ldots, e_n)$ of $\mathfrak{n}$ we know the Lie brackets

$$[e_j, e_k] = N^l_{jk} e_l \tag{10}$$

(summation over repeated indices applies). We wish to extend the nilpotent algebra $\mathfrak{n}$ to all possible indecomposable solvable Lie algebras $\mathfrak{s}$ having $\mathfrak{n}$ as their nilradical. Thus, we add further elements $f_1, \ldots, f_f$ to the basis $(e_1, \ldots, e_n)$ which together form a basis of $\mathfrak{s}$. The derived algebra of a solvable Lie algebra is contained in the nilradical (see [27]), i.e.

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{n}. \tag{11}$$

It follows that the Lie brackets on $\mathfrak{s}$ take the form

$$[f_a, e_j] = (A_a)^k_j e_k, \qquad 1 \leqslant a \leqslant f, \quad 1 \leqslant j \leqslant n, \tag{12}$$

$$[f_a, f_b] = \gamma^j_{ab} e_j, \qquad 1 \leqslant a, \quad b \leqslant f. \tag{13}$$

The matrix elements of the matrices $A_a$ must satisfy certain linear relations following from the Jacobi relations between the elements $(f_a, e_j, e_k)$. The Jacobi identities between the triples $(f_a, f_b, e_j)$ will provide linear expressions for the structure constants $\gamma^j_{ab}$ in terms of the matrix elements of the commutators of the matrices $A_a$ and $A_b$.

Since $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{s}$ no nontrivial linear combination of the matrices $A_i$ is a nilpotent matrix, i.e. they are *linearly nil independent*.

Let us now consider the adjoint representation of $\mathfrak{s}$, restrict it to the nilradical $\mathfrak{n}$ and find $\mathrm{ad}|_{\mathfrak{n}}(f_a)$. It follows from the Jacobi identities that $\mathrm{ad}|_{\mathfrak{n}}(f_a)$ is a derivation of $\mathfrak{n}$. In other words, finding all sets of matrices $A_a$ in (12) satisfying the Jacobi identities is equivalent to finding all sets of outer nil-independent derivations of $\mathfrak{n}$

$$D^1 = \mathrm{ad}|_{\mathfrak{n}}(f_1), \ldots, D^f = \mathrm{ad}|_{\mathfrak{n}}(f_f). \tag{14}$$

Furthermore, in view of (11), the commutators $[D^a, D^b]$ must be inner derivations of $\mathfrak{n}$. This requirement determines the Lie brackets (13), i.e. the structure constants $\gamma_{ab}^j$, up to elements in the center $C(\mathfrak{n})$ of $\mathfrak{n}$.

Different sets of derivations may correspond to isomorphic Lie algebras, so redundancies must be eliminated. The equivalence is generated by the following transformations:

(i) We may add any inner derivation to $D^a$.
(ii) We may perform a change of basis in $\mathfrak{n}$ such that the Lie brackets (10) are not changed.
(iii) We can change the basis in the space $\mathrm{span}\{D^1, \ldots, D^f\}$.

## 3. Classification of solvable Lie algebras with the nilradical $\mathfrak{n}_{n,2}$

### 3.1. Nilpotent algebra $\mathfrak{n}_{n,2}$ and its structure

The Lie algebra $\mathfrak{n} = \mathfrak{n}_{n,2}$ is defined by the Lie brackets (1) of the introduction. We shall mostly consider $n \geqslant 6$ (the final result is the same for $n = 5$ but there is a small peculiarity in the computation). The dimensions of the subalgebras in the characteristic series are

$$\mathrm{DS} = [n, n-2, 0], \qquad \mathrm{CS} = [n, n-2, n-3, \ldots, 1, 0], \qquad \mathrm{US} = [1, 2, \ldots, n-2, n]. \tag{15}$$

Its maximal Abelian ideal $\mathfrak{a}$ coincides with the derived algebra $\mathfrak{n}^{(1)} = \mathfrak{n}^2$, i.e. $\mathfrak{a} = \mathrm{span}\{e_1, \ldots, e_{n-2}\}$.

In order to find all non-nilpotent derivations of $\mathfrak{n}$ we first consider the structure of automorphisms of $\mathfrak{n}_{n,2}$. There exists a flag of ideals which is invariant under any automorphism

$$\mathfrak{n} \supset \mathfrak{n}_{\mathfrak{n}^{n-2}} \supset \mathfrak{n}^2 \supset \mathfrak{n}^3 \supset \cdots \supset \mathfrak{n}^{n-1}, \tag{16}$$

where each element in the flag has codimension one in the previous one. (We recall that $\mathfrak{n}_{\mathfrak{n}^{n-2}}$ is the centralizer of $\mathfrak{n}^{n-2}$ in $\mathfrak{n}$.) In any basis respecting the flag, e.g. the one used in the Lie brackets (1), any automorphism will be represented by a triangular matrix.

Furthermore, the whole algebra $\mathfrak{n}$ is generated via multiple commutators of the elements $e_{n-1}$ and $e_n$, e.g. $e_{n-3} = [[e_{n-1}, e_n], e_n]$. That means that due to the definition of an automorphism (8) the knowledge of

$$\Phi(e_{n-1}) = \sum_{k=1}^{n-1} \phi_k e_k, \qquad \Phi(e_n) = \sum_{k=1}^{n} \psi_k e_k \tag{17}$$

in principle amounts to full knowledge of $\Phi$. It remains to establish which choices of $\phi_k$, $1 \leqslant k \leqslant n-1$ and $\psi_k$, $1 \leqslant k \leqslant n$ are consistent with the definition (8) of an automorphism.

Because of the triangular structure respecting (16) we immediately have

$$[\Phi(e_j), \Phi(e_k)] = 0, \quad 1 \leqslant j, \quad k \leqslant n-2,$$
$$[\Phi(e_1), \Phi(e_{n-1})] = [\Phi(e_2), \Phi(e_{n-1})] = 0,$$
$$[\Phi(e_1), \Phi(e_n)] = 0$$

and relation

$$[\Phi(e_k), \Phi(e_n)] = \Phi(e_{k-1}), \qquad 2 \leqslant k \leqslant n-1 \tag{18}$$

can be viewed as a definition of $\Phi(e_{k-1})$, $2 \leqslant k \leqslant n-1$ in accordance with (17). Consequently it remains to check

$$[\Phi(e_k), \Phi(e_{n-1})] = \Phi(e_{k-2}), \qquad 3 \leqslant k \leqslant n-2$$

or, equivalently,

$$[\Phi(e_{n-1}), \Phi(e_k)] = -[\Phi(e_n), [\Phi(e_n), \Phi(e_k)]], \qquad 1 \leqslant k \leqslant n-2 \tag{19}$$

(the change in the index range is just for convenience, the added two relations are satisfied trivially). Since any automorphism $\Phi$ restricted to $\mathfrak{n}^2 = \mathrm{span}\{e_1, \dots, e_{n-2}\}$ is a regular (invertible) map, we can write (19) as a relation between restrictions to $\mathfrak{n}^2$ of adjoint operators

$$\mathrm{ad}|_{\mathfrak{n}^2}(\Phi(e_{n-1})) = -(\mathrm{ad}|_{\mathfrak{n}^2}(\Phi(e_n)))^2. \tag{20}$$

Writing down the matrices of the operators we find

$$\mathrm{ad}|_{\mathfrak{n}^2}(\Phi(e_{n-1})) = \begin{pmatrix} 0 & 0 & -\phi_{n-1} & 0 & 0 & \dots & 0 \\ & 0 & 0 & -\phi_{n-1} & 0 & \dots & 0 \\ & & 0 & 0 & -\phi_{n-1} & \dots & 0 \\ & & & \ddots & \ddots & \ddots & \\ & & & & 0 & 0 & -\phi_{n-1} \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}$$

and

$$\mathrm{ad}|_{\mathfrak{n}^2}(\Phi(e_n)) = \begin{pmatrix} 0 & -\psi_n & -\psi_{n-1} & 0 & 0 & \dots & 0 \\ & 0 & -\psi_n & -\psi_{n-1} & 0 & \dots & 0 \\ & & 0 & -\psi_n & -\psi_{n-1} & \dots & 0 \\ & & & \ddots & \ddots & \ddots & \\ & & & & 0 & -\psi_n & -\psi_{n-1} \\ & & & & & 0 & -\psi_n \\ & & & & & & 0 \end{pmatrix}.$$

Consequently, the condition (20) gives us two constraints on $\phi_k, \psi_k$, namely

$$\phi_{n-1} = \psi_n^2, \qquad \psi_{n-1} = 0. \tag{21}$$

Note that here the case $n = 5$ differs—the matrices above become $3 \times 3$ and only the first condition remains. Accordingly, the dimension of the group of automorphisms and the algebra of derivations is by one higher than in the generic case. Even so, the final conclusion about the number and structure of the solvable Lie algebras with the nilradical $\mathfrak{n}_{5,2}$ also fits into the general pattern shown below.

To sum up, we have found that automorphisms $\Phi$ of $\mathfrak{n}$ are uniquely determined by a set of $2n - 3$ parameters

$$\phi_k, \psi_k, \psi_n, \qquad 1 \leqslant k \leqslant n-2, \tag{22}$$

where $\psi_n \neq 0$ and $\phi_k, \psi_k$ are arbitrary. Such an automorphism acts on the basis elements $e_k$ in the following way:

$$\begin{aligned}
\Phi(e_k) &= \sum_{j=1}^{k-1} (\cdots) e_j + (\psi_n)^{n-k+1} e_k, \qquad 1 \leqslant k \leqslant n-2,\\
\Phi(e_{n-1}) &= \sum_{j=1}^{n-2} \phi_j e_j + \psi_n^2 e_{n-1},\\
\Phi(e_n) &= \sum_{j=1}^{n-2} \psi_j e_j + \psi_n e_n,
\end{aligned} \tag{23}$$

where the coefficient of the $e_k$ term in $\Phi(e_k)$ was found from (8) and $\cdots$ denote some rather complicated functions of the parameters $\phi_k, \psi_k, \psi_n$ which can be deduced from equation (8) but we shall not need them in the following.

The derivations of $\mathfrak{n}$ are now easily found by considering automorphisms infinitesimally close to identity, i.e. differentiating one-parameter subgroups in $\mathrm{Aut}(\mathfrak{n})$. We find that the algebra of derivations is $2n-3$ dimensional. An arbitrary derivation $D$ depends on $2n-3$ parameters $c_k, d_k, d_n$, $1 \leqslant k \leqslant n-2$ and has the form

$$\begin{aligned}
D(e_k) &= \sum_{j=1}^{k-1} (\cdots) e_j + (n-k+1) d_n e_k, \qquad 1 \leqslant k \leqslant n-2,\\
D(e_{n-1}) &= \sum_{j=1}^{n-2} c_j e_j + 2 d_n e_{n-1},\\
D(e_n) &= \sum_{j=1}^{n-2} d_j e_j + d_n e_n,
\end{aligned} \tag{24}$$

where $\cdots$ denote some linear functions of the parameters $c_k, d_k, d_n$ (again their explicit knowledge is not needed in the remainder of the paper).

### 3.2. *Construction of solvable Lie algebras with nilradical* $\mathfrak{n}_{n,2}$

As was explained in subsection 2.2, to find all solvable Lie algebras with nilradical $\mathfrak{n}_{n,2}$ we must find all nonequivalent nil-independent sets $\{D^1, \ldots, D^f\}$ of derivations $\mathfrak{n}_{n,2}$.

Looking at (24) we immediately recognize that we can have at most one nil-independent derivation—such that $d_n \neq 0$. If there would be more of them, say $D$ and $\tilde{D}$ then obviously by taking a linear combination $\tilde{d}_n D - d_n \tilde{D}$ we obtain a nilpotent operator (namely one represented by a strictly upper triangular matrix). Therefore any solvable but not nilpotent Lie algebra with the nilradical $\mathfrak{n}_{n,2}$ must be $n+1$ dimensional. The question which remains is how many such algebras are nonisomorphic.

By proper choice of the multiple of $D$ and adding suitable inner derivations we can transform $D$ into the form

$$\begin{aligned}
D(e_k) &= \sum_{j=1}^{k-1} (\cdots) e_j + (n-k+1) e_k, \qquad 1 \leqslant k \leqslant n-2,\\
D(e_{n-1}) &= \sum_{j=1}^{n-3} c_j e_j + 2 e_{n-1},\\
D(e_n) &= e_n.
\end{aligned} \tag{25}$$

There are $n-1$ nontrivial inner derivations $\mathrm{ad}(e_k), 2 \leqslant k \leqslant n$ and one choice of scaling, so we are able to remove $n$ parameters in a non-nilpotent outer derivation (24). There are still $n-3$ parameters left in equation (25).

Next we perform a change of basis in $\mathfrak{n}$ such that the Lie brackets (1) are preserved, i.e. conjugate the derivation $D$ by a suitable automorphism $\Phi$

$$D \to \tilde{D} = \Phi^{-1} \circ D \circ \Phi. \tag{26}$$

Our aim is to diagonalize the action of $D$, if possible. We find it convenient to perform this in $n-3$ steps, setting one parameter $c_k$ equal to 0 in each step. Thus our $\Phi$ will be expressed as

$$\Phi = \Phi_{n-3} \circ \Phi_{n-2} \circ \cdots \circ \Phi_1, \tag{27}$$

where the automorphisms $\Phi_k$ are constructed as follows.

Let us assume that for a given $k \leqslant n-3$ we have already set $c_j = 0$ for all $k < j \leqslant n-2$ (assuming of course the form (25) for $D$). We construct an automorphism $\Phi_k$ defined by

$$\Phi_k(e_{n-1}) = \alpha_k e_k + e_{n-1}, \qquad \Phi_k(e_n) = e_n,$$

where $\alpha_k$ is to be determined. We have

$$D(\Phi_k(e_{n-1})) = D(e_{n-1}) + \alpha_k D(e_k) = 2e_{n-1} + c_k e_k + (n-k+1)\alpha_k e_k + \sum_{j=1}^{k-1}(\cdots)e_j$$

$$= 2\left(e_{n-1} + \frac{1}{2}\left(c_k + (n-k+1)\alpha_k\right) e_k\right) + \sum_{j=1}^{k-1}(\cdots)e_j.$$

We find that

$$D(\Phi_k(e_{n-1})) = 2\Phi_k(e_{n-1}) + \sum_{j=1}^{k-1}(\cdots)e_j$$

precisely when

$$\alpha_k = \frac{c_k}{k-n+1}.$$

By this choice of $\alpha_k$ we set $c_k$ to 0 and proceed to the next step, namely elimination of $c_{k-1}$.

To conclude, we are able to eliminate all $c_k$'s using suitably chosen automorphisms $\Phi_k$ in equation (27), i.e. we have found that up to addition of inner derivations, conjugation by automorphisms, and rescaling there exists just one nil-independent set of outer derivations, consisting of a unique element $D$

$$D(e_k) = (n-k+1)e_k, \qquad 1 \leqslant k \leqslant n. \tag{28}$$

Consequently, we have

**Theorem 1.** *For the given nilradical $\mathfrak{n}_{n,2}, n \geqslant 5$ there exists precisely one solvable non-nilpotent Lie algebra $\mathfrak{s}_{n+1}$ with the nilradical $\mathfrak{n}_{n,2}$. It has dimension* $\dim \mathfrak{s}_{n+1} = n+1$ *and its Lie brackets are as follows:*

$$\begin{aligned}
&\mathfrak{s}_{n+1} = \mathrm{span}\{e_1, \ldots, e_k, f_1\}, \\
&[e_j, e_k] = 0, \qquad 1 \leqslant j, \quad k \leqslant n-2, \\
&[e_1, e_{n-1}] = [e_2, e_{n-1}] = 0, \\
&[e_k, e_{n-1}] = e_{k-2}, \qquad 3 \leqslant k \leqslant n-2, \\
&[e_1, e_n] = 0, \\
&[e_k, e_n] = e_{k-1}, \qquad 2 \leqslant k \leqslant n-1, \\
&[e_k, f_1] = (n-k+1)e_k, \qquad 1 \leqslant k \leqslant n.
\end{aligned} \tag{29}$$

*The dimensions of the characteristic series are*

$$\mathrm{DS} = [n+1, n, n-2, 0], \qquad \mathrm{CS} = [n+1, n], \quad \mathrm{US} = [0].$$

Above we have proved theorem 1 for $n \geqslant 6$. For $n = 5$ the proof requires a slight modification (see a comment below equation (21)) but proceeds in a very similar way. One just has to construct one more automorphism eliminating an additional parameter in the non-nilpotent derivation $D$.

## 4. Generalized Casimir invariants

### *4.1. Definitions and methods of computation*

The term *Casimir operator*, or *Casimir invariant*, is usually reserved for elements of the center of the enveloping algebra of a Lie algebra $\mathfrak{g}$ [20, 43]. These operators are in one-to-one correspondence with polynomial invariants characterizing orbits of the coadjoint representation of $\mathfrak{g}$ [29] (or of the corresponding Lie group $G$). On the other hand, in the representation theory of solvable Lie algebras the invariants of the coadjoint representation are not necessarily polynomials. They can be rational functions, or even transcendental ones. In that case we call them *generalized Casimir invariants*. For algebraic Lie algebras, which include semisimple, perfect and also nilpotent Lie algebras it is possible to choose a basis for all invariants of the coadjoint representation consisting entirely of polynomials [1, 2].

Two different systematic methods of constructing invariants of group actions exist, in particular of the coadjoint representation of a Lie group $G$. The first method is an infinitesimal one. A basis for the coadjoint representation of the Lie algebra $\mathfrak{g}$ of the Lie group $G$ is given by the first-order differential operators

$$\hat{X}_k = x_a c_{kb}^a \frac{\partial}{\partial x_b}, \tag{30}$$

where $c_{ij}^k$ are the structure constants of Lie algebra $\mathfrak{g}$ in the basis $(x_1, \ldots, x_N)$. In equation (30), the quantities $x_a$ are commuting independent variables which can be identified with coordinates in the basis of the space $\mathfrak{g}^*$ dual to the basis $(x_1, \ldots, x_N)$ of the algebra $\mathfrak{g}$.

The invariants of the coadjoint representation, i.e. the generalized Casimir invariants, are solutions of the following system of partial differential equations:

$$\hat{X}_k I(x_1, \ldots, x_N) = 0, \qquad k = 1, \ldots, N. \tag{31}$$

Traditionally, the system (31) is solved using the method of characteristics.

The number of functionally independent solutions of the system (31) is

$$n_I = N - r, \tag{32}$$

where $r$ is the generic rank of the antisymmetric matrix

$$C = \begin{pmatrix} 0 & c_{12}^b x_b & \ldots & c_{1N}^b x_b \\ -c_{12}^b x_b & 0 & \ldots & c_{2N}^b x_b \\ \vdots & & & \vdots \\ -c_{1,N-1}^b x_b & \ldots & 0 & c_{N-1,N}^b x_b \\ -c_{1N}^b x_b & \ldots & -c_{N-1,N}^b x_b & 0 \end{pmatrix}. \tag{33}$$

Since $C$ is antisymmetric, its rank is even. Hence $n_I$ has the same parity as $N$.

This method of calculating the invariants of Lie group actions is a standard one and goes back to the 19th century. For a brief history with references to the original literature we refer to Olver's book [39]. To our knowledge this method was first adapted to the construction of (generalized) Casimir operators in [1, 2, 5]. It has been extensively applied to low-dimensional Lie algebras (for $n \leqslant 5$ and nilpotent $n = 6$ in [40], solvable $n = 6$ in [7, 38] with four-dimensional nilradicals and in [14] with five-dimensional nilradicals), certain solvable rigid Lie algebras [12, 13], solvable Lie algebras with Heisenberg nilradical [45], the nilradicals $\mathfrak{n}_{n,1}$ [47], triangular nilradicals [49], certain inhomogeneous classical Lie algebras [15], certain affine Lie algebras [16] and other specific solvable Lie algebras [3, 4].

The second method of calculating invariants of group actions is called the method of moving frames. It goes back to Cartan [18, 19] and its recent formulation is due to Fels and Olver [21, 22]. A related method was also applied to the inhomogeneous classical groups [41]). Boyko *et al* adapted the method of moving frames to the case of coadjoint representations. They presented an algebraic algorithm for calculating (generalized) Casimir operators and applied it to a large number of solvable Lie algebras [7–11].

We shall apply the method of moving frames to calculate the invariants of the coadjoint action of the groups corresponding to the nilpotent Lie algebras $\mathfrak{n}_{n,2}$ and the solvable Lie algebra $\mathfrak{s}_{n+1}$ of theorem 1.

The method of moving frames as we apply it can be roughly divided into the following steps.

(i) Integration of the coadjoint action of the Lie algebra $\mathfrak{g}$ on its dual $\mathfrak{g}^*$ as given by the vector fields (30) to the (local) action of the group $G$.
This is usually realized by choosing a convenient (local) parameterization of $G$ in terms of one-parametric subgroups, e.g.

$$g(\vec{\alpha}) = \exp(\alpha_N x_N) \cdot \ldots \cdot \exp(\alpha_2 x_2) \cdot \exp(\alpha_1 x_1) \in G, \qquad \vec{\alpha} = (\alpha_1, \ldots, \alpha_N) \tag{34}$$

and correspondingly composing the flows $\Psi^{\alpha_k}_{\hat{X}_k}$ of the vector fields $\hat{X}_k$ defined in (30)

$$\frac{\mathrm{d}\Psi^{\alpha_k}_{\hat{X}_k}(p)}{\mathrm{d}\alpha_\mathrm{k}} = \hat{X}_k\big(\Psi^{\alpha_k}_{\hat{X}_k}(p)\big), \qquad p \in \mathfrak{g}^*, \tag{35}$$

i.e. we have

$$\Psi(g(\vec{\alpha})) = \Psi^{\alpha_N}_{\hat{X}_N} \circ \cdots \Psi^{\alpha_2}_{\hat{X}_2} \circ \Psi^{\alpha_1}_{\hat{X}_1}. \tag{36}$$

For a given point $p \in \mathfrak{g}^*$ with coordinates $x_k = x_k(p), \vec{x} = (x_1, \ldots, x_N)$ we denote the coordinates of the transformed point $\Psi(g(\vec{\alpha}))p$ by $\tilde{x}_k$

$$\tilde{x}_k \equiv \Psi_k(\vec{\alpha})(\vec{x}) = x_k\,(\Psi(g(\vec{\alpha}))p)\,. \tag{37}$$

We consider $\tilde{x}_k$ to be a function of both the group parameters $\vec{\alpha}$ and the coordinates $\vec{x}$ of the original point $p$.

(ii) Choice of a section cutting through the orbits of the action $\Psi$.
We need to choose in a smooth way a single point on each of the (generic) orbits of the action of the group $G$. Typically this is done as follows: we find a subset of $r$ coordinates, say $(x_{\pi(i)})_{i=1}^r$, on which the group $G$ acts transitively, at least locally in an open neighborhood of chosen values $\big(x^0_{\pi(i)}\big)_{i=1}^r$. Here $\pi$ denotes a suitable injection $\pi : \{1, \ldots, r\} \to \{1, \ldots, N\}$ and $r$ is the rank of the matrix $C$ in equation (33). Points whose coordinates satisfy

$$x_{\pi(i)} = x^0_{\pi(i)}, \qquad 1 \leqslant i \leqslant r \tag{38}$$

form our desired section $\Sigma$, intersecting each generic orbit once.

(iii) Construction of invariants.
For a given point $p \in \mathfrak{g}^*$ we find group elements transforming $p$ into $\tilde{p} \in \Sigma$ by the action $\Psi$. We express as many of their parameters as possible (i.e., $r$ of them) in terms of the original coordinates $\vec{x}$ and substitute them back into equation (37). This gives us $\tilde{x}_k$ as functions of $\vec{x}$ only. Out of them, $\tilde{x}_{\pi(i)}, i = 1, \ldots, r$ have the prescribed fixed values $x^0_{\pi(i)}$. The remaining $N - r$ functions $\tilde{x}_k$ are by construction invariant under the coadjoint action of $G$, i.e. define the sought after invariants of the coadjoint representation.

Technically, as we shall see in our particular case below (cf equation (43)), it may not be necessary to evaluate all the functions $\tilde{x}_k$ so that a suitable choice of the basis in $\mathfrak{g}$ can substantially simplify the whole procedure. This happens when only a smaller subset of say $r_0$ group parameters $\alpha_k$ enters into the computation of $N-r+r_0$ functions $\tilde{x}_k$, $k = 1, \ldots, N-r+r_0$ (possibly after a re-arrangement of $x_k$'s). In this case the other parameters can be ignored throughout the computation. They are specified by the remaining equations

$$\tilde{x}_i = x^0_i, \qquad N - r + r_0 + 1 \leqslant i \leqslant N \tag{39}$$

but do not enter into the expressions for $\tilde{x}_k$, $1 \leqslant k \leqslant N - r + r_0$ which define our invariants.

We shall remark that the invariants found using either of the methods above may not be in the most convenient form. That can be remedied once we find them. For example, as we already mentioned, the generalized Casimir invariants of a nilpotent Lie algebra can be always chosen as polynomials, i.e. proper Casimir invariants. As we shall see below the method of moving frames may naturally give us nonpolynomial ones. Nevertheless, it is usually quite easy to construct polynomials out of them.

### *4.2. Casimir invariants of the Lie algebra $\mathfrak{n}_{n,2}$*

The differential operators corresponding to the basis elements of $\mathfrak{n}_{n,2}$ are

$$\begin{aligned} &\hat{E}_1 = 0, \qquad \hat{E}_2 = e_1 \frac{\partial}{\partial e_n}, \qquad \hat{E}_k = e_{k-2}\frac{\partial}{\partial e_{n-1}} + e_{k-1}\frac{\partial}{\partial e_n}, \qquad 3 \leqslant k \leqslant n-2, \\ &\hat{E}_{n-1} = -\sum_{k=3}^{n-2} e_{k-2}\frac{\partial}{\partial e_k} + e_{n-2}\frac{\partial}{\partial e_n}, \qquad \hat{E}_n = -\sum_{k=2}^{n-1} e_{k-1}\frac{\partial}{\partial e_k}. \end{aligned} \tag{40}$$

The form of $\hat{E}_k$, $1 \leqslant k \leqslant n$ implies that the invariants do not depend on $e_{n-1}, e_n$. Using equation (32) we find that the nilpotent Lie algebra $\mathfrak{n}_{n,2}$ has $n - 4$ functionally independent invariants but it is rather complicated to directly solve the remaining two partial differential equations defining the invariants, namely

$$\hat{E}_{n-1} I(e_1, e_2, \ldots, e_{n-2}) = 0, \qquad \hat{E}_n I(e_1, e_2, \ldots, e_{n-2}) = 0$$

(in particular, it is not too difficult to solve it for given $n = 5, 6, 7, 8, \ldots$ but it is more involved to deduce a general formula valid for arbitrary $n$ out of the solutions thus obtained).

Let us employ the method of moving frames. First, we construct the flows of the vector fields $\hat{E}_{n-1}$, $\hat{E}_n$ acting on the space spanned by $e_1, \ldots, e_{n-2}$ only. We find

$$e_k\left(\Psi^{\alpha_{n-1}}_{\hat{E}_{n-1}} p\right) = \sum_{j=0}^{[\frac{k-1}{2}]} \frac{(-1)^j}{j!} \alpha^j_{n-1} e_{k-2j}(p), \qquad 1 \leqslant k \leqslant n-2 \tag{41}$$

(where [] denotes the integer part) and

$$e_k\left(\Psi^{\alpha_n}_{\hat{E}_n} p\right) = \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \alpha^j_n e_{k-j}(p), \qquad 1 \leqslant k \leqslant n-2. \tag{42}$$

Combining these two expressions together we find (in the notation of the previous subsection)

$$\tilde{e}_k = \sum_{l=0}^{k-1} \sum_{m=0}^{[\frac{k-l-1}{2}]} \frac{(-1)^{l+m}}{l!m!} \alpha_n^l \alpha_{n-1}^m e_{k-l-2m}, \qquad 1 \leqslant k \leqslant n-2. \tag{43}$$

We see that these $n-2$ functions involve only the group parameters $\alpha_{n-1}, \alpha_n$. We can easily determine them out of two equations defining our section $\Sigma$. We choose them to be

$$0 = \tilde{e}_2 = e_2 - \alpha_n e_1, \qquad 0 = \tilde{e}_3 = e_3 - \alpha_n e_2 + \frac{\alpha_n^2}{2} e_1 - \alpha_{n-1} e_1. \tag{44}$$

We find

$$\alpha_{n-1} = \frac{1}{e_1^2}\left(e_1 e_3 - \frac{1}{2} e_2^2\right), \qquad \alpha_n = \frac{e_2}{e_1}. \tag{45}$$

Substituting these back into remaining equations (43) and multiplying all of them by $e_1^{k-2}$ (in order to get polynomial expressions) we find the invariants

$$\xi_0 = e_1,$$

$$\xi_j = e_1^{j+1} \tilde{e}_{j+3} = e_1^{j+1} \sum_{l=0}^{j+2} \sum_{m=0}^{[\frac{j-l}{2}+1]} \frac{(-1)^{l+m}}{l!m!} \alpha_n^l \alpha_{n-1}^m e_{j+3-l-2m}, \qquad 1 \leqslant j \leqslant n-5, \tag{46}$$

where the substitution (45) is assumed. Performing it explicitly we arrive at the following theorem.

**Theorem 2.** *The nilpotent Lie algebra $\mathfrak{n}_{n,2}$ has $n-4$ functionally independent Casimir invariants. They can be chosen to be the following polynomials:*

$$\xi_0 = e_1,$$

$$\xi_j = \sum_{l=0}^{j+2} \sum_{m=0}^{[\frac{j-l}{2}+1]} \sum_{q=0}^{m} \frac{(-1)^{l+m+q}}{2^q l!(m-q)!q!} e_1^{j+1-l-m-q} e_2^{l+2q} e_3^{m-q} e_{j+3-l-2m}, \tag{47}$$

*where $1 \leqslant j \leqslant n-5$. All other Casimir invariants are functions of $\xi_0, \ldots, \xi_{n-5}$.*

We note that $\xi_j$ is for $j \geqslant 1$ a homogeneous polynomial of degree $j+2$. For reader's convenience we list a few lowest order invariants explicitly (note that the dimension $n$ of $\mathfrak{n}_{n,2}$ does not enter directly into the formulae, it just specifies where the list terminates)

$$\begin{aligned}
\xi_1 &= e_1^2 e_4 - e_1 e_2 e_3 + \tfrac{1}{3} e_2^3, \\
\xi_2 &= e_1^3 e_5 - e_1^2 e_2 e_4 - \tfrac{1}{2} e_1^2 e_3^2 + e_1 e_2^2 e_3 - \tfrac{1}{4} e_2^4, \\
\xi_3 &= e_1^4 e_6 - e_2 e_1^3 e_5 - e_1^3 e_3 e_4 + e_2^2 e_1^2 e_4 + e_1^2 e_2 e_3^2 - e_1 e_2^3 e_3 + \tfrac{1}{5} e_2^5, \\
\xi_4 &= e_1^5 e_7 - e_1^4 e_2 e_6 + e_1^3 e_2^2 e_5 - e_1^4 e_3 e_5 + e_1^3 e_2 e_3 e_4 - \tfrac{2}{3} e_1^2 e_2^3 e_4 \\
&\quad + \tfrac{2}{3} e_1 e_2^4 e_3 + \tfrac{1}{3} e_1^3 e_3^3 - e_1^2 e_2^2 e_3^2 - \tfrac{1}{9} e_2^6.
\end{aligned} \tag{48}$$

### *4.3. The generalized Casimir invariants of the Lie algebra $\mathfrak{s}_{n+1}$*

Let us now consider the $(n+1)$-dimensional solvable Lie algebra $\mathfrak{s}_{n+1}$ of theorem 1. The operators $\hat{E}_i$ representing elements in the nilradical $\mathfrak{n}_{n,2}$ will each contain an additional term involving a derivative with respect to $f_1$ and there is one additional operator, namely

$$\hat{F}_1 = -\sum_{k=1}^{n} (n-k+1) e_k \frac{\partial}{\partial e_k}. \tag{49}$$

However, from the form of these operators, namely $\hat{E}_1$, we see that the invariants cannot depend on $f_1$. Moreover, they can only depend on the invariants (47) of $\mathfrak{n}_{n,2}$. To find the invariants of the algebra $\mathfrak{s}_{n+1}$ we represent the non-nilpotent element $f_1 \in \mathfrak{s}_{n+1}$ by the appropriate 'truncated' differential operator acting only on $(e_1, \ldots, e_{n-2})$

$$\hat{F}_{1T} = -\sum_{k=1}^{n-2}(n-k+1)e_k\frac{\partial}{\partial e_k}. \tag{50}$$

We must then solve the equation

$$\hat{F}_{1T} I(\xi_0, \ldots, \xi_{n-5}) = 0. \tag{51}$$

We can proceed directly, using an easily established formula

$$\hat{F}_{1T}\, y(\xi_0, \ldots, \xi_{n-5}) = n\xi_0\frac{\partial y}{\partial \xi_0} + \sum_{j=1}^{n-5}(j+2)(n-1)\xi_j\frac{\partial y}{\partial \xi_j} \tag{52}$$

(in order to deduce equation (52) it is sufficient to determine how $\hat{F}_{1T}$ acts on each monomial in equation (47)).

Let us instead employ the method of moving frames, finding the flow of the vector field $F_{1T}$

$$e_k\left(\Psi^{\alpha_{n+1}}_{\hat{F}_{1T}} p\right) = \exp(-(n-k+1)\alpha_{n+1})e_k(p), \qquad 1 \leqslant k \leqslant n-2. \tag{53}$$

The full action of the group $S_{n+1}$ on the space with coordinates $e_1, \ldots, e_{n-2}$ gives

$$\tilde{e}_k = \exp(-(n-k+1)\alpha_{n+1})\sum_{l=0}^{k-1}\sum_{m=0}^{[\frac{k-l-1}{2}]}\frac{(-1)^{l+m}}{l!m!}\alpha_n^l\alpha_{n-1}^m e_{k-l-2m}, \qquad 1 \leqslant k \leqslant n-2. \tag{54}$$

We choose our section $\Sigma$ in the truncated space to be $\{(1, 0, 0, e_4, \ldots, e_{n-2})\}$, i.e. we have one more equation in addition to equation (44)

$$1 = \tilde{e}_1 = \exp(-n\alpha_{n+1})e_1. \tag{55}$$

Solving it we find

$$\exp(-\alpha_{n+1}) = \left(\frac{1}{e_1}\right)^{\frac{1}{n}} \tag{56}$$

and substituting it together with equation (45) back into (54) we find invariants which can be succinctly expressed in the form

$$\tilde{e}_k = \frac{\xi_{k-3}}{e_1^{\frac{jn+2n-j-2}{n}}}, \qquad 4 \leqslant k \leqslant n-2. \tag{57}$$

By taking suitable powers (in order to express invariants as ratios of polynomials) we arrive at a theorem.

**Theorem 3.** *The $(n+1)$-dimensional solvable Lie algebra $\mathfrak{s}_{n+1}$ of theorem 1 has $n-5$ functionally independent invariants. They can be chosen in the form*

$$\chi_j = \frac{\xi_j^n}{\xi_0^{(n-1)(j+2)}}, \qquad 1 \leqslant j \leqslant n-5, \tag{58}$$

*i.e. they are rational in $\xi_k$ and consequently in $e_k$.*

## 5. Conclusions

Let us sum up the main results of this paper and compare them with those obtained for indecomposable solvable Lie algebras $\mathfrak{s}$ with other nilradicals of dimension $\dim \mathrm{NR}(\mathfrak{s}) = n$.

(i) The nilradical $\mathfrak{n}_{n,2}$ (this paper). The series exists for all $n \geqslant 5$. It is possible to add precisely one non-nilpotent element to form a solvable Lie algebra. Its action on the nilradical is a diagonal one. The algebra $\mathfrak{n}_{n,2}$ has $n-4$ functionally independent Casimir operators. Its solvable extension has $n-5$ functionally independent generalized Casimir invariants, all of them can be chosen as rational functions of the elements of the nilradical.

(ii) The nilradical $\mathfrak{n}_{n,1}$ ([47]). The series exists for all $n \geqslant 4$. It is possible to add at most $f_{\max} = 2$ nil-independent element to $\mathfrak{n}_{n,1}$. The algebra $\mathfrak{n}_{n,1}$ has $n-2$ functionally independent Casimir invariants. For solvable Lie algebras we have $n-2-f$ independent generalized Casimir invariants. For $f = 2$ they can be chosen to be rational functions, for $f = 1$ they either can be chosen rational or they might involve logarithms. In both cases they depend only on the elements of the nilradical.

(iii) Abelian algebras $\mathfrak{a}_n$ as nilradicals ([36–38]). They exist for all $n \geqslant 1$. The Abelian algebra $\mathfrak{a}_n$ has $n$ functionally independent Casimir operators (i.e., the basis elements of $\mathfrak{a}_n$). For $n = 1$ we can add just one non-nilpotent element. The obtained solvable Lie algebra has no generalized Casimir invariants. For $n = 2$ over the field $F = \mathbb{C}$ we can add only one non-nilpotent element $f_1$ and we obtain a solvable algebra with one generalized Casimir invariant. Depending on the action of $f_1$ on the nilradical the invariant can be chosen rational or involves logarithms. For $n = 2$ and $F = \mathbb{R}$ we can add one or two nil-independent elements. The obtained solvable algebras have just one independent invariant or none at all. For $n \geqslant 3$ we can add $f$ elements where $1 \leqslant f \leqslant n-1$. The number of independent generalized Casimir invariants of the solvable Lie algebras is $n-f$ and in general they may involve logarithms (i.e., cannot be expressed as rational). In all cases the generalized Casimir invariants depend on the elements of the nilradical alone. For $F = \mathbb{R}$ the generalized Casimir operators may involve other functions than logarithms, for instance inverse trigonometric ones. Implicit examples are in [40] and explicit ones in [7, 8].

(iv) Heisenberg algebras $\mathfrak{h}(N)$ as nilradicals ([45]). The dimension of the Heisenberg algebra $\mathfrak{h}(N)$ in $N$ dimensions is $n = 2N+1$ with $N \geqslant 1$. We can add up to $N+1$ nil-independent elements. The nilpotent algebra $\mathfrak{h}(N)$ has only one Casimir invariant, corresponding to the one-dimensional center of $\mathfrak{h}(N)$ spanned by $e_1$ (for any $N$). Two types of solvable extensions exist. If one of the non-nilpotent elements, say $f_1$, of the solvable Lie algebra does not commute with the center of $\mathfrak{h}(N)$, i.e. with $e_1$, then all $f_a$'s must commute among each other $[f_a, f_b] = 0$ and the solvable Lie algebra has $f-1$ independent generalized Casimir invariants. They can be chosen to be rational functions and depend both on elements of the nilradical $e_i$ and on the elements $f_a$. If we have

$$[f_a, e_1] = 0, \qquad 1 \leqslant a \leqslant f$$

then the number of generalized Casimir invariants is $f+1-\gamma$ where $\gamma$ is the rank of the matrix $\gamma^1_{ab}$ in

$$[f_a, f_b] = \gamma^1_{ab} e_1. \tag{59}$$

They can be chosen to be rational functions of elements $e_i$ and $f_a$.

(v) Triangular nilradicals $\mathfrak{t}(N)$ ([48]). These can be represented by the set of all strictly upper triangular matrices. The dimension is $\dim \mathfrak{t}(N) = n = \frac{N(N-1)}{2}$. For $N = 3$ we have $\mathfrak{t}(3) = \mathfrak{h}(1)$, so the series really starts with $N = 4$ and hence $n = 6$. It is possible to add

$f$ linearly independent elements to $\mathfrak{t}(N)$ with $1 \leqslant f \leqslant N-1$. The nilpotent algebra $\mathfrak{t}(N)$ has $\left[\frac{N}{2}\right]$ independent Casimir invariants. Let us denote by $S(N, f)$ the solvable Lie algebras that have $\mathfrak{t}(N)$ as their nilradical and $f$ added non-nilpotent elements $f_a$. The Casimir invariants of $\mathfrak{t}(N)$ and $S(4, f)$ with $f = 1, 2, 3$ were calculated in [49] using the infinitesimal method. Formulae for the case of $S(N, N-1)$ and $S(N, 1)$ were also presented [49] for all $N$ but the result was not proven and left as a conjecture. This result was later proven by Boyko *et al* [9, 10] using the method of moving frames. The algebras $S(N, N-1)$ have $\left[\frac{N-1}{2}\right]$ independent generalized Casimir invariants. They can be chosen rational and involve both $e_i$ and $f_a$. The algebras $S(N, 1)$ have either $\left[\frac{N}{2}\right]+1$ or $\left[\frac{N}{2}\right]-1$ generalized Casimir invariants; they can all be chosen rational and depend on $e_i$ and $f_1$. Boyko *et al* also calculated [11] the Casimir invariants for $S(N, f)$, $2 \leqslant f \leqslant N-2$ for the case when the non-nilpotent elements act diagonally. The invariants for a non-diagonal action of the nil-independent elements and even their number are known only for $S(4, 2)$ where there are either two invariants, or none [49].

This list of known series of solvable Lie algebras with given nilradicals covers with the notable exception of Abelian nilradicals only nilpotent algebras with one-dimensional center. This is not totally unexpected since larger center means that for given choice of derivations $D^1, \ldots, D^f$ there is more freedom in the choice of structure constants $\gamma_{ab}^{j}$ in equation (13) and consequently the classification becomes more intricate to perform and the results may be more complicated. It would be of interest to fully classify at least one such series.

As was mentioned in the introduction such a list can never exhaust all possible solvable Lie algebras but it is possible to further extend it by considering other possible series of nilradicals. Such lists may be useful in testing some general hypotheses concerning solvable Lie algebras, e.g. the structure of their generalized Casimir invariants.

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## Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras

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We construct all solvable Lie algebras with a specific $n$-dimensional nilradical $\mathfrak{n}_{n,3}$ which contains the previously studied filiform $(n-2)$-dimensional nilpotent algebra $\mathfrak{n}_{n-2,1}$ as a subalgebra but not as an ideal. Rather surprisingly it turns out that the classification of such solvable algebras can be deduced from the classification of solvable algebras with the nilradical $\mathfrak{n}_{n-2,1}$. Also the sets of invariants of coadjoint representation of $\mathfrak{n}_{n,3}$ and its solvable extensions are deduced from this reduction. In several cases they have polynomial bases, i.e. the invariants of the respective solvable algebra can be chosen to be Casimir invariants in its enveloping algebra.



### 1. Introduction

The current article belongs to a series of papers initiated by Rubin and Winternitz in [1] and continued throughout the years with his various collaborators in [2–7]. All these papers dealt with the problem of classification of all solvable Lie algebras with the given $n$-dimensional nilradical, e.g. Abelian, Heisenberg algebra, the algebra of strictly upper triangular matrices etc., for arbitrary finite dimension $n$. Other similar series have been recently investigated by different groups in [8] (naturally graded nilradicals with maximal nilindex and a Heisenberg subalgebra of codimension one) and [9] (a certain series of quasi-filiform nilradicals).

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As is well known, the problem of classification of all solvable (including nilpotent) Lie algebras in an arbitrarily large finite dimension is presently unsolved and is generally believed to be unsolvable. All known full classifications terminate at relatively low dimensions, e.g. the classification of nilpotent algebras is available at most in dimension 8 [10,11], for the solvable ones in dimension 6 [12,13]. The unifying idea behind the series [1–7] is a belief that the knowledge of full classification of all solvable extensions of certain series of nilradicals can be very useful for both theoretical considerations – e.g. testing various hypotheses about general structure of solvable Lie algebras – and practical purposes – e.g. when a generalization of a given algebra or its nilradical to higher dimensions appears in some physical theory.

In this paper we shall consider the nilradical

$$\mathfrak{n}_{n,3} = \mathrm{span}\{x_1, \ldots, x_n\}, \quad n \geqslant 5,$$

with the following nonvanishing Lie brackets

$$\begin{aligned} &[x_2, x_n] = x_1, \\ &[x_3, x_{n-1}] = x_1, \\ &[x_k, x_{n-1}] = x_{k-1}, \quad 4 \leqslant k \leqslant n-2, \\ &[x_{n-1}, x_n] = x_2. \end{aligned} \tag{1}$$

When $n = 5$, the only remaining nonvanishing Lie brackets are

$$[x_2, x_5] = [x_3, x_4] = x_1, \quad [x_4, x_5] = x_2. \tag{2}$$

The $n$-dimensional nilpotent Lie algebra $\mathfrak{n}_{n,3}$ is nilpotent of degree of nilpotency[1] equal to $n-3$ and with $(n-2)$-dimensional maximal Abelian ideal. It has one-dimensional center $C(\mathfrak{n}_{n,3}) = \mathrm{span}\{x_1\}$.

Later it will become important for our investigation that it contains as a subalgebra the nilpotent algebra $\mathfrak{n}_{n-2,1}$

$$[y_k, y_{n-2}] = y_{k-1}, \quad 2 \leqslant k \leqslant n-3, \tag{3}$$

whose solvable extensions were investigated in [6]. Namely, we have $\tilde{\mathfrak{n}}_{n-2,1}$ spanned by $x_1, x_3, \ldots, x_{n-1}$. Similarly, $\mathfrak{n}_{n,3}$ also contains $\tilde{\mathfrak{n}}_{6,3}$ spanned by $x_1, x_2, x_3, x_4, x_{n-1}, x_n$. Here, tildes were used to denote these particular embeddings of algebras of the type (3) and (1), respectively, into the $n$-dimensional nilradical $\mathfrak{n}_{n,3}$. We stress that neither $\tilde{\mathfrak{n}}_{n-2,1}$ nor $\tilde{\mathfrak{n}}_{6,3}$ are ideals.

In general, the knowledge of solvable extensions of a subalgebra of the given nilradical does not provide much help in the classification of all solvable extensions of the nilradical. That is because the outer derivations of the nilradical need not to leave the subalgebra invariant – indeed, it is not invariant even with respect to inner derivations. However, in the particular case of the nilradical $\mathfrak{n}_{n,3}$ considered here all the classification can be reduced to the cases of $\mathfrak{n}_{n-2,1}$ already investigated in [6] and $\mathfrak{n}_{6,3}$.

In the following we shall firstly find out the general form of an automorphism and a derivation of $\mathfrak{n}_{n,3}$. Next, we use this knowledge in the construction of all solvable extensions of the nilradical $\mathfrak{n}_{n,3}$. Finally, we deduce generalized Casimir invariants of both $\mathfrak{n}_{n,3}$ and its solvable extensions.

Throughout the paper we shall use the same notation as in [7]. We have attempted to make the present paper self-contained but if any doubts arise about chosen conventions, etc. the reader may consult [7] as a suitable reference. Also, if the reader desires to get a more general background information about the classification of solvable Lie algebras, the construction of Casimir invariants and so on, we refer him to the review parts of [7] and the literature cited there.

## 2. Automorphisms and derivations of the nilradical $\mathfrak{n}_{n,3}$

In the computations below we shall assume that $n \geqslant 7$. The results for $n = 5, 6$ are derived in Sections 3.1 and 3.2.

[1] Also called the nilindex. It is the largest value of $k$ for which the $k$th power $\mathfrak{g}^k = [\mathfrak{g}, [\mathfrak{g}, \ldots, [\mathfrak{g}, \mathfrak{g}] \ldots]]$ of $\mathfrak{g}$ is nonvanishing. Equivalently, it can be defined as the number of nonvanishing ideals in the lower central series (5) including $\mathfrak{g}^1 = \mathfrak{g}$.

The nilpotent algebra $\mathfrak{n} = \mathfrak{n}_{n,3}$ has the following complete flag of ideals

$$0 \subset \mathfrak{n}^{n-3} \subset \mathfrak{n}^{n-4} \subset \mathfrak{z}_2 \subset \mathfrak{z}_3 \cap \mathfrak{n}^2 \subset \cdots \subset \mathfrak{z}_{n-5} \cap \mathfrak{n}^2 \subset \mathfrak{n}^2 \subset (\mathfrak{z}_2)_\mathfrak{n} \subset (\mathfrak{n}^{n-4})_\mathfrak{n} \subset \mathfrak{n}, \qquad (4)$$

where

- $\mathfrak{n}^k$ are elements of the lower central series, defined recursively by:

$$\mathfrak{n}^1 = \mathfrak{n}, \quad \mathfrak{n}^k = [\mathfrak{n}^{k-1}, \mathfrak{n}], \quad k \geqslant 2, \qquad (5)$$

- $\mathfrak{z}_k$ are elements of the upper central series - that means that $\mathfrak{z}_k$ is the unique ideal in $\mathfrak{n}$ such that $\mathfrak{z}_k/\mathfrak{z}_{k-1}$ is the center of $\mathfrak{n}/\mathfrak{z}_{k-1}$; the recursion starts from the center of $\mathfrak{n}$, i.e. $\mathfrak{z}_1 = C(\mathfrak{n})$,
- and $(\mathfrak{n}^{n-4})_\mathfrak{n}$ is the centralizer of $\mathfrak{n}^{n-4}$ in $\mathfrak{n}$, i.e.

$$(\mathfrak{n}^{n-4})_\mathfrak{n} = \{x \in \mathfrak{n} | [x, y] = 0, \ \forall y \in \mathfrak{n}^{n-4}\}.$$

By construction, the flag (4) is invariant with respect to any automorphism of the Lie algebra $\mathfrak{n}$, i.e. in the basis respecting the flag any automorphism will be represented by an upper triangular matrix. Because derivations of $\mathfrak{n}$ can be viewed as infinitesimal automorphisms (i.e. elements of the Lie algebra of the matrix Lie group of automorphisms of $\mathfrak{n}$), the same triangular form holds also for them.

Therefore, we find it convenient to change the basis $(x_k)$ of $\mathfrak{n}$ defined in Eq. (1) to a seemingly less natural (i.e. Lie brackets appear more cumbersome) basis $(e_k)$ whose essential advantage over the original one is that it respects the flag (4), i.e. the $k$th subspace in the flag is $\mathrm{span}\{e_1, \ldots, e_k\}$ for all $k$. Namely, we take

$$e_1 = x_1, \ e_2 = x_3, \ e_3 = x_2, \ e_4 = x_4, \ \ldots, e_{n-2} = x_{n-2}, \ e_{n-1} = x_n, \ e_n = x_{n-1}. \qquad (6)$$

The nonvanishing Lie brackets now become

$$\begin{aligned} &[e_2, e_n] = e_1, \\ &[e_3, e_{n-1}] = e_1, \\ &[e_4, e_n] = e_2, \\ &[e_k, e_n] = e_{k-1}, \quad 5 \leqslant k \leqslant n-2, \\ &[e_{n-1}, e_n] = -e_3. \end{aligned} \qquad (7)$$

The important subalgebras isomorphic to $\mathfrak{n}_{n-2,1}, \mathfrak{n}_{6,3}$ are now expressed as

$$\tilde{\mathfrak{n}}_{n-2,1} = \mathrm{span}\{e_1, e_2, e_4, \ldots, e_{n-2}, e_n\}, \quad \tilde{\mathfrak{n}}_{6,3} = \mathrm{span}\{e_1, e_2, e_3, e_4, e_{n-1}, e_n\},$$

respectively. The ideals in the derived,[2] lower central and upper central series are

$$\begin{aligned} &\mathfrak{n}^2 = \mathfrak{n}^{(1)} = \mathrm{span}\{e_1, \ldots, e_{n-3}\}, \quad \mathfrak{n}^{(2)} = 0, \\ &\mathfrak{n}^k = \mathrm{span}\{e_1, e_2, e_4, \ldots, e_{n-k-1}\}, \quad 3 \leqslant k \leqslant n-5, \\ &\mathfrak{n}^{n-4} = \mathrm{span}\{e_1, e_2\}, \quad \mathfrak{n}^{n-3} = \mathrm{span}\{e_1\}, \quad \mathfrak{n}^{n-2} = 0, \\ &\mathfrak{z}_1 = \mathfrak{n}^{n-3}, \quad \mathfrak{z}_2 = \mathrm{span}\{e_1, e_2, e_3\}, \\ &\mathfrak{z}_k = \mathrm{span}\{e_1, \ldots, e_{k+1}, e_{n-1}\}, \quad 3 \leqslant k \leqslant n-4, \quad \mathfrak{z}_{n-3} = \mathfrak{n}. \end{aligned}$$

In order to find the structure of an arbitrary automorphism of $\mathfrak{n}_{n,3}$ we consider its matrix in the basis (6)

$$\Phi(e_k) = e_j \Phi_{jk} \qquad (8)$$

[2] The elements $\mathfrak{n}^{(k)}$ of the derived series are defined recursively by:

$$\mathfrak{n}^{(0)} = \mathfrak{n}, \quad \mathfrak{n}^{(k)} = [\mathfrak{n}^{(k-1)}, \mathfrak{n}^{(k-1)}], \ k \geqslant 1.$$

(summation over repeated indices applies throughout the paper unless otherwise stated). As mentioned above, such a matrix must be necessarily upper triangular because the flag (4) is preserved. It is also obvious that the knowledge of its last three columns, i.e. of $\Phi(e_{n-2})$, $\Phi(e_{n-1})$ and $\Phi(e_n)$, is sufficient for the knowledge of the whole matrix $\Phi$ due to the definition of an automorphism

$$\Phi([x,y]) = [\Phi(x), \Phi(y)], \quad \forall x, y \in \mathfrak{n}$$

and the Lie brackets (7) – we can recover all $\Phi(e_k)$, $1 \leqslant k \leqslant n-3$ through multiple brackets of $\Phi(e_{n-2})$, $\Phi(e_{n-1})$ and $\Phi(e_n)$. A natural question is the following: Under which conditions do the relations

$$\Phi(e_{n-2}) = \alpha e_{n-2} + \sum_{k=1}^{n-3} \phi_k e_k,$$

$$\Phi(e_{n-1}) = \beta e_{n-1} + \gamma e_{n-2} + \sum_{k=1}^{n-3} \psi_k e_k,$$

$$\Phi(e_n) = \kappa e_n + \lambda e_{n-1} + \mu e_{n-2} + \sum_{k=1}^{n-3} \rho_k e_k$$

give rise to an automorphism of $\mathfrak{n}_{n,3}$?

Obviously, we must have $\alpha\beta\kappa \neq 0$ to have an invertible map. The preservation of $\mathfrak{z}_3$ implies $\gamma = 0$, $\psi_k = 0, k = 5, \ldots, n-3$. The remaining conditions are found as follows

- $0 = \Phi([e_{n-2}, e_{n-1}])$ implies $\phi_3 = 0$,
- $0 = \Phi([[e_{n-1}, e_n], e_n])$ leads to $\psi_4 = \frac{\lambda}{\kappa}\beta$,
- $0 = \Phi([[e_{n-1}, e_n], e_{n-1}]) + \Phi((-ad_{e_n})^{n-4} e_{n-2})$ leads to $\alpha = \beta^2\kappa^{5-n}$.

All other Lie brackets are either used to define $\Phi(e_k)$, $1 \leqslant k \leqslant n-3$ or are preserved trivially. Therefore, we conclude that any automorphism $\Phi$ of $\mathfrak{n}_{n,3}$ is defined in terms of $2n$ parameters which have been denoted by $\beta, \kappa, \lambda, \psi_1, \psi_2, \psi_3, \phi_1, \phi_2, \phi_4, \ldots, \phi_{n-3}, \rho_1, \ldots, \rho_{n-3}$. It acts on the generators of the Lie algebra $\mathfrak{n}_{n,3}$ in the following way:

$$\begin{aligned}
\Phi(e_{n-2}) &= \beta^2\kappa^{5-n} e_{n-2} + \sum_{k=4}^{n-3} \phi_k e_k + \phi_2 e_2 + \phi_1 e_1,\\
\Phi(e_{n-1}) &= \beta e_{n-1} + \frac{\lambda}{\kappa}\beta e_4 + \sum_{k=1}^{3} \psi_k e_k, \qquad (9)\\
\Phi(e_n) &= \kappa e_n + \lambda e_{n-1} + \mu e_{n-2} + \sum_{k=1}^{n-3} \rho_k e_k.
\end{aligned}$$

Taking automorphisms infinitesimally close to the unity, i.e. constructing the Lie algebra of the group of automorphisms, we find the algebra of derivations $\mathfrak{Der}(\mathfrak{n}_{n,3})$. It consists of all linear maps $D$ which act on the generators $e_{n-2}, e_{n-1}, e_n$ as follows:

$$\begin{aligned}
D(e_{n-2}) &= (2c_{n-1} + (5-n)d_n)e_{n-2} + \sum_{k=4}^{n-3} b_k e_k + b_2 e_2 + b_1 e_1,\\
D(e_{n-1}) &= c_{n-1}e_{n-1} + d_{n-1}e_4 + \sum_{k=1}^{3} c_k e_k, \qquad (10)\\
D(e_n) &= \sum_{k=1}^{n} d_k e_k;
\end{aligned}$$

the action of $D$ on the remaining basis elements $e_1, \ldots, e_{n-3}$ is uniquely determined using multiple brackets and the Leibniz's law

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

The $2n$-dimensional algebra of derivations $\mathfrak{Der}(\mathfrak{n}_{n,3})$ contains a $(n-1)$-dimensional ideal of inner derivations $\mathfrak{Inn}(\mathfrak{n}_{n,3})$ having the form

$$\begin{aligned} D(e_{n-2}) &= -c_3 e_{n-3}, \\ D(e_{n-1}) &= c_3 e_3 + c_1 e_1, \\ D(e_n) &= \sum_{k=1}^{n-3} d_k e_k. \end{aligned} \tag{11}$$

Indeed, such a derivation $D$ can be expressed as

$$D = \operatorname{ad}\left(d_1 e_2 + c_1 e_3 + d_2 e_4 + \sum_{k=4}^{n-3} d_k e_{k+1} - d_3 e_{n-1} + c_3 e_n\right). \tag{12}$$

Because $e_1$ spans the kernel of ad, i.e. the center of $\mathfrak{n}_{n,3}$, derivations of the form (11) exhaust all inner derivations.

## 3. Construction of solvable Lie algebras with the nilradical $\mathfrak{n}_{n,3}$

Firstly, we recall how the knowledge of automorphisms and derivations of a given nilpotent Lie algebra $\mathfrak{n}$ can be employed in the construction of all solvable Lie algebras $\mathfrak{s}$ with the nilradical $\mathfrak{n}$.

Let us consider a basis of $\mathfrak{s}$ in the form $(e_1, \ldots, e_n, f_1, \ldots, f_p)$ where $(e_1, \ldots, e_n)$ is a basis of $\mathfrak{n}$ with prescribed Lie brackets. Since $\mathfrak{n}$ is an ideal in $\mathfrak{s}$ and the derived algebra of $\mathfrak{s}$ falls into $\mathfrak{n}$ we necessarily have Lie brackets of the form

$$[f_a, e_j] = (A_a)_j^k e_k, \quad [f_a, f_b] = \gamma_{ab}^j e_j. \tag{13}$$

Furthermore, $\mathfrak{n}$ must be the maximal nilpotent ideal of $\mathfrak{s}$, i.e. any nonvanishing linear combination of the matrices $A_a$ must be non-nilpotent.

The algebra $\mathfrak{s}$ does not change if we transform its basis. Since the structure of $\mathfrak{n}$ is fixed we allow only such transformations that the Lie brackets in $\mathfrak{n}$ are not altered, i.e.

$$e_k \to \tilde{e}_k = e_j \Phi_{jk}, \quad f_a \to \tilde{f}_a = f_b \Xi_{ba} + e_k \Psi_{ka}, \tag{14}$$

where $\Phi$ is a matrix of an automorphism of $\mathfrak{n}$ in the original basis $(e_1, \ldots, e_n)$, $\Xi$ is a regular matrix and $\Psi$ is arbitrary.

We represent all non-nilpotent elements $f_a$ in the basis of $\mathfrak{s}$ by the corresponding operators in $\mathfrak{Der}(\mathfrak{n}) \subset \mathfrak{gl}(\mathfrak{n})$,

$$f_a \in \mathfrak{s} \to D_a = \operatorname{ad}_{f_a}|_{\mathfrak{n}} \in \mathfrak{Der}(\mathfrak{n}). \tag{15}$$

We note that under this mapping of $f_a$'s to outer derivations we lose some information – from the knowledge of $D_a, D_b$ we can reconstruct the Lie bracket $[f_a, f_b]$ only modulo the kernel of this map, i.e. modulo elements in the center of $\mathfrak{n}$. Nevertheless, the construction of all non-equivalent sets of $(D_1, \ldots, D_p)$ is crucial in the construction of all solvable Lie algebras $\mathfrak{s}$ with the nilradical $\mathfrak{n}$.

Because Eq. (15) defines a homomorphism of $\mathfrak{s}$ into $\mathfrak{Der}(\mathfrak{n})$ we can translate properties of $f_a$'s to $D_a$'s. In particular, a commutator of any $D_a, D_b$ must be an inner derivation and no nontrivial linear combination of $D_a$'s can be nilpotent. That means that $(D_1, \ldots, D_p)$ must span an Abelian subalgebra $\mathfrak{a}$ in the factor algebra $\mathfrak{Der}(\mathfrak{n})/\mathfrak{Inn}(\mathfrak{n})$ such that no nonvanishing element of $\mathfrak{a}$ is nilpotent. The subalgebras conjugated under any automorphism of $\mathfrak{n}$ are equivalent. Therefore, in an abstract formulation we can say that the Lie brackets of solvable extensions of $\mathfrak{n}$ are determined modulo elements in the center of $\mathfrak{n}$ by conjugacy classes of Abelian subalgebras $\mathfrak{a}$ of the factor algebra $\mathfrak{Der}(\mathfrak{n})/\mathfrak{Inn}(\mathfrak{n})$ such that no element of $\mathfrak{a}$ is represented by a nilpotent operator on $\mathfrak{n}$. Now the practical issue is how one can conveniently construct these classes for particular $\mathfrak{n} = \mathfrak{n}_{n,3}$?

Let us start by considering one additional basis element $f_1 \equiv f$, i.e. one derivation $D$. The elements of $\mathfrak{Der}(\mathfrak{n}_{n,3})/\mathfrak{Inn}(\mathfrak{n}_{n,3})$ can be uniquely represented by outer derivations of the form

$$\begin{aligned} D(e_{n-2}) &= (2c_{n-1} + (5-n)d_n)e_{n-2} + \sum_{k=4}^{n-4} b_k e_k + b_2 e_2 + b_1 e_1, \\ D(e_{n-1}) &= c_{n-1}e_{n-1} + d_{n-1}e_4 + c_3 e_3 + c_2 e_2, \\ D(e_n) &= d_n e_n + d_{n-1}e_{n-1} + d_{n-2}e_{n-2} \end{aligned} \tag{16}$$

(the action on $e_1, \ldots, e_{n-3}$ follows from the Leibniz's law). Above, a suitable inner derivation (11) was added to an arbitrary derivation, eliminating $n-1$ parameters. We mention that the form (16) of the representative of the coset $[D]$ is not invariant under conjugation by an automorphism

$$D \to D_\Phi = \Phi^{-1} \circ D \circ \Phi,$$

so that we may be forced to use a representative $\Phi(D)'$ of the coset $[\Phi(D)]$ different from $\Phi(D)$. Such a change of representative amounts to an addition of an inner derivation and is understood in all simplifications below whenever we employ an automorphism. Due to the triangular shape of $D$ we see that the sought-after Abelian subalgebras are at most two-dimensional since any higher dimensional subalgebra in $\mathfrak{Der}(\mathfrak{n}_{n,3})/\mathfrak{Inn}(\mathfrak{n}_{n,3})$ will necessarily involve nonvanishing nilpotent elements.

Next, we find all possible canonical forms of the coset (16) up to conjugation by automorphisms and rescaling. In order to reduce the problem to the one already investigated in [6] we realize that the derivation of the form (16) leaves

$$\tilde{\mathfrak{n}}_{n-2,1} = \mathrm{span}\{e_1, e_2, e_4, \ldots, e_{n-2}, e_n\}$$

invariant if and only if $d_{n-1} = 0$. We conjugate a given derivation $D$ by the automorphism defined by

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1} + \frac{d_{n-1}}{d_n - c_{n-1}} e_4, \quad \Phi(e_n) = e_n + \frac{d_{n-1}}{d_n - c_{n-1}} e_{n-1},$$

whenever possible, i.e. when $d_n \neq c_{n-1}$. Now we have $\hat{d}_{n-1} = 0$, i.e. $D_\Phi \equiv \widehat{D}$ leaves $\tilde{\mathfrak{n}}_{n-2,1}$ invariant. The case when none of the conjugate derivations $D_\Phi$ leaves $\tilde{\mathfrak{n}}_{n-2,1}$ invariant, which necessarily means that $d_n = c_{n-1}, \ d_{n-1} \neq 0$, will be dealt with later on.

Provided we set $d_{n-1} = 0$, the outer derivation (16) restricted to $\tilde{\mathfrak{n}}_{n-2,1}$ has the same structure as in [6, Eq. (25)]. Consequently, we may consider all solvable extensions of $\tilde{\mathfrak{n}}_{n-2,1}$ and then extend these to solvable extensions of $\mathfrak{n}_{n,3}$, i.e. determine the parameters $c_{n-1}, c_3, c_2$. In this way we obtain all solvable extensions of $\mathfrak{n}_{n,3}$ except the case $d_n = c_{n-1}, \ d_{n-1} \neq 0$.

The value of the parameter $c_{n-1}$ is fixed by the structure of the solvable extension of $\tilde{\mathfrak{n}}_{n-2,1}$. Namely, in relation to parameters $\alpha, \beta$ introduced below in Theorem 1 we have

$$c_{n-1} = \frac{1}{2}\left(\beta + (n-5)\alpha\right), \quad d_n = \alpha.$$

When $c_{n-1} \neq 0$ any derivation $D$ can be brought to $D_\phi$ with $c_2 = 0$ using an automorphism $\Phi$ specified by

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1} - \frac{c_2}{c_{n-1}} e_2, \quad \Phi(e_n) = e_n.$$

When $c_{n-1} = 0$ we cannot eliminate nonvanishing $c_2$ by any automorphism but we can bring it to 1 by rescaling of $e_k$'s provided such scaling remains available by the structure of the solvable extension of the subalgebra $\tilde{\mathfrak{n}}_{n-2,1}$. It turns out that for $c_{n-1} = 0$ two non-conjugate extensions of a derivation of $\tilde{\mathfrak{n}}_{n-2,1}$ exist, namely those determined by $c_2 = 0, 1$.

A similar consideration can be applied also to the parameter $c_3$. When $d_n \neq 0$ any derivation $D$ can be brought to $D_\phi$ with $c_3 = 0$ using the automorphism $\Phi$ specified by

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1} - \frac{c_3}{d_n} e_3, \quad \Phi(e_n) = e_n.$$

When $d_n = 0$ we cannot eliminate nonvanishing $c_3$ by any automorphism. Whether or not $c_3$ can be rescaled depends on the residual automorphisms still available – if the diagonal part of automorphisms is completely fixed by the structure of the solvable extension of the subalgebra $\tilde{\mathfrak{n}}_{n-2,1}$ nothing can be done, otherwise we can scale $c_3$ to 1 using an automorphism

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1}, \quad \Phi(e_n) = \frac{1}{c_3} e_n.$$

To sum up, the extension to a derivation of the nilradical $\mathfrak{n}_{n,3}$ is unique up to a conjugation when $d_n \neq 0$ and $c_{n-1} \neq 0$; otherwise, several non-equivalent extensions do exist.

We recall the main classification theorem of [6]:

**Theorem 1.** *Let $\mathbb{F}$ be the field of real or complex numbers. Any solvable Lie algebra $\tilde{\mathfrak{s}}$ over the field $\mathbb{F}$ with the nilradical $\mathfrak{n}_{m,1}$ has dimension $\dim \tilde{\mathfrak{s}} = m + 1$, or $\dim \tilde{\mathfrak{s}} = m + 2$. Three types of solvable Lie algebras of dimension $\dim \tilde{\mathfrak{s}} = m + 1$ exist for any $m \geqslant 4$. They are represented by the following:*

1. $[\tilde{f}, \tilde{e}_k] = ((m-k-1)\alpha + \beta)\tilde{e}_k,\ k \leqslant m-1, [\tilde{f}, \tilde{e}_m] = \alpha \tilde{e}_m$. *The classes of mutually nonisomorphic algebras of this type are*

$$\begin{aligned}
&\tilde{\mathfrak{s}}_{m+1,1}(\beta): \quad \alpha = 1, \quad \beta \in \mathbb{F}\backslash\{0, m-2\},\\
&\quad DS = [m+1, m, m-2, 0], \quad CS = [m+1, m], \quad US = [0],\\
&\tilde{\mathfrak{s}}_{m+1,2}: \quad \alpha = 1, \quad \beta = 0,\\
&\quad DS = [m+1, m-1, m-3, 0], \quad CS = [m+1, m-1], \quad US = [0],\\
&\tilde{\mathfrak{s}}_{m+1,3}: \quad \alpha = 1, \quad \beta = 2-m,\\
&\quad DS = [m+1, m, m-2, 0], \quad CS = [m+1, m], \quad US = [1],\\
&\tilde{\mathfrak{s}}_{m+1,4}: \quad \alpha = 0, \quad \beta = 1,\\
&\quad DS = [m+1, m-1, 0], \quad CS = [m+1, m-1], \quad US = [0].
\end{aligned}$$

2. $\tilde{\mathfrak{s}}_{m+1,5}: \quad [\tilde{f}, \tilde{e}_k] = (m-k)\tilde{e}_k,\ k \leqslant m-1, [\tilde{f}, \tilde{e}_m] = \tilde{e}_m + \tilde{e}_{m-1}$.
$$DS = [m+1, m, m-2, 0], \quad CS = [m+1, m], \quad US = [0].$$
3. $\tilde{\mathfrak{s}}_{m+1,6}(a_3, \ldots, a_{m-1}): [\tilde{f}, \tilde{e}_k] = \tilde{e}_k + \sum_{l=1}^{k-2} a_{k-l+1}\tilde{e}_l,\ k \leqslant m-1, [f, \tilde{e}_m] = 0, a_j \in \mathbb{F}$, *at least one $a_j$ satisfies $a_j \neq 0$.*
*Over $\mathbb{C}$: the first nonzero $a_j$ satisfies $a_j = 1$.*
*Over $\mathbb{R}$: the first nonzero $a_j$ for even $j$ satisfies $a_j = 1$. If all $a_j = 0$ for $j$ even, then the first nonzero $a_j$ ($j$ odd) satisfies $a_j = \pm 1$. We have*
$$DS = [m+1, m-1, 0], \quad CS = [m+1, m-1], \quad US = [0].$$

*For each $m \geqslant 4$ precisely one solvable Lie algebra $\tilde{\mathfrak{s}}_{m+2}$ of $\dim \tilde{\mathfrak{s}} = m + 2$ with the nilradical $\mathfrak{n}_{m,1}$ exists. It is represented by a basis $(\tilde{e}_1, \ldots, \tilde{e}_m, \tilde{f}_1, \tilde{f}_2)$ and the Lie brackets involving $f_1$ and $f_2$ are*

$$\begin{aligned}
&[\tilde{f}_1, \tilde{e}_k] = (m-1-k)\tilde{e}_k, \quad 1 \leqslant k \leqslant m-1, \quad [\tilde{f}_1, \tilde{e}_m] = \tilde{e}_m,\\
&[\tilde{f}_2, \tilde{e}_k] = \tilde{e}_k, \quad 1 \leqslant k \leqslant m-1, \quad [\tilde{f}_2, \tilde{e}_m] = 0, \quad [\tilde{f}_1, \tilde{f}_2] = 0.
\end{aligned}$$

*For this algebra we have*

$$DS = [m+2, m, m-2, 0], \quad CS = [m+2, m], \quad US = [0].$$

Above, we used the abbreviations *DS*, *CS* and *US* for (ordered) lists of integers denoting the dimensions of subalgebras in the derived, lower central and upper central series, respectively. We listed the last (then repeated) entry only once (e.g. we write $CS = [m, m-1]$ rather than $CS = [m, m-1, m-1, m-1, \ldots]$).

We must point out, however, that there is a caveat in the presented theorem. If we work over the field $\mathbb{R}$ the group of automorphisms of $\mathfrak{n}_{n-2,1}$ used in the derivation of Theorem 1 in [6] is slightly larger than the one we have available for the subalgebra $\tilde{\mathfrak{n}}_{n-2,1}$, i.e. inherited from automorphisms of $\mathfrak{n}_{n,3}$. In other words, the available automorphisms form a group only locally isomorphic to the group of automorphisms of $\mathfrak{n}_{n-2,1}$. Namely, the sign of $\alpha = \beta^2\kappa^{5-n}$ in Eq. (9) is restricted – for given $n$ we have $\operatorname{sgn}\alpha = (\operatorname{sgn}\kappa)^{n-5}$. As a consequence, for our purposes we must for $n$ even consider $[\tilde{f}, \tilde{e}_m] = \tilde{e}_m \pm \tilde{e}_{m-1}$ in $\tilde{\mathfrak{s}}_{m+1,5} (m = n-2)$. All other results in Theorem 1 hold irrespective of this constraint on allowed automorphisms.

The corresponding solvable extensions of the nilradical $\mathfrak{n}_{n,3}$ are summarized in Theorem 2 below.

Coming back to the case $d_n = c_{n-1}, d_{n-1} \neq 0$, we first rescale $D$ to get $d_n = c_{n-1} = 1$ and by scaling of $e_k$'s we set $d_{n-1} = 1$. Using the automorphism

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1}, \quad \Phi(e_n) = e_n + \frac{d_{n-2}}{n-6} e_{n-2},$$

we get rid of $d_{n-2}$; it is possible since $n \neq 6$. We get $D$ which preserves the subalgebra $\tilde{\mathfrak{n}}_{6,3}$. Therefore, it is enough to consider its solvable extensions (with $d_n = c_{n-1} = 1$) and then extend these to solvable algebras with the nilradical $\mathfrak{n}_{n,3}$. It turns out that such an enlargement is unique up to conjugation, i.e. fully determined by $d_n = c_{n-1} = 1, d_{n-1} = 1, d_{n-2} = 0$, the remaining parameters in Eq. (16) vanish.

Finally, the two-dimensional Abelian subalgebras $\mathfrak{a}$ in $\mathfrak{Der}(\mathfrak{n}_{n,3})/\mathfrak{Inn}(\mathfrak{n}_{n,3})$ are easily obtained using the results of the previous analysis. Such subalgebras must contain two linearly independent elements $D'_1, D'_2$, whose diagonal parameters can be chosen to have the values $c_{n-1} = 1, d_n = -1$ and $c_{n-1} = 1, d_n = 0$, respectively. Due to the chosen values for $D_1$ we can always go over to $\widetilde{D}_1 = (D'_1)_\Phi, \widetilde{D}_2 = (D'_2)_\Phi$ where $\widetilde{D}_1$ was diagonalized by a suitable automorphism $\Phi$. The restriction $[\widetilde{D}_1, \widetilde{D}_2] \in \mathfrak{Inn}(\mathfrak{n}_{n,3})$ now restricts $\widetilde{D}_2$ to be also diagonal. Therefore, all elements of $\mathfrak{a}$ act diagonally on $\mathfrak{n}_{n,3}$ in the chosen basis and can be expressed e.g. in the basis defined by $D_1$ ($c_{n-1} = 0, d_n = 1$) and $D_2$ ($c_{n-1} = 1, d_n = 0$). The corresponding non-nilpotent elements $f_1, f_2$ in $\mathfrak{s}$ in general satisfy

$$[f_1, f_2] = \alpha e_1 \in C(\mathfrak{n})$$

but a simple redefinition $f_1 \to f_1 + \frac{\alpha}{2} e_1$ gives an isomorphic solvable algebra $\mathfrak{s}$ with $[f_1, f_2] = 0$.

To sum up, we have the following theorem.

**Theorem 2.** *Let $\mathbb{F}$ be the field of real or complex numbers and n be an integer number greater or equal to 7. Any solvable Lie algebra $\mathfrak{s}$ over the field $\mathbb{F}$ with the nilradical $\mathfrak{n}_{n,3}$ has dimension* $\dim \mathfrak{s} = n+1$ *or* $\dim \mathfrak{s} = n+2$.

*Five types of solvable Lie algebras of dimension* $\dim \mathfrak{s} = n+1$ *with the nilradical $\mathfrak{n}_{n,3}$ exist. They are represented by the following:*

1. $[f, e_1] = (\alpha + 2\beta)e_1,\ [f, e_2] = 2\beta e_2,\ [f, e_3] = (\alpha+\beta)e_3,$
   $[f, e_k] = ((3-k)\alpha + 2\beta)e_k,\ 4 \leqslant k \leqslant n-2,$
   $[f, e_{n-1}] = \beta e_{n-1},\ [f, e_n] = \alpha e_n.$
   *The classes of mutually nonisomorphic algebras of this type are*

$$\begin{aligned}
&\mathfrak{s}_{n+1,1}(\beta): \quad \alpha = 1, \quad \beta \in \mathbb{F} \backslash \left\{0, -\frac{1}{2}, \frac{n-5}{2}\right\},\\
&\quad DS = [n+1, n, n-3, 0], \quad CS = [n+1, n], \quad US = [0],\\
&\mathfrak{s}_{n+1,2}: \quad \alpha = 1, \quad \beta = \frac{n-5}{2},\\
&\quad DS = [n+1, n-1, n-4, 0], \quad CS = [n+1, n-1], \quad US = [0],\\
&\mathfrak{s}_{n+1,3}: \quad \alpha = 1, \quad \beta = 0,\\
&\quad DS = [n+1, n-1, n-4, 0], \quad CS = [n+1, n-1], \quad US = [0],\\
&\mathfrak{s}_{n+1,4}: \quad \alpha = 1, \quad \beta = -\frac{1}{2},
\end{aligned}$$

$DS = [n+1, n, n-3, 0], \quad CS = [n+1, n], \quad US = [1],$

$\mathfrak{s}_{n+1,5}: \quad \alpha = 0, \quad \beta = 1,$

$DS = [n+1, n-1, 1, 0], \quad CS = [n+1, n-1], \quad US = [0].$

2. $\mathfrak{s}_{n+1,6}(\epsilon)$ :
$[f, e_1] = (n-3)e_1,\ [f, e_2] = (n-4)e_2,\ [f, e_3] = \left(\frac{n}{2} - 1\right) e_3,$
$[f, e_k] = (n-1-k)e_k,\ 4 \leqslant k \leqslant n-2,$
$[f, e_{n-1}] = \frac{n-4}{2} e_{n-1},\ [f, e_n] = e_n + \epsilon e_{n-2},$
*where* $\epsilon = 1$ *over* $\mathbb{C}$, *whereas over* $\mathbb{R}$ $\epsilon = 1$ *for n odd*, $\epsilon = \pm 1$ *for n even*.

$DS = [n+1, n, n-3, 0], \quad CS = [n+1, n], \quad US = [0].$

3. $\mathfrak{s}_{n+1,7}$ :
$[f, e_1] = e_1,\ [f, e_2] = 0,\ [f, e_3] = e_3 - e_1,$
$[f, e_k] = (3-k)e_k,\ 4 \leqslant k \leqslant n-2,$
$[f, e_{n-1}] = e_2,\ [f, e_n] = e_n.$

$DS = [n+1, n-1, n-4, 0], \quad CS = [n+1, n-1], \quad US = [0].$

4. $\mathfrak{s}_{n+1,8}(a_2, a_3, \ldots, a_{n-3})$ :
$[f, e_1] = e_1,\ [f, e_2] = e_2,\ [f, e_3] = \frac{1}{2} e_3,$
$[f, e_k] = e_k + \sum_{l=4}^{k-2} a_{k-l+1} e_l + a_{k-2} e_2 + a_{k-1} e_1,\ 4 \leqslant k \leqslant n-2,$
$[f, e_{n-1}] = \frac{1}{2} e_{n-1} + a_2 e_3,\ [f, e_n] = 0,$
$a_j \in \mathbb{F}$, *at least one* $a_j$ *satisfies* $a_j \neq 0$ *and*:

- *when* $\mathbb{F} = \mathbb{C}$ *the first nonzero* $a_j$ *satisfies* $a_j = 1$.
- *when* $\mathbb{F} = \mathbb{R}$ *the first nonzero* $a_j$ *for even j satisfies* $a_j = 1$. *If all* $a_j = 0$ *for j even, then the first nonzero* $a_j$ *(j odd) satisfies* $a_j = \pm 1$.

$DS = [n+1, n-1, 1, 0], \quad CS = [n+1, n-1], \quad US = [0].$

5. $\mathfrak{s}_{n+1,9}$ :
$[f, e_1] = 3e_1,\ [f, e_2] = 2e_2,\ [f, e_3] = 2e_3 - e_2,$
$[f, e_k] = (5-k)e_k,\ 4 \leqslant k \leqslant n-2,$
$[f, e_{n-1}] = e_{n-1} + e_4,\ [f, e_n] = e_n + e_{n-1}.$

$DS = [n+1, n, n-3, 0], \quad CS = [n+1, n], \quad US = [0].$

*Exactly one solvable Lie algebra* $\mathfrak{s}_{n+2}$ *of* $\dim \mathfrak{s} = n+2$ *with the nilradical* $\mathfrak{n}_{n,3}$ *exists. It is presented in a basis* $(e_1, \ldots, e_n, f_1, f_2)$ *where the Lie brackets involving* $f_1$ *and* $f_2$ *are*

$$\begin{aligned}
&[f_1, e_1] = e_1, \quad [f_2, e_1] = 2e_1,\\
&[f_1, e_2] = 0, \quad [f_2, e_2] = 2e_2,\\
&[f_1, e_3] = e_3, \quad [f_2, e_3] = e_3,\\
&[f_1, e_k] = (3-k)e_k, \quad [f_2, e_k] = 2e_k, \quad 4 \leqslant k \leqslant n-2,\\
&[f_1, e_{n-1}] = 0, \quad [f_2, e_{n-1}] = e_{n-1},\\
&[f_1, e_n] = e_n, \quad [f_2, e_n] = 0, \quad [f_1, f_2] = 0.
\end{aligned}$$

*For this algebra we have*

$DS = [n+2, n, n-3, 0], \quad CS = [n+2, n], \quad US = [0].$

We note that the class $\mathfrak{s}_{n+1,8}(a_2, a_3, \ldots, a_{n-3})$ encompasses both extensions of $\tilde{\mathfrak{s}}_{m+1,7}(a_3, \ldots, a_{m-1})$ and an extension of $\tilde{\mathfrak{s}}_{m+1,4}$ with $c_3 \neq 0$ in Eq. (16). The parameter brought to $\pm 1$ was selected in the most convenient form for presentation and consequently is equivalent but slightly different from a direct extension of $\tilde{\mathfrak{s}}_{m+1,7}(a_3, \ldots, a_{m-1})$ to the nilradical $\mathfrak{n}_{n,3}$ – for that choice the non-equivalent values of parameters would be more cumbersome to write down.

Next, we investigate the classification of solvable extensions of $\mathfrak{n}_{n,3}$ in low dimensions $n = 6, 5$. Results in these dimensions somewhat differ from the general ones presented in Theorem 2.

*3.1. Dimension $n = 6$*

When $n = 6$ the results are as follows: all the algebras presented in Theorem 2 exist (with $e_{n-2} \equiv e_4$) but they do not exhaust all the possibilities. The reason for this is that in this particular dimension we have $[f, e_{n-2}] = (2c_5 - d_6)e_{n-2} + \ldots$ Therefore, if $d_6 = c_5$ then also $[f, e_{n-2}] = d_6 e_{n-2} + \cdots$ That implies that if we have $d_6 = c_5 \to 1,\ d_5 \neq 0,\ d_4 \neq 0$ in the derivation (16) then we can set to zero neither $d_5$ nor $d_4$ by any choice of automorphism $\Phi$ and we are left with only one scaling available - preferably used to set $d_5 \to 1$.

That means that for the 6-dimensional nilradical $\mathfrak{n}_{6,3}$ we have solvable extensions $\mathfrak{s}_{7,1}(\beta)$, $\mathfrak{s}_{7,2}$, $\mathfrak{s}_{7,3}$, $\mathfrak{s}_{7,4}$, $\mathfrak{s}_{7,5}$, $\mathfrak{s}_{7,6}(\epsilon)$, $\mathfrak{s}_{7,7}$, $\mathfrak{s}_{7,8}(1, a_3)$, $\mathfrak{s}_{7,8}(0, \epsilon)$, $\mathfrak{s}_{7,9}$, $\mathfrak{s}_8$ where $\epsilon = 1$ over $\mathbb{C}$ and $\epsilon = \pm 1$ over $\mathbb{R}$, whose structure is as described in Theorem 2 above and one additional class of algebras, differing from $\mathfrak{s}_{7,9}$ by one additional nonvanishing parameter $\alpha$

- $\mathfrak{s}_{7,10}(\alpha),\ \alpha \neq 0$ :
  $[f, e_1] = 3e_1,\ [f, e_2] = 2e_2,\ [f, e_3] = 2e_3 - e_2,$
  $[f, e_4] = e_4,\ [f, e_5] = e_5 + e_4,\ [f, e_6] = e_6 + e_5 + \alpha e_4,$
  $$DS = [7, 6, 3, 0], \quad CS = [7, 6], \quad US = [0].$$

*3.2. Dimension $n = 5$*

When $n = 5$, the investigation must be performed in a different way. Namely, there is no $\tilde{\mathfrak{n}}_{3,1}$ subalgebra – it has collapsed to the Heisenberg algebra which has different properties. Nevertheless, by a rather straightforward, if repetitive, computation (essentially linear algebra of $5 \times 5$ matrices) one can construct all solvable extensions of $\mathfrak{n}_{5,3}$. Since this was done already in [12] for one non-nilpotent element and for two elements the result can be derived from the previous one, we shall only list the results and compare them to their higher dimensional analogues. In order to make our comparison as simple as possible we work in a basis analogous to Eq. (6), namely

$$e_1 = x_1, \quad e_2 = x_3, \quad e_3 = x_2, \quad e_4 = x_5, \quad e_5 = x_4. \tag{17}$$

The nonvanishing Lie brackets are

$$[e_2, e_5] = e_1, \quad [e_3, e_4] = e_1, \quad [e_4, e_5] = -e_3. \tag{18}$$

Although the structure of the nilradical is quite different from the other elements of the series, the set of solvable extensions is rather similar. We get analogues of all solvable algebras in Theorem 2 with some changes in the structure of $\mathfrak{s}_{n+1,6}, \mathfrak{s}_{n+1,8}, \mathfrak{s}_{n+1,9}$; in addition, the two algebras $\mathfrak{s}_{n+1,2}$ and $\mathfrak{s}_{n+1,3}$ become identical when $n = 5$. The fact that the algebras $\mathfrak{s}_{n+1,6}, \mathfrak{s}_{n+1,8}, \mathfrak{s}_{n+1,9}$ must be modified when $n = 5$ can be inferred already from Theorem 2 since the Lie brackets as presented there cannot be made sense of if $n = 5$. These structurally different analogues are distinguished by primes below.

Explicitly, assuming the structure of $\mathfrak{n}_{5,3}$ in the form (18), we have the following Lie brackets with non-nilpotent element(s) and dimensions of the characteristic series

- $\mathfrak{s}_{6,1}(\beta),\ \beta \in \mathbb{F} \backslash \left\{0, -\frac{1}{2}\right\}$ :
  $[f, e_1] = (1 + 2\beta)e_1,\ [f, e_2] = 2\beta e_2,\ [f, e_3] = (\beta + 1)e_3,\ [f, e_4] = \beta e_4, [f, e_5] = e_5,$
  $$DS = [6, 5, 2, 0], \quad CS = [6, 5], \quad US = [0].$$
- $\mathfrak{s}_{6,2}$ : $[f, e_1] = e_1,\ [f, e_2] = 0,\ [f, e_3] = e_3,\ [f, e_4] = 0,\ [f, e_5] = e_5,$

$DS = [6, 3, 0], \quad CS = [6, 3], \quad US = [0].$

- $\mathfrak{s}_{6,4} : [f, e_1] = 0,\ [f, e_2] = -e_2,\ [f, e_3] = \frac{1}{2}e_3,\ [f, e_4] = -\frac{1}{2}e_4,\ [f, e_5] = e_5,$

  $DS = [6, 5, 2, 0], \quad CS = [6, 5], \quad US = [1].$
- $\mathfrak{s}_{6,5} : [f, e_1] = 2e_1,\ [f, e_2] = 2e_2,\ [f, e_3] = e_3,\ [f, e_4] = e_4,\ [f, e_5] = 0,$

  $DS = [6, 4, 1, 0], \quad CS = [6, 4], \quad US = [0].$
- $\mathfrak{s}'_{6,6} : [f, e_1] = 2e_1,\ [f, e_2] = e_2, [f, e_3] = \frac{3}{2}e_3,\ [f, e_4] = \frac{1}{2}e_4, [f, e_5] = e_5 + e_2,$

  $DS = [6, 5, 2, 0], \quad CS = [6, 5], \quad US = [0].$
- $\mathfrak{s}_{6,7} : [f, e_1] = e_1,\ [f, e_2] = 0,\ [f, e_3] = e_3 - e_1,\ [f, e_4] = e_2,\ [f, e_5] = e_5,$

  $DS = [6, 4, 1, 0], \quad CS = [6, 4, 3], \quad US = [0].$
- $\mathfrak{s}'_{6,8} : [f, e_1] = 2e_1,\ [f, e_2] = 2e_2,\ [f, e_3] = e_3,\ [f, e_4] = -e_3 + e_4,\ [f, e_5] = 0,$

  $DS = [6, 4, 1, 0], \quad CS = [6, 4], \quad US = [0].$
- $\mathfrak{s}'_{6,9} : [f, e_1] = 3e_1, [f, e_2] = 2e_2 - e_3, [f, e_3] = 2e_3, [f, e_4] = e_4 + e_5, [f, e_5] = e_5,$

  $DS = [6, 5, 2, 0], \quad CS = [6, 5], \quad US = [0].$
- $\mathfrak{s}_7 : [f_1, e_1] = e_1,\ [f_1, e_2] = 0,\ [f_1, e_3] = e_3,\ [f_1, e_4] = 0,\ [f_1, e_5] = e_5,$
  $[f_2, e_1] = 2e_1,\ [f_2, e_2] = 2e_2,\ [f_2, e_3] = e_3,\ [f_2, e_4] = e_4,\ [f_2, e_5] = 0,$
  $[f_1, f_2] = 0,$

  $DS = [7, 5, 2, 0], \quad CS = [7, 5], \quad US = [0].$

We note that in several cases the characteristic series are different from the ones in Theorem 2. This difference in behavior is due to the structural difference between $\mathfrak{n}_{n-2,1}$ and the Heisenberg algebra.

## 4. Generalized Casimir invariants

We proceed to construct generalized Casimir invariants, i.e. invariants of the coadjoint representation, of the nilpotent algebra $\mathfrak{n}_{n,3}$ and its solvable extensions. We recall that a basis for the coadjoint representation of the Lie algebra $\mathfrak{g}$ is given by the first order differential operators

$$\widehat{X}_k = \mathrm{x}_a c^a_{kb} \frac{\partial}{\partial \mathrm{x}_b} \tag{19}$$

acting on functions on the vector space $\mathfrak{g}^*$. Here, $c^k_{ij}$ are the structure constants of the Lie algebra $\mathfrak{g}$ in the given basis $(x_1, \ldots, x_N)$ and the quantities $\mathrm{x}_a$ are coordinates in the basis of the space $\mathfrak{g}^*$ dual to the basis $(x_1, \ldots, x_N)$ of the algebra $\mathfrak{g}$. That means that $\mathrm{x}_a$ are linear functionals on $\mathfrak{g}^*$, i.e. $\mathrm{x}_a \in (\mathfrak{g}^*)^*$, and through the canonical isomorphism of vector spaces $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ one can identify $\mathrm{x}_a \simeq x_a$. In what follows we shall not typographically distinguish between $\mathrm{x}_a$ and $x_a$, the meaning - vector in algebra vs. linear functional on the dual space - shall be clear from the context.

Invariants of the coadjoint representation, i.e. generalized Casimir invariants, are functions $I$ on $\mathfrak{g}^*$ which satisfy the following system of partial differential equations

$$\widehat{X}_k I(x_1, \ldots, x_N) = 0, \quad k = 1, \ldots, N. \tag{20}$$

Several methods exist for construction of invariants of the coadjoint representation, most widely used ones are direct solution of Eq. (20) by the method of characteristics (see e.g. [14–17]) and the method of moving frames (see [18–23]).

However, we shall use a different approach. We reduce Eq. (20) to the ones encountered and solved in [6] for the subalgebra $\tilde{\mathfrak{n}}_{n-2,1}$ and its solvable extensions.

Considering first the nilpotent algebra $\mathfrak{n}_{n,3}$ we have the operators (19) in the form

$$\begin{aligned}
&\widehat{E}_1 = 0, \quad \widehat{E}_2 = e_1 \frac{\partial}{\partial e_n}, \quad \widehat{E}_3 = e_1 \frac{\partial}{\partial e_{n-1}}, \quad \widehat{E}_4 = e_2 \frac{\partial}{\partial e_n}, \\
&\widehat{E}_k = e_{k-1} \frac{\partial}{\partial e_n}, \quad 5 \leqslant k \leqslant n-2, \quad \widehat{E}_{n-1} = -e_1 \frac{\partial}{\partial e_3} - e_3 \frac{\partial}{\partial e_n}, \\
&\widehat{E}_n = -e_1 \frac{\partial}{\partial e_2} - e_2 \frac{\partial}{\partial e_4} - \sum_{k=5}^{n-2} e_{k-1} \frac{\partial}{\partial e_k} + e_3 \frac{\partial}{\partial e_{n-1}}.
\end{aligned} \tag{21}$$

It is evident that any solution $I$ of Eq. (20) cannot depend[3] on $e_3, e_{n-1}$ because of $\widehat{E}_{n-1} I = \widehat{E}_3 I = \widehat{E}_2 I = 0$. Consequently, all considered operators $\widehat{E}_j$ can be truncated to act on functions of $\tilde{e}_1 = e_1, \tilde{e}_2 = e_2, \tilde{e}_3 = e_4, \ldots, \tilde{e}_{n-3} = e_{n-2}, \tilde{e}_{n-2} = e_n$ only. Then $\widehat{E}_{3T}, \widehat{E}_{n-1T}$ vanish and the remaining operators are exactly those present in the investigation of invariants of $\mathfrak{n}_{n-2,1}$ in [6]. Therefore, the generalized Casimir invariants of $\mathfrak{n}_{n,3}$ are the same as the ones for $\mathfrak{n}_{n-2,1}$ once written in an appropriate basis.

Similarly, when we consider the solvable extensions of $\mathfrak{n}_{n,3}$, the operators $\widehat{E}_j$ in (21) get additional $\frac{\partial}{\partial f}$ or $\frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2}$ terms and one ($\widehat{F}$) or two ($\widehat{F}_1, \widehat{F}_2$) additional operators are present in Eq. (20).

Let us first consider the case with $\widehat{F}$ only. When the derivation $D$ defining $f$ is such that $2c_{n-1} + d_n \neq 0$, we have $\widehat{E}_1 = (2c_{n-1} + d_n) e_1 \frac{\partial}{\partial f}$ which excludes the dependence of $I$ on $f$. When $2c_{n-1} + d_n = 0$ the situation is only slightly more complicated – the operators $\widehat{E}_2, \widehat{E}_4$ together again exclude the dependence of $I$ on both $f$ and $e_n$. In both cases, we can restrict all operators (21) and $\widehat{F}$ to $\mathfrak{n}_{n,3}$ and then to $\mathfrak{n}_{n-2,1}$, reducing the computation to the corresponding solvable extension of $\mathfrak{n}_{n-2,1}$.

In the second case we have two additional operators $\widehat{F}_1, \widehat{F}_2$ and $\frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2}$ terms in $\widehat{E}_j$. Now the operators $\widehat{E}_1, \widehat{E}_2, \widehat{E}_3, \widehat{E}_4$ are used in the same way to show that any invariant $I$ cannot depend on $f_1, f_2$.

Altogether, the construction of generalized Casimir invariants was fully reduced to the one for the nilradical $\mathfrak{n}_{n-2,1}$.

As proved in [6], invariants of the Lie algebra $\mathfrak{n}_{m,1}$ and its solvable extensions are as follows:

**Theorem 3.** *The nilpotent Lie algebra $\mathfrak{n}_{m,1}$ has $m-2$ functionally independent invariants. They can be chosen to be the following polynomials:*

$$\begin{aligned}
&\tilde{\xi}_0 = \tilde{e}_1, \\
&\tilde{\xi}_k = \frac{(-1)^k k}{(k+1)!} \tilde{e}_2^{k+1} + \sum_{j=0}^{k-1} (-1)^j \frac{\tilde{e}_2^j \, \tilde{e}_{k+2-j} \tilde{e}_1^{k-j}}{j!}, \quad 1 \leqslant k \leqslant m-3.
\end{aligned} \tag{22}$$

*The algebras $\tilde{\mathfrak{s}}_{m+1,1}(\beta), \ldots, \tilde{\mathfrak{s}}_{m+1,5}$ have $m-3$ invariants each. Their form is*

1. $\tilde{\mathfrak{s}}_{m+1,1}(\beta)$, $\tilde{\mathfrak{s}}_{m+1,2}$ *and* $\tilde{\mathfrak{s}}_{m+1,5}$ :

$$\tilde{\chi}_k = \frac{\tilde{\xi}_k}{\tilde{\xi}_0^{(k+1)\frac{m-3+\beta}{m-2+\beta}}}, \quad 1 \leqslant k \leqslant m-3. \tag{23}$$

*For $\tilde{\mathfrak{s}}_{m+1,2}$ and $\tilde{\mathfrak{s}}_{m+1,5}$ we have $\beta = 0$ and $\beta = 1$, respectively in Eq.* (23).

2. $\tilde{\mathfrak{s}}_{m+1,3}$ :

$$\tilde{\chi}_1 = \tilde{\xi}_0, \quad \tilde{\chi}_k = \frac{\tilde{\xi}_k^2}{\tilde{\xi}_1^{k+1}}, \quad 2 \leqslant k \leqslant m-3. \tag{24}$$

[3] Neither can $I$ depend on $e_n$.

3. $\tilde{\mathfrak{s}}_{m+1,4}$ :

$$\tilde{\chi}_k = \frac{\tilde{\xi}_k}{\tilde{\xi}_0^{k+1}}, \quad 1 \leqslant k \leqslant m-3. \tag{25}$$

4. $\tilde{\mathfrak{s}}_{m+1,7}(a_3, \ldots, a_{m-1})$ :

$$\tilde{\chi}_k = \sum_{q=0}^{\left[\frac{k+1}{2}\right]} (-1)^q \frac{(\ln \tilde{\xi}_0)^q}{q!} \left( \sum_{i_1+\cdots+i_q=k-2q+1} a_{i_1+3}a_{i_2+3}\ldots a_{i_q+3} + \sum_{j+i_1+\cdots+i_q=k-2q-1} \frac{\tilde{\xi}_{j+1}}{\tilde{\xi}_0^{j+2}} a_{i_1+3}a_{i_2+3}\ldots a_{i_q+3} \right), \quad 1 \leqslant k \leqslant m-3. \tag{26}$$

*The summation indices take the values* $0 \leqslant j, i_1, \ldots, i_q \leqslant k+1$.

*The Lie algebra* $\tilde{\mathfrak{s}}_{m+2}$ *has* $m-4$ *functionally independent invariants that can be chosen to be*

$$\tilde{\chi}_k = \frac{\tilde{\xi}_{k+1}}{\tilde{\xi}_1^{\frac{k+2}{2}}}, \quad 1 \leqslant k \leqslant m-4. \tag{27}$$

The results for $\mathfrak{n}_{n,3}$ and its solvable extensions are now as follows:

**Theorem 4.** *Let* $n \geqslant 6$. *The nilpotent Lie algebra* $\mathfrak{n}_{n,3}$ *has* $n-4$ *functionally independent invariants. They can be chosen to be the following polynomials*

$$\xi_0 = e_1,$$
$$\xi_k = \frac{(-1)^k k}{(k+1)!} e_2^{k+1} + \sum_{j=0}^{k-1} (-1)^j \frac{e_2^j \, e_{k+3-j} \, e_1^{k-j}}{j!}, \quad 1 \leqslant k \leqslant n-5. \tag{28}$$

*The algebras* $\mathfrak{s}_{n+1,1}(\beta), \ldots, \mathfrak{s}_{n+1,9}$ *have* $n-5$ *invariants each. Their form is*

1. $\mathfrak{s}_{n+1,1}(\beta)$, $\mathfrak{s}_{n+1,2}$, $\mathfrak{s}_{n+1,3}$, $\mathfrak{s}_{n+1,6}$, $\mathfrak{s}_{n+1,7}$ *and* $\mathfrak{s}_{n+1,9}$ :

$$\chi_k = \frac{\xi_k}{\xi_0^{(k+1)\frac{2\beta}{1+2\beta}}}, \quad 1 \leqslant k \leqslant n-5. \tag{29}$$

*For* $\mathfrak{s}_{n+1,2}$ *is* $\beta = \frac{n-5}{2}$, *for* $\mathfrak{s}_{n+1,3}$ *and* $\mathfrak{s}_{n+1,7}$ *we have* $\beta = 0$, *for* $\mathfrak{s}_{n+1,6}(\epsilon)$ *we have* $\beta = \frac{n-4}{2}$ *and for* $\mathfrak{s}_{n+1,9}$ *is* $\beta = 1$, *respectively in Eq.* (29).

2. $\mathfrak{s}_{n+1,4}$ :

$$\chi_1 = \xi_0, \quad \chi_k = \frac{\xi_k^2}{\xi_1^{k+1}}, \quad 2 \leqslant k \leqslant n-5. \tag{30}$$

3. $\mathfrak{s}_{n+1,5}$ :

$$\chi_k = \frac{\xi_k}{\xi_0^{k+1}}, \quad 1 \leqslant k \leqslant n-5. \tag{31}$$

4. $\mathfrak{s}_{n+1,8}(a_2, a_3, \ldots, a_{n-3})$ :

$$\chi_k = \sum_{q=0}^{\left[\frac{k+1}{2}\right]} (-1)^q \frac{(\ln \xi_0)^q}{q!} \left( \sum_{i_1+\cdots+i_q=k-2q+1} a_{i_1+3}a_{i_2+3}\ldots a_{i_q+3} \right. \tag{32}$$

$$+ \sum_{j+i_1+\cdots+i_q=k-2q-1} \frac{\xi_{j+1}}{\xi_0^{j+2}} a_{i_1+3}a_{i_2+3}\ldots a_{i_q+3} \Bigg), \quad 1 \leqslant k \leqslant n-5.$$

*The summation indices take the values* $0 \leqslant j, i_1, \ldots, i_q \leqslant k+1$.

*When* $n = 6$ *the Lie algebra* $\mathfrak{s}_{7,10}(\alpha)$ *has one invariant which can be chosen in the form* $\frac{2e_4e_1-e_2^2}{e_1^{4/3}}$, *i.e. coincides with the one for* $\mathfrak{s}_{7,9}$.

*The Lie algebra* $\mathfrak{s}_{n+2}$ *has* $n-6$ *functionally independent invariants that can be chosen to be*

$$\chi_k = \frac{\xi_{k+1}}{\xi_1^{\frac{k+2}{2}}}, \quad 1 \leqslant k \leqslant n-6. \tag{33}$$

We point out that the algebras $\mathfrak{s}_{n+1,3}$ and $\mathfrak{s}_{n+1,7}$ are examples of solvable non-nilpotent Lie algebras with a polynomial basis of invariants, i.e. their bases of invariants can be chosen in the form of Casimir operators in the enveloping algebra of $\mathfrak{s}_{n+1,3}$ and $\mathfrak{s}_{n+1,7}$ (the same holds also for $\tilde{s}_{m+1,1}(3-m)$ of [6]). If ever a hypothesis concerning a criterion for the existence of polynomial basis of invariants of solvable algebras is presented, these examples can be easily used as simple tests of its plausibility.

For 5-dimensional nilradical $\mathfrak{n}_{5,3}$ we have solvable algebras $\mathfrak{s}_{6,1}(\beta), \mathfrak{s}_{6,2}, \mathfrak{s}_{6,5}, \mathfrak{s}'_{6,6}, \mathfrak{s}_{6,7}, \mathfrak{s}'_{6,8}, \mathfrak{s}'_{6,9}$ with no invariants and $\mathfrak{s}_{6,4}$ which has two invariants. They can be chosen in the polynomial form

$$e_1, \quad 2e_1^2f - 2e_1e_2e_5 + e_1e_3e_4 + e_2e_3^2.$$

The algebra $\mathfrak{s}_7$ has one invariant

$$\frac{(f_2-2f_1)e_1^2 + (2e_2e_5 - e_3e_4)e_1 - e_2e_3^2}{e_1^2}.$$

We observe that invariants of the solvable Lie algebras with the nilradical $\mathfrak{n}_{5,3}$ (if nonconstant) depend on elements outside of $\mathfrak{n}_{5,3}$, i.e. $f$ or $f_1, f_2$. This is related to the fact that there is no $\tilde{\mathfrak{n}}_{3,1}$ subalgebra – it degenerates to the Heisenberg algebra, the properties of which are markedly different.

## 5. Conclusions

We have fully classified all solvable Lie algebras with the nilradical $\mathfrak{n}_{n,3}$ in arbitrary dimension $n$ and constructed their generalized Casimir invariants.

There are two general lessons to be learned from this computation. Firstly, it turned out that the knowledge of all solvable extensions of a suitable subalgebra $\tilde{\mathfrak{n}}$ of the given nilpotent algebra $\mathfrak{n}$ may lead to a significant simplification of the whole computation and is definitively worth investigating if such subalgebras are identified in $\mathfrak{n}$. This can hold notwithstanding the fact that not all automorphisms of $\mathfrak{n}$ preserve the subalgebra $\tilde{\mathfrak{n}}$. Of course, it was important in our investigation that the structure of the subalgebra was restrictive enough, i.e. we expect that a similar simplification can be achieved probably for subalgebras with high enough degree of nilpotency, e.g. filiform or quasi-filiform.

Secondly, it was of profound importance that (almost) all automorphisms of $\tilde{\mathfrak{n}}$ could be obtained as a restriction of automorphisms of $\mathfrak{n}$. In our case we had a local isomorphism of $Aut(\tilde{\mathfrak{n}})$ and $Aut(\mathfrak{n})|_{\tilde{\mathfrak{n}}}$; the two differ topologically by the absence of some connected components of $Aut(\tilde{\mathfrak{n}})$ in $Aut(\mathfrak{n})|_{\tilde{\mathfrak{n}}}$. This minor difference could be easily taken into account and the classification of all solvable extensions of $\tilde{\mathfrak{n}}$ with respect to this restricted group of automorphisms acting on $\tilde{\mathfrak{n}}$ was obtained by inspection from previously known results [6]. On the other hand, had the $Aut(\tilde{\mathfrak{n}})$ and $Aut(\mathfrak{n})|_{\tilde{\mathfrak{n}}}$ been locally non-isomorphic, the knowledge of solvable extensions of $\tilde{\mathfrak{n}}$ would not be of much use in the study of solvable extensions of $\mathfrak{n}$. A simple example of this is the maximal Abelian ideal $\mathfrak{a}$ of $\mathfrak{n}$. Its group of automorphisms *per se* is typically much larger than the automorphisms inherited from $\mathfrak{n}$, i.e. many transformations used in $\mathfrak{a}$ are not allowed in $\mathfrak{n}$ and, at the same time, most of solvable extensions of $\mathfrak{a}$ cannot be enlarged to solvable extensions of $\mathfrak{n}$ – the Lie brackets in $\mathfrak{n}$ simply do not allow that.

Therefore, the particular properties of the subalgebra and its immersion into the whole nilradical are of crucial importance for the whole setup to work.

Finally, we have seen that although the considered series of nilpotent algebras can be rather naturally constructed starting from dimension $n = 5$, the 5-dimensional one has substantially different properties. They reflect themselves also in possible solvable extensions and their invariants.

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## 2.4

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# On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras

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**Abstract**

We establish an improved upper estimate on the dimension of any solvable Lie algebra $\mathfrak{s}$ with its nilradical isomorphic to a given nilpotent Lie algebra $\mathfrak{n}$. Next we consider Levi decomposable algebras with a given nilradical $\mathfrak{n}$ and investigate restrictions on possible Levi factors originating from the structure of characteristic ideals of $\mathfrak{n}$. We present a new perspective on Turkowski's classification of Levi decomposable algebras up to dimension 9.

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## 1. Introduction

The aim of this paper is to establish some general properties of solvable and Levi extensions of nilpotent Lie algebras.

As is well known, the problem of classification of all solvable (including nilpotent) Lie algebras in an arbitrarily large finite dimension is presently unsolved and is generally believed to be unsolvable, at least unless some completely new ideas emerge and a new understanding of the notion 'classification' itself develops. The problem stems from an obvious fact that the number of solvable Lie algebras in higher dimensions increases drastically, and infinite parametrized families of such nonisomorphic algebras arise already in very low dimensions. This behavior is in stark contrast with the theory of semisimple algebras where only finitely many algebras exist in any given dimension and their full classification was completed already long time ago [1, 2]. Because any Lie algebra is a semidirect sum of its maximal solvable ideal and a semisimple subalgebra by the theorem of Levi [3], the difficulty in the classification of solvable algebras also shows up in the classification of all types of non-semisimple algebras.

All known full classifications of non-semisimple algebras terminate at relatively low dimensions. First low-dimensional classifications were established already by Lie and his contemporaries in [4–6], their results are reviewed e.g. in [7]. Newer results since the mid-20th century are the classifications of nilpotent algebras in dimension 6 [8], dimension 7 [9–12], dimension 8 [13] and partially in dimension 9 [14], and of solvable algebras in dimension 5 [15], dimension 6 [16, 17] and some partial results in dimension 7 [18]. Algebras of semidirect sum type, i.e. Levi decomposable algebras, were classified up to dimension 8 in [19] and in dimension 9 in [20].

As a possible stopgap solution, the idea of a classification of solvable extensions of certain particular classes of nilpotent Lie algebras, i.e. of all solvable, non-nilpotent algebras with the given nilradical, of arbitrarily large dimension emerged. It is based on a belief that the knowledge of full classification of all solvable extensions of certain series of nilradicals can be very useful for both theoretical considerations—e.g. testing various hypotheses concerning the general structure of solvable Lie algebras—and practical purposes—e.g. when a generalization of a given algebra or its nilradical to higher dimensions is needed in some physical situation. Such need arises for example in the construction of superintegrable systems from a given solvable Lie algebra and its Casimir invariants which was introduced in [21]. Another application comes from the construction of cosmological models in higher dimensions, now fashionable e.g. in string cosmology, using algebraic methods [22, 23]. Lie algebras of Killing vectors are in many cases solvable[1] as was realized already in [24] using the classification of homogeneous spaces [25]. Higher dimensional solvable Lie algebras and their semidirect sums with semisimple algebras therefore appear naturally in such constructions in higher spacetime dimensions, see e.g. [26]. If some of the properties of the resulting spacetimes ought to resemble the behavior of their low-dimensional counterparts then it is natural to expect that also their algebras of Killing vectors should have some properties in common, e.g. to belong to one common series of algebras.

Gradually a series of classifications of solvable extensions was performed. The first one was done by Winternitz together with Rubin in [27]. The series then continued throughout the years in [28–32]. All these papers dealt with the problem of classification of all solvable Lie algebras with the given $n$-dimensional nilradical, e.g. Abelian algebra, Heisenberg algebra, the algebra of strictly upper triangular matrices, etc, for arbitrary finite dimension $n$. Similar sequences have also recently been investigated by other research groups in [33] (naturally graded nilradicals with maximal nilindex and a Heisenberg subalgebra of codimension 1) and [34] (a certain sequence of quasi-filiform nilradicals). A recent paper [35] generalized results of [30, 31] in the sense that all solvable extensions of $\mathbb{N}$-graded filiform nilradicals were classified.

Levi decomposable algebras with a fixed structure of their nilradical were considered in [36] for Heisenberg nilradicals.

This paper builds on our experience gained in [30–32]. In those papers we analyzed in detail the structure of solvable extensions of particular chosen sequences of nilradicals. Here, we use methods and ideas developed in [30, 31] in a different direction, namely, to give at least some general estimate on the dimension of any solvable Lie algebra with a given nilradical.

Next we investigate Levi decomposable algebras with a given nilradical. Our main concern is the formulation of some necessary conditions for the existence of nontrivial Levi extension(s). We formulate these conditions in terms of characteristic ideals of the nilradical, in particular in terms of lower central series.

The structure of our paper is as follows. After an introduction of notation and a brief review of known facts in sections 2 and 3 we present an improved upper bound on the dimension

[1] It was probably the first appearance of solvable groups and algebras in physics other than their use in the solution of ODEs using the reduction method of Lie.

of any solvable Lie algebra with the given nilradical in section 4. In section 5, we study the structure of Levi decomposable algebras with the given nilradicals and provide a novel perspective on the classification of Turkowski. Finally, we introduce some open questions.

Throughout the paper the analysis is done over the fields of complex and real numbers unless indicated otherwise.

## 2. Notation

We shall often need to refer to the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{1}$$

for a particular triple $x, y, z$ of vectors in $\mathfrak{g}$. For brevity, we speak about the Jacobi identity $(x, y, z)$ in such a case. A Lie bracket of two vector subspaces is defined to be the whole span

$$[\mathfrak{h}, \tilde{\mathfrak{h}}] = \mathrm{span}\{[x, \tilde{x}] | x \in \mathfrak{h}, \tilde{x} \in \tilde{\mathfrak{h}}\}.$$

For a given Lie algebra $\mathfrak{g}$ we consider the following three series of ideals.

The *derived series* $\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \cdots \supset \mathfrak{g}^{(k)} \supset \cdots$ is defined recursively as

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \qquad \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \qquad k \geqslant 1. \tag{2}$$

If the derived series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is a *solvable Lie algebra*.

The *lower central series*, which is of particular importance for our considerations in this paper, $\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots \supset \mathfrak{g}^k \supset \cdots$ is again defined recursively as

$$\mathfrak{g}^1 = \mathfrak{g}, \qquad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \qquad k \geqslant 2. \tag{3}$$

If the lower central series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ is called a *nilpotent Lie algebra*. The largest value of $K$ for which we have $\mathfrak{g}^K \neq 0$ is the *degree of nilpotency* of the nilpotent Lie algebra $\mathfrak{g}$.

By definition, a nilpotent Lie algebra is also solvable. An Abelian Lie algebra is nilpotent of degree 1.

Because we have $[\mathfrak{g}^j, \mathfrak{g}^k] \subset \mathfrak{g}^{j+k}$ due to the Jacobi identity, the lower central series defines a *natural filtration* on the Lie algebra $\mathfrak{g}$.

The *upper central series* is $\mathfrak{z}_1 \subset \cdots \subset \mathfrak{z}_k \subset \cdots \subset \mathfrak{g}$ where $\mathfrak{z}_1$ is the *center* of $\mathfrak{g}$ and $\mathfrak{z}_k$ are defined recursively: $\mathfrak{z}_k$ is the unique ideal in $\mathfrak{g}$ such that $\mathfrak{z}_k/\mathfrak{z}_{k-1}$ is the center of $\mathfrak{g}/\mathfrak{z}_{k-1}$. The upper central series terminates, i.e. a number $L$ exists such that $\mathfrak{z}_L = \mathfrak{g}$, if and only if $\mathfrak{g}$ is nilpotent [37].

We denote by $\mathfrak{cent}(\mathfrak{h})$ the *centralizer* of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$:

$$\mathfrak{cent}(\mathfrak{h}) = \{x \in \mathfrak{g} | [x, y] = 0, \ \forall\, y \in \mathfrak{h}\}.$$

We recall that an *automorphism* $\Phi$ of a given Lie algebra $\mathfrak{g}$ is a bijective linear map $\Phi : \mathfrak{g} \to \mathfrak{g}$ such that for any pair $x, y$ of vectors in $\mathfrak{g}$

$$\Phi([x, y]) = [\Phi(x), \Phi(y)]. \tag{4}$$

All automorphisms of $\mathfrak{g}$ form a Lie group $\mathrm{Aut}(\mathfrak{g})$. Its Lie algebra is the algebra of *derivations* of $\mathfrak{g}$, i.e. of linear maps $D : \mathfrak{g} \to \mathfrak{g}$ such that for any pair $x, y$ of vectors in $\mathfrak{g}$

$$D([x, y]) = [D(x), y] + [x, D(y)]. \tag{5}$$

If a vector $z \in \mathfrak{g}$ exists such that $D = \mathrm{ad}(z)$, i.e. $D(x) = [z, x], \ \forall\, x \in \mathfrak{g}$, the derivation $D$ is called an *inner derivation*, any other one is an *outer derivation*. The space of inner derivations is denoted by $\mathfrak{Inn}(\mathfrak{g})$, of all derivations $\mathfrak{Der}(\mathfrak{g})$.

The ideals in the derived, lower and upper central series as well as their centralizers are invariant with respect to any automorphism and any derivation, i.e. they belong among the *characteristic ideals*.

We denote by $\dot{+}$ the direct sum of vector spaces.

## 3. Solvable Lie algebras with a given nilradical

Any solvable Lie algebra $\mathfrak{s}$ contains a unique maximal nilpotent ideal $\mathfrak{n} = \mathrm{NR}(\mathfrak{s})$, the *nilradical* of $\mathfrak{s}$. We assume that $\mathfrak{n}$ is known. That is, in some basis $(e_1, \ldots, e_n)$ of $\mathfrak{n}$ we are given the Lie brackets

$$[e_j, e_k] = N_{jk}{}^l e_l \tag{6}$$

(summation over repeated indices applies throughout the paper).

Let us consider an extension of the nilpotent algebra $\mathfrak{n}$ to a solvable Lie algebra $\mathfrak{s}$, $\mathfrak{n} \subsetneq \mathfrak{s}$ having $\mathfrak{n}$ as its nilradical. We call any such $\mathfrak{s}$ a *solvable extension* of the nilpotent Lie algebra $\mathfrak{n}$. By definition, any such solvable extensions $\mathfrak{s}$ is non-nilpotent.

We can assume without loss of generality that the structure of $\mathfrak{s}$ is expressed in terms of linearly independent vectors $f_1, \ldots, f_f \in \mathfrak{s}$ added to the basis $(e_1, \ldots, e_n)$ so that together they form a basis of $\mathfrak{s}$. The derived algebra of a solvable Lie algebra is contained in the nilradical (see [37]), i.e.

$$[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{n}. \tag{7}$$

It follows that the Lie brackets in $\mathfrak{s}$ take the form

$$[f_a, e_j] = (A_a)^k_j e_k, \qquad 1 \leqslant a \leqslant f, \qquad 1 \leqslant j \leqslant n, \tag{8}$$

$$[f_a, f_b] = \gamma^j_{ab} e_j, \qquad 1 \leqslant a < b \leqslant f. \tag{9}$$

The matrix elements of the matrices $A_a$ must satisfy certain linear relations following from the Jacobi identities $(f_a, e_j, e_k)$. The Jacobi identities $(f_a, f_b, e_j)$ provide linear expressions for the structure constants $\gamma^j_{ab}$ in terms of matrix elements of the commutators of matrices $A_a$ and $A_b$. Finally, the Jacobi identities $(f_a, f_b, f_c)$ imply some bilinear compatibility conditions on $\gamma^j_{ab}$ and $A_a$ (which may become trivial for a particular choice of $\mathfrak{n}$).

By inspection of equation (8) we realize that the matrices $A_a$ are matrices representing $f_a$ in the adjoint representation of $\mathfrak{s}$ restricted to the nilradical $\mathfrak{n}$:

$$A_a = \mathrm{ad}(f_a)|_{\mathfrak{n}}.$$

For any choice of $a$, the operator $\mathrm{ad}(f_a)|_{\mathfrak{n}}$ is a derivation of $\mathfrak{n}$. It must be an outer derivation—if the contrary held, then $\mathfrak{n} \dot{+} \mathrm{span}\{f_a\}$ would be a nilpotent ideal in $\mathfrak{s}$ contradicting the maximality of the nilradical $\mathfrak{n}$. In fact, the maximality of $\mathfrak{n}$ implies that no nontrivial linear combination of the operators $\mathrm{ad}(f_a)|_{\mathfrak{n}}$ can be a nilpotent matrix, i.e. $\mathrm{ad}(f_1)|_{\mathfrak{n}}, \ldots, \mathrm{ad}(f_f)|_{\mathfrak{n}}$ must be *linearly nilindependent*. A nilpotent algebra $\mathfrak{n}$ which possesses only nilpotent derivations and consequently is not a nilradical of any solvable Lie algebra is called *characteristically nilpotent*.

In other words, finding all sets of matrices $A_a$ in equation (8) satisfying the Jacobi identity is equivalent to finding all sets of outer nilindependent derivations of $\mathfrak{n}$:

$$D_1 = \mathrm{ad}(f_1)|_{\mathfrak{n}}, \ldots, D_f = \mathrm{ad}(f_f)|_{\mathfrak{n}}. \tag{10}$$

Furthermore, in view of equation (7), the commutators $[D_a, D_b]$ must be inner derivations of $\mathfrak{n}$. The structure constants $\gamma^j_{ab}$ in the Lie brackets (9) are determined through the consequence of equation (9):

$$[D_a, D_b] = \gamma^j_{ab} \mathrm{ad}(e_j)|_{\mathfrak{n}} \tag{11}$$

up to elements in the center $\mathfrak{z}_1$ of $\mathfrak{n}$. The consistency of $\gamma_{ab}^j$ and $D_i$ is then subject to the constraint

$$\gamma_{ab}^j D_c(e_j) + \gamma_{bc}^j D_a(e_j) + \gamma_{ca}^j D_b(e_j) = 0 \tag{12}$$

coming from the Jacobi identity $(f_a, f_b, f_c)$. We remark that the lhs of equation (12) is valued in the center of $\mathfrak{n}$ because the derivations $D_a$ themselves satisfy the Jacobi identity.

Different sets of derivations $D_a$ (and their accompanying constants $\gamma_{ab}^j$) may correspond to isomorphic Lie algebras. The equivalence between sets of derivations $D_a$ is generated by the following transformations.

(i) We may add any inner derivation to any $D_a$.
(ii) We may simultaneously conjugate all $D_a$ by an automorphism $\Phi$ of $\mathfrak{n}$, $D_a \to \Phi^{-1} \circ D_a \circ \Phi$.
(iii) We can change the basis in the space $\mathrm{span}\{D_1, \ldots, D_f\}$.

The corresponding changes in $\gamma_{ab}^j$ are of no interest to us in this paper so we do not explicitly write them out here.

## 4. Upper bound on the dimension of a solvable extension of the given nilradical

In this section we derive the following upper bound on the maximal number $f$ of non-nilpotent elements $f_a$ that we can add to a given nilradical $\mathfrak{n}$.

**Theorem 1.** *Let $\mathfrak{n}$ be a nilpotent Lie algebra and $\mathfrak{s}$ a solvable Lie algebra with the nilradical $\mathfrak{n}$. Let $\dim \mathfrak{n} = n$, $\dim \mathfrak{s} = n + f$. Then $f$ satisfies*

$$f \leqslant \dim \mathfrak{n} - \dim \mathfrak{n}^2. \tag{13}$$

In order to derive the estimate (13) we start by choosing a convenient basis $\mathcal{E}$ of the nilpotent Lie algebra $\mathfrak{n}$. We choose first some complement $\mathfrak{m}_1$ of $\mathfrak{n}^2$ in $\mathfrak{n}$,

$$\mathfrak{n} = \mathfrak{n}^2 \dot{+} \mathfrak{m}_1,$$

and denote $m_1 = \dim \mathfrak{m}_1$. We construct a basis $\mathcal{E}_{\mathfrak{m}_1} = (e_{n-m_1+1}, \ldots, e_n)$ of $\mathfrak{m}_1$. In the next step, we recall that

$$\mathfrak{n}^2 = [\mathfrak{n}, \mathfrak{n}] = [\mathfrak{m}_1 \dot{+} \mathfrak{n}^2, \mathfrak{m}_1 \dot{+} \mathfrak{n}^2] = [\mathfrak{m}_1, \mathfrak{m}_1] + \mathfrak{n}^3$$

(the last sum is not necessarily direct). Consequently, we can choose a complement $\mathfrak{m}_2$ of $\mathfrak{n}^3$ in $\mathfrak{n}^2$ such that $\mathfrak{m}_2 \subset [\mathfrak{m}_1, \mathfrak{m}_1]$ and its basis $\mathcal{E}_{\mathfrak{m}_2}$ in the form of some subset of Lie brackets of vectors in $\mathcal{E}_{\mathfrak{m}_1}$, i.e. $\mathcal{E}_{\mathfrak{m}_2} = (e_{n-m_1-m_2+1}, \ldots, e_{n-m_1})$ where $m_2 = \dim \mathfrak{m}_2$ and for any $k \in \{n - m_1 - m_2 + 1, n - m_1\}$ a pair $y_k, z_k \in \mathcal{E}_{\mathfrak{m}_1}$ (not necessarily unique) exists such that $e_k = [y_k, z_k]$.

Proceeding by induction we have

$$\mathfrak{n}^k = [\mathfrak{m}_{k-1} \dot{+} \mathfrak{n}^k, \mathfrak{m}_1 \dot{+} \mathfrak{n}^2] = [\mathfrak{m}_{k-1}, \mathfrak{m}_1] + \mathfrak{n}^{k+1}$$

and we can construct a complement $\mathfrak{m}_k$ of $\mathfrak{n}^{k+1}$ in $\mathfrak{n}^k$,

$$\mathfrak{n}^k = \mathfrak{n}^{k+1} \dot{+} \mathfrak{m}_k,$$

$\mathfrak{m}_k \subset [\mathfrak{m}_{k-1}, \mathfrak{m}_1]$, $m_k = \dim \mathfrak{m}_k$ and a basis $\mathcal{E}_{\mathfrak{m}_k} = \left(e_{n+1-\sum_{i=1}^{k} m_i}, \ldots, e_{n-\sum_{i=1}^{k-1} m_i}\right)$ of $\mathfrak{m}_k$ such that

$$\forall e_j \in \mathcal{E}_{\mathfrak{m}_k} \quad \exists y_j \in \mathcal{E}_{\mathfrak{m}_{k-1}}, \, z_j \in \mathcal{E}_{\mathfrak{m}_1} : \quad e_j = [y_j, z_j]. \tag{14}$$

Together the elements of the bases $\mathcal{E}_{\mathfrak{m}_k}$ form a basis $\mathcal{E} = (e_1, \ldots, e_n)$ of the whole nilpotent algebra $\mathfrak{n}$. The main advantage of working in the basis $\mathcal{E}$ lies in the fact that any automorphism

$\phi$, or any derivation $D$, is fully specified once its action on the elements of the basis $\mathcal{E}_{\mathfrak{m}_1}$ of $\mathfrak{m}_1$ is known due to the definition of an automorphism, equation (4), or a derivation, equation (5), together with equation (14), respectively.

In particular, this implies that the matrix of any automorphism $\Phi$ of $\mathfrak{n}$ is upper block triangular in the basis $\mathcal{E}$:

$$\Phi = \begin{pmatrix} \Phi_{\mathfrak{m}_K\mathfrak{m}_K} & \dots & \Phi_{\mathfrak{m}_K\mathfrak{m}_2} & \Phi_{\mathfrak{m}_K\mathfrak{m}_1} \\ & \ddots & & \vdots \\ & & \Phi_{\mathfrak{m}_2\mathfrak{m}_2} & \Phi_{\mathfrak{m}_2\mathfrak{m}_1} \\ & & & \Phi_{\mathfrak{m}_1\mathfrak{m}_1} \end{pmatrix}, \tag{15}$$

and its diagonal blocks $\Phi_{\mathfrak{m}_k\mathfrak{m}_k}, k = 2, \dots, K$, can be expressed as functions of the elements of the lowest diagonal block $\Phi_{\mathfrak{m}_1\mathfrak{m}_1}$ only by the repeated use of $\Phi([e_k, e_j]) = [\Phi(e_k), \Phi(e_j)]$. The same applies to any derivation $D$ of $\mathfrak{n}$:

$$D = \begin{pmatrix} D_{\mathfrak{m}_K\mathfrak{m}_K} & \dots & D_{\mathfrak{m}_K\mathfrak{m}_2} & D_{\mathfrak{m}_K\mathfrak{m}_1} \\ & \ddots & \vdots & \vdots \\ & & D_{\mathfrak{m}_2\mathfrak{m}_2} & D_{\mathfrak{m}_2\mathfrak{m}_1} \\ & & & D_{\mathfrak{m}_1\mathfrak{m}_1} \end{pmatrix}. \tag{16}$$

Due to the relation $D([e_k, e_j]) = [D(e_k), e_j] + [e_k, D(e_j)]$ we conclude that elements of the diagonal blocks $D_{\mathfrak{m}_k\mathfrak{m}_k}, k = 2, \dots, K$, are linear functions of elements of $D_{\mathfrak{m}_1\mathfrak{m}_1}$. For example, for $e_j \in \mathfrak{m}_2$, $e_j = [e_k, e_l]$ where $e_k, e_l \in \mathfrak{m}_1$ we have

$$\begin{aligned} D(e_j) &= \sum_{p=n-m_1+1}^{n} \left(D^p{}_k[e_p, e_l] + D^p{}_l[e_k, e_p]\right) \bmod \mathfrak{n}^3 \\ &= \sum_{p=n-m_1+1}^{n} \sum_{q=n-m_1-m_2+1}^{n-m_1} \left(D^p{}_k N_{pl}{}^q + D^p{}_l N_{kp}{}^q\right) e_q \bmod \mathfrak{n}^3, \end{aligned}$$

i.e.

$$D^q{}_j = \sum_{p=n-m_1+1}^{n} \left(D^p{}_k N_{pl}{}^q + D^p{}_l N_{kp}{}^q\right)$$

showing that any $D_{\mathfrak{m}_2\mathfrak{m}_2}$-block element $D^q{}_j$ can be expressed in terms of $D_{\mathfrak{m}_1\mathfrak{m}_1}$-block elements together with the structure constants $N_{ab}{}^c$ of $\mathfrak{n}$ and that its dependence on $D_{\mathfrak{m}_1\mathfrak{m}_1}$ is a linear one.

Extending the same argument, we see that in general $D_{\mathfrak{m}_j\mathfrak{m}_k}, k \leqslant j = 2, \dots, K$, is a linear function of elements in the last column blocks $D_{\mathfrak{m}_1\mathfrak{m}_1}, \dots, D_{\mathfrak{m}_{j-k+1}\mathfrak{m}_1}$.

We note that inner derivations have a strictly upper triangular block structure because inner derivations by definition map $\mathfrak{n}^k \to \mathfrak{n}^{k+1}$. Consequently, any set of outer derivations $\{D_1, \dots, D_f\}$ such that $[D_j, D_k] \in \mathfrak{Inn}(\mathfrak{n})$ for all $j, k = 1, \dots, f$ must necessarily have commuting $\mathfrak{m}_1\mathfrak{m}_1$-submatrices:

$$[(D_j)_{\mathfrak{m}_1\mathfrak{m}_1}, (D_k)_{\mathfrak{m}_1\mathfrak{m}_1}] = 0. \tag{17}$$

A derivation $D$ is nilpotent if and only if its submatrix $D_{\mathfrak{m}_1\mathfrak{m}_1}$ is nilpotent. One of these implications is obvious; the other is derived as follows. From a consequence of equation (5)

$$D^n[x, y] = \sum_{j=0}^{n} \binom{n}{j} [D^j x, D^{n-j} y]$$

together with the assumed existence of $N \in \mathbb{N}$ such that $(D_{\mathfrak{m}_1\mathfrak{m}_1})^N = 0$ we deduce the existence of $M \in \mathbb{N}$ such that $(D_{\mathfrak{m}_k\mathfrak{m}_k})^M = 0$ for all $k = 1, \ldots, K$, i.e. the block upper triangular matrix of $D^M$ has vanishing diagonal blocks and is consequently nilpotent, implying also the nilpotency of $D$ itself.

This equivalence implies that derivations $D_1, \ldots, D_f$ are linearly nilindependent if and only if their submatrices $(D_1)_{\mathfrak{m}_1\mathfrak{m}_1}, \ldots, (D_k)_{\mathfrak{m}_1\mathfrak{m}_1}$ are linearly independent. Together with equation (17) this means that the number $f$ of linearly nilindependent outer derivations of the given nilpotent algebra $\mathfrak{n}$ commuting to inner derivations is bounded from above by the maximal number of linearly independent commuting matrices of dimension $m_1 \times m_1$. This number is $m_1 = \dim \mathfrak{n} - \dim \mathfrak{n}^2$, finishing the derivation of the estimate (13).

The estimate (13) allows us to construct also a lower bound on the dimension of the nilradical of a given solvable Lie algebra $\mathfrak{s}$. We have

$$\dim \mathfrak{s} + \dim \mathfrak{n}^2 \leqslant 2 \dim \mathfrak{n}$$

and $\mathfrak{s}^{(2)} = (\mathfrak{s}^2)^2 \subset \mathfrak{n}^2$ because $\mathfrak{s}^2 \subset \mathfrak{n}$. Altogether, we find

$$\dim \mathfrak{n} \geqslant \tfrac{1}{2}(\dim \mathfrak{s} + \dim \mathfrak{s}^{(2)}). \tag{18}$$

In our experience, this estimate is often less accurate than the trivial estimate $\dim \mathfrak{n} \geqslant \dim \mathfrak{s}^2$. Nevertheless, the bound (18) can be useful in some particular cases.

## 5. Levi decomposable algebras with the given nilradical

Let us now shift our attention to nonsolvable algebras with the given nilradical $\mathfrak{n}$. As was demonstrated by Levi [3], any such algebra $\mathfrak{g}$ can be written in the form

$$\mathfrak{g} = \mathfrak{r} \dotplus \mathfrak{p}, \qquad \mathfrak{r} \supset \mathfrak{n}, \qquad [\mathfrak{r}, \mathfrak{g}] \subset \mathfrak{n}, \qquad [\mathfrak{p}, \mathfrak{p}] = \mathfrak{p} \tag{19}$$

where $\mathfrak{r}$ is the *radical*, i.e. the maximal solvable ideal of $\mathfrak{g}$, and $\mathfrak{p}$ is a semisimple Lie algebra, called the *Levi factor*, unique up to automorphisms of $\mathfrak{g}$. We recall that $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ provides a representation of $\mathfrak{p}$ on $\mathfrak{r}$ and this fact is a cornerstone in the construction and classification of algebras of this type. We shall consider only the case when $\mathfrak{g}$ is *indecomposable* in the sense that it cannot be decomposed into a direct sum of ideals. This assumption in particular implies that the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ of $\mathfrak{p}$ on $\mathfrak{r}$ is faithful, i.e. a monomorphism into $\mathfrak{Der}(\mathfrak{r})$.

The classification of algebras of the type (19) with $\mathfrak{p} \neq 0$, $\mathfrak{r} \neq 0$, i.e. of *Levi decomposable* algebras, was considered in [19] and [20]. The approach used in [19] was to consider a given semisimple algebra $\mathfrak{p}$ and all its possible representations $\rho$ on a vector space $V$ of the chosen dimension. For each $\rho$, all solvable algebras $\mathfrak{r}$ compatible with the representation $\rho$ were found by an explicit evaluation of the Jacobi identity with unknown structure constants ${c_{ij}}^k$ of the radical $\mathfrak{r}$, and classified into equivalence classes.

In [20] also some general properties of Levi decomposable algebras were found and used in the construction of all nine-dimensional Levi decomposable algebras[2]. These properties are a direct consequence of the complete reducibility of representations of semisimple Lie algebras, namely

(i) if the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ of $\mathfrak{p}$ is irreducible then $\mathfrak{r}$ is Abelian;

(ii) if $\mathfrak{r}$ is solvable non-nilpotent, then there exists a complement $\mathfrak{q}$ of $\mathfrak{n}$ in $\mathfrak{r}$, i.e.

$$\mathfrak{r} = \mathfrak{n} \dotplus \mathfrak{q}$$

such that $\mathrm{ad}(p)|_{\mathfrak{q}} = 0$ for all $p \in \mathfrak{p}$, i.e. $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ must necessarily contain a copy of the trivial representation.

[2] With one additional algebra missing, as was shown in [43].

In view of this property, it is of interest to study and classify Levi decomposable algebras with nilpotent radicals first;

(iii) the set of all elements belonging to the trivial representation

$$\{x \in \mathfrak{n} | \mathrm{ad}(p)x = 0, \forall p \in \mathfrak{p}\}$$

is a subalgebra of $\mathfrak{n}$.

In this section we intend to provide several more stringent, yet easy to verify, restrictions on the structure of $\mathfrak{n}$ obtained from the compatibility of the nilradical structure with the given representation of $\mathfrak{p}$. We shall call any Levi decomposable algebra $\mathfrak{g}$ with the nilradical $\mathfrak{n}$ (radical $\mathfrak{r}$) a *Levi extension* of $\mathfrak{n}$ (of $\mathfrak{r}$, respectively). In most of this section we suppose that the nilradical coincides with the radical and that the Levi factor acts faithfully on $\mathfrak{n}$, i.e. $\mathfrak{g}$ is indecomposable.

Because all ideals in the characteristic series and their centralizers are invariant with respect to any derivation, in particular with respect to $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$, we can use Lie's theorem to easily deduce the following proposition.

**Proposition 1.** *If a complete flag*

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathfrak{n}$$

*of codimension 1 subspaces can be built out of ideals in the characteristic series and their centralizers, then no Levi decomposable algebra*

$$\mathfrak{g} = \mathfrak{n} \dotplus \mathfrak{p}$$

*such that* $[\mathfrak{p}, \mathfrak{n}] \neq 0$ *exists.*

Using this criterion one can immediately establish, without further considerations of the structure of the representations of $\mathfrak{p}$, that out of low-dimensional nilpotent algebras (dimension at most 5), the following can never appear as a nilradical of a Levi decomposable algebra[3]:

- $\dim \mathfrak{n} = 4$ :
  $A_{4,1}$: $[e_2, e_4] = e_1,\ [e_3, e_4] = e_2$; the characteristic flag is
  $$0 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{cent}(\mathfrak{n}^2) \subset \mathfrak{n};$$
- $\dim \mathfrak{n} = 5$ :
  $A_{5,2}$: $[e_2, e_5] = e_1,\ [e_3, e_5] = e_2,\ [e_4, e_5] = e_3$; the characteristic flag is
  $$0 \subset \mathfrak{n}^4 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{cent}(\mathfrak{n}^3) \subset \mathfrak{n};$$
  $A_{5,5}$: $[e_3, e_4] = e_1,\ [e_2, e_5] = e_1,\ [e_3, e_5] = e_2$; the characteristic flag is
  $$0 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{z}_2 \subset \mathfrak{cent}(\mathfrak{n}^2) \subset \mathfrak{n};$$
  $A_{5,6}$: $[e_2, e_5] = e_1,\ [e_3, e_4] = e_1,\ [e_3, e_5] = e_2,\ [e_4, e_5] = e_3$, with the same flag as $A_{5,2}$.

It is rather interesting that all other indecomposable nilpotent algebras $A_{3,1}, A_{5,1}, A_{5,3}, A_{5,4}$ of dimension $2 \leqslant n \leqslant 5$ do show up as nilradicals in Turkowski's list of Levi decomposable algebras of dimension $\leqslant 8$, i.e. they do admit Levi extension(s). In the case of six-dimensional nilpotent algebras, the same argument allows us to immediately exclude from the list of Levi extendable nilradicals the algebras $A_{6,1}, A_{6,2}, A_{6,6}, A_{6,7}, A_{6,11}, A_{6,16}, A_{6,17}, A_{6,19}, A_{6,20}, A_{6,21}, A_{6,22}$. In this case, however, not all of the remaining algebras allow a Levi extension, as a brief look into [20] tells us. According to Turkowski, only four algebras $A_{6,3}, A_{6,4}, A_{6,5}, A_{6,12}$ out of 22 indecomposable six-dimensional nilpotent algebras

[3] Our notation follows [7] for nilpotent algebras and [19, 20] for their Levi extensions.

contained in the list in [7] allow a Levi extension. The structural reasons for that will be given below.

We can improve on proposition 1 by considering the following type of ideals: let $\mathfrak{i}, \mathfrak{j}$ be characteristic ideals in a Lie algebra $\mathfrak{g}$ and let $\mathfrak{k} = \{x \in \mathfrak{g} | [x, y] \in \mathfrak{j}, \ \forall y \in \mathfrak{i}\}$ be a subspace in $\mathfrak{g}$. Then $\mathfrak{k}$ is a characteristic ideal because

$$[Dx, y] = D[x, y] - [x, Dy] \in \mathfrak{j}$$

by the virtue of the definition of $\mathfrak{k}$ and the characteristic property of $\mathfrak{i}, \mathfrak{j}$, i.e. $Dy \in \mathfrak{i}$, $D[x, y] \in \mathfrak{j}$; consequently, $\mathfrak{k}$ is closed under every derivation, i.e. is itself characteristic. Such ideals, if identified, can be used to refine the sequence of characteristic subspaces in proposition 1. Unfortunately, it appears to be of no help in the case of six-dimensional nilradicals—we were not able to identify any such additional subspace.

Concerning the decomposable nilpotent algebras, we note that e.g. $A_{1,1} \oplus A_{3,1}$, i.e. centrally extended Heisenberg algebra, appears in [19] only as a nilradical of a Levi extendable five-dimensional solvable radical, but not a nilpotent radical of a seven-dimensional Levi decomposable algebra. The reason is that the Levi extension $\mathfrak{sl}(2) \dotplus (A_{1,1} \oplus A_{3,1})$ is decomposable. Similarly also for some other decomposable nilpotent algebras.

Now we employ an analysis similar to the one in section 4 in order to present a more refined necessary criteria on the interplay between the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ of the semisimple algebra $\mathfrak{p}$ on the nilpotent radical $\mathfrak{n}$ and the structure of $\mathfrak{n}$.

The complete reducibility of representations of semisimple Lie algebras allows us to deduce the existence of complementary $\mathrm{ad}(\mathfrak{p})$-invariant subspaces $\tilde{\mathfrak{m}}_j$ of $\mathfrak{n}^{j+1}$ in $\mathfrak{n}^j$:

$$\mathfrak{n}^j = \tilde{\mathfrak{m}}_j \dotplus \mathfrak{n}^{j+1}, \qquad \mathrm{ad}(\mathfrak{p})\tilde{\mathfrak{m}}_j \subset \tilde{\mathfrak{m}}_j, \qquad j = 1, \ldots, K. \tag{20}$$

Let us now explore whether these subspaces can be taken in the same form as the one used in section 4 (or as close to it as possible). We can take

$$\mathfrak{m}_1 = \tilde{\mathfrak{m}}_1.$$

Now the commutator of two $\mathrm{ad}(\mathfrak{p})$-invariant subspaces is again an $\mathrm{ad}(\mathfrak{p})$-invariant subspace (see the definition of a derivation, equation (5)). In particular, $[\mathfrak{m}_1, \mathfrak{m}_1]$ is an $\mathrm{ad}(\mathfrak{p})$-invariant subspace of $\mathfrak{n}^2$ and we have

$$\mathfrak{n}^2 = [\mathfrak{m}_1, \mathfrak{m}_1] + \mathfrak{n}^3.$$

Since both $[\mathfrak{m}_1, \mathfrak{m}_1]$ and $\mathfrak{n}^3$ are $\mathrm{ad}(\mathfrak{p})$-invariant, so is their intersection $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$. By the complete reducibility of $\mathrm{ad}(\mathfrak{p})$, there is an $\mathrm{ad}(\mathfrak{p})$-invariant complement of $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$ in $[\mathfrak{m}_1, \mathfrak{m}_1]$ which we denote by $\mathfrak{m}_2$. Altogether, we have

$$\mathfrak{n}^2 = \mathfrak{m}_2 \dotplus \mathfrak{n}^3, \qquad \mathfrak{m}_2 \subset [\mathfrak{m}_1, \mathfrak{m}_1], \qquad \mathrm{ad}(\mathfrak{p})\mathfrak{m}_2 \subset \mathfrak{m}_2.$$

Continuing in the same way, we can construct a sequence of subspaces $\mathfrak{m}_j$ such that

$$\mathfrak{n} = \mathfrak{m}_K \dotplus \mathfrak{m}_{K-1} \dotplus \cdots \dotplus \mathfrak{m}_1, \tag{21}$$

where

$$\mathfrak{n}^j = \mathfrak{m}_j \dotplus \mathfrak{n}^{j+1}, \qquad \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1], \qquad \mathrm{ad}(\mathfrak{p})\mathfrak{m}_j \subset \mathfrak{m}_j. \tag{22}$$

The only minor difference between the decompositions constructed here and in section 4 is that now we cannot in general find a basis of $\mathfrak{m}_1$ such that bases of $\mathfrak{m}_j$ are obtained by simple commutations (cf equation (14))—in the present case taking linear combinations of the commutators may be necessary. Nevertheless, all the essential arguments presented in section 4 can be taken over here.

We have established that in any basis of the nilradical $\mathfrak{n}$ which respects the decomposition (21) the matrices of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ have a block diagonal form. If any of the blocks is one dimensional then it necessarily corresponds to the trivial representation $\rho(p) = 0$, $\forall p \in \mathfrak{p}$. Similarly as in section 4, the $\mathfrak{m}_j\mathfrak{m}_j$-submatrices of the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$, $j > 1$, i.e. the matrices of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j}$, are fully determined by $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ through linear relations coming from the definition of a derivation (equation (5)).

For the same reason, the kernel of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is also the kernel of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$. Therefore, the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ of the Levi factor $\mathfrak{p}$ is faithful if and only if $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is faithful.

In the particular case when $\mathfrak{m}_j$ is one dimensional we can easily find a simple relation between the representations $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ and $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j+1}}$. Let $\mathfrak{m}_j = \mathrm{span}\{x\}$. We have

$$\mathrm{ad}(p)[x, y] = [x, \mathrm{ad}(p)y], \qquad \forall y \in \mathfrak{m}_1, \qquad \forall p \in \mathfrak{p}. \tag{23}$$

Now let us assume that $y$ belongs to a nontrivial irreducible representation $\rho$ contained in $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ and that $[x, y] \neq 0$. Let $V$ be the space generated by repeated applications of $\mathrm{ad}(p)$, $p \in \mathfrak{p}$ on $y$, i.e. $V$ is the representation space of the representation $\rho$. Consider $\mathrm{ad}(x)V \subset [\mathfrak{m}_j, \mathfrak{m}_1]$. Due to equation (23) the subspace $\mathrm{ad}(x)V$ is by construction invariant with respect to $\mathrm{ad}(\mathfrak{p})$. The kernel of $\mathrm{ad}(x) : V \to \mathrm{ad}(x)V$ is an invariant subspace of $V$. By assumption the representation $\rho$ is irreducible and $\mathrm{ad}(x)V \neq 0$. Therefore, $\mathrm{ad}(x)V$ is a vector space isomorphic to $V$ and an irreducible representation $\rho'$ equivalent to $\rho$ is contained in the decomposition of the representation $\mathrm{ad}(\mathfrak{p})|_{[\mathfrak{m}_j,\mathfrak{m}_1]}$ into irreducible representations.

To sum up, let $\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_L$ be the decomposition of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ into irreducible representations. Then the induced representation on $[\mathfrak{m}_j, \mathfrak{m}_1]$ is a direct sum of irreducible representations equivalent to the ones contained in some subset[4] $\{\rho_a, a \in J \subset \{1, \ldots, L\}\}$. The same necessarily holds also for $\mathfrak{m}_{j+1} \subset [\mathfrak{m}_j, \mathfrak{m}_1]$.

If $\dim \mathfrak{m}_j > 1$ then the relation between the representations on $\mathfrak{m}_1, \mathfrak{m}_j$ and $\mathfrak{m}_{j+1}$ takes a more complicated form. Because the commutators of $e_a \in \mathfrak{m}_j, e_b \in \mathfrak{m}_1$ transform under action of any block-diagonal derivation $D$ by

$$D[e_a, e_b] = \sum_{c=n+1-\sum_{i=1}^{j} m_i}^{n-\sum_{i=1}^{j-1} m_i} D^c{}_a[e_c, e_b] + \sum_{d=n+1-m_1}^{n} D^d{}_b[e_a, e_d], \tag{24}$$

where $D^c{}_a$ are components of the matrix of $D|_{\mathfrak{m}_j} : \mathfrak{m}_j \to \mathfrak{m}_j$ and $D^d{}_b$ are components of $D|_{\mathfrak{m}_1} : \mathfrak{m}_1 \to \mathfrak{m}_1$, the commutator subspace $[\mathfrak{m}_j, \mathfrak{m}_1]$ transforms under a certain subset of irreducible factors in the tensor representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j} \otimes \mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ and the same is true also for $\mathfrak{m}_{j+1} \subset [\mathfrak{m}_j, \mathfrak{m}_1]$.

We sum up these conclusions in the following theorem.

**Theorem 2.** *Let $\mathfrak{g}$ be an indecomposable Lie algebra with a nilpotent radical $\mathfrak{n}$ and a nontrivial Levi decomposition*

$$\mathfrak{g} = \mathfrak{n} \dot{+} \mathfrak{p}.$$

*There exists a decomposition of $\mathfrak{n}$ into a direct sum of $\mathrm{ad}(\mathfrak{p})$-invariant subspaces*

$$\mathfrak{n} = \mathfrak{m}_K \dot{+} \mathfrak{m}_{K-1} \dot{+} \cdots \dot{+} \mathfrak{m}_1,$$

*where*

$$\mathfrak{n}^j = \mathfrak{m}_j \dot{+} \mathfrak{n}^{j+1}, \qquad \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1], \qquad \mathrm{ad}(\mathfrak{p})\mathfrak{m}_j \subset \mathfrak{m}_j,$$

*such that $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is a faithful representation of $\mathfrak{p}$ on $\mathfrak{m}_1$. For $j = 2, \ldots, K$ the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j}$ of $\mathfrak{p}$ on the subspace $\mathfrak{m}_j$ can be decomposed into some subset of irreducible components of the tensor representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j-1}} \otimes \mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$.*

[4] It may only be a subset because some of the commutators in equation (23) may vanish.

*If any of the subspaces $\mathfrak{m}_j$ is one dimensional, then $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ must contain a copy of the trivial representation corresponding to the subspace $\mathfrak{m}_j$. When $j < K$, the representation of $\mathfrak{p}$ on $\mathfrak{m}_{j+1}$ can be decomposed into a sum of irreducible representations, each of which is equivalent to an irreducible representation contained in the decomposition of $\mathfrak{m}_1$.*

*In particular, when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible and $\dim \mathfrak{m}_j = 1$, $1 < j < K$, then the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j+1}}$ on $\mathfrak{m}_{j+1}$ is equivalent to $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$.*

We remark that it was shown in [44] that when the radical $\mathfrak{r}$ of a Levi decomposable algebra $\mathfrak{g}$ has a one-dimensional center, then the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ contains a copy of the trivial representation. This result is contained in our theorem as a particular subcase when $\dim \mathfrak{m}_K = 1$.

Theorem 2 gives us a simple dimensional necessary criterion on possible Levi extensions of $\mathfrak{n}$. Namely, a faithful representation of dimension $m_1 = \dim \mathfrak{n} - \dim \mathfrak{n}^2$ must exist. For example, there cannot be any Levi extension of the Heisenberg algebra $A_{3,1}$ ($[e_2, e_3] = e_1$) with a Levi factor other than $\mathfrak{sl}(2)$ and similarly for any other nilradical such that $\dim \mathfrak{n} - \dim \mathfrak{n}^2 = 2$.

In particular, if $\mathfrak{n}$ is *filiform*, i.e. a nilpotent Lie algebra of maximal degree of nilpotency, $K = n - 1$ [11, 38–40], we have $m_1 = \dim \mathfrak{n} - \dim \mathfrak{n}^2 = 2$ and $m_j = \dim \mathfrak{n}^j - \dim \mathfrak{n}^{j+1} = 1$ for $j = 2, \ldots, n-1$. When $\dim \mathfrak{n} = 3$ we have the Heisenberg algebra which possesses a Levi extension. When $\dim \mathfrak{n} \geqslant 4$ the existence of a Levi extension would imply that the one-dimensional subspace $\mathfrak{m}_3$ must carry an equivalent copy of the two-dimensional irreducible representation of $\mathfrak{sl}(2)$ on $\mathfrak{m}_1$, i.e. a clear contradiction. Therefore, no Levi decomposable algebra with a filiform nilradical $\mathfrak{n}$ ($\dim \mathfrak{n} \geqslant 4$) exists as was already derived by other means in [40], lemma 25, and independently in [45], corollary 1.

In the same way we can also explain the prevalence of Levi factors isomorphic to $\mathfrak{sl}(2)$ in Turkowski's lists of real Levi decomposable algebras. Whenever we can identify a nontrivial two-dimensional representation in the subspace $\mathfrak{m}_1$ the Levi factor $\mathfrak{so}(3)$ is immediately ruled out, e.g. for the nilpotent algebra $\mathbf{A_{5,3}}$ with the dimensions of the invariant subspaces $m_1 = 2$, $m_2 = 1$, $m_3 = 2$. Even when $\mathfrak{m}_1$ can carry a three-dimensional irreducible representation there may be further restrictions. They come from the fact that for $\mathfrak{so}(3)$ we have the following decomposition of the three-dimensional irreducible representation $\mathbf{3}$ tensored with itself

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \dotplus \mathbf{3} \dotplus \mathbf{1}$$

where $\mathbf{5} \dotplus \mathbf{1}$ is the symmetric part and $\mathbf{3}$ the antisymmetric part. Because $[e_i, e_j]$ is antisymmetric, only $\mathbf{3}$ remains in $[V, V]$ where $V = \mathbf{3}$. Consequently, the algebra

$$\mathbf{A_{5,1}} : [e_3, e_5] = e_1, \qquad [e_4, e_5] = e_2$$

with $\mathfrak{n}^2 = \mathfrak{z}_1 = \mathrm{span}\{e_1, e_2\}$ cannot have a Levi extension by the Levi factor $\mathfrak{so}(3)$. If the contrary held, we would have $\mathfrak{m}_1 = \mathbf{3}$, $\mathfrak{m}_2 = \mathbf{1} \dotplus \mathbf{1}$ but $\mathbf{1} \dotplus \mathbf{1}$ is not contained in $[\mathbf{3}, \mathbf{3}] \simeq \mathbf{3}$ of $[\mathfrak{m}_1, \mathfrak{m}_1]$.

Let us now apply these ideas to six-dimensional nilpotent radicals. Let us consider the nilpotent algebra

$$\mathbf{A_{6,15}} : [e_1, e_2] = e_3 + e_5, \qquad [e_1, e_3] = e_4, \qquad [e_1, e_4] = e_6, \qquad [e_2, e_5] = e_6$$

which does not appear in Turkowski's list of Levi decomposable algebras as a radical. It has an incomplete flag of characteristic ideals

$$0 \subset \mathfrak{n}^4 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{z}_3 \subset \mathfrak{n}$$

in which only a five-dimensional ideal is missing. Therefore, if any Levi decomposable algebra with the radical $\mathbf{A_{6,15}}$ exists then the action of the Levi factor $\mathfrak{p}$ on the four-dimensional ideal $\mathfrak{z}_3$

is trivial as is seen by the dimensional analysis and we have a decomposition of the subspaces $\mathfrak{m}_i$ into irreducible representations as follows:

$$\mathfrak{m}_1 = \mathbf{2}_1 \dotplus \mathbf{1}_1, \qquad \mathfrak{m}_2 = \mathbf{1}_2, \qquad \mathfrak{m}_3 = \mathbf{1}_3, \qquad \mathfrak{m}_4 = \mathbf{1}_4,$$

where the boldface numbers stand for representation spaces of irreducible representations of $\mathfrak{p}$ of that dimension and indices specify into which $\mathfrak{m}_i$ space they belong. The representation space $\mathbf{1}_1$ coincides with the one-dimensional subspace $\mathfrak{m}_1 \cap \mathfrak{z}_3$. By theorem 2 we have

$$\mathfrak{m}_3 = [\mathbf{1}_1, \mathfrak{m}_2] = [\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_2], \qquad \mathfrak{m}_4 = [\mathbf{1}_1, \mathfrak{m}_3] = [\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_3]. \tag{25}$$

At the same time, $\mathfrak{z}_3$ of the algebra $\mathbf{A_{6,15}}$ is Abelian and contains both $\mathfrak{m}_2$ and $\mathfrak{m}_3$ which leads to a contradiction with equation (25). Therefore, no Levi extension of $\mathbf{A_{6,15}}$ exists. The same argument can also be applied to the algebra $\mathbf{A_{6,17}}$.

The case of algebras $\mathbf{A_{6,8}}$, $\mathbf{A_{6,9}}$ is more involved. They can be both viewed as an extension of the five-dimensional algebra $\mathbf{A_{5,3}}$:

$$[e_3, e_4] = e_2, \qquad [e_3, e_5] = e_1, \qquad [e_4, e_5] = e_3$$

by one element $e_6$ which has only non-vanishing commutators with $e_4, e_5$, spanning a one-dimensional subspace in the center $\mathfrak{z}_1 = \operatorname{span}\{e_1, e_2\}$. In a suitable basis we have $[e_6, e_4] = e_2$ in $\mathbf{A_{6,8}}$ and $[e_6, e_4] = e_1$ in $\mathbf{A_{6,9}}$, respectively. Whereas the five-dimensional algebra $\mathbf{A_{5,3}}$ does possess a Levi extension by $\mathfrak{sl}(2)$, neither $\mathbf{A_{6,8}}$ nor $\mathbf{A_{6,9}}$ do. The reason is that the additional element $e_6$ presents an obstruction which can be identified in the following way. We have $\mathfrak{n}^2 = \operatorname{span}\{e_1, e_2, e_3\}$, $\mathfrak{n}^3 = \mathfrak{z}_1 = \operatorname{span}\{e_1, e_2\}$, $\mathfrak{z}_2 = \operatorname{span}\{e_1, e_2, e_3, e_6\}$. By dimensional analysis we find that the only hypothetically permissible Levi extension is by the Levi factor $\mathfrak{p} = \mathfrak{sl}(2)$ and the structure of the representations must be as follows:

$$\mathfrak{m}_1 = \mathbf{2_1} \oplus \mathbf{1_1}, \qquad \mathfrak{m}_2 = \mathbf{1_2}, \qquad \mathfrak{m}_3 = \mathbf{2_3},$$

where $\mathbf{1_1} \subset \mathfrak{z}_2$. The basis respecting the characteristic subspaces can be chosen without loss of generality in the form

$$\begin{aligned}
&\tilde{e}_1 = e_1, \qquad \tilde{e}_2 = e_2, \qquad n^3 = \mathfrak{m}_3 = \mathbf{2_3} = \operatorname{span}\{e_1, e_2\},\\
&\tilde{e}_3 = e_3 \bmod \mathfrak{n}^3, \qquad \mathfrak{m}_2 = \mathbf{1_2} = \operatorname{span}\{\tilde{e}_3\},\\
&\tilde{e}_6 = e_6 \bmod \mathfrak{n}^2, \qquad \mathbf{1_1} = \operatorname{span}\{\tilde{e}_6\},\\
&\tilde{e}_4 = e_4 \bmod \mathfrak{z}_2, \qquad \tilde{e}_5 = e_5 \bmod \mathfrak{z}_2, \qquad \mathbf{2_1} = \operatorname{span}\{\tilde{e}_4, \tilde{e}_5\},
\end{aligned}$$

where e.g. $\bmod \mathfrak{n}^2$ stands for some (unknown) element of $\mathfrak{n}^2$. Because both $\tilde{e}_6$ and $\tilde{e}_3$ belong to the trivial representation of $\mathfrak{sl}(2)$, we can add a suitable multiple of $\tilde{e}_3$ to $\tilde{e}_6$ to set

$$\tilde{e}_6 = e_6 \bmod \mathfrak{n}^3$$

without altering the block diagonal structure of $\operatorname{ad}(\mathfrak{p})$ acting on $\mathfrak{m}_1 \dotplus \mathfrak{m}_2 \dotplus \mathfrak{m}_3$. Now we have $\mathbf{1_1} = \operatorname{span}\{\tilde{e}_6\}$ and $[\mathbf{1_1}, \mathbf{2_1}] \subset [\mathbf{1_1}, \mathfrak{n}] = V$ where $V$ is a certain one-dimensional subspace of the center $\mathfrak{m}_3$ ($V = \operatorname{span}\{e_2\}$ for $\mathbf{A_{6,8}}$, $V = \operatorname{span}\{e_1\}$ for $\mathbf{A_{6,9}}$). That means we have arrived at a contradictory conclusion that a one-dimensional subspace must carry a two-dimensional representation and consequently no Levi extension of algebras $\mathbf{A_{6,8}}$ and $\mathbf{A_{6,9}}$ exists. The same also holds for $\mathbf{A_{6,10}}$ which is just another real form of the complex version of $\mathbf{A_{6,8}}$.

A similar but somewhat simpler argument shows that $\mathbf{A_{6,13}}$:

$$[e_1, e_2] = e_5, \qquad [e_1, e_3] = e_4, \qquad [e_1, e_4] = e_6, \qquad [e_2, e_5] = e_6, \tag{26}$$

does not possess a nontrivial Levi decomposition. Namely, we have

$$\mathfrak{n}^2 = \mathfrak{z}_2 = \operatorname{span}\{e_4, e_5, e_6\}, \qquad \mathfrak{n}^3 = \mathfrak{z}_1 = \operatorname{span}\{e_6\}, \qquad \mathfrak{cent}(\mathfrak{n}^2) = \operatorname{span}\{e_3, e_4, e_5, e_6\}.$$

By dimensional analysis alone we have the following structure of irreducible representations of hypothetical $\mathfrak{p}$:

$$\mathfrak{m}_1 = \mathbf{2_1} \dotplus \mathbf{1_1}, \qquad \mathfrak{m}_3 = \mathbf{1_3}$$

and two options for $\mathfrak{m}_2$: either $\mathfrak{m}_2 = \mathbf{2_2}$ or $\mathfrak{m}_2 = \mathbf{1} \dotplus \mathbf{1}$. Out of the two, $\mathfrak{m}_2 = \mathbf{1} \dotplus \mathbf{1}$ cannot be found in the antisymmetrized tensor product of $\mathbf{2_1} \dotplus \mathbf{1_1}$ with itself; therefore, it must be $\mathfrak{m}_2 = \mathbf{2_2} = [\mathbf{2_1}, \mathbf{1_1}]$. On the other hand, from the Lie brackets (26) we have

$$[\mathbf{2_1}, \mathbf{1_1}] \subset [\mathfrak{cent}(\mathfrak{n}^2), \mathfrak{n}] = \mathrm{span}\{e_4, e_6\}$$

which splits into $\mathfrak{m}_3$ and a one-dimensional subspace of $\mathfrak{m}_2$. Therefore, $[\mathbf{2_1}, \mathbf{1_1}]$ must be simultaneously a trivial representation and a two-dimensional irreducible representation, a contradiction showing that no Levi extension of $\mathbf{A_{6,13}}$ exists.

That leaves only the nilpotent algebras $\mathbf{A_{6,14}}$:

$$[e_1, e_3] = e_4, \qquad [e_1, e_4] = e_6, \qquad [e_2, e_3] = e_5, \qquad [e_2, e_5] = \epsilon e_6, \qquad \epsilon^2 = 1,$$

and $\mathbf{A_{6,18}}$:

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_4, \qquad [e_1, e_4] = e_6, \qquad [e_2, e_3] = e_5, \qquad [e_2, e_4] = e_6,$$

unexplained. $\mathbf{A_{6,14}}$ has the characteristic ideals

$$\mathfrak{n}^2 = \mathfrak{z}_2 = \mathrm{span}\{e_4, e_5, e_6\}, \qquad \mathfrak{n}^3 = \mathfrak{z}_1 = \mathrm{span}\{e_6\}, \qquad \mathfrak{cent}(\mathfrak{n}^2) = \mathrm{span}\{e_3, e_4, e_5, e_6\}$$

which seem to allow the representation structure of the Levi factor $\mathfrak{sl}(2)$ in the form

$$\mathfrak{m}_1 = \mathbf{2_1} \dotplus \mathbf{1_1}, \qquad \mathfrak{m}_2 = \mathbf{2_2}, \qquad \mathfrak{m}_3 = \mathbf{1_3}$$

with $\mathfrak{cent}(\mathfrak{n}^2) = \mathbf{1_1} \dotplus \mathfrak{m}_2 \dotplus \mathfrak{m}_3$. We did not find any obvious obstruction to this representation structure considering dimension only. Therefore, in order to exclude the possible existence of a Levi extension in this case we have to consider the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ in more detail. We can assume that

$$\begin{aligned}
\mathbf{2_1} &= \mathrm{span}\{\tilde{e}_1 = e_1 \bmod \mathfrak{cent}(\mathfrak{n}^2), \tilde{e}_2 = e_2 \bmod \mathfrak{cent}(\mathfrak{n}^2)\},\\
\mathbf{1_1} &= \mathrm{span}\{\tilde{e}_3 = e_3 \bmod \mathfrak{n}^2\},\\
\mathbf{2_2} &= \mathrm{span}\{\tilde{e}_4 = e_4 \bmod \mathfrak{n}^3, \tilde{e}_5 = e_5 \bmod \mathfrak{n}^3\},\\
\mathbf{1_2} &= \mathrm{span}\{\tilde{e}_6 = e_6\}
\end{aligned}$$

and consider $\mathfrak{p} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2)$ represented in this representation space. We have by assumption

$$\mathrm{ad}(p)\tilde{e}_1 = \tilde{e}_1, \qquad \mathrm{ad}(p)\tilde{e}_2 = -\tilde{e}_2, \qquad \mathrm{ad}(p)\tilde{e}_3 = 0$$

together with the consequences of equation (5):

$$\mathrm{ad}(p)\tilde{e}_4 = \mathrm{ad}(p)[\tilde{e}_1, \tilde{e}_3] = \tilde{e}_4, \qquad \mathrm{ad}(p)\tilde{e}_5 = \mathrm{ad}(p)[\tilde{e}_2, \tilde{e}_3] = -\tilde{e}_5$$

and finally

$$\mathrm{ad}(p)\tilde{e}_6 = \mathrm{ad}(p)[\tilde{e}_1, \tilde{e}_4] = 2\tilde{e}_6$$

in a clear violation of

$$\mathfrak{m}_3 = \mathrm{span}\{\tilde{e}_6\} = \mathbf{1_3}.$$

Similarly for $\mathbf{A_{6,18}}$ where the characteristic ideals are

$$\mathfrak{n}^2 = \mathfrak{z}_3 = \mathrm{span}\{e_3, e_4, e_5, e_6\}, \qquad \mathfrak{n}^3 = \mathfrak{z}_2 = \mathrm{span}\{e_4, e_5, e_6\},$$

and $\mathfrak{n}^4 = \mathfrak{z}_1 = \mathrm{span}\{e_6\}$, i.e. a similar representation structure

$$\mathfrak{m}_1 = \mathbf{2_1}, \qquad \mathfrak{m}_2 = \mathbf{1_2}, \qquad \mathfrak{m}_3 = \mathbf{2_3}, \qquad \mathfrak{m}_4 = \mathbf{1_4}$$

naively seems possible. Now we have

$$\mathfrak{m}_1 = \mathrm{span}\{\tilde{e}_1 = e_1 \bmod \mathfrak{n}^2, \tilde{e}_2 = e_2 \bmod \mathfrak{n}^2\},$$
$$\mathfrak{m}_2 = \mathrm{span}\{\tilde{e}_3 = e_3 \bmod \mathfrak{n}^3\},$$
$$\mathfrak{m}_3 = \mathrm{span}\{\tilde{e}_4 = e_4 \bmod \mathfrak{n}^4, \tilde{e}_5 = e_5 \bmod \mathfrak{n}^4\},$$
$$\mathfrak{m}_4 = \mathrm{span}\{\tilde{e}_6 = e_6\}.$$

If the representation of $\mathfrak{sl}(2)$ on $\mathfrak{n}$ exists we can again consider $\mathfrak{p} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{sl}(2)$ acting on $\mathfrak{n}$. We have again

$$\mathrm{ad}(p)\tilde{e}_1 = \tilde{e}_1, \qquad \mathrm{ad}(p)\tilde{e}_2 = -\tilde{e}_2$$

together with the consequences

$$\mathrm{ad}(p)\tilde{e}_3 = \mathrm{ad}(p)[\tilde{e}_1, \tilde{e}_2] = 0, \qquad \mathrm{ad}(p)\tilde{e}_4 = \mathrm{ad}(p)[\tilde{e}_1, \tilde{e}_3] = \tilde{e}_4,$$
$$\mathrm{ad}(p)\tilde{e}_5 = \mathrm{ad}(p)[\tilde{e}_2, \tilde{e}_3] = -\tilde{e}_5$$

and

$$\mathrm{ad}(p)\tilde{e}_6 = \mathrm{ad}(p)[\tilde{e}_1, \tilde{e}_4] = 2\tilde{e}_6$$

demonstrating the incompatibility of $\mathfrak{sl}(2)$ with $\mathbf{A_{6,18}}$.

To sum up, we have shown in detail that for all but two six-dimensional nilpotent algebras $\mathfrak{n}$ in the list [7] which do not have any nontrivial Levi extension, obstructions precluding its existence can be identified by dimensional arguments, without the knowledge of derivations of $\mathfrak{n}$.

Concerning the $\mathfrak{so}(3)$ Levi factor acting on six-dimensional indecomposable nilpotent radicals, we may consider only the four algebras which have not been excluded by the previous analysis (and appear as nilpotent radicals in Turkowski's list [20]). We can establish on dimensional grounds that $\mathbf{A_{6,12}}$ with the flag $0 \subsetneq \mathfrak{n}^3 \subsetneq \mathfrak{n}^2 \subsetneq \mathfrak{cent}(\mathfrak{n}^2) \subsetneq \mathfrak{n}$ of dimensions $(0, 1, 2, 4, 6)$ cannot have a Levi extension by $\mathfrak{so}(3)$. $\mathbf{A_{6,3}}$ with dimensions $m_1 = m_2 = 3$ is the only one to allow a three-dimensional representation of $\mathfrak{so}(3)$ (in fact two copies of it) in its Levi extension, identified as $L_{9,11}$. Out of the remaining two Levi extendable algebras $\mathbf{A_{6,4}}$, $\mathbf{A_{6,5}}$ with $m_1 = 4$, $m_2 = 2$ only the second one has a Levi extension $L_{9,4}$ with the Levi factor $\mathfrak{so}(3)$ (with the four-dimensional bispinor representation of $\mathfrak{so}(3)$ acting on $\mathfrak{m}_1$ and the trivial representation on $\mathfrak{m}_2$). The fact that $\mathbf{A_{6,4}}$ does not have a nontrivial Levi extension by $\mathfrak{so}(3)$ cannot be found by the dimensional analysis alone and the detailed structure of the representation acting on it must again be considered (similarly as in the case of algebras $\mathbf{A_{6,14}}$, $\mathbf{A_{6,18}}$ above).

Let us now turn our attention to non-nilpotent radicals. Let us assume that $\mathfrak{r}$ is a solvable Lie algebra with the nilradical $\mathfrak{n}$. The existence of a Levi extension $\mathfrak{g}$ of the non-nilpotent radical $\mathfrak{r}$ with a Levi factor $\mathfrak{p}$ imposes restrictions that go beyond the ones originating from the existence of $\mathfrak{g}' = \mathfrak{n} \dotplus \mathfrak{p}$. (On the other hand, we have already observed that $\mathfrak{g}'$ may be decomposable and consequently not included in the lists in [19, 20] even if $\mathfrak{g}$ is indecomposable.)

Let us assume that $\mathfrak{r} = \mathfrak{n} \dotplus \mathfrak{q}$, $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{q}} = 0$ as is always possible to achieve by the theorem of Turkowski. Then we have

$$\mathrm{ad}(p)[x, y] = [\mathrm{ad}(p)x, y] + [x, \mathrm{ad}(p)y] = 0$$

for any $x, y \in \mathfrak{q}$ and $p \in \mathfrak{p}$, i.e. the subspace $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{n}$ must be a representation space of the trivial representation (if $[\mathfrak{q}, \mathfrak{q}]$ is nonvanishing). Furthermore, for any $z \in \mathfrak{n}$, $y \in \mathfrak{q}$, $p \in \mathfrak{p}$ we have

$$\mathrm{ad}(p)[y, z] = [y, \mathrm{ad}(p)z],$$

i.e. similarly as in the proof of theorem 2 we see that $\mathrm{ad}(y)$, $y \in \mathfrak{q}$ maps any representation subspace $V \subset \mathfrak{n}$ of an irreducible representation of $\mathfrak{p}$ either to a representation space of an equivalent representation (including $V$ itself) or to zero.

Another restriction comes from the fact that $[\mathfrak{p}, \mathfrak{r}] \subset \mathfrak{n}$ which for the corresponding derivations acting on $\mathfrak{n}$ means that

$$[\mathrm{ad}(p)|_{\mathfrak{n}}, \mathrm{ad}(x)|_{\mathfrak{n}}] \in \mathfrak{Inn}(\mathfrak{n}), \qquad \forall p \in \mathfrak{p}, \qquad x \in \mathfrak{r}.$$

This in turn implies that the $\mathfrak{m}_1\mathfrak{m}_1$-blocks of $\mathrm{ad}(p)|_{\mathfrak{n}}$ and $\mathrm{ad}(x)|_{\mathfrak{n}}$ commute.

We collect some of these results into the following theorem.

**Theorem 3.** *Let $\mathfrak{g}$ be a Levi decomposable Lie algebra which cannot be decomposed into a direct sum of ideals, $\mathfrak{p}$ its Levi factor, $\mathfrak{r}$ its radical, $\mathfrak{n}$ its nilradical. Let $\mathfrak{n} = \sum_{k=1}^{K} \mathfrak{m}_k$ be the decomposition (21) of the nilradical $\mathfrak{n}$. Then for any $p \in \mathfrak{p}$ and $x \in \mathfrak{r}$ the submatrices $(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}$ and $(\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}$ of $\mathrm{ad}(p)|_{\mathfrak{n}}$ and $\mathrm{ad}(x)|_{\mathfrak{n}}$, respectively, commute:*

$$[(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}, (\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}] = 0.$$

*In particular, if the restriction of $\mathrm{ad}(\mathfrak{p})$ to $\mathfrak{m}_1$ is irreducible and $\mathfrak{g}$ is an algebra over $\mathbb{C}$ then $\dim \mathfrak{r} - \dim \mathfrak{n} \leqslant 1$. When equality holds then the $\mathfrak{m}_1\mathfrak{m}_1$-block of the derivation $\mathrm{ad}(f_1)$ ($f_1 \in \mathfrak{r} \setminus \mathfrak{n}$) is a nonvanishing multiple of the unit operator.*

The proof of the statements in the particular case when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible is a direct consequence of Schur's lemma.

Theorem 3 can be used in explaining the particular values of parameters of solvable radicals $\mathfrak{r}$ allowing Levi extension. For example, in the algebra $g_{6,54}$ in Mubarakzyanov's classification of solvable algebras [16] there are two parameters whereas its Levi extension $L^{p}_{9,49}$ in [20] has only one. The reason is that in order to have $(\mathrm{ad}(f_1))_{\mathfrak{m}_1\mathfrak{m}_1}$ commuting with $(\mathrm{ad}(\mathfrak{p}))_{\mathfrak{m}_1\mathfrak{m}_1}$ one of the parameters must be equal to 1. For the same reason the four parameters in the algebra $\mathcal{N}^{\alpha\beta\gamma\delta}_{6,1}$ of [17] are reduced to the values $\gamma = \alpha = p, \beta = \delta = q$ in the Levi extension $L^{p,q}_{9,28}$ of [20], and similarly for other parametric families in Turkowski's classifications [19, 20].

## 6. Conclusions

We have established an improved upper bound on the dimension of any solvable extension of a given nilpotent Lie algebra. The new estimate (13)

$$f \leqslant \dim \mathfrak{n} - \dim \mathfrak{n}^2, \qquad \text{where} \qquad f = \dim \mathfrak{s} - \dim \mathfrak{n}$$

is different from the one derived by Mubarakzyanov in [41]

$$f \leqslant \dim \mathfrak{n} \tag{27}$$

and improved in [42] to

$$f \leqslant \dim \mathfrak{n} - \dim C(\mathfrak{s}). \tag{28}$$

There are at least two advantages to the estimate (13) over (28):

- the bound (13) is in most cases more restrictive than (28)
- and it does not depend on the knowledge of the structure of the whole solvable Lie algebra $\mathfrak{s}$, contrary to the bound (28).

The bound (13) is saturated for many classes of nilpotent Lie algebras whose solvable extensions were previously investigated—e.g. Abelian [28], naturally graded filiform $\mathfrak{n}_{n,1}$, $\mathcal{Q}_n$ [30, 33], a decomposable central extension of $\mathfrak{n}_{n,1}$ in [34] and triangular in [29].

On the other hand, it is obvious that even the improved bound (13) cannot give a precise estimate on the maximal dimension of a solvable extension in all cases. In particular, we have always $\dim \mathfrak{n} - \dim \mathfrak{n}^2 \geqslant 2$, i.e. characteristically nilpotent Lie algebras cannot be easily detected using equation (13). Similarly, the bound (13) is not saturated in the case of Heisenberg nilradicals $\mathfrak{h}$ [27] where the maximal number of non-nilpotent elements is in fact equal to $\frac{\dim \mathfrak{h}+1}{2} < \dim \mathfrak{h} - 1$. It remains an open problem to further improve the estimate (13) if it is possible.

An interesting observation arises from the investigation of numerous nilradicals in [27–35]. In all these cases the maximal solvable extension of the given nilradical over the field of complex numbers turns out to be unique up to isomorphism. Therefore, we formulate it as a conjecture.

**Conjecture 1.** *Let $\mathfrak{n}$ be a complex nilpotent Lie algebra, not characteristically nilpotent. Let $\mathfrak{s}, \tilde{\mathfrak{s}}$ be solvable Lie algebras with the nilradical $\mathfrak{n}$ of maximal dimension in the sense that no such solvable algebra of larger dimension exists. Then, $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are isomorphic.*

It would be of interest to establish whether this conjecture holds in general or requires some supplementary assumptions on the structure of $\mathfrak{n}$.

Next we have investigated the structure of Levi decomposable algebras. We have formulated several general properties that the nilradical of any Levi decomposable algebra must necessarily satisfy and applied these to the lists of Levi decomposable algebras in the papers [19, 20] by Turkowski. It turns out that dimensional analysis of the three characteristic series and their centralizers is enough to determine whether a given five-dimensional nilpotent algebra has a nontrivial Levi extension. In the case of six-dimensional nilpotent algebras this is no longer a sufficient criterion and more involved considerations were necessary. Nevertheless, we were able to explain the absence of a Levi extension for all but two six-dimensional indecomposable nilpotent algebras by abstract, mostly dimensional, considerations, without an explicit construction of derivations. This indicates that techniques developed here can be of significant help in this kind of analysis also in higher dimensions. Especially, if a particular choice of the Levi factor and the nilradical is desired by some application, our results can be easily used to estimate whether such a Levi extension may exist.

The results and methods used in this section can be applied to Levi extensions of arbitrary dimension. One particular immediate consequence of them is that no filiform algebra can be a nilradical of a Levi decomposable algebra. More generally, the same also holds for any nilpotent algebra $\mathfrak{n}$ such that $\dim \mathfrak{n} - \dim \mathfrak{n}^2 = 2$ and $j \in \mathbb{N}$ exists such that $\dim \mathfrak{n}^j - \dim \mathfrak{n}^{j+1} = \dim \mathfrak{n}^{j+1} - \dim \mathfrak{n}^{j+2} = 1$.

It remains an open problem to find some structurally interesting series of nilradicals in arbitrary dimension allowing the classification of its nontrivial Levi extensions other than $\mathfrak{n}$ being Abelian or Heisenberg.

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# Chapter 3

# Symmetries of differential equations with anticommuting variables

In this chapter we present two papers

[e] A.M. Grundland, A.J. Hariton and L. Šnobl, Invariant solutions of the supersymmetric sine-Gordon equation, J. Phys. A: Math. Theor. 42 (2009) 335203,

[f] A.M. Grundland, A.J. Hariton and L. Šnobl, Invariant solutions of supersymmetric nonlinear wave equations, J. Phys. A: Math. Theor. 44 (2011) 085204,

in which we study a generalization of the symmetry analysis reviewed in Section 1.3.5 to supersymmetric differential equations.

In particular, we determine the Lie superalgebra of infinitesimal symmetries of the considered equations, identify conjugation classes of 1–parameter subgroups and perform corresponding inequivalent symmetry reductions to ODEs. When possible, we find explicit solutions.

The main conceptual observation in these two papers is that the symmetry reduction may not always work when anticommuting variables are involved. The reason is that in this case not all vector fields can be rectified to the form $X = \frac{\partial}{\partial x}$ where $x$ is an ordinary, commuting variable.

The paper [f] was included in the IOPselect collection by decision of the editors of Journal of Physics A: Mathematical and Theoretical.

## 3.1

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# Invariant solutions of the supersymmetric sine-Gordon equation

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**Abstract**

A comprehensive symmetry analysis of the $\mathcal{N} = 1$ supersymmetric sine-Gordon equation is performed. Two different forms of the supersymmetric system are considered. We begin by studying a system of partial differential equations corresponding to the coefficients of the various powers of the anticommuting independent variables. Next, we consider the super-sine-Gordon equation expressed in terms of a bosonic superfield involving anticommuting independent variables. In each case, a Lie (super)algebra of symmetries is determined and a classification of all subgroups having generic orbits of codimension 1 in the space of independent variables is performed. The method of symmetry reduction is systematically applied in order to derive invariant solutions of the supersymmetric model. Several types of algebraic, hyperbolic and doubly periodic solutions are obtained in explicit form.

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### 1. Introduction

The purpose of this paper is to obtain Lie point symmetries and group-invariant solutions of the minimal ($\mathcal{N} = 1$) supersymmetric extension of the (1 + 1)-dimensional sine-Gordon equation:

$$\varphi_{xt} = \sin\varphi. \tag{1}$$

The symmetry reduction method (SRM) is systematically applied in order to derive invariant solutions of the $\mathcal{N} = 1$ supersymmetric extension of the model (1).

 

The classical sine-Gordon equation (1) has applications in various areas of physics including, among others, nonlinear field theory, solid-state physics (evolution of magnetic flux in Josephson junctions, Bloch wall motion of magnetic crystals, etc), nonlinear optics (self–induced transparency, fiber optics), elementary particle theory and fluid dynamics; see [1–6] and references therein. A broad review of recent developments in the theory involved as well as their applications can be found for example in [2, 6–8] and bibliographies therein. The sine-Gordon equation (1) also has great significance in mathematics, especially in the soliton theory of surfaces. Analytic nonpertubative techniques for solving equation (1) exist, including, among others, the inverse scattering method and the Darboux–Bäcklund transformations. Multiple soliton solutions of (1) have found a wide variety of applications. The Bäcklund transformation for the sine-Gordon equation (1) linking different analytic descriptions of constant negative curvature surfaces in $\mathbb{R}^3$ was established a century ago by Bianchi [9] and then by Steuerwald [10]. They were the first to find solutions of the structural equations (i.e. the Gauss–Weingarten and the Gauss–Codazzi–Mainardi equations). In particular, they constructed pseudospherical surfaces for the sine-Gordon equation (1) by means of the auto-Bäcklund transformation. It was demonstrated [11] that these surfaces can be described either by the Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 + (1 + (u_x)^2 + (u_y)^2)^2 = 0 \tag{2}$$

(where $z = u(x, y)$ is the graph of a surface in $\mathbb{R}^3$) or by the sine-Gordon equation (1) for the angle $\varphi(x, t)$ between asymptotic directions. The surfaces associated with equations (1) and (2) are characterized by the Gaussian curvature $K = -1$. The explicit form of the correspondence between these two integrable models is known [11].

In recent publications (see e.g. [12–17]), a superspace extension of the Lagrangian formulation has been established for the supersymmetric sine-Gordon (SSG) equation. The associated linear spectral problem was thoroughly discussed by many authors (see e.g. [13, 17] and references therein). It was shown [15] that the equation of motion appears as the compatibility condition of a set of Riccati equations. The supersymmetric sine-Gordon equation admits an infinite number of conservation laws, and a connection was established [15, 16] between its super-Bäcklund and super-Darboux transformations. Consequently, it was shown in [16] that the Darboux transformation is related to the super-Bäcklund transformation, and the latter was used to construct multi-super soliton solutions. The SSG equation was shown to be equivalent to the super $\mathbb{C}P^1$ sigma model [18, 19]. The prolongation method of Wahlquist and Estabrook was used to find an infinite-dimensional superalgebra and the associated super Lax pairs [20].

In physics, the supersymmetric sine-Gordon equation is a useful example of a nonlinear integrable supersymmetric theory, on which conjectures concerning the properties of such theories can be tested. These involve, among others, the computations of the $S$-matrix [21, 22]. In addition, $\mathcal{N} = 2$ supersymmetric sine-Gordon models arise in certain reductions of superstring worldsheet theories on particular backgrounds, e.g. the Pohlmeyer reduction on $AdS_2 \times S^2$ [23].

The purpose of this paper is to study the symmetry properties of the supersymmetric sine-Gordon system (more precisely, of the equations of motion of the $\mathcal{N} = 1$ supersymmetric sine-Gordon model) and to construct various classes of invariant solutions of this model. In order to accomplish this, we use a generalized version of the prolongation method which encompasses commuting and anticommuting variables. The total derivatives with respect to these variables are adapted in such a way that they are consistent with the standard definitions. We then use a generalized version of the SRM in order to obtain group-invariant solutions of the supersymmetric sine-Gordon model. These solutions complement the multi-super soliton solutions found recently.

This paper is organized as follows. In section 2, we recall the supersymmetric sine-Gordon equation, constructed in such a way that it is invariant under two independent supersymmetry transformations. In section 3, we decompose the supersymmetric sine-Gordon equation into three partial differential equations involving the component fields of the superfield and proceed to determine a Lie symmetry algebra of this system. Next, we focus on the SSG equation expressed explicitly in terms of the odd superspace variables $\theta_1$ and $\theta_2$ and the bosonic superfield $\Phi$. In section 4, we compute in detail the Lie superalgebra of symmetries of this equation using a generalized version of the prolongation method. The subalgebra classification of this superalgebra is performed in section 5, and a discussion of the invariant solutions of the SSG equation is the subject of section 6. Finally, in section 7, we provide a summary of the results and list some possible future developments.

## 2. Supersymmetric extension

We are interested in the supersymmetric sine-Gordon equation [14–16] constructed on the four-dimensional superspace $\{(x, t, \theta_1, \theta_2)\}$. Here, $x$ and $t$ represent the even (bosonic) coordinates on the two-dimensional super-Minkowski space $\mathbb{R}^{(1,1|2)}$, while the quantities $\theta_1$ and $\theta_2$ are anticommuting odd coordinates.

We replace the real-valued function $\varphi(x, t)$ in equation (1) by the real scalar bosonic superfield $\Phi\,(x, t, \theta_1, \theta_2)$. Such a superfield can be decomposed into its component fields as

$$\Phi(x, t, \theta_1, \theta_2) = \tfrac{1}{2}u(x, t) + \theta_1\phi(x, t) + \theta_2\psi(x, t) + \theta_1\theta_2 F(x, t), \tag{3}$$

where $\phi$ and $\psi$ are the odd-valued functions (fields) and $u$ and $F$ are the even-valued functions (fields). The supersymmetric extension of equation (1) is constructed in such a way that it is invariant under the two independent supersymmetry transformations:

$$x \to x - \underline{\eta}_1\theta_1, \qquad \theta_1 \to \theta_1 + \underline{\eta}_1 \qquad \text{and} \qquad t \to t - \underline{\eta}_2\theta_2, \qquad \theta_2 \to \theta_2 + \underline{\eta}_2, \tag{4}$$

where $\underline{\eta_1}$ and $\underline{\eta_2}$ are the odd parameters (in general, we use the convention that underlined letters represent odd parameters). These transformations are generated by the infinitesimal supersymmetry generators:

$$Q_x = \partial_{\theta_1} - \theta_1\partial_x \qquad \text{and} \qquad Q_t = \partial_{\theta_2} - \theta_2\partial_t, \tag{5}$$

which satisfy the anticommutation relations

$$\{Q_x, Q_x\} = -2\partial_x, \qquad \{Q_t, Q_t\} = -2\partial_t, \qquad \{Q_x, Q_t\} = 0. \tag{6}$$

In order to make our superfield theory invariant under the actions $Q_x$ and $Q_t$, we write the equation in terms of the covariant derivatives

$$D_x = \partial_{\theta_1} + \theta_1\partial_x \qquad \text{and} \qquad D_t = \partial_{\theta_2} + \theta_2\partial_t, \tag{7}$$

which possess the property that they square to the generators of spacetime translations and anticommute with the supersymmetry generators:

$$\begin{aligned} D_x^2 = \partial_x, \qquad D_t^2 = \partial_t, \qquad &\{D_x, D_t\} = \{D_x, Q_x\} = \{D_x, Q_t\} \\ &= \{D_t, Q_x\} = \{D_t, Q_t\} = 0. \end{aligned} \tag{8}$$

The superspace Lagrangian density of the supersymmetric model is

$$\mathcal{L}(\Phi) = \tfrac{1}{2}D_x\Phi D_t\Phi - \cos\Phi, \tag{9}$$

and the corresponding Euler–Lagrange superfield equation is given by

$$D_x D_t \Phi = \sin \Phi. \tag{10}$$

Equation (10) is invariant under the supersymmetry transformations (4), and we therefore refer to it as the supersymmetric sine-Gordon (SSG) equation. Once it is expanded out in terms of the component fields $\frac{1}{2}u(x,t), \phi(x,t), \psi(x,t), F(x,t)$, one finds that the scalar part of equation (10) is in fact algebraic and restricts $F$ to be the following function of $u$ [14]:

$$F = -\sin\left(\frac{u}{2}\right). \tag{11}$$

Up to this point, the presentation has been formulated in the language usually used in physics, not yet mathematically well defined. The mathematically sound formulation is based on the notion of supermanifolds in the sense of [24, 25] and can be described as follows.

One starts by considering a real Grassmann algebra $\Lambda$ generated by a finite or infinite number of generators $(\xi_1, \xi_2, \ldots)$. The number of Grassmann generators of $\Lambda$ is not directly relevant for applications; essentially the only assumption is that 'there are at least as many independent ones as are needed in any formula encountered'. The Grassmann algebra $\Lambda$ has a naturally defined parity $\tilde{1} = 0, \tilde{\xi}_i = 1, \widetilde{(ab)} = \tilde{a}\tilde{b}$ and can be split into even and odd parts:

$$\Lambda = \Lambda_{\text{even}} + \Lambda_{\text{odd}}. \tag{12}$$

The spaces $\Lambda$ and $\Lambda_{\text{even}}$ replace the field of real numbers in the context of supersymmetry. Elements of $\Lambda$ are called supernumbers, while elements of its even/odd part are even/odd supernumbers. For instance, in equation (4) we have parameters $\underline{\eta_1}, \underline{\eta_2} \in \Lambda_{\text{odd}}$. Sometimes we may also employ a different split:

$$\Lambda = \Lambda_{\text{body}} + \Lambda_{\text{soul}} \tag{13}$$

where $\Lambda_{\text{body}} = \wedge^0[\xi_1, \xi_2, \ldots] \simeq \mathbb{R}$ and $\Lambda_{\text{soul}} = \sum_{k\geqslant 1} \wedge^k[\xi_1, \xi_2, \ldots]$. The bodiless elements in $\Lambda_{\text{soul}}$ are obviously non-invertible because of the $\mathbb{Z}_0^+$-grading of the Grassmann algebra. If the number of Grassmann generators $K$ is finite, bodiless elements are nilpotent of degree at most $K$. In what follows, we shall assume that $K$ is arbitrarily large but finite—this assumption will allow us to use rigorous theorems of [26].

Next, one considers a $\mathbb{Z}_2$-graded real vector space $V$, with even basis elements $u_i, i = 1, \ldots, N$, and odd basis elements $\upsilon_\mu, \mu = 1, \ldots, M$, and constructs $W = \Lambda \otimes_{\mathbb{R}} V$. The space of interest to us is its even part:

$$W_{\text{even}} = \left\{ \sum_i a_i u_i + \sum_\mu \underline{\alpha}_\mu \upsilon_\mu | a_i \in \Lambda_{\text{even}}, \underline{\alpha}_\mu \in \Lambda_{\text{odd}} \right\}.$$

Obviously, $W_{\text{even}}$ is a $\Lambda_{\text{even}}$ module and can be identified with $\Lambda_{\text{even}}^{\times N} \times \Lambda_{\text{odd}}^{\times M}$. To the original basis consisting of $u_i$ and $\upsilon_\mu$ (although $\upsilon_\mu \notin W_{\text{even}}$ !), we associate the corresponding functionals

$$E_j : W_{\text{even}} \to \Lambda_{\text{even}} : E_j\left(\sum_i a_i v_i + \sum_\mu \underline{\alpha}_\mu \upsilon_\mu\right) = a_j,$$

$$\Upsilon_\nu : W_{\text{even}} \to \Lambda_{\text{odd}} : \Upsilon_\nu\left(\sum_i a_i v_i + \sum_\mu \underline{\alpha}_\mu \upsilon_\mu\right) = \underline{\alpha}_\nu$$

and view them as the coordinates (even and odd, respectively) on $W_{\text{even}}$. Any topological space locally diffeomorphic to a suitable $W_{\text{even}}$ is called a supermanifold. The transition functions to even and odd coordinates between different charts on supermanifold are assumed to be even- and odd-valued superanalytic or at least $G^\infty$ functions on $W_{\text{even}}$. For comprehensive definitions of the classes of 'supersmooth' functions $G^\infty$ and superanalytic functions $G^\omega$, we refer the reader to consult e.g. [24], definition 2.5—here we only note that superanalytic functions are those that can be expanded into convergent power series in even and odd coordinates, whereas the definition of the $G^\infty$ function is a more involved analog on supermanifold of $C^\infty$ functions on manifolds. Any $G^\infty$ function can be expanded into products of odd coordinates (i.e. Taylor-like expansion) but the coefficients, being functions of even coordinates, may not necessarily be analytic.

In our context, the super-Minkowski space $\mathbb{R}^{(1,1|2)}$ should be understood as such a supermanifold, globally diffeomorphic to $\Lambda_{\text{even}}^{\times 2} \times \Lambda_{\text{odd}}^{\times 2}$ with even coordinates $x, t$ and odd coordinates $\theta_1, \theta_2$. The supersymmetry transformation (4) can be viewed as a particular change of coordinates on $\mathbb{R}^{(1,1|2)}$ which transforms solutions of equation (10) into solutions of the same equation in new coordinates.

A bosonic, also called even, superfield is a $G^\infty$ function $\Phi : \mathbb{R}^{(1,1|2)} \to \Lambda_{\text{even}}$. It can be expanded in powers of odd coordinates $\theta_1, \theta_2$ giving decomposition (3), with

$$u, F : \Lambda_{\text{even}}^{\times 2} \to \Lambda_{\text{even}},$$
$$\phi, \psi : \Lambda_{\text{even}}^{\times 2} \to \Lambda_{\text{odd}}.$$

The partial derivatives with respect to the odd coordinate (for a detailed description see [24], definitions 2.5 and 5.6) satisfy the usual operational rules, namely $\partial_{\theta_i}\theta_j = \delta^i_j$, together with the graded product rule:

$$\partial_{\theta_i}(fg) = (\partial_{\theta_i} f)g + (-1)^{\tilde{f}}(\partial_{\theta_i} g). \tag{14}$$

The operations $\partial_{\theta_i}, Q_{x,t}, D_{x,t}$ in equations (5) and (7) switch the parity of the function acted on. For instance, $\partial_{\theta_1}\Phi$ becomes an odd superfield $\partial_{\theta_1}\Phi : \mathbb{R}^{(1,1|2)} \to \Lambda_{\text{odd}}$ whose component decomposition is

$$\partial_{\theta_1}\Phi(x, t, \theta_1, \theta_2) = \phi(x, t) + \theta_2 F(x, t).$$

## 3. Lie symmetry properties of the supersymmetric sine-Gordon system in component form

When decomposed in terms of the various powers of $\theta_1$ and $\theta_2$, the SSG equation (10) is seen to be equivalent to a system of three partial differential equations for the fields $u, \phi$ and $\psi$. That is, the coefficients of the powers $\theta_1, \theta_2$ and $\theta_1\theta_2$ combine to form the following system of coupled equations for the component fields [14]:

$$\begin{aligned} &\text{(i)} \quad u_{xt} = -\sin u + 2\phi\psi \sin\left(\frac{u}{2}\right),\\ &\text{(ii)} \quad \phi_t = -\psi \cos\left(\frac{u}{2}\right),\\ &\text{(iii)} \quad \psi_x = \phi \cos\left(\frac{u}{2}\right). \end{aligned} \tag{15}$$

In order to determine the Lie point symmetry algebra $\mathfrak{g}$ of the system (15), we restrict our consideration to Lie groups and use an infinitesimal approach. We adapt the method of prolongation of vector fields described in the book by Olver [27] to the case where the equations of the system contain both even- and odd-valued functions [28, 29]. We begin by writing the set of partial differential equations (15) in the form

$$\Delta_k(x, t, u, \phi, \psi, u_{xt}, \phi_t, \psi_x) = 0, \qquad k = 1, 2, 3, \tag{16}$$

where

$$\Delta_1(x,t,u,\phi,\psi,u_{xt},\phi_t,\psi_x) = u_{xt} + \sin u - 2\phi\psi\sin\left(\frac{u}{2}\right),$$
$$\Delta_2(x,t,u,\phi,\psi,u_{xt},\phi_t,\psi_x) = \phi_t + \psi\cos\left(\frac{u}{2}\right), \tag{17}$$
$$\Delta_3(x,t,u,\phi,\psi,u_{xt},\phi_t,\psi_x) = \psi_x - \phi\cos\left(\frac{u}{2}\right).$$

A symmetry group $G$ of the system (16) is a (local) group of transformations acting on the cartesian product of supermanifolds:

$$X \times U$$

with even coordinates $(x, t, u)$ and odd coordinates $(\phi, \psi)$, whose associated action on the functions $u(x,t)$, $\Phi(x,t)$, $\psi(x,t)$ maps solutions of (16) to solutions of (16). Assuming that $G$ is a super Lie group in the sense of [26], one can associate with it its Lie algebra of even left-invariant vector fields $\mathcal{G}$, whose elements are the infinitesimal symmetries of the system (16). In particular, a local one-parameter subgroup of $G$ consists of a family of transformations

$$g_\varepsilon : \quad \tilde{x}^i = X^i(x,u,\varepsilon), \qquad \tilde{u}^\alpha = U^\alpha(x,u,\varepsilon), \tag{18}$$

where $x = (x^1, x^2) = (x, t)$ are the independent variables and $u = (u^1, u^2, u^3) = (u, \phi, \psi)$ are the dependent ones. $\varepsilon \in \Lambda_{\text{even}}$ is a group parameter whose range may be restricted depending on the values of $x, t, u, \phi, \psi$. Such a local subgroup is generated by a vector field of the form

$$\mathbf{v} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \Phi(x,u)^\alpha\frac{\partial}{\partial u^\alpha}, \tag{19}$$

where

$$\xi^i(x,u) = \frac{\partial}{\partial\varepsilon}X^i\bigg|_{\varepsilon=0}, \qquad \Phi^\alpha(x,u) = \frac{\partial}{\partial\varepsilon}U^\alpha\bigg|_{\varepsilon=0}. \tag{20}$$

The advantage of working with the Lie algebra $\mathfrak{g}$ instead of directly with the super Lie group $G$ is that the equations defining the infinitesimal symmetries are linear.

In order to determine the infinitesimal symmetries of a system of partial differential equations, it is useful to make use of the concept of the prolongation of a group action. The idea is that a transformation of coordinates $x^i \to \tilde{x}^i$, $u^\alpha \to \tilde{u}^\alpha$ induces a transformation of the derivatives:

$$\frac{\partial u^\alpha}{\partial x^i} \longrightarrow \frac{\partial\tilde{u}^\alpha}{\partial\tilde{x}^i}. \tag{21}$$

In order to make use of this concept, we define the multi-index $J = (j_1, \ldots, j_p)$, where $j_i = 0, 1, \ldots$ and $|J| = j_1 + \cdots + j_p$. The space of coordinates on $X \times U$ is extended to the jet bundle

$$\mathcal{J}_k = \left\{\left(x^i, u^\alpha, u^\alpha_J\right) \big| |J| \leqslant k\right\}, \tag{22}$$

which includes the coordinates and all derivatives of the dependent variables of order less than or equal to $k$. In our setting the jet bundle (22) is a supermanifold on which we define total derivatives

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} + \sum_{\alpha,J} u^\alpha_{J_i}\frac{\partial}{\partial u^\alpha_J} \tag{23}$$

where $J_i = (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_n)$. More generally, for $J = (j_1, j_2, \ldots, j_n)$, we define

$$\mathcal{D}_J = \underbrace{\mathcal{D}_1\mathcal{D}_1\cdots\mathcal{D}_1}_{j_1}\cdots\underbrace{\mathcal{D}_n\mathcal{D}_n\cdots\mathcal{D}_n}_{j_n}. \tag{24}$$

The prolongation of a group action to the jet bundle $\mathcal{J}_k$ in turn induces a prolongation of the generating infinitesimal vector field in the Lie algebra. For the vector field $\mathbf{v}$ given by (19), the $k$th-order prolongation of $\mathbf{v}$ is

$$pr^{(k)}(\mathbf{v}) = \mathbf{v} + \sum_{\alpha,|J|\neq 0} \phi_J^\alpha(x, u^{(k)}) \frac{\partial}{\partial u_J^\alpha}, \tag{25}$$

where $\phi_J^\alpha(x, u^{(k)})$ are given by the formula

$$\phi_J^\alpha = \mathcal{D}_J \left( \phi^\alpha - \xi^i \frac{\partial u^\alpha}{\partial x^i} \right) + \xi^i u_{J_i}^\alpha \tag{26}$$

or, equivalently, by the recursive formula

$$\phi_{J_j}^\alpha = \mathcal{D}_j \phi_J^\alpha - \sum_i (\mathcal{D}_j \xi^i) u_{J_i}^\alpha. \tag{27}$$

The symmetry criterion (theorem 2.31 in [27]) assumes that $G$ is a connected Lie group of transformations acting locally on $X \times U$ through the transformations

$$\tilde{x}_i = X^i(x, u, g), \qquad \tilde{u}^\alpha = U^\alpha(x, u, g),$$

where $g \in G$ and $\Delta_\nu(x, u^{(n)})$ is a non-degenerate system of partial differential equations (meaning that the system is locally solvable with respect to highest derivatives and is of maximal rank at every point $\left(x_0, u_0^{(n)}\right) \in X \times U^{(n)}$). Then $G$ is a symmetry group of $\Delta = 0$ if and only if

$$[pr^{(k)}(\mathbf{v})](\Delta) = 0 \qquad \text{whenever} \quad \Delta = 0 \tag{28}$$

for each infinitesimal generator $\mathbf{v}$ of $G$.

Using the results of [26], one finds that the same criterion can be used also in the case of the super Lie group $G$ and its Lie algebra of even left-invariant vector fields.

For the purpose of determining the Lie algebra of symmetries of the system (16), let us write a vector field of the form

$$\begin{aligned}\mathbf{v} = {} & \xi(x, t, u, \phi, \psi)\partial_x + \tau(x, t, u, \phi, \psi)\partial_t + \mathcal{U}(x, t, u, \phi, \psi)\partial_u \\ & + \Sigma(x, t, u, \phi, \psi)\partial_\phi + \Psi(x, t, u, \phi, \psi)\partial_\psi,\end{aligned} \tag{29}$$

where $\xi$, $\tau$ and $\mathcal{U}$ are the $\Lambda_{\text{even}}$-valued functions while $\Sigma$ and $\Psi$ are the $\Lambda_{\text{odd}}$-valued so that $\mathbf{v}$ is an even vector field. We consider a second prolongation of the vector field (29) which is of the form

$$\mathrm{pr}^{(2)}(\mathbf{v}) = \mathbf{v} + \mathcal{U}^{xt}\partial_{u_{xt}} + \Sigma^t \partial_{\phi_t} + \Psi^x \partial_{\psi_x} + (\mathcal{U}^x \partial_{u_x} + \mathcal{U}^t \partial_{u_t} + \cdots), \tag{30}$$

where the terms in the parentheses do not contribute in what follows, namely in equation (33). The coefficients $\mathcal{U}^{xt}$, $\Sigma^t$ and $\Psi^x$ are the known functions of the components $\xi, \ldots, \Psi$ of the vector field $\mathbf{v}$ and their derivatives with respect to the independent and dependent variables $x, \ldots, \psi$ (as given by the general prolongation formula (26) or (27)). We use upper indices in coefficients $\mathcal{U}^{xt}$, $\Sigma^t$, etc in order to distinguish them from partial derivatives, e.g. $\mathcal{U}_{xt} = \partial_t \partial_x \mathcal{U}$. The first-order coefficients are given by

$$\begin{aligned}\Sigma^t = {} & \Sigma_t + \Sigma_u u_t + \Sigma_\phi \phi_t + \Sigma_\psi \psi_t - \xi_t \phi_x - \xi_u u_t \phi_x - \xi_\phi \phi_x \phi_t \\ & - \xi_\psi \phi_x \psi_t - \tau_t \phi_t - \tau_u u_t \phi_t - \tau_\psi \phi_t \psi_t,\end{aligned} \tag{31}$$

and

$$\begin{aligned}\Psi^x = {} & \Psi_x + \Psi_u u_x + \Psi_\phi \phi_x + \Psi_\psi \psi_x - \xi_x \psi_x - \xi_u u_x \psi_x + \xi_\phi \phi_x \psi_x \\ & - \tau_x \psi_t - \tau_u u_x \psi_t + \tau_\phi \phi_x \psi_t + \tau_\psi \psi_x \psi_t.\end{aligned} \tag{32}$$

The second-order coefficient, $\mathcal{U}^{xt}$, is much involved and will not be presented here.

According to the symmetry criterion, the vector field (29) is an infinitesimal generator of the symmetry group of the system of differential equations (16) if and only if

$$\mathrm{pr}^{(2)}(\mathbf{v})[\Delta_k(x,t,u,\phi,\psi,u_{xt},\phi_t,\psi_x)] = 0, \qquad k = 1,2,3, \tag{33}$$

whenever $\Delta_l(x,t,u,\phi,\psi,u_{xt},\phi_t,\psi_x) = 0, l = 1,2,3$.

The condition $\mathrm{pr}^{(2)}(\mathbf{v})[\Delta_k] = 0$, when applied to the system (15), leads to the following conditions on the coefficients:

$$\begin{aligned}
&\text{(i)} \quad \mathcal{U}^{xt} = \mathcal{U}\left(-\cos u + \cos\left(\frac{u}{2}\right)\phi\psi\right) + \Sigma\left(2\sin\left(\frac{u}{2}\right)\psi\right) + \Psi\left(-2\sin\left(\frac{u}{2}\right)\phi\right),\\
&\text{(ii)} \quad \Sigma^t = \frac{1}{2}\mathcal{U}\sin\left(\frac{u}{2}\right)\psi - \Psi\cos\left(\frac{u}{2}\right),\\
&\text{(iii)} \quad \Psi^x = -\frac{1}{2}\mathcal{U}\sin\left(\frac{u}{2}\right)\phi + \Sigma\cos\left(\frac{u}{2}\right),
\end{aligned} \tag{34}$$

whenever $u, \phi, \psi$ satisfy the system (15).

Substituting the prolongation formulas for $\mathcal{U}^{xt}$, $\Sigma^t$ and $\Psi^x$ into (34) and imposing the condition that $\Delta_k = 0, k = 1,2,3$, i.e. substituting for $u_{xt}, \phi_t, \psi_x$ and their derivatives, we equate the coefficients of the various monomials in the various remaining derivatives of $u, \phi$ and $\psi$ with respect to $x$ and $t$ (i.e. those unconstrained by equation (15)). We obtain a series of determining equations which impose restrictions on the coefficients $\xi, \tau, \mathcal{U}$, $\Sigma$ and $\Psi$ of the vector field (29). Solving the determining equations, we see that the coefficients must be

$$\begin{aligned}
&\xi(x) = C_1x + C_2, \qquad \tau(t) = -C_1t + C_3, \qquad \mathcal{U} = 0,\\
&\Sigma(\phi) = -\tfrac{1}{2}C_1\phi, \qquad \Psi(\psi) = \tfrac{1}{2}C_1\psi,
\end{aligned} \tag{35}$$

where $C_1, C_2$ and $C_3$ are arbitrary even parameters. Thus, we have determined that the Lie algebra $\mathfrak{g}$ is spanned by the following three vector fields:

$$P_x = \partial_x, \qquad P_t = \partial_t, \qquad D = 2x\partial_x - 2t\partial_t - \phi\partial_\phi + \psi\partial_\psi, \tag{36}$$

where $\partial_x = \partial/\partial x$, etc. We have two translations, $P_x$ and $P_t$, in the $x$ and $t$ directions respectively, and the dilation $D$ acting on the independent and dependent variables. We note that although the method may in general yield a super Lie algebra (for explicit examples see e.g. [30–32]), in our particular case of the supersymmetric sine-Gordon system (15) the result is just a Lie algebra acting on the supermanifold $X \times U$. In fact, the Lie algebra in question whose nonzero commutation relations are

$$[P_x, D] = 2P_x, \qquad [P_t, D] = -2P_t \tag{37}$$

is $ISO(1,1)$, which is also the symmetry Lie algebra of the ordinary sine-Gordon equation (in $(1+1)$-dimensional Minkowski space described in the light-cone coordinates, the Lorentz boost takes the form of a dilation). This represents the Poincaré invariance of the sine-Gordon equation, supersymmetric or otherwise. This algebra is also identified as $A_{3,4}\,(E(1,1))$ in [33] where its non-conjugate one-dimensional subalgebras are found to be

$$L_1 = \{D\}, \qquad L_2 = \{P_x\}, \qquad L_3 = \{P_t\}, \qquad L_4 = \{P_x + P_t\}, \qquad L_5 = \{P_x - P_t\}. \tag{38}$$

One can now proceed to apply the SRM in order to obtain invariant solutions of the supersymmetric system (15). First, we find for each of the subalgebras listed in (38) the associated four invariants along with the appropriate change of variable that has to be substituted into thesystem (15) in order to obtain the set of reduced ordinary differential equations. In each case, the invariant involving only the independent variables, the so-called

**Table 1.** Invariants and change of variables for subalgebras of the Lie algebra $\mathfrak{g}$ spanned by the vector fields (36).

| Subalgebra | Invariants | Relations and change of variable |
|---|---|---|
| $L_1 = \{D\}$ | $\sigma = xt, u, t^{-1/2}\phi, t^{1/2}\psi$ | $u = u(\sigma), \phi = t^{1/2}\Theta(\sigma), \psi = t^{-1/2}\Omega(\sigma)$ |
| $L_2 = \{P_x\}$ | $\sigma = t, u, \phi, \psi$ | $u = u(t), \phi = \phi(t), \psi = \psi(t)$ |
| $L_3 = \{P_t\}$ | $\sigma = x, u, \phi, \psi$ | $u = u(x), \phi = \phi(x), \psi = \psi(x)$ |
| $L_4 = \{P_x + P_t\}$ | $\sigma = x - t, u, \phi, \psi$ | $u = u(\sigma), \phi = \phi(\sigma), \psi = \psi(\sigma)$ |
| $L_5 = \{P_x - P_t\}$ | $\sigma = x + t, u, \phi, \psi$ | $u = u(\sigma), \phi = \phi(\sigma), \psi = \psi(\sigma)$ |

**Table 2.** Reduced equations obtained for subalgebras of the Lie algebra $\mathfrak{g}$ spanned by the vector fields (36).

| Subalgebra | Reduced equations |
|---|---|
| $L_1 = \{D\}$ | $\sigma u_{\sigma\sigma} + u_\sigma = -\sin u + 2\sin\left(\frac{u}{2}\right)\Theta\Omega, \quad \frac{1}{2}\Theta + \sigma\Theta_\sigma = -\cos\left(\frac{u}{2}\right)\Omega,$ $\Omega_\sigma = \cos\left(\frac{u}{2}\right)\Theta$ |
| $L_2 = \{P_x\}$ | $-\sin u + 2\sin\left(\frac{u}{2}\right)\phi\psi = 0, \quad \phi_t = -\cos\left(\frac{u}{2}\right)\psi, \quad \cos\left(\frac{u}{2}\right)\phi = 0$ |
| $L_3 = \{P_t\}$ | $-\sin u + 2\sin\left(\frac{u}{2}\right)\phi\psi = 0, \quad \cos\left(\frac{u}{2}\right)\psi = 0, \quad \psi_x = \cos\left(\frac{u}{2}\right)\phi$ |
| $L_4 = \{P_x + P_t\}$ | $-u_{\sigma\sigma} = -\sin u + 2\sin\left(\frac{u}{2}\right)\phi\psi, \quad \phi_\sigma = \cos\left(\frac{u}{2}\right)\psi, \quad \psi_\sigma = \cos\left(\frac{u}{2}\right)\phi$ |
| $L_5 = \{P_x - P_t\}$ | $u_{\sigma\sigma} = -\sin u + 2\sin\left(\frac{u}{2}\right)\phi\psi, \quad \phi_\sigma = -\cos\left(\frac{u}{2}\right)\psi, \quad \psi_\sigma = \cos\left(\frac{u}{2}\right)\phi$ |

symmetry variable, is labeled by the symbol $\sigma$. The invariants and the change of variables are listed in table 1, while the systems of reduced ordinary differential equations are listed in table 2. Because the reduced ODE systems in table 2 form a subset of cases investigated in section 6, we postpone their discussion there.

## 4. Symmetries of the SSG equation

When we performed the group-theoretical analysis of the supersymmetric sine-Gordon system in the component form (15), we noted that the resulting symmetry algebra did not essentially differ from the purely bosonic case, i.e. equation (1). That is, the supersymmetry algebra was not recovered in this way. In order to overcome this shortcoming of the method, we now turn our attention to the superfield version of the model represented by the SSG equation (10). This equation can be rewritten in the form

$$\theta_1\theta_2\Phi_{xt} - \theta_2\Phi_{t\theta_1} + \theta_1\Phi_{x\theta_2} - \Phi_{\theta_1\theta_2} = \sin\Phi, \tag{39}$$

where each successive subscript (from left to right) indicates a successive partial derivative (for example, $\Phi_{\theta_1\theta_2}$ represents $\partial_{\theta_2}(\partial_{\theta_1}\Phi)$). In order to determine the Lie superalgebra of symmetries of equation (39), we employ the generalized method of prolongations so as to include also the two independent odd variables $\theta_1$ and $\theta_2$. Such procedure was proposed and used in [34, 35].

We consider transformations on the supermanifold

$$\mathbb{R}^{(1,1|2)} \times \Lambda_{\text{even}}.$$

We write a generator of symmetry transformation in the form of an even vector field on this manifold:

$$\begin{aligned}\mathbf{v} = \xi(x,t,\theta_1,\theta_2,\Phi)\partial_x + \tau(x,t,\theta_1,\theta_2,\Phi)\partial_t + \rho(x,t,\theta_1,\theta_2,\Phi)\partial_{\theta_1}\\ +\sigma(x,t,\theta_1,\theta_2,\Phi)\partial_{\theta_2} + \Pi(x,t,\theta_1,\theta_2,\Phi)\partial_\Phi,\end{aligned} \tag{40}$$

where $\xi$, $\tau$ and $\Pi$ are supposed to be even, i.e. $\Lambda_{\text{even}}$-valued functions, while $\rho$ and $\sigma$ are odd, i.e. $\Lambda_{\text{odd}}$ valued. Here, we adopt the ordering convention that the odd coefficients in the expression (in this case $\rho$ and $\sigma$) precede the odd derivatives ($\partial_{\theta_1}$ and $\partial_{\theta_2}$ respectively). We generalize the total derivatives $\mathcal{D}_x$, $\mathcal{D}_t$, $\mathcal{D}_{\theta_1}$ and $\mathcal{D}_{\theta_2}$ as

$$\begin{aligned}\mathcal{D}_x = {} & \partial_x + \Phi_x\partial_\Phi + \Phi_{xx}\partial_{\Phi_x} + \Phi_{xt}\partial_{\Phi_t} + \Phi_{x\theta_1}\partial_{\Phi_{\theta_1}} + \Phi_{x\theta_2}\partial_{\Phi_{\theta_2}} + \Phi_{xxx}\partial_{\Phi_{xx}} + \Phi_{xxt}\partial_{\Phi_{xt}} \\ & + \Phi_{xx\theta_1}\partial_{\Phi_{x\theta_1}} + \Phi_{xx\theta_2}\partial_{\Phi_{x\theta_2}} + \Phi_{xtt}\partial_{\Phi_{tt}} + \Phi_{xt\theta_1}\partial_{\Phi_{t\theta_1}} + \Phi_{xt\theta_2}\partial_{\Phi_{t\theta_2}} + \Phi_{x\theta_1\theta_2}\partial_{\Phi_{\theta_1\theta_2}},\end{aligned} \tag{41}$$

and

$$\begin{aligned}\mathcal{D}_{\theta_1} = {} & \partial_{\theta_1} + \Phi_{\theta_1}\partial_\Phi + \Phi_{x\theta_1}\partial_{\Phi_x} + \Phi_{t\theta_1}\partial_{\Phi_t} + \Phi_{\theta_2\theta_1}\partial_{\Phi_{\theta_2}} + \Phi_{xx\theta_1}\partial_{\Phi_{xx}} + \Phi_{xt\theta_1}\partial_{\Phi_{xt}} \\ & + \Phi_{x\theta_2\theta_1}\partial_{\Phi_{x\theta_2}} + \Phi_{tt\theta_1}\partial_{\Phi_{tt}} + \Phi_{t\theta_2\theta_1}\partial_{\Phi_{t\theta_2}},\end{aligned} \tag{42}$$

while $\mathcal{D}_t$ and $\mathcal{D}_{\theta_2}$ are defined in analogy with $\mathcal{D}_x$ and $\mathcal{D}_{\theta_1}$ respectively. Here, we note that the chain rule for an odd-valued function $f(g(x))$ is [36, 37]

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial g}. \tag{43}$$

The graded interchangeability of mixed derivatives (i.e. with proper respect to the ordering of odd variables) of course holds. The second prolongation of the vector field (40) is given by

$$\begin{aligned}\mathrm{pr}^{(2)}\mathbf{v} = {} & \xi\partial_x + \tau\partial_t + \rho\partial_{\theta_1} + \sigma\partial_{\theta_2} + \Pi\partial_\Phi + \Pi^x\partial_{\Phi_x} + \Pi^t\partial_{\Phi_t} + \Pi^{\theta_1}\partial_{\Phi_{\theta_1}} + \Pi^{\theta_2}\partial_{\Phi_{\theta_2}} \\ & + \Pi^{xx}\partial_{\Phi_{xx}} + \Pi^{xt}\partial_{\Phi_{xt}} + \Pi^{x\theta_1}\partial_{\Phi_{x\theta_1}} + \Pi^{x\theta_2}\partial_{\Phi_{x\theta_2}} + \Pi^{tt}\partial_{\Phi_{tt}} + \Pi^{t\theta_1}\partial_{\Phi_{t\theta_1}} \\ & + \Pi^{t\theta_2}\partial_{\Phi_{t\theta_2}} + \Pi^{\theta_1\theta_2}\partial_{\Phi_{\theta_1\theta_2}}.\end{aligned} \tag{44}$$

Applying the second prolongation (44) to equation (39), we obtain the following condition:

$$\begin{aligned}& \rho(\theta_2\Phi_{xt} + \Phi_{x\theta_2}) - \sigma(\theta_1\Phi_{xt} + \Phi_{t\theta_1}) - \Pi(\cos\Phi) + \Pi^{xt}(\theta_1\theta_2) \\ & \quad + \Pi^{t\theta_1}(\theta_2) - \Pi^{x\theta_2}(\theta_1) - \Pi^{\theta_1\theta_2} = 0.\end{aligned} \tag{45}$$

Note that proper respect to the ordering of odd terms is essential, e.g. $\Pi^{t\theta_1}$ is odd. We see that we only need to calculate the coefficients $\Pi^x$, $\Pi^t$, $\Pi^{\theta_1}$, $\Pi^{\theta_2}$, $\Pi^{xt}$, $\Pi^{t\theta_1}$, $\Pi^{x\theta_2}$ and $\Pi^{\theta_1\theta_2}$ in equation (44). They are found from the superspace version of the formulas for the first and second prolongations of vector fields (see equation (27)):

$$\Pi^A = \mathcal{D}_A\Pi - \sum_B \mathcal{D}_A\zeta^B\Phi_B, \qquad \Pi^{AB} = \mathcal{D}_B\Pi^A - \sum_C \mathcal{D}_B\zeta^C\Phi_{AC}, \tag{46}$$

where

$$A, B, C \in \{x, t, \theta_1, \theta_2\}, \qquad \zeta^A = (\xi, \tau, \rho, \sigma). \tag{47}$$

The derivation of these formulas is performed in the same way as in the bosonic case, working with infinitesimal transformations and keeping track of ordering properties. Explicitly, the coefficients are given as follows:

$$\begin{aligned}\Pi^x = {} & \Pi_x + \Pi_\Phi\Phi_x - \xi_x\Phi_x - \xi_\Phi(\Phi_x)^2 - \tau_x\Phi_t - \tau_\Phi\Phi_x\Phi_t - \rho_x\Phi_{\theta_1} \\ & - \rho_\Phi\Phi_x\Phi_{\theta_1} - \sigma_x\Phi_{\theta_2} - \sigma_\Phi\Phi_x\Phi_{\theta_2},\end{aligned}$$

$$\begin{aligned}\Pi^t = {} & \Pi_t + \Pi_\Phi\Phi_t - \xi_t\Phi_x - \xi_\Phi\Phi_x\Phi_t - \tau_t\Phi_t - \tau_\Phi(\Phi_t)^2 - \rho_t\Phi_{\theta_1} \\ & - \rho_\Phi\Phi_t\Phi_{\theta_1} - \sigma_t\Phi_{\theta_2} - \sigma_\Phi\Phi_t\Phi_{\theta_2},\end{aligned}$$

$$\begin{aligned}\Pi^{\theta_1} = {} & \Pi_{\theta_1} + \Pi_\Phi\Phi_{\theta_1} - \xi_{\theta_1}\Phi_x - \xi_\Phi\Phi_x\Phi_{\theta_1} - \tau_{\theta_1}\Phi_t - \tau_\Phi\Phi_t\Phi_{\theta_1} \\ & - \rho_{\theta_1}\Phi_{\theta_1} - \sigma_{\theta_1}\Phi_{\theta_2} + \sigma_\Phi\Phi_{\theta_1}\Phi_{\theta_2},\end{aligned}$$

$$
\begin{aligned}
\Pi^{\theta_2} = {} & \Pi_{\theta_2} + \Pi_\Phi \Phi_{\theta_2} - \xi_{\theta_2}\Phi_x - \xi_\Phi \Phi_x \Phi_{\theta_2} - \tau_{\theta_2}\Phi_t - \tau_\Phi \Phi_t \Phi_{\theta_2} - \rho_{\theta_2}\Phi_{\theta_1} \\
& - \rho_\Phi \Phi_{\theta_1}\Phi_{\theta_2} - \sigma_{\theta_2}\Phi_{\theta_2}, \\
\Pi^{xt} = {} & \Pi_{xt} + \Pi_{x\Phi}\Phi_t + \Pi_{t\Phi}\Phi_x + \Pi_{\Phi\Phi}\Phi_x\Phi_t + \Pi_\Phi \Phi_{xt} - \xi_{xt}\Phi_x - \xi_{x\Phi}\Phi_x\Phi_t - \xi_x\Phi_{xt} \\
& - \xi_{t\Phi}(\Phi_x)^2 - \xi_{\Phi\Phi}(\Phi_x)^2\Phi_t - 2\xi_\Phi\Phi_x\Phi_{xt} - \xi_t\Phi_{xx} - \xi_\Phi\Phi_t\Phi_{xx} - \tau_{xt}\Phi_t \\
& - \tau_{t\Phi}\Phi_x\Phi_t - \tau_t\Phi_{xt} - \tau_{x\Phi}(\Phi_t)^2 - \tau_{\Phi\Phi}(\Phi_t)^2\Phi_x - 2\tau_\Phi\Phi_t\Phi_{xt} - \tau_x\Phi_{tt} \\
& - \tau_\Phi\Phi_x\Phi_{tt} - \rho_{xt}\Phi_{\theta_1} - \rho_{x\Phi}\Phi_t\Phi_{\theta_1} - \rho_{t\Phi}\Phi_x\Phi_{\theta_1} - \rho_x\Phi_{t\theta_1} - \rho_t\Phi_{x\theta_1} \\
& - \rho_{\Phi\Phi}\Phi_x\Phi_t\Phi_{\theta_1} - \rho_\Phi\Phi_{xt}\Phi_{\theta_1} - \rho_\Phi\Phi_{t\theta_1}\Phi_x - \rho_\Phi\Phi_{x\theta_1}\Phi_t - \sigma_{xt}\Phi_{\theta_2} \\
& - \sigma_{x\Phi}\Phi_t\Phi_{\theta_2} - \sigma_{t\Phi}\Phi_x\Phi_{\theta_2} - \sigma_x\Phi_{t\theta_2} - \sigma_t\Phi_{x\theta_2} - \sigma_{\Phi\Phi}\Phi_x\Phi_t\Phi_{\theta_2} - \sigma_\Phi\Phi_{xt}\Phi_{\theta_2} \\
& - \sigma_\Phi\Phi_{t\theta_2}\Phi_x - \sigma_\Phi\Phi_{x\theta_2}\Phi_t, \\
\Pi^{t\theta_1} = {} & \Pi_{t\theta_1} + \Pi_{t\Phi}\Phi_{\theta_1} + \Pi_{\theta_1\Phi}\Phi_t + \Pi_{\Phi\Phi}\Phi_t\Phi_{\theta_1} + \Pi_\Phi\Phi_{t\theta_1} - \xi_{t\theta_1}\Phi_x - \xi_{t\Phi}\Phi_x\Phi_{\theta_1} - \xi_t\Phi_{x\theta_1} \\
& - \xi_{\theta_1\Phi}\Phi_x\Phi_t - \xi_{\Phi\Phi}\Phi_x\Phi_t\Phi_{\theta_1} - \xi_\Phi\Phi_t\Phi_{x\theta_1} - \xi_\Phi\Phi_x\Phi_{t\theta_1} - \xi_{\theta_1}\Phi_{xt} - \xi_\Phi\Phi_{xt}\Phi_{\theta_1} \\
& - \tau_{t\theta_1}\Phi_t - \tau_{t\Phi}\Phi_t\Phi_{\theta_1} - \tau_t\Phi_{t\theta_1} - \tau_{\theta_1\Phi}(\Phi_t)^2 - \tau_{\Phi\Phi}(\Phi_t)^2\Phi_{\theta_1} - 2\tau_\Phi\Phi_t\Phi_{t\theta_1} \\
& - \tau_{\theta_1}\Phi_{tt} - \tau_\Phi\Phi_{tt}\Phi_{\theta_1} - \rho_{t\theta_1}\Phi_{\theta_1} - \rho_{\theta_1\Phi}\Phi_t\Phi_{\theta_1} - \rho_{\theta_1}\Phi_{t\theta_1} - \sigma_{t\theta_1}\Phi_{\theta_2} \\
& + \sigma_{t\Phi}\Phi_{\theta_1}\Phi_{\theta_2} - \sigma_t\Phi_{\theta_1\theta_2} - \sigma_{\theta_1\Phi}\Phi_t\Phi_{\theta_2} + \sigma_{\Phi\Phi}\Phi_t\Phi_{\theta_1}\Phi_{\theta_2} + \sigma_\Phi\Phi_{t\theta_1}\Phi_{\theta_2} \\
& - \sigma_\Phi\Phi_t\Phi_{\theta_1\theta_2} - \sigma_{\theta_1}\Phi_{t\theta_2} + \sigma_\Phi\Phi_{\theta_1}\Phi_{t\theta_2}, \\
\Pi^{x\theta_2} = {} & \Pi_{x\theta_2} + \Pi_{x\Phi}\Phi_{\theta_2} + \Pi_{\theta_2\Phi}\Phi_x + \Pi_{\Phi\Phi}\Phi_x\Phi_{\theta_2} + \Pi_\Phi\Phi_{x\theta_2} - \xi_{x\theta_2}\Phi_x - \xi_{x\Phi}\Phi_x\Phi_{\theta_2} \\
& - \xi_x\Phi_{x\theta_2} - \xi_{\theta_2\Phi}(\Phi_x)^2 - \xi_{\Phi\Phi}(\Phi_x)^2\Phi_{\theta_2} - 2\xi_\Phi\Phi_x\Phi_{x\theta_2} - \xi_{\theta_2}\Phi_{xx} - \xi_\Phi\Phi_{xx}\Phi_{\theta_2} \\
& - \tau_{x\theta_2}\Phi_t - \tau_{x\Phi}\Phi_t\Phi_{\theta_2} - \tau_x\Phi_{t\theta_2} - \tau_{\theta_2\Phi}\Phi_x\Phi_t - \tau_{\Phi\Phi}\Phi_x\Phi_t\Phi_{\theta_2} - \tau_\Phi\Phi_x\Phi_{t\theta_2} \\
& - \tau_\Phi\Phi_t\Phi_{x\theta_2} - \tau_{\theta_2}\Phi_{xt} - \tau_\Phi\Phi_{xt}\Phi_{\theta_2} - \rho_{x\theta_2}\Phi_{\theta_1} + \rho_{x\Phi}\Phi_{\theta_2}\Phi_{\theta_1} + \rho_x\Phi_{\theta_1\theta_2} \\
& - \rho_{\theta_2\Phi}\Phi_x\Phi_{\theta_1} + \rho_{\Phi\Phi}\Phi_x\Phi_{\theta_2}\Phi_{\theta_1} + \rho_\Phi\Phi_{x\theta_2}\Phi_{\theta_1} + \rho_\Phi\Phi_x\Phi_{\theta_1\theta_2} - \rho_{\theta_2}\Phi_{x\theta_1} \\
& + \rho_\Phi\Phi_{\theta_2}\Phi_{x\theta_1} - \sigma_{x\theta_2}\Phi_{\theta_2} - \sigma_{\theta_2\Phi}\Phi_x\Phi_{\theta_2} - \sigma_{\theta_2}\Phi_{x\theta_2}, \\
\Pi^{\theta_1\theta_2} = {} & \Pi_{\theta_1\theta_2} - \Pi_{\theta_1\Phi}\Phi_{\theta_2} + \Pi_{\theta_2\Phi}\Phi_{\theta_1} - \Pi_{\Phi\Phi}\Phi_{\theta_1}\Phi_{\theta_2} + \Pi_\Phi\Phi_{\theta_1\theta_2} - \xi_{\theta_1\theta_2}\Phi_x + \xi_{\theta_1\Phi}\Phi_x\Phi_{\theta_2} \\
& + \xi_{\theta_1}\Phi_{x\theta_2} - \xi_{\theta_2\Phi}\Phi_x\Phi_{\theta_1} + \xi_{\Phi\Phi}\Phi_x\Phi_{\theta_1}\Phi_{\theta_2} + \xi_\Phi\Phi_{\theta_1}\Phi_{x\theta_2} - \xi_\Phi\Phi_x\Phi_{\theta_1\theta_2} - \xi_{\theta_2}\Phi_{x\theta_1} \\
& - \xi_\Phi\Phi_{\theta_2}\Phi_{x\theta_1} - \tau_{\theta_1\theta_2}\Phi_t + \tau_{\theta_1\Phi}\Phi_t\Phi_{\theta_2} + \tau_{\theta_1}\Phi_{t\theta_2} - \tau_{\theta_2\Phi}\Phi_t\Phi_{\theta_1} + \tau_{\Phi\Phi}\Phi_t\Phi_{\theta_1}\Phi_{\theta_2} \\
& + \tau_\Phi\Phi_{\theta_1}\Phi_{t\theta_2} - \tau_\Phi\Phi_t\Phi_{\theta_1\theta_2} - \tau_{\theta_2}\Phi_{t\theta_1} - \tau_\Phi\Phi_{\theta_2}\Phi_{t\theta_1} - \rho_{\theta_1\theta_2}\Phi_{\theta_1} + \rho_{\theta_1\Phi}\Phi_{\theta_1}\Phi_{\theta_2} \\
& - \rho_{\theta_1}\Phi_{\theta_1\theta_2} - \sigma_{\theta_1\theta_2}\Phi_{\theta_2} + \sigma_{\theta_2\Phi}\Phi_{\theta_1}\Phi_{\theta_2} - \sigma_{\theta_2}\Phi_{\theta_1\theta_2}.
\end{aligned} \tag{48}
$$

Substituting the above formulas into equation (45) and replacing each term $\Phi_{\theta_1\theta_2}$ in the resulting expression by the terms $\theta_1\theta_2\Phi_{xt} - \theta_2\Phi_{t\theta_1} + \theta_1\Phi_{x\theta_2} - \sin\Phi$, we obtain a series of determining equations for the functions $\xi, \tau, \rho, \sigma$ and $\Pi$. The general solution of these determining equations is given by

$$
\begin{aligned}
&\xi(x,\theta_1) = -2C_1x + C_2 - \underline{D_1}\theta_1, \qquad \tau(t,\theta_2) = 2C_1t + C_3 - \underline{D_2}\theta_2, \\
&\rho(\theta_1) = -C_1\theta_1 + \underline{D_1}, \qquad \sigma(\theta_2) = C_1\theta_2 + \underline{D_2}, \qquad \Pi = 0,
\end{aligned} \tag{49}
$$

where the parameters $C_1, C_2, C_3 \in \Lambda_{\text{even}}$, while $\underline{D_1}, \underline{D_2} \in \Lambda_{\text{odd}}$. Thus, the algebra of infinitesimal transformations is the even part of the Lie superalgebra $\mathfrak{S}$ over $\Lambda$ spanned by the following generators:

$$
\begin{aligned}
&L = -2x\partial_x + 2t\partial_t - \theta_1\partial_{\theta_1} + \theta_2\partial_{\theta_2}, \qquad P_x = \partial_x, \qquad P_t = \partial_t, \\
&Q_x = -\theta_1\partial_x + \partial_{\theta_1}, \qquad Q_t = -\theta_2\partial_t + \partial_{\theta_2}.
\end{aligned} \tag{50}
$$

**Table 3.** Supercommutation table for the Lie superalgebra $\mathfrak{S}$ spanned by the vector fields (50).

| | $L$ | $P_x$ | $P_t$ | $Q_x$ | $Q_t$ |
|---|---|---|---|---|---|
| $L$ | 0 | $2P_x$ | $-2P_t$ | $Q_x$ | $-Q_t$ |
| $P_x$ | $-2P_x$ | 0 | 0 | 0 | 0 |
| $P_t$ | $2P_t$ | 0 | 0 | 0 | 0 |
| $Q_x$ | $-Q_x$ | 0 | 0 | $-2P_x$ | 0 |
| $Q_t$ | $Q_t$ | 0 | 0 | 0 | $-2P_t$ |

The even generators $L, P_x$ and $P_t$ represent a dilation and translations in space and time respectively, while the odd generators $Q_x$ and $Q_t$ are simply the generators of supersymmetric transformations identified in section 2. This means that we have recovered the full super-Poincaré algebra in (1 + 1) dimensions which was expected. The commutation (and anticommutation in the case of two odd generators) relations of the Lie super algebra $\mathfrak{S}$ generated by the vector fields (50) are given in table 3.

## 5. One-dimensional subalgebras of the symmetry algebra of the SSG equation

In this section, we classify the one-dimensional subalgebras of the Lie algebra of infinitesimal transformations $\mathfrak{S}_{\text{even}}$ into conjugacy classes under the action of the super Lie group $\exp(\mathfrak{S}_{\text{even}})$ generated by $\mathfrak{S}_{\text{even}}$. Such a classification is of importance for us because conjugate subgroups necessarily lead to invariant solutions equivalent in the sense that they can be transformed by a suitable symmetry from one to the other; therefore, there is no need to compute reductions with respect to algebras which are conjugate to each other. On the other hand, for our purposes it is not of particular importance to establish exactly one representative of each class, as long as the procedure of reduction has the same form for all the representatives, differing by a choice of parameters only.

We recall why $\mathfrak{S}_{\text{even}}$ is the algebra we are interested in. It would be inconsistent to consider the $\mathbb{R}$ span of the generators (50) because we multiply the odd generators $Q_x$ and $Q_t$ by the odd parameters $\underline{\eta_1}$ and $\underline{\eta_2}$ respectively in equation (4). Therefore, one is naturally led to consideration of $\mathfrak{S}_{\text{even}}$ which is a supermanifold in the sense presented in section 2. It means that $\mathfrak{S}_{\text{even}}$ contains sums of any even combination of $P_x, P_t, L$ (i.e. multiplied by even parameters in $\Lambda_{\text{even}}$, including real numbers), and odd combination of $Q_x$ and $Q_t$ (i.e. multiplied by odd parameters in $\Lambda_{\text{odd}}$). At the same time, $\mathfrak{S}_{\text{even}}$ is a $\Lambda_{\text{even}}$ Lie module.

This leads to the following complication.

- For a given $X \in \mathfrak{S}$, the subalgebras $\mathfrak{X}, \mathfrak{X}'$ spanned by $X$ and by $X' = aX, a \in \Lambda_{\text{even}}\backslash\mathbb{R}$ are in general not isomorphic, $\mathfrak{X}' \subset \mathfrak{X}$.

It seems that the subalgebras obtained from other ones through multiplication by nilpotent elements of $\Lambda_{\text{even}}$ do not give us anything new for the purpose of symmetry reduction—they may allow a bit more freedom in the choice of invariants, but we then encounter the problem of non-standard invariants which we will discuss at the end of section 6.

Similarly, it does not appear to be particularly useful to consider a subalgebra of the form e.g. $\{P_x + \underline{\eta_1}\,\underline{\eta_2} P_t\}$ (although the reduction for this case can easily be reconstructed by substituting $\varepsilon = \underline{\eta_1}\,\underline{\eta_2}$ in the subalgebra $\mathfrak{S}_4$ and the corresponding formulas below).

Therefore, we will assume throughout the computation of the non-isomorphic one-dimensional subalgebras that the nonzero even parameters are invertible, i.e. behave essentially like ordinary real numbers.

The Lie algebra $\mathfrak{S}_{\text{even}}$ can be decomposed into the semi-direct sum:

$$\mathfrak{S} = \{L\} \supsetplus \{P_x, P_t, Q_x, Q_t\}. \tag{51}$$

In order to classify this Lie superalgebra, we make use of the techniques for semi-direct sums of algebras described in [38] (section 4.4) and generalize them to superalgebras involving both even and odd generators. Here, we identify the components $F$ and $N$ of the semi-direct sum as

$$F = \{L\}, \qquad N = \{P_x, P_t, Q_x, Q_t\}.$$

The trivial subalgebras of $F$ are simply $F_1 = \{0\}$ and $F_2 = \{L\}$. We begin by considering the splitting one-dimensional subalgebras.

For $F_1 = \{0\}$, all one-dimensional subspaces of the form

$$\{\alpha P_x + \beta P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}, \qquad \alpha, \beta \in \Lambda_{\text{even}}, \quad \underline{\mu}, \underline{\nu} \in \Lambda_{\text{odd}} \tag{52}$$

are invariant subalgebras, i.e. subalgebras of $N$ invariant under the action of $F_1$.

Under the action of the one-parameter group generated by the generator

$$Y = kL + mP_x + nP_t + \underline{\eta} Q_x + \underline{\lambda} Q_t, \tag{53}$$

where $k, m, n \in \Lambda_{\text{even}}$ and $\underline{\eta}, \underline{\lambda} \in \Lambda_{\text{odd}}$, the one-dimensional subalgebra (52) transforms under the Baker–Campbell–Hausdorff formula

$$X \longrightarrow \mathrm{Ad}_{\exp(Y)} X = X + [Y, X] + \frac{1}{2!}[Y, [Y, X]] + \frac{1}{3!}[Y, [Y, [Y, X]]] + \cdots \tag{54}$$

to

$$\left(\mathrm{e}^{2k}\alpha + 2\underline{\eta}\underline{\mu}\,\mathrm{e}^{k}\frac{(\mathrm{e}^{k}-1)}{k}\right)P_x + \left(\mathrm{e}^{-2k}\beta + 2\underline{\lambda}\underline{\nu}\,\mathrm{e}^{-2k}\frac{(\mathrm{e}^{k}-1)}{k}\right)P_t + \mathrm{e}^{k}\underline{\mu}Q_x + \mathrm{e}^{-k}\underline{\nu}Q_t. \tag{55}$$

If $k$ is bodiless (see equation (13)), then we interpret $\frac{\mathrm{e}^k-1}{k}$ as its well-defined limit $\frac{\mathrm{e}^k-1}{k} = \sum_{j=0}^{\infty}\frac{1}{(j+1)!}k^j$.

We note that the action (54) with $Y = kL$ on the even generators $\alpha P_x + \beta P_t$ together with an overall rescaling of the subalgebra generator can always be used to bring one of the coefficients $\alpha, \beta$ to 1 under the assumption that at least one of them was invertible supernumber. The other can be scaled to either $\pm 1$ or bodiless even supernumber. Note that here the assumption of the finite number of Grassmann generators of $\Lambda$ is essential—it guarantees a cutoff in the sum $\ln(1+\gamma) = \sum_{j\geqslant 1}\frac{(-1)^{j-1}}{j}\gamma^j$ for bodiless $\gamma \in \Lambda_{\text{even}}$ so that one does not have to worry about its convergence; i.e. $k \in \Lambda_{\text{even}}$ such that $\mathrm{e}^k = 1+\gamma$ exists for every even bodiless $\gamma$.

Once the coefficients $P_x, P_t$ are brought to the simple form, one uses any remaining freedom to simplify the coefficients of $Q_x, Q_t$. As is seen from equation (55) not much can be accomplished—only rescaling by $\exp(k)$, $k \in \Lambda_{\text{even}}$ (and an overall rescaling if both $\alpha = \beta = 0$), may still be available.

Considering first the subalgebras containing only the generators $P_x, P_t$, we obtain essentially the same subalgebras as for the system in the component form described in section 3.

(i) If $\beta = 0$, $\underline{\mu} = 0$, $\underline{\nu} = 0$, we have the subalgebra $\{P_x\}$ which is not conjugate to any other subalgebra.
(ii) If $\alpha = 0$, $\underline{\mu} = 0$, $\underline{\nu} = 0$, we have the subalgebra $\{P_t\}$.
(iii) The subalgebra $\{\alpha P_x + \beta P_t\}$, where $a, b \in \Lambda_{\text{even}}$ such that $a^{-1}$ exists, can be brought to the form

$$\{P_x + \varepsilon P_t\}, \tag{56}$$

where $\varepsilon = \pm 1$ or $\varepsilon$ is bodiless. If $a$ is bodiless, then we get similarly

$$\{P_t + \omega P_x\},$$

where $\omega$ is bodiless. As mentioned at the beginning of this section, we shall consider only subalgebra (56) with $\epsilon = \pm 1$ in what follows.

Next we complement the generators $P_x$, $P_t$ by $Q_x$, $Q_t$. This leads to the following types of non-conjugate subalgebra:

(iv) $\{\underline{\mu} Q_x\}$,

(v) $\{\underline{\nu} Q_t\}$,

(vi) $\{P_x + \underline{\mu} Q_x\}$ where algebras with $\underline{\mu}$ and $\mathrm{e}^k \underline{\mu}$, $k \in \Lambda_{\text{even}}$, are isomorphic,

(vii) $\{P_t + \underline{\mu} Q_x\}$ where algebras with $\underline{\mu}$ and $\mathrm{e}^k \underline{\mu}$, $k \in \Lambda_{\text{even}}$, are isomorphic,

(viii) $\{P_x + \underline{\nu} Q_t\}$ where algebras with $\underline{\nu}$ and $\mathrm{e}^k \underline{\nu}$, $k \in \Lambda_{\text{even}}$, are isomorphic,

(ix) $\{P_t + \underline{\nu} Q_t\}$ where algebras with $\underline{\nu}$ and $\mathrm{e}^k \underline{\nu}$, $k \in \Lambda_{\text{even}}$, are isomorphic,

(x) $\{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$,

(xi) $\{P_x + \varepsilon P_t + \underline{\nu} Q_t\}$,

(xii) $\{\underline{\mu} Q_x + \underline{\nu} Q_t\}$ where both $\underline{\mu}$ and $\underline{\nu}$ can be simultaneously rescaled by $a \in \Lambda_{\text{even}}$ and then one of them by $\mathrm{e}^k$, $k \in \Lambda_{\text{even}}$,

(xiii) $\{P_x + \underline{\mu} Q_x + \underline{\nu} Q_t\}$ where algebras defined by $(\underline{\mu}, \underline{\nu})$ and $(\mathrm{e}^k \underline{\mu}, \mathrm{e}^{3k} \underline{\nu})$, $k \in \Lambda_{\text{even}}$, are isomorphic,

(xiv) $\{P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$ where algebras defined by $(\underline{\mu}, \underline{\nu})$ and $(\mathrm{e}^{3k} \underline{\mu}, \mathrm{e}^k \underline{\nu})$, $k \in \Lambda_{\text{even}}$, are isomorphic,

(xv) $\{P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$.

For $F_2 = \{L\}$, the only splitting one-dimensional subalgebra is $\{L\}$ itself.

Next, we then look for non-splitting subalgebras of $\mathfrak{S}$ of the form

$$V = \{L + \alpha P_x + \beta P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}, \tag{57}$$

but an easy calculation using the Baker–Campbell–Hausdorff formula (54) shows that all such algebras are conjugate to $\{L\}$. Thus, there are no separate conjugacy classes of non-splitting one-dimensional subalgebras of $\mathfrak{S}$.

Therefore, the one-dimensional subalgebra classification (under the restrictions mentioned at the beginning of this section) is

$$\begin{gathered}
\mathfrak{S}_1 = \{L\}, \quad \mathfrak{S}_2 = \{P_x\}, \quad \mathfrak{S}_3 = \{P_t\}, \quad \mathfrak{S}_4 = \{P_x + \varepsilon P_t\}, \quad \mathfrak{S}_5 = \{\underline{\mu} Q_x\}, \\
\mathfrak{S}_6 = \{P_x + \underline{\mu} Q_x\}, \quad \mathfrak{S}_7 = \{P_t + \underline{\mu} Q_x\}, \quad \mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}, \\
\mathfrak{S}_9 = \{\underline{\nu} Q_t\}, \quad \mathfrak{S}_{10} = \{P_x + \underline{\nu} Q_t\}, \quad \mathfrak{S}_{11} = \{P_t + \underline{\nu} Q_t\}, \\
\mathfrak{S}_{12} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\}, \quad \mathfrak{S}_{13} = \{\underline{\mu} Q_x + \underline{\nu} Q_t\}, \quad \mathfrak{S}_{14} = \{P_x + \underline{\mu} Q_x + \underline{\nu} Q_t\}, \\
\mathfrak{S}_{15} = \{P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}, \quad \mathfrak{S}_{16} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}.
\end{gathered} \tag{58}$$

Any parameter, if present, is assumed to be nonvanishing. The underlined parameters belong to $\Lambda_{\text{odd}}$, $\varepsilon = \pm 1$ (although also $\varepsilon \in \Lambda_{\text{even}}$ bodiless can in principle be considered).

This classification will allow us to use the SRM in order to determine invariant solutions of the SSG equation (39).

## 6. Invariant solutions of the supersymmetric sine-Gordon equation

We now proceed to apply a modified version of the SRM to the SSG equation (39) in order to obtain invariant solutions of the model. Considering in turn each of the one-dimensional subalgebras described in section 5, we begin by constructing, where possible, a set of four independent invariants of the specific subalgebra. In each case, the even invariant is labeled by $\sigma$ and the odd invariant(s) by $\tau$ (or $\tau_1$, $\tau_2$). For the subalgebras $\mathfrak{S}_5$, $\mathfrak{S}_9$, $\mathfrak{S}_{13}$, $\mathfrak{S}_{14}$, $\mathfrak{S}_{15}$ and $\mathfrak{S}_{16}$, the structure of the invariants is non-standard and will be discussed at the end of this section.

The bosonic superfield $\Phi$ is expanded in terms of its various odd invariants. The dependence of $\Phi$ on each odd variable $\tau_i$ must be at most linear (as $(\tau_i)^2 = 0$). Substituting this decomposition into the SSG equation (39), we obtain a reduced partial differential equation for the superfield $\Phi$ which in turn leads to a system of differential constraints between its component even and odd functions. For instance, if the invariants are given by $\sigma$, $\tau_1$, $\tau_2$, $\Phi$, the superfield $\Phi$ can be decomposed into the form

$$\Phi = \mathcal{A}(\sigma, \tau_1, \tau_2) = \alpha(\sigma) + \tau_1\eta(\sigma) + \tau_2\lambda(\sigma) + \tau_1\tau_2\beta(\sigma), \tag{59}$$

where $\alpha$ and $\beta$ are the even-valued functions of $\sigma$ while $\eta$ and $\lambda$ are the odd-valued functions of $\sigma$. Substitution into the SSG equation (39) allows us to determine the differential constraints between the functions $\alpha$, $\beta$, $\eta$ and $\lambda$. In general, the reduced supersymmetric equation will contain the term $\sin\mathcal{A}$ which can be expanded in the form

$$\sin\mathcal{A} = \sin\alpha + \tau_1\eta\cos\alpha + \tau_2\lambda\cos\alpha + \tau_1\tau_2(\beta\cos\alpha + \eta\lambda\sin\alpha), \tag{60}$$

as identified from the series

$$\sin\mathcal{A} = \mathcal{A} - \frac{1}{3!}\mathcal{A}^3 + \frac{1}{5!}\mathcal{A}^5 - \cdots \tag{61}$$

The results are summarized in tables 4 and 5. In table 4, we list the one-dimensional subalgebras and their respective invariants and superfields. In table 5, we present the systems of differential constraints resulting from each symmetry reduction and assumed form of the superfield. In what follows, we deal separately with each case described above by performing an analysis of the various solutions of the obtained differential constraints. The resulting expressions are then substituted into the superfield formula for $\Phi$, from which we obtain group-invariant solutions.

The subalgebra $\mathfrak{S}_1 = \{L\}$ leads to the reduction

$$\Phi(x, t, \theta_1, \theta_2) = \alpha(\sigma) + t^{1/2}\theta_1\mu(\sigma) + t^{-1/2}\theta_2\nu(\sigma) + \theta_1\theta_2\beta(\sigma), \tag{62}$$

where $\sigma = xt$ and the functions $\alpha$, $\beta$, $\mu$, $\nu$ satisfy

$$\begin{aligned}
&\sigma\alpha_{\sigma\sigma} + \alpha_\sigma + \frac{1}{2}\sin(2\alpha) - C_0\sigma^{-1/2}\sin\alpha = 0,\\
&\nu_{\sigma\sigma} + \tan\alpha\,\alpha_\sigma\nu_\sigma + \frac{1}{2\sigma}\nu_\sigma + \frac{1}{\sigma}\cos^2\alpha\,\nu = 0,\\
&\mu - \frac{1}{\cos\alpha}\nu_\sigma = 0,\\
&\beta + \sin\alpha = 0,\\
&(\sigma^{1/2}\mu\nu)_\sigma = 0,
\end{aligned} \tag{63}$$

where $C_0$ denotes the nilpotent even constant equal to $\sigma^{1/2}\mu\nu$. These equations are equivalent to the ones listed in table 5 but written in the form more convenient for further simplification. This reduction is also equivalent to that found for the SSG equation in the component form in table 2.

We find it convenient to consider the first equation in (63), namely

$$\sigma\alpha_{\sigma\sigma} + \alpha_\sigma + \tfrac{1}{2}\sin(2\alpha) - C_0\sigma^{-1/2}\sin\alpha = 0$$

**Table 4.** Invariants and change of variables for subalgebras of the Lie superalgebra $\mathfrak{S}$ spanned by the vector fields (50).

| Subalgebra | Invariants | Superfield |
|---|---|---|
| $\mathfrak{S}_1 = \{L\}$ | $\sigma = xt, \tau_1 = t^{1/2}\theta_1,$ $\tau_2 = t^{-1/2}\theta_2, \Phi$ | $\Phi = \mathcal{A}(\sigma, \tau_1, \tau_2) = \alpha(\sigma) + \tau_1\mu(\sigma)$ $+\tau_2\nu(\sigma) + \tau_1\tau_2\beta(\sigma)$ |
| $\mathfrak{S}_2 = \{P_x\}$ | $t, \theta_1, \theta_2, \Phi$ | $\Phi = \mathcal{A}(t, \theta_1, \theta_2) = \alpha(t) + \theta_1\mu(t)$ $+\theta_2\nu(t) + \theta_1\theta_2\beta(t)$ |
| $\mathfrak{S}_3 = \{P_t\}$ | $x, \theta_1, \theta_2, \Phi$ | $\Phi = \mathcal{A}(x, \theta_1, \theta_2) = \alpha(x) + \theta_1\mu(x)$ $+\theta_2\nu(x) + \theta_1\theta_2\beta(x)$ |
| $\mathfrak{S}_4 = \{P_x + \varepsilon P_t\}$ | $\sigma = x - \varepsilon t, \theta_1, \theta_2, \Phi$ | $\Phi = \mathcal{A}(\sigma, \theta_1, \theta_2) = \alpha(\sigma) + \theta_1\mu(\sigma)$ $+\theta_2\nu(\sigma) + \theta_1\theta_2\beta(\sigma)$ |
| $\mathfrak{S}_6 = \{P_x + \underline{\mu} Q_x\}$ | $t, \tau = \theta_1 - \underline{\mu} x, \theta_2, \Phi$ | $\Phi = \mathcal{A}(t, \tau, \theta_2) = \alpha(t) + \tau\eta(t)$ $+\theta_2\lambda(t) + \tau\theta_2\beta(t)$ |
| $\mathfrak{S}_7 = \{P_t + \underline{\mu} Q_x\}$ | $\sigma = x + \underline{\mu}\theta_1 t,$ $\tau = \theta_1 - \underline{\mu} t, \theta_2, \Phi$ | $\Phi = \mathcal{A}(\sigma, \tau, \theta_2) = \alpha(\sigma)$ $+\theta_2\lambda(\sigma) + \tau\theta_2\beta(\sigma)$ |
| $\mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$ | $\sigma = \varepsilon x - t + \underline{\mu} t\theta_1,$ $\tau = \theta_1 - \varepsilon\underline{\mu} t, \theta_2, \Phi$ | $\Phi = \mathcal{A}(\sigma, \tau, \theta_2) = \alpha(\sigma)$ $+\theta_2\lambda(\sigma) + \tau\theta_2\beta(\sigma)$ |
| $\mathfrak{S}_{10} = \{P_x + \underline{\nu} Q_t\}$ | $\sigma = t + \underline{\nu}\theta_2 x,$ $\tau = \theta_2 - \underline{\nu} x, \theta_1, \Phi$ | $\Phi = \mathcal{A}(\sigma, \tau, \theta_1) = \alpha(\sigma)$ $+\theta_1\lambda(\sigma) + \tau\theta_1\beta(\sigma)$ |
| $\mathfrak{S}_{11} = \{P_t + \underline{\nu} Q_t\}$ | $x, \theta_1, \tau = \theta_2 - \underline{\nu} t, \Phi$ | $\Phi = \mathcal{A}(x, \tau, \theta_1) = \alpha(x) + \tau\eta(x)$ $+\theta_1\lambda(x) + \tau\theta_1\beta(x)$ |
| $\mathfrak{S}_{12} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\}$ | $\sigma = t - \varepsilon x + \underline{\nu} x\theta_2,$ $\tau = \theta_2 - \underline{\nu} x, \theta_1, \Phi$ | $\Phi = \mathcal{A}(\sigma, \tau, \theta_1) = \alpha(\sigma) + \tau\eta(\sigma)$ $+\theta_1\lambda(\sigma) + \tau\theta_1\beta(\sigma)$ |

as a complex ordinary differential equation. Then under the transformation

$$\alpha = \mathrm{i} \ln y, \tag{64}$$

it becomes

$$y_{\sigma\sigma} = \frac{1}{y}(y_\sigma)^2 - \frac{1}{\sigma} y_\sigma + \frac{1}{4\sigma}(y^{-1} - y^3) - \frac{C_0}{2\sigma^{3/2}}(1 - y^2). \tag{65}$$

In the case where $C_0 = 0$, we can rescale the independent variable $\sigma$ to $z = \pm 2\mathrm{i}\sigma$ and we obtain the following form of equation (65):

$$y_{zz} = \frac{1}{y}(y_z)^2 - \frac{1}{z} y_z \pm \frac{\mathrm{i}}{8z}\left(y^3 - \frac{1}{y}\right). \tag{66}$$

The solution of the reduced system (63) can be expressed in terms of $y$ through the transformation (64). Under the assumption that $C_0 = 0$, the odd-valued functions $\mu$ and $\nu$ have to satisfy the following differential equations:

$$\nu_{\sigma\sigma} = -\left(\frac{1}{2\sigma} + \frac{1 - y^2}{y(1 + y^2)} y_\sigma\right)\nu_\sigma - \frac{1}{4\sigma}\left(y + \frac{1}{y}\right)^2 \nu, \qquad \mu = \frac{2y}{1 + y^2}\nu_\sigma, \tag{67}$$

together with the constraint $\mu\nu = 0$.

**Table 5.** Reduced equations obtained for subalgebras of the Lie superalgebra $\mathfrak{S}$ spanned by the vector fields (50).

| Subalgebra | Reduced equations |
|---|---|
| $\mathfrak{S}_1 = \{L\}$ | $\beta + \sin\alpha = 0, \quad \nu_\sigma - \mu\cos\alpha = 0,$ <br> $\sigma\mu_\sigma + \frac{1}{2}\mu + \nu\cos\alpha = 0, \quad \alpha_\sigma + \sigma\alpha_{\sigma\sigma} - \beta\cos\alpha - \mu\nu\sin\alpha = 0$ |
| $\mathfrak{S}_2 = \{P_x\}$ | $\beta + \sin\alpha = 0, \quad \mu\cos\alpha = 0,$ <br> $\mu_t + \nu\cos\alpha = 0, \quad \beta\cos\alpha + \mu\nu\sin\alpha = 0$ |
| $\mathfrak{S}_3 = \{P_t\}$ | $\beta + \sin\alpha = 0, \quad \nu_x - \mu\cos\alpha = 0,$ <br> $\nu\cos\alpha = 0, \quad \beta\cos\alpha + \mu\nu\sin\alpha = 0$ |
| $\mathfrak{S}_4 = \{P_x + \varepsilon P_t\}$ | $\beta + \sin\alpha = 0, \quad \nu_\sigma - \mu\cos\alpha = 0,$ <br> $\varepsilon\mu_\sigma - \nu\cos\alpha = 0, \quad \varepsilon\alpha_{\sigma\sigma} + \beta\cos\alpha + \mu\nu\sin\alpha = 0$ |
| $\mathfrak{S}_6 = \{P_x + \underline{\mu}Q_x\}$ | $\beta + \sin\alpha = 0, \quad \underline{\mu}\beta - \eta\cos\alpha = 0,$ <br> $\eta_t + \lambda\cos\alpha = 0, \quad \underline{\mu}\eta_t + \beta\cos\alpha + \eta\lambda\sin\alpha = 0$ |
| $\mathfrak{S}_7 = \{P_t + \underline{\mu}Q_x\}$ | $\beta + \sin\alpha = 0, \quad \lambda_\sigma - \eta\cos\alpha = 0,$ <br> $\underline{\mu}\alpha_\sigma - \lambda\cos\alpha = 0, \quad \underline{\mu}\eta_\sigma + \beta\cos\alpha + \eta\lambda\sin\alpha = 0$ |
| $\mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu}Q_x\}$ | $\beta + \sin\alpha = 0, \quad \varepsilon\lambda_\sigma - \eta\cos\alpha = 0,$ <br> $\eta_\sigma + \underline{\mu}\alpha_\sigma - \lambda\cos\alpha = 0, \quad \varepsilon\alpha_{\sigma\sigma} + \underline{\mu}\eta_\sigma + \beta\cos\alpha + \eta\lambda\sin\alpha = 0$ |
| $\mathfrak{S}_{10} = \{P_x + \underline{\nu}Q_t\}$ | $\beta - \sin\alpha = 0, \quad \lambda_\sigma + \eta\cos\alpha = 0,$ <br> $\underline{\nu}\alpha_\sigma + \lambda\cos\alpha = 0, \quad \underline{\nu}\eta_\sigma - \beta\cos\alpha - \eta\lambda\sin\alpha = 0$ |
| $\mathfrak{S}_{11} = \{P_t + \underline{\nu}Q_t\}$ | $\beta - \sin\alpha = 0, \quad \underline{\nu}\beta + \eta\cos\alpha = 0,$ <br> $\eta_x - \lambda\cos\alpha = 0, \quad \underline{\nu}\eta_x - \beta\cos\alpha - \eta\lambda\sin\alpha = 0$ |
| $\mathfrak{S}_{12} = \{P_x + \varepsilon P_t + \underline{\nu}Q_t\}$ | $\beta - \sin\alpha = 0, \quad \lambda_\sigma + \eta\cos\alpha = 0,$ <br> $\underline{\nu}\alpha_\sigma + \varepsilon\eta_\sigma + \lambda\cos\alpha = 0, \quad \varepsilon\alpha_{\sigma\sigma} + \underline{\nu}\eta_\sigma - \beta\cos\alpha - \eta\lambda\sin\alpha = 0$ |

On the other hand, taking $\alpha = 0$ in equation (63), we obtain the following particular solution of the SSG equation:

$$\Phi(x,t,\theta_1,\theta_2) = \left[\frac{\underline{\underline{D_1}}}{\sqrt{x}}\cos(2\sqrt{xt}) - \frac{\underline{\underline{D_2}}}{\sqrt{x}}\sin(2\sqrt{xt})\right]\theta_1 + \left[\frac{\underline{\underline{D_1}}}{\sqrt{t}}\sin(2\sqrt{xt}) + \frac{\underline{\underline{D_2}}}{\sqrt{t}}\cos(2\sqrt{xt})\right]\theta_2 \tag{68}$$

representing a nonsingular periodic solution with the damping factor $t^{-1/2}$ (where $t \neq 0$).

For the subalgebra $\mathfrak{S}_2 = \{P_x\}$, the reduced equations in table 5 are equivalent to the corresponding ones obtained in component form, i.e. listed in table 2. The only nonvanishing solutions which we obtain are

$$\Phi(x,t,\theta_1,\theta_2) = k\pi, \tag{69}$$

where $k \in \mathbb{Z}$ and

$$\Phi(x,t,\theta_1,\theta_2) = \left(k + \tfrac{1}{2}\right)\pi + \theta_1\underline{\mu_0} + \theta_2\underline{\mu_0}\varphi(t) + (-1)^{k+1}\theta_1\theta_2, \tag{70}$$

where $k \in \mathbb{Z}$, $\underline{\mu_0}$ is an odd supernumber and $\varphi$ is an arbitrary even-valued function of $t$.

Similarly, in the case of the subalgebra $\mathfrak{S}_3 = \{P_t\}$ the reduced equations in table 5 are equivalent to the corresponding ones obtained in the component form in table 2. The only nonzero solutions are

$$\Phi(x,t,\theta_1,\theta_2) = k\pi, \tag{71}$$

where $k \in \mathbb{Z}$ and

$$\Phi(x,t,\theta_1,\theta_2) = \left(k+\tfrac{1}{2}\right)\pi + \theta_1 \underline{\nu_0}\varphi(x) + \theta_2 \underline{\nu_0} + (-1)^{k+1}\theta_1\theta_2, \tag{72}$$

where $k \in \mathbb{Z}$, $\underline{\nu_0}$ is an odd supernumber and $\varphi$ is an arbitrary even-valued function of $x$.

The subalgebra $\mathfrak{S}_4 = \{P_x + \varepsilon P_t\}$ leads to the last reduction which was obtained also in the component form in table 2. The reduced equations for the superfield

$$\Phi(x,t,\theta_1,\theta_2) = \alpha(\sigma) + \theta_1\mu(\sigma) + \theta_2\nu(\sigma) + \theta_1\theta_2\beta(\sigma), \tag{73}$$

where $\sigma = x - \varepsilon t$, are equivalent to the following set of equations for functions $\alpha$, $\nu$, $\mu$, $\beta$:

$$\begin{aligned}
&\varepsilon\alpha_{\sigma\sigma} - \tfrac{1}{2}\sin(2\alpha) + K_0 \sin\alpha = 0,\\
&\nu_{\sigma\sigma} + \tan\sigma\, \nu_\sigma\alpha_\sigma - \varepsilon\cos^2\alpha\, \nu = 0,\\
&\mu - \frac{1}{\cos\alpha}\nu_\sigma = 0,\\
&\beta + \sin\alpha = 0,\\
&(\mu\nu)_\sigma = 0,
\end{aligned} \tag{74}$$

where we denoted the nilpotent constant $\mu\nu$ by $K_0$. The resulting solutions are traveling wave solutions in both the even and odd fields. We recall that the equation for $\alpha$, namely

$$\varepsilon\alpha_{\sigma\sigma} - \tfrac{1}{2}\sin(2\alpha) + K_0\sin\alpha = 0, \tag{75}$$

appears in the reduction of the double sine-Gordon equation in $(2+1)$ dimensions [5] but with real $K_0$.

Considering the different values of $\epsilon$ separately, in the case $\epsilon = 1$, we make the substitution $\alpha = -\mathrm{i}\ln v$ into equation (75) followed by an integration which leads to the equation

$$(v_\sigma)^2 - \tfrac{1}{4}(v^4 - 4K_0v^3 + 8K_1v^2 - 4K_0v + 1) = 0, \tag{76}$$

solved by an elliptic integral in terms of a P-Weierstrass function. When $K_0 = 0$, equation (75) reduces to the reduced sine-Gordon equation and its traveling wave solutions are well known and represent classical periodic, nonperiodic and kink solutions [39, 40]. For example, in the special case where $K_0 = 0$ and $K_1 = 0$, we obtain the following particular wave solution which is expressed in terms of Jacobi elliptic functions

$$\begin{aligned}
&\alpha(\sigma) = \arccos\left(\mathrm{cn}(\sigma, i)\right), \qquad \sigma = x - t,\\
&\mu(\sigma) = \underline{D_1}\left[\frac{\left(1 - \frac{\mathrm{sn}^2(\sigma,i)}{(1+\mathrm{dn}(\sigma,i))^2}\right)}{\left(1 + \frac{\mathrm{sn}(\sigma,i)}{1+\mathrm{dn}(\sigma,i)}\right)^2} + \frac{\left(1 + \frac{\mathrm{sn}(\sigma,i)}{1+\mathrm{dn}(\sigma,i)}\right)^2}{\left(1 - \frac{\mathrm{sn}^2(\sigma,i)}{(1+\mathrm{dn}(\sigma,i))^2}\right)}\right],\\
&\nu(\sigma) = \underline{D_1}\left[-\frac{\left(1 - \frac{\mathrm{sn}^2(\sigma,i)}{(1+\mathrm{dn}(\sigma,i))^2}\right)}{\left(1 + \frac{\mathrm{sn}(\sigma,i)}{1+\mathrm{dn}(\sigma,i)}\right)^2} + \frac{\left(1 + \frac{\mathrm{sn}(\sigma,i)}{1+\mathrm{dn}(\sigma,i)}\right)^2}{\left(1 - \frac{\mathrm{sn}^2(\sigma,i)}{(1+\mathrm{dn}(\sigma,i))^2}\right)}\right],
\end{aligned} \tag{77}$$

where $\underline{D_1}$ is an arbitrary odd supernumber. Physically, this represents an elliptic traveling wave.

Another type of traveling wave solution is obtained for $\epsilon = -1$. A particular explicit solution of the reduced equations (74) takes the form

$$\Phi(x,t,\theta_1,\theta_2) = \arcsin(\tanh\sigma) + \theta_1\frac{\underline{D_1}}{\cosh\sigma} + \theta_2\underline{D_1}\tanh\sigma - \theta_1\theta_2\tanh\sigma, \tag{78}$$

where $\sigma = x + t$ and $\underline{D_1}$ is an arbitrary odd supernumber. This represents a bump function in the $\theta_1$ direction and a kink in the $\theta_2$ direction (i.e. in the corresponding odd components).

For the cases of the subalgebras $\mathfrak{S}_6 = \{P_x + \underline{\mu} Q_x\}$ and $\mathfrak{S}_{11} = \{P_t + \underline{\nu} Q_t\}$, the only nonzero solution which we obtain is

$$\Phi(x,t,\theta_1,\theta_2) = k\pi, \qquad \text{where} \quad k \in \mathbb{Z}. \tag{79}$$

For the subalgebra $\mathfrak{S}_7 = \{P_t + \underline{\mu} Q_x\}$, we obtain the solutions

$$\Phi(x,t,\theta_1,\theta_2) = k\pi, \tag{80}$$

where $k \in \mathbb{Z}$ and

$$\Phi(x,t,\theta_1,\theta_2) = \left(k+\tfrac{1}{2}\right)\pi + \theta_1 \underline{\mu}\underline{\lambda_0}\psi(\sigma) + \theta_2 \underline{\lambda_0} + (-1)^{k+1}(\theta_1 - \underline{\mu} t)\theta_2, \tag{81}$$

where $k \in \mathbb{Z}$, $\underline{\lambda_0}$ is an odd supernumber and $\psi$ is an arbitrary even-valued function of $\sigma = x + \underline{\mu}\theta_1 t$.

For the subalgebra $\mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$, we were able to obtain an explicit solution only if we assume that $\lambda$ and $\eta$ are multiples of $\underline{\mu}$. Then the equation for $\alpha$ does not involve odd unknowns and can be solved in terms of elliptic functions. (We note that in this case the reduced equations become very similar to those for $\mathfrak{S}_4$, see table 5, but not identical—they differ by the $\mu\alpha_\sigma$ term in $\eta_\sigma + \underline{\mu}\alpha_\sigma - \lambda\cos\alpha = 0$.) We find

$$\Phi(x,t,\theta_1,\theta_2) = \alpha(\sigma) + (\theta_1 - \varepsilon\underline{\mu} t)\eta(\sigma) + \theta_2\lambda(\sigma) + (\theta_1 - \varepsilon\underline{\mu} t)\theta_2\beta(\sigma), \tag{82}$$

where $\sigma = \varepsilon x - t + \underline{\mu} t\theta_1$ and the even-valued function $\alpha$ is given in terms of the Jacobi elliptic function

$$\alpha = \arcsin[k\,\mathrm{sn}(\sqrt{-\varepsilon}\sigma, k)], \tag{83}$$

where the modulus $k$ is restricted by the relation $|k| < 1$. The latter condition ensures that the elliptic solutions possess one real and one purely imaginary period when restricted to real $\sigma$. The even-valued function $\beta$ is given by

$$\beta = -k\,\mathrm{sn}(\sqrt{-\varepsilon}\sigma, k). \tag{84}$$

The odd-valued function $\lambda$ is given by $\lambda = \underline{\mu} g(\sigma)$, where $g$ is an even-valued function of $\sigma$ which obeys the linear ordinary differential equation

$$g_{\sigma\sigma} + (\tan\alpha)g_\sigma - \varepsilon(\cos^2\alpha)g + \varepsilon(\cos\alpha)\alpha_\sigma = 0, \tag{85}$$

and the odd-valued function $\eta$ is given by $\eta = \underline{\mu} f(\sigma)$, where the even-valued function $f$ is given by

$$f = \frac{\varepsilon}{\cos\alpha} g_\sigma. \tag{86}$$

The subalgebra $\mathfrak{S}_{10} = \{P_x + \underline{\nu} Q_t\}$ leads to the solutions

$$\Phi(x,t,\theta_1,\theta_2) = k\pi, \tag{87}$$

where $k \in \mathbb{Z}$ and

$$\Phi(x,t,\theta_1,\theta_2) = \left(k+\tfrac{1}{2}\right)\pi + \theta_2 \underline{\nu}\underline{\lambda_0}\psi(\sigma) + \theta_1 \underline{\lambda_0} + (-1)^{k}(\theta_2 - \underline{\nu} x)\theta_1, \tag{88}$$

where $k \in \mathbb{Z}$, $\underline{\lambda_0}$ is an odd supernumber and $\psi$ is an arbitrary even-valued function of $\sigma$.

For the subalgebra $\mathfrak{S}_{12} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\}$, we find a solution similarly as in the case of $\mathfrak{S}_8$. We have

$$\Phi(x,t,\theta_1,\theta_2) = \alpha(\sigma) + (\theta_2 - \underline{\nu} x)\eta(\sigma) + \theta_1\lambda(\sigma) + (\theta_2 - \underline{\nu} x)\theta_1\beta(\sigma), \tag{89}$$

where $\sigma = t - \varepsilon x + \underline{\nu} x\theta_2$ and the even-valued function $\alpha$ is given by

$$\alpha = \arcsin[k\,\mathrm{sn}(\sqrt{-\varepsilon}\sigma, k)], \tag{90}$$

where the modulus $k$ is restricted by the relation $|k| < 1$. The even-valued function $\beta$ is given by

$$\beta = k\,\mathrm{sn}(\sqrt{-\varepsilon}\sigma, k). \tag{91}$$

The odd-valued function $\lambda$ is given by $\lambda = \underline{\nu} g(\sigma)$ where $g$ is an even-valued function of $\sigma$ which obeys the linear differential equation for $g$:

$$g_{\sigma\sigma} + (\tan\alpha)\alpha_\sigma g_\sigma - \varepsilon(\cos^2\alpha)g - \varepsilon(\cos\alpha)\alpha_\sigma = 0, \tag{92}$$

and the odd-valued function $\eta$ is given by $\eta = \underline{\nu} f(\sigma)$ where the even-valued function $f$ is given by

$$f = -\frac{1}{\cos\alpha} g_\sigma. \tag{93}$$

Let us now turn our attention to those subalgebras whose invariants possess a non-standard structure. Such subalgebras are distinguished by the fact that each of them admits an invariant expressed in terms of an arbitrary function of the superspace variables, multiplied by an odd supernumber. Such invariants are nilpotent and this causes complications in the computation. This aspect can be illustrated by means of the following example. The subalgebra $\mathfrak{S}_5 = \{\underline{\mu} Q_x\}$ generates the first of the two one-parameter group transformations described in equation (4). Its invariants are $t, \theta_2, \Phi$ and any quantity of the form

$$\tau = \underline{\mu} f(x, t, \theta_1, \theta_2, \Phi), \tag{94}$$

where $f$ is an arbitrary function which can be either even or odd valued. It is an open question as to whether or not for a particular choice of function $f$ a substitution of these invariants into the SSG equation (39) can lead to a reduced system of equations expressible in terms of the invariants. It is clearly not possible for an arbitrary function $f$. For example, in the case when $\tau = \underline{\mu} x\theta_1$, the system (39) transforms into the equation

$$\underline{\mu} x\theta_2 \mathcal{A}_{t\tau} + \underline{\mu} x \mathcal{A}_{\tau\theta_2} + \sin\mathcal{A} = 0, \tag{95}$$

for the field

$$\Phi = \mathcal{A}(t, \tau, \theta_2). \tag{96}$$

The presence of the variable $x$ in equation (95) clearly demonstrates that we do not obtain a reduced equation expressible in terms of the invariants.

On the other hand, if we attempt the reduction with respect to all the vector fields $\mu Q_x, \mu \in \Lambda_{\mathrm{odd},}$ we immediately find that such vector fields do not form a subalgebra and we have to reduce with respect to the subalgebra generated by $\{Q_x, P_x\}$. That leads to $\Phi(t, \theta_2)$ and substituting into equation (39) we find the reduction

$$\sin\Phi = 0,$$

which allows again only the trivial solution

$$\Phi = k\pi, \qquad k \in \mathbb{Z}. \tag{97}$$

The other five subalgebras having non-standard invariants display similar features, and we list below the invariants expressed in terms of an arbitrary function of the superspace variables for each case.

| Subalgebra | Non-standard invariant |
|---|---|
| $\mathfrak{S}_9 = \{\underline{\nu} Q_t\}$ | $\underline{\nu} f(x, t, \theta_1, \theta_2, \Phi)$ |
| $\mathfrak{S}_{13} = \{\underline{\mu} Q_x + \underline{\nu} Q_t\}$ | $\underline{\mu}\underline{\nu} f(x, t, \theta_1, \theta_2, \Phi)$ |

| | |
|---|---|
| $\mathfrak{S}_{14} = \{P_x + \underline{\mu} Q_x + \underline{\nu} Q_t\}$ | $\underline{\mu}\underline{\nu} f(t, \theta_1, \theta_2, \Phi)$ |
| $\mathfrak{S}_{15} = \{P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$ | $\underline{\mu}\underline{\nu} f(x, \theta_1, \theta_2, \Phi)$ |
| $\mathfrak{S}_{16} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$ | $\underline{\mu}\underline{\nu} f(\theta_1, \theta_2, \Phi)$ |

where, in each case, $f$ is an arbitrary function of its arguments.

From a more general perspective, the problem of non-standard invariants can be expressed as follows. We recall that the construction of invariant solutions is based on a proper (local) choice of canonical coordinates on the space of independent and dependent coordinates such that it 'straightens the flow' of the generator of the one-parametric subgroup, i.e. we get one coordinate corresponding to the group parameter and the remaining ones are invariant with respect to the flow. In order to perform the reduction, we have to assume that the group parameter can be expressed as a function of original independent coordinates (i.e. the orbits of the subgroup action are of codimension 1 in the space of independent variables). We choose the proper number of invariant coordinates as our new dependent coordinates and interpret them as functions of the remaining ones and the group parameter.

Once the differential equation(s) possessing the symmetry is expressed in these canonical coordinates, due to its symmetry, we are guaranteed that the group parameter coordinate drops out of the equation(s) and we have an equation(s) with one less independent variable. If this(these) equation(s) is(are) still a partial differential equation(s) too difficult to tackle directly, we can repeat the procedure and further reduce (provided of course that the reduced equation has some symmetries).

Now the source of the problem becomes clear: in the commutative case we can always locally straighten the flow of any nonvanishing vector field, as is well known from the differential geometry. On the other hand, once we allow anticommuting variables, we are not always able to find such a coordinate transformation, as we have just seen—although we have found the proper number of invariants, the transformation is obviously non-invertible. Therefore, we are led to the conclusion that in the case of anticommuting independent variables not all symmetry generators allow a symmetry reduction; we have to restrict our attention only to those which can be written as a partial derivative with respect to even coordinate in some suitable coordinate system on the supermanifold $X \times U$ (of course, a possibility is not excluded that in some particular case a solution constructed out of non-standard invariants may exist—but its existence and the consistency of the reduction is not guaranteed).

## 7. Conclusions

In this paper, we have performed a group-theoretical analysis of the (1 + 1)-dimensional supersymmetric sine-Gordon model. This was accomplished using two different approaches.

In the first one, the decomposition (3) of the bosonic superfield was substituted into the SSG equation and decomposed into a system of partial differential equations for the component fields. Next, we focused directly on the SSG equation expressed in terms of a bosonic superfield involving odd, anticommuting independent variables. In each case, we have determined a Lie (super)algebra of symmetries of the supersymmetric system and classified all of its one-dimensional subalgebras. In the case of the SSG equation (10), the superalgebra of symmetries was computed through the use of a generalized version of the prolongation method.

For the decomposed system (15), no odd symmetry generators were obtained and the Lie algebra of symmetries was just a realization of the Poincaré algebra in (1 + 1) dimensions on the superspace. On the other hand, the Lie superalgebra of symmetries of the SSG equation (10) is the full super-Poincaré algebra in (1 + 1) dimensions.

Through the use of the SRM, we have constructed exact analytic solutions of the SSG equation. The reductions in the component decomposition were found to be a special subset of reductions in superspace. Solutions included constant, algebraic, trigonometric, hyperbolic and doubly periodic solutions in terms of Jacobi elliptic functions. In some cases, the reductions lead to systems of coupled ordinary differential equations whose full solution is unknown and we had to content ourselves with some particular explicit solutions or solutions expressed in terms of an arbitrary solution of given inhomogeneous linear ordinary differential equation (e.g. equation (92)).

In the superspace formulation, we have encountered one complication not present in the ordinary bosonic case; namely not all generators allow the corresponding reduction. The reason for this is that there may be no canonical coordinates on the superspace straightening the flow of such a vector field. Presently, we do not know about any simple criteria that would allow us to immediately identify such problematic vector fields, i.e. without computation of invariants.

We note that the groups of symmetries found were those one could guess at the beginning from the structure of the SSG equation. Unfortunately, that is very often the case with such an investigation—an involved and lengthy computation is needed in order to exclude the possibility of hidden, unexpected, symmetries but otherwise brings nothing new.

An interesting open question is whether the supersymmetries can be somehow directly detected in the component form using the methods of symmetry analysis of differential equations (if one does not know that the equation was constructed to be supersymmetric, of course). That is, does supersymmetry demonstrate itself in some way e.g. as a contact or conditional symmetry? If such a detection was possible, it might be systematically applied to find enhanced hidden supersymmetry in some models, which would be of significant practical importance.

Also, it would be of interest to apply the method in section 4 to other physically relevant nonlinear supersymmetric models. For example, it should be possible to perform a similar analysis on a model with more supersymmetries, e.g. the $N = 2$ supersymmetric sine-Gordon model. Given the computational complexities involved, namely the form of the prolongations defined by formulas (46), it seems rather necessary to use computer algebra systems to deal with the expressions in these cases and we defer such investigation to future work.

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## 3.2

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# Invariant solutions of supersymmetric nonlinear wave equations

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**Abstract**

Systematic group-theoretical analyses of two supersymmetric nonlinear wave equations, namely the supersymmetric sinh-Gordon and polynomial Klein–Gordon equations, are performed. In each case, a generalization of the method of prolongations is used to determine the Lie superalgebra of symmetries, and the method of symmetry reduction is applied in order to obtain invariant solutions of the supersymmetric equations under consideration. In the case of the supersymmetric sinh-Gordon equation, the results are compared with those previously found for the supersymmetric sine-Gordon equation. The presence of non-standard invariants is discussed for the supersymmetric sinh-Gordon and polynomial Klein–Gordon equations.

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## 1. Introduction

Recently, there has been much interest in the study of supersymmetric extensions of both classical and quantum mechanical models [1–4]. Various techniques have been used in order to obtain supersolitonic solutions, including the inverse scattering method, Bäcklund transformations and their Riccati forms, Darboux transformations for the odd and even superfields, Lax formalism in superspace, and generalized versions of the symmetry reduction method [5–11]. Integrable models which were studied using these methods include (among others) the Korteweg–de Vries equation [12–14], the sine-Gordon equation [1], Liouville theory [4], the Schrödinger equation [15] and sigma models [16]. A number of solitonic and super multi-solitonic solutions were determined by a Crum-type transformation [17] and it was found [2, 3] that there exist infinitely many local conserved quantities. A connection was established between the super-Darboux transformations and super-Bäcklund transformations, which allows one to construct $N$ supersoliton solutions.

In our previous article [18], the Lie symmetry superalgebra of the supersymmetric sine-Gordon equation was determined by means of a generalization of the prolongation method, and its subalgebras were classified. A number of invariant solutions of the model were found, including constant, algebraic, hyperbolic and doubly periodic solutions expressed in terms of elliptic functions. It was also found that some of the subalgebras had invariants which possess a non-standard structure in the sense that they do not admit symmetry reduction in the classical sense. In this context, it would be worthwhile to extend this analysis to other nonlinear wave equations.

The purpose of this paper is to perform a systematic group-theoretical study of the supersymmetric versions of two particular nonlinear equations. First, we consider the sinh-Gordon equation

$$u_{xt} = \sinh u. \tag{1}$$

A supersymmetric version of this equation has already been studied (see e.g. [19–21]), but to our knowledge a systematic group-theoretical analysis of the supersymmetric sinh-Gordon equation has never been performed. In this paper, we determine the Lie symmetry superalgebra, subalgebra classification and invariant solutions of this supersymmetric equation. Next, we use the superspace and superfield formalism to construct a supersymmetric extension of the following polynomial generalization of the Klein–Gordon equation:

$$u_{xt} + au + bu^3 + cu^5 = 0, \tag{2}$$

for certain choices of constant parameters $a$, $b$ and $c$. Through the use of the transformation

$$\Psi = \frac{2}{k}\sin^{-1} u, \tag{3}$$

it has been demonstrated [22, 23] that the double sine-Gordon equation

$$(\partial_t^2 - \nabla^2)\Psi + M^2k^{-1}\sin k\Psi + p(2k)^{-1}\sin 2k\Psi = 0 \tag{4}$$

can be reduced to equation (2). As far as we are aware, equation (2) has never been supersymmetrized before (although supersymmetric versions of certain Klein–Gordon-type equations do exist, see e.g. [24]). We perform a systematic group-theoretical analysis of our constructed supersymmetric model.

This paper is organized as follows. In section 2 we discuss the supersymmetric version of the sinh-Gordon equation. We use a generalized form of the prolongation method to calculate the Lie superalgebra of symmetries, and then proceed to classify the subalgebras and determine invariant solutions of the supersymmetric sinh-Gordon equation through the symmetry reduction method. In section 3, we repeat the same procedure for the supersymmetric version of the polynomial Klein–Gordon equation. For both equations we consider the fact that some subalgebras possess non-standard invariant structures. Finally, in section 4 we present our conclusions, final remarks and possibilities for future research.

## 2. Supersymmetric extension of the sinh-Gordon equation

Let us first consider the sinh-Gordon equation

$$u_{xt} = \sinh u. \tag{5}$$

In order to supersymmetrize equation (5), we extend the space of independent variables $\{x, t\}$ to the superspace $\{x, t, \theta_1, \theta_2\}$, where $\theta_1$ and $\theta_2$ are independent Grassmann-odd variables. We also replace the classical real field $u(x,t)$ by the Grassmann-even superfield:

$$\Phi(x, t, \theta_1, \theta_2) = \tfrac{1}{2}u(x, t) + \theta_1\phi(x, t) + \theta_2\psi(x, t) + \theta_1\theta_2F(x, t), \tag{6}$$

where $\phi$ and $\psi$ are two new odd fields and $F$ is a new even field. We intend to construct the supersymmetric extension of (5) in such a way that it is invariant under the following two supersymmetry transformations:

$$x \to x - \underline{\eta}_1\theta_1, \quad \theta_1 \to \theta_1 + \underline{\eta}_1 \qquad \text{and} \qquad t \to t + \underline{\eta}_2\theta_2, \quad \theta_2 \to \theta_2 + \underline{\eta}_2, \tag{7}$$

where $\underline{\eta_1}$ and $\underline{\eta_2}$ are arbitrary constant odd parameters (where we use the convention that underlined constants represent Grassmann-odd parameters). The choice of signs in equation (7) is not arbitrary; it is caused by the requirement that the supersymmetric version of the sinh-Gordon equation be written as an equation for real superfield, i.e. that the functions $u(x,t), \phi(x,t), \psi(x,t), F(x,t)$ take values in a real Grassmann ring and can therefore be physically interpreted as real bosonic and fermionic fields.

The two transformations (7) are generated by the infinitesimal supersymmetry operators

$$Q_x = \partial_{\theta_1} - \theta_1\partial_x \qquad \text{and} \qquad Q_t = \partial_{\theta_2} + \theta_2\partial_t, \tag{8}$$

respectively. In order to make the generalized model invariant under the supersymmetry generators $Q_x$ and $Q_t$, we introduce the covariant derivatives

$$D_x = \partial_{\theta_1} + \theta_1\partial_x \qquad \text{and} \qquad D_t = \partial_{\theta_2} - \theta_2\partial_t, \tag{9}$$

which possess the property that each derivative $D_i$ anticommutes with both supersymmetry operators $Q_j$. Also,

$$\{Q_x, Q_x\} = -2\partial_x, \qquad \{Q_t, Q_t\} = 2\partial_t, \qquad \{Q_x, Q_t\} = \{D_x, D_t\} = 0. \tag{10}$$

Thus, if we write our supersymmetric equation in terms of the superfield $\Phi$ and its covariant derivatives of various orders, it will indeed be invariant under the transformations generated by $Q_x$ and $Q_t$. The superspace Lagrangian density of the supersymmetric model is

$$\mathcal{L}(\Phi) = \tfrac{1}{2}D_x\Phi D_t\Phi + \cosh\Phi, \tag{11}$$

and the corresponding Euler–Lagrange superfield equation (the supersymmetric sinh-Gordon equation) takes the form

$$D_x D_t\Phi = \sinh\Phi. \tag{12}$$

In terms of the partial derivatives with respect to the independent variables, this equation can be re-written as

$$-\theta_1\theta_2\Phi_{xt} + \theta_2\Phi_{t\theta_1} + \theta_1\Phi_{x\theta_2} - \Phi_{\theta_1\theta_2} - \sinh\Phi = 0. \tag{13}$$

In this paper, we use the convention that for partial derivatives involving odd variables,

$$\partial_{\theta_i}(fg) = (\partial_{\theta_i}f)g + (-1)^{\deg(f)}f(\partial_{\theta_i}g), \tag{14}$$

where

$$\deg(f) = \begin{cases} 0 & \text{if } f \text{ is even} \\ 1 & \text{if } f \text{ is odd} \end{cases} \tag{15}$$

and the notation

$$f_{\theta_1\theta_2} = \partial_{\theta_2}(\partial_{\theta_1}f). \tag{16}$$

The even (super)numbers, variables, fields etc. are assumed to be elements of the even part $\Lambda_{\text{even}}$ of the underlying abstract real Grassmann ring $\Lambda = \wedge[\xi_1, \xi_2, \ldots]$; the odd (super)numbers, variables fields, etc. lie in its odd part $\Lambda_{\text{odd}}$. We shall assume throughout the paper that the function $u(x,t)$ in equation (6) (and $\alpha(\sigma)$ closely related to $u(x,t)$, used in section 2.2) has values in the invertible subset of $\Lambda_{\text{even}}$ plus $\{0\}$, i.e. nonvanishing nilpotent values of $u(x,t)$ are ruled out. This technical assumption allows us to perform necessary simplifications in our calculations without splitting off of singular subcases.

**Table 1.** Supercommutation table for the Lie superalgebra $\mathfrak{S}$ spanned by the vector fields (19).

| | **L** | $\mathbf{P_x}$ | $\mathbf{P_t}$ | $\mathbf{Q_x}$ | $\mathbf{Q_t}$ |
|---|---|---|---|---|---|
| **L** | 0 | $2P_x$ | $-2P_t$ | $Q_x$ | $-Q_t$ |
| $\mathbf{P_x}$ | $-2P_x$ | 0 | 0 | 0 | 0 |
| $\mathbf{P_t}$ | $2P_t$ | 0 | 0 | 0 | 0 |
| $\mathbf{Q_x}$ | $-Q_x$ | 0 | 0 | $-2P_x$ | 0 |
| $\mathbf{Q_t}$ | $Q_t$ | 0 | 0 | 0 | $2P_t$ |

### *2.1. Symmetries of the supersymmetric sinh-Gordon equation*

We apply the generalized version of the method of prolongation of vector fields, as considered for systems of partial differential equations involving Grassmann-odd variables (see [18]). That is, we postulate an even vector field of the form

$$\mathbf{v} = \xi(x,t,\theta_1,\theta_2,\Phi)\partial_x + \tau(x,t,\theta_1,\theta_2,\Phi)\partial_t + \rho(x,t,\theta_1,\theta_2,\Phi)\partial_{\theta_1} + \sigma(x,t,\theta_1,\theta_2,\Phi)\partial_{\theta_2} + \Lambda(x,t,\theta_1,\theta_2,\Phi)\partial_\Phi, \tag{17}$$

where $\xi$, $\tau$ and $\Lambda$ are the even functions, while $\rho$ and $\sigma$ are odd. We use the generalized total derivatives defined in [18] in order to calculate the coefficients of the second prolongation of the vector field (17). In our case, the prolongation coefficients are exactly the same as those found for the supersymmetric sine-Gordon equation [18]. We apply the second prolongation to the supersymmetric sinh-Gordon equation (13) in order to obtain conditions relating various prolongation coefficients to each other.

Substituting the formulae for the prolongation coefficients into this condition and replacing each term $\Phi_{\theta_1\theta_2}$ in the resulting expression by the terms $-\theta_1\theta_2\Phi_{xt} + \theta_2\Phi_{t\theta_1} + \theta_1\Phi_{x\theta_2} - \sinh\Phi$, we obtain a set of determining equations for the functions $\xi$, $\tau$, $\rho$, $\sigma$ and $\Lambda$. The general solution of these determining equations is found to be

$$\begin{aligned} &\xi(x,\theta_1) = -2C_1x + C_2 - \underline{D_1}\theta_1, \qquad \tau(t,\theta_2) = 2C_1t + C_3 + \underline{D_2}\theta_2,\\ &\rho(\theta_1) = -C_1\theta_1 + \underline{D_1}, \qquad \sigma(\theta_2) = C_1\theta_2 + \underline{D_2}, \qquad \Lambda = 0, \end{aligned} \tag{18}$$

where $C_1$, $C_2$, $C_3$ are bosonic constants, while $\underline{D_1}$ and $\underline{D_2}$ are fermionic constants. Thus, we find that the superalgebra $\mathfrak{S}$ of symmetries of the supersymmetric sinh-Gordon equation (13) is the Poincaré superalgebra $P(1|1)$ generated by the following five infinitesimal vector fields:

$$\begin{aligned} &L = -2x\partial_x + 2t\partial_t - \theta_1\partial_{\theta_1} + \theta_2\partial_{\theta_2}, \qquad P_x = \partial_x, \qquad P_t = \partial_t,\\ &Q_x = -\theta_1\partial_x + \partial_{\theta_1}, \qquad Q_t = \theta_2\partial_t + \partial_{\theta_2}. \end{aligned} \tag{19}$$

The generators $P_x$ and $P_t$ represent translations in space and time respectively, while $L$ generates a dilation in both even and odd independent variables. In addition, we recover the supersymmetry transformations $Q_x$ and $Q_t$ which we identified previously in (8). As we could expect by analogy with the non-supersymmetric case, no additional symmetries are obtained and this superalgebra is almost identical to the one which was found for the supersymmetric sine-Gordon equation; it differs only by the sign of $\{Q_t, Q_t\}$. This sign difference arises from the choice of supersymmetry generators and supercovariant derivatives in equations (8) and (9). The commutation (anticommutation in the case of two odd operators) relations of the superalgebra $\mathfrak{S}$ of the supersymmetric sinh-Gordon equation are given in table 1.

The Lie superalgebra $\mathfrak{S}$ can be decomposed into the semi-direct sum

$$\mathfrak{S} = \{L\} \ni \{P_x, P_t, Q_x, Q_t\}. \tag{20}$$

The classification of the one-dimensional subalgebras into conjugacy classes is similar to that found for the sine-Gordon equation (see [18] for details) and is given as follows:

$$
\begin{aligned}
&\mathfrak{S}_1 = \{L\},\\
&\mathfrak{S}_2 = \{P_x\},\\
&\mathfrak{S}_3 = \{P_t\},\\
&\mathfrak{S}_4 = \{P_x + \varepsilon P_t\},\\
&\mathfrak{S}_5 = \{\underline{\mu} Q_x\},\ \text{ where } \underline{\mu} \text{ and } \tilde{\underline{\mu}} = k\underline{\nu} \text{ represent the same conjugacy class}\\
&\text{for any invertible even supernumber } k\\
&\mathfrak{S}_6 = \{P_x + \underline{\mu} Q_x\},\ \text{ where } \underline{\mu} \text{ and } \tilde{\underline{\mu}} = e^k\underline{\nu} \text{ represent the same conjugacy class}\\
&\text{for any even supernumber } k\\
&\mathfrak{S}_7 = \{P_t + \underline{\mu} Q_x\},\ \text{ where } \underline{\mu} \text{ and } \tilde{\underline{\nu}} = e^k\underline{\nu} \text{ represent the same conjugacy class}\\
&\text{for any even supernumber } k\\
&\mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\},\\
&\mathfrak{S}_9 = \{\underline{\nu} Q_t\},\ \text{ where } \underline{\mu} \text{ and } \tilde{\underline{\mu}} = k\underline{\nu} \text{ represent the same conjugacy class}\\
&\text{for any invertible even supernumber } k\\
&\mathfrak{S}_{10} = \{P_x + \underline{\nu} Q_t\},\ \text{ where } \underline{\nu} \text{ and } \tilde{\underline{\nu}} = e^k\underline{\nu} \text{ represent the same conjugacy class}\\
&\text{for any even supernumber } k\\
&\mathfrak{S}_{11} = \{P_t + \underline{\nu} Q_t\},\ \text{ where } \underline{\nu} \text{ and } \tilde{\underline{\nu}} = e^k\underline{\nu} \text{ represent the same conjugacy class}\\
&\text{for any even supernumber } k\\
&\mathfrak{S}_{12} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\},\\
&\mathfrak{S}_{13} = \{\underline{\mu} Q_x + \underline{\nu} Q_t\},\ \text{ where } (\underline{\mu}, \underline{\nu}) \text{ and } (\tilde{\underline{\mu}}, \tilde{\underline{\nu}}) = (e^k\underline{\mu}, e^k\underline{\nu}) \text{ represent the}\\
&\text{same the conjugacy class for any even supernumber } k\\
&\mathfrak{S}_{14} = \{P_x + \underline{\mu} Q_x + \underline{\nu} Q_t\},\ \text{ where } (\underline{\mu}, \underline{\nu}) \text{ and } (\tilde{\underline{\mu}}, \tilde{\underline{\nu}}) = (e^k\underline{\mu}, e^{3k}\underline{\nu}) \text{ represent the}\\
&\text{same conjugacy class for any even supernumber } k\\
&\mathfrak{S}_{15} = \{P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\},\ \text{ where } (\underline{\mu}, \underline{\nu}) \text{ and } (\tilde{\underline{\mu}}, \tilde{\underline{\nu}}) = (e^{3k}\underline{\mu}, e^k\underline{\nu}) \text{ represent the}\\
&\text{same conjugacy class for any even supernumber } k\\
&\mathfrak{S}_{16} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}.
\end{aligned}
\tag{21}
$$

These subalgebras allow us to determine invariant solutions of the supersymmetric sinh-Gordon equation (13).

### 2.2. *Invariant solutions of the supersymmetric sinh-Gordon equation*

We now proceed to apply a modified version of the symmetry reduction method to the supersymmetric sinh-Gordon equation (13) in order to obtain its invariant solutions. Passing systematically through each subalgebra in the classification, we construct (whenever possible) a set of four functionally independent invariants. For the subalgebras $\mathfrak{S}_5$, $\mathfrak{S}_9$, $\mathfrak{S}_{13}$, $\mathfrak{S}_{14}$, $\mathfrak{S}_{15}$ and $\mathfrak{S}_{16}$, the invariants possess a non-standard structure, which will be discussed at the end of this section. They are the same as those found for the supersymmetric sine-Gordon equation [18]. For the remaining subalgebras, the bosonic superfield $\Phi$ is expressed in terms of the invariants. That is, if the independent invariants are given by $\sigma$, $\tau_1$, $\tau_2$, where $\sigma$ is an

**Table 2.** Invariants and change of variable for subalgebras of the Lie superalgebra $\mathfrak{S}$ spanned by the vector fields (19).

| Subalgebra | Invariants | Superfield |
|---|---|---|
| $\mathfrak{S}_1 = \{L\}$ | $\sigma = xt,\ \tau_1 = t^{1/2}\theta_1,$ $\tau_2 = t^{-1/2}\theta_2,\ \Phi$ | $\Phi = \mathcal{A}(\sigma,\tau_1,\tau_2)$ $= \alpha(\sigma) + \tau_1\eta(\sigma) + \tau_2\lambda(\sigma) + \tau_1\tau_2\beta(\sigma)$ |
| $\mathfrak{S}_2 = \{P_x\}$ | $t, \theta_1, \theta_2, \Phi$ | $\Phi = \mathcal{A}(t,\theta_1,\theta_2) = \alpha(t)+\theta_1\eta(t)+\theta_2\lambda(t)+\theta_1\theta_2\beta(t)$ |
| $\mathfrak{S}_3 = \{P_t\}$ | $x, \theta_1, \theta_2, \Phi$ | $\Phi = \mathcal{A}(x,\theta_1,\theta_2)$ $= \alpha(x) + \theta_1\eta(x) + \theta_2\lambda(x) + \theta_1\theta_2\beta(x)$ |
| $\mathfrak{S}_4 = \{P_x + \varepsilon P_t\}$ | $\sigma = x - \varepsilon t, \theta_1, \theta_2, \Phi$ | $\Phi = \mathcal{A}(\sigma,\theta_1,\theta_2)$ $= \alpha(\sigma) + \theta_1\eta(\sigma) + \theta_2\lambda(\sigma) + \theta_1\theta_2\beta(\sigma)$ |
| $\mathfrak{S}_6 = \{P_x + \underline{\mu} Q_x\}$ | $t, \tau_1 = \theta_1 - \underline{\mu} x, \theta_2, \Phi$ | $\Phi = \mathcal{A}(t,\tau_1,\theta_2) = \alpha(t)+\tau_1\eta(t)+\theta_2\lambda(t)+\tau_1\theta_2\beta(t)$ |
| $\mathfrak{S}_7 = \{P_t + \underline{\mu} Q_x\}$ | $\sigma = x + \underline{\mu}\theta_1 t,$ $\tau_1 = \theta_1 - \underline{\mu} t, \theta_2, \Phi$ | $\Phi = \mathcal{A}(\sigma,\tau_1,\theta_2)$ $= \alpha(\sigma) + \tau_1\eta(\sigma) + \theta_2\lambda(\sigma) + \tau_1\theta_2\beta(\sigma)$ |
| $\mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$ | $\sigma = \varepsilon x - t + \underline{\mu} t\theta_1,$ $\tau_1 = \theta_1 - \varepsilon\underline{\mu} t, \theta_2, \Phi$ | $\Phi = \mathcal{A}(\sigma,\tau_1,\theta_2)$ $= \alpha(\sigma) + \tau_1\eta(\sigma) + \theta_2\lambda(\sigma) + \tau_1\theta_2\beta(\sigma)$ |
| $\mathfrak{S}_{10} = \{P_x + \underline{\nu} Q_t\}$ | $\sigma = t - \underline{\nu}\theta_2 x,$ $\theta_1, \tau_2 = \theta_2 - \underline{\nu} x, \Phi$ | $\Phi = \mathcal{A}(\sigma,\theta_1,\tau_2)$ $= \alpha(\sigma) + \theta_1\eta(\sigma) + \tau_2\lambda(\sigma) + \theta_1\tau_2\beta(\sigma)$ |
| $\mathfrak{S}_{11} = \{P_t + \underline{\nu} Q_t\}$ | $x, \theta_1, \tau_2 = \theta_2 - \underline{\nu} t, \Phi$ | $\Phi = \mathcal{A}(x,\theta_1,\tau_2)$ $= \alpha(x) + \theta_1\eta(x) + \tau_2\lambda(x) + \theta_1\tau_2\beta(x)$ |
| $\mathfrak{S}_{12} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\}$ | $\sigma = t - \varepsilon x - \underline{\nu} x\theta_2,$ $\theta_1, \tau_2 = \theta_2 - \underline{\nu} x, \Phi$ | $\Phi = \mathcal{A}(\sigma,\theta_1,\tau_2)$ $= \alpha(\sigma) + \theta_1\eta(\sigma) + \tau_2\lambda(\sigma) + \theta_1\tau_2\beta(\sigma)$ |

even invariant while $\tau_1$ and $\tau_2$ are odd invariants, then the superfield $\Phi$ can be written in the form

$$\Phi = \mathcal{A}(\sigma,\tau_1,\tau_2) = \alpha(\sigma) + \tau_1\eta(\sigma) + \tau_2\lambda(\sigma) + \tau_1\tau_2\beta(\sigma), \tag{22}$$

where $\alpha$ and $\beta$ are even-valued functions, while $\eta$ and $\lambda$ are odd-valued functions to be determined. When this decomposition is substituted into the supersymmetric sinh-Gordon equation, we obtain a reduced system of ordinary differential equations for the functions $\alpha$, $\eta$, $\lambda$ and $\beta$. In general, the term $\sinh\mathcal{A}$ can be expanded into the form

$$\sinh\mathcal{A} = (\sinh\alpha) + \tau_1\eta(\cosh\alpha) + \tau_2\lambda(\cosh\alpha) + \tau_1\tau_2\left(\beta(\cosh\alpha) - \eta\lambda(\sinh\alpha)\right), \tag{23}$$

as identified by the series

$$\sinh\mathcal{A} = \mathcal{A} + \frac{1}{3!}\mathcal{A}^3 + \frac{1}{5!}\mathcal{A}^5 + \cdots. \tag{24}$$

We summarize our results as follows. In table 2, we list the invariants of the respective one-dimensional subalgebras together with the form of their superfield solutions. In table 3, we present the respective reduced systems of ordinary differential equations for $\alpha$, $\eta$, $\lambda$ and $\beta$.

For the sake of simplicity we unify the notation as follows: $\alpha$ and $\beta$ are even functions of their arguments, while $\eta$ and $\lambda$ are odd functions of their argument.

Subalgebras $\mathfrak{S}_5 = \{\underline{\mu} Q_x\}$, $\mathfrak{S}_9 = \{\underline{\nu} Q_t\}$, $\mathfrak{S}_{13} = \{\underline{\mu} Q_x + \underline{\nu} Q_t\}$, $\mathfrak{S}_{14} = \{P_x + \underline{\mu} Q_x + \underline{\nu} Q_t\}$, $\mathfrak{S}_{15} = \{P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$, $\mathfrak{S}_{16} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$ have invariants which possess non-standard structures and will be discussed at the end of this section.

**Table 3.** Reduced equations obtained for subalgebras of the Lie superalgebra $\mathfrak{S}$ spanned by the vector fields (19).

| Subalgebra | Reduced equations |
|---|---|
| $\mathfrak{S}_1 = \{L\}$ | $\beta + \sinh\alpha = 0,\ \lambda_\sigma - \eta\cosh\alpha = 0,$ $\sigma\eta_\sigma + \frac{1}{2}\eta - \lambda\cosh\alpha = 0,\ \alpha_\sigma + \sigma\alpha_{\sigma\sigma} + \beta\cosh\alpha - \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_2 = \{P_x\}$ | $\beta + \sinh\alpha = 0,\ \eta\cosh\alpha = 0,$ $\eta_t - \lambda\cosh\alpha = 0,\ \beta\cosh\alpha - \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_3 = \{P_t\}$ | $\beta + \sinh\alpha = 0,\ \lambda_x - \eta\cosh\alpha = 0,$ $\lambda\cosh\alpha = 0,\ \beta\cosh\alpha - \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_4 = \{P_x + \varepsilon P_t\}$ | $\beta + \sinh\alpha = 0,\ \lambda_\sigma - \eta\cosh\alpha = 0,$ $\varepsilon\eta_\sigma + \lambda\cosh\alpha = 0,\ \varepsilon\alpha_{\sigma\sigma} - \beta\cosh\alpha + \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_6 = \{P_x + \underline{\mu} Q_x\}$ | $\beta + \sinh\alpha = 0,\ \underline{\mu}\beta - \eta\cosh\alpha = 0,$ $\eta_t - \lambda\cosh\alpha = 0,\ \underline{\mu}\eta_t - \beta\cosh\alpha + \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_7 = \{P_t + \underline{\mu} Q_x\}$ | $\beta + \sinh\alpha = 0,\ \lambda_\sigma - \eta\cosh\alpha = 0,$ $\underline{\mu}\alpha_\sigma + \lambda\cosh\alpha = 0,\ \underline{\mu}\eta_\sigma - \beta\cosh\alpha + \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$ | $\beta + \sinh\alpha = 0,\ \varepsilon\lambda_\sigma - \eta\cosh\alpha = 0,$ $\eta_\sigma + \underline{\mu}\alpha_\sigma + \lambda\cosh\alpha = 0,\ \varepsilon\alpha_{\sigma\sigma} + \underline{\mu}\eta_\sigma - \beta\cosh\alpha + \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_{10} = \{P_x + \underline{\nu} Q_t\}$ | $\beta + \sinh\alpha = 0,\ \underline{\nu}\alpha_\sigma - \eta\cosh\alpha = 0,$ $\eta_\sigma - \lambda\cosh\alpha = 0,\ \underline{\nu}\lambda_\sigma - \beta\cosh\alpha + \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_{11} = \{P_t + \underline{\nu} Q_t\}$ | $\beta + \sinh\alpha = 0,\ \lambda_x - \eta\cosh\alpha = 0,$ $\underline{\nu}\beta + \lambda\cosh\alpha = 0,\ \underline{\nu}\lambda_x - \beta\cosh\alpha + \eta\lambda\sinh\alpha = 0$ |
| $\mathfrak{S}_{12} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\}$ | $\beta + \sinh\alpha = 0,\ \underline{\nu}\alpha_\sigma - \varepsilon\lambda_\sigma - \eta\cosh\alpha = 0,$ $\eta_\sigma - \lambda\cosh\alpha = 0,\ \varepsilon\alpha_{\sigma\sigma} + \underline{\nu}\lambda_\sigma - \beta\cosh\alpha + \eta\lambda\sinh\alpha = 0$ |

For the subalgebras $\mathfrak{S}_2 = \{P_x\}$, $\mathfrak{S}_3 = \{P_t\}$, $\mathfrak{S}_6 = \{P_x + \underline{\mu} Q_x\}$, $\mathfrak{S}_7 = \{P_t + \underline{\mu} Q_x\}$, $\mathfrak{S}_{10} = \{P_x + \underline{\nu} Q_t\}$ and $\mathfrak{S}_{11} = \{P_t + \underline{\nu} Q_t\}$, the only solution of the reduced equations is the null solution $\Phi = 0$.

Subalgebra $\mathfrak{S}_1 = \{L\}$ leads to the solution

$$\Phi = \alpha(\sigma) + t^{1/2}\theta_1\eta(\sigma) + t^{-1/2}\theta_2\lambda(\sigma) + \theta_1\theta_2\beta(\sigma), \tag{25}$$

where the symmetry variable is $\sigma = xt$, the functions $\alpha$ and $\lambda$ satisfy the following ordinary differential equations:

$$\begin{aligned} &\alpha_{\sigma\sigma} + \sigma^{-1}\alpha_\sigma - \tfrac{1}{2}\sigma^{-1}\sinh(2\alpha) - C_0\sigma^{-3/2}\sinh\alpha = 0, \\ &\lambda_{\sigma\sigma} + \left(\tfrac{1}{2}\sigma^{-1} - (\tanh\alpha)\alpha_\sigma\right)\lambda_\sigma - \sigma^{-1}\cosh^2\alpha\lambda = 0, \end{aligned} \tag{26}$$

and $\eta$, $\beta$ are expressed as

$$\begin{aligned} \eta &= \frac{1}{\cosh\alpha}\lambda_\sigma, \\ \beta &= -\sinh\alpha, \end{aligned} \tag{27}$$

subject to the condition that

$$\lambda\eta = C_0\sigma^{-1/2}, \tag{28}$$

where $C_0$ is a nilpotent even constant. This represents a nontrivial scaling-invariant solution, where the ordinary differential equation for $\alpha$ does not have the Painlevé property and its solution in a closed form is unknown.

The reduction with respect to the subalgebra $\mathfrak{S}_4 = \{P_x + \varepsilon P_t\}$ implies that

$$(\eta\lambda)_\sigma = 0,$$

i.e. $\eta\lambda = C_0$ is an even nilpotent constant. The bosonic part of the equations of motion becomes

$$\varepsilon\alpha_{\sigma\sigma} + \sinh\alpha\cosh\alpha + C_0\sinh\alpha = 0. \tag{29}$$

Firstly, we restrict ourselves to $C_0 = 0$. This choice allows us to find a solution of the equation for $\alpha(\sigma)$ in the implicit form (31). Consequently, we find the following solution of the supersymmetric sinh-Gordon equation (13):

$$\Phi = \alpha(\sigma) + \theta_1\eta(\sigma) + \theta_2\lambda(\sigma) + \theta_1\theta_2\beta(\sigma), \tag{30}$$

where the symmetry variable is $\sigma = x - \varepsilon t$. The function $\alpha$ is expressed in terms of the elliptic function $F$:

$$\frac{\pm\sqrt{\frac{2\varepsilon\cosh^2\alpha - \varepsilon - 4C_1}{4C_1+\varepsilon}}\, F\left(\cosh\alpha, \sqrt{\frac{2\varepsilon}{4C_1+\varepsilon}}\right)}{\sqrt{2C_1 - \varepsilon\cosh^2\alpha + \frac{1}{2}\varepsilon}} = \sigma + C_2. \tag{31}$$

The function $\lambda$ has the form $\lambda = \underline{K} f(\sigma)$, where $\underline{K}$ is an odd constant and $f$ is an even function which satisfies the equation

$$\underline{K}[f_{\sigma\sigma} - (\tanh\alpha)\alpha_\sigma f_\sigma + \varepsilon(\cosh^2\alpha) f] = 0. \tag{32}$$

The functions $\lambda$ and $\beta$ are defined as follows:

$$\eta = \frac{\underline{K}}{\cosh\alpha} f_\sigma, \qquad \beta = -\sinh\alpha. \tag{33}$$

This represents a travelling wave expressed in terms of elliptic functions. We observe that, by choosing $\underline{K} = 0$, we can make $\Phi$ in equation (30) into a purely bosonic nontrivial solution, i.e. $\eta = \lambda = 0$.

When we set $\eta\lambda = C_0 \neq 0$ in equation (29) we find a more complicated implicit solution $\alpha(\sigma)$ :

$$\int_{a_0}^{a=\alpha(\sigma)} \frac{\pm 2e^a}{\sqrt{-e^{4a} - 4C_0e^{3a} + (4C_1 - 2)e^{2a} - 4C_0e^a - 1}}\mathrm{d}a - \sigma = 0, \tag{34}$$

where $C_1, a_0$ are the integration constants. The integral in equation (34) can be converted via the substitution $y = e^a$ to the elliptic integral, giving an equation

$$\int_{y_0}^{y(\sigma)} \frac{\pm 2\mathrm{d}y}{\sqrt{-y^4 - 4C_0y^3 + (4C_1 - 2)y^2 - 4C_0y - 1}} - \sigma = 0. \tag{35}$$

The general solution of equation (35) is well known when $C_0, C_1$ are ordinary real numbers (see e.g. [25] p 453). The solution $y$ can be expressed as a rational Weierstrass elliptic function:

$$y - y_0 = \tfrac{1}{4} f'(y_0)\{\mathcal{P}(\sigma, g_2, g_3) - \tfrac{1}{24} f''(y_0)\}^{-1}, \tag{36}$$

where the invariants of the elliptic Weierstrass function are

$$g_2 = \tfrac{4}{3} - 4C_0^2 + \tfrac{4}{3}C_1(C_1 - 1), \qquad g_3 = \tfrac{4}{9}C_1 - \tfrac{8}{27} + \tfrac{2}{3}C_0^2C_1 - \tfrac{7}{3}C_0^2 - \tfrac{8}{27}C_1^3 + \tfrac{4}{9}C_1^2, \tag{37}$$

and the function $f$ is defined by $f(y) = -y^4 - 4C_0y^3 + (4C_1 - 2)y^2 - 4C_0y - 1$. Depending on the values of $C_0$ and $C_1$, this can lead to doubly periodic solutions expressed in terms of the Jacobi elliptic functions $sn(\xi, k)$, $cn(\xi, k)$ and $dn(\xi, k)$. Due to the presence of the nilpotent even constant $C_0$, the constants $g_2, g_3$ and consequently also the modulus $k$ must be considered in the whole ring of even supernumbers $\Lambda_{\text{even}}$ and cannot be restricted to be real or complex numbers only. The properties of such a generalization of elliptic functions are, as far as we know, not yet fully understood and an understanding going much further than the standard references, e.g. [26], would be required for a full analysis and explicit construction of the solution of the reduced equations. Nevertheless, under the assumption that these functions can be consistently generalized to Grassmann ring-valued parameters, i.e. in $\Lambda_{\text{even}}$, we conjecture that the solution of equation (35) retains the form (36) even for $C_0$ nilpotent.

The odd fields $\lambda, \eta$ are then solutions of the following homogeneous coupled linear ordinary differential equations:

$$\lambda_\sigma - \eta\cosh\alpha = 0, \qquad \varepsilon\eta_\sigma + \lambda\cosh\alpha = 0$$

constrained by the condition $\eta\lambda = C_0$.

When reducing with respect to the subalgebra $\mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$, i.e. considering the equations

$$\begin{aligned} &\varepsilon\alpha_{\sigma\sigma} + \underline{\mu}\eta_\sigma + \sinh\alpha\cosh\alpha + \eta\lambda\sinh\alpha = 0, \\ &\varepsilon\lambda_\sigma - \eta\cosh\alpha = 0, \qquad \eta_\sigma + \underline{\mu}\alpha_\sigma + \lambda\cosh\alpha = 0 \end{aligned} \tag{38}$$

we arrive at the constraint

$$(\eta\lambda)_\sigma = -(\alpha_\sigma)\underline{\mu}\lambda. \tag{39}$$

A general solution of the coupled set of reduced equations (38) is not known. If we assume that both $\eta$ and $\lambda$ are multiples of $\underline{\mu}$ then equation (39) holds trivially and the differential equation for $\alpha$ again becomes

$$\varepsilon\alpha_{\sigma\sigma} + \sinh\alpha\cosh\alpha = 0.$$

Its general solution is therefore the same as in equation (31). Consequently, we arrive at the solution of the supersymmetric sinh-Gordon equation (13)

$$\Phi = \alpha(\sigma) + (\theta_1 - \varepsilon\underline{\mu}t)\eta(\sigma) + \theta_2\lambda(\sigma) + (\theta_1 - \varepsilon\underline{\mu}t)\theta_2\beta(\sigma), \tag{40}$$

where the symmetry variable is $\sigma = \varepsilon x - t + \underline{\mu}t\theta_1$, the function $\alpha$ is defined by equation (31), $\lambda = \underline{\mu} f(\sigma)$, where $f$ is an even function which satisfies the inhomogeneous linear ordinary differential equation:

$$\underline{\mu}\left[f_{\sigma\sigma} - (\tanh\alpha)\alpha_\sigma f_\sigma + \varepsilon(\cosh^2\alpha)f + \varepsilon(\cosh\alpha)\alpha_\sigma\right] = 0, \tag{41}$$

$$\begin{aligned} \eta &= \frac{\varepsilon\underline{\mu}}{\cosh\alpha} f_\sigma, \\ \beta &= -\sinh\alpha. \end{aligned} \tag{42}$$

This represents a travelling simple wave involving $x, t$ modified by the odd variable $\theta_1$. In this case, a solution with $\eta = \lambda = 0$ is not present.

The reduction with respect to the subalgebra $\mathfrak{S}_{12} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\}$ proceeds similarly to the case $\mathfrak{S}_8 = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$. Under similar assumptions on $\lambda, \eta$, i.e. both of them being a multiple of $\underline{\nu}$ we find the solution

$$\Phi = \alpha(\sigma) + \theta_1\eta(\sigma) + (\theta_2 - \underline{\nu}x)\lambda(\sigma) + \theta_1(\theta_2 - \underline{\nu}x)\beta(\sigma), \tag{43}$$

where the symmetry variable is $\sigma = t - \varepsilon x - \underline{\nu}x\theta_2$, the function $\alpha$ is defined by equation (31), $\eta = \underline{\nu} f(\sigma)$, where $f$ is an even function which satisfies the inhomogeneous linear ordinary differential equation (ODE):

$$\underline{\nu}[f_{\sigma\sigma} - (\tanh\alpha)\alpha_\sigma f_\sigma + \varepsilon(\cosh^2\alpha)f - \varepsilon(\cosh\alpha)\alpha_\sigma] = 0, \tag{44}$$

$$\begin{aligned} \lambda &= \frac{\underline{\nu}}{\cosh\alpha} f_\sigma, \\ \beta &= -\sinh\alpha. \end{aligned} \tag{45}$$

The solution represents a travelling simple wave involving $x$, $t$ modified by the odd variable $\theta_2$. We note that the solutions for $\mathfrak{S}_8$ and $\mathfrak{S}_{12}$ are very similar; one can be obtained from the other upon simultaneous interchange of $x$ and $t$, $\theta_1$ and $\theta_2$, $\eta$ and $\lambda$, $\underline{\mu}$ and $\underline{\nu}$ and changes of signs which can be deduced from the difference in $Q_x$ and $Q_t$.

The elliptic function $F$ in equation (31) possesses one real and one purely imaginary period provided that the modulus

$$k = \frac{2\varepsilon}{4C_1 + \varepsilon} \tag{46}$$

is such that $0 < k^2 < 1$. This implies that either $C > \frac{1}{4}$ or $C < -\frac{3}{4}$ when $\varepsilon = 1$ and similarly $C < -\frac{1}{4}$ or $C < -\frac{3}{4}$ when $\varepsilon = -1$.

To sum up our results so far, for subalgebras $\mathfrak{S}_1, \mathfrak{S}_4, \mathfrak{S}_8, \mathfrak{S}_{12}$ we have obtained consistent reduced systems of equations which we were able to solve case by case under some additional assumptions about the form of the solution (where the solution may be implicit or involve a solution of a known linear ODE whose coefficients depend on previously found, i.e. in principle known, functions). The subalgebras $\mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_6, \mathfrak{S}_7, \mathfrak{S}_{10}$ and $\mathfrak{S}_{11}$ allow consistent systems of reduced equations but their solution in each case is the null solution $\Phi = 0$.

Those subalgebras whose invariants possess a non-standard structure, i.e. $\mathfrak{S}_5 = \{\underline{\mu} Q_x\}$, $\mathfrak{S}_9 = \{\underline{\nu} Q_t\}$, $\mathfrak{S}_{13} = \{\underline{\mu} Q_x + \underline{\nu} Q_t\}$, $\mathfrak{S}_{14} = \{P_x + \underline{\mu} Q_x + \underline{\nu} Q_t\}$, $\mathfrak{S}_{15} = \{P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$ and $\mathfrak{S}_{16} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$ are the same as those found for the supersymmetric sine-Gordon equation [18]. Such subalgebras are distinguished by the fact that each of them admits an invariant expressed in terms of an arbitrary function of the superspace variables, multiplied by an odd constant. Such invariants are nilpotent and this causes complications in the computation. This aspect can be illustrated by means of the following example. The subalgebra $\mathfrak{S}_5 = \{\underline{\mu} Q_x\}$ generates the first of the two one-parameter group transformations described in equation (7). Its invariants are $t$, $\theta_2$, $\Phi$ and any quantity of the form

$$\tau = \underline{\mu} f(x, t, \theta_1, \theta_2, \Phi), \tag{47}$$

where $f$ is an arbitrary function which can be either bosonic or fermionic. It is an open question as to whether or not a substitution of these invariants into the supersymmetric sinh-Gordon equation (13) can lead to a reduced system of equations expressible in terms of the invariants. This is clearly not possible for every function $f$. For example, in the case where $\tau = \underline{\mu} x\theta_1$, the system (13) transforms into the equation

$$\underline{\mu} x\theta_2 \mathcal{A}_{t\tau} + \underline{\mu} x \mathcal{A}_{\tau\theta_2} + \sinh \mathcal{A} = 0, \tag{48}$$

for the field

$$\Phi = \mathcal{A}(t, \tau, \theta_2). \tag{49}$$

The presence of the variable $x$ in equation (48) demonstrates that we do not obtain a reduced equation expressible in terms of the invariants. On the other hand, if we would like to perform the reduction with respect to the vector field $Q_x$ (i.e. without $\underline{\mu}$) we immediately find that it is not a subalgebra and we have to reduce with respect to the two-dimensional subalgebra $\{Q_x, P_x\}$. That leads to $\Phi(t, \theta_2)$ and substituting into equation (13), we find the reduction

$$\sinh \Phi = 0,$$

which again allows for only the null solution

$$\Phi = 0. \tag{50}$$

These non-standard invariants arise from the fact that, in the case where we allow both even and odd variables, it is not always possible to find a coordinate transformation which rectifies the vector fields.

It should be noted that non-standard invariants exist also in the case of the $N = 2$ supersymmetric Korteweg–de Vries equation [14]:

$$\begin{aligned} A_t + A_{xxx} - 3a\theta_1\theta_2 A_x A_{xx} - (a+2)\theta_1 A A_{xx\theta_2} - (a+2)(\theta_1\theta_2 A A_{xxx} - \theta_2 A A_{xx\theta_1}) \\ + (2a+1)\theta_2 A_x A_{x\theta_1} + (a+2)\left(A_x A_{\theta_1\theta_2} + A A_{x\theta_1\theta_2}\right) - (2a+1)\theta_1 A_x A_{x\theta_2} \\ - (a-1)\left(\theta_1 A_{\theta_2} A_{xx} - \theta_2 A_{\theta_1} A_{xx} + A_{\theta_1} A_{x\theta_2} - A_{\theta_2} A_{x\theta_1}\right) - 3aA^2 A_x = 0, \end{aligned} \quad (51)$$

where $A(x,t,\theta_1,\theta_2) = u(x,t) + \theta_1\rho^1(x,t) + \theta_2\rho^2(x,t) + \theta_1\theta_2 v(x,t)$ is a bosonic superfield. Here, the Lie symmetry superalgebra $\mathfrak{g}$ of equation (51) is spanned by the generators [14]:

$$\begin{aligned} &\mathcal{C}_1 = \partial_x, \qquad \mathcal{C}_2 = \partial_t, \qquad \mathcal{C}_3 = x\partial_x + 3t\partial_t + \tfrac{1}{2}\theta_1\partial_{\theta_1} + \tfrac{1}{2}\theta_2\partial_{\theta_2} - A\partial_A, \\ &\mathcal{A}_1 = \theta_1\partial_x - \partial_{\theta_1}, \qquad \mathcal{A}_2 = \theta_2\partial_x - \partial_{\theta_2}. \end{aligned} \quad (52)$$

There exist subalgebras of $\mathfrak{g}$ for which the invariants possess a non-standard structure. For example, if we take the subalgebra $\underline{\mu}\mathcal{A}_1 = \{\underline{\mu}\theta_1\partial_x - \underline{\mu}\partial_{\theta_1}\}$, the invariants are $t$, $\theta_2$, $\Phi$ and any quantity of the form

$$\tau = \underline{\mu} f(x,t,\theta_1,\theta_2,\Phi), \quad (53)$$

where $f$ is an arbitrary function which can be either bosonic or fermionic. Other examples include the subalgebra $\underline{\mu}\mathcal{A}_1 + \underline{\nu}\mathcal{A}_2 = \{(\underline{\mu}\theta_1 + \underline{\nu}\theta_2)\partial_x - \underline{\mu}\partial_{\theta_1} - \underline{\nu}\partial_{\theta_2}\}$, for which the non-standard invariant is $\underline{\mu}\underline{\nu} f(x,t,\theta_1,\theta_2,\Phi)$ and the subalgebra $\mathcal{C}_1 - \underline{\mu}\mathcal{A}_1 - \underline{\nu}\mathcal{A}_2 = \{(1 - \underline{\mu}\theta_1 - \underline{\nu}\theta_2)\partial_x + \underline{\mu}\partial_{\theta_1} + \underline{\nu}\partial_{\theta_2}\}$, for which the non-standard invariant is $\underline{\mu}\underline{\nu} f(t,\theta_1,\theta_2,\Phi)$.

## 3. Supersymmetric extension of the polynomial Klein–Gordon equation

Let us now focus on the polynomial form of the Klein–Gordon equation

$$u_{xt} + au + bu^3 + cu^5 = 0, \quad (54)$$

where $a$, $b$ and $c$ are parameters. It is natural to attempt to the supersymmetrize equation (54) in a way similar to what we did for the sinh-Gordon equation (and previously for the sine-Gordon equation), but this turns out to be quite difficult in general. However, the supersymmetrization is straightforward in the case where $a = 3b^2/16c$, where equation (54) becomes

$$u_{xt} + \left(\frac{3b^2}{16c}\right)u + bu^3 + cu^5 = 0. \quad (55)$$

Proceeding in analogy with the case of the sine/sinh-Gordon equation, we replace the bosonic field $u(x,t)$ with the bosonic superfield:

$$\Phi(x,t,\theta_1,\theta_2) = u(x,t) + \theta_1\phi(x,t) + \theta_2\psi(x,t) + \theta_1\theta_2 F(x,t), \quad (56)$$

where $\theta_1$ and $\theta_2$ are independent fermionic variables, $\phi$ and $\psi$ are fermionic-valued fields and $F$ is a bosonic-valued function. We construct the supersymmetric generalization of equation (55) in such a way that it is invariant under the transformations

$$x \to x - \underline{\eta}_1\theta_1, \quad \theta_1 \to \theta_1 + \underline{\eta}_1 \qquad \text{and} \qquad t \to t - \underline{\eta}_2\theta_2, \quad \theta_2 \to \theta_2 + \underline{\eta}_2, \quad (57)$$

which are generated by the supersymmetry operators

$$Q_x = \partial_{\theta_1} - \theta_1\partial_x \qquad \text{and} \qquad Q_t = \partial_{\theta_2} - \theta_2\partial_t. \quad (58)$$

Here, the operator $Q_t$ differs from its counterpart in equation (8) for the case of the sinh-Gordon equation by a sign in the $\partial_t$ term. Again, we define covariant derivatives

$$D_x = \partial_{\theta_1} + \theta_1 \partial_x \qquad \text{and} \qquad D_t = \partial_{\theta_2} + \theta_2 \partial_t, \tag{59}$$

which possess the property that they anticommute with the supersymetry operators (58):

$$\{Q_x, Q_x\} = -2\partial_x, \qquad \{Q_t, Q_t\} = -2\partial_t, \qquad \{Q_x, Q_t\} = \{D_x, D_t\} = 0. \tag{60}$$

The supersymmetric polynomial Klein–Gordon model is given by the following superequation expressed in terms of the superfield (56) and the covariant derivatives (59):

$$D_x D_t \Phi + \sqrt{\frac{3b^2}{16c}}\Phi + \sqrt{\frac{c}{3}}\Phi^3 = 0. \tag{61}$$

This superequation is equivalent to the following system of partial differential equations for the fields $u$, $\phi$, $\psi$ and $F$:

$$F = \sqrt{\frac{3b^2}{16c}}u + \sqrt{\frac{c}{3}}u^3, \tag{62}$$

$$\psi_x + \sqrt{\frac{3b^2}{16c}}\phi + \sqrt{3c}u^2\phi = 0, \tag{63}$$

$$\phi_t - \sqrt{\frac{3b^2}{16c}}\psi - \sqrt{3c}u^2\psi = 0, \tag{64}$$

$$u_{xt} + \sqrt{\frac{3b^2}{16c}}F + \sqrt{3c}u^2 F - 2\sqrt{3c}u\phi\psi = 0, \tag{65}$$

where the first equation fixes the function $F$, so that the fourth equation becomes

$$u_{xt} + \frac{3b^2}{16c}u + bu^3 + cu^5 - 2\sqrt{3c}u\phi\psi = 0. \tag{66}$$

The supersymmetric polynomial Klein–Gordon equation (61) can be rewritten in the convenient form:

$$\theta_1\theta_2\Phi_{xt} - \theta_2\Phi_{t\theta_1} + \theta_1\Phi_{x\theta_2} - \Phi_{\theta_1\theta_2} + \sqrt{\frac{3b^2}{16c}}\Phi + \sqrt{\frac{c}{3}}\Phi^3 = 0. \tag{67}$$

### *3.1. Symmetries of the supersymmetric polynomial Klein–Gordon equation*

We now proceed to determine the Lie superalgebra of symmetries of the supersymmetric polynomial Klein–Gordon equation (67). In the case when $b \neq 0$, we obtain the same symmetry superalgebra as for the sine-Gordon case

$$\begin{aligned} &P_x = \partial_x, \qquad P_t = \partial_t, \qquad L = -2x\partial_x + 2t\partial_t - \theta_1\partial_{\theta_1} + \theta_2\partial_{\theta_2}, \\ &Q_x = -\theta_1\partial_x + \partial_{\theta_1}, \qquad Q_t = -\theta_2\partial_t + \partial_{\theta_2}. \end{aligned} \tag{68}$$

Since the classification of this superalgebra has already been determined in [18], we will not repeat the details here. However, in the specific case when $b = 0$, the Lie symmetry superalgebra is enlarged and it is generated by

$$\begin{aligned} &P_x = \partial_x, \qquad P_t = \partial_t, \qquad L_1 = 2x\partial_x + \theta_1\partial_{\theta_1} - \tfrac{1}{2}\Phi\partial_\Phi, \\ &L_2 = 2t\partial_t + \theta_2\partial_{\theta_2} - \tfrac{1}{2}\Phi\partial_\Phi, \qquad Q_x = -\theta_1\partial_x + \partial_{\theta_1}, \qquad Q_t = -\theta_2\partial_t + \partial_{\theta_2} \end{aligned} \tag{69}$$

**Table 4.** Supercommutation table for the Lie superalgebra $\mathfrak{g}$ spanned by the vector fields (69).

| | $\mathbf{L_1}$ | $\mathbf{L_2}$ | $\mathbf{P_x}$ | $\mathbf{P_t}$ | $\mathbf{Q_x}$ | $\mathbf{Q_t}$ |
|---|---|---|---|---|---|---|
| $\mathbf{L_1}$ | 0 | 0 | $-2P_x$ | 0 | $-Q_x$ | 0 |
| $\mathbf{L_2}$ | 0 | 0 | 0 | $-2P_t$ | 0 | $-Q_t$ |
| $\mathbf{P_x}$ | $2P_x$ | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{P_t}$ | 0 | $2P_t$ | 0 | 0 | 0 | 0 |
| $\mathbf{Q_x}$ | $Q_x$ | 0 | 0 | 0 | $-2P_x$ | 0 |
| $\mathbf{Q_t}$ | 0 | $Q_t$ | 0 | 0 | 0 | $-2P_t$ |

whose commutation (anticommutation in the case of two odd operators) relations are given in table 4.

The superalgebra generated by (69) differs from the one generated by (68) in the sense that it contains two dilations, $L_1$ and $L_2$, rather than a single one. The generator $L$ in superalgebra (68) can be recovered by taking the difference $L_2 - L_1$. In addition, we note that the supersymmetric equation (67) is invariant under the discrete transformation

$$x \to t, \qquad t \to x, \qquad \theta_1 \to \theta_2, \qquad \theta_2 \to -\theta_1. \tag{70}$$

The Lie superalgebra $\mathfrak{g}$ can be decomposed into the direct sum

$$\mathfrak{g} = \{L_1, P_x, Q_x\} \oplus \{L_2, P_t, Q_t\}, \tag{71}$$

and the classification of the one-dimensional subalgebras into conjugacy classes is given as follows

$\mathfrak{g}_1 = \{L_1\}$, $\mathfrak{g}_2 = \{L_2\}$,
$\mathfrak{g}_3 = \{L_1 + aL_2, a \neq 0\}$, $\mathfrak{g}_4 = \{P_x\}$,
$\mathfrak{g}_5 = \{P_t\}$, $\mathfrak{g}_6 = \{P_x + \varepsilon P_t, \varepsilon = \pm 1\}$,
$\mathfrak{g}_7 = \{\underline{\mu} Q_x\}$, $\mathfrak{g}_8 = \{P_x + \underline{\mu} Q_x\}$,
$\mathfrak{g}_9 = \{P_t + \underline{\mu} Q_x\}$, $\mathfrak{g}_{10} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x, \varepsilon = \pm 1\}$,
$\mathfrak{g}_{11} = \{\underline{\nu} Q_t\}$, $\mathfrak{g}_{12} = \{P_x + \underline{\nu} Q_t\}$,
$\mathfrak{g}_{13} = \{P_t + \underline{\nu} Q_t\}$, $\mathfrak{g}_{14} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t, \varepsilon = \pm 1\}$,
$\mathfrak{g}_{15} = \{\underline{\mu} Q_x + \underline{\nu} Q_t\}$, $\mathfrak{g}_{16} = \{P_x + \underline{\mu} Q_x + \underline{\nu} Q_t\}$,
$\mathfrak{g}_{17} = \{P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\}$, $\mathfrak{g}_{18} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t, \varepsilon = \pm 1\}$,
$\mathfrak{g}_{19} = \{L_1 + \varepsilon P_t + \underline{\nu} Q_t, \varepsilon = \pm 1\}$, $\mathfrak{g}_{20} = \{L_2 + \varepsilon P_x + \underline{\mu} Q_x, \varepsilon = \pm 1\}$,
$\mathfrak{g}_{21} = \{L_1 + \varepsilon P_t, \varepsilon = \pm 1\}$, $\mathfrak{g}_{22} = \{L_1 + \underline{\nu} Q_t\}$,
$\mathfrak{g}_{23} = \{L_2 + \varepsilon P_x, \varepsilon = \pm 1\}$, $\mathfrak{g}_{24} = \{L_2 + \underline{\mu} Q_x\}$.

For some values of parameters the algebras in a given class may become isomorphic, in the same fashion as in list (21).

Many members of the subalgebra list can be transformed into another element of the list by applying the discrete transformation (70). Specifically, we obtain the pairs $\mathfrak{g}_1$ and $\mathfrak{g}_2$, $\mathfrak{g}_4$ and $\mathfrak{g}_5$, $\mathfrak{g}_7$ and $\mathfrak{g}_{11}$, $\mathfrak{g}_8$ and $\mathfrak{g}_{13}$, $\mathfrak{g}_9$ and $\mathfrak{g}_{12}$, $\mathfrak{g}_{10}$ and $\mathfrak{g}_{14}$, $\mathfrak{g}_{16}$ and $\mathfrak{g}_{17}$, $\mathfrak{g}_{19}$ and $\mathfrak{g}_{20}$, $\mathfrak{g}_{21}$ and $\mathfrak{g}_{23}$, $\mathfrak{g}_{22}$ and $\mathfrak{g}_{24}$. The reduced solutions for these pairs are also related by transformation (70), as will become obvious below.

### *3.2. Invariant solutions of the supersymmetric polynomial Klein–Gordon equation*

We now make use of the symmetry reduction method in order to obtain invariant solutions of the supersymmetric polynomial Klein–Gordon equation for the case $b = 0$. The invariants

**Table 5.** Invariants and change of variable for subalgebras of the Lie superalgebra $\mathfrak{g}$ spanned by the vector fields (69).

| Subalgebra | Invariants | Superfield |
|---|---|---|
| $\mathfrak{g}_1 = \{L_1\}$ | $\sigma = t,\ \tau_1 = x^{-1/2}\theta_1,$ $\tau_2 = \theta_2,\ \mathcal{A} = x^{1/4}\Phi$ | $\Phi = x^{-1/4}\mathcal{A}(t, \tau_1, \theta_2)$, where $\mathcal{A} = \alpha(t) + \tau_1\eta(t) + \theta_2\lambda(t) + \tau_1\theta_2\beta(t)$ |
| $\mathfrak{g}_2 = \{L_2\}$ | $\sigma = x,\ \tau_1 = \theta_1,$ $\tau_2 = t^{-1/2}\theta_2,\ \mathcal{A} = t^{1/4}\Phi$ | $\Phi = t^{-1/4}\mathcal{A}(x, \theta_1, \tau_2)$, where $\mathcal{A} = \alpha(x) + \theta_1\eta(x) + \tau_2\lambda(x) + \theta_1\tau_2\beta(x)$ |
| $\mathfrak{g}_3 = \{L_1 + aL_2\}$ | $\sigma = t^{-\frac{1}{a}}x,\ \tau_1 = t^{-\frac{1}{2a}}\theta_1,$ $\tau_2 = t^{-\frac{1}{2}}\theta_2,\ \mathcal{A} = t^{\frac{a+1}{4a}}\Phi$ | $\Phi = t^{-\frac{a+1}{4a}}\mathcal{A}(\sigma, \tau_1, \tau_2)$, where $\mathcal{A} = \alpha(\sigma) + \tau_1\eta(\sigma) + \tau_2\lambda(\sigma) + \tau_1\tau_2\beta(\sigma)$ |
| $\mathfrak{g}_4 = \{P_x\}$ | $t, \theta_1, \theta_2, \Phi$ | $\Phi = \alpha(t) + \theta_1\eta(t) + \theta_2\lambda(t) + \theta_1\theta_2\beta(t)$ |
| $\mathfrak{g}_5 = \{P_t\}$ | $x, \theta_1, \theta_2, \Phi$ | $\Phi = \alpha(x) + \theta_1\eta(x) + \theta_2\lambda(x) + \theta_1\theta_2\beta(x)$ |
| $\mathfrak{g}_6 = \{P_x + \varepsilon P_t\}$ | $\sigma = x - \varepsilon t, \theta_1, \theta_2, \Phi$ | $\Phi = \alpha(\sigma) + \theta_1\eta(\sigma) + \theta_2\lambda(\sigma) + \theta_1\theta_2\beta(\sigma)$ |
| $\mathfrak{g}_8 = \{P_x + \underline{\mu} Q_x\}$ | $t, \tau_1 = \theta_1 - \underline{\mu} x, \theta_2, \Phi$ | $\Phi = \alpha(t) + \tau_1\eta(t) + \theta_2\lambda(t) + \tau_1\theta_2\beta(t)$ |
| $\mathfrak{g}_9 = \{P_t + \underline{\mu} Q_x\}$ | $\sigma = x + \underline{\mu}\theta_1 t,$ $\tau_1 = \theta_1 - \underline{\mu} t, \theta_2, \Phi$ | $\Phi = \alpha(\sigma) + \tau_1\eta(\sigma) + \theta_2\lambda(\sigma) + \tau_1\theta_2\beta(\sigma)$ |
| $\mathfrak{g}_{10} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$ | $\sigma = \varepsilon x - t + \underline{\mu} t\theta_1,$ $\tau_1 = \theta_1 - \varepsilon\underline{\mu} t, \theta_2, \Phi$ | $\Phi = \alpha(\sigma) + \tau_1\eta(\sigma) + \theta_2\lambda(\sigma) + \tau_1\theta_2\beta(\sigma)$ |
| $\mathfrak{g}_{12} = \{P_x + \underline{\nu} Q_t\}$ | $\sigma = t + \underline{\nu}\theta_2 x,$ $\theta_1, \tau_2 = \theta_2 - \underline{\nu} x, \Phi$ | $\Phi = \alpha(\sigma) + \theta_1\eta(\sigma) + \tau_2\lambda(\sigma) + \theta_1\tau_2\beta(\sigma)$ |
| $\mathfrak{g}_{13} = \{P_t + \underline{\nu} Q_t\}$ | $x, \theta_1, \tau_2 = \theta_2 - \underline{\nu} t, \Phi$ | $\Phi = \alpha(x) + \theta_1\eta(x) + \tau_2\lambda(x) + \theta_1\tau_2\beta(x)$ |
| $\mathfrak{g}_{14} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\}$ | $\sigma = t - \varepsilon x + \underline{\nu} x\theta_2,$ $\theta_1, \tau_2 = \theta_2 - \underline{\nu} x, \Phi$ | $\Phi = \alpha(\sigma) + \theta_1\eta(\sigma) + \tau_2\lambda(\sigma) + \theta_1\tau_2\beta(\sigma)$ |
| $\mathfrak{g}_{19} = \{L_1 + \varepsilon P_t + \underline{\nu} Q_t\}$ | $\sigma = t + \frac{1}{2}\underline{\nu}\theta_2 \ln x - \frac{1}{2}\varepsilon \ln x,$ $\tau_1 = x^{-1/2}\theta_1,$ $\tau_2 = \theta_2 - \frac{1}{2}\underline{\nu} \ln x,\ \mathcal{A} = x^{1/4}\Phi$ | $\Phi = x^{-1/4}\mathcal{A}(\sigma, \tau_1, \tau_2)$, where $\mathcal{A} = \alpha(\sigma) + \tau_1\eta(\sigma) + \tau_2\lambda(\sigma) + \tau_1\tau_2\beta(\sigma)$ |
| $\mathfrak{g}_{20} = \{L_2 + \varepsilon P_x + \underline{\mu} Q_x\}$ | $\sigma = x + \frac{1}{2}\underline{\mu}\theta_1 \ln t - \frac{1}{2}\varepsilon \ln t,$ $\tau_1 = \theta_1 - \frac{1}{2}\underline{\mu} \ln t,$ $\tau_2 = t^{-1/2}\theta_2,\ \mathcal{A} = t^{1/4}\Phi$ | $\Phi = t^{-1/4}\mathcal{A}(\sigma, \tau_1, \tau_2)$, where $\mathcal{A} = \alpha(\sigma) + \tau_1\eta(\sigma) + \tau_2\lambda(\sigma) + \tau_1\tau_2\beta(\sigma)$ |
| $\mathfrak{g}_{21} = \{L_1 + \varepsilon P_t\}$ | $\sigma = t - \frac{1}{2}\varepsilon \ln x,\ \tau_1 = x^{-1/2}\theta_1,$ $\tau_2 = \theta_2,\ \mathcal{A} = x^{1/4}\Phi$ | $\Phi = x^{-1/4}\mathcal{A}(\sigma, \tau_1, \theta_2)$, where $\mathcal{A} = \alpha(\sigma) + \tau_1\eta(\sigma) + \theta_2\lambda(\sigma) + \tau_1\theta_2\beta(\sigma)$ |
| $\mathfrak{g}_{22} = \{L_1 + \underline{\nu} Q_t\}$ | $\sigma = t + \frac{1}{2}\underline{\nu}\theta_2 \ln x,\ \tau_1 = x^{-1/2}\theta_1,$ $\tau_2 = \theta_2 - \frac{1}{2}\underline{\nu} \ln x,\ \mathcal{A} = x^{1/4}\Phi$ | $\Phi = x^{-1/4}\mathcal{A}(\sigma, \tau_1, \tau_2)$, where $\mathcal{A} = \alpha(\sigma) + \tau_1\eta(\sigma) + \tau_2\lambda(\sigma) + \tau_1\tau_2\beta(\sigma)$ |
| $\mathfrak{g}_{23} = \{L_2 + \varepsilon P_x\}$ | $\sigma = x - \frac{1}{2}\varepsilon \ln t,\ \tau_1 = \theta_1,$ $\tau_2 = t^{-1/2}\theta_2,\ \mathcal{A} = t^{1/4}\Phi$ | $\Phi = t^{-1/4}\mathcal{A}(\sigma, \theta_1, \tau_2)$, where $\mathcal{A} = \alpha(\sigma) + \theta_1\eta(\sigma) + \tau_2\lambda(\sigma) + \theta_1\tau_2\beta(\sigma)$ |
| $\mathfrak{g}_{24} = \{L_2 + \underline{\mu} Q_x\}$ | $\sigma = x + \frac{1}{2}\underline{\mu}\theta_1 \ln t,\ \tau_1 = \theta_1 - \frac{1}{2}\underline{\mu} \ln t,$ $\tau_2 = t^{-1/2}\theta_2,\ \mathcal{A} = t^{1/4}\Phi$ | $\Phi = t^{-1/4}\mathcal{A}(\sigma, \tau_1, \tau_2)$, where $\mathcal{A} = \alpha(\sigma) + \tau_1\eta(\sigma) + \tau_2\lambda(\sigma) + \tau_1\tau_2\beta(\sigma)$ |

and change of variable corresponding to each of the one-dimensional subalgebras, as well as the corresponding reduced systems, are given in tables 5 and 6.

**Table 6.** Reduced equations obtained for subalgebras of the Lie superalgebra $\mathfrak{g}$ spanned by the vector fields (69).

| Subalgebra | Reduced equations |
|---|---|
| $\mathfrak{g}_1 = \{L_1\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda - 4\sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_t - \sqrt{3c}\alpha^2\lambda = 0,\ \alpha_t - 4\sqrt{3c}\alpha^2\beta + 8\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_2 = \{L_2\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_x + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta + 4\sqrt{3c}\alpha^2\lambda = 0,\ \alpha_x - 4\sqrt{3c}\alpha^2\beta + 8\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_3 = \{L_1 + aL_2\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_\sigma + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\frac{a+3}{4a}\eta + \frac{1}{a}\sigma\eta_\sigma + \sqrt{3c}\alpha^2\lambda = 0,\ \frac{1}{a}\sigma\alpha_{\sigma\sigma} + \frac{a+5}{4a}\alpha_\sigma - \sqrt{3c}\alpha^2\beta + 2\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_4 = \{P_x\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_t - \sqrt{3c}\alpha^2\lambda = 0,\ \alpha^2\beta - 2\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_5 = \{P_t\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_x + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\sqrt{3c}\alpha^2\lambda = 0,\ \alpha^2\beta - 2\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_6 = \{P_x + \varepsilon P_t\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_\sigma + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_\sigma + \sqrt{3c}\varepsilon\alpha^2\lambda = 0,\ \alpha_{\sigma\sigma} - \sqrt{3c}\varepsilon\alpha^2\beta + 2\sqrt{3c}\varepsilon\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_8 = \{P_x + \underline{\mu}Q_x\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \underline{\mu}\beta + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_t - \sqrt{3c}\alpha^2\lambda = 0,\ \underline{\mu}\eta_t - \sqrt{3c}\alpha^2\beta + 2\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_9 = \{P_t + \underline{\mu}Q_x\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_\sigma + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\underline{\mu}\alpha_\sigma + \sqrt{3c}\alpha^2\lambda = 0,\ \underline{\mu}\eta_\sigma - \sqrt{3c}\alpha^2\beta + 2\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{10} = \{P_x + \varepsilon P_t + \underline{\mu}Q_x\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_\sigma + \sqrt{3c}\varepsilon\alpha^2\eta = 0,$ <br> $\eta_\sigma + \underline{\mu}\alpha_\sigma + \sqrt{3c}\alpha^2\lambda = 0,\ \varepsilon\alpha_{\sigma\sigma} + \underline{\mu}\eta_\sigma - \sqrt{3c}\alpha^2\beta + 2\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{12} = \{P_x + \underline{\nu}Q_t\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \underline{\nu}\alpha_\sigma - \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_\sigma - \sqrt{3c}\alpha^2\lambda = 0,\ \underline{\nu}\lambda_\sigma - \sqrt{3c}\alpha^2\beta + 2\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{13} = \{P_t + \underline{\nu}Q_t\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_x + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\underline{\nu}\beta + \sqrt{3c}\alpha^2\lambda = 0,\ \underline{\nu}\lambda_x - \sqrt{3c}\alpha^2\beta + 2\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{14} = \{P_x + \varepsilon P_t + \underline{\nu}Q_t\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \underline{\nu}\alpha_\sigma + \varepsilon\lambda_\sigma - \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_\sigma - \sqrt{3c}\alpha^2\lambda = 0,\ \varepsilon\alpha_{\sigma\sigma} + \underline{\nu}\lambda_\sigma - \sqrt{3c}\alpha^2\beta + 2\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{19} = \{L_1 + \varepsilon P_t + \underline{\nu}Q_t\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda + 2\varepsilon\lambda_\sigma + 2\underline{\nu}\alpha_\sigma - 4\sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_\sigma - \sqrt{3c}\alpha^2\lambda = 0,\ \alpha_{\sigma\sigma} + \frac{1}{2}\varepsilon\alpha_\sigma + \varepsilon\underline{\nu}\lambda_\sigma - 2\sqrt{3c}\varepsilon\alpha^2\beta + 4\sqrt{3c}\varepsilon\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{20} = \{L_2 + \varepsilon P_x + \underline{\mu}Q_x\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_\sigma + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta + 2\underline{\mu}\alpha_\sigma + 2\varepsilon\eta_\sigma + 4\sqrt{3c}\alpha^2\lambda = 0,$ <br> $\alpha_{\sigma\sigma} + \frac{1}{2}\varepsilon\alpha_\sigma + \varepsilon\underline{\mu}\eta_\sigma - 2\sqrt{3c}\varepsilon\alpha^2\beta + 4\sqrt{3c}\varepsilon\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{21} = \{L_1 + \varepsilon P_t\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda + 2\varepsilon\lambda_\sigma - 4\sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_\sigma - \sqrt{3c}\alpha^2\lambda = 0,\ \alpha_\sigma + 2\varepsilon\alpha_{\sigma\sigma} - 4\sqrt{3c}\alpha^2\beta + 8\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{22} = \{L_1 + \underline{\nu}Q_t\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda + 2\underline{\nu}\alpha_\sigma - 4\sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta_\sigma - \sqrt{3c}\alpha^2\lambda = 0,\ \alpha_\sigma + 2\underline{\nu}\lambda_\sigma - 4\sqrt{3c}\alpha^2\beta + 8\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{23} = \{L_2 + \varepsilon P_x\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_\sigma + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta + 2\varepsilon\eta_\sigma + 4\sqrt{3c}\alpha^2\lambda = 0,\ \alpha_\sigma + 2\varepsilon\alpha_{\sigma\sigma} - 4\sqrt{3c}\alpha^2\beta + 8\sqrt{3c}\alpha\eta\lambda = 0$ |
| $\mathfrak{g}_{24} = \{L_2 + \underline{\mu}Q_x\}$ | $\beta - \sqrt{\frac{c}{3}}\alpha^3 = 0,\ \lambda_\sigma + \sqrt{3c}\alpha^2\eta = 0,$ <br> $\eta + 2\underline{\mu}\alpha_\sigma + 4\sqrt{3c}\alpha^2\lambda = 0,\ \alpha_\sigma + 2\underline{\mu}\eta_\sigma - 4\sqrt{3c}\alpha^2\beta + 8\sqrt{3c}\alpha\eta\lambda = 0$ |

In addition to the invariants listed below, we obtain the following non-standard invariants for the following one-dimensional subalgebras:

$$\begin{aligned}
\mathfrak{g}_7 &= \{\underline{\mu} Q_x\} : \underline{\mu} f(x,t,\theta_1,\theta_2,\Phi),\\
\mathfrak{g}_{11} &= \{\underline{\nu} Q_t\} : \underline{\nu} f(x,t,\theta_1,\theta_2,\Phi),\\
\mathfrak{g}_{15} &= \{\underline{\mu} Q_x + \underline{\nu} Q_t\} : \underline{\mu}\underline{\nu} f(x,t,\theta_1,\theta_2,\Phi),\\
\mathfrak{g}_{16} &= \{P_x + \underline{\mu} Q_x + \underline{\nu} Q_t\} : \underline{\mu}\underline{\nu} f(t,\theta_1,\theta_2,\Phi),\\
\mathfrak{g}_{17} &= \{P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\} : \underline{\mu}\underline{\nu} f(x,\theta_1,\theta_2,\Phi),\\
\mathfrak{g}_{18} &= \{P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t\} : \underline{\mu}\underline{\nu} f(\theta_1,\theta_2,\Phi).
\end{aligned} \tag{72}$$

These subalgebras do not lead to standard reductions or invariant solutions. However, for each of the other one-dimensional subalgebras (i.e. the ones listed in tables 5 and 6), we discuss the reduction and the corresponding solutions (whenever possible).

For the subalgebra $\mathfrak{g}_1 = \{L_1\}$ the solution to the reduced equations is

$$\alpha(t) = \frac{\varepsilon}{(-16ct+c_0)^{1/4}}, \tag{73}$$

$$\eta(t) = \frac{\underline{\gamma}}{(-16ct+c_0)^{3/4}}, \tag{73}$$

$$\lambda(t) = \frac{4\sqrt{3c}\underline{\gamma}}{(-16ct+c_0)^{5/4}}, \tag{73}$$

$$\beta(t) = \frac{\sqrt{3c}\varepsilon}{3(-16ct+c_0)^{3/4}}, \tag{73}$$

where $\varepsilon^2 = 1$, $c_0$ is a bosonic constant and $\underline{\gamma}$ is a fermionic constant. This leads to the following solution of the supersymmetric polynomial Klein–Gordon equation:

$$\begin{aligned}
\Phi(x,t,\theta_1,\theta_2) = {} & \frac{\varepsilon}{x^{1/4}(-16ct+c_0)^{1/4}} + \frac{\theta_1\underline{\gamma}}{x^{3/4}(-16ct+c_0)^{3/4}} + \frac{4\sqrt{3c}\theta_2\underline{\gamma}}{x^{1/4}(-16ct+c_0)^{5/4}}\\
& + \frac{\sqrt{3c}\varepsilon\theta_1\theta_2}{3x^{3/4}(-16ct+c_0)^{3/4}}.
\end{aligned} \tag{74}$$

This solution decreases monotonically as $x$ increases, and it has poles at $x = 0$ and $t = c_0/(16c)$.

For the subalgebra $\mathfrak{g}_2 = \{L_2\}$ the solution to the reduced equations is

$$\begin{aligned}
\alpha(x) &= \frac{\varepsilon}{(-16cx+c_0)^{1/4}},\\
\eta(x) &= -\frac{4\sqrt{3c}\underline{\gamma}}{(-16ct+c_0)^{5/4}},\\
\lambda(x) &= \frac{\underline{\gamma}}{(-16ct+c_0)^{3/4}},\\
\beta(x) &= \frac{\sqrt{3c}\varepsilon}{3(-16cx+c_0)^{3/4}},
\end{aligned} \tag{75}$$

where $\varepsilon^2 = 1$, $c_0$ is a bosonic constant and $\underline{\gamma}$ is a fermionic constant. This leads to the following solution of the supersymmetric polynomial Klein–Gordon equation:

$$\begin{aligned}
\Phi(x,t,\theta_1,\theta_2) = {} & \frac{\varepsilon}{t^{1/4}(-16cx+c_0)^{1/4}} - \frac{4\sqrt{3c}\theta_1\underline{\gamma}}{t^{1/4}(-16cx+c_0)^{5/4}} + \frac{\theta_2\underline{\gamma}}{t^{3/4}(-16cx+c_0)^{3/4}}\\
& + \frac{\sqrt{3c}\varepsilon\theta_1\theta_2}{3t^{3/4}(-16cx+c_0)^{3/4}}.
\end{aligned} \tag{76}$$

This solution decreases with time, and has poles at $x = c_0/(16c)$ and $t = 0$.

For the subalgebra $\mathfrak{g}_3 = \{L_1 + aL_2\}$, the value of $\sigma^{\left(\frac{a+3}{4}\right)}\eta\lambda$ is a nilpotent bosonic constant which we call $C_1$. The equation for $\alpha$ then decouples

$$\frac{1}{a}\sigma\alpha_{\sigma\sigma} + \frac{a+5}{4a}\alpha_\sigma - c\alpha^5 + 2\sqrt{3c}\alpha C_1\sigma^{\left(-\frac{a+3}{4}\right)} = 0. \tag{77}$$

The remaining equations reduce to

$$4\sigma\lambda_{\sigma\sigma}\alpha + (a+3)\alpha\lambda_\sigma - 8\sigma\lambda_\sigma\alpha_\sigma - 12ac\alpha^5\lambda = 0 \tag{78}$$

together with the constraint

$$\sqrt{3}\sigma^{\frac{a+3}{4}}\lambda_\sigma\lambda = -3\sqrt{c}\alpha^2 C_1. \tag{79}$$

The function $\eta$ is then expressed as

$$\eta = -\frac{1}{\sqrt{3c}\alpha^2}\lambda_\sigma.$$

Solutions of the reduced system of equations (77)–(79) are not known.

For the subalgebra $\mathfrak{g}_4 = \{P_x\}$ the solution to the reduced equations is either

$$\alpha = 0, \qquad \eta = \underline{\eta_0} \text{ is a constant}, \qquad \lambda(t) \text{ is arbitrary}, \qquad \beta = 0, \tag{80}$$

or

$$\alpha \text{ is a nilpotent bosonic}, \qquad \eta = \underline{\eta_0} \text{ is a constant}, \qquad \beta = 0, \tag{81}$$

and $\alpha$ and $\lambda$ are subject to the constraint $\alpha\eta\lambda = 0$. This leads to the following solution of the supersymmetric polynomial Klein–Gordon equation:

$$\Phi(x, t, \theta_1, \theta_2) = \theta_1\underline{\eta_0} + \theta_2\lambda(t), \tag{82}$$

or

$$\Phi(x, t, \theta_1, \theta_2) = \alpha(t) + \theta_1\underline{\eta_0} + \theta_2\lambda(t), \tag{83}$$

where $\alpha(t)$ and $\lambda(t)$ are subject to the constraint $\alpha\eta\lambda = 0$. The arbitrary functions within the solution allow us to consider waves of various shapes, including bumps, kinks and elliptic solutions, depending on the initial conditions.

For the subalgebra $\mathfrak{g}_5 = \{P_t\}$ the solution to the reduced equations is either

$$\alpha = 0, \qquad \eta(t) \text{ is arbitrary}, \qquad \lambda = \underline{\lambda_0} \text{ is a constant}, \qquad \beta = 0, \tag{84}$$

or

$$\alpha \text{ is a nilpotent bosonic}, \qquad \lambda = \underline{\lambda_0} \text{ is a constant}, \qquad \beta = 0, \tag{85}$$

and $\alpha$ and $\eta$ are subject to the constraint $\alpha\eta\lambda = 0$. This leads to the following solution of the supersymmetric polynomial Klein–Gordon equation:

$$\Phi(x, t, \theta_1, \theta_2) = \theta_1\eta(t) + \theta_2\underline{\lambda_0}, \tag{86}$$

or

$$\Phi(x, t, \theta_1, \theta_2) = \alpha(t) + \theta_1\eta(t) + \theta_2\underline{\lambda_0}, \tag{87}$$

where $\alpha(t)$ and $\eta(t)$ are subject to the constraint $\alpha\eta\lambda = 0$. Again, the arbitrary functions allow for a certain freedom in the shape of the solution.

For the subalgebra $\mathfrak{g}_6 = \{P_x + \varepsilon P_t\}$, the reduced equations imply the condition $\eta\lambda = -C_1$ and $\alpha$ is determined from the quadrature

$$\int \frac{\mathrm{d}\alpha}{\left(\frac{\varepsilon c}{3}\alpha^6 + 4\sqrt{3c}C_1\alpha^2 + \alpha_0\right)^{1/2}} = \sigma - \sigma_0. \tag{88}$$

Next, $\lambda$ is, at least formally, determined through the equation

$$\left(\frac{2\alpha_\sigma}{\sqrt{3c}\alpha^3}\right)\lambda_\sigma - \left(\frac{1}{\sqrt{3c}\alpha^2}\right)\lambda_{\sigma\sigma} + \varepsilon\sqrt{3c}\lambda = 0, \tag{89}$$

and $\eta$ and $\beta$ are then easily found in terms of $\alpha$ and $\lambda$. The integration constant for equations (88) and (89) must be chosen so that the constraint $\eta\lambda = -C_1$ is satisfied.

Subalgebra $\mathfrak{g}_8 = \{P_x + \underline{\mu} Q_x\}$ produces the same results as for subalgebra $\mathfrak{g}_4$.

For the subalgebra $\mathfrak{g}_9 = \{P_t + \underline{\mu} Q_x\}$, we get either

$$\alpha = 0, \qquad \underline{\mu}\eta \text{ is a constant}, \qquad \lambda = \underline{\lambda_0} \text{ is a constant}, \qquad \beta = 0, \tag{90}$$

or

$$\alpha \text{ is a nilpotent bosonic}, \qquad \lambda = \underline{\lambda_0} \text{ is a constant}, \qquad \beta = 0, \tag{91}$$

and $\eta$ is determined from the equation

$$\underline{\mu}\eta_\sigma + 2\sqrt{3c}\alpha\eta\lambda = 0. \tag{92}$$

For the subalgebra $\mathfrak{g}_{10} = \{P_x + \varepsilon P_t + \underline{\mu} Q_x\}$, the functions $\alpha$ and $\lambda$ must satisfy the coupled system of nonlinear second-order ordinary differential equations

$$\varepsilon\alpha\alpha_{\sigma\sigma} - c\alpha^6 + 2\varepsilon\lambda\lambda_\sigma - \sqrt{3c}\alpha^3\underline{\mu}\lambda = 0, \tag{93}$$

and

$$\varepsilon\alpha\lambda_{\sigma\sigma} - 3\alpha^5 c\lambda - 2\varepsilon\alpha_\sigma\lambda_\sigma - \sqrt{3c}\alpha^3\alpha_\sigma\underline{\mu} = 0. \tag{94}$$

The functions $\eta$ and $\beta$ are then expressed in terms of $\alpha$, $\lambda$ and their derivatives.

For the subalgebra $\mathfrak{g}_{12} = \{P_x + \underline{\nu} Q_t\}$, we get either

$$\alpha = 0, \qquad \eta = \underline{\eta_0} \text{ is a constant}, \qquad \underline{\nu}\lambda \text{ is a constant}, \qquad \beta = 0, \tag{95}$$

or

$$\alpha \text{ is a nilpotent bosonic}, \qquad \eta = \underline{\eta_0} \text{ is a constant}, \qquad \beta = 0, \tag{96}$$

and $\lambda$ is determined from the equation

$$\underline{\nu}\lambda_\sigma + 2\sqrt{3c}\alpha\eta\lambda = 0. \tag{97}$$

Subalgebra $\mathfrak{g}_{13} = \{P_t + \underline{\nu} Q_t\}$ produces the same results as for subalgebra $\mathfrak{g}_5$.

For the subalgebra $\mathfrak{g}_{14} = \{P_x + \varepsilon P_t + \underline{\nu} Q_t\}$, the functions $\alpha$ and $\eta$ must satisfy the coupled system of nonlinear second-order ordinary differential equations

$$\varepsilon\alpha\alpha_{\sigma\sigma} - c\alpha^6 + 2\eta\eta_\sigma + \sqrt{3c}\varepsilon\alpha^3\underline{\nu}\eta = 0, \tag{98}$$

and

$$\varepsilon\alpha\eta_{\sigma\sigma} - 3\alpha^5 c\eta - 2\varepsilon\alpha_\sigma\eta_\sigma + \sqrt{3c}\alpha^3\alpha_\sigma\underline{\nu} = 0. \tag{99}$$

The functions $\lambda$ and $\beta$ are then expressed in terms of $\alpha$, $\eta$ and their derivatives.

For the subalgebra $\mathfrak{g}_{19} = \{L_1 + \varepsilon P_t + \underline{\nu} Q_t\}$, the functions $\alpha$ and $\eta$ must satisfy the coupled system of nonlinear second-order ordinary differential equations:

$$6\sqrt{c}\alpha^2\alpha_{\sigma\sigma} + 12\sqrt{3}c\alpha^4\underline{\nu}\eta + 3\sqrt{c}\varepsilon\alpha^2\alpha_\sigma - 12c^{3/2}\varepsilon\alpha^7 + 24\sqrt{c}\varepsilon\alpha\eta\eta_\sigma - \sqrt{3}\underline{\nu}\eta_\sigma = 0, \tag{100}$$

and

$$2\varepsilon\alpha\eta_{\sigma\sigma} - 4\varepsilon\alpha_\sigma\eta_\sigma + 2\sqrt{3c}\underline{\nu}\alpha^3\alpha_\sigma - 12c\alpha^5\eta + \alpha\eta_\sigma = 0. \tag{101}$$

The functions $\lambda$ and $\beta$ are then expressed in terms of $\alpha$, $\eta$ and their derivatives.

For the subalgebra $\mathfrak{g}_{20} = \{L_2 + \varepsilon P_x + \underline{\mu} Q_x\}$, the functions $\alpha$ and $\lambda$ must satisfy the coupled system of nonlinear second-order ordinary differential equations (similar to $\mathfrak{g}_{19}$):

$$6\sqrt{c}\alpha^2\alpha_{\sigma\sigma} - 12\sqrt{3}c\alpha^4\underline{\mu}\lambda + 3\sqrt{c}\varepsilon\alpha^2\alpha_\sigma - 12c^{3/2}\varepsilon\alpha^7 + 24\sqrt{c}\varepsilon\alpha\lambda\lambda_\sigma + \sqrt{3}\underline{\mu}\lambda_\sigma = 0, \quad (102)$$

and

$$2\varepsilon\alpha\lambda_{\sigma\sigma} - 4\varepsilon\alpha_\sigma\lambda_\sigma - 2\sqrt{3c}\underline{\mu}\alpha^3\alpha_\sigma - 12c\alpha^5\lambda + \alpha\lambda_\sigma = 0. \quad (103)$$

The functions $\eta$ and $\beta$ are then expressed in terms of $\alpha$, $\lambda$ and their derivatives.

For the subalgebra $\mathfrak{g}_{21} = \{L_1 + \varepsilon P_t\}$, the functions $\alpha$ and $\eta$ are determined from the coupled system of nonlinear second-order ordinary differential equations

$$\alpha_\sigma + 2\varepsilon\alpha_{\sigma\sigma} - 4c\alpha^5 + \frac{8}{\alpha}\eta\eta_\sigma = 0, \quad (104)$$

and

$$\alpha\eta_\sigma + 2\varepsilon\alpha\eta_{\sigma\sigma} - 4\varepsilon\alpha_\sigma\eta_\sigma - 12c\alpha^5\eta = 0. \quad (105)$$

For the specific case where $\eta = \underline{K} f(\sigma)$, where $\underline{K}$ is a fermionic constant and $f$ a bosonic-valued function of $\sigma$, equation (104) becomes

$$\alpha_{\sigma\sigma} + \tfrac{1}{2}\varepsilon\alpha_\sigma - 2\varepsilon c\alpha^5 = 0. \quad (106)$$

In general, this equation does not possess the Painlevé property. If we consider the specific case of the supersymmetric equation (67) where $c = -\frac{1}{8}\varepsilon$ (and $b = 0$), and we make the change of variable $\rho = \frac{1}{2}\varepsilon\sigma$, equation (106) becomes

$$\alpha_{\rho\rho} + \alpha_\rho + \alpha^5 = 0. \quad (107)$$

If we define $\alpha = \gamma(\xi)$, where $\xi = e^\sigma$, then equation (107) becomes

$$\xi\gamma_{\xi\xi} + 2\gamma_\xi + \xi^{-1}\gamma^5 = 0, \quad (108)$$

which is the Emden–Fowler-type equation. In the literature, one can find existence theorems for solutions of such equations [27].

For the subalgebra $\mathfrak{g}_{22} = \{L_1 + \underline{\nu} Q_t\}$, the functions $\alpha$ and $\eta$ are determined from the equations

$$\alpha_\sigma + \frac{2}{\sqrt{3c}}\frac{\underline{\nu}\eta_{\sigma\sigma}}{\alpha^2} - \frac{4}{\sqrt{3c}}\frac{\underline{\nu}\alpha_\sigma\eta_\sigma}{\alpha^3} - 4c\alpha^5 + \frac{8}{\alpha}\eta\eta_\sigma = 0, \quad (109)$$

and

$$\eta_\sigma + 2\sqrt{3c}\underline{\nu}\alpha^2\alpha_\sigma - 12c\alpha^4\eta = 0. \quad (110)$$

In fact, using (110) and its differential consequence, we may equivalently write a simpler equation

$$\alpha_\sigma + 32\sqrt{3c}\alpha\alpha_\sigma\underline{\nu}\eta + 96\sqrt{3}c^{3/2}\alpha^6\underline{\nu}\eta - 4c\alpha^5 = 0 \quad (111)$$

instead of equation (109). Substituting equation (111) back into equation (110) we also simplify it to

$$\eta_\sigma - 12c\alpha^4\eta + 8\sqrt{3}c^{3/2}\underline{\nu}\alpha^7 = 0. \quad (112)$$

For the specific case where $\eta = \underline{\nu} f(\sigma)$, where $f$ a bosonic-valued function of $\sigma$, equation (109) becomes

$$\alpha_\sigma - 4c\alpha^5 = 0, \quad (113)$$

from where we obtain the solution

$$\alpha(\sigma) = \frac{1}{2}\left(-\frac{1}{c(\sigma - \sigma_0)}\right)^{1/4}. \quad (114)$$

Equation (110) then becomes an inhomogeneous linear first-order ordinary differential equation for $f(\sigma)$:

$$\left(f_\sigma + \frac{3}{4}\frac{f}{\sigma-\sigma_0} + \frac{1}{16}\sqrt{3}c^{3/2}\left(-\frac{1}{c\,(\sigma-\sigma_0)}\right)^{7/4}\right)\underline{\nu} = 0. \tag{115}$$

For the subalgebra $\mathfrak{g}_{23} = \{L_2 + \varepsilon P_x\}$, the functions $\alpha$ and $\lambda$ are determined from the equations

$$\alpha_\sigma + 2\varepsilon\alpha_{\sigma\sigma} - 4c\alpha^5 + \frac{8}{\alpha}\lambda\lambda_\sigma = 0, \tag{116}$$

and

$$-\alpha\lambda_\sigma - 2\varepsilon\alpha\lambda_{\sigma\sigma} + 4\varepsilon\alpha_\sigma\lambda_\sigma + 12c\alpha^5\lambda = 0. \tag{117}$$

For the specific case where $\lambda = \underline{K}f(\sigma)$, where $\underline{K}$ is a fermionic constant and $f$ a bosonic-valued function of $\sigma$, equation (116) becomes

$$\alpha_{\sigma\sigma} + \tfrac{1}{2}\varepsilon\alpha_\sigma - 2\varepsilon c\alpha^5 = 0, \tag{118}$$

which is the same as equation (106) for subalgebra $\mathfrak{g}_{21}$ above. Thus, we obtain the same result for the same particular cases.

For the subalgebra $\mathfrak{g}_{24} = \{L_2 + \underline{\mu}Q_x\}$, the functions $\alpha$ and $\lambda$ are determined from the equations

$$\alpha_\sigma - 32\alpha\sqrt{3c}\alpha_\sigma\underline{\mu}\lambda - 96\sqrt{3}c^{3/2}\alpha^6\underline{\mu}\lambda - 4c\alpha^5 = 0 \tag{119}$$

and

$$\lambda_\sigma - 8\sqrt{3}c^{3/2}\alpha^7\underline{\mu} - 12c\alpha^4\lambda = 0. \tag{120}$$

For the specific case where $\lambda = \underline{\mu}f(\sigma)$, and $f$ is a bosonic-valued function of $\sigma$, equation (119) becomes (cf equation (113))

$$\alpha_\sigma - 4c\alpha^5 = 0, \tag{121}$$

from which we obtain $\alpha(\sigma)$ in the form (114). Similarly as above, equation (120) becomes a linear inhomogeneous first-order ODE

$$\left(f_\sigma + \frac{3}{4}\frac{f}{\sigma-\sigma_0} - \frac{1}{16}\sqrt{3}c^{3/2}\left(-\frac{1}{c\,(\sigma-\sigma_0)}\right)^{7/4}\right)\underline{\mu} = 0. \tag{122}$$

## 4. Conclusions

We have determined the Lie superalgebra of symmetries of the supersymmetric sinh-Gordon model and found that it is similar to that of the supersymmetric sine-Gordon equation which we had previously determined. Through the use of the symmetry reduction method we have constructed several new analytic solutions, including doubly periodic solutions in terms of Jacobi elliptic functions. These implicitly defined solutions represent traveling waves defined in terms of $x$ and $t$, modified for certain cases by the fermionic independent variables $\theta_1$ and $\theta_2$. There were fewer classes of nonvanishing invariant solutions for the supersymmetric sinh-Gordon equation than for its supersymmetric sine-Gordon counterpart. This is due to the fact that, in contrast to trigonometric functions (such as sin and cos) hyperbolic functions have very few roots. The solutions of the supersymmetric sinh-Gordon equation can also be of use in determining solutions of the super-Korteweg–de Vries equations due to the link which exists between the two supersymmetric models [28].

A supersymmetric extension of the polynomial Klein–Gordon equation was constructed for specific cases of the constant parameters $a$, $b$ and $c$. When $a = b = 0$, the symmetries are enhanced; namely the Lie symmetry superalgebra was found to contain two (rather than one) dilation operators. The model was also found to be invariant under a parity transformation[4] linking the independent variables $x$ with $t$ and $\theta_1$ with $\theta_2$. A number of interesting solutions were found for this supersymmetric model as well, including rational solutions with poles and solutions expressed in terms of arbitrary functions. In particular, we obtained a solution which decreases over time and another which decreases monotonically with $x$, in both their bosonic and fermionic components. The arbitrary function solutions allow us to consider waves of various shapes, including bumps, kinks, elliptic functions. In addition, we obtain radical solutions with movable poles. Some reductions were found to lead to solutions where the bosonic variables are defined implicitly by Emden–Fowler-type equations, and the fermionic variables are determined in terms of the bosonic ones.

This study was well worth performing since it complements the analysis which had already been done for the supersymmetric sine-Gordon equation. We have obtained interesting new solutions which are distinct from the ones obtained through other methods, such as multi-solitons. We have also found that both the supersymmetric sinh-Gordon equation and the supersymmetric version of the polynomial Klein–Gordon equation admit non-standard invariants.

One open problem is to determine whether all integrable supersymmetric systems possess non-standard invariants in this way. Also, could we apply the group-theoretical methods used in this paper to other integrable equation of mathematical physics? Such equations may include, among others, the supersymmetric Schrödinger equation (motivated by supersymmetric quantum mechanics [15, 29]) and the supersymmetric Sawada–Kotera equation [30]. These will be the subject of future investigations.

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[4] Recall that $x$ and $t$ in our notation have the physical interpretation of the light-cone coordinates due to the presence of $\partial_{xt}u$ in the considered equations (1) and (2).

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# Chapter 4

# Aspects of Poisson–Lie T-dual models

In this chapter we present three papers

[**g**] L. Hlavatý and L. Šnobl, Poisson-Lie T-plurality as canonical transformation, Nucl. Phys. B 768 (2007) 209-218,

[**h**] C. Albertsson, L. Hlavatý and L. Šnobl, On the Poisson-Lie T-plurality of boundary conditions, J. Math. Phys. 49 (2008) 032301,

[**i**] L. Hlavatý and L. Šnobl, Description of D-branes invariant under the Poisson-Lie T-plurality, J. High Energy Phys. 07 (2008) 122.

In the first paper [g] we generalize to Poisson–Lie T–plurality the Sfetsos' proof [88] that Poisson–Lie T–duality can be interpreted as canonical transformation.

In the other two papers we investigate how the boundary conditions for open strings transform under T–plurality. The boundary conditions are expressed using the so–called gluing matrices of [100, 101, 102]. In the paper [h] we derive a formula for transformation of gluing matrices and contemplate consistency requirements that should be imposed on the gluing matrices. In paper [i] we further advance our analysis and provide a definitive answer what should be the constraints imposed on the gluing matrices so that the T–plurality transformations preserve these constraints. Finally, we show that our local, gluing matrix formulation is equivalent to a D–brane picture of C. Klimčík and P. Ševera [83].

A crucial computational tool used in all three papers [g,h,i] is the transformation of on-shell right–invariant fields under Poisson–Lie T–plurality.

The paper [g] was honored by the CTU Rector's Prize for outstanding research accomplishments in 2006.

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NUCLEAR PHYSICS B

# Poisson–Lie T-plurality as canonical transformation

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**Abstract**

We generalize the prescription realizing classical Poisson–Lie T-duality as canonical transformation to Poisson–Lie T-plurality. The key ingredient is the transformation of left-invariant fields under Poisson–Lie T-plurality. Explicit formulae realizing canonical transformation are presented and the preservation of canonical Poisson brackets and Hamiltonian density is shown.


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*Keywords:* Poisson–Lie T-plurality; Sigma models; Canonical transformation

## 1. Introduction

Poisson–Lie T-duality and T-plurality is already quite an old subject. It was introduced in 1995 when Klimčík and Ševera in [1–3] proposed Poisson–Lie T-duality as an approach solving certain problems in T-duality with respect to non-Abelian groups of isometries (especially that the original T-duality worked only in one direction). Already in [1,3] they considered the possibility of what is now called Poisson–Lie T-plurality. This is related to the fact that the Lie algebra of Drinfel'd double may be decomposable into more than one pair of subalgebras whose transposition corresponds to duality. On the other hand, further development (like the explicit formulation of canonical transformation in [4,5]) focused only on Poisson–Lie T-duality and almost no explicit formulae and no examples of genuine Poisson–Lie T-plurality were known until 2002 when von Unge considered T-plurality of conformal quantum sigma models (on one-loop

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level) in [6] and coined the current phrase "Poisson–Lie T-plurality". By that time classifications of Drinfel'd doubles in low dimension, e.g. [7], became available, facilitating construction of more examples and study of their properties (see [8] and references therein).

Gradually, the need arose for generalization of formulae previously derived in the Poisson–Lie T-duality context to the general plurality case. One of these is the formulation of Poisson–Lie T-plurality as canonical transformation (derived for the duality case by Sfetsos in [4,5]).

In this paper we shall derive the explicit canonical formulation of Poisson–Lie T-plurality. As we shall show, the key point is the transformation of the extremal left and right-invariant fields, which can be derived in a direct way, and which will enable us to find the transformation of the canonical variables of the dualizable $\sigma$-models and prove that they really constitute a canonical transformation.

One of possible applications of our results is in the study of the worldsheet boundary conditions. Recently, Poisson–Lie T-duality transformation of worldsheet boundary conditions of the dualizable $\sigma$-models was derived in [9]. The key formulae there were the transformations of left-invariant fields by the Poisson–Lie T-duality obtained from canonical formulation of T-dual $\sigma$-models [4,5]. Using the formulae derived in this paper one can easily generalize the results of [9] to the T-plurality case. Detailed discussion of them shall be the subject of future work.

A note concerning the conventions: We are using in the current paper the conventions introduced in [4,5] in order to be able to compare with results therein. Unfortunately, this notation is not the same as the one used in [1,2] and all our previous papers. The two notations are equivalent upon substitutions $g, l, \ldots \leftrightarrow g^{-1}, l^{-1}, \ldots$ accompanied by the worldsheet parity transformation $x_+ \leftrightarrow x_-$.

## 2. Elements of Poisson–Lie T-plurality

For simplicity we shall consider the $\sigma$-models without spectator fields, i.e. with target manifold isomorphic to the group of generalized isometries. The inclusion of spectators is straightforward, see [4,6]. The classical action of $\sigma$-model without spectators reads

$$S_{\mathcal{E}}[\phi] = \frac{1}{2} \int d^2x\, \partial_+ \phi^\mu \mathcal{E}_{\mu\nu}(\phi) \partial_- \phi^\nu \tag{1}$$

where $\mathcal{E}$ is a tensor on a Lie group $G$ and the functions $\phi^\mu : V \subset \mathbb{R}^2 \to \mathbb{R}$, $\mu = 1, 2, \ldots, \dim G$ are obtained by the composition $\phi^\mu = y^\mu \circ g$ of a map $g : V \subset \mathbb{R}^2 \to G$ and a coordinate map $y$ of a neighborhood $U_g$ of an element $g(x_+, x_-) \in G$. Further on we shall use formulation of the $\sigma$-models in terms of left-invariant fields $g^{-1}\partial_\pm g$. The tensor $\mathcal{E}$ can be written as

$$\mathcal{E}_{\mu\nu} = e^{L\,a}_\mu(g) E_{ab}(g) e^{L\,b}_\nu(g) \tag{2}$$

where

- $e^{L\,a}_\mu$ are components of left-invariant forms (vielbeins) $g^{-1}\,dg = dy^\mu\, e^{L\,a}_\mu(g) T_a$,
- $T_a$ are basis elements of $\mathfrak{g}$, i.e. Lie algebra of $G$,
- $E_{ab}(g)$ are matrix elements of a $G$-dependent bilinear non-degenerate form on $\mathfrak{g}$ in the basis $\{T_a\}$.

The action of the $\sigma$-model then reads[1]

$$S[g] = \frac{1}{2}\int d^2x\, L_+(g)\cdot E(g)\cdot L_-^t(g) \tag{3}$$

where

$$L_\pm(g)^a \equiv \left(g^{-1}\partial_\pm g\right)^a = \partial_\pm \phi^\mu e_\mu^{L\,a}(g), \quad g^{-1}\partial_\pm g = L_\pm(g)\cdot T. \tag{4}$$

The $\sigma$-models that can be transformed by the Poisson–Lie T-plurality are formulated (see [1, 2]) on a Drinfel'd double $D \equiv (G|\tilde{G})$—a Lie group whose Lie algebra $\mathfrak{d}$ admits a decomposition $\mathfrak{d} = \mathfrak{g} \dotplus \tilde{\mathfrak{g}}$ into a pair of subalgebras maximally isotropic with respect to a symmetric ad-invariant non-degenerate bilinear form $\langle\,.\,,\,.\,\rangle$. The matrices $E(g)$ for such $\sigma$-models are of the form

$$E(g) = \left(E_0^{-1} + \Pi(g)\right)^{-1}, \quad \Pi(g) = b^t(g)\cdot a(g) = -\Pi(g)^t, \tag{5}$$

where $E_0$ is a constant matrix and $a(g)$, $b(g)$ are submatrices of the adjoint representation of the subgroup $G$ on the Lie algebra $\mathfrak{d}$ satisfying

$$gTg^{-1} \equiv \mathrm{Ad}(g) \triangleright T = a^{-1}(g)\cdot T, \qquad g\tilde{T}g^{-1} \equiv \mathrm{Ad}(g)\triangleright \tilde{T} = b^t(g)\cdot T + a^t(g)\cdot\tilde{T}, \tag{6}$$

and $\tilde{T}^a$ are elements of dual basis in the dual algebra $\tilde{\mathfrak{g}}$, i.e. $\langle T_a, \tilde{T}^b\rangle = \delta_a^b$. The matrix $a(g)$ also relates the left- and right-invariant fields on G

$$\partial_\pm g g^{-1} = R_\pm(g)\cdot T, \qquad L_\pm(g) = R_\pm(g)\cdot a(g). \tag{7}$$

The equations of motion of the dualizable $\sigma$-models can be written as Bianchi identities for the left-invariant fields $\tilde{L}_\pm(\tilde{h})$ on the dual algebra $\tilde{\mathfrak{g}}$

$$\begin{aligned} &\tilde{L}_+(\tilde{h})\cdot\tilde{T} \equiv \tilde{h}^{-1}\partial_+\tilde{h} = L_+(g)\cdot E(g)\cdot a^t(g)\cdot\tilde{T},\\ &\tilde{L}_-(\tilde{h})\cdot\tilde{T} \equiv \tilde{h}^{-1}\partial_-\tilde{h} = -L_-(g)\cdot E^t(g)\cdot a^t(g)\cdot\tilde{T}. \end{aligned} \tag{8}$$

This is a consequence of the fact that the equations of motion of the dualizable $\sigma$-model can be formulated as the equations on the Drinfel'd double [1]

$$\left\langle l^{-1}\partial_\pm l, \mathcal{E}^\mp\right\rangle = 0, \tag{9}$$

where $l = \tilde{h}g \in D$, $\tilde{h}\in\tilde{G}$, $g\in G$ and

$$\mathcal{E}^+ = \mathrm{span}(T + E_0\cdot\tilde{T}), \qquad \mathcal{E}^- = \mathrm{span}\left(T - E_0^t\cdot\tilde{T}\right)$$

are two orthogonal subspaces in $\mathfrak{d}$. (The unique decomposition $l = \tilde{h}g$ on $D$ exists for a general Drinfel'd double only in the vicinity of the group unit. For the so-called perfect Drinfel'd doubles it is defined globally and we shall consider only these. Otherwise all the constructions considered would hold only locally.)

In general there are several decompositions (Manin triples) of a Drinfel'd double. Let $\hat{\mathfrak{g}} \dotplus \bar{\mathfrak{g}}$ be another decomposition of the Lie algebra $\mathfrak{d}$ into maximal isotropic subalgebras. Then another $\sigma$-model can be defined. The dual bases of $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}$, $\bar{\mathfrak{g}}$ are related by the linear transformation

$$\begin{pmatrix} T \\ \tilde{T}\end{pmatrix} = \begin{pmatrix} K & Q \\ R & S\end{pmatrix}\begin{pmatrix}\hat{T}\\ \bar{T}\end{pmatrix}, \tag{10}$$

[1] Central dot means matrix multiplication and we consider $L_\pm$ as a row vector whereas $T$ is a column vector with components $T_a$. $L^t$ denotes transposition. Later on we shall also use the notation $M^{-t} = (M^{-1})^t$ for matrices.

where the matrices $K$, $Q$, $R$, $S$ are chosen in such a way that the structure of the Lie algebra $\mathfrak{d}$

$$[T_a, T_b] = f_{ab}{}^c T_c,$$
$$[\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}{}_c \tilde{T}^c,$$
$$[\tilde{T}^a, T_b] = f_{bc}{}^a \tilde{T}^c - \tilde{f}^{ac}{}_b T_c \tag{11}$$

transforms to the similar one where $T \to \hat{T}$, $\tilde{T} \to \bar{T}$ and the structure constants $f$, $\tilde{f}$ of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are replaced by the structure constants $\hat{f}$, $\bar{f}$ of $\hat{\mathfrak{g}}$ and $\bar{\mathfrak{g}}$. The duality of both bases requires

$$\begin{pmatrix} K & Q \\ R & S \end{pmatrix}^{-1} = \begin{pmatrix} S^t & Q^t \\ R^t & K^t \end{pmatrix}. \tag{12}$$

The other $\sigma$-model is defined analogously to (3)–(5) where

$$\hat{E}(\hat{g}) = \left(\hat{E}_0^{-1} + \hat{\Pi}(\hat{g})\right)^{-1}, \quad \hat{\Pi}(\hat{g}) = \hat{b}^t(\hat{g}) \cdot \hat{a}(\hat{g}) = -\hat{\Pi}(\hat{g})^t, \tag{13}$$

$$\hat{E}_0 = (K + E_0 \cdot R)^{-1} \cdot (Q + E_0 \cdot S) = \left(S^t \cdot E_0 - Q^t\right) \cdot \left(K^t - R^t \cdot E_0\right)^{-1}, \tag{14}$$

and classical solutions of the two $\sigma$-models are related by two possible decompositions of $l \in D$,

$$l = \tilde{h} g = \bar{h} \hat{g}. \tag{15}$$

The explicit examples of solutions of the $\sigma$-models related by the Poisson–Lie T-plurality are given in [10].

## 3. Poisson–Lie transformation of extremal left-invariant fields

As mentioned in the introduction, the formulae for transformation of left-invariant fields evaluated on solutions of equations of motion (hence extremal) by the Poisson–Lie T-duality were found in [9]. We are going to derive the extension of these formulae in an alternative way.

Let us write the left-invariant field $l^{-1}\partial_+ l$ on the Drinfel'd double in terms of $L_+(g)$ and $\tilde{L}_+(\tilde{h})$

$$\begin{aligned} l^{-1}\partial_+ l &= (\tilde{h} g)^{-1}\left(\partial_+(\tilde{h} g)\right) = L_+(g) \cdot T + \tilde{L}_+(\tilde{h}) \cdot g^{-1}\tilde{T} g \\ &= L_+(g) \cdot T + \tilde{L}_+(\tilde{h}) \cdot \left[b(g) \cdot T + a^{-t}(g) \cdot \tilde{T}\right], \end{aligned} \tag{16}$$

where $a(g)$ and $b(g)$ are the matrices introduced in (6).

Using the equations of motion (8) and the expression (5) for $E(g)$ we get

$$\begin{aligned} l^{-1}\partial_+ l &= L_+(g) \cdot T + L_+(g) \cdot E(g) \cdot \left[a^t(g) \cdot b(g) \cdot T + \tilde{T}\right] \\ &= L_+(g) \cdot E(g) \cdot \left[E_0^{-1} \cdot T + \tilde{T}\right]. \end{aligned} \tag{17}$$

On the other hand, from the decomposition $l = \bar{h}\hat{g}$ we find by a similar procedure

$$l^{-1}\partial_+ l = \hat{L}_+(\hat{g}) \cdot \hat{E}(\hat{g}) \cdot \left[\hat{E}_0^{-1} \cdot \hat{T} + \bar{T}\right]. \tag{18}$$

Inserting (10) and (14) into (17) and comparing coefficients of $\hat{T}$ and $\bar{T}$ with those in (18) we obtain the formula for transformation of the left-invariant fields under the Poisson–Lie T-plurality

$$\hat{L}_+(\hat{g}) = L_+(g) \cdot E(g) \cdot \left[S + E_0^{-1} \cdot Q\right] \cdot \hat{E}^{-1}(\hat{g}). \tag{19}$$

In the same way we can derive

$$\hat{L}_-(\hat{g}) = L_-(g) \cdot E^t(g) \cdot \left[S - E_0^{-t} \cdot Q\right] \cdot \hat{E}^{-t}(\hat{g}). \tag{20}$$

This agrees with the formulae obtained in [9] for Poisson–Lie T-duality, i.e. for $Q = R = \mathbf{1}$, $K = S = 0$, $\tilde{L}_\pm(\tilde{g}) = \hat{L}_\pm(\hat{g})$, which in our notation (i.e. $L_\pm$ rows) read

$$\begin{aligned}\tilde{L}_+^t(\tilde{g}) &= \tilde{E}^{-t}(\tilde{g}) \cdot E_0^{-t} \cdot E^t(g) \cdot L_+^t(g),\\ \tilde{L}_-^t(\tilde{g}) &= -\big(\tilde{E}(\tilde{g})\big)^{-1} \cdot E_0^{-1} \cdot E(g) \cdot L_-^t(g).\end{aligned} \tag{21}$$

The transformations of right-invariant extremal fields can be easily obtained from the relation (7).

## 4. Transformation of canonical variables

In the present section we are going to generalize the formulae for canonical transformation obtained in [4,5] for the Poisson–Lie T-duality to the general T-plurality case.

Recall that the time and space coordinates on the worldsheet are $\tau = x_+ + x_-$, $\sigma = x_+ - x_-$, i.e. $\partial_\tau = \frac{1}{2}(\partial_+ + \partial_-)$, $\partial_\sigma = \frac{1}{2}(\partial_+ - \partial_-)$. The canonical momentum is defined by

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \phi^\mu)} = \frac{1}{2}\big(\mathcal{E}_{\mu\nu}\partial_-\phi^\nu + \mathcal{E}_{\nu\mu}\partial_+\phi^\nu\big). \tag{22}$$

It turns out, similarly as above, to be advantageous to use a momentum in local frame, defined as

$$\mathcal{P}_a = v_a^{L\mu}(g)\mathcal{P}_\mu, \tag{23}$$

where $v^L = (e^L)^{-1}$. We shall denote by $\mathcal{P}$ the column vector with the components $\mathcal{P}_a$ so that

$$\mathcal{P} = \frac{1}{2}\big(E(g) \cdot L_-^t(g) + E^t(g) \cdot L_+^t(g)\big). \tag{24}$$

We also define

$$L_\sigma = \frac{1}{2}\big(L_+(g) - L_-(g)\big). \tag{25}$$

For the future reference let us quote the inverse relations

$$\begin{aligned}L_+(g) &= 2\big(\mathcal{P}^t + L_\sigma \cdot E^t(g)\big) \cdot \big(E(g) + E^t(g)\big)^{-1},\\ L_-(g) &= 2\big(\mathcal{P}^t - L_\sigma \cdot E(g)\big) \cdot \big(E(g) + E^t(g)\big)^{-1}.\end{aligned} \tag{26}$$

Defining the similar quantities $\hat{\mathcal{P}}$, $\hat{L}_\sigma$ for the model after the Poisson–Lie T-plurality transformation and using (19), (20) we find

$$\hat{\mathcal{P}} = \frac{1}{2}\big(\big(Q^t \cdot E_0^{-t} + S^t\big) \cdot E^t(g) \cdot L_+^t(g) - \big(Q^t \cdot E_0^{-1} - S^t\big) \cdot E(g) \cdot L_-^t(g)\big), \tag{27}$$

$$\begin{aligned}\hat{L}_\sigma = \frac{1}{2}\big(&L_+(g) \cdot E(g) \cdot \big(E_0^{-1} \cdot Q + S\big) \cdot \hat{E}(\hat{g})^{-1}\\ &+ L_-(g) \cdot E^t(g) \cdot \big(E_0^{-t} \cdot Q - S\big) \cdot \hat{E}(\hat{g})^{-t}\big),\end{aligned} \tag{28}$$

which, as we shall show, becomes the transformation of the canonical variables[2]

$$\hat{\mathcal{P}} = \big(Q^t \cdot \Pi(g) + S^t\big) \cdot \mathcal{P} + Q^t \cdot L_\sigma^t, \tag{29}$$

$$\hat{L}_\sigma = \mathcal{P}^t \cdot \big[\big(S - \Pi(g) \cdot Q\big) \cdot \hat{\tilde{\Pi}}(\hat{\tilde{g}}) + R - \Pi(g) \cdot K\big] + L_\sigma \cdot \big(Q \cdot \hat{\tilde{\Pi}}(\hat{\tilde{g}}) + K\big). \tag{30}$$

This agrees with the formulae obtained in [4] for Poisson–Lie T-duality,[3] i.e. for $Q = R = \mathbf{1}$, $K = S = 0$, but generalizes the results from [4,5] to any T-plurality transformation.

In order to deduce (29), (30) we shall first list a few useful formulae. Because of their complexity we shall suppress the $g$, $\tilde{g}$-dependence in the proof (i.e. till the end of this section) and also the explicit dot $\cdot$ for matrix multiplication. This does not lead to any difficulty because the derivation of (29), (30) from (27), (28) is purely algebraic.

We have matrix identities valid for any matrix $A$ (whenever the expressions make sense)

$$\begin{aligned} &A^{-1}\big(A^{-1} + A^{-t}\big)^{-1} = \big(A + A^t\big)^{-1} A^t, \\ &A^{-t}\big(A^{-1} + A^{-t}\big)^{-1} = \big(A + A^t\big)^{-1} A, \\ &A^{-1}\big(A^{-1} + A^{-t}\big)^{-1} A^{-t} = \big(A + A^t\big)^{-1} = A^{-t}\big(A^{-1} + A^{-t}\big)^{-1} A^{-1}. \end{aligned} \tag{31}$$

Directly from the definition (5) of $E$ we have

$$E^{-1} + E^{-t} = E_0^{-1} + E_0^{-t} \tag{32}$$

and its consequences due to (31)

$$\begin{aligned} &E\big(E + E^t\big)^{-1} E^t = \big(E^{-1} + E^{-t}\big)^{-1} = \big(E_0^{-1} + E_0^{-t}\big)^{-1}, \\ &E^t\big(E + E^t\big)^{-1} E = \big(E^{-1} + E^{-t}\big)^{-1} = \big(E_0^{-1} + E_0^{-t}\big)^{-1}. \end{aligned} \tag{33}$$

Finally, combining (33) and (31) (using first $A = E$ and then $A = E_0^{-1}$) together with (5) (in the second equality) we get

$$\begin{aligned} \big(E + E^t\big)^{-1}\big(E E_0^{-1} - E^t E_0^{-t}\big) &= E^{-t}\big(E_0^{-1} + E_0^{-t}\big)^{-1} E_0^{-1} - E^{-1}\big(E_0^{-1} + E_0^{-t}\big)^{-1} E_0^{-t} \\ &= E_0^{-t}\big(E_0^{-1} + E_0^{-t}\big)^{-1} E_0^{-1} - E_0^{-1}\big(E_0^{-1} + E_0^{-t}\big)^{-1} E_0^{-t} \\ &\quad - \Pi\big(E_0^{-1} + E_0^{-t}\big)^{-1}\big(E_0^{-1} + E_0^{-t}\big) \\ &= -\Pi \end{aligned} \tag{34}$$

and similarly its transpose

$$\big(E_0^{-t} E^t - E_0^{-1} E\big)\big(E + E^t\big)^{-1} = \Pi. \tag{35}$$

Now, when we substitute the relations (26) into the formula (27), we get

[2] We slightly abuse the terminology here: Strictly speaking the canonical variables are $\mathcal{P}_\mu$, $\phi^\mu$ and $\hat{\mathcal{P}}_\mu$, $\hat{\phi}^\mu$, respectively. Because the plurality transformation of $\phi^\mu$ defined via (15) (where $h$, $\tilde{h}$ are constructed via (8)) is non-local, we write instead the transformation of its space derivative $\partial_\sigma \phi^\mu$ and also we use for convenience the local frame versions instead of coordinate versions of these. Nevertheless, as we show later on, this does not lead to any non-localities in the Hamiltonian or the Poisson brackets.

[3] And, as another consistency check, reduces to identity transformation when $K = S = \mathbf{1}$, $R = Q = 0$.

$$\hat{\mathcal{P}} = \big[\big(Q^t E_0^{-t} + S^t\big)E^t\big(E+E^t\big)^{-1} - \big(Q^t E_0^{-1} - S^t\big)E\big(E+E^t\big)^{-1}\big]\mathcal{P} + Q^t\big[E_0^{-t}E^t\big(E+E^t\big)^{-1}E + E_0^{-1}E\big(E+E^t\big)^{-1}E^t\big]L_\sigma^t$$

(the terms involving $S^t(\cdots)L_\sigma^t$ cancel each other). Using (33) we simplify the coefficient of $L_\sigma^t$, getting the desired $Q^t L_\sigma^t$ term in (29). The terms of the form $S^t(\cdots)\mathcal{P}$ give $S^t\mathcal{P}$. The remaining $Q^t(\cdots)\mathcal{P}$ terms are simplified using (35)

$$Q^t\big(E_0^{-t}E^t - E_0^{-1}E\big)\big(E+E^t\big)^{-1}\mathcal{P} = Q^t \Pi \mathcal{P}.$$

Therefore, the formula (29) is proven.

Similarly, we substitute the relations (26) together with the definition of $\hat{E}$ (13), (14), i.e.

$$\hat{E} = \big((Q+E_0S)^{-1}(K+E_0R) + \hat{\Pi}\big)^{-1} = \big(\big(E_0^tS - Q\big)^{-1}\big(K - E_0^tR\big) - \hat{\Pi}\big)^{-t}$$

into the formula (28). We get

$$\begin{aligned}\hat{L}_\sigma = {} & \mathcal{P}^t\big(E+E^t\big)^{-1}\big[E\big(E_0^{-1}Q+S\big)\big((Q+E_0S)^{-1}(K+E_0R)+\hat{\Pi}\big) \\ & + E^t\big(E_0^{-t}Q - S\big)\big(\big(E_0^tS-Q\big)^{-1}\big(K-E_0^tR\big)-\hat{\Pi}\big)\big] \\ & + L_\sigma\big[E^t\big(E+E^t\big)^{-1}E\big(E_0^{-1}Q+S\big)\big((Q+E_0S)^{-1}(K+E_0R)+\hat{\Pi}\big) \\ & - E\big(E+E^t\big)^{-1}E^t\big(E_0^{-t}Q-S\big)\big(\big(E_0^tS-Q\big)^{-1}\big(K-E_0^tR\big)-\hat{\Pi}\big)\big].\end{aligned}$$

We note that

$$\big(E_0^{-1}Q+S\big)(Q+E_0S)^{-1} = E_0^{-1}, \qquad \big(E_0^{-t}Q-S\big)\big(E_0^tS-Q\big)^{-1} = -E_0^{-t}$$

and using relations (31)–(34) we simplify the expression for $\hat{L}_\sigma$ to the desired form (30) which finishes the proof of the formulae (29), (30).

## 5. Poisson–Lie T-plurality as canonical transformation

In order to show that (29), (30) is really a canonical transformation we shall use the expressions for Poisson brackets of $\mathcal{P}_a$ and

$$\mathcal{J}^a = L_\sigma^a + \Pi(g)^{ab}\mathcal{P}_b, \quad \text{i.e. } \mathcal{J} = L_\sigma^t + \Pi(g)\cdot\mathcal{P} \tag{36}$$

introduced in [5], namely

$$\begin{aligned}&\big\{\mathcal{J}^a,\mathcal{J}^b\big\} = \tilde{f}^{ab}{}_c\mathcal{J}^c\delta(\sigma-\sigma'), \\ &\{\mathcal{P}_a,\mathcal{P}_b\} = f_{ab}{}^c\mathcal{P}_c\delta(\sigma-\sigma'), \\ &\big\{\mathcal{J}^a,\mathcal{P}_b\big\} = \big(f_{bc}{}^a\mathcal{J}^c - \tilde{f}^{ac}{}_b\mathcal{P}_c\big)\delta(\sigma-\sigma') + \delta_b^a\delta'(\sigma-\sigma').\end{aligned} \tag{37}$$

These Poisson brackets are equivalent to the canonical ones

$$\begin{aligned}&\{\mathcal{P}_\mu,\mathcal{P}_\nu\} = \big\{\partial_\sigma\phi^\mu,\partial_\sigma\phi^\nu\big\} = 0, \\ &\big\{\partial_\sigma\phi^\mu,\mathcal{P}_\nu\big\} = \delta_\nu^\mu\delta'(\sigma-\sigma').\end{aligned} \tag{38}$$

Further, using the definition (36) we note that the transformation of the canonical momentum (29) can be written as

$$\hat{\mathcal{P}} = S^t\cdot\mathcal{P} + Q^t\cdot\mathcal{J} \tag{39}$$

which reminds of the inverse of the transformation (10) of the basis elements of the Drinfel'd double. From this one can conjecture that

$$\hat{\mathcal{J}} = R^t \cdot \mathcal{P} + K^t \cdot \mathcal{J} \tag{40}$$

and a simple calculation using the definition (36) of $\mathcal{J}$ proves that (40) is indeed equivalent to (30).

To prove the invariance of the Poisson brackets (37) (and thus of (38)) under the Poisson–Lie T-plurality transformations (29), (30) or (39), (40) it is useful to note that their structure strongly reminds of the Lie structure (11) of the Drinfel'd double, namely that the Poisson brackets (37) can be written in the compact form

$$\{\mathcal{Y}_\alpha, \mathcal{Y}_\beta\} = \mathcal{F}_{\alpha\beta}{}^{\gamma} \mathcal{Y}_\gamma \delta(\sigma - \sigma') + \mathcal{B}_{\alpha\beta} \delta'(\sigma - \sigma') \tag{41}$$

where $\alpha, \beta, \gamma = 1, \ldots, \dim \mathfrak{d}$,

$$\mathcal{Y} = \begin{pmatrix} \mathcal{P} \\ \mathcal{J} \end{pmatrix}, \tag{42}$$

$\mathcal{F}_{\alpha\beta}{}^{\gamma}$ are structure coefficients of the Drinfel'd double and $\mathcal{B}_{\alpha\beta}$ are matrix elements of the bilinear form $\langle .\,, .\rangle$ in the basis $T_a, \tilde{T}^a$ of $\mathfrak{d}$. From this compact form it is clear that the Poisson brackets (41) are form-invariant under the transformation (39), (40) that is an analog of the transformation (10) of bases of the Drinfel'd double which transforms $f, \tilde{f}$ to $\hat{f}, \bar{f}$ and preserves the duality of bases, i.e. $\mathcal{B}_{\alpha\beta}$. Consequently, the canonical Poisson brackets are invariant, i.e. (38) is transformed by Poisson–Lie T-plurality to

$$\begin{aligned} \{\hat{\mathcal{P}}_\mu, \hat{\mathcal{P}}_\nu\} &= \{\partial_\sigma \hat{\phi}^\mu, \partial_\sigma \hat{\phi}^\nu\} = 0, \\ \{\partial_\sigma \hat{\phi}^\mu, \hat{\mathcal{P}}_\nu\} &= \delta^\mu_\nu \delta'(\sigma - \sigma'). \end{aligned} \tag{43}$$

Finally, we compute the Hamiltonian density

$$\mathcal{H} = \partial_\tau \phi^\mu \mathcal{P}_\mu - \mathcal{L} \tag{44}$$

where the Lagrangian density is deduced from the action (3)

$$\mathcal{L} = \frac{1}{2} L_+(g) \cdot E(g) \cdot L_-^t(g) = \frac{1}{4}\big(L_+(g) \cdot E(g) \cdot L_-^t(g) + L_-(g) \cdot E^t(g) \cdot L_+^t(g)\big)$$

and we have used an obvious identity valid for any column vector $x$ and matrix $A$

$$x^t A x = x^t A^t x = \frac{1}{2} x^t \big(A + A^t\big) x. \tag{45}$$

We recall that due to the definition (22), (24) of the canonical momentum we have

$$\begin{aligned} \partial_\tau \phi^\mu \mathcal{P}_\mu &= \frac{1}{2}\big(\partial_+ \phi^\mu + \partial_- \phi^\mu\big) \mathcal{P}_\mu \\ &= \frac{1}{4}\big(L_+(g) + L_-(g)\big) \cdot \big(E(g) \cdot L_-^t(g) + E^t(g) \cdot L_+^t(g)\big). \end{aligned}$$

Substituting into the definition of Hamiltonian density (44) we find

$$\mathcal{H} = \frac{1}{4}\big(L_-(g) \cdot E(g) \cdot L_-^t(g) + L_+(g) \cdot E(g) \cdot L_+^t(g)\big) \tag{46}$$

where the substitution for the left-invariant fields $L_-(g), L_+(g)$ in terms of the canonical variables (26) is understood. Performing explicitly the substitution (26) we get for the Hamiltonian density the formula

$$\mathcal{H} = \frac{1}{2}\big(\mathcal{P}^t - L_\sigma \cdot B\big) \cdot G^{-1} \cdot \big(\mathcal{P} + B \cdot L_\sigma^t\big) + \frac{1}{2} L_\sigma \cdot G \cdot L_\sigma^t \tag{47}$$

used in [4,5] where $G$, $B$ are symmetric and antisymmetric part of $E(g)$, respectively, i.e.

$$G = \frac{1}{2}\big(E(g) + E^t(g)\big), \qquad B = \frac{1}{2}\big(E(g) - E^t(g)\big).$$

The Hamiltonian density of the $\sigma$-model obtained by T-plurality transformation can be written analogously as

$$\hat{\mathcal{H}} = \frac{1}{4}\big(\hat{L}_-(\hat{g}) \cdot \hat{E}(\hat{g}) \cdot \hat{L}_-^t(\hat{g}) + \hat{L}_+(\hat{g}) \cdot \hat{E}(\hat{g}) \cdot \hat{L}_+^t(\hat{g})\big),$$

where we assume, as above, that the left-invariant fields $\hat{L}_-(\hat{g})$, $\hat{L}_+(\hat{g})$ are expressed in terms of the new canonical variables. Using the transformation of the left-invariant fields (19), (20) we find

$$\hat{\mathcal{H}} = \frac{1}{4}\big(L_-(g) \cdot E^t(g) \cdot \big(S - E_0^{-t} Q\big) \cdot \hat{E}(\hat{g})^{-t} \cdot \big(S^t - Q^t E_0^{-1}\big) \cdot E(g) \cdot L_-^t(g)$$
$$+ L_+(g) \cdot E(g) \cdot \big(S + E_0^{-1} Q\big) \cdot \hat{E}(\hat{g})^{-t} \cdot \big(S^t + Q^t E_0^{-t}\big) \cdot E^t(g) \cdot L_+^t(g)\big).$$

Due to the identity (45) we can replace $\hat{E}^{-t}(\hat{g})$ by $\hat{E}^{-t}(\hat{g}) + \hat{E}^{-1}(\hat{g}) = \hat{E}_0^{-t} + \hat{E}_0^{-1}$. From the definition of $\hat{E}_0$ (14) and the duality of bases (12), i.e.

$$QR^t = \mathbf{1} - KS^t, \qquad RS^t = -SR^t, \qquad KQ^t = -QK^t, \qquad RQ^t = \mathbf{1} - SK^t$$

we get

$$\hat{\mathcal{H}} = \frac{1}{8}\big(L_-(g) \cdot E^t(g) \cdot \big(E_0^{-1} + E_0^{-t}\big) \cdot E(g) \cdot L_-^t(g)$$
$$+ L_+(g) \cdot E(g) \cdot \big(E_0^{-1} + E_0^{-t}\big) \cdot E^t(g) \cdot L_+^t(g)\big).$$

Using the relation (32) we replace $E_0^{-1} + E_0^{-t}$ by $E^{-1}(g) + E^{-t}(g)$ and employ once again the identity (45), getting the final result

$$\hat{\mathcal{H}} = \frac{1}{4}\big(L_-(g) \cdot E(g) \cdot L_-^t(g) + L_+(g) \cdot E(g) \cdot L_+^t(g)\big).$$

Consequently, we find that the Hamiltonian density is preserved under Poisson–Lie T-plurality transformation,

$$\hat{\mathcal{H}} = \mathcal{H}. \tag{48}$$

We could have equivalently used the form of the Hamiltonian density (47) together with the transformation of canonical variables (29), (30). In the approach we used the computation of $\hat{\mathcal{H}}$ in terms of original canonical variables $\mathcal{P}$, $L_\sigma$, or equivalently the left-invariant fields $L_-(g)$, $L_+(g)$, is significantly algebraically simpler.

## 6. Conclusions

We have derived a transformation of the canonical structure of dualizable $\sigma$-models, more precisely their (pseudo)canonical variables, Poisson brackets and Hamiltonian densities under the Poisson–Lie T-plurality. It turned out that by a suitable choice of the variables the Poisson brackets acquire a rather symmetric form that can be turned into the compact form (41). This expression is explicitly form-invariant with respect to the choice of basis in the Drinfel'd double on which the $\sigma$-models are defined. This proves the invariance of the canonical structure under the Poisson–Lie T-plurality because its transformations follow from various decompositions of the Drinfel'd double, i.e. special transformations of its bases that turn one decomposition into another.

The explicit formulae for transformations of extremal left and right-invariant fields (19), (20) and canonical variables (29), (30), (39), (40) can be used for further investigation of particular properties of $\sigma$-models related by the Poisson–Lie T-plurality transformations, for example their boundary conditions.

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# 4.2

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# On the Poisson-Lie *T*-plurality of boundary conditions

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Conditions for the gluing matrix defining consistent boundary conditions of two-dimensional nonlinear $\sigma$-models are analyzed and reformulated. Transformation properties of the right-invariant fields under the Poisson-Lie $T$-plurality are used to derive a formula for the transformation of the boundary conditions. Examples of transformation of $D$-branes in two and three dimensions are presented. We investigate obstacles arising in this procedure and propose possible solutions. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2832622]

## I. INTRODUCTION

$T$-duality of strings may be realized as a canonical transformation acting on the fields in a two-dimensional nonlinear $\sigma$-model. This model describes the worldsheet theory of a string propagating on some target manifold equipped with a background tensor field $\mathcal{F}_{\mu\nu}$ which is a convenient rearrangement of the metric and the Kalb-Ramond B-field. For open strings, the worldsheet has boundaries, by definition confined to $D$-branes; hence, the action is subject to boundary conditions. Imposing extra symmetries, e.g., conformal invariance, restricts these conditions. They determine the dynamics of the ends of the string and hence the embedding of $D$-branes in the target space. Applying duality transformations yields the dual boundary conditions and hence the geometry of $D$-branes in the dual target.

Traditional $T$-duality requires the presence of an isometry group leaving the $\sigma$-model invariant, a rather severe restriction. In the Poisson-Lie $T$-duality,[1] isometries are not necessary, provided that the two dual target spaces are both Poisson-Lie group manifolds (or at least Poisson-Lie groups act freely on them) whose Lie algebras constitute a *Drinfel'd double*. That is, they are maximally isotropic Lie subalgebras in the decomposition of a Lie bialgebra $\mathfrak{d}=\mathfrak{g}+\tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}}\equiv\mathfrak{g}^*$. The background $\mathcal{F}_{\mu\nu}$ is related to the Poisson structure on the target manifold and satisfies the *Poisson-Lie condition*, a restriction that replaces the traditional isometry condition.

Recently, the transformation of worldsheet boundary conditions under the Poisson-Lie $T$-duality was derived in Ref. 2. The key formulas were transformations of left-invariant fields[1]

$$\tilde{L}^t_{\mp}(\tilde{g})=\tilde{E}^{-t}(\tilde{g})\cdot E_0^{-t}\cdot E^t(g)\cdot L^t_{\mp}(g), \tag{1}$$

[1]The dot denotes matrix multiplication, $t$ denotes transposition, $E^{-t}\equiv(E^t)^{-1}$, where $E$ is a general background field in the Lie algebra frame, and $E_0$ is a constant matrix.

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$$\tilde{L}^t_{=}(\tilde{g}) = -(\tilde{E}(\tilde{g}))^{-1} \cdot E_0^{-1} \cdot E(g) \cdot L^t_{=}(g), \tag{2}$$

obtained from the canonical transformations derived in Refs. 3 and 4. Here, $g$ and $\tilde{g}$ are elements of the groups corresponding to $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, respectively, and the subscripts $+\!\!+$ and $=$ refer to the worldsheet lightcone coordinates.

Poisson-Lie $T$-plurality[5] is a further generalization of $T$-duality, where the mutually dual target spaces do not necessarily belong to the same Lie algebra decomposition of the Drinfel'd double (i.e., they belong to different *Manin triples*).

In Refs. 6 and 7 we found classical solutions of $\sigma$-models in curved backgrounds by applying Poisson-Lie $T$-plurality transformations to flat $\sigma$-models. Unfortunately, we were not able to control the boundary conditions necessary for string solutions in the curved background or, more precisely, to identify the conditions for the flat solution that transform to suitable conditions in the curved background.

Our goal here is to derive a transformation of boundary conditions under the Poisson-Lie $T$-plurality that could enable us to control the boundary conditions in the transformed $\sigma$-model. Analogs of the formulas (1) and (2) for the Poisson-Lie $T$-plurality were derived in Ref. 8 so that we can easily write down the transformation of the boundary conditions. As the $\sigma$-models investigated in Refs. 6 and 7 and other papers of ours are formulated in terms of right-invariant fields $\partial_{\pm} g g^{-1}$, we shall use this formulation here.[2]

In Sec. II, we review the Poisson-Lie $T$-plurality and introduce the framework necessary for the subsequent analysis. In Sec. III, we list and discuss the set of boundary conditions required to define consistent $\sigma$-models, describing them in terms of a gluing matrix. In Sec. IV, we derive the $T$-plurality transformation of the gluing matrix and show that it does not automatically yield well-defined boundary conditions on the $T$-plural side. In Secs. V and VI, we analyze two explicit examples, one three dimensional and one two dimensional, demonstrating how different $D$-branes transform under the Poisson-Lie $T$-plurality. In the process, we discuss the conditions necessary to eliminate any interdependence of the gluing matrices on coordinates of the different target spaces involved. Finally, Sec. VII contains our conclusions.

## II. ELEMENTS OF POISSON-LIE *T*-PLURALITY

The classical action of the $\sigma$-model under consideration is

$$S_{\mathcal{F}}[\phi] = \int_{\Sigma} d^2x \partial_- \phi^{\mu} \mathcal{F}_{\mu\nu}(\phi) \partial_+ \phi^{\nu}, \tag{3}$$

where $\mathcal{F}$ is a tensor on a Lie group $G$ and the functions $\phi^{\mu}:\Sigma \subset \mathbb{R}^2 \to \mathbb{R}$, $\mu = 1,2,\ldots,\dim G$ are obtained by the composition $\phi^{\mu} = y^{\mu} \circ g$ of a map $g:\Sigma \to G$ and components of a coordinate map $y$ of a neighborhood $U_g$ of an element $g(x_+,x_-) \in G$. For the purpose of this paper, we shall assume that the worldsheet $\Sigma$ has the topology of a strip infinite in timelike direction, $\Sigma = \langle 0,\pi \rangle \times \mathbb{R}$.

On a Lie group $G$, the tensor $\mathcal{F}$ can be written as

$$\mathcal{F}_{\mu\nu} = e_{\mu}{}^{a}(g) F_{ab}(g) e_{\nu}{}^{b}(g), \tag{4}$$

where $e_{\mu}{}^{a}(g)$ are components of the right-invariant Maurer-Cartan forms $dgg^{-1}$ and $F_{ab}(g)$ are matrix elements of bilinear nondegenerate form $F(g)$ on $\mathfrak{g}$, the Lie algebra of $G$. The action of the $\sigma$-model then reads

[2]Left-invariant fields were used in Ref. 8.

$$S_F[g] = \int_\Sigma d^2x \rho_-(g) \cdot F(g) \cdot \rho_+(g)^t, \tag{5}$$

where the right-invariant vector fields $\rho_\pm(g)$ are given by[3]

$$\rho_\pm(g)^a \equiv (\partial_\pm g g^{-1})^a = \partial_\pm \phi^\mu e_\mu{}^a(g), \quad (\partial_\pm g g^{-1}) = \rho_\pm(g) \cdot T, \tag{6}$$

and $T_a$ are basis elements of $\mathfrak{g}$. [Note that $\rho_\pm(g)$ is written in a condensed notation; in full detail, it would read $\rho_\pm(g(x_+,x_-), \partial_\pm g(x_+,x_-))$ since it is a map $\Sigma \to \mathfrak{g}$.]

The $\sigma$-models that are transformable under the Poisson-Lie $T$-duality can be formulated (see Refs. 1 and 9) on a Drinfel'd double $D \equiv (G|\tilde{G})$, a Lie group whose Lie algebra $\mathfrak{d}$ admits a decomposition $\mathfrak{d} = \mathfrak{g} \dot{+} \tilde{\mathfrak{g}}$ into a pair of subalgebras maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form $\langle .,. \rangle$. The matrices $F_{ab}(g)$ for the dualizable $\sigma$-models are of the form[1]

$$F(g) = (E_0^{-1} + \Pi(g))^{-1}, \quad \Pi(g) = b(g) \cdot a(g)^{-1} = -\Pi(g)^t, \tag{7}$$

where $E_0$ is a constant matrix, $\Pi$ defines the Poisson structure on the group $G$, and $a(g), b(g)$ are submatrices of the adjoint representation of $G$ on $\mathfrak{d}$. They satisfy

$$gTg^{-1} \equiv \mathrm{Ad}(g) \triangleright T = a^{-1}(g) \cdot T, \quad g\tilde{T}g^{-1} \equiv \mathrm{Ad}(g) \triangleright \tilde{T} = b^t(g) \cdot T + a^t(g) \cdot \tilde{T}, \tag{8}$$

where $\tilde{T}^a$ are elements of dual basis in the dual algebra $\tilde{\mathfrak{g}}$, i.e., $\langle T_a, \tilde{T}^b \rangle = \delta_a^b$. The matrix $a(g)$ relates the left- and right-invariant fields on $G$ via

$$(g^{-1}\partial_\pm g) = L_\pm(g) \cdot T, \quad L_\pm(g) = \rho_\pm(g) \cdot a(g). \tag{9}$$

The equations of motion of the dualizable $\sigma$-models can be written as Bianchi identities for the right-invariant fields $\tilde{\rho}_\pm(\tilde{h})$ on the dual algebra $\tilde{\mathfrak{g}}$ satisfying[9]

$$\tilde{\rho}_+(\tilde{h}) \cdot \tilde{T} \equiv (\partial_+ \tilde{h}\tilde{h}^{-1}) = -\rho_+(g) \cdot F(g)^t \cdot a^{-t}(g) \cdot \tilde{T}, \tag{10}$$

$$\tilde{\rho}_-(\tilde{h}) \cdot \tilde{T} \equiv (\partial_- \tilde{h}\tilde{h}^{-1}) = +\rho_-(g) \cdot F(g) \cdot a^{-t}(g) \cdot \tilde{T}. \tag{11}$$

This is a consequence of the fact that the equations of motion of the dualizable $\sigma$-model can be written as the following equations on the Drinfel'd double:[1]

$$\langle \partial_\pm l l^{-1}, \mathcal{E}^\pm \rangle = 0, \tag{12}$$

where $l = g\tilde{h}$ and $\mathcal{E}^\pm$ are two orthogonal subspaces in $\mathfrak{d}$. On the other hand, the solution $g(x_+,x_-)$ of the equations of motion of the action (5) gives us a flat connection (10) and (11), which is therefore locally pure gauge, and the gauge potential $\tilde{h}(x_+,x_-)$ is determined up to right multiplication by a constant element $\tilde{h}_0$. Therefore, we find $l(x_+,x_-) = g(x_+,x_-) \cdot \tilde{h}(x_+,x_-)$, the so-called lift of the solution $g(x_+,x_-)$ to the Drinfel'd double, determined up to the constant shift

$$l \to l\tilde{h}_0, \quad \tilde{h}_0 \in \tilde{G}. \tag{13}$$

In general, as was realized already in Ref. 1 and then further developed in Ref. 5, there are several decompositions (Manin triples) of a Drinfel'd double. Let $\hat{\mathfrak{g}} \dot{+} \bar{\mathfrak{g}}$ be another decomposition of the Lie algebra $\mathfrak{d}$. The pairs of dual bases of $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}, \bar{\mathfrak{g}}$ are related by the linear transformation

[3] Note that while matrix multiplication is denoted by dot, for group multiplication we use concatenation.

$$\begin{pmatrix} T \\ \widetilde{T} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \hat{T} \\ \bar{T} \end{pmatrix}, \tag{14}$$

where the duality of both bases requires

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} = \begin{pmatrix} s^t & q^t \\ r^t & p^t \end{pmatrix}, \tag{15}$$

i.e.,

$$\begin{aligned} p \cdot s^t + q \cdot r^t &= \mathbf{1}, \\ p \cdot q^t + q \cdot p^t &= 0, \\ r \cdot s^t + s \cdot r^t &= 0. \end{aligned} \tag{16}$$

The $\sigma$-model obtained by the plurality transformation is then defined analogously to the original one, namely, by substituting

$$\hat{F}(\hat{g}) = (\hat{E}_0^{-1} + \hat{\Pi}(\hat{g}))^{-1}, \quad \hat{\Pi}(\hat{g}) = \hat{b}(\hat{g}) \cdot \hat{a}(\hat{g})^{-1} = -\hat{\Pi}(\hat{g})^t, \tag{17}$$

$$\hat{E}_0 = (p + E_0 \cdot r)^{-1} \cdot (q + E_0 \cdot s) = (s^t \cdot E_0 - q^t) \cdot (p^t - r^t \cdot E_0)^{-1} \tag{18}$$

into (4) and (5). Solutions of the two $\sigma$-models are related by two possible decompositions of $l \in D$, namely,

$$l = g\widetilde{h} = \hat{g}\bar{h}. \tag{19}$$

The transformed solution $\hat{g}$ is determined by the original solution $g(x_+, x_-)$ up to a choice of constant shift (13).

### III. BOUNDARY CONDITIONS AND *D*-BRANES

The properties of $D$-branes in the groups $G$ and $\hat{G}$ can be derived from the so-called gluing operators $\mathcal{R}$ and $\hat{\mathcal{R}}$, respectively; the number of their $-1$ eigenvalues determines the number of Dirichlet directions and hence the dimension of the $D$-branes. Moreover, the explicit form of the operator in principle yields the embedding of a brane in the target space.

We impose the boundary conditions for open strings in the form of the gluing operator $\mathcal{R}$ relating the left and right derivatives of field $g:\Sigma \to G$ on the boundary of $\Sigma$,

$$\partial_- g|_{\sigma=0,\pi} = \mathcal{R}\partial_+ g|_{\sigma=0,\pi}, \quad \sigma \equiv x_+ - x_-. \tag{20}$$

As we have to work with several choices of coordinates, we denote the matrices corresponding to the operator $\mathcal{R}$ in the bases of coordinate derivatives as $R_\phi, R_\lambda$, etc., e.g.,

$$\partial_- \phi|_{\sigma=0,\pi} = \partial_+ \phi \cdot R_\phi|_{\sigma=0,\pi} \tag{21}$$

or

$$\partial_- \lambda|_{\sigma=0,\pi} = \partial_+ \lambda \cdot R_\lambda|_{\sigma=0,\pi}, \tag{22}$$

where $\partial_- \phi, \partial_- \lambda$ are row vectors of the derivatives of the respective coordinates (therefore, matrices of operators in our notation may differ by a transposition from expressions in other papers). Nevertheless, we suppress the indices $\phi, \lambda$ in expressions valid in any choice of coordinates, $R$ having the obvious meaning of the matrix of the gluing operator, the tensor $\mathcal{F}$ is assumed to be expressed in the same coordinates, etc.

We define the Dirichlet projector $\mathcal{Q}$ that projects vectors onto the space normal to the $D$-brane, which is identified with the eigenspace of $\mathcal{R}$ with the eigenvalue $-1$, and the Neumann projector $\mathcal{N}$, which projects onto the tangent space of the brane. The corresponding matrices $Q$, $N$ are given by the axioms

$$Q^2 = Q, \quad Q \cdot R = -Q, \quad N = \mathbf{1} - Q. \tag{23}$$

In so-called adapted coordinates $\lambda^{\alpha}$ (where $\alpha = 1, \ldots, \dim G$), the gluing matrix can be written as[10]

$$R_{\lambda} = \begin{pmatrix} R_m{}^n & 0 \\ 0 & -\delta_i^j \end{pmatrix}, \quad m,n = 1, \ldots, p+1, \quad i,j, = p+2, \ldots, \dim G. \tag{24}$$

If the B-field of the model vanishes, one can choose $R_m{}^n = \delta_m{}^n$. In such coordinates, the terminology becomes clearer as $\lambda^i$ become coordinates in the Dirichlet directions,

$$\partial_\tau \lambda^i = \tfrac{1}{2}(\partial_+ + \partial_-)\lambda^i = 0,$$

whereas $\lambda^m$ are Neumann directions. This is a traditional misnomer; it is actually a generalization of the Neumann boundary conditions

$$\partial_\sigma \lambda^m = \tfrac{1}{2}(\partial_+ - \partial_-)\lambda^m = 0$$

to the cases with nonvanishing B-field (a better notation might be free boundary conditions, but we shall stick to the traditional "Neumann").

To obtain the corresponding boundary conditions written in terms of right-invariant fields $\rho_{\pm}(g)$, we must first express the gluing operator in the group coordinates $y$ as

$$R_{\phi} = T(y) \cdot R_{\lambda} \cdot T(y)^{-1},$$

where

$$T(y)_{\mu}{}^{\alpha} = \frac{\partial \lambda^{\alpha}}{\partial y^{\mu}}(y),$$

and then transform it into the basis of the Lie algebra of right-invariant fields,

$$R_{\rho} = e^{-1}(g) \cdot R_{\phi} \cdot e(g) = e^{-1}(g) \cdot T(y) \cdot R_{\lambda} \cdot T(y)^{-1} \cdot e(g), \tag{25}$$

where $e(g)$ are the right-invariant vielbeins on $G$ introduced in Eq. (6). The boundary conditions may then be expressed in terms of the right-invariant fields as

$$\rho_-(g)|_{\sigma=0,\pi} = \rho_+(g) \cdot R_{\rho}|_{\sigma=0,\pi}. \tag{26}$$

Of course, not every operator-valued function on the target space, in our case the group $G$, can be interpreted as a gluing operator, giving consistent boundary conditions for the $\sigma$-model in question. There are several restrictions on $\mathcal{R}$ derived, e.g., in Ref. 10. We shall briefly recall how these conditions arise and rewrite them in a slightly more compact but equivalent form.

First, in order that the adapted coordinates exist in a particular point, we must impose

$$R \cdot Q = Q \cdot R. \tag{27}$$

This is essentially a part of the definition of $Q$; otherwise, $Q$ is not fully determined because to define a projector we need to specify its image and its kernel. Equation (23) defines the image of $Q$ to be an eigenspace of $R$, while Eq. (27) implies that the kernel is the sum of all the remaining (generalized) eigenspaces of $R$. On the other hand, condition (27) is a restriction on $R$ since it tells us that the geometrical[4] and algebraic[5] multiplicities of the eigenvalue $-1$ are equal. If this condition does not hold, one cannot find adapted coordinates (24), and the boundary conditions cannot be split into Dirichlet and (generalized) Neumann directions.

The distribution defined by the image of the Neumann projector must be integrable in order to be a tangent space to a submanifold, i.e., the brane. We find using the Frobenius theorem on integrability of distributions that the distribution must be in involution. When expressed in terms of the matrix $N$ of the Neumann projector, this condition reads in any coordinates,

$$N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}{}^\rho = 0. \tag{28}$$

In an arbitrary, noncoordinate frame, e.g., when expressed in terms of the right-invariant fields, the condition (28) appears more complicated. It may in general be expressed using covariant derivatives but for simplicity we shall use only the coordinate expression (28).

Since our $\sigma$-models are studied with applications to string theory in mind, they are often viewed as gauge fixed Polyakov actions. This imposes a further constraint on the solutions, in the form of a vanishing stress tensor

$$\mathcal{T}_{++} = \mathcal{T}_{--} = 0$$

(the trace $\mathcal{T}_{+-}$ vanishes automatically). Enforcing this condition not only in the bulk but also on the boundary leads to the consistency condition that the gluing operator preserves the metric on the target space; in other words, it is orthogonal with respect to the metric. If this condition were not satisfied, the $\sigma$-model would not allow generic string solutions. Explicitly, we have

$$R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \tag{29}$$

where the metric is written as $\mathcal{G}=(\mathcal{F}+\mathcal{F}^t)/2$. Equivalently, in the Lie algebra frame $\{T_a\}$, we express the metric as $(F+F^t)/2$ and consequently we have

$$R_\rho \cdot (F + F^t) \cdot R_\rho^t = (F + F^t). \tag{30}$$

We moreover require that what we identified as Dirichlet and Neumann directions are indeed orthogonal with respect to the metric on the target space,

$$N \cdot \mathcal{G} \cdot Q^t = 0. \tag{31}$$

When the metric on the target space is positive (or negative) definite, this is an automatic consequence of (29). In the pseudo-Riemannian signature, it is an additional constraint weeding out pathological configurations.

Finally, a crucial condition follows from the field variation of the action. Since the boundary conditions should be such that the variation of the action vanishes not only in the bulk but also on the boundary (that is why we impose the boundary conditions in the first place), we find by inspection of the boundary term arising in the variation that under the assumption of locality[6] we must impose

$$\delta\phi \cdot N_\phi \cdot \ (\mathcal{F} \cdot \partial_+ \phi^t - \mathcal{F}^t \cdot \partial_- \phi^t)|_{\sigma=0,\pi} = 0,$$

which after the use of Eq. (21) becomes

$$\delta\phi \cdot N_\phi \cdot (\mathcal{F} - \mathcal{F}^t \cdot R_\phi^t) \cdot \ \partial_+ \phi^t|_{\sigma=0,\pi} = 0. \tag{32}$$

Because $\delta\phi = \delta\phi \cdot N_\phi$ (i.e., $\delta\phi$ is tangent to the brane) and since $\partial_+ \phi^t$ are not further restricted, we find

[4] i.e., the dimension of the eigenspace

[5] i.e., the multiplicity of the root of the characteristic polynomial

[6] That is, the integrand itself, not only the integral $\int_{\partial\Sigma}(\cdots)$, vanishes.

$$N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0, \tag{33}$$

which, using Eqs. (27) and (31) as well as the following consequences of the definition of the projectors (23):

$$N \cdot (\mathbf{1} + R) = \mathbf{1} + R, \quad N \cdot (\mathbf{1} - R) = \mathbf{1} - R - 2Q, \tag{34}$$

can be rewritten in an equivalent form originally deduced and used in Ref. 10,

$$N \cdot \mathcal{F} \cdot N^t - N \cdot \mathcal{F}^t \cdot N^t \cdot R^t = 0. \tag{35}$$

In fact, once we impose condition (27), the pair of conditions (31) and (35) is equivalent to condition (33). For example, assuming (33), we can establish (31) as follows:

$$2N \cdot \mathcal{G} \cdot Q^t = N \cdot (\mathcal{F} + \mathcal{F}^t) \cdot Q^t = N \cdot (\mathcal{F} \cdot Q^t - \mathcal{F}^t \cdot R^t \cdot Q^t) = N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) \cdot Q^t = 0,$$

where we have used first Eq. (23) and then Eq. (33). Moreover, once we have established that the condition (31) holds, we know that Eqs. (33) and (35) are equivalent.

To summarize, we are lead to the following conditions on a consistent gluing operator $\mathcal{R}$:

$$\begin{gathered} Q^2 = Q, \quad N = \mathbf{1} - Q, \quad R \cdot Q = Q \cdot R = -Q, \\ N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}{}^\rho = 0, \\ R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \\ N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0. \end{gathered} \tag{36}$$

Next, we investigate whether or not these conditions are preserved under the Poisson-Lie $T$-plurality. As we shall see by investigation of explicit examples, they are not preserved in general.

## IV. POISSON-LIE $T$-PLURALITY TRANSFORMATIONS OF RIGHT-INVARIANT FIELDS AND BOUNDARY CONDITIONS

The derivation of Poisson-Lie $T$-plurality transformations of left-invariant fields was presented in Ref. 8 but we find it instructive to repeat it here for the right-invariant fields. In particular, we derive the formulas generalizing Eqs. (1) and (2).

Let us write the right-invariant field $(\partial_+ l l^{-1})$ on the Drinfel'd double in terms of $\rho_+(g)$ and $\tilde{\rho}_+(\tilde{h})$,

$$(\partial_+ l l^{-1}) = (\partial_+ (g\tilde{h})(g\tilde{h})^{-1}) = \rho_+(g) \cdot T + \tilde{\rho}_+(\tilde{h}) \cdot g\tilde{T}g^{-1} = \rho_+(g) \cdot T + \tilde{\rho}_+(\tilde{h}) \cdot [b^t(g) \cdot T + a^t(g) \cdot \tilde{T}]. \tag{37}$$

Using the equations of motion (10) and the formula (7) for $F(g)$, we get

$$(\partial_+ l l^{-1}) = \rho_+(g) \cdot T - \rho_+(g) \cdot F(g)^t \cdot [a^{-t}(g) \cdot b^t(g) \cdot T + \tilde{T}] = \rho_+(g) \cdot F(g)^t \cdot [E_0^{-t} \cdot T - \tilde{T}]. \tag{38}$$

Similarly, from the decomposition $l = \hat{g}\bar{h}$, we get

$$(\partial_+ l l^{-1}) = \hat{\rho}_+(\hat{g}) \cdot \hat{F}(\hat{g})^t \cdot [\hat{E}_0^{-t} \cdot \hat{T} - \bar{T}]. \tag{39}$$

Substituting the relation (14) into Eq. (38) and comparing coefficients of $\hat{T}$ and $\bar{T}$ with those in (39), we find the transformation of right-invariant fields under the Poisson-Lie $T$-plurality,

$$\hat{\rho}_+(\hat{g}) = -\rho_+(g) \cdot F^t(g) \cdot [(E_0^t)^{-1} \cdot q - s] \cdot \hat{F}^{-t}(\hat{g}). \tag{40}$$

In the same way, we can derive

$$\hat{\rho}_-(\hat{g}) = \rho_-(g) \cdot F(g) \cdot [E_0^{-1} \cdot q + s] \cdot \hat{F}^{-1}(\hat{g}). \tag{41}$$

Formulas (1) and (2) for $T$-duality are obtained if $q=\mathbf{1}$, $s=0$, $F(g)=E(g^{-1})$, $\rho_+(g)=-L_=(g^{-1})$, and $\rho_-(g)=-L_{\#}(g^{-1})$, in agreement with the alternative version for the $\sigma$-model action used in Ref. 2,

$$S_E[g] = \int_\Sigma d^2x L_{\#}(g) \cdot E(g) \cdot L_=^t(g). \tag{42}$$

Substituting Eqs. (40) and (41) into the gluing condition (26), we find the $T$-plural boundary condition

$$\hat{\rho}_-(\hat{g})|_{\sigma=0,\pi} = \hat{\rho}_+(\hat{g}) \cdot \hat{R}_\rho|_{\sigma=0,\pi}, \tag{43}$$

where the $T$-plural gluing matrix is given by

$$\hat{R}_\rho = \hat{F}^t(\hat{g}) \cdot M_-^{-1} \cdot F^{-t}(g) \cdot R_\rho(g) \cdot F(g) \cdot M_+ \cdot \hat{F}^{-1}(\hat{g}) \tag{44}$$

and

$$M_+ \equiv s + E_0^{-1} \cdot q, \quad M_- \equiv s - E_0^{-t} \cdot q. \tag{45}$$

Equation (44) defines the transformation of the gluing matrix $R_\rho$ under the Poisson-Lie $T$-plurality. For the Poisson-Lie $T$-duality, i.e., for $q=r=\mathbf{1}$, $p=s=0$, the map (44) reduces (up to transpositions due to the different notations for matrices) to the duality map found in Ref. 2,

$$\tilde{R} = -\tilde{E}^{-1} \cdot E_0^{-1} \cdot E \cdot R \cdot E^{-t} \cdot E_0^t \cdot \tilde{E}^t. \tag{46}$$

An obvious problem is that the transformed gluing matrix $\hat{R}_\rho$ may depend not only on $\hat{g}$ but also on $g$, i.e., after performing the lift into the double $g\tilde{h}=\hat{g}\bar{h}$, it may depend on the new dual group elements $\bar{h} \in \bar{G}$, which contradicts any reasonable geometric interpretation of the dual boundary conditions. Nevertheless, as we shall see in Sec. V, if $g$ and $\hat{g}$ represent the maps $\Sigma \to G$ and $\Sigma \to \hat{G}$ related by the plurality transformation, the boundary conditions (26) and (43) are equivalent in the sense that they result in the same conditions on arbitrary functions [see e.g., (85)] occurring in solutions of the Euler-Lagrange equation of the action (5).

The $T$-plural counterparts of the Dirichlet and Neumann projectors may be consistently introduced in the same manner as for the $T$-dual case,[2] letting the relations $\hat{R}\cdot\hat{Q}=\hat{Q}\cdot\hat{R}=-\hat{Q}$ and $\hat{N}=\mathbf{1}-\hat{Q}$ define $\hat{Q}$ and $\hat{N}$ on $\hat{G}$. When the conditions (36) are satisfied also for $\hat{R},\hat{Q},\hat{N}$, then given a nonlinear $\sigma$-model on $G$ with well-defined boundary conditions, we find a $\sigma$-model on $\hat{G}$ with well-defined boundary conditions.

The conformal condition (29) is preserved under the Poisson-Lie $T$-plurality, i.e., Eq. (30) implies

$$\hat{R}_\phi \cdot \hat{\mathcal{G}} \cdot \hat{R}_\phi{}^t = \hat{\mathcal{G}}, \quad \hat{R}_\rho \cdot \hat{G}(g) \cdot \hat{R}_\rho{}^t = \hat{G}(g). \tag{47}$$

This is seen by using Eqs. (30) and (44), as well as the identities

$$F(g)^{-t} \cdot G(g) \cdot F(g)^{-1} = E_0^{-1} + E_0^{-t} = M_\pm \cdot (\hat{E}_0^{-1} + \hat{E}_0^{-t}) \cdot M_\pm^t, \tag{48}$$

which follow from Eqs. (16)–(18).

Imposing the condition (33) on the $T$-plural model and working in the basis of right-invariant fields, we may substitute Eq. (44) in the left-hand side of Eq. (33) to obtain

$$\hat{N}\cdot(\hat{F}-\hat{F}^t\cdot\hat{R}_\rho{}^t)=\hat{N}\cdot(\hat{F}-\hat{F}^t\cdot\hat{F}^{-t}\cdot(s+E_0^{-1}\cdot q)^t\cdot C^t\cdot(s-E_0^{-t}\cdot q)^{-t}\hat{F}), \tag{49}$$

where we have defined $C\equiv F^{-t}(g)\cdot R_\rho(g)\cdot F(g)$. This simplifies to

$$\hat{N}\cdot((s-E_0^{-t}\cdot q)^t-(s+E_0^{-1}\cdot q)^t\cdot C^t)\cdot(s-E_0^{-t}\cdot q)^{-t}\cdot\hat{F}.$$

The last two terms are by construction regular matrices and can be omitted while investigating when expression (49) vanishes. Consequently, the $T$-plural version of condition (33) has the form

$$\hat{N}\cdot((s-E_0^{-t}\cdot q)^t-(s+E_0^{-1}\cdot q)^t\cdot C^t)=0. \tag{50}$$

To gain a better understanding of Eq. (50), consider the particular case of originally purely Neumann boundary conditions, i.e., free endpoints. In this case $R_\rho(g)=F^t(g)\cdot F^{-1}(g)$, i.e., $C=\mathbf{1}$, and the transformation (44) is well defined (i.e., $\hat{R}_\rho$ is a function of $\hat{g}$ only) on any of the groups in any decomposition of the Drinfel'd double. This means that any $T$-plural $\hat{R}$ depends on the coordinates on the respective group $\hat{G}$ only. In this case, condition (50) further simplifies to

$$\hat{N}\cdot q^t=0, \tag{51}$$

where again regular matrices have been omitted in the product. We conclude that in the case of Poisson-Lie $T$-duality, where $q=\mathbf{1}$, the dual gluing operator satisfies condition (33) only if it is completely Dirichlet, in which case the dual version of (33) is trivially satisfied.

A possible solution to this problem, considered already in Ref. 11, comes from the fact that condition (33) is modified if the endpoints of the string are electrically charged. Let us modify the action by boundary terms

$$S_{\mathcal{F}}[\phi]\to S_{\mathcal{F}}[\phi]+S_{\text{boundary}}[\phi], \tag{52}$$

where

$$S_{\text{boundary}}[\phi]=q_0\int_{\sigma=0}A_\mu\frac{\partial\phi^\mu}{\partial\tau}d\tau+q_\pi\int_{\sigma=\pi}A_\mu\frac{\partial\phi^\mu}{\partial\tau}d\tau \tag{53}$$

corresponds to electrical charges $q_0,q_\pi$ associated with the two endpoints of the string interacting with electric field(s) present on the respective $D$-branes. In order to make the following derivation easily comprehensible, let us assume that the potential $A_\mu$ can be in an arbitrary but smooth way extended to the neighborhood of the respective brane[7] and denote the field strength of the potential $A_\mu$ by[8]

$$\Delta_{\mu\nu}=\frac{1}{2}\left(\frac{\partial A_\nu}{\partial y^\mu}-\frac{\partial A_\mu}{\partial y^\nu}\right),\quad \text{i.e.,}\quad \Delta=dA. \tag{54}$$

Consequently, the equations of motion in the bulk obtained by the variation of the action are left unchanged but we find on the boundary

$$\delta\phi\cdot N_\phi\cdot(\mathcal{F}-\mathcal{F}^t\cdot R_\phi^t+q_0\Delta\cdot(1+R_\phi^t))\cdot\partial_+\phi^t|_{\sigma=0}=0 \tag{55}$$

together with

[7]Generalization to the case when this is not possible will be explained below.

[8]Recall that $y^\mu$ are coordinates on $G$ and $\phi^\mu=y^\mu\circ g$.

$$\delta\phi \cdot N_\phi \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t_\phi - q_\pi \Delta \cdot (1 + R^t_\phi)) \cdot \partial_+ \phi^t|_{\sigma=\pi} = 0, \tag{56}$$

instead of (32). Therefore, by similar arguments as before, we find the following conditions instead of (33):

$$N \cdot \ (\mathcal{F} - \mathcal{F}^t \cdot R^t + q_0 \Delta \cdot (1 + R^t))|_{\sigma=0} = 0,$$

$$N \cdot \ (\mathcal{F} - \mathcal{F}^t \cdot R^t - q_\pi \Delta \cdot (1 + R^t))|_{\sigma=\pi} = 0.$$

Because these conditions should hold irrespective of which of the two endpoints lies on the considered brane (i.e., on any given brane a string may begin, end, or both,) we see that the endpoints are oppositely charged (and by proper choice of convention for $A_\mu$, we set the charge to unity),

$$q_0 = - q_\pi = 1. \tag{57}$$

This means that condition (33) modified by the presence of electric charge at the endpoints reads

$$N \cdot ((\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t) = 0. \tag{58}$$

In fact, recalling Eq. (34) and writing

$$N \cdot (\Delta - \Delta^t \cdot R^t) = N \cdot \Delta \cdot (\mathbf{1} + R^t) = N \cdot \Delta \cdot N^t \cdot (\mathbf{1} + R^t), \tag{59}$$

we see that only derivatives of $A_\mu$ along the brane are relevant in the variation of the action $S_{\mathcal{F}}[\phi]+S_{\text{boundary}}[\phi]$, i.e., the resulting condition (58) does not depend on the way we extend $A_\mu$ outside the brane. If such an extension is impossible, the definition (54) of $\Delta$ is obviously meaningless and must be corrected in the following way. We introduce the embedding $\iota$ of the brane $\mathcal{B}$,

$$\iota:\mathcal{B} \to G, \quad \mathcal{B} \simeq \iota(\mathcal{B}) \subset G$$

and construct the electric field on the brane as

$$\Delta_{\mathcal{B}} = d_{\mathcal{B}} A \in \Omega^2(\mathcal{B}). \tag{60}$$

Then, we may pointwise extend $\Delta_{\mathcal{B}}|_p$ to a two-form $\Delta|_{\iota(p)}$ with values in $\Omega^2_{\iota(p)}(G)$ [i.e., a two-form on $G$ in the point $\iota(p)$],

$$\Delta\ (V,W)|_{\iota(p)} = \Delta_{\mathcal{B}}\ (\mathcal{N}(V),\mathcal{N}(W))|_p, \quad p \in \mathcal{B}, V, W \in T_{\iota(p)}G \tag{61}$$

[where the natural identification $T_p\mathcal{B} \simeq \iota_*(T_p\mathcal{B}) = \text{Im}(\mathcal{N})|_{\iota(p)}$ is assumed]. With this understanding in mind, condition (58) remains the same as before but supplemented by a consequence of (61)

$$\Delta = N \cdot \Delta \cdot N^t. \tag{62}$$

Consequently, even if the target group $G$ is foliated by $D$-branes and $\Delta$ constructed as a collection of $\Delta$'s on different branes may be well defined and smooth on $G$ (or its open subset), $\Delta$ may nonetheless not be closed—only its restrictions $\Delta|_{\mathcal{B}}$ to the respective branes need to be closed in order to allow the potential $A_\mu$ along the brane.

In the following, we shall use condition (58) to look for a suitable background electric field strength $\Delta$ such that the boundary equations of motion are satisfied in the transformed models. Taking into account (59), we see that (58) determines $\Delta = N \cdot \Delta \cdot N^t$ uniquely and generically smoothly (except when $N$ changes rank). The self-consistency of such a procedure of course requires that $\Delta$ found in this way is closed along the branes, i.e.,

$$N_\kappa{}^\nu N_\lambda{}^\rho N_\mu{}^\sigma \partial_{[\nu}\,\Delta_{\;\rho\sigma]} = 0 \tag{63}$$

and hence[9] gives rise to the potential $A_\mu$.

We should note that the case of free endpoints, i.e., purely Neumann boundary condition $R_\rho(g) = F^t(g)\cdot F^{-1}(g)$, was investigated in Ref. 11. The approach used there was based on symplectic geometry and it was shown that the Poisson-Lie $T$-dual configuration corresponds to $D$-branes as symplectic leaves of the Poisson structure on the dual group $\widetilde{G}$ [once one fixes one end of the dual string at the origin of $\widetilde{G}$ using the freedom of a constant shift (13)] and that the correction $\Delta$ in this case exists and is obtained from the Semenov-Tian-Shansky symplectic form on the Drinfel'd double as a symplectic form on the symplectic leaves and is therefore closed along the branes. These results are in accord with the analysis here. Also, it is clear from the conclusions of Ref. 11 that in this particular case, the integrability condition (28) is automatically satisfied on the dual since the symplectic leaves are submanifolds.

## V. THREE-DIMENSIONAL EXAMPLE

As mentioned in Sec. I, there are several explicitly solvable $\sigma$-models whose solutions are related by the Poisson-Lie $T$-plurality. We can construct their gluing matrices corresponding to $D$-branes and check the equivalence of Eqs. (26) and (43). Here, we present a three-dimensional example, where one of the solutions is flat with vanishing B-field, while the $T$-plural one is curved and torsionless. They are given by a six-dimensional Drinfel'd double with decompositions into, on the one hand, the Bianchi 5 and Bianchi 1 algebras and, on the other hand, the Bianchi $6_0$ and Bianchi 1 algebras. On Bianchi 5, the background is given by

$$E_0 = F(g) = \begin{pmatrix} 0 & 0 & \kappa \\ 0 & \kappa & 0 \\ \kappa & 0 & 0 \end{pmatrix}, \quad \kappa \in \mathbb{R}. \tag{64}$$

The right-invariant vielbein in a convenient parametrization $g = g(y^\mu)$ of the solvable group corresponding to Bianchi 5 is

$$e(g) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-y^1} & 0 \\ 0 & 0 & e^{-y^1} \end{pmatrix}, \tag{65}$$

so that the tensor field of the conformal $\sigma$-model that lives on this group reads

$$\mathcal{F}_{\mu\nu}(y) = \begin{pmatrix} 0 & 0 & \kappa e^{-y^1} \\ 0 & \kappa e^{-2y^1} & 0 \\ \kappa e^{-y^1} & 0 & 0 \end{pmatrix}. \tag{66}$$

The metric of this model is indefinite and flat. The general solution of the equations of motion is[7]

[9] up to possible topological obstructions which we shall neglect here

$$\phi^1(x_+,x_-) = -\ln(-W_1 - Y_1),$$

$$\phi^2(x_+,x_-) = -\frac{W_2 + Y_2}{W_1 + Y_1}, \tag{67}$$

$$\phi^3(x_+,x_-) = W_3 + Y_3 + \frac{(W_2 + Y_2)^2}{2(W_1 + Y_1)},$$

where $W_j = W_j(x_+)$ and $Y_j = Y_j(x_-)$ are arbitrary functions.

The $\sigma$-model related to that on Bianchi 5 by the Poisson-Lie $T$-plurality lives on the solvable group corresponding to Bianchi $6_0$ and its tensor field obtained from

$$\hat{E}_0 = \hat{F}(\hat{g}) = \begin{pmatrix} \frac{1}{\kappa} & \frac{1}{\kappa} & \frac{\kappa}{2} \\ \frac{1}{\kappa} & \frac{1}{\kappa} & -\frac{\kappa}{2} \\ \frac{\kappa}{2} & -\frac{\kappa}{2} & 0 \end{pmatrix} \tag{68}$$

and

$$\hat{e}(\hat{g}) = \begin{pmatrix} \cosh \hat{y}^3 & -\sinh \hat{y}^3 & 0 \\ -\sinh \hat{y}^3 & \cosh \hat{y}^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{69}$$

reads

$$\hat{\mathcal{F}}_{\mu\nu}(\hat{y}) = \begin{pmatrix} \frac{1}{\kappa}e^{-2\hat{y}^3} & \frac{1}{\kappa}e^{-2\hat{y}^3} & \frac{\kappa}{2}e^{\hat{y}^3} \\ \frac{1}{\kappa}e^{-2\hat{y}^3} & \frac{1}{\kappa}e^{-2\hat{y}^3} & -\frac{\kappa}{2}e^{\hat{y}^3} \\ \frac{\kappa}{2}e^{\hat{y}^3} & -\frac{\kappa}{2}e^{\hat{y}^3} & 0 \end{pmatrix}. \tag{70}$$

The Ricci tensor of this metric is nontrivial so that the background is curved but has a zero Gauss curvature.

The transformation (14) between the bases of decompositions of the Lie algebra of the Drinfel'd double into Bianchi $5\dot{+}$Bianchi 1 and Bianchi $6_0\dot{+}$Bianchi 1 is given by the matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \tag{71}$$

and the coordinate transformation on the Drinfel'd double that follows from this reads (see Ref. 7)

$$\hat{y}^1 = -y^3 + \tfrac{1}{2}\tilde{h}_2,$$

$$\hat{y}^2 = y^3 + \tfrac{1}{2}\tilde{h}_2, \tag{72}$$

$$\hat{y}^3 = -y^1,$$

$$\bar{h}_1 = -\tfrac{1}{2}\tilde{h}_3 + y^2,$$

$$\bar{h}_2 = \tfrac{1}{2}\tilde{h}_3 + y^2, \tag{73}$$

$$\bar{h}_3 = -\tilde{h}_1 + \tilde{h}_2 y^2,$$

where $y$, $\tilde{h}$, $\hat{y}$, $\bar{h}$ are coordinates on the respective subgroups $G$, $\tilde{G}$, $\hat{G}$, $\bar{G}$ that correspond to the different decompositions of the Drinfel'd double. Inserting Eq. (67) and the solution of Eqs. (10) and (11), into Eq. (72), we obtain the solution[7] of the equations of motion for the $\sigma$-model in the curved background given by $\hat{F}$,

$$\hat{\phi}^1(x_+,x_-) = \frac{1}{2}\kappa[Y_1(x_-)W_2(x_+) - Y_2(x_-)W_1(x_+)] - [W_3(x_+) + Y_3(x_-)] - \frac{1}{2}\frac{[W_2(x_+) + Y_2(x_+)]^2}{(W_1(x_+) + Y_1(x_-)]}$$
$$+ \frac{1}{2}\kappa(\alpha(x_+) + \beta(x_-)),$$

$$\hat{\phi}^2(x_+,x_-) = \frac{1}{2}\kappa[Y_1(x_-)W_2(x_+) - Y_2(x_-)W_1(x_+)] + [W_3(x_+) + Y_3(x_-)] + \frac{1}{2}\frac{(W_2(x_+) + Y_2(x_-))^2}{W_1(x_+) + Y_1(x_-)}$$
$$+ \frac{1}{2}\kappa(\alpha(x_+) + \beta(x_-)), \tag{74}$$

$$\hat{\phi}^3(x_+,x_-) = \ln(-W_1(x_+) - Y_1(x_-)),$$

where $\alpha$, $\beta$ satisfy (primes denote differentiation)

$$\alpha' = W_1 W_2' - W_2 W_1',$$
$$\beta' = Y_2 Y_1' - Y_1 Y_2'. \tag{75}$$

### A. *D*-branes

In the following, we analyze examples of $D$-branes for which the adapted coordinates $\lambda^\alpha$ of the flat model are equal to those that bring the metric of the flat model to the diagonal form

$$F_{kl}(\lambda) = \begin{pmatrix} -\kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix},$$

namely,

$$\lambda^1(y) = \lambda_0^1 - \tfrac{1}{\sqrt{2}}\left[y^3 + \tfrac{1}{2}(y^2)^2 e^{-y^1} + e^{-y^1}\right],$$

$$\lambda^2(y) = \lambda_0^2 + y^2 e^{-y^1}, \tag{76}$$

$$\lambda^3(y) = \lambda_0^3 + \tfrac{1}{\sqrt{2}}\left[y^3 + \tfrac{1}{2}(y^2)^2 e^{-y^1} - e^{-y^1}\right].$$

In these coordinates, the gluing matrices $R_\lambda$ by assumption become diagonal.[10]

- $D2$-branes. The Dirichlet projector is zero (and the Neumann projector is the identity) in this case and as the tensor $\mathcal{F}$ is symmetric, it follows from Eq. (33) that the gluing matrices are

$$R_\lambda = R_\phi = R_\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{77}$$

The conditions (36) are trivially satisfied. The condition (26), or equivalently (22), then gives the boundary conditions for the solution (67),

$$W_j'(x_+)|_{\sigma=0,\pi} = Y_j'(x_-)|_{\sigma=0,\pi}, \quad j = 1,2,3. \tag{78}$$

From Eq. (44), we get

$$\hat{R}_\rho = \hat{R}_\phi = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{79}$$

This matrix has eigenvalues $(-1,1,1)$ and the eigenvector corresponding to the eigenvalue $-1$ is spacelike in the (curved) metric (70) so that the $D2$-brane is transformed to a $D1$-brane. The Dirichlet projector obtained from Eqs. (23) and (27) is

$$\hat{Q} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{80}$$

and the conditions (36) are satisfied for the matrix (79). Using Eqs. (74) and (78), one can verify that

$$\partial_- \hat{\phi}|_{\sigma=0,\pi} = \partial_+ \hat{\phi} \cdot \hat{R}_\phi|_{\sigma=0,\pi}, \tag{81}$$

which is equivalent to Eq. (43). Note that unlike the $D1$-branes and $D0$-branes discussed below, in this case neither the matrix $R_\rho$ nor $\hat{R}_\rho$ depends on elements of the groups $G$ and $\hat{G}$.

- $D1$-branes. We have chosen the branes as coordinate planes of the flat coordinates, i.e.,

$$R_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{82}$$

which in $y$ coordinates gives the $y$-dependent gluing matrix

$$R_\phi = \begin{pmatrix} \frac{(y^2)^2}{2} & \frac{1}{2}y^2[(y^2)^2-2] & -\frac{1}{4}e^{-y^1}[(y^2)^2-2]^2 \\ -y^2 & 1-(y^2)^2 & \frac{1}{2}e^{-y^1}y^2[(y^2)^2-2] \\ -e^{y^1} & -e^{y^1}y^2 & \frac{(y^2)^2}{2} \end{pmatrix}. \tag{83}$$

The Dirichlet projector obtained from Eqs. (23) and (27) is

$$Q = \begin{pmatrix} \frac{1}{4}[2-(y^2)^2] & \frac{1}{4}[2(y^2)-(y^2)^3] & \frac{1}{8}e^{-y^1}[(y^2)^2-2]^2 \\ \frac{1}{2}(y^2)^2 & \frac{1}{2}(y^2)^2 & -\frac{1}{4}e^{-y^1}(y^2)[(y^2)^2-2] \\ \frac{1}{2}e^{y^1} & \frac{1}{2}e^{y^1}(y^2) & \frac{1}{4}[2-(y^2)^2] \end{pmatrix}, \tag{84}$$

and the conditions (36) are satisfied. The condition (26) then gives

$$\begin{aligned} W_1'(x_+)|_{\sigma=0,\pi} &= Y_3'(x_-)|_{\sigma=0,\pi}, \\ W_2'(x_+)|_{\sigma=0,\pi} &= Y_2'(x_-)|_{\sigma=0,\pi}, \\ W_3'(x_+)|_{\sigma=0,\pi} &= Y_1'(x_-)|_{\sigma=0,\pi}. \end{aligned} \tag{85}$$

From Eq. (44), we obtain $\hat{R}_\rho$ and $\hat{R}_\phi$, which, however, are too complicated to be displayed here. The matrix $\hat{R}_\phi$ depends on the coordinates on both $\hat{G}$ and $G$ and consequently on $\bar{G}$; nevertheless, we have checked that the boundary condition (43) for the solution (74) implies again the relations (85). In this sense, the conditions (26) and (43) are equivalent.

The eigenvalues of $\hat{R}_\phi$ are $1, -1+(y^2)^2 \pm \sqrt{(y^2)^4-2(y^2)^2}$ so that for $y^2 \neq 0$, the projectors are $\hat{Q}=0, \hat{N}=\mathbf{1}$, and the condition (33) is not satisfied.

On the other hand, the hypersurface $y^2=0$ does not coincide with a $D1$-brane in the original model since the tangent vector $\partial_{y^2}|_{y^2=0}$ is Neumann. Consequently, if at a given time the endpoint of a string is located at $y^2=0$, it might not stay there at later times. We conclude that in this case, the transformed $D$-brane configuration is not well defined due to the dependence of $\hat{R}_\rho$ on the coordinates on $\bar{G}$.

- $D0$-branes. We choose

$$R_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{86}$$

so that

$$R_\phi = \begin{pmatrix} \frac{(y^2)^2}{2} & \frac{(y^2)^3}{2}+y^2 & -\frac{1}{4}e^{-y^1}[(y^2)^2+2]^2 \\ -y^2 & -(y^2)^2-1 & \frac{1}{2}e^{-y^1}y^2[(y^2)^2+2] \\ -e^{y^1} & -e^{y^1}y^2 & \frac{(y^2)^2}{2} \end{pmatrix}. \tag{87}$$

The Dirichlet projector is

$$Q = \begin{pmatrix} \frac{1}{4}[2-(y^2)^2] & \frac{1}{4}[-(y^2)^3 - 2(y^2)] & \frac{1}{8}e^{-y^1}[(y^2)^2+2]^2 \\ \frac{(y^2)}{2} & \frac{1}{2}[(y^2)^2+2] & -\frac{1}{4}e^{-y^1}(y^2)[(y^2)^2+2] \\ \frac{1}{2}e^{y^1} & \frac{1}{2}e^{y^1}(y^2) & \frac{1}{4}[2-(y^2)^2] \end{pmatrix}, \tag{88}$$

and the conditions (36) are satisfied. The condition (26) yields

$$\begin{aligned} W_1'(x_+)|_{\sigma=0,\pi} &= -Y_3'(x_-)|_{\sigma=0,\pi}, \\ W_2'(x_+)|_{\sigma=0,\pi} &= -Y_2'(x_-)|_{\sigma=0,\pi}, \\ W_3'(x_+)|_{\sigma=0,\pi} &= -Y_1'(x_-)|_{\sigma=0,\pi}. \end{aligned} \tag{89}$$

The matrix $\hat{R}_\phi$ is again rather complicated and depends on the coordinates of both $G$ and $\hat{G}$, but once again using Eqs. (74) and (89), one can verify that conditions (26) and (43) are equivalent in the sense explained above. The eigenvalues of $\hat{R}_\phi$ are $-1,1+(y^2)^2 \pm \sqrt{(y^2)^4+2(y^2)^2}$ and the Dirichlet projector $\hat{Q}$ obtained from Eqs. (23) and (27) reads

$$\hat{Q} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{e^{2y^1+\hat{y}^3}}{2(y^2)^2+4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{e^{2y^1+\hat{y}^3}}{2(y^2)^2+4} \\ \frac{1}{4}e^{-2y^1-\hat{y}^3}((y^2)^2+2) & -\frac{1}{4}e^{-2y^1-\hat{y}^3}((y^2)^2+2) & \frac{1}{2} \end{pmatrix}. \tag{90}$$

Due to (72) and (73), namely, $y^2=\frac{1}{2}(\bar{h}_1+\bar{h}_2)$, the projector $\hat{Q}$ depends both on $\hat{G}$ and $\bar{G}$. The conditions (36) are again satisfied only for $y^2=0$ but now the tangent vector $\partial_{y^2}|_{y^2=0}$ is Dirichlet. We can therefore consistently restrict ourselves in the original model to $D0$-branes inside the hypersurface $y^2=0$. Their plural counterparts are given by a gluing matrix of the form

$$\hat{R}_\phi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}e^{-\hat{y}^3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}e^{-\hat{y}^3} \\ -e^{\hat{y}^3} & e^{\hat{y}^3} & 0 \end{pmatrix}, \tag{91}$$

where we have used the coordinate (72). Its eigenvalues are $(-1,1,1)$ and the eigenvector corresponding to the eigenvalue $-1$ is spacelike so that the matrix (91) defines a $D1$-brane in the dual model.

- $D(-1)$-branes. We have

$$R_\lambda = R_\phi = R_\rho = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{92}$$

The Dirichlet projector is the identity in this case so that the conditions (36) are trivially satisfied. The condition (26) then gives the boundary conditions for the solution (67),

$$W'_j(x_+)|_{\sigma=0,\pi} = -Y'_j(x_-)|_{\sigma=0,\pi}, \quad j=1,2,3. \tag{93}$$

From Eq. (44), we find

$$\hat{R}_\rho = \hat{R}_\phi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{94}$$

and the $T$-plural Dirichlet projector is

$$\hat{Q} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{95}$$

The conditions (36) are satisfied and the condition (93) implies both Eqs. (26) and (43). The matrix (94) has eigenvalues $(-1,-1,1)$, where $+1$ corresponds to a spacelike direction. Hence, we get a Euclidean $D0$-brane. Similarly to the $D2$-brane case, also here neither $R_\rho$ nor $\hat{R}_\rho$ depends on elements of the groups $G$ and $\hat{G}$.

### B. Gluing matrices that produce $\hat{R}$ dependent only on $\hat{G}$

The lesson we have learned from the previous subsection is that in some cases the transformation of coordinates (72) may cure the problem of dependence of the gluing matrix $\hat{R}$ on elements of the group $\bar{G}$. In particular, in our three-dimensional example, it turned out that if $D0$-branes in Bianchi 5 are contained in the hypersurface of constant $y^2$ located at $y^2=0$, then due to Eq. (72) the plural gluing matrices are well defined.

In the present section, we address the problem of coordinate cross dependence from another point of view. We shall assume that the plural gluing matrix depends on elements of $\hat{G}$ only, i.e., it is independent of the dual coordinates on $\bar{G}$, and we derive the gluing matrices on both sides of the plurality that make this assumption possible. Inspecting the transformation formula (44) for the gluing operator, we find that the $T$-plural gluing matrix $\hat{R}_\rho$ is a function on $\hat{G}$ if and only if the matrix-valued function

$$C(g) = F^{-t}(g) \cdot R_\rho(g) \cdot F(g)$$

extended to a function on the whole Drinfel'd double as $C_D(l)=C(g)$, where $l=g\tilde{h}$, satisfies

$$C_D(\hat{g}\bar{h}) = C_D(\hat{g}). \tag{96}$$

In our particular setting, where the relations between original and new coordinates on the Drinfel'd double $D$ are given by Eqs. (72) and (73), we find that (only) the following combinations of $\hat{y}$'s can be written in terms of the original $y$'s:

$$\hat{y}^2 - \hat{y}^1 = 2y^3, \quad \hat{y}^3 = -y^1.$$

Consequently, if the original gluing matrix has the form

$$R_\rho(g) = F^t(g) \cdot C \cdot F^{-1}(g), \tag{97}$$

where $C=C(y^1,y^3)$, then the gluing matrices $\hat{R}_\rho$ given by Eq. (44) and $\hat{R}_\phi$, $\hat{R}_\lambda$ given by Eq. (25) can be expressed as functions on $\hat{G}$ only, i.e., they are well defined. The condition (30) that $R_\rho$ of the form (97) preserve the metric yields

$$C \cdot (E_0^{-1} + E_0^{-t}) \cdot C^t = (E_0^{-1} + E_0^{-t}). \tag{98}$$

In other words, the matrices $C$ belong to the representation of the group $O(n, \dim G-n)$ given by the constant symmetric matrix $(E_0^{-1}+E_0^{-t})$ with signature $n$.

For $E_0$ of the form (64), we get the following possibilities:

$$C = \begin{pmatrix} -\frac{\alpha^2}{2\beta} & \alpha & \beta \\ -\epsilon\frac{\alpha}{\beta} & \epsilon & 0 \\ \frac{1}{\beta} & 0 & 0 \end{pmatrix}, \tag{99}$$

$$C = \begin{pmatrix} \frac{(\alpha+\epsilon)^2}{4\beta} & \frac{1-\alpha^2}{2\gamma} & -\frac{(\alpha-\epsilon)^2\beta}{2\gamma^2} \\ -\frac{(\alpha+\epsilon)\gamma}{2\beta} & \alpha & \frac{(\alpha-\epsilon)\beta}{\gamma} \\ -\frac{\gamma^2}{2\beta} & \gamma & \beta \end{pmatrix}, \tag{100}$$

$$C = \begin{pmatrix} \frac{1}{\beta} & \alpha & -\frac{\alpha^2\beta}{2} \\ 0 & \epsilon & -\epsilon\alpha\beta \\ 0 & 0 & \beta \end{pmatrix}, \tag{101}$$

where $\epsilon = \pm 1$ and $\alpha$, $\beta$, $\gamma$ are arbitrary functions of $y^1$ and $y^3$. In addition, the matrices $R_\phi$ and $\hat{R}_\phi$ calculated from Eqs. (25), (44), and (97) must satisfy the conditions (36) so that further restrictions on the matrices $C$ are imposed.

- Case (99): The conditions (36) for $R$ are satisfied only if $\alpha=0$. The gluing matrices then read

$$R_\rho = \begin{pmatrix} 0 & 0 & \frac{1}{\beta} \\ 0 & \epsilon & 0 \\ \beta & 0 & 0 \end{pmatrix}, \quad \epsilon = \pm 1, \quad \beta = \beta(y^1, y^3), \tag{102}$$

$$\hat{R}_\rho = \frac{1}{2}\begin{pmatrix} -\epsilon & -\epsilon & \beta \\ -\epsilon & -\epsilon & -\beta \\ \frac{2}{\beta} & -\frac{2}{\beta} & 0 \end{pmatrix}, \quad \epsilon = \pm 1, \quad \beta = \beta\left(-\hat{y}^3, \frac{\hat{y}^2 - \hat{y}^1}{2}\right). \tag{103}$$

  The conditions (36) are satisfied for $\hat{R}$ as well. This corresponds to the transformation of $D1$-branes to $D0$-branes for $\epsilon=+1$ and $D0$-branes to $D1$-branes for $\epsilon=-1$.
- Case (100): The conditions (36) are satisfied for $R$ only if $\alpha=-\epsilon-2\beta$. The gluing matrices then read

$$R_\rho = \begin{pmatrix} \beta & \gamma & -\dfrac{\gamma^2}{2\beta} \\ -\dfrac{2(\beta+\epsilon)\beta}{\gamma} & -\epsilon - 2\beta & \gamma \\ -\dfrac{2(\beta+\epsilon)^2\beta}{\gamma^2} & -\dfrac{2(\beta+\epsilon)\beta}{\gamma} & \beta \end{pmatrix}, \quad \beta = \beta(y^1,y^3), \quad \gamma = \gamma(y^1,y^3), \tag{104}$$

$$\hat{R}_\rho = \begin{pmatrix} \dfrac{(\beta+\epsilon)\beta\kappa^2 + \gamma(3\beta+\epsilon)\kappa + 2\gamma^2}{2\kappa\gamma} & \dfrac{(\beta+\epsilon)\beta\kappa^2 + \gamma(\beta+\epsilon)\kappa - 2\gamma^2}{2\kappa\gamma} & -\dfrac{(2\gamma+\kappa(\beta+\epsilon))(\beta+\epsilon)\beta}{\kappa\gamma^2} \\ \dfrac{-(\beta+\epsilon)\beta\kappa^2 + \gamma(\beta+\epsilon)\kappa + 2\gamma^2}{2\kappa\gamma} & \dfrac{-(\beta+\epsilon)\beta\kappa^2 + \gamma(3\beta+\epsilon)\kappa - 2\gamma^2}{2\kappa\gamma} & \dfrac{(\kappa(\beta+\epsilon) - 2\gamma)(\beta+\epsilon)\beta}{\kappa\gamma^2} \\ -\dfrac{\gamma(\gamma+\kappa\beta)}{2\beta} & \dfrac{\gamma(\gamma-\kappa\beta)}{2\beta} & \beta \end{pmatrix}, \tag{105}$$

where $\beta=\beta(-\hat{y}^3,(\hat{y}^2-\hat{y}^1)/2)$, $\gamma=\gamma(-\hat{y}^3,(\hat{y}^2-\hat{y}^1)/2)$. For $\epsilon=-1$, the dependence of $\beta$ and $\gamma$ on $y^1$, $y^3$ is constrained by the condition (28) that yields

$$e^{y^1}\gamma^2\left(\gamma\frac{\partial\beta}{\partial y^3} - \beta\frac{\partial\gamma}{\partial y^3}\right) = 2\beta^2\left(\gamma\frac{\partial\beta}{\partial y^1} + \frac{\partial\gamma}{\partial y^1} - \beta\frac{\partial\gamma}{\partial y^1}\right). \tag{106}$$

For $\epsilon=1$, we do not get any constraint on the functions $\beta$, $\gamma$.

The condition (33) is not satisfied for the matrix $\hat{R}$ unless we replace $\hat{\mathcal{F}}$ by $\hat{\mathcal{F}}+\hat{\Delta}$, where $\hat{N}\cdot\hat{\Delta}\cdot\hat{N}^t=\hat{\Delta}$.

For $\epsilon=1$,

$$\hat{\Delta} = \begin{pmatrix} 0 & -\dfrac{\beta}{\gamma} & -\dfrac{\gamma e^{-\hat{y}^3}}{2+2\beta} \\ \dfrac{\beta}{\gamma} & 0 & -\dfrac{\gamma e^{-\hat{y}^3}}{2+2\beta} \\ \dfrac{\gamma e^{-\hat{y}^3}}{2+2\beta} & \dfrac{\gamma e^{-\hat{y}^3}}{2+2\beta} & 0 \end{pmatrix}, \tag{107}$$

and it is closed along the branes for arbitrary $\beta$, $\gamma$. This case corresponds to the transformation of $D0$-branes to $D1$-branes.

For $\epsilon=-1$,

$$\hat{\Delta} = \begin{pmatrix} 0 & -\dfrac{\beta-1}{\gamma} & -\dfrac{\gamma e^{-\hat{y}^3}}{2\beta} \\ \dfrac{\beta-1}{\gamma} & 0 & -\dfrac{\gamma e^{-\hat{y}^3}}{2\beta} \\ \dfrac{\gamma e^{-\hat{y}^3}}{2\beta} & \dfrac{\gamma e^{-\hat{y}^3}}{2\beta} & 0 \end{pmatrix}, \tag{108}$$

and it is closed along the branes due to (106). This case corresponds to the transformation of $D1$-branes to $D2$-branes.

- Case (101): The conditions (36) for both $R$ and $\hat{R}$ are satisfied if $\beta=\pm 1$ and $\alpha=0$. This corresponds to the transformation of $D2$-branes to $D1$-branes and $D(-1)$-branes to $D0$-branes,

$$R_\rho = \pm\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \hat{R}_\rho = \pm\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{109}$$

as presented in Sec. V A and

$$R_\rho = \pm\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \hat{R}_\rho = \pm\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{110}$$

which correspond to transformations of $D1$-branes to $D2$-branes and of $D0$-branes to $D(-1)$-branes.

Besides that, the conditions (36) for $R$ are also satisfied for

$$\beta = -\epsilon, \quad \frac{\partial\alpha}{\partial y^1} = 0. \tag{111}$$

However, to satisfy the condition (27) for $\hat{R}$, i.e., $\hat{R}\cdot\hat{Q}=\hat{Q}\cdot\hat{R}=-\hat{Q}$ with $\beta=-\epsilon=-1$, we must set $\alpha=0$. Thus, for $\beta=-\epsilon=-1$, we see that in general (i.e., for $\alpha\neq 0$) the Poisson-Lie $T$-plurality does not preserve the condition (27).

If $\alpha\neq 0$ and $\beta=-\epsilon=1$, then we have $\hat{Q}=0$ and the condition (27) holds trivially. We can satisfy the condition (33) by replacing $\hat{\mathcal{F}}$ by $\hat{\mathcal{F}}+\hat{\Delta}$, where

$$\hat{\Delta} = \begin{pmatrix}0 & \frac{1}{2}\alpha & 0\\ -\frac{1}{2}\alpha & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{112}$$

This form is closed due to (111). The gluing matrices in this case,

$$R_\rho = \begin{pmatrix}1 & 0 & 0\\ \alpha & -1 & 0\\ -\frac{\alpha^2}{2} & \alpha & 1\end{pmatrix}, \quad \hat{R}_\rho = \begin{pmatrix}1-\frac{\alpha\kappa}{4} & -\frac{\alpha\kappa}{4} & \frac{\alpha}{\kappa}-\frac{\alpha^2}{4}\\ \frac{\alpha\kappa}{4} & \frac{\alpha\kappa}{4}+1 & \frac{\alpha(\alpha\kappa+4)}{4\kappa}\\ 0 & 0 & 1\end{pmatrix}, \tag{113}$$

correspond to the transformation of $D1$-branes to $D2$-branes.

We remark that in three dimensions, the integrability condition (28) is nontrivial only if the rank of the Neumann projector $N$ is equal to 2; otherwise, the distribution $\Delta=\mathrm{Im}(N)$ is integrable on dimensional grounds. In two dimensions, investigated below, the condition (28) is always trivially satisfied.

## VI. TWO-DIMENSIONAL EXAMPLE

The only $\sigma$-models with two-dimensional targets that can be transformed under $T$-plurality with nonisomorphic decompositions of a Drinfel'd double are generated by the semi-Abelian four-dimensional Drinfel'd double of Ref. 12. It has decompositions into two different pairs of maximally isotropic Lie subalgebras, namely, the semi-Abelian Manin triple with basis $T_1, T_2, \tilde{T}^1, \tilde{T}^2$ and Lie brackets (only nontrivial brackets are displayed)

$$[T_1,T_2]=T_2, \quad [\tilde{T}^2,T_1]=\tilde{T}^2, \quad [\tilde{T}^2,T_2]=-\tilde{T}^1, \tag{114}$$

and the so-called type B non-Abelian Manin triple with basis $\hat{T}_1, \hat{T}_2, \bar{T}^1, \bar{T}^2$ and Lie brackets

$$[\hat{T}_1,\hat{T}_2]=\hat{T}_2, \quad [\bar{T}^1,\bar{T}^2]=\bar{T}^1,$$
$$[\hat{T}_1,\bar{T}^1]=\hat{T}_2, \quad [\hat{T}_1,\bar{T}^2]=-\hat{T}_1-\bar{T}^2, \quad [\hat{T}_2,\bar{T}^2]=\bar{T}^1. \tag{115}$$

A simple transformation between the bases of these two decompositions is given by

$$\hat{T}_1=-T_1+T_2, \quad \hat{T}_2=\tilde{T}^1+\tilde{T}^2,$$
$$\bar{T}^1=\tilde{T}^2, \quad \bar{T}^2=T_1, \tag{116}$$

which corresponds to the transformation matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{117}$$

The coordinate transformation on the Drinfel'd double that follows from this reads

$$\hat{x}^1=-\ln(-x^2+1), \quad \hat{x}^2=-\frac{\tilde{x}^1}{x^2-1},$$
$$\bar{x}^1=\frac{\tilde{x}^1\exp(x^1)+x^2\tilde{x}^2-\tilde{x}^2}{x^2-1}, \quad \bar{x}^2=-\ln(-x^2+1)+x^1. \tag{118}$$

We shall consider examples of two-dimensional $\sigma$-models given by the matrices

$$E_0=\begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}, \quad \hat{E}_0=\begin{pmatrix} \kappa & 1 \\ -1 & 1 \end{pmatrix}, \tag{119}$$

where $\kappa$ is a real constant. The corresponding tensors $\mathcal{F}$, $\hat{\mathcal{F}}$ are calculated from Eqs. (7) and (17), where

$$g=\exp(x^2T_2)\exp(x^1T_1), \quad \hat{g}=\exp(\hat{x}^2\hat{T}_2)\exp(\hat{x}^1\hat{T}_1). \tag{120}$$

They read

$$\mathcal{F}(x^\mu)=\begin{pmatrix} \kappa(x^2)^2+1 & -\kappa x^2 \\ -\kappa x^2 & \kappa \end{pmatrix}, \tag{121}$$

$$\hat{\mathcal{F}}(\hat{x}^\mu)=\frac{1}{-2e^{\hat{x}^1}\kappa+\kappa+e^{2\hat{x}^1}(\kappa+1)}\begin{pmatrix} (\hat{x}^2)^2+\kappa & -\kappa+e^{\hat{x}^1}(\kappa+1)-\hat{x}^2 \\ \kappa-e^{\hat{x}^1}(\kappa+1)-\hat{x}^2 & 1 \end{pmatrix}. \tag{122}$$

Unfortunately, the metrics of both models are curved and we are not able to solve the equations of motion. Nevertheless, we can at least find the gluing matrices that satisfy the conditions (36). Moreover, we require that the gluing matrices depend only on the coordinates where the $\sigma$-models live so that we have to take $R_\rho$ of the form (97), with $C$ depending only on $x^2$ in order to satisfy Eq. (96).

The condition (98) restricts $C$ to the form

$$C = \begin{pmatrix} \epsilon_1\sqrt{1-\gamma^2\kappa} & \epsilon_2\gamma\kappa \\ \gamma & -\epsilon_1\epsilon_2\sqrt{1-\gamma^2\kappa} \end{pmatrix}, \tag{123}$$

where $\gamma$ is an arbitrary function of $x^2$ and $\epsilon_1$, $\epsilon_2=\pm 1$. The conditions (27) and (31) are then satisfied for all corresponding matrices $R$. The condition (33) is satisfied only if $\epsilon_2=1$ or $\epsilon_2=-1$, $\gamma=0$.

If $\epsilon_2=-1$, $\gamma=0$, then the conditions (36) are satisfied for the transformed $\sigma$-model as well. The gluing matrices are

$$R_\rho = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_1 \end{pmatrix}, \quad \hat{R}_\rho = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & -\epsilon_1 \end{pmatrix}, \tag{124}$$

so that the boundary conditions for the $\sigma$-model on $G$ are purely Dirichlet or purely Neumann. Interpretation of the boundary condition for the $\sigma$-model on $\hat{G}$ as either usual $D0$-branes or Euclidean (spacelike) $D1$-branes depends on the signature of the metric, i.e., on the sign of $\kappa$.

If $\epsilon_2=1$, then

$$R_\rho = \begin{pmatrix} -\epsilon_1\epsilon_2\sqrt{1-\gamma^2\kappa} & \gamma \\ \epsilon_2\gamma\kappa & \epsilon_1\sqrt{1-\gamma^2\kappa} \end{pmatrix}, \tag{125}$$

The transformed gluing matrix $\hat{R}_\rho$ is easily obtained from (44) but it is too complicated to display here. The conditions (27) and (31) are satisfied for all these matrices $\hat{R}_\rho$. The condition (33) can be always satisfied by replacing $\hat{\mathcal{F}}$ by $\hat{\mathcal{F}}+\hat{\Delta}$, where

$$\hat{\Delta} = \begin{pmatrix} 0 & \hat{\Delta}_{12} \\ -\hat{\Delta}_{12} & 0 \end{pmatrix}, \tag{126}$$

$$\hat{\Delta}_{12} = \frac{1+\gamma\kappa-\epsilon_1\sqrt{1-\gamma^2\kappa}+e^{\hat{x}^1}(\gamma(1-\kappa)+2\epsilon_1\sqrt{1-\gamma^2\kappa})}{\gamma\kappa+e^{2\hat{x}^1}(\gamma(-1+\kappa)-2\epsilon_1\sqrt{1-\gamma^2\kappa})+2e^{\hat{x}^1}(-(\gamma\kappa)+\epsilon_1\sqrt{1-\gamma^2\kappa})}. \tag{127}$$

In the case when the denominator of $\hat{\Delta}_{12}$ vanishes, i.e., for $\hat{x}^1$ satisfying (recall that $\gamma$ is a function of $\hat{x}^1$)

$$\frac{\epsilon_3-\epsilon_1\gamma\kappa+\sqrt{1-\gamma^2\kappa}}{\epsilon_1\gamma(1-\kappa)+2\sqrt{1-\gamma^2\kappa}} = e^{\hat{x}^1}, \tag{128}$$

where $\epsilon_3=\pm 1$, we get

$$\hat{R}_\rho = -\epsilon_1\epsilon_3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{129}$$

The eigenvalues of $R_\rho$ are $+1$, $-1$ corresponding to either usual $D0$-branes or Euclidean $D1$-branes, while the boundary conditions for the $\sigma$-model on $\hat{G}$ are purely Neumann except for $\hat{x}^1$ satisfying (128) with $\epsilon_3=\epsilon_1$ in which case they are purely Dirichlet and $\hat{\Delta}_{12}$ becomes singular.[10]

## VII. CONCLUSIONS

We have derived a formula (44) for the transformation of boundary conditions under the Poisson-Lie $T$-plurality. The examples in Sec. V A confirm that the formula works for solutions of

[10] whereas when (128) holds with $\epsilon_3=-\epsilon_1$, we have $\hat{R}_\rho=\mathbf{1}$ and the singularity of $\hat{\Delta}_{12}$ is only apparent—it becomes an expression of the form $\frac{0}{0}$ with a finite and well-defined limit.

the equations of motion of the $\sigma$-models. This is not surprising since it was derived using these equations. The problem is that the transformed gluing matrix may depend on elements of the original group (and hence on elements of the dual group $\bar{G}$), so that only special forms of gluing matrices are transformable under the Poisson-Lie $T$-plurality.

To ensure that the gluing matrices transformed by the Poisson-Lie $T$-plurality depend only on the coordinates of the groups where the $\sigma$-models live, we can restrict them to the form (97)

$$R_\rho = F^t(g) \cdot C \cdot F^{-1}(g).$$

The matrix $C$ must be constant or depend only on a particular subset of coordinates on $G$ that transform into coordinates on $\hat{G}$.

Another problem is that not all conditions (36) for consistent $D$-branes are preserved under the Poisson-Lie $T$-plurality. We have proven that the condition (29), i.e., $R \cdot \mathcal{G} \cdot R^t = \mathcal{G}$, is always preserved. In Euclidean signature, this implies the preservation of conditions (27) and (31), i.e., $R \cdot Q = Q \cdot R$ and $N \cdot \mathcal{G} \cdot Q^t = 0$. As we have seen in the investigation of the matrix $C$ of the form (101) in Sec. V B, it is not necessarily so in the case of indefinite signature. In that case, the transformed gluing matrix may become nondiagonalizable (in the sense of nondiagonal Jordan canonical form) and consequently the projector on $(-1)$-eigenspace cannot satisfy $R \cdot Q = Q \cdot R$. Nevertheless, when such an obstruction did not arise, the conditions (27) and (31) were satisfied in all cases investigated here also in the indefinite signature. Similarly, the integrability condition (28) was preserved in all examples.

On the other hand, we have seen explicitly that the condition (33), i.e., $N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0$, is not preserved in general under the Poisson-Lie $T$-plurality and that in the transformed background it must be modified by the presence of an electric field constrained to the branes and interacting with oppositely charged endpoints of the string. We have moreover seen in several cases with nonconstant matrix $C$ in (97) that the closedness (63) of this additional electric field is intimately related to the integrability of the Neumann distribution (28) in the original model. It is an open question whether and how this behavior can be proven in general or whether it happens just in the low dimensions investigated here.

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# 4.3

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# Description of D-branes invariant under the Poisson-Lie T-plurality

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ABSTRACT: We write the conditions for open strings with charged endpoints in the language of gluing matrices. We identify constraints imposed on the gluing matrices that are essential in this setup and investigate the question of their invariance under the Poisson-Lie T-plurality transformations. We show that the chosen set of constraints is equivalent to the statement that the lifts of D-branes into the Drinfel'd double are right cosets with respect to a maximally isotropic subgroup and therefore it is invariant under the Poisson-Lie T-plurality transformations.

KEYWORDS: D-branes, String Duality.

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## 1. Introduction

In our previous paper [1], the transformation of worldsheet boundary conditions for non-linear sigma models under the Poisson-Lie T-plurality [2, 3] was investigated and a formula for the transformation of gluing matrices was presented there. Boundary conditions were formulated in terms of a so-called gluing matrix that was subjected to a set of constraints originally formulated for supersymmetric models in [4, 5]. Abelian T-duality of such models (and also of their purely bosonic analogues) was studied in [6] and later also partially extended to Poisson-Lie T-duality context in [7]. Unfortunately, we have shown in [1] that some of the constraints are not preserved under the Poisson-Lie transformation (even in the simplest non-Abelian T-duality context).

In this paper we present a restricted set of constraints for the gluing matrix that does not disqualify the interpretation of corresponding boundary condition in terms of D-branes and simultaneously preserves its validity under the Poisson-Lie transformations. It means that well defined D-branes formulated in this way transform into well defined D-branes again under the Poisson-Lie T-plurality.

The existence of such description was to be expected because there exists a different, geometric formulation of the same problem based on the geometry of D-branes lifted into the Drinfel'd double by C. Klimčík and P. Ševera in [8, 9]. The open problem was how to express their formulation in the language of gluing matrices, i.e., how their boundary conditions manifest themselves on the level of original $\sigma$-models.

The paper is structured as follows. Firstly, we review and modify the formulation of boundary conditions in terms of gluing matrices (or operators) $\mathcal{R}$. Secondly, we recall some of the basic properties of Poisson-Lie T-duality and plurality and how the gluing matrices transform. Thirdly, we demonstrate a few examples we have used in the search for consistency constraints on $\mathcal{R}$ preserved under Poisson-Lie transformations. Next, we rewrite the constraints on $\mathcal{R}$ in an equivalent form suitable for further computations (i.e. without projectors). Finally, we lift the D-branes into the Drinfel'd double, study how the boundary conditions manifest themselves there, demonstrate the connection with the description in [8] and show the invariance of our constraints.

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## 2. Boundary conditions and D-branes

We investigate the boundary conditions for equations of motion of nonlinear sigma models given by the action[1]

$$S_{\mathcal{F}}[\phi] = \int_{\Sigma} d^2x\, \partial_-\phi^\mu \mathcal{F}_{\mu\nu}(\phi)\partial_+\phi^\nu = \int_{\Sigma} d^2x\, \partial_-\phi \cdot \mathcal{F}(\phi) \cdot \partial_+\phi^t \tag{2.1}$$

where $\mathcal{F}$ is a tensor field on a Lie group $G$ and the functions $\phi^\mu : \Sigma \subset \mathbb{R}^2 \to \mathbb{R}$, $\mu = 1, 2, \ldots, \dim G$ are obtained by the composition $\phi^\mu = y^\mu \circ g$ of a map $g : \Sigma \to G$ and components of a coordinate map $y$ of a neighborhood $U_g$ of an element $g(x_+, x_-) \in G$. For the purpose of this paper we shall assume that the worldsheet $\Sigma$ has a topology of a strip infinite in $\tau \equiv x_+ + x_-$ direction, $\Sigma = \mathbb{R} \times \langle 0, \pi \rangle$ and $x_+, x_-$ are light-cone coordinates on $\Sigma$.

We impose the boundary conditions for open strings in the form of the gluing operator $\mathcal{R}$ relating left and right derivatives of the field $g : \Sigma \to G$ on the boundary of $\Sigma$,

$$\partial_- g|_{\sigma=0,\pi} = \mathcal{R}\, \partial_+ g|_{\sigma=0,\pi}, \qquad \sigma \equiv x_+ - x_-. \tag{2.2}$$

We denote the matrices corresponding to the operator $\mathcal{R}$ on $T_gG$ in the bases of coordinate derivatives $\partial_{y^\mu}$ as $R$, e.g.,[2]

$$\partial_-\phi|_{\sigma=0,\pi} = \partial_+\phi \cdot R|_{\sigma=0,\pi}. \tag{2.3}$$

The explicit form of the operator $\mathcal{R}$ in principle yields the embedding of a brane in the target space which is in this case the Lie group $G$.

When varying the action (2.1) we shall impose vanishing of boundary terms

$$\delta\phi \cdot (\mathcal{G} \cdot \partial_\sigma \phi^t + \mathcal{H} \cdot \partial_\tau \phi^t)|_{\sigma=0,\pi} = 0, \tag{2.4}$$

[1] We use a bit unusual notation that $\partial_\pm\phi^\mu$ form row vectors of the derivatives of $\phi$, therefore matrices of operators in our notation may differ by a transposition from expressions in other papers. The dot denotes matrix multiplication, $t$ denotes transposition, $X^{-t} \equiv (X^t)^{-1}$.

[2] Similarly we shall distinguish operators from their matrices by the calligraphic script used. This does not apply to tensorial expressions $\mathcal{F}, \mathcal{G}, \mathcal{H}$.

where $\mathcal{G}$ and $\mathcal{H}$ are symmetric and antisymmetric part of the tensor field $\mathcal{F}$. We shall assume that the ends of an open string move along a D-brane — submanifold $\mathcal{D} \subset G$ – so that both $\delta\phi|_{\sigma=0,\pi} \in T_g\mathcal{D}$ and $\partial_\tau\phi|_{\sigma=0,\pi} \in T_g\mathcal{D}$. Let $\mathcal{N}$ be a projector $T_gG \to T_g\mathcal{D}$ so that

$$\delta\phi|_{\sigma=0,\pi} = \mathcal{N}(\delta\phi|_{\sigma=0,\pi}), \quad \partial_\tau\phi|_{\sigma=0,\pi} = \mathcal{N}(\partial_\tau\phi|_{\sigma=0,\pi}). \tag{2.5}$$

From eqs. (2.3) and (2.5) we may express the defining properties of $\mathcal{N}$ as

$$\mathcal{N} \circ (\mathcal{R} + id) = (\mathcal{R} + id), \ \mathcal{N}^2 = \mathcal{N}, \ \mathrm{Ran}\,\mathcal{N} = \mathrm{Ran}\,(\mathcal{R} + id),$$

i.e.

$$(R + \mathbf{1}) \cdot N = R + \mathbf{1}, \ N^2 = N, \ \mathrm{rank}\,N = \mathrm{rank}\,(R + \mathbf{1}). \tag{2.6}$$

We should stress that these properties do not specify the projector $\mathcal{N}$ uniquely since its kernel is not determined. As it will become clear later on, we may consider all such projectors equivalent for any sensible use in physics.

We can rewrite the equation (2.4) as

$$\delta\phi \cdot N \cdot (\mathcal{F} \cdot \partial_+\phi^t - \mathcal{F}^t \cdot \partial_-\phi^t)|_{\sigma=0,\pi} = 0, \tag{2.7}$$

which after the use of eq. (2.3) becomes

$$\delta\phi \cdot N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) \cdot \partial_+\phi^t|_{\sigma=0,\pi} = 0. \tag{2.8}$$

Because $\delta\phi \cdot N$ and $\partial_+\phi^t$ are not further restricted, we find

$$N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0. \tag{2.9}$$

Besides that there are conditions for $N$ and $R$

$$\begin{aligned} N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}{}^\rho &= 0, \\ R \cdot \mathcal{G} \cdot R^t &= \mathcal{G} \end{aligned} \tag{2.10}$$

that follow from the condition that the projectors $\mathcal{N}$ in all points of $G$ define integrable distribution and that the stress tensor of the action vanishes on the boundary (see e.g. [1, 6, 10]).

In our previous paper [1], we have used the formulation first presented in [5], i.e. we have defined D-branes by virtue of Dirichlet projector $\mathcal{Q}$ that projects tangent vectors in a point of $G$ onto the space normal to the D-brane going through this point and the normal space was identified with the eigenspace of $\mathcal{R}$ with the eigenvalue $-1$, i.e.,

$$Q^2 = Q, \ Q \cdot R = -Q. \tag{2.11}$$

The Neumann projector $\mathcal{N}$, which projects onto the tangent space of the brane was then defined as complementary to $\mathcal{Q}$, i.e.

$$N := \mathbf{1} - Q.$$

The eq. (2.11) is then equivalent to

$$N^2 = N,\ N \cdot (R + \mathbf{1}) = R + \mathbf{1}.$$

In order to get an agreement with eq. (2.6) we had to assume that the geometrical and algebraic multiplicities of the eigenvalue $-1$ are equal. This gave another condition that relates $R$ and $Q$

$$Q \cdot R = R \cdot Q \tag{2.12}$$

so that we got the following set of conditions (equivalent to those in [5])

$$\begin{aligned} Q^2 = Q,\ Q \cdot R = -Q,\ \operatorname{rank} Q &= \dim\ker(R + \mathbf{1}) && (2.13)\\ Q \cdot R &= R \cdot Q, && (2.14)\\ N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}{}^\rho &= 0, && (2.15)\\ R \cdot \mathcal{G} \cdot R^t &= \mathcal{G}, && (2.16)\\ N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) &= 0. && (2.17) \end{aligned}$$

We found in our previous work that the constraints for a consistent gluing operator $\mathcal{R}$ derived above are not in general preserved under the Poisson-Lie transformations (see section 5.2, case (100) in [1]).

The situation improved a bit when we admitted that the endpoints of the string are electrically charged so that the action must be modified by boundary terms. Such an extension in the context of Poisson-Lie T-duality of open strings was already introduced in [8], in the gluing matrix language was firstly mentioned in [5]. We have

$$S_{\mathcal{F}}[\phi] \to S_{\mathcal{F}}[\phi] + S_{\text{boundary}}[\phi] \tag{2.18}$$

where

$$S_{\text{boundary}}[\phi] = q_0 \int_{\sigma=0} A_\mu \frac{\partial \phi^\mu}{\partial \tau} \mathrm{d}\tau - q_0 \int_{\sigma=\pi} A_\mu \frac{\partial \phi^\mu}{\partial \tau} \mathrm{d}\tau \tag{2.19}$$

corresponds to electrical charges $q_0, -q_0$ associated with the two endpoints of the string interacting with electric field(s) present on the respective D-branes. The condition (2.9) is then modified to the form [1]

$$N \cdot \big( (\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t \big) = 0, \tag{2.20}$$

where in local coordinates adapted to the brane[3] we have

$$\Delta_{\mu\nu} = \frac{1}{2} \left( \frac{\partial A_\nu}{\partial y^\mu} - \frac{\partial A_\mu}{\partial y^\nu} \right), \tag{2.21}$$

$\mu, \nu \leq \dim(\text{brane})$ (the remaining components of $\Delta$ do not contribute to the eq. (2.20)). For computational simplicity we assume that $\Delta$ can be smoothly extended into a neighborhood of the brane. Because the values of $\Delta$ are physically relevant only along the D-brane we

[3] i.e., $\frac{\partial}{\partial y^\mu}$, $\mu = 1, \ldots, \dim(\text{brane})$ are tangential to the brane and the remaining vectors $\frac{\partial}{\partial y^\kappa}$, $\kappa >$ dim(brane) are transversal.



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may impose a supplementary restriction on $\Delta$ that fixes its extension into the transversal directions

$$\Delta = N \cdot \Delta \cdot N^t. \tag{2.22}$$

The exactness of $\Delta$ along the brane (2.21) is locally equivalent to its closeness written in arbitrary coordinates as

$$N_\kappa{}^\nu N_\lambda{}^\rho N_\mu{}^\sigma \partial_{[\nu} \Delta_{\rho\sigma]} = 0. \tag{2.23}$$

Unfortunately, neither this generalized formulation of D-branes defined by the gluing operator and interaction with the charges is preserved under the Poisson-Lie T-plurality or Poisson-Lie T-duality in the sense that there are cases when the set of conditions (2.13)–(2.16), (2.20) and (2.23) holds for a $\sigma$-model with boundary conditions given by $\mathcal{R}$ but not for a model and boundary conditions obtained by the Poisson-Lie transformation (See section 5.2, case (101) in [1]).

This problem forces us to reconsider the necessity of conditions (2.13)–(2.16). Namely, motivated by explicit examples in [1] we revisit the condition (2.14). If this condition holds (as is always the case when $\mathcal{G}$ is positive/negative definite but not in general) then there is a canonical choice of the projector $\mathcal{N}$, namely, such that $\mathcal{N}$ is an orthogonal projector with respect to the metric $\mathcal{G}$. On the other hand, if the condition (2.14) does not hold, one cannot choose the projector $\mathcal{N}$ uniquely and also it is not possible to find the so-called adapted coordinates [5], i.e. the boundary conditions cannot be split into Dirichlet and (generalized) Neumann directions. Although such boundary conditions may appear strange, we don't see any reason why they should be a priori excluded from consideration.

Moreover, we shall prove that if we relax the condition (2.14) and reformulate the other ones in such a way that the $\sigma$-model with boundary conditions is given by $(\mathcal{F}, \mathcal{R}, \Delta)$ satisfying

$$R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \tag{2.24}$$

$$(R + \mathbf{1}) \cdot N = (R + \mathbf{1}), \quad N^2 = N, \quad \operatorname{rank} N = \operatorname{rank}(R + \mathbf{1}) \tag{2.25}$$

$$N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}^{\ \rho} = 0, \tag{2.26}$$

$$N \cdot \left( (\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t \right) = 0, \tag{2.27}$$

$$N \cdot \Delta \cdot N^t = \Delta, \tag{2.28}$$

$$N_\kappa{}^\nu N_\lambda{}^\rho N_\mu{}^\sigma \partial_{[\nu} \Delta_{\rho\sigma]} = 0. \tag{2.29}$$

then these conditions are preserved by the Poisson-Lie transformation.

## 3. Elements of Poisson-Lie T-plurality and transformation of boundary conditions

The Poisson-Lie T-plurality was described in many papers (e.g. [2, 3, 11]) and we sketch here only its main features, mainly to set the notation. The tensor field $\mathcal{F}$ on the Lie group $G$ can be written as

$$\mathcal{F}_{\mu\nu} = e_\mu{}^a(g) F_{ab}(g) e_\nu{}^b(g) \tag{3.1}$$

where the vielbeins $e_\mu{}^a(g)$ are components of the right-invariant Maurer-Cartan forms $dgg^{-1}$ and $F_{ab}(g)$ are matrix elements of bilinear nondegenerate form $F(g)$ on $\mathfrak{g}$, the Lie algebra of $G$. The action of the $\sigma$-model then reads

$$S_{F,A}[g] = \int_\Sigma d^2x\, \rho_-(g) \cdot F(g) \cdot \rho_+(g)^t + \int_{\sigma=0} A - \int_{\sigma=\pi} A, \tag{3.2}$$

where the right-invariant vector fields $\rho_\pm(g)$ are given by

$$\rho_\pm(g)^a \equiv (\partial_\pm g g^{-1})^a = \partial_\pm \phi^\mu\, e_\mu{}^a(g), \quad (\partial_\pm g g^{-1}) = \rho_\pm(g) \cdot T = \partial_\pm \phi \cdot e(g) \cdot T, \tag{3.3}$$

$T_a$ are basis elements of the Lie algebra $\mathfrak{g}$ and $A$ is the 1-form introduced in (2.19).

Similarly, the boundary conditions (2.2) may be expressed in terms of the right-invariant fields, as

$$\rho_-(g)|_{\sigma=0,\pi} = \rho_+(g) \cdot R_\rho|_{\sigma=0,\pi}, \tag{3.4}$$

where

$$R_\rho = e^{-1}(g) \cdot R \cdot e(g). \tag{3.5}$$

The $\sigma$-models that are transformable under Poisson-Lie T-duality can be formulated on a Drinfel'd double $D \equiv (G|\tilde{G})$, a Lie group whose Lie algebra $\mathfrak{d}$ admits a decomposition $\mathfrak{d} = \mathfrak{g} \dot{+} \tilde{\mathfrak{g}}$ into a pair of subalgebras maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form $\langle\, .\, ,\, .\, \rangle$. The matrices $F_{ab}(g)$ for the dualizable $\sigma$-models are of the form

$$F(g) = (E_0^{-1} + \Pi(g))^{-1}, \quad \Pi(g) = b(g) \cdot a(g)^{-1} = -\Pi(g)^t, \tag{3.6}$$

where $E_0$ is a constant matrix, $\Pi$ defines the Poisson structure on the group $G$, and $a(g), b(g)$ are submatrices of the adjoint representation of $G$ on $\mathfrak{d}$

$$gTg^{-1} \equiv Ad(g) \triangleright T = a^{-1}(g) \cdot T, \quad g\tilde{T}g^{-1} \equiv Ad(g) \triangleright \tilde{T} = b^t(g) \cdot T + a^t(g) \cdot \tilde{T}, \tag{3.7}$$

where $\tilde{T}^a$ are elements of dual basis in the dual algebra $\tilde{\mathfrak{g}}$, i.e., $\langle\, T_a, \tilde{T}^b \,\rangle = \delta_a^b$.

The bulk equations of motion of the dualizable $\sigma$-models can be written as Bianchi identities for the $\tilde{\mathfrak{g}}$-valued fields

$$(\rho_+)_a = -\rho_+(g)^b F(g)_{cb} (a(g)^{-1})_a^c, \qquad (\rho_-)_a = \rho_-(g)^b F(g)_{bc} (a(g)^{-1})_a^c.$$

These fields can be consequently integrated in terms of suitable $\tilde{h} : \Sigma \to \tilde{G}$,

$$\begin{aligned} \tilde{\rho}_+(\tilde{h})_a = (\partial_+ \tilde{h}\, \tilde{h}^{-1})_a &= -\rho_+(g)^b F(g)_{cb} (a(g)^{-1})_a^c, \\ \tilde{\rho}_+(\tilde{h})_a = (\partial_- \tilde{h}\, \tilde{h}^{-1})_a &= \rho_-(g)^b F(g)_{bc} (a(g)^{-1})_a^c. \end{aligned} \tag{3.8}$$

This procedure defines the lift $l : \Sigma \to D$ of the solution $g : \Sigma \to G$ into the Drinfel'd double. As a consequence, the lift satisfies [2],

$$\langle\, \partial_\pm l l^{-1}\, , \mathcal{E}^\pm \,\rangle = 0, \tag{3.9}$$

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where $l = g\tilde{h}$ and $\mathcal{E}^{\pm}$ are two orthogonal subspaces in $\mathfrak{d}$, spanned by $T + E_0 \cdot \tilde{T}$, $T - E_0^t \cdot \tilde{T}$, respectively. On the other hand, starting from a solution $l$ in the Drinfel'd double we find a corresponding solution $g$ by constructing the decomposition $l = g\tilde{h}$.

In general, there are several decompositions (Manin triples) of a Drinfel'd double that enable to transform one $\sigma$-model and its solutions into others. Let $\hat{\mathfrak{g}} \dot{+} \bar{\mathfrak{g}}$ be another decomposition of the Lie algebra $\mathfrak{d}$. The pairs of dual bases of $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}, \bar{\mathfrak{g}}$ are related by the linear transformation

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix}, \tag{3.10}$$

where the duality of both bases requires

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} = \begin{pmatrix} s^t & q^t \\ r^t & p^t \end{pmatrix}, \tag{3.11}$$

i.e.,

$$\begin{aligned} p \cdot s^t + q \cdot r^t &= \mathbf{1}, \\ p \cdot q^t + q \cdot p^t &= 0, \\ r \cdot s^t + s \cdot r^t &= 0. \end{aligned} \tag{3.12}$$

The $\sigma$-model obtained by the plurality transformation is then defined analogously to the original one, namely by substituting

$$\widehat{F}(\hat{g}) = (\widehat{E}_0^{-1} + \widehat{\Pi}(\hat{g}))^{-1}, \quad \widehat{\Pi}(\hat{g}) = \widehat{b}(\hat{g}) \cdot \widehat{a}(\hat{g})^{-1} = -\widehat{\Pi}(\hat{g})^t, \tag{3.13}$$

$$\widehat{E}_0 = (p + E_0 \cdot r)^{-1} \cdot (q + E_0 \cdot s) = (s^t \cdot E_0 - q^t) \cdot (p^t - r^t \cdot E_0)^{-1} \tag{3.14}$$

into (3.1), (3.2). Solutions of the two $\sigma$-models are related by two possible decompositions of $l \in D$, namely

$$l = g\tilde{h} = \hat{g}\bar{h}. \tag{3.15}$$

For $p = s = 0,\ q = r = \mathbf{1}$ we get the so-called Poisson-Lie T-duality where $\hat{G} = \tilde{G},\ G' = G,\ \widehat{E}_0 = E_0^{-1}$. If $G$ is non-Abelian and $\tilde{G}$ is Abelian we call it non-Abelian T-duality.

The corresponding transformation of the gluing matrix $R_\rho$ under the Poisson-Lie T-plurality was found in [1] in the form

$$\widehat{R_\rho} = \widehat{F}^t(\hat{g}) \cdot M_-^{-1} \cdot F^{-t}(g) \cdot R_\rho(g) \cdot F(g) \cdot M_+ \cdot \widehat{F}^{-1}(\hat{g}), \tag{3.16}$$

where

$$M_+ \equiv s + E_0{}^{-1} \cdot q, \quad M_- \equiv s - E_0{}^{-t} \cdot q. \tag{3.17}$$

An obvious drawback of the formula (3.16) is that the transformed gluing matrix $\widehat{R_\rho}$ may depend not only on $\hat{g}$ but also on $g$, i.e., after performing the lift into the double $g\tilde{h} = \hat{g}\bar{h}$ it may depend on the new dual group elements $\bar{h} \in \bar{G}$, which contradicts any reasonable geometric interpretation of the transformed boundary conditions. A solution of this problem is that we admit gluing matrices only in the form

$$R_\rho(g) = F^t(g) \cdot C \cdot F^{-1}(g), \tag{3.18}$$

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where $C$ is a constant matrix.[4] Then $\widehat{R_\rho}$ depends only on $\hat{g}$.

The condition (2.24) requiring that $R_\rho$ of the form (3.18) preserves the metric then restricts the form of the matrix $C$ by

$$C \cdot (E_0^{-1} + E_0^{-t}) \cdot C^t = (E_0^{-1} + E_0^{-t}). \tag{3.19}$$

It is an easy exercise [1] to show that eq. (3.19) is preserved under the Poisson-Lie transformation (3.16).

## 4. Examples of three-dimensional $\sigma$-models

The conditions (2.24)–(2.29) can be used in the following way. Let us assume that the tensor $\mathcal{F}$ is given. For the given metric $\mathcal{G}$, i.e. symmetric part of $\mathcal{F}$, we find admissible gluing operators $\mathcal{R}$ from eq. (2.24), i.e. operators orthogonal with respect to $\mathcal{G}$. Then the projector $N$ is determined from eqs. (2.25) and the condition of integrability (2.26) is checked. Finally, the 2-form $\Delta$ is obtained from (2.27), (2.28) and we check the condition (2.29), namely, that it is closed on the brane. The same procedure is then repeated for the dual or plural model with $\widehat{F}$ and $\widehat{R_\rho}$ given by (3.13) and (3.16).

As an example we shall investigate the Poisson-Lie transformations of the $\sigma$-models formulated on the Drinfel'd doubles $D \equiv (G|\tilde{G})$, where $G$ is the Lie group corresponding to one of the nine three-dimensional Lie algebras *Bianchi* 1 - *Bianchi* 9 (for notation see e.g. [12]) and $\tilde{G}$ is the Abelian Lie group corresponding to *Bianchi* 1. We shall denote these Drinfel'd doubles $(X|1)$ where $X$ is the number of the Bianchi algebra.

The matrix $\Pi$ vanishes for Abelian $\tilde{G}$ so that $F(g) = E_0$ and

$$\mathcal{F}_{\mu\nu} = e_\mu{}^a(g)(E_0)_{ab}\, e_\nu{}^b(g). \tag{4.1}$$

We choose the constant matrix $E_0$ as

$$E_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.2}$$

so that we work with an indefinite metric on $G$.

Our task is to choose gluing operators $\mathcal{R}$ producing $\Delta$ and $N$ that satisfy the conditions (2.24)–(2.29) and check whether the transformed gluing operators $\widehat{\mathcal{R}}$ which are expressed in the non-coordinate frame of the right-invariant fields by (3.16), produce $\widehat{\Delta}$ and $\widehat{N}$ satisfying the conditions (2.24)–(2.29) if and only if the original ones do.

The generic solution of eq. (2.24) for $E_0$ given by (4.2) is

$$R_\rho = \begin{pmatrix} \beta & \gamma & -\frac{\gamma^2}{2\beta} \\ \frac{(\alpha-\epsilon)\beta}{\gamma} & \alpha & -\frac{(\alpha+\epsilon)\gamma}{2\beta} \\ -\frac{(\alpha-\epsilon)^2\beta}{2\gamma^2} & \frac{1-\alpha^2}{2\gamma} & \frac{(\alpha+\epsilon)^2}{4\beta} \end{pmatrix}, \tag{4.3}$$

[4] In general, one can admit $C$ dependent on some combinations of coordinates of $G$ that transform by Poisson-Lie T-plurality to coordinates on $\widehat{G}$ (see [1]).

where $\epsilon = \pm 1$, and $\alpha, \beta, \gamma$ are real constants such that $\beta, \gamma \neq 0$ .

Note that the conditions (2.25), (2.27), (2.28) can be calculated even in the non-coordinate frame where $\mathcal{F} = E_0$, therefore $N_\rho$ and $\Delta_\rho$ are independent of $G$. Moreover, the condition (2.26) holds for all ranks of $\mathcal{N}$ but two and the condition (2.29) holds for all ranks of $\mathcal{N}$ but three on dimensional grounds.

Solving eq. (2.25) for the above given matrix $R_\rho$ and $\epsilon = 1$ we get the identity projector $\mathcal{N} = id$, and for $\epsilon = -1$ we get $N = e(g) \cdot N_\rho \cdot e(g)^{-1}$ where[5]

$$N_\rho = \begin{pmatrix} \frac{n_1\beta^2}{\alpha\gamma+\gamma} + 1 & \frac{n_2\beta^2}{\alpha\gamma+\gamma} & -\frac{\beta^2\left(n_2\beta(\alpha-2\gamma-1)+2\left(n_1\beta^2+\alpha\gamma+\gamma\right)\right)}{2(\alpha+1)^2\gamma^2} \\ \frac{n_1\beta(\alpha-2\gamma-1)}{2(\alpha+1)\gamma} & \frac{n_2\beta(\alpha-2\gamma-1)}{2(\alpha+1)\gamma} + 1 & -\frac{\beta(\alpha-2\gamma-1)\left(n_2\beta(\alpha-2\gamma-1)+2\left(n_1\beta^2+\alpha\gamma+\gamma\right)\right)}{4(\alpha+1)^2\gamma^2} \\ n_1 & n_2 & \frac{\beta(-\alpha n_2+2\gamma n_2+n_2-2n_1\beta)}{2(\alpha+1)\gamma} \end{pmatrix} \tag{4.4}$$

and $n_1, n_2$ are arbitrary constants. The rank of the latter projector is 2.

For $\epsilon = 1$, the condition (2.26) is satisfied trivially as the distribution of tangent vector spaces of the space filling D-branes is identical with the tangent spaces of the manifold. The conditions (2.27), (2.28) yield

$$\Delta_\rho = \begin{pmatrix} 0 & -\frac{2\gamma}{\alpha+2\beta+1} & \frac{\alpha-2\beta+1}{\alpha+2\beta+1} \\ \frac{2\gamma}{\alpha+2\beta+1} & 0 & -\frac{2(\alpha-1)\beta}{\gamma(\alpha+2\beta+1)} \\ -\frac{\alpha-2\beta+1}{\alpha+2\beta+1} & \frac{2(\alpha-1)\beta}{\gamma(\alpha+2\beta+1)} & 0 \end{pmatrix}, \quad \Delta = e(g) \cdot \Delta_\rho \cdot e(g)^t. \tag{4.5}$$

The form of $e(g)$ and therefore also the condition (2.29) depend on $G$.

The results for $\epsilon = 1$ are:

- For *Bianchi* 1,2,$6_0, 7_0$ the condition (2.29) is satisfied for any gluing matrix of the form (4.3).
- For *Bianchi* 3,4,5,$6_a, 7_a$ the condition (2.29) is satisfied if and only if $\alpha = 1$.

If $\epsilon = -1$, the results are:

- For *Bianchi* 1,5 the condition (2.26) is satisfied for any gluing matrix of the form (4.3).
- For *Bianchi* $3, 6_a$ the condition (2.26) is satisfied if and only if $\beta = -1,\ \gamma = \pm 2$ or $\alpha = \frac{\gamma + 2\gamma\beta \pm 2\beta}{\gamma \mp 2\beta}$.
- For *Bianchi* $6_0$ the condition (2.26) is satisfied if and only if $\alpha = 1 + 2\beta \pm 2\gamma$.
- For *Bianchi* $2, 4, 7_0, 7_a$ the condition (2.26) is never satisfied.

It is too complicated to check the conditions (2.26) and (2.29) for the simple groups that correspond to *Bianchi* $8, 9$ and the generic solution of eq. (3.19). Nevertheless, we

[5]This holds for generic values of $\alpha, \beta, \gamma$. Cases $\epsilon = 1,\ \alpha = -1-2\beta$ and $\epsilon = -1,\ \alpha = 1-2\beta \pm 4\sqrt{-\beta},\ \alpha = -1$ when forms of $\mathcal{N}$ are different were analyzed separately and the invariance under T-duality was also confirmed.



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can calculate them at least for a particular gluing matrix

$$R_\rho = \begin{pmatrix} 0 & 0 & \frac{1}{\beta} \\ 0 & 1 & -\frac{\alpha}{\beta} \\ \beta & \alpha & -\frac{\alpha^2}{2\beta} \end{pmatrix} \tag{4.6}$$

that is a special solution of eq. (3.19). Solving eq. (2.25) for the above given matrix $R_\rho$ we get the projector

$$N_\rho = \begin{pmatrix} \frac{n}{\beta}+1 & 0 & \frac{n+\beta}{\beta^2} \\ -\frac{n\alpha}{2\beta} & 1 & -\frac{\alpha(n+\beta)}{2\beta^2} \\ -n & 0 & -\frac{n}{\beta} \end{pmatrix}, \tag{4.7}$$

where $n$ is an arbitrary constant. Rank of this projector is 2 so that the condition (2.29) is satisfied trivially and

- For *Bianchi* 8 the condition (2.26) is satisfied if and only if $\alpha = \pm 2\sqrt{\beta^2-1}$.
- For *Bianchi* 9 the condition (2.26) is never satisfied.

### 4.1 Non-Abelian T-duality

As a next step, we shall investigate the constraints for the dual gluing matrices obtained by the Poisson-Lie T-duality that interchanges $G$ and $\tilde{G}$. We have proven in [1] that the so-called conformal condition (2.24) is preserved by the transformation (3.16) so it is not necessary to check it. For the models on the Drinfel'd doubles $(X|1)$, the Poisson-Lie T-duality reduces to the non-Abelian T-duality and the gluing matrices of the dual models are

$$\widehat{R_\rho} = -\widehat{F}^t(\hat{g}) \cdot E_0^t \cdot C \cdot E_0^{-1} \cdot \widehat{F}^{-1}(\hat{g}) = -\left(\mathbf{1} - E_0^{-t} \cdot \hat{\Pi}(\hat{g})\right)^{-1} \cdot C \cdot \left(\mathbf{1} + E_0^{-1} \cdot \hat{\Pi}(\hat{g})\right). \tag{4.8}$$

They depend on the choice of $G$ which determines the matrices $\widehat{\Pi}$. The projectors $\widehat{\mathcal{N}}$ are obtained from (2.25) and it turns out that the rank of the projector $\widehat{\mathcal{N}}$ is independent of $G$. For $\epsilon = 1$ it is equal to 2 while for $\epsilon = -1$ it is equal to 3. It means that for $\epsilon = 1$ the nontrivial condition is (2.26) while for $\epsilon = -1$ it is the condition (2.29).

For the matrix (4.3) and $\epsilon = 1$ we get:

- *Bianchi* 1,2,$6_0, 7_0$: The condition (2.26) for $\widehat{\mathcal{R}}$ is satisfied for any gluing matrix of the form (4.3).
- *Bianchi* 3,4,5,$6_a, 7_a$: The condition (2.26) for $\widehat{\mathcal{R}}$ is satisfied if and only if $\alpha = 1$.

For the matrix (4.3) and $\epsilon = -1$ we get:

- *Bianchi* 1,5: The condition (2.29) for $\widehat{\mathcal{R}}$ is satisfied for any gluing matrix of the form (4.3).
- *Bianchi* $3, 6_a$: The condition (2.29) for $\widehat{\mathcal{R}}$ is satisfied if and only if $\beta = -1,\ \gamma = \pm 2$ or $\alpha = \frac{\gamma + 2\gamma\beta \pm 2\beta}{\gamma \mp 2\beta}$.
- *Bianchi* $6_0$: The condition (2.29) for $\widehat{\mathcal{R}}$ is satisfied if and only if $\alpha = 1 + 2\beta \pm 2\,\gamma$.

- *Bianchi* $2, 4, 7_0, 7_a$: The condition (2.29) for $\widehat{\mathcal{R}}$ is never satisfied.

For the matrix (4.6) the projectors $\widehat{\mathcal{N}}$ obtained from (2.25) have the rank equal to 3 so that the condition (2.26) is satisfied trivially and for:

- *Bianchi* 8 the condition (2.29) is satisfied if and only if $\alpha = \pm 2\sqrt{\beta^2 - 1}$.
- *Bianchi* 9 the condition (2.29) is never satisfied.

Comparing the above given results with those in the previous subsection we see that the conditions (2.24)–(2.29) are preserved under the non-Abelian T-duality. We have also checked in examples that the conditions are preserved under the Poisson-Lie T-plurality as well.

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## 5. Invariance of the constraints for the boundary conditions under the Poisson-Lie T-plurality

As we have noted in section 2, it is not a priori clear what kind of constraints should be imposed on the gluing operator $\mathcal{R}$ so that on one hand it properly defines the boundary conditions as D-branes and on the other hand these constraints are preserved under the Poisson-Lie T-plurality. The examples in the previous section indicate that we may have managed to establish the right set of constraints, namely (2.24)–(2.29). We have shown in [1] that (2.24) is preserved under Poisson-Lie T-plurality. It remains to be shown that the others are invariant under the Poisson-Lie transformations as well.

### 5.1 An alternative formulation of the consistency conditions on the gluing operator

As it is difficult to find the Poisson-Lie transformation of the projector $\mathcal{N}$ it is convenient to reformulate the conditions (2.25)–(2.29) without its explicit use, i.e., using the gluing operator $\mathcal{R}$ only. This will also prove that the conditions (2.25)–(2.29) do not depend on the non-unique choice of the projector $\mathcal{N}$ and that we don't have to impose the condition (2.28).

For this purpose we recall eq. (2.6)

$$\mathrm{Ran}\,\mathcal{N} = \mathrm{Ran}\,(\mathcal{R} + id)$$

which means that any condition of the form

$$\mathcal{A} \circ \mathcal{N} = 0, \quad \text{i.e.,} \quad N \cdot A = 0$$

can be equivalently written as

$$\mathcal{A} \circ (\mathcal{R} + id) = 0, \quad \text{i.e.,} \quad (R + \mathbf{1}) \cdot A = 0.$$

Consequently, the condition (2.27) can be equivalently written as

$$(R + \mathbf{1}) \cdot \big((\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t\big) = 0. \tag{5.1}$$

Similarly, the condition (2.29), which when expressed in the basis-free form reads

$$\mathrm{d}\Delta(\mathcal{N}(X), \mathcal{N}(Y), \mathcal{N}(Z)) = 0, \ \forall X, Y, Z \in T_g\mathcal{D},$$

can be equivalently written as

$$(R+\mathbf{1})_\kappa{}^\nu (R+\mathbf{1})_\lambda{}^\rho (R+\mathbf{1})_\mu{}^\sigma \partial_{[\nu}\Delta_{\rho\sigma]} = 0. \tag{5.2}$$

Besides that, we recall that the condition (2.26) is just a statement that the distribution (of non-constant dimension)

$$\Lambda : g \in G \to \mathrm{Ran}\,(\mathcal{R} + id)|_g \subseteq T_gG$$

is in involution,

$$[\Lambda, \Lambda] \subseteq \Lambda \tag{5.3}$$

and consequently by Frobenius Theorem completely integrable. Such a statement is obviously independent of the particular choice of the projector $\mathcal{N}$ (although it doesn't have the nice form $0 = \ldots$ of eq. (2.26)).

Finally, we look for the the 2-form $\Delta$. We notice that by virtue of eq. (2.24) the matrix

$$(R+\mathbf{1}) \cdot \left(\mathcal{F} - \mathcal{F}^t \cdot R^t\right)$$

is skew-symmetric and consequently has the form

$$(R+\mathbf{1}) \cdot M \cdot (R+\mathbf{1})^t$$

for some antisymmetric matrix $M$ related to $\mathcal{F}, R$ (and, in general, non-unique). Therefore, the condition (5.1) takes the form

$$(R+\mathbf{1}) \cdot (\Delta + M) \cdot (R+\mathbf{1})^t = 0 \tag{5.4}$$

and, when considered as an equation for $\Delta$, has a solution, e.g. $\Delta = -M$.

Moreover, we can show that the condition (5.2) doesn't depend on the particular choice of a solution of the equation (5.4). It suffices to consider

$$\Upsilon = (R+\mathbf{1}) \cdot \Delta \cdot (R+\mathbf{1})^t = (R+\mathbf{1}) \cdot \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)$$

and compute the expression

$$\partial_\vartheta \Upsilon_{[\mu\nu}(R+\mathbf{1})_{\lambda]}{}^\vartheta$$

using the two ways of expressing $\Upsilon$. Due to the integrability condition (5.3) written in terms of generators $(R+\mathbf{1})_\nu{}^\sigma\partial_\sigma$ of the distribution $\Lambda$ there exist functions $\gamma_{\mu\nu}{}^\kappa$ such that

$$\partial_\vartheta (R+\mathbf{1})_{[\nu}{}^\sigma (R+\mathbf{1})_{\mu]}{}^\vartheta = \gamma_{\mu\nu}{}^\kappa (R+\mathbf{1})_\kappa{}^\sigma.$$

Using this fact one finds by comparison of different expressions for $\partial_\vartheta \Upsilon_{[\mu\nu}(R+\mathbf{1})_{\lambda]}{}^\vartheta$ that

$$\begin{aligned}(R+\mathbf{1})_{[\mu}{}^\rho (R+\mathbf{1})_\nu{}^\sigma (R+\mathbf{1})_{\lambda]}{}^\vartheta \partial_\vartheta \Delta_{\rho\sigma} = \\ \partial_\vartheta \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta - \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} \frac{\partial}{\partial y^\vartheta} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta\end{aligned}$$

(note that index $\vartheta$ in $\frac{\partial}{\partial y^\vartheta} \equiv \partial_\vartheta$ is not antisymmetrized, the antisymmetrization on the right hand side involves $\mu, \nu, \lambda$ only).

To sum up, we have found that an equivalent formulation of the condition (5.2) which doesn't depend on the particular choice of $\Delta$ exists and has the form

$$\partial_\vartheta \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta - \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} \frac{\partial}{\partial y^\vartheta} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta = 0. \tag{5.5}$$

We mention that although the functions $\gamma_{\mu\nu}{}^\kappa$ do not appear in the final expression their existence was important in intermediate steps, i.e. the conditions (5.2) and (5.5) are equivalent only if the integrability condition (5.3) holds.

Watchful reader may notice that we have not imposed the condition (2.28) yet. This condition restricts the field strength $\Delta$ only to the physically relevant degrees of freedom and it is reasonable to apply it from this viewpoint. On the other hand, it requires the knowledge of the explicit form of the projector $\mathcal{N}$ which we want to avoid. Under the assumption that the conditions (5.3), (5.5) hold we take any projector $\mathcal{N}$ and any $\Delta$ satisfying (5.1) and construct

$$\tilde{\Delta} = N \cdot \Delta \cdot N^t$$

which also satisfies the conditions (5.1), (5.2) and in addition it satisfies the condition (2.28). The influence of $\Delta$ and $\tilde{\Delta}$ on the motion of strings, i.e. extrema of the action (2.18), is exactly the same. Therefore we may consider $\Delta$ and $\tilde{\Delta}$ physically equivalent and forget the condition (2.28) altogether.

In summary we may write all conditions defining a consistent gluing operator $\mathcal{R}$ as

$$R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \tag{5.6}$$

$$[\Lambda, \Lambda] \subseteq \Lambda, \qquad \Lambda(g) = \mathrm{Ran}(\mathcal{R} + id)|_g, \tag{5.7}$$

$$\begin{aligned} &\partial_\vartheta \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta - \\ &- \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} \frac{\partial}{\partial y^\vartheta} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta = 0. \end{aligned} \tag{5.8}$$

Given such an operator $\mathcal{R}$ we can find the field strength $\Delta$ (using eq. (5.1)) and the projector $\mathcal{N}$ such that conditions (2.24)–(2.29) hold. Both $\mathcal{N}$ and $\Delta$ are in general non-unique but lead to the same dynamics of the strings on the classical level, i.e., the extrema of the action (2.18).

### 5.2 Lift of D-branes to the Drinfel'd double

We can define the lift of a D-brane $\mathcal{D} \subset G$ given by (2.3) to the Drinfel'd double as an integral manifold of the distribution generated by

$$\partial_\tau l|_{\sigma=0,\pi} = \partial_- l|_{\sigma=0,\pi} + \partial_+ l|_{\sigma=0,\pi}. \tag{5.9}$$

From $l = g\tilde{h}$, (3.8) and (3.7) we get

$$\begin{aligned} \partial_\tau l\, l^{-1} &= (\rho_-(g) + \rho_+(g)) \cdot T + (\tilde{\rho}_-(\tilde{h}) + \tilde{\rho}_+(\tilde{h})) \cdot Ad(g)(\tilde{T}) \\ &= (\rho_-(g) + \rho_+(g)) \cdot T + (\rho_-(g) \cdot F(g) - \rho_+(g) \cdot F^t(g))(a^{-t}(g) \cdot b^t(g) \cdot T + \tilde{T}) \end{aligned}$$

On the boundary we get from (3.4), (3.6) and (3.18)

$$\begin{aligned}\partial_\tau l\, l^{-1}|_{\sigma=0,\pi} &= \rho_+(g)|_{\sigma=0,\pi} \cdot F^t(g) \cdot \\ &\quad [(F^{-t}(g) + C \cdot F^{-t}(g)) \cdot T + (C - \mathbf{1}) \cdot (a^{-t}(g) \cdot b^t(g) \cdot T + \tilde{T})] \\ &= \rho_+(g)|_{\sigma=0,\pi} \cdot F^t(g) \cdot [(E_0^{-t} + C \cdot E_0^{-1}) \cdot T + (C - \mathbf{1}) \cdot \tilde{T}] \end{aligned} \tag{5.10}$$

As $\rho_+(g)|_{\sigma=0,\pi}$ is arbitrary and $F(g)$ is invertible we see that the vectors tangent to the lifted D-branes pulled to the unit of the Drinfel'd double form the vector subspace $V_{\mathcal{D}}$ of $\mathfrak{d}$

$$V_{\mathcal{D}} = \mathrm{span}(A^{ab} T_b + B^a{}_b \tilde{T}^b), \tag{5.11}$$

where the matrices $A$ and $B$ are

$$A = E_0^{-t} + C \cdot E_0^{-1}, \quad B = C - \mathbf{1}. \tag{5.12}$$

This subspace is isotropic because

$$\langle (A \cdot T + B \cdot \tilde{T})^t, A \cdot T + B \cdot \tilde{T} \rangle = C \cdot E_0^{-t} \cdot C^t - E_0^{-t} + C \cdot E_0^{-1} \cdot C^t - E_0^{-1} = 0 \tag{5.13}$$

due to (3.19). Moreover one can see that the subspace is maximally isotropic as the block matrix

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} E_0^{-t} + C \cdot E_0^{-1} \\ C - \mathbf{1} \end{pmatrix} \tag{5.14}$$

has the same rank as the block matrix

$$\begin{pmatrix} E_0^{-t} + E_0^{-1} \\ C - \mathbf{1} \end{pmatrix}, \tag{5.15}$$

whose rank is dim $\mathfrak{g}$, because $E_0^{-t} + E_0^{-1} = E_0^{-1} \cdot (E_0 + E_0^t) \cdot E_0^{-t} = E_0^{-1} \cdot \mathcal{G}(e) \cdot E_0^{-t}$ is an invertible matrix.

The space $V_{\mathcal{D}}$ is invariant under the Poisson-Lie transformation by construction, nevertheless, one may check it directly from the transformation properties of $T, \tilde{T}, E_0$ and $C$. We shall show that the condition (5.8) for admissible gluing matrix $R$ is equivalent to a statement that the isotropic subspace $V_{\mathcal{D}}$ is also a subalgebra.

First of all we shall rewrite the matrices occurring in (5.8) in terms of the matrices (5.12) defining the space $V_{\mathcal{D}}$.

$$R + \mathbf{1} = \mathcal{F}^t \cdot (A_c + B_c \cdot \Pi_c), \quad \mathcal{F}^t \cdot R^t - \mathcal{F} = B_c{}^t \cdot \mathcal{F}, \tag{5.16}$$

where

$$A_c = e^{-t}(g) \cdot A \cdot e^{-1}(g), \quad B_c = e^{-t}(g) \cdot B \cdot e^t(g), \quad \Pi_c = e^{-t}(g) \cdot \Pi(g) \cdot e^{-1}(g). \tag{5.17}$$

The condition (5.8) then acquires the form

$$\left[\mathcal{F}^t \cdot (A_c + B_c \cdot \Pi_c)\right]_{[\lambda}{}^{\rho} \left[\mathcal{F}^t \cdot \left(A_c \cdot \partial_\rho B_c{}^t - \partial_\rho A_c \cdot B_c{}^t - B_c \cdot \partial_\rho \Pi_c \cdot B_c{}^t\right) \cdot \mathcal{F}\right]_{\mu\nu]} = 0. \tag{5.18}$$

(Many terms occurring during derivation of this expression cancel by total antisymmetrization in $\lambda, \mu, \nu$.) Using (3.1) and the fact that both $e(g)$ and $F(g)$ are invertible we can simplify the above equation to

$$[(A + B \cdot \Pi(g)) \cdot e^{-1}(g)]^{[a\rho} \left( \left[ 2\,(A + B \cdot \Pi(g)) \cdot e^{-1}(g) \cdot \partial_\rho e(g) - B \cdot \partial_\rho \Pi(g) \right] \cdot B^t \right)^{bc]} = 0. \tag{5.19}$$

(The antisymmetrization involves only the indices $a, b, c$.) For the derivatives of $e$ we can use Maurer-Cartan equations, and derivatives of $\Pi(g)$ are

$$\partial_\rho \Pi^{ik} = -(a^{-1})_j{}^i \, \widetilde{f}^{jm}{}_n \, \overset{\mathrm{L}}{e}_\rho{}^n (a^{-1})_m{}^k, \tag{5.20}$$

where $\overset{\mathrm{L}}{e}_\mu{}^n$ are components of the left-invariant form $\overset{\mathrm{L}}{e}(g) = e(g) \cdot a(g)$. All that gives

$$(A + B \cdot \Pi(g))^{[ai} \left[ f_{ij}{}^k (A + B \cdot \Pi(g))^{bj} B^{c]}{}_k + a_i{}^r(g) \tilde{f}_r^{jk} (B \cdot a^{-t}(g))^b{}_j (B \cdot a^{-t}(g))^{c]}{}_k \right] = 0. \tag{5.21}$$

(where we again antisymmetrize in $a, b, c$ only). We define a mixed product on the Drinfel'd double

$$\langle\langle\, X, Y, Z \,\rangle\rangle := \langle\, [X, Y], Z \,\rangle. \tag{5.22}$$

It is totally antisymmetric and Ad-invariant. In terms of this mixed product we can write the above condition as

$$\langle\langle\, (A \cdot T + B \cdot \Pi(g) \cdot T)^{[a}, (A \cdot T - B \cdot \Pi(g) \cdot T + B \cdot \tilde{T})^b, (B \cdot \tilde{T})^{c]} \,\rangle\rangle = 0. \tag{5.23}$$

The antisymmetry of the mixed product and antisymmetrization in indices $a, b, c$ imply

$$\langle\langle\, X^{[a}, Y^b, Z^{c]} \,\rangle\rangle = \langle\langle\, X^{[a}, Z^b, Y^{c]} \,\rangle\rangle = \langle\langle\, Z^{[a}, X^b, Y^{c]} \,\rangle\rangle \tag{5.24}$$

that allows to rewrite the left-hand side of (5.23) as

$$\begin{gathered}\langle\langle\, (A \cdot T)^{[a}, (A \cdot T)^b, (B \cdot \tilde{T})^{c]} \,\rangle\rangle + \langle\langle\, (A \cdot T)^{[a}, (B \cdot \tilde{T})^b, (B \cdot \tilde{T})^{c]} \,\rangle\rangle \\ -\langle\langle (B \cdot \Pi(g) \cdot T)^{[a}, (B \cdot \Pi(g) \cdot T)^b, (B \cdot \tilde{T})^{c]} \rangle\rangle + \langle\langle (B \cdot \Pi(g) \cdot T)^{[a}, (B \cdot \tilde{T})^b, (B \cdot \tilde{T})^{c]} \rangle\rangle.\end{gathered}$$

The last two terms drop out by isotropy of the subalgebra $\tilde{\mathfrak{g}}$ because they are equal to

$$\begin{aligned}&-\frac{1}{3}\langle\langle (B \cdot \Pi(g) \cdot T - B \cdot \tilde{T})^{[a}, (B \cdot \Pi(g) \cdot T - B \cdot \tilde{T})^b, (B \cdot \Pi(g) \cdot T - B \cdot \tilde{T})^{c]} \rangle\rangle \\ &\qquad = \frac{1}{3}\langle\langle\, (B \cdot a^{-t}(g) \cdot \tilde{T})^{[a}, (B \cdot a^{-t}(g) \cdot \tilde{T})^b, (B \cdot a^{-t}(g) \cdot \tilde{T})^{c]} \,\rangle\rangle = 0.\end{aligned}$$

The first two terms give

$$\frac{1}{3}\langle\langle\, (A \cdot T + B \cdot \tilde{T})^{[a}, (A \cdot T + B \cdot \tilde{T})^b, (A \cdot T + B \cdot \tilde{T})^{c]} \,\rangle\rangle = 0 \tag{5.25}$$

and we can drop the antisymmetrization because of antisymmetry of (5.22). Then eq. (5.25) becomes exactly the statement that the maximal isotropic subspace $V_{\mathcal{D}}$ is a subalgebra of the Drinfel'd double, i.e., that for any $v_1, v_2, v_3 \in V_{\mathcal{D}}$ we have

$$\langle\langle\, v_1, v_2, v_3 \,\rangle\rangle = 0.$$



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To sum up, we conclude that the condition (5.8) is in the case of Poisson-Lie dualizable models equivalent to the statement that the maximally isotropic subspace $V_{\mathcal{D}}$ is a subalgebra. Therefore, the condition (5.8) is Poisson-Lie invariant.

We also see that the lifts of D-branes into the Drinfel'd double $D$ acquire the form of cosets $\mathfrak{D}l$ where $\mathfrak{D}$ is the Lie subgroup of $D$ with Lie algebra $V_{\mathcal{D}}$ and $l \in D$. This demonstrates that the gluing matrix formalism naturally leads to D-branes in Drinfel'd double as devised by C. Klimčík and P. Ševera in [8]. Obviously, the D-brane in Drinfel'd double $\mathfrak{D}l$ is an embedded submanifold of $D$ whenever the condition (5.8) is satisfied, irrespective of the condition (5.7). That leads us to a natural hypothesis that in our case of dualizable gluing operators the distribution $\Lambda : g \in G \to \mathrm{Ran}\,(\mathcal{R} + id)|_g$ is integrable by virtue of the condition (5.8) alone.

In order to show that the distribution $\Lambda$ is integrable we define a coset projection map

$$\pi : \ D \to G : \ l = g\tilde{h} \mapsto g.$$

The D-brane in $G$ passing through $g_0$ is then obtained as $\pi(\mathfrak{D}g_0\tilde{h}_0)$ for some $\tilde{h}_0 \in \tilde{G}$ provided that it is well-defined. That it is indeed so can be seen from the fact that for any $l, l' \in D$ such that $\pi(l) = \pi(l')$ we obviously have

$$\pi \circ R_l = \pi \circ R_{l'} \tag{5.26}$$

and consequently for any $\mathfrak{D}l_1, \mathfrak{D}l_2$ such that $\pi(\mathfrak{D}l_1) \cap \pi(\mathfrak{D}l_2) \neq \emptyset$ we find that intersecting D-branes in $G$ coincide, i.e. $\pi(\mathfrak{D}l_1) = \pi(\mathfrak{D}l_2)$, and are submanifolds. Consequently, $\{\pi(\mathfrak{D}l) | l \in D\}$ form a foliation (of non-constant dimension) of the group $G$ and the distribution $\Lambda$ consisting of tangent spaces to this foliation is by definition integrable.

For a more explicit derivation it is sufficient to consider a basis of right-invariant vector fields on $D$ extended from a basis $(e_k(e))$ of $V_{\mathcal{D}}$ by

$$e_k(l) = (R_l)_* e_k(e)$$

and project them by $\pi_*$

$$E_k(g) = \pi_* e_k(g\tilde{h}).$$

Such $E_k$ are well-defined vector fields on $G$, i.e. don't depend on the choice of $\tilde{h}$, due to eq. (5.26), and define the distribution $\Lambda|_g = \mathrm{span}\{E_k(g)\}$ by construction of the lift. Because $e_k$ close under the commutator, also $E_k$ do so due to $\pi_*([e_j, e_k]) = [\pi_*(e_j), \pi_*(e_k)]$ and consequently the distribution $\Lambda$ is integrable.

A further question arises concerning the generality of our description, i.e. whether any D-brane configuration described in the language of [8] can be expressed in terms of gluing matrices. Let us suppose that we are given an arbitrary maximally isotropic subalgebra $V_{\mathcal{D}}$ of the Drinfel'd double algebra $\mathfrak{d}$, i.e.

$$V_{\mathcal{D}} = \mathrm{span}\{K^{ab} T_b + L^a{}_b \tilde{T}^b\}$$

where $K, L$ are arbitrary matrices such that

$$K \cdot L^t + L \cdot K^t = 0$$

and rank $(K, L) = \dim G$. Does a matrix $C$ exist such that there is an equivalent description

$$V_{\mathcal{D}} = \mathrm{span}\{A^{ab}T_b + B^a{}_b\tilde{T}^b\}$$

where

$$A = E_0^{-t} + C \cdot E_0^{-1}, \quad B = C - \mathbf{1} \ ?$$

The answer is positive provided $L - K \cdot E_0^{-1}$ is regular (invertible) matrix. Indeed, we are looking for an invertible matrix $S$ such that $S \cdot L = A,\ S \cdot K = B$. We find

$$S = \left(E_0^{-t} + E_0^{-t}\right) \cdot \left(L - K \cdot E_0^{-1}\right)^{-1},$$

and

$$C = \left(E_0^{-t} + E_0^{-t}\right) \cdot \left(L - K \cdot E_0^{-1}\right)^{-1} \cdot K + \mathbf{1}.$$

Such matrix $C$ satisfies the condition (3.19). The singular case when $C$ doesn't exist and we cannot use the description based on gluing matrices occurs if and only if there is $v \in V_{\mathcal{D}},\ v \neq 0$ such that $\langle v, \mathcal{E}^- \rangle = 0$, i.e.,

$$v \in V_{\mathcal{D}} \cap \mathcal{E}^+ \neq 0. \tag{5.27}$$

This is rather exceptional since both $V_{\mathcal{D}}$ and $\mathcal{E}^+$ are $(\dim G)$-dimensional subspaces in $(2 \dim G)$-dimensional vector space $\mathfrak{d}$.

## 6. Conclusions

We have revisited the bosonic version of conditions (2.13)–(2.17) formulated in [5] for the gluing matrices defining boundary conditions for open strings. We have investigated them from the point of view of their invariance under the Poisson-Lie transformations defined by the formulas (3.13), (3.14) and (3.16).

We have seen that in order to keep the conditions invariant under the Poisson-Lie transformations, it is necessary to introduce the electromagnetic field $\Delta$ on the D-branes where the boundary conditions are imposed as in [8]. Besides that we have relaxed the condition (2.14) for the so-called Dirichlet projector $\mathcal{Q}$ that projects onto the space normal to the D-brane as it is not invariant under the Poisson-Lie transformations. We suggest that the proper set of constraints for the gluing matrices is (2.24)–(2.29). The invariance of these constraints under the Poisson-Lie transformations was firstly checked in many examples; some of them were presented in section 4. The invariance was proved in section 5.

Of course, one may imagine also other possible generalizations of the conditions (2.13)–(2.17). One possible approach (in supersymmetric setting) appeared in [14] where the condition (2.16) was not strictly enforced whereas the splitting into Dirichlet and Neumann directions due to (2.13)–(2.14) was retained (together with a stringent restriction $R^2 = 1$). However, that paper dealt with Abelian T-duality only. In the context of Poisson-Lie T-duality it seems that the condition (2.16), i.e. (2.24), has a natural geometric interpretation, namely the isotropy of lifted D-branes (5.13), and it was essential in most of our derivations. That's why we consider it indispensable in our setting. The condition (2.26) is an integrability statement, needed for interpretation of D-branes as submanifolds. The conditions (2.27), (2.29) are equivalent to the vanishing of the boundary term in the variation of action (3.2) and as such are also necessary (as long as one keeps the action in the form (3.2)). The condition (2.28) restricts the field strength $\Delta$ to a specific choice from a physically equivalent set — the physics is not at all influenced by it but it is useful for the uniqueness of $\Delta$. To sum up we believe that all the conditions (2.24)–(2.29) should be imposed in Poisson-Lie T-duality context.

To prove the Poisson-Lie invariance of the constraints (2.24)–(2.29) it was necessary to reformulate them to the form (5.6)–(5.8) that does not contain the (non-unique) projector $\mathcal{N}$. In the end it turned out that the constraints for the gluing matrices

$$R_\rho(g) = F^t(g) \cdot C \cdot F^{-1}(g), \tag{6.1}$$

where $C$ is a constant matrix which satisfies

$$C \cdot (E_0^{-1} + E_0^{-t}) \cdot C^t = (E_0^{-1} + E_0^{-t}), \tag{6.2}$$

are equivalent to the condition that the subspace

$$V_{\mathcal{D}} = \operatorname{span}\left(\left(E_0^{-t} + C \cdot E_0^{-1}\right) \cdot T + (C - \mathbf{1}) \cdot \tilde{T}\right) \tag{6.3}$$

is a maximally isotropic subalgebra. This statement is clearly invariant under the Poisson-Lie transformations because the choice of $V_{\mathcal{D}}$ is independent of the decomposition of the Lie algebra of the Drinfel'd double into the sum of the isotropic subalgebras (Manin triple).

On the other hand, if $V_{\mathcal{D}}$ is a maximally isotropic subalgebra and

$$V_{\mathcal{D}} \cap \mathcal{E}^+ = 0$$

then there is a unique matrix $C$ such that $V_{\mathcal{D}}$ can be written in the form (6.3) and the condition (6.2) is satisfied. The gluing matrix (6.1) then satisfies the consistency conditions (5.6)–(5.8) or equivalently (2.24)–(2.29) where the suitable field strength $\Delta$ is found as a solution of

$$(R + \mathbf{1}) \cdot \Delta \cdot (R + \mathbf{1})^t = (R + \mathbf{1}) \cdot \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)$$

and the projector $\mathcal{N}$ is defined by eq. (2.5).

This means that we have shown that the current version of the formulation of transformable boundary conditions in terms of gluing matrices is equivalent to the description originally discovered by C. Klimčík and P. Ševera in [8]. Both approaches can be considered complementary. In their original formulation the invariance of the description is clear from its geometric formulation in the Drinfel'd double and also some of the geometric properties of the lifted D-branes are immediately obvious. However, it may be quite tedious to work out the explicit form of the boundary conditions in the $\sigma$-models on the groups $G, \hat{G}$. (e.g. in the original paper [8] only the Poisson-Lie T-duals of free boundary conditions were worked out in any detail. More complicated D-branes in WZW models found in this way were given in [13].) On the other hand, in our approach these are easy to write down but it required some calculation to show that both the original and transformed boundary conditions satisfy the same consistency requirements (5.6)–(5.8).

Finally, we would like to recall that we have expressed the conditions on gluing matrix in a form independent of the projector $\mathcal{N}$, i.e. (5.6)–(5.8), and that this derivation does not depend at all on the particular structure of Poisson-Lie transformable models or on the fact that we consider group targets. We believe that this formulation may be of use also in other investigations of the properties of gluing matrices.

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