Review of the dissertation thesis
Properties and applications of geometric flows

submitted by
Jiří Minarčík
Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

The thesis focuses on curve-shortening flows with high codimensions, which have not received much attention so far. It covers a wide range of topics, including basic mathematical analysis of the solutions and the curves themselves, minimal surface generating flows, homotopy, framed curvature flows, and more.

Chapter 0 serves as an introduction to the topic. The author explains the significance of various moving boundary problems in science, engineering, and mathematics through the use of multiple examples. When discussing the geometry of curves and surfaces, it is crucial to determine which geometric structures to include. The following section describes the mathematical formulation of intrinsic and extrinsic flow, along with some important examples. The paper then focuses on curve flows, which are extrinsic flows to curves, and details the extensively used parametric methods. Finally, the paper derives time evolution equations for both local and global quantities. Local quantities include length, curvature, tortuosity, and Frenet's frame, while global quantities are their integrals. The chapter is well-organized.

Chapter 1 presents a mathematical analysis of curve-shortening flow in three-dimensional space. Gage and Hamilton, as well as Grayson, have provided results for the mathematical analysis of curve-shortening flows in the plane. In essence, their findings demonstrate that the time evolution of a curve-shortening flow for an arbitrary Jordan curve leads to a convex solution in finite time, which gradually rounds and degenerates to a single point in finite time. Similar results have been extended to mean curvature flows in high-dimensional spaces. However, these results are only valid for problems with codimension one and cannot be applied to high codimension problems. This chapter focuses on curve-shortening flows in three-dimensional space, which are problems with codimension two, as the first step towards analyzing problems with high-codimensions. The results of this chapter by the author are briefly summarized below. First, the time evolution equations for the local length, curvature, torsion, and Frenet frame are presented, followed by the derivation of the time evolution equations for the important global quantities: the length, the integral of the curvature, and the integral of the torsion. Using these, an evaluation from above of the length is obtained, and consequently, an evaluation from above of the existence time of the solution of the curve-shortening flow is obtained. Furthermore, an evaluation of the curvature is produced and compared for the plane case and the space case, and in particular, it is pointed out that the problem in space is more mathematically challenging. Next, a new type of comparison principle is proved by comparing curvilinear shortening flows with moving surfaces. In particular, by choosing a special type of moving surface, the author succeeds in giving various upper estimates of the maximum existence time of the solution. The article introduces the notion of convexity for spatial curves and discusses convexity under curve-shortening flows. It is demonstrated that, unlike planar curves, convexity is not generally preserved. However, if the orthogonal projection of the initial curve is convex, then the orthogonal projection of the solution of the curve-shortening flow preserves convexity. Additionally, it has been demonstrated that sphericity is maintained for spherical curves (spatial curves that travel along a particular sphere). The spherical avoidance principle also applies to spherical curves, which states that if the initial state is simple, then simplicity is maintained over time. The mathematical analysis of curve-shortening flows in space presented in this chapter is original and interesting. These results alone could suffice for a doctoral dissertation.

Chapter 2 introduces a novel concept of minimal surface generating flow and provides a mathematical analysis. The chapter also discusses various mathematical and numerical analysis methods that have been proposed in the past. However, this new approach offers a unique perspective. When a spatial curve evolves over time, its trajectory can be traced, and a surface can be considered by taking two parameters: one in the spatial direction of the curve and the other in the temporal direction. This surface is referred to as a trajectory surface. After calculating the basic geometric quantities of the trajectory surface, such as the first fundamental form, second fundamental form, Gauss curvature, and mean curvature, they are used to define a minimal surface generating flow. The flow is defined as the one that generates a minimal surface trajectory. The basic properties of minimal surface generating flow are then investigated, including conservation of $\tau^{1/2}$ (where τ is the torsion), monotonically decreasing length, and inequality evaluation of the maximum principle. Utilizing these results, the evaluation of length and curved area is derived, and the maximum existence time of the solution of minimal surface generating flow is evaluated from above. Numerical results for these new concepts are not given, and therefore, interesting numerical studies, such as what minimal surfaces can be obtained when minimal surface generating flow is computed numerically, remain untouched. However, I believe that the concepts given in this chapter are quite original, mathematically interesting, and very well described.

Chapters 1 and 2 focus on curve-shortening flow and minimal surface generating flow, respectively. Chapter 3 develops a topological discussion of most curve flows in space. To prepare for this, the concept of Frenet frame dependent flow is introduced, followed by the definition of nondegenerate homotopy, which plays an important role in this chapter. Afterward, the concept of a tangent turning signature is introduced to classify locally convex curves. It is shown to be invariant with respect to nondegenerate homotopy and well-defined on the appropriate equivalence class. It is also shown to be conserved under arbitrary Frenet frame dependent geometric flows as a system of arguments. The discussion in this chapter differs significantly from the others as it is based on topological concepts. The author's ability to approach geometric flows from both analytical and alternative perspectives demonstrates his advanced mathematical knowledge.

Chapter 4 introduces the concept of framed curvature flow and provides a mathematical analysis. In traditional geometric flow, motion is determined by velocities in the tangent, normal, and binormal directions. However, in framed curvature flow, a new Frenet frame is adopted by giving an angle function θ and rotating the normal and binormal directions by θ. The flow is defined as a framed flow. To prepare for the mathematical analysis of the framed flow, we first derive the time evolution equations for the new Frenet frame, curvature, and torsion. Based on these equations, we demonstrate the time-local existence of the framed flow. It has been demonstrated that if the solution's maximum existence time is finite, then the maximum of the curvature or the second-order derivative of the angle function diverges at that time. Framed flow generates singularities of a type that do not appear in ordinary flows. Therefore, the author first defines various singularities and provide basic examples of each. For the global analysis, the author derives an objective evaluation of the time evolution of the length to obtain the maximum time of existence of the solution. Additionally, the author derives an evaluation of the total area. These evaluations, along with the maximum existence time of the solution, lead to a global evaluation of the total area. Various interesting trajectory surfaces can be obtained by appropriately defining the angle function. It has been demonstrated that constant mean curvature surfaces and constant Gauss curvature surfaces can be constructed using framed flows. The concepts presented in this chapter, as well as in the previous chapters, are highly original and commendable, providing a new framework for the mathematical analysis of curvilinear flows.

Appendix A analyzes the time evolution of filament networks instead of curves. A discrete method is used to compute energy gradients, enabling the handling of branching topological changes. Appendix B provides a brief overview of the numerical algorithms and presents numerical results for some of the new flows discussed in this thesis.

This paper ventures into the field of time evolution of spatial curves, a classical subject that has been little studied. Starting from traditional mathematical issues, such as the existence of solutions to curve-shortening flows and the elucidation of their long-time behavior, the paper introduces new concepts, such as minimal surface generating flow and framed flow. The dissertation is of an extremely high level and makes truly essential advances in the mathematical analysis of moving boundary problems.

In view of the foregoing, I have every reason to recommend the candidate to the Committee for the Defense of the Doctor of Philosophy Degree.

March 13, 2024

Koya Sakakibara

Kanazawa University, Japan