LABORATOIRE BORDELAIS DE RECHERCHE EN INFORMATIQUE
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To whom it may concern,

Bordeaux, 22 février 2024

Subject: *Supervisor's report of doctoral thesis*

Candidate : Jana Lepšová

Thesis : Substitutive structures in combinatorics, number theory, and discrete geometry

The thesis proposes advances on the following different subjects : combinatorics on words, numeration systems and discrete geometry (theory of aperiodic tilings). A common thread that connects the 5 chapters that follow the introduction and preliminaries is the notion of substitutions which the author uses to cleverly express results that remained unnoticed until today.

The Chapter 3 on Complement numeration systems is such an example. The author recalls the well-known two's complement notation of integers which is the most common method of representing signed (positive, negative, and zero) integers on computers. The author proposes a way to generalize the two's complement using Fibonacci numbers instead of powers of two. This part was published in the journal *RAIRO - Theoretical Informatics and Applications* in 2023 where it was proved that the numeration system behaves well with respect to addition (the automaton which performs the addition of Zeckendorff representation of integers also works for their Complement version). The author further extends it by proposing a Complement Numeration System for every simple Parry number. This part is new and the author is encouraged to publish it independently.

The Chapter 4 proposes advances in the subject of Combinatorics on words, more precisely on the Critical exponents of Arnoux–Rauzy words. The chapter introduces in a very pedagogical way the Sturmian words and Arnoux–Rauzy words. Then it proves a result on the critical exponents of regular Arnoux–Rauzy words and what are called d-bonacci words. The results presented in the chapter were presented at the *International Conference on Combinatorics on Words (WORDS 2023)* and are published in the volume 13899 of the *Lecture Notes in Comput. Sci.*

The Chapter 5 considers the Faithful representation of Sturmian morphisms. Sturmian morphisms are substitutions that map Sturmian sequences to Sturmian sequences. The set of Sturmian morphisms is a monoid which is generated by a finite set of morphisms. These morphisms can be encoded into what are called incidence matrices, but two morphisms

may have the same incidence matrix. In this chapter, the candidate presents a way to encode bijectively the morphism into matrices. These are called faithful representations. Such faithful representation is then used to tackly open problems in combinatorics on words with respect to the Square roots and the intercepts of fixed points of Sturmian morphisms. The results presented in this chapter were published in the journal *European Journal of Combinatorics* in 2023 independently of the Ph.D. advisors with coauthors Edita Pelantová and Štěpán Starosta.

The Chapter 6 is about Dumont–Thomas numeration systems for $\mathbb{Z}$. Extending the approach of Dumont–Thomas with admissible sequences, the author proposes a way to extend the usual Dumont–Thomas numeration system to negative integers in the spirit of complement numeration system. Inspired from the more complicated notion of Bratelli-Vershik diagram and its adic shift map, this is a very nice and simple idea which will certainly have application in the future. Most of the results presented in this chapter are part of a prepublication submitted to the journal *Integers*. In a subsection, not part of the prepublication, the author of the thesis considers the Dumont–Thomas numeration system for $\mathbb{Z}$ when the substitution is the canonical substitution associated to a simple Parry number. The author proves that such numeration system are positional (the value of the representation can be computed as a linear combination of predefined values whose coefficients are given by the representation itself). This raises an interesting open question which is to characterize the substitutions for which the Dumont–Thomas numeration system is positional.

The Chapter 7 is about an automatic characterization of an aperiodic Wang shift. The author introduces a set of 16 Wang tiles and shows that the set of valid tilings of the plane with these tiles is topologically conjugate to the set of tilings obtained from a self-similar set of 19 Wang tiles which was deduced from the aperiodic set of Jeandel–Rao tiles. Thus we can say that the 16 tiles are a simplification of the 19 tiles, still keeping the same properties (aperiodicity and minimality of the Wang shift). Then, the authors proves the existence of automatic characterization of that simplified Wang shift. The proof is surprising as it is based on the Dumont–Thomas numeration system for $\mathbb{Z}^2$ associated to the Fibonacci substitution. Thus Chapter 7 illustrates that the theory developped in Chapter 3 and 6 have an interesting application in discrete geometry, and in particular, in the study of aperiodic tilings. These results were presented in the *International Conference on Combinatorics on Words (WORDS 2021)* and are published in the volume 12847 of the *Lecture Notes in Comput. Sci.*

The candidate proved her ability to deal with complex mathematical problems. She is able to write code to experiment and solve problems. The candidate is the author of the Python package `colored-arnoux-rauzy-sequences` now available in the Python Package Index at `https://pypi.org/project/colored-arnoux-rauzy-sequences/` which allows to reproduce the computations that support the results presented in Chapter 4. Also, the candidate is the author of a new module for morphic words that was integrated in 2021 in the open source software in SageMath (`https://github.com/sagemath/sage/issues/31378`). The Ph.D. was performed in a cotutelle agreement between Czech Technical University in Prague and Université de Bordeaux during which the candidate

took part in many courses, seminars, exams, working group, coding sessions, in both Universities in a period still affected by the COVID pandemic. The candidate quickly learned the French language at an impressive speed. The submitted thesis is of very high quality. Each of the main 5 chapters is associated to a publication (2 published in the proceedings of conferences, 2 published in journals and 1 submitted to a journal). Some chapters even contain additional unpublished results which deserve publication. Therefore, the thesis presents a valuable and considerable contribution to the domain of combinatorics on words, numeration systems and discrete geometry.

The thesis fulfills all requirements for Ph.D. theses. Hence, I recommend the candidate to the committe for the doctoral theses to bestow her the title of Doctor of Philosophy.

Sincerely,

Sébastien Labbé
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