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Report on Jana Lepšová's doctoral thesis, entitled "Substitutive structures in combinatorics, number theory and discrete geometry"

The focus of the doctoral research developed by Jana Lepšová concerns the representation of numbers in non-standard numeration systems, combinatorics on words and aperiodic tilings with Wang tiles. All these subjects are part of very active research domains and communities.

This dissertation consists of an introductory chapter highlighting the Fibonacci sequence as the guiding thread of the work, a chapter presenting all the needed basic notions, five chapters reporting the investigated problems and the related obtained results, and a concluding chapter summarizing the six open questions posed by Jana Lepšová throughout the text. I describe the chapters 3 to 8 in more details hereafter.

In Chapter 3, Jana Lepšová introduces the notion of complement numeration systems. These numeration systems are positional numeration systems that allow us to represent all integers in a similar way as is done with the two's complement numeration system. Jana Lepšová starts by studying the Fibonacci complement numeration system. This system is a generalization of the Zeckendorf numeration system, which may be seen as the Bertrand numeration system built from the golden ratio. She then extends this idea to define complement numeration systems generalizing any Bertrand numeration systems built from a simple Parry number. In doing so, she is able to provide a unified proof of previously obtained results for the two's complement numeration system and the Fibonacci complement numeration system. Finally she presents her results for computing addition within the Fibonacci complement numeration system thanks to a finite automaton. For that purpose, she cleverly extends the Berstel adder initially built for the Zeckendorf numeration system to be able to deal with negative integers as well.

Chapter 4 is concerned with the computation of the critical exponent and the asymptotic critical exponent of Arnoux-Rauzy words. Two contributions in this area are provided by Jana Lepšová. The first one is a formula for the critical exponents of regular Arnoux-Rauzy words. The second one is a proof that the minimal critical exponents among all regular d-ary Arnoux-Rauzy words are reached by the d-bonacci word and that they coincide for $2 \leq d \leq 15$ (coincidence for $d \in \{2, 3\}$ was previously known). Jana Lepšová also reports that subsequently to her work on the subject, a stronger result has been obtained by Dvořáková and Pelantová which states that the minimal critical exponents among all d-ary episturmian words (a class that strictly contains the Arnoux-Rauzy words) are reached by the d-bonacci word and that they coincide for all d.

In Chapter 5, Jana Lepšová introduces a new representation of the special sturmian monoid by 3×3 matrices. This representation is called *faithful* as, unlike in the classical representation of morphisms by adjacency matrices, distinct morphisms have distinct such representations. Moreover, the faithful representation has the property to preserve the presentation of the special sturmian monoid. An algorithm for finding the faithful representation of a morphism in an efficient way is obtained and two applications are then described. First, the faithful representations can be used to find the intercepts of the fixed points of all morphisms belonging to the conjugacy class of a given morphism in the special sturmian monoid. The second application concerns square roots of sturmian words as were introduced by Peltomäki and Whiteland. Namely, it is proved that the square root of a sturmian word that is fixed by a primitive morphism in the special sturmian monoid is fixed by one of the first four integer powers of this morphism.

In Chapter 6, Jana Lepšová resumes her study of non-standard numeration systems from Chapter 3. Here, she introduces and studies the notion of Dumont-Thomas numeration systems for all integers. Then she shows that this new framework can be seen as a further generalization of complement numeration systems associated with simple Parry numbers. It is also shown how to use these numeration systems in a multidimensional setting. This makes the link with the following chapter since the representation of 2-dimensional integer points in the Fibonacci complement numeration system will be used to understand a specific aperiodic Wang shift.

Aperiodic tilings with Wang tiles are the subject of Chapter 7. Jana Lepšová defines a new Wang shift and proves that is turns out to be a topological conjugate of another Wang shift based on 19 Wang tiles previously defined by Labbé. These 19 Wang tiles are based on the minimal set of 11 Wang tiles obtained by Jeandel and Rao, and their definition was motivated by the very good properties that they exhibit, namely that the associated shift is minimal, aperiodic and self-similar, and it enjoys several nice characterizations. The new set of 16 Wang tiles defined by Jana Lepšová offers a further remarkable property: it is automatic with respect to the Fibonacci complement numeration system.

The main results of this dissertation have already been reported in two journal papers published in 2023, namely in *RAIRO - Theoretical Informatics and Applications* and *European Journal of Combinatorics*, two papers published in the LNCS proceedings of the WORDS 2021 and WORDS 2023 conferences, and one paper submitted for publication. Furthermore, one of these articles stems from research conducted without the collaboration of her thesis supervisors, which demonstrates Jana Lepšová's ability to work with various collaborators. The present dissertation also includes additional results that are not part of these already written papers, and only the proofs of these additional results are provided.

The numerous problems studied and solved by Jana Lepšová in this dissertation significantly contribute to the fields of numeration systems, combinatorics on words and symbolic dynamical systems. The questions she addresses are challenging and motivated by numerous appropriate references to the literature. It is quite remarkable that in addressing some natural questions, Jana Lepšová has been able developed new tools that are not only relevant for the study of the questions at hand but also prove to be very interesting in themselves, namely the compelling notions of complement numeration systems and of faithful representations of morphisms. Several intriguing open problems are presented throughout the text and summarized in the concluding chapter, testifying of Jana Lepšová's scientific curiosity. Lastly, I would also like to emphasize that the text is very well-written and enriched with many examples facilitating the understanding of all relevant concepts.

In conclusion, this is an excellent dissertation and I strongly recommend the competent authorities to allow Jana Lepšová to defend her doctoral work.

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