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MULTIVARIATE FOURIER–WEYL TRANSFORMS AND THEIR APPLICATIONS

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This thesis is the result of my own work, except where explicit reference is made to the work of others and has not been submitted for another qualification to this or any other university.

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Abstract

The thesis summarizes publications that contain the author's contribution to the field of multivariate Fourier transforms and their applications within mathematical physics. The corresponding research was conducted following the award of the doctoral degree (Ph.D.). The Fourier–Weyl transforms comprise discrete transforms constructed on finite fragments of the Weyl group invariant lattices. Ten types of Weyl orbit functions, restricted to the fundamental domain of the affine Weyl groups and their even subgroups, induce the forward and backward discrete transforms. The related 2D and 3D interpolation problems and cubature formulas are developed. The transforms are linked to the Kac–Walton formulas and Kac–Peterson matrices in conformal field theory and applied to vibration models in solid state physics. The entire author's published body of work in this field comprises 16 articles in impacted journals and several conference proceedings. The presented nine articles, published in impacted journals, are chosen to represent the entire collection of the author's publications in the field and contain key notions and original concepts of the author's contribution.

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Introduction

Continuous and discrete symmetries of physical systems are fundamental for construction and mathematization of the related physical theories and models. Group theory forms a central part of mathematical description and analysis of the inherent symmetries. The classical theory of Lie groups and Lie algebras [29, 49] establishes an essential connection between group theory and modern physics. Multivariate exponential functions associated to crystallographic root systems of complex simple Lie algebras and their corresponding affine Weyl groups constitute a standard segment of Lie theory and its application in mathematical physics [4, 5, 48, 53, 88]. Character functions of irreducible representations are determined for any dominant integral form via the Weyl character formula as ratios of two antisymmetric multivariate exponential functions [29]. The character functions are directly related to elements of finite order in the corresponding Lie group [25]. General continuous harmonic analysis on symmetric spaces of the related generalized hypergeometric functions of the crystallographic root systems contains Weyl orbit polynomials as special, explicitly constructed cases [36]. The Weyl orbit functions are embedded in conformal field theory [26] and appear implicitly in solid state physics [96].

Linked to the crystallographic root systems and their induced finite reflection groups, the Weyl orbit functions represent multivariate generalizations of the classical trigonometric functions [5, 53, 88]. Symmetric sums of exponential functions, directly connected to multivariate versions of Chebyshev polynomials [5], are named C -functions in [56] and serve as generalizations of the cosine function. Antisymmetric S -functions from the Weyl character formula lead to generalizations of the sine function [57] and E -functions, related to even subgroups of the Weyl groups [60], produce specific versions of the exponential function. Symmetry and antisymmetry properties of the C - and S -functions with respect to their inherent Weyl groups together with the translation invariance by shifts from the dual root lattice permit restrictions of these functions to the fundamental domains of their affine Weyl groups [48, 53]. The fundamental domains in the form of the Weyl alcoves constitute generalizations of the one-dimensional interval as domain for the classical cosine and sine functions [48, 53]. The fundamental domains of the affine Weyl group necessitate extension by their reflected images to form domains for the E -functions [A3]. The set of multivariate generalization of the cosine, sine and exponential functions is further enriched by the concept of sign homomorphisms [78].

The sign homomorphisms of the crystallographic root systems with two lengths of roots induce two transitional types of multidimensional generalizations of cosine and sine functions [44, 78]. Hybrids of C -functions and S -functions, named S^s - and S^l -functions,

acquire both symmetry and antisymmetry behavior on the boundaries of the fundamental domain. The sign homomorphisms also produce two additional even subgroups of the Weyl groups and five novel related types of E -functions, named $E^{s\pm-}$, $E^{l\pm-}$ and E^{e-} -functions [34]. The resulting ten types of the multivariate functions retain similar properties as their one-dimensional precursors and permit formulation of continuous multivariate generalizations of Fourier–Weyl transforms [34, 56, 57, 60]. Formation of the discrete Fourier–Weyl transforms requires a detailed description of the sets of points of the transforms jointly with the finite sets of labels of the corresponding Weyl orbit functions [40]. The first implicit attempts to develop the discrete Fourier–Weyl transforms are taken in the review articles [56, 57, 60, 84]. The paper [80] formulates the discrete orthogonality of the C -, S - and E -functions on the refinement of the dual weight lattice using group theoretical arguments.

The dual root system and the dual affine Weyl group first appear in the description of the labels of the C - and S -functions and the corresponding fully explicit form of the dual weight-lattice Fourier–Weyl transforms [A1]. The precise description of the fundamental domains of E -functions and their dual versions lead to the explicit discrete even Fourier–Weyl transforms of E -functions [A3]. The boundary behavior of the hybrid orbit functions with respect to both point and label domains results in the corresponding hybrid Fourier–Weyl transforms in [44]. Shifting weight and dual weight lattices by admissible shifts together with the related compact description of the point and label sets further enriched the collection of Fourier–Weyl transforms on the shifted refinement of the dual weight lattice in [A2]. Taking into account direct products of the Weyl groups opens novel possibilities for the types of E -functions and the related discrete transforms [A4]. Challenging description of the discrete transforms for the remaining five types of E -functions is attempted for bivariate cases in [34]. Fully general characterization of the point and label sets of all six types of even Fourier–Weyl transforms is achieved recently in [40].

The multivariate antisymmetric and symmetric cosine, sine and exponential functions and their alternating versions are introduced in [58, 59, 61, 62] as determinants and permanents [77] of matrices with entries comprising univariate cosine, sine and exponential functions, respectively. The symmetric generalizations of cosine and sine functions are related in [42] to the Weyl orbit functions associated to the root systems of types B_n and C_n . The lower-dimensional bivariate cases are detailed in [43, A5], the trivariate alternating exponentials are analyzed in [46]. The discrete orthogonality relations of these functions are a consequence of ubiquitous univariate discrete cosine and sine transforms and their Cartesian direct product multidimensional generalizations. The eight underlying distinct types of discrete cosine and sine transforms obey miscellaneous boundary conditions [7, 103]. The first four transforms I–IV are generalized to the multidimensional symmetric trigonometric functions in [59], the four symmetric cosine transforms of types V–VIII are developed in [A6]. Utilization of the symmetric cosine and sine transforms to 2D and 3D interpolation methods [8, 21] is demonstrated in [43, A5, A6]. The antisymmetric and symmetric cosine functions also permit construction of the multivariate Chebyshev polynomial analysis [2, 36, 47, 73, 74, 85].

The classical Chebyshev polynomials are extensively exploited orthogonal polynomials interlaced with efficient methods of numerical integration and approximation [30,95]. Both classes of antisymmetric and symmetric cosine functions are based on the one-dimensional cosine functions and admit a multidimensional generalization of the Chebyshev polynomials of the first and third kinds [A6]. For the two-dimensional case, the resulting polynomials become special cases of bivariate analogues of Jacobi polynomials [63–65]. The multivariate symmetric cosine Chebyshev polynomials inherit crucial properties from the symmetric cosine functions and this link provides tools to generalize powerful numerical integration formulas of the classical Chebyshev polynomials. The main objective of cubature formulas is replacing integration by finite summing over the set of the generalized Chebyshev nodes [15]. The exact equality of the finite sum over the nodes and the approximated integral holds for polynomials up to a specific degree, which depends on the number of nodes. Gaussian cubature formulas require the lowest bound of the number of nodes and achieve the maximal degree of precision [28,69–71,98]. Among sixteen types of the symmetric cosine cubature formulas in [A6] four types are Gaussian.

Besides the ubiquitous dual weight–lattice discretization, the dual root–lattice discretization and the corresponding dual root Fourier–Weyl transforms are recently developed in [41]. Existence of the dual root Fourier–Weyl transforms permits construction of the transforms for point sets, which are constructed by subtracting the point sets of the dual root–lattice transform from the point sets of the dual weight–lattice transform. Among these subtractively constructed point sets stands out a triangular honeycomb lattice fragment with armchair boundaries. The concept of extended Weyl orbit functions and the induced honeycomb Fourier–Weyl transforms are developed in [A9]. The real–valued Hartley orbit functions [40,41,A9] and the corresponding discrete Hartley–Weyl transforms constitute generalizations of the discrete version of the univariate Hartley transform [6,93]. The discrete Hartley–Weyl transforms are developed on the dual weight lattice [40], dual root lattice [41] and honeycomb lattice [A9]. Modifications of the results in [A1], which produce the weight–lattice discretization and the corresponding Fourier–Weyl transforms, are achieved in [A8]. The dual weight and weight–lattice Fourier–Weyl transforms are closely linked to conformal field theory.

Conformal field theories with the Lie group symmetry regularly utilize the antisymmetric Weyl orbit functions and their discrete Fourier–Weyl transforms [26,111]. A correspondence between the dual weight discretization of Weyl orbit functions and affine modular data associated with the Wess–Zumino–Novikov–Witten conformal field theories is developed in [A7]. Products of the discretized orbit functions are analogous with truncations of tensor products which determine interactions in the conformal field models. A significant tool for description of the tensor products, which leads to an efficient algorithm for calculation of the fusion coefficients, is the Kac–Walton formula [26]. The generalization of the Kac–Walton formula for the dual weight lattice Fourier–Weyl transforms from [A1] and the related Galois symmetries are developed in [A7]. The Wess–Zumino–Novikov–Witten conformal models depend on the weight–lattice discretization of Weyl orbit functions rather than the dual weight–lattice discretization from [A1]. Three additional generalizations of

the Kac–Peterson unitary and symmetric S –matrices resulting from the symmetric and hybrid Weyl orbit functions are constructed in [A8]. Apart from conformal field theory, the Fourier–Weyl and Hartley–Weyl transforms found also direct applications to eigenvibrations of mechanical models in solid state physics.

The 2D and 3D mechanical vibration models based on the Fourier–Weyl transforms retain Weyl group symmetries and determine vibrations of the Weyl group invariant lattices. General cases of the mechanical vibration models in solid state physics constitute fundamental stepping stones for their quantum field versions [27]. Dispersion relations of these models are systematically derived in solid state physics assuming solutions in exponential form while imposing periodic Born–von Kármán boundary conditions [50]. The Hartley orbit functions represent multidimensional generalizations of the cosine and sine standing waves solutions of the one-dimensional beaded string subjected to von Neumann and Dirichlet conditions, respectively. The Hartley orbit functions form Weyl group invariant solutions with significantly expanded selection of boundary conditions. The spectral analysis of initial conditions provided by the Fourier–Weyl transforms enables calculation of time evolving exact solutions of the mechanical models. A special case of such mechanical models, the mechanical graphene model [17], is currently intensively investigated [54] in connection with the relevant graphene material [11,27]. Characteristics of the triangular graphene dots, which form the point sets of the honeycomb Fourier–Weyl and Hartley–Weyl transforms, are extensively theoretically and experimentally studied [96]. Transversal eigenvibrations of the equilateral triangular sheets of the mechanical graphene and the wave functions of the quantum particle on the same triangular honeycomb point set [96] are determined by the honeycomb Hartley orbit functions from [A9].

The contribution of the author of the thesis to the original results, contained in the included articles [A1–A9], ranges from theoretical research to calculation of model examples [A1–A3, A5, A7, A8], preparation of figures [A1–A5, A7–A9] and execution of symbolic [A1–A3, A7–A9] and numerical computations [A5]. The most significant theoretical results, which were obtained with the essential contribution of the author, encompass

- Theorem 3.3, Propositions 5.3 and 5.4 in [A1],
- Theorems 3.3, 3.4 and 4.1 in [A2],
- Propositions 2.1 and 2.3, Theorem 3.2, Proposition 5.1 in [A3],
- Sections 3.1.2 and 3.2.2 in [A4],
- Propositions 3.1 and 3.2 in [A5],
- Sections 3.2 and 3.4 in [A6],
- Theorems 4.1 and 4.5 in [A8],
- Sections V and VI in [A9].

The thesis is organized as follows. In Chapter 1, the notation is established and relevant notions from the theory of root systems of Lie algebras, Weyl groups and affine Weyl groups are summarized. In Chapter 2, the four types of discrete Fourier–Weyl transforms are detailed and the even transforms and semisimple even transforms are compiled. In Chapter 3, the symmetric exponential and cosine transforms are outlined, the related 2D and 3D interpolation problems presented and the developed cosine cubature formulas listed. In Chapter 4, the modified multiplication of orbit functions and generalized Kac–Walton formulas are summarized, the weight lattice discretization and the inherent Kac–Peterson matrices presented. Application of the Fourier–Weyl transforms to the transversal vibration models of Weyl group invariant lattices is developed. In the Conclusion, the conclusions and follow up problems are presented. In the section Included Publications, the nine included articles [A1–A9] are listed.

Chapter 1

Affine Weyl groups

The purpose of the chapter is to establish notation of the thesis and recall pertinent notions from the theory of Lie groups, Lie algebras and their related Weyl groups and affine Weyl groups [49, 108]. The classical material is extended where necessary by definitions and properties from [A1, A2, A7]. The starting point is the classification of the compact, simple, connected and simply connected Lie groups and their corresponding complex simple Lie algebras [49, 108] which consists of four infinite series A_n ($n \geq 1$), B_n ($n \geq 3$), C_n ($n \geq 2$), D_n ($n \geq 4$) and five exceptional cases E_6, E_7, E_8, F_4, G_2 .

1.1 Root systems and Weyl groups

To each complex simple Lie algebra from the four infinite series and the five exceptional cases corresponds the set of vectors

$$\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R}^n,$$

that are called **simple roots** [5, 49, 108]. Each set of simple roots Δ constitutes a non-orthogonal basis of the Euclidean space \mathbb{R}^n with the standard scalar product denoted as $\langle \cdot, \cdot \rangle$. There are two types of the sets of simple roots Δ . The first type of Δ , referred to also as **simply-laced**, consists of the roots of one length only and comprises the series A_n ($n \geq 1$), D_n ($n \geq 4$) and three special cases E_6, E_7, E_8 . The second type contains roots with two different lengths and is represented by the series B_n ($n \geq 3$), C_n ($n \geq 2$) and two exceptional cases F_4, G_2 . For the cases of Δ with two different root-lengths, the set Δ is disjointly decomposed into a set Δ_s of **short simple roots** and a set Δ_l of **long simple roots**,

$$\Delta = \Delta_s \cup \Delta_l. \tag{1.1}$$

Every simple root $\alpha_i \in \Delta$ induces a **reflection** r_i given by the standard formula for any $a \in \mathbb{R}^n$,

$$r_i a = a - \frac{2 \langle a, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i.$$

The set of reflections $r_i, i \in \{1, \dots, n\}$ generates a finite **Weyl group** W of orthogonal operators. Action of the Weyl group W on the set Δ generates the **root system** Π ,

$$\Pi = W\Delta.$$

The current notion of the root system coincides with a more general notion of root systems from the theory of Coxeter groups [4], the root systems corresponding to the complex simple Lie algebras are called **irreducible** and **crystallographic** in [48].

The set of simple roots Δ determines a specific **partial ordering** \leq_{Δ} on \mathbb{R}^n defined for any $\lambda, \nu \in \mathbb{R}^n$ as $\nu \leq_{\Delta} \lambda$ if and only if

$$\lambda - \nu = k_1\alpha_1 + \cdots + k_n\alpha_n, \quad k_i \in \mathbb{Z}^{\geq 0}, \quad i \in \{1, \dots, n\}.$$

There exists a unique root $\xi \in \Pi$ that is maximal with respect to the ordering \leq_{Δ} restricted to the root system Π , this **highest root** $\xi \in \Pi$ is expressed as a linear combination of the simple roots

$$\xi = m_1\alpha_1 + \cdots + m_n\alpha_n,$$

with non-negative integer coefficients $m_1, \dots, m_n \in \mathbb{N}$. The numbers m_1, \dots, m_n are called the **marks** and are listed in Table 1 in [A1]. The numbers $q_1, \dots, q_n \in \mathbb{N}$ associated to the marks via relation

$$q_i = \frac{m_i \langle \alpha_i, \alpha_i \rangle}{2}, \quad i \in \{1, \dots, n\}, \quad (1.2)$$

are called the **comarks**. The comarks are tabulated in Table I in [A8].

A circle inversion of simple roots $\alpha_i \in \Delta$,

$$\alpha_i^{\vee} = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle},$$

leads to a set of the vectors $\Delta^{\vee} = \{\alpha_1^{\vee}, \dots, \alpha_n^{\vee}\}$ that is also a set of simple roots of some complex simple Lie algebra. The set Δ^{\vee} generates the entire **dual root system** via action of the Weyl group,

$$\Pi^{\vee} = W\Delta^{\vee},$$

that contains the **highest dual root** $\eta \in \Pi^{\vee}$ with respect to the ordering $\leq_{\Delta^{\vee}}$ of the form

$$\eta = m_1^{\vee}\alpha_1^{\vee} + \cdots + m_n^{\vee}\alpha_n^{\vee}.$$

The expansion coefficients of the highest dual root $m_1^{\vee}, \dots, m_n^{\vee}$, named the **dual marks** are listed in Table 1 in [A1].

Setting additionally the zero mark and dual mark $m_0 = m_0^{\vee} = 1$, the **Coxeter number** m is a common value of the sum of the marks and the dual marks,

$$m = \sum_{i=0}^n m_i = \sum_{i=0}^n m_i^{\vee}. \quad (1.3)$$

Setting also the zero comark $q_0 = 1$, the **dual Coxeter number** g is the sum of the comarks,

$$g = \sum_{i=0}^n q_i. \quad (1.4)$$

Explicit form of the the sets of simple roots and their relation to their highest root is encoded in extended Coxeter-Dynkin diagrams in Figure 1 in [A1].

There are four significant Weyl group invariant lattices: the root lattice, the dual weight lattice, the dual root lattice and the weight lattice. The **root lattice** Q is the integer span of the set of simple roots Δ ,

$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n.$$

The **dual weight lattice** P^\vee is \mathbb{Z} -dual to the root lattice Q ,

$$P^\vee = \{\omega^\vee \in \mathbb{R}^n \mid \langle \omega^\vee, \alpha_i \rangle \in \mathbb{Z}, \forall \alpha_i \in \Delta\} = \mathbb{Z}\omega_1^\vee + \cdots + \mathbb{Z}\omega_n^\vee,$$

where the vectors ω_i^\vee are called the **dual fundamental weights** and are determined by the duality formula,

$$\langle \omega_i^\vee, \alpha_j \rangle = \delta_{ij}.$$

The **dual root lattice** Q^\vee is the integer span of the set of dual simple roots Δ^\vee ,

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee.$$

The **weight lattice** P is \mathbb{Z} -dual to the dual root lattice Q^\vee ,

$$P = \{\omega \in \mathbb{R}^n \mid \langle \omega, \alpha_i^\vee \rangle \in \mathbb{Z}, \forall \alpha_i^\vee \in \Delta^\vee\} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n, \quad (1.5)$$

where the vectors ω_i are called the **fundamental weights** and are determined by the duality formula,

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

The cone of the **dominant weights** is given as

$$P_+ = \mathbb{Z}^{\geq 0}\omega_1 + \cdots + \mathbb{Z}^{\geq 0}\omega_n, \quad (1.6)$$

and the cone of the **strictly dominant weights** is determined by

$$P_{++} = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n. \quad (1.7)$$

Any vector $\varrho^\vee \in \mathbb{R}^n$ is an **admissible shift** if the shifted dual weight lattice $\varrho^\vee + P^\vee$ remains W -invariant,

$$W(\varrho^\vee + P^\vee) = \varrho^\vee + P^\vee. \quad (1.8)$$

Any vector $\varrho \in \mathbb{R}^n$ is a **dual admissible shift** if the shifted weight lattice $\varrho + P$ remains W -invariant,

$$W(\varrho + P) = \varrho + P. \quad (1.9)$$

Any admissible shifts $\varrho^\vee \in P^\vee$ that preserve the dual weight lattice $\varrho^\vee + P^\vee = P^\vee$ and any dual admissible shifts $\varrho \in P$ that preserve the weight lattice $\varrho + P = P$ are called **trivial**. Any two admissible shifts which differ by a corresponding trivial shift are equivalent. The non-trivial admissible shifts and dual admissible shifts are classified up to the equivalence in Table 1 in [A2].

The **Cartan matrix** C is determined by its entries C_{ij} via relation

$$C_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle. \quad (1.10)$$

The determinant c of the Cartan matrix C determines the **index of connection** of the root system Π and the orders of the quotient groups P/Q and P^\vee/Q^\vee ,

$$c = \det C = |P/Q| = |P^\vee/Q^\vee|. \quad (1.11)$$

The **Gram determinant of the α^\vee -basis** d determines the order of the quotient group P/Q^\vee ,

$$d = \det \langle \alpha_i^\vee, \alpha_j^\vee \rangle = |P/Q^\vee| = \frac{2}{\langle \alpha_1, \alpha_1 \rangle} \cdots \frac{2}{\langle \alpha_n, \alpha_n \rangle} \det C, \quad (1.12)$$

and coincides with the number c for the simply-laced systems. The numbers d are for the systems with two root-lengths summarized in Table I in [A8].

The **affine reflection** r_0 and the **reflection** r_ξ with respect to the highest root $\xi \in \Pi$ are defined for any $a \in \mathbb{R}^n$ as

$$r_0 a = r_\xi a + \frac{2\xi}{\langle \xi, \xi \rangle}, \quad r_\xi a = a - \frac{2\langle a, \xi \rangle}{\langle \xi, \xi \rangle} \xi. \quad (1.13)$$

The **set of generators** R , comprising reflections r_1, \dots, r_n and the affine reflection r_0 ,

$$R = \{r_0, r_1, \dots, r_n\}, \quad (1.14)$$

generates the **affine Weyl group** W^{aff} . The **group of shifts** Q^\vee by the elements of the dual root lattice forms a normal subgroup of W^{aff} and the affine Weyl group is of the following semidirect product form,

$$W^{\text{aff}} = Q^\vee \rtimes W. \quad (1.15)$$

The **dual affine reflection** r_0^\vee and the **reflection** r_η with respect to the dual highest root $\eta \in \Pi^\vee$ are defined for any $a \in \mathbb{R}^n$ as

$$r_0^\vee a = r_\eta a + \frac{2\eta}{\langle \eta, \eta \rangle}, \quad r_\eta a = a - \frac{2\langle a, \eta \rangle}{\langle \eta, \eta \rangle} \eta. \quad (1.16)$$

The **set of generators** R^\vee , comprising reflections $r_1^\vee \equiv r_1, \dots, r_n^\vee \equiv r_n$ and the dual affine reflection r_0^\vee ,

$$R^\vee = \{r_0^\vee, r_1^\vee, \dots, r_n^\vee\},$$

generates the **dual affine Weyl group** \widehat{W}^{aff} . The **group of shifts** Q by the elements of the root lattice forms a normal subgroup of \widehat{W}^{aff} and the dual affine Weyl group is of the following semidirect product form,

$$\widehat{W}^{\text{aff}} = Q \rtimes W. \quad (1.17)$$

1.2 Fundamental Domains and Homomorphisms

For any $w^{\text{aff}} \in W^{\text{aff}}$, a unique Weyl group element $w \in W$ and a unique shift $T(q^\vee) \in W^{\text{aff}}$ by $q^\vee \in Q^\vee$ exist such that it holds

$$w^{\text{aff}} a = T(q^\vee) w a = w a + q^\vee.$$

The **retraction homomorphism** $\psi : W^{\text{aff}} \rightarrow W$ of the semidirect product (1.15) and the **mapping** $\tau : W^{\text{aff}} \rightarrow Q^\vee$ are given for any $w^{\text{aff}} = T(q^\vee)w \in W^{\text{aff}}$ by

$$\psi(w^{\text{aff}}) = w, \quad (1.18)$$

$$\tau(w^{\text{aff}}) = q^\vee. \quad (1.19)$$

For any $w^{\text{aff}} \in \widehat{W}^{\text{aff}}$, a unique Weyl group element $w \in W$ and a unique shift $T(q) \in \widehat{W}^{\text{aff}}$ by $q \in Q$ exist such that

$$w^{\text{aff}}a = T(q)wa = wa + q.$$

The **dual retraction homomorphism** $\widehat{\psi} : \widehat{W}^{\text{aff}} \rightarrow W$ of the semidirect product (1.17) and the **mapping** $\widehat{\tau} : \widehat{W}^{\text{aff}} \rightarrow Q$ are given for any $T(q)w \in \widehat{W}^{\text{aff}}$ by

$$\widehat{\psi}(w^{\text{aff}}) = w, \quad (1.20)$$

$$\widehat{\tau}(w^{\text{aff}}) = q. \quad (1.21)$$

The **augmented dual affine Weyl group** $\widehat{W}_M^{\text{aff}}$ is defined for any scaling factor $M \in \mathbb{N}$ by relation

$$\widehat{W}_M^{\text{aff}} = MQ \rtimes W. \quad (1.22)$$

Each admissible shift ϱ^\vee generates a **shift homomorphism** from the dual affine Weyl group $\widehat{\theta}_{\varrho^\vee} : \widehat{W}^{\text{aff}} \rightarrow \{\pm 1\}$ defined in [A2] for any $w^{\text{aff}} \in \widehat{W}^{\text{aff}}$ via the mapping (1.21) as

$$\widehat{\theta}_{\varrho^\vee}(w^{\text{aff}}) = e^{2\pi i \langle \widehat{\tau}(w^{\text{aff}}), \varrho^\vee \rangle}. \quad (1.23)$$

To trivial admissible shifts corresponds the trivial shift homomorphism $\widehat{\theta}_{\varrho^\vee}(w^{\text{aff}}) = 1$. The **dual shift homomorphism** $\theta_\varrho : W^{\text{aff}} \rightarrow \{\pm 1\}$ is defined in [A2] for any dual admissible shift ϱ and for any $w^{\text{aff}} \in W^{\text{aff}}$ via the mapping (1.19) as

$$\theta_\varrho(w^{\text{aff}}) = e^{2\pi i \langle \tau(w^{\text{aff}}), \varrho \rangle}. \quad (1.24)$$

Any homomorphism $\sigma : W \mapsto \{\pm 1\}$ is called a **sign homomorphism** [44]. There exist exactly two sign homomorphisms for the simply-laced roots systems and four for the systems with two root-lengths. The **identity** $\mathbf{1}$ and the **determinant** σ^e **sign homomorphisms**, which exist for all Weyl groups W , are given on the generating reflections r_i , $\alpha_i \in \Delta$ as

$$\begin{aligned} \mathbf{1}(r_i) &= 1, \\ \sigma^e(r_i) &= -1. \end{aligned}$$

For the root systems with two lengths of roots, the **short** and **long sign homomorphisms** σ^s and σ^l are defined via decomposition (1.1) as

$$\begin{aligned} \sigma^s(r_i) &= \begin{cases} -1, & \alpha_i \in \Delta_s, \\ 1, & \alpha_i \in \Delta_l, \end{cases} \\ \sigma^l(r_i) &= \begin{cases} -1, & \alpha_i \in \Delta_l, \\ 1, & \alpha_i \in \Delta_s. \end{cases} \end{aligned}$$

Defining the composition \cdot of the sign homomorphisms pointwise [40], the resulting two-element and four-element abelian groups are isomorphic to \mathbb{Z}_2 and the Klein four-group, respectively.

Composing the retraction homomorphism (1.18), the dual shift homomorphism (1.24) and up to four sign homomorphisms yields the γ_ρ^σ -**homomorphism** as a mapping

$$\gamma_\rho^\sigma : W^{\text{aff}} \rightarrow \{\pm 1\},$$

defined for any $w^{\text{aff}} \in W^{\text{aff}}$ as

$$\gamma_\rho^\sigma(w^{\text{aff}}) = \theta_\rho(w^{\text{aff}}) \cdot [\sigma \circ \psi(w^{\text{aff}})]. \quad (1.25)$$

The values of the γ_ρ^σ -homomorphism on the generators from the generator set R are for trivial and non-trivial admissible shifts summarized in Table II in [A2]. The dual retraction homomorphism (1.20), the shift homomorphism (1.23) and up to four sign homomorphisms lead to the **homomorphism** $\widehat{\gamma}_{\rho^\vee}^\sigma : \widehat{W}^{\text{aff}} \rightarrow \{\pm 1\}$ given for any $w^{\text{aff}} \in \widehat{W}^{\text{aff}}$ as

$$\widehat{\gamma}_{\rho^\vee}^\sigma(w^{\text{aff}}) = \widehat{\theta}_{\rho^\vee}(w^{\text{aff}}) \cdot [\sigma \circ \widehat{\psi}(w^{\text{aff}})]. \quad (1.26)$$

The values of the $\widehat{\gamma}_{\rho^\vee}^\sigma$ -homomorphism on the generators from the generator set R^\vee are listed in Table III in [A2].

The **fundamental domain** F of the affine Weyl group W^{aff} consists of exactly one point of each W^{aff} -orbit. The fundamental domain F is chosen as the convex hull of the nodes $\left\{0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n}\right\}$,

$$F = \left\{ \sum_{i=1}^n y_i \omega_i^\vee \mid \sum_{i=0}^n y_i m_i = 1, y_0, \dots, y_n \geq 0 \right\}. \quad (1.27)$$

The order of the isotropy subgroup $\text{Stab}_{W^{\text{aff}}}(a)$ of any point $a \in \mathbb{R}^n$,

$$\text{Stab}_{W^{\text{aff}}}(a) = \left\{ w^{\text{aff}} \in W^{\text{aff}} \mid w^{\text{aff}} a = a \right\},$$

defines for any $M \in \mathbb{N}$ a **function** $h_M : \mathbb{R}^n \rightarrow \mathbb{N}$ and a **function** $\varepsilon : \mathbb{R}^n \rightarrow \mathbb{N}$ by

$$h_M(a) = \left| \text{Stab}_{W^{\text{aff}}}\left(\frac{a}{M}\right) \right|, \quad \varepsilon(a) = \frac{|W|}{h_1(a)}. \quad (1.28)$$

The **dual fundamental domain** F^\vee of the dual affine Weyl group \widehat{W}^{aff} is the convex hull of nodes $\left\{0, \frac{\omega_1^\vee}{m_1^\vee}, \dots, \frac{\omega_n^\vee}{m_n^\vee}\right\}$,

$$F^\vee = \left\{ \sum_{i=1}^n z_i \omega_i^\vee \mid \sum_{i=0}^n z_i m_i^\vee = 1, z_0, \dots, z_n \geq 0 \right\}. \quad (1.29)$$

The order of the isotropy subgroup $\text{Stab}_{\widehat{W}^{\text{aff}}}(b)$ of a point $b \in \mathbb{R}^n$ defines for any $M \in \mathbb{N}$ a **function** $h_M^\vee : \mathbb{R}^n \rightarrow \mathbb{N}$ by the relation

$$h_M^\vee(b) = \left| \text{Stab}_{\widehat{W}^{\text{aff}}}\left(\frac{b}{M}\right) \right|. \quad (1.30)$$

For any sign homomorphism σ and any admissible dual shift ϱ the **generalized fundamental domain** $F^\sigma(\varrho) \subset F$ is given as

$$F^\sigma(\varrho) = \{a \in F \mid \gamma_\varrho^\sigma(\text{Stab}_{W^{\text{aff}}}(a)) = \{1\}\}$$

and the subset $R^\sigma(\varrho) \subset R$ of generators of W^{aff} is given as

$$R^\sigma(\varrho) = \{r \in R \mid \gamma_\varrho^\sigma(r) = -1\}. \quad (1.31)$$

Subsets $H^\sigma(\varrho)$ of the boundaries of F are defined as

$$H^\sigma(\varrho) = \{a \in F \mid (\exists r \in R^\sigma(\varrho))(ra = a)\} \quad (1.32)$$

and Proposition 2.7 in [A2] asserts that

$$F^\sigma(\varrho) = F \setminus H^\sigma(\varrho).$$

Each subset $R^\sigma(0)$ of the set of generators R determines the decomposition of the sum of marks and comarks

$$m^\sigma = \sum_{r_i \in R^\sigma(0)} m_i, \quad (1.33)$$

$$q^\sigma = \sum_{r_i \in R^\sigma(0)} q_i. \quad (1.34)$$

For any sign homomorphism σ and any admissible shift ϱ^\vee the **generalized dual fundamental domain** $F^{\sigma^\vee}(\varrho^\vee) \subset F^\vee$ is given as

$$F^{\sigma^\vee}(\varrho^\vee) = \{a \in F^\vee \mid \widehat{\gamma}_{\varrho^\vee}^\sigma(\text{Stab}_{\widehat{W}^{\text{aff}}}(a)) = \{1\}\} \quad (1.35)$$

and the subset $R^{\sigma^\vee}(\varrho^\vee) \subset R^\vee$ of generators of \widehat{W}^{aff} is given as

$$R^{\sigma^\vee}(\varrho^\vee) = \{r \in R^\vee \mid \widehat{\gamma}_{\varrho^\vee}^\sigma(r) = -1\}. \quad (1.36)$$

Subsets $H^{\sigma^\vee}(\varrho^\vee)$ of the boundaries of F^\vee are defined as

$$H^{\sigma^\vee}(\varrho^\vee) = \{a \in F^\vee \mid (\exists r \in R^{\sigma^\vee}(\varrho^\vee))(ra = a)\} \quad (1.37)$$

and Proposition 2.8 in [A2] asserts that

$$F^{\sigma^\vee}(\varrho^\vee) = F^\vee \setminus H^{\sigma^\vee}(\varrho^\vee).$$

1.3 Even Weyl Groups and Fundamental Domains

The sign homomorphism σ^e is explicitly determined as the determinant of the Weyl group element $w \in W$,

$$\sigma^e(w) = \det w.$$

The kernel of the determinant homomorphism σ^e forms a normal subgroup $W^e \subset W$,

$$W^e = \{w \in W \mid \det w = 1\}.$$

The **even Weyl group** W^e exists for all root systems and comprises exactly a half of the elements of W ,

$$|W| = 2|W^e|.$$

The infinite **even affine Weyl group** $W_e^{\text{aff}} \subset W^{\text{aff}}$ is the semidirect product of the dual root lattice translation group Q^\vee and of the even Weyl group W^e ,

$$W_e^{\text{aff}} = Q^\vee \rtimes W^e. \quad (1.38)$$

The infinite **dual even affine Weyl group** $\widehat{W}_e^{\text{aff}} \subset \widehat{W}^{\text{aff}}$ is the semidirect product of the root lattice translation group Q and of the even Weyl group W^e ,

$$\widehat{W}_e^{\text{aff}} = Q \rtimes W^e. \quad (1.39)$$

The **even fundamental domain** F^e of the even affine Weyl group W_e^{aff} consists of exactly one point of each W_e^{aff} -orbit. The fundamental domain F^e is chosen in [A3] as the fundamental domain F extended by a reflected open interior of the form $r_j \text{int}(F)$, with some fixed $j \in \{1, \dots, n\}$,

$$F^e = F \cup r_j \text{int}(F). \quad (1.40)$$

The order of the even isotropy subgroup $\text{Stab}_{W_e^{\text{aff}}}(a)$ of any point $a \in \mathbb{R}^n$, defines for any $M \in \mathbb{N}$ an **even function** $h_M^e : \mathbb{R}^n \rightarrow \mathbb{N}$ and an **even function** $\varepsilon^e : \mathbb{R}^n \rightarrow \mathbb{N}$ by

$$h_M^e(a) = \left| \text{Stab}_{W_e^{\text{aff}}}\left(\frac{a}{M}\right) \right|, \quad \varepsilon^e(a) = \frac{|W^e|}{h_1^e(a)}. \quad (1.41)$$

The **dual even fundamental domain** $F^{e\vee}$ of the dual affine Weyl group $\widehat{W}_e^{\text{aff}}$ is the dual fundamental domain extended by its reflected open interior [A3],

$$F^{e\vee} = F^\vee \cup r_j \text{int}(F^\vee). \quad (1.42)$$

The order of the isotropy subgroup $\text{Stab}_{\widehat{W}_e^{\text{aff}}}(b)$ of a point $b \in \mathbb{R}^n$ defines for any $M \in \mathbb{N}$ an **even function** $h_M^{e\vee} : \mathbb{R}^n \rightarrow \mathbb{N}$ by the relation

$$h_M^{e\vee}(b) = \left| \text{Stab}_{\widehat{W}_e^{\text{aff}}}\left(\frac{b}{M}\right) \right|. \quad (1.43)$$

Chapter 2

Fourier–Weyl Transforms

The chapter collects the results concerning the development of the discrete orbit function transforms. The first major contribution of the author to this field is presented in paper [A1], where the two standard cases of discrete Fourier–Weyl transforms of the orbit C –functions and S –functions are developed. The two basic transforms are generalized and exemplified in papers [35, 44]. The paper [A2] further generalizes the four types of transforms for the shifted lattices. Another direction in generalizing the transforms lies in taking even subgroups of the Weyl groups as the symmetrizing tool. The essential discrete E –transform is achieved in the author’s publication [A3]. The E –transform is additionally generalized and exemplified in papers [34, 40]. Two fundamental possibilities of extending the E –transforms to reducible root systems are achieved in [A4]. In the following sections, the crucial notions from the representative papers [A1–A4] are concisely summarized.

2.1 (Anti)symmetric and Hybrid Transforms

Two families of complex **orbit functions** $\varphi_b^\sigma : \mathbb{R}^n \rightarrow \mathbb{C}$ for any root system together with two additional families for the systems with two root-lengths are labeled by the labels $b \in \mathbb{R}^n$ and determined by the sign homomorphisms σ via signed symmetrization of exponential functions over the Weyl group W ,

$$\varphi_b^\sigma(a) = \sum_{w \in W} \sigma(w) e^{2\pi i \langle wb, a \rangle}, \quad a \in \mathbb{R}^n. \quad (2.1)$$

Using the Hartley kernel functions

$$\text{cas } a = \cos a + \sin a,$$

the real-valued **Hartley orbit functions**, introduced in [40, 41, A9], are given by

$$\zeta_b^\sigma(a) = \sum_{w \in W} \sigma(w) \text{cas } 2\pi \langle wb, a \rangle, \quad a \in \mathbb{R}^n. \quad (2.2)$$

In [A1], the two families φ_b^1 and $\varphi_b^{\sigma^e}$ are called the C –**functions** and S –**functions**, respectively, and the following notation is used,

$$\Phi_b = \varphi_b^1, \quad \varphi_b = \varphi_b^{\sigma^e}. \quad (2.3)$$

For ϱ an admissible dual shift and $b \in \varrho + P$, for any $w^{\text{aff}} \in W^{\text{aff}}$ and $a \in \mathbb{R}^n$ the argument symmetry, containing the γ_ϱ^σ -homomorphism (1.25), is of the form

$$\varphi_b^\sigma(w^{\text{aff}} a) = \gamma_\varrho^\sigma(w^{\text{aff}}) \cdot \varphi_b^\sigma(a). \quad (2.4)$$

The orbit functions φ_b^σ have common zero points on the boundary $H^\sigma(\varrho)$,

$$\varphi_b^\sigma(a') = 0, \quad a' \in H^\sigma(\varrho). \quad (2.5)$$

For ϱ^\vee an admissible shift and a point $a \in \frac{1}{M}(\varrho^\vee + P^\vee)$, $M \in \mathbb{N}$ together with any $w^{\text{aff}} \in \widehat{W}^{\text{aff}}$ and $b \in \mathbb{R}^n$, the label symmetry of orbit functions is determined by the dual $\widehat{\gamma}_\varrho^\sigma$ -homomorphism (1.26),

$$\varphi_{Mw^{\text{aff}}(\frac{b}{M})}^\sigma(a) = \widehat{\gamma}_{\varrho^\vee}^\sigma(w^{\text{aff}}) \varphi_b^\sigma(a). \quad (2.6)$$

The orbit functions φ_b^σ are zero on the magnified boundary $MH^{\sigma^\vee}(\varrho^\vee)$,

$$\varphi_b^\sigma(a) = 0, \quad b \in MH^{\sigma^\vee}(\varrho^\vee). \quad (2.7)$$

Discrete values of both points $a \in \frac{1}{M}(\varrho^\vee + P^\vee)$ and labels $b \in \varrho + P$ of the orbit functions $\varphi_b^\sigma(a)$ are due to the argument symmetries restricted to the **set of points** $F_M^\sigma(\varrho, \varrho^\vee)$,

$$F_M^\sigma(\varrho, \varrho^\vee) = \left[\frac{1}{M}(\varrho^\vee + P^\vee) \right] \cap F^\sigma(\varrho), \quad (2.8)$$

and due to the label symmetries restricted to the **set of labels** $\Lambda_M^\sigma(\varrho, \varrho^\vee)$,

$$\Lambda_M^\sigma(\varrho, \varrho^\vee) = (\varrho + P) \cap MF^{\sigma^\vee}(\varrho^\vee). \quad (2.9)$$

Different notations are used for special cases of the sets $F_M^\sigma(\varrho, \varrho^\vee)$; for $F_M^1(0, 0)$ and $F_M^{\sigma^e}(0, 0)$, the symbols F_M and \widetilde{F}_M are used in [A1, A7],

$$\begin{aligned} F_M &= F_M^1(0, 0), \\ \widetilde{F}_M &= F_M^{\sigma^e}(0, 0). \end{aligned} \quad (2.10)$$

Different notations are also used for special cases of the sets $\Lambda_M^\sigma(\varrho, \varrho^\vee)$; for $\Lambda_M^1(0, 0)$ and $\Lambda_M^{\sigma^e}(0, 0)$, the symbols Λ_M and $\widetilde{\Lambda}_M$ are used in [A1] and the symbols P_+^M and P_{++}^M in [A7],

$$\begin{aligned} \Lambda_M &= P_+^M = \Lambda_M^1(0, 0), \\ \widetilde{\Lambda}_M &= P_{++}^M = \Lambda_M^{\sigma^e}(0, 0). \end{aligned} \quad (2.11)$$

Theorem 3.4 in [A2] states that the cardinalities of the sets of labels and the sets of points coincide for each case,

$$|\Lambda_M^\sigma(\varrho, \varrho^\vee)| = |F_M^\sigma(\varrho, \varrho^\vee)|.$$

The explicit form of the point and label sets, crucial for theoretical implications as well applications, is for each case of admissible shifts $\varrho^\vee = \varrho_1^\vee \omega_1^\vee + \cdots + \varrho_n^\vee \omega_n^\vee$ and admissible dual shifts $\varrho = \varrho_1 \omega_1 + \cdots + \varrho_n \omega_n$ determined via the symbols

$$u_i^{\sigma, \varrho} \in \begin{cases} \mathbb{N}, & r_i \in R^\sigma(\varrho), \\ \mathbb{Z}^{\geq 0}, & r_i \in R \setminus R^\sigma(\varrho), \end{cases} \quad (2.12)$$

and the symbols

$$t_i^{\sigma, \varrho^\vee} \in \begin{cases} \mathbb{N}, & r_i \in R^{\sigma^\vee}(\varrho^\vee), \\ \mathbb{Z}^{\geq 0}, & r_i \in R^\vee \setminus R^{\sigma^\vee}(\varrho^\vee). \end{cases}$$

The set of points $F_M^\sigma(\varrho, \varrho^\vee)$ is calculated explicitly as

$$F_M^\sigma(\varrho, \varrho^\vee) = \left\{ \sum_{i=1}^n \frac{u_i^{\sigma, \varrho} + \varrho_i^\vee}{M} \omega_i^\vee \mid \sum_{i=0}^n u_i^{\sigma, \varrho} m_i = M \right\}, \quad (2.13)$$

and the set of labels $\Lambda_M^\sigma(\varrho, \varrho^\vee)$ as

$$\Lambda_M^\sigma(\varrho, \varrho^\vee) = \left\{ \sum_{i=1}^n (t_i^{\sigma, \varrho^\vee} + \varrho_i) \omega_i \mid \sum_{i=0}^n t_i^{\sigma, \varrho^\vee} m_i^\vee = M \right\}.$$

Explicit counting formulas for cardinalities of the sets $F_M^1(0, 0)$ and $F_M^{\sigma^e}(0, 0)$, denoted as F_M and \tilde{F}_M , are derived in Theorem 3.3 and Proposition 3.5 in [A1]. Explicit counting formulas for cardinalities of the sets $F_M^{\sigma^s}(0, 0)$ and $F_M^{\sigma^l}(0, 0)$, denoted as F_M^s and F_M^l , are contained in Theorem 5.2 in [44]. The remaining cases related to non-trivial shifts are derived in Theorem 3.3 in [A2].

The vector space $\mathcal{F}_M^\sigma(\varrho, \varrho^\vee)$ of complex functions $f : F_M^\sigma(\varrho, \varrho^\vee) \rightarrow \mathbb{C}$ is equipped with a scalar product containing the weight function (1.28). This **weighted scalar product** is of the following form for any $f, g \in \mathcal{F}_M^\sigma(\varrho, \varrho^\vee)$,

$$\langle f, g \rangle_{F_M^\sigma(\varrho, \varrho^\vee)} = \sum_{a \in F_M^\sigma(\varrho, \varrho^\vee)} \varepsilon(a) f(a) \overline{g(a)}. \quad (2.14)$$

The orthogonality relations of orbit functions in the Hilbert space $\mathcal{F}_M^\sigma(\varrho, \varrho^\vee)$ are summarized in Theorem 4.1 in [A2]. Using the numbers (1.11) and functions (1.30), the orthogonality relations are for any labels $b, b' \in \Lambda_M^\sigma(\varrho, \varrho^\vee)$ of the form

$$\langle \varphi_b^\sigma, \varphi_{b'}^\sigma \rangle_{F_M^\sigma(\varrho, \varrho^\vee)} = c |W| M^n h_M^\vee(b) \delta_{b, b'}. \quad (2.15)$$

The **forward Fourier–Weyl transform** calculates for any $f \in \mathcal{F}_M^\sigma(\varrho, \varrho^\vee)$ its spectral transform $\hat{f} : \Lambda_M^\sigma(\varrho, \varrho^\vee) \rightarrow \mathbb{C}$ by prescribing for any $b \in \Lambda_M^\sigma(\varrho, \varrho^\vee)$ the value

$$\hat{f}(b) = \frac{\langle f, \varphi_b^\sigma \rangle_{F_M^\sigma(\varrho, \varrho^\vee)}}{\langle \varphi_b^\sigma, \varphi_b^\sigma \rangle_{F_M^\sigma(\varrho, \varrho^\vee)}} = (c |W| M^n h_M^\vee(b))^{-1} \sum_{a \in F_M^\sigma(\varrho, \varrho^\vee)} \varepsilon(a) f(a) \overline{\varphi_b^\sigma(a)}. \quad (2.16)$$

Due to the orthogonality relations, the **backward Fourier–Weyl transform** returns the original function $f \in \mathcal{F}_M^\sigma(\varrho, \varrho^\vee)$,

$$f(a) = \sum_{b \in \Lambda_M^\sigma(\varrho, \varrho^\vee)} \hat{f}(b) \varphi_b^\sigma(a), \quad a \in F_M^\sigma(\varrho, \varrho^\vee). \quad (2.17)$$

The corresponding Plancherel formula is of the form

$$\sum_{a \in F_M^\sigma(\varrho, \varrho^\vee)} \varepsilon(a) |f(a)|^2 = c |W| M^n \sum_{b \in \Lambda_M^\sigma(\varrho, \varrho^\vee)} h_M^\vee(b) |\hat{f}(b)|^2.$$

2.2 Even Transforms

A family of complex **even orbit functions** $\Xi_b : \mathbb{R}^n \rightarrow \mathbb{C}$ for any root system, labeled by the labels $b \in \mathbb{R}^n$, is determined via symmetrization of exponential functions over the even Weyl group W^e ,

$$\Xi_b(a) = \sum_{w \in W^e} e^{2\pi i \langle wb, a \rangle}. \quad (2.18)$$

The argument symmetry of the even orbit function with respect to the even affine Weyl group W_e^{aff} for $w^{\text{aff}} \in W_e^{\text{aff}}$, $b \in P$ and $a \in \mathbb{R}^n$ is of the form

$$\Xi_b(w^{\text{aff}}a) = \Xi_b(a). \quad (2.19)$$

For a point $a \in \frac{1}{M}P^\vee$, $M \in \mathbb{N}$ together with any $w^{\text{aff}} \in \widehat{W}_e^{\text{aff}}$ and $b \in \mathbb{R}^n$, the label symmetry of even orbit functions is of the form

$$\Xi_{Mw^{\text{aff}}(\frac{b}{M})}(a) = \Xi_b(a). \quad (2.20)$$

Discrete values of both points $a \in \frac{1}{M}P^\vee$ and labels $b \in P$ of the even orbit functions $\Xi_b(a)$ are due to the argument symmetries restricted to the **even set of points** F_M^e ,

$$F_M^e = \frac{1}{M}P^\vee \cap F^e, \quad (2.21)$$

and due to the label symmetries restricted to the **even set of labels** Λ_M^e ,

$$\Lambda_M^e = P \cap MF^{e\vee}. \quad (2.22)$$

The cardinalities of the even sets of labels and the even sets of points coincide for each case [A3],

$$|\Lambda_M^e| = |F_M^e|.$$

The explicit form of the even point and label sets, determined by relation (31) in [A3], is reformulated in current notation as

$$F_M^e = [F_M^1(0, 0)] \cup [r_j F_M^{\sigma^e}(0, 0)],$$

with $F_M^1(0, 0)$ given explicitly by (2.13) and

$$r_j F_M^{\sigma^e}(0, 0) = \left\{ \sum_{\substack{i=1 \\ i \neq j}}^n \frac{s'_i}{M} \omega_i^\vee + \frac{s'_j}{M} (\omega_j^\vee - \alpha_j^\vee) \mid s'_0, \dots, s'_n \in \mathbb{N}, \sum_{i=0}^n s'_i m_i = M \right\}. \quad (2.23)$$

Explicit counting formulas for cardinalities of the sets F_M^e are derived in Theorem 3.2 in [A3].

The vector space \mathcal{F}_M^e of complex functions $f : F_M^e \rightarrow \mathbb{C}$ is equipped with a scalar product containing the even weight function (1.41). This **even weighted scalar product** is of the following form for any $f, g \in \mathcal{F}_M^e$,

$$\langle f, g \rangle_{F_M^e} = \sum_{a \in F_M^e} \varepsilon^e(a) f(a) \overline{g(a)}. \quad (2.24)$$

The orthogonality relations of even orbit functions in the Hilbert space \mathcal{F}_M^e are summarized in Proposition 5.1 in [A3]. Using the numbers (1.11) and even functions (1.43), the orthogonality relations are for any labels $b, b' \in \Lambda_M^e$ of the form

$$\langle \Xi_b, \Xi_{b'} \rangle_{F_M^e} = c |W^e| M^n h_M^{e\vee}(b) \delta_{b,b'}. \quad (2.25)$$

The **forward even Fourier–Weyl transform** calculates for any $f \in \mathcal{F}_M^e$ its even spectral transform $\widehat{f}: \Lambda_M^e \rightarrow \mathbb{C}$ by prescribing for any $b \in \Lambda_M^e$ the value

$$\widehat{f}(b) = \frac{\langle f, \Xi_b \rangle_{F_M^e}}{\langle \Xi_b, \Xi_b \rangle_{F_M^e}} = (c |W^e| M^n h_M^{e\vee}(b))^{-1} \sum_{a \in F_M^e} \varepsilon^e(a) f(a) \overline{\Xi_b(a)}. \quad (2.26)$$

Due to the orthogonality relations (2.25), the **backward even Fourier–Weyl transform** returns the original function $f \in \mathcal{F}_M^e$,

$$f(a) = \sum_{b \in \Lambda_M^e} \widehat{f}(b) \Xi_b(a), \quad a \in F_M^e. \quad (2.27)$$

The corresponding even Plancherel formula is of the form

$$\sum_{a \in F_M^e} \varepsilon^e(a) |f(a)|^2 = c |W^e| M^n \sum_{b \in \Lambda_M^e} h_M^{e\vee}(b) |\widehat{f}(b)|^2.$$

2.3 Semisimple Even Transforms

For two compact simply connected, connected simple Lie groups G_1, G_2 of rank n_1 and n_2 with irreducible root systems Δ_1 and Δ_2 , their corresponding Weyl groups are denoted by W_1 and W_2 , their Cartan matrices by C_1 and C_2 , their weight lattices by P_1 and P_2 , their dual weight lattices by P_1^\vee and P_2^\vee , their root lattices by Q_1 and Q_2 , their dual root lattices by Q_1^\vee and Q_2^\vee . The corresponding affine Weyl groups and their dual versions are for $i = 1, 2$ denoted by

$$\begin{aligned} W_i^{\text{aff}} &= Q_i^\vee \rtimes W_i, \\ \widehat{W}_i^{\text{aff}} &= Q_i \rtimes W_i, \end{aligned}$$

and their fundamental domains by F_1, F_2 and F_1^\vee, F_2^\vee , respectively.

For the semisimple Lie group $G = G_1 \times G_2$ of rank $n_1 + n_2$, the corresponding Weyl group and root system are $W = W_1 \times W_2$ and $\Delta = \Delta_1 \times \Delta_2$, the Cartan matrix is $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$, the weight lattice $P = P_1 \times P_2$, the dual weight lattice $P^\vee = P_1^\vee \times P_2^\vee$, the root lattice $Q = Q_1 \times Q_2$, the dual root lattice $Q^\vee = Q_1^\vee \times Q_2^\vee$. The affine Weyl group $W^{\text{aff}} = W_1^{\text{aff}} \times W_2^{\text{aff}} = Q^\vee \rtimes W$ has the fundamental domain of the form $F = F_1 \times F_2$. Two possibilities to define even Weyl subgroup of $W = W_1 \times W_2$ exist [A4]. The first defines the **direct product even Weyl group** $W^{ee} \subset W$ as a direct product of the corresponding even Weyl groups $W_1^e \subset W_1$ and $W_2^e \subset W_2$,

$$W^{ee} = W_1^e \times W_2^e. \quad (2.28)$$

The direct product even Weyl group comprises one-fourth of elements of the entire Weyl group W ,

$$|W^{ee}| = \frac{1}{4}|W_1||W_2|.$$

The **direct product even affine group** W_{ee}^{aff} and the **direct product dual even affine group** $\widehat{W}_{ee}^{\text{aff}}$ are given by

$$\begin{aligned} W_{ee}^{\text{aff}} &= Q^\vee \times W^{ee}, \\ \widehat{W}_{ee}^{\text{aff}} &= Q \times W^{ee}, \end{aligned}$$

and their fundamental domains are the products of the corresponding even fundamental domains,

$$\begin{aligned} F^{ee} &= F_1^e \times F_2^e, \\ F^{ee\vee} &= F_1^{e\vee} \times F_2^{e\vee}. \end{aligned}$$

The **direct product even orbit functions** Ξ_b^{ee} corresponding to W^{ee} are for any $b = (b_1, b_2) \in \mathbb{R}^{n_1+n_2}$ and $a = (a_1, a_2) \in \mathbb{R}^{n_1+n_2}$, with $a_1, b_1 \in \mathbb{R}^{n_1}$ and $a_2, b_2 \in \mathbb{R}^{n_2}$, of the form

$$\Xi_b^{ee}(a) = \sum_{w \in W^{ee}} e^{2\pi i \langle wa, b \rangle} = \Xi_{b_1}(a_1) \Xi_{b_2}(a_2),$$

where the functions Ξ_{b_1} and Ξ_{b_2} are even orbit functions defined by W_1^e and W_2^e , respectively. The contour plots of several lowest Ξ^{ee} -functions of $A_1 \times A_1$ are depicted in Figure 1 in [A4]. For two arbitrary $M_1, M_2 \in \mathbb{N}$, discrete values of both points $a \in \frac{1}{M_1}P_1^\vee \times \frac{1}{M_2}P_2^\vee$ and labels $b \in P$ of the even orbit functions $\Xi_b^{ee}(a)$ are due to the argument symmetries restricted to the **direct product even set of points** $F_{M_1 M_2}^{ee}$,

$$F_{M_1 M_2}^{ee} = \left(\frac{1}{M_1}P_1^\vee \times \frac{1}{M_2}P_2^\vee \right) \cap F^{ee},$$

and due to the label symmetries restricted to the **direct product even set of labels** $\Lambda_{M_1 M_2}^{ee}$,

$$\Lambda_{M_1 M_2}^{ee} = P \cap (M_1 F_1^{\vee e} \times M_2 F_2^{\vee e}).$$

For any $b = (b_1, b_2) \in \mathbb{R}^{n_1+n_2}$ and $a = (a_1, a_2) \in \mathbb{R}^{n_1+n_2}$, with $a_1, b_1 \in \mathbb{R}^{n_1}$ and $a_2, b_2 \in \mathbb{R}^{n_2}$, the **direct product even functions** $\varepsilon^{ee}, h_{M_1 M_2}^{ee\vee}(b) : \mathbb{R}^{n_1+n_2} \rightarrow \mathbb{C}$ are defined via relations

$$\begin{aligned} \varepsilon^{ee}(a) &= \varepsilon^e(a_1) \varepsilon^e(a_2), \\ h_{M_1 M_2}^{ee\vee}(b) &= h_{M_1}^{e\vee}(b_1) h_{M_2}^{e\vee}(b_2). \end{aligned}$$

The vector space $\mathcal{F}_{M_1 M_2}^{ee}$ of complex functions $f : F_{M_1 M_2}^{ee} \rightarrow \mathbb{C}$ is equipped with the **direct product even weighted scalar product** of the following form for any $f, g \in \mathcal{F}_{M_1 M_2}^{ee}$,

$$\langle f, g \rangle_{\mathcal{F}_{M_1 M_2}^{ee}} = \sum_{a \in F_{M_1 M_2}^{ee}} \varepsilon^{ee}(a) f(a) \overline{g(a)}. \quad (2.29)$$

The discrete orthogonality relations of direct product even orbit functions hold for any $b, b' \in \Lambda_{M_1 M_2}^{ee}$,

$$\langle \Xi_b^{ee}, \Xi_{b'}^{ee} \rangle_{F_{M_1 M_2}^{ee}} = \det C |W^{ee}| M_1^{n_1} M_2^{n_2} h_{M_1 M_2}^{ee\vee}(b) \delta_{b, b'}. \quad (2.30)$$

The **forward direct product even Fourier–Weyl transform** calculates for any $f \in \mathcal{F}_{M_1 M_2}^{ee}$ its spectral transform $\widehat{f}: \Lambda_{M_1 M_2}^{ee} \rightarrow \mathbb{C}$ by prescribing for any $b \in \Lambda_{M_1 M_2}^{ee}$ the value

$$\widehat{f}(b) = (\det C |W^{ee}| M_1^{n_1} M_2^{n_2} h_{M_1 M_2}^{ee\vee}(b))^{-1} \sum_{a \in F_{M_1 M_2}^{ee}} \varepsilon^{ee}(a) f(a) \overline{\Xi_b^{ee}(a)}. \quad (2.31)$$

Due to the orthogonality relations (2.30), the **backward direct product even Fourier–Weyl transform** returns the original function $f \in \mathcal{F}_{M_1 M_2}^{ee}$,

$$f(a) = \sum_{b \in \Lambda_{M_1 M_2}^{ee}} \widehat{f}(b) \Xi_b^{ee}(a), \quad a \in F_{M_1 M_2}^{ee}. \quad (2.32)$$

The second possibility uses the **full even Weyl group** $W^e \subset W$ as

$$W^e = \{w \in W_1 \times W_2 \mid \det w = 1\}. \quad (2.33)$$

The full even Weyl group comprises one-half of elements of the entire Weyl group W ,

$$|W^e| = \frac{1}{2} |W_1| |W_2|.$$

The **full even affine group** W_e^{aff} and the **full dual even affine group** $\widehat{W}_e^{\text{aff}}$ are given by

$$\begin{aligned} W_e^{\text{aff}} &= Q^\vee \rtimes W^e, \\ \widehat{W}_e^{\text{aff}} &= Q \rtimes W^e, \end{aligned}$$

and their fundamental domains are given by taking any fixed generating reflection r_1 of the group W_1 ,

$$\begin{aligned} F^e &= F_1 \times F_2 \cup \text{int}(r_1 F_1 \times F_2), \\ F^{e\vee} &= F_1^\vee \times F_2^\vee \cup \text{int}(r_1 F_1^\vee \times F_2^\vee). \end{aligned}$$

The **full even orbit functions** Ξ_b^e corresponding to W^e are for any $b = (b_1, b_2) \in \mathbb{R}^{n_1+n_2}$ and $a = (a_1, a_2) \in \mathbb{R}^{n_1+n_2}$, with $a_1, b_1 \in \mathbb{R}^{n_1}$ and $a_2, b_2 \in \mathbb{R}^{n_2}$, of the form

$$\Xi_b^e(a) = \sum_{w \in W^e} e^{2\pi i \langle wa, b \rangle} = \Xi_{b_1}(a_1) \Xi_{b_2}(a_2) + \Xi_{r_1 b_1}(a_1) \Xi_{r_2 b_2}(a_2),$$

where r_2 is some generating reflection of the group W_2 and the functions Ξ_{b_1} and Ξ_{b_2} are even orbit functions defined by W_1^e and W_2^e , respectively. The contour plots of several lowest Ξ^e –functions of $A_1 \times A_1$ are depicted in Figure 2 in [A4]. For arbitrary $M \in \mathbb{N}$, discrete values of both points $a \in \frac{1}{M} P^\vee$ and labels $b \in P$ of the full even orbit functions $\Xi_b^e(a)$ are due to the argument symmetries restricted to the **full even set of points** F_M^e ,

$$F_M^e = \frac{1}{M} P^\vee \cap F^e,$$

and due to the label symmetries restricted to the **full even set of labels** Λ_M^e ,

$$\Lambda_M^e = P \cap MF^{\vee e}.$$

The **full even functions** $\varepsilon^e, h_M^{e\vee}(b) : \mathbb{R}^{n_1+n_2} \rightarrow \mathbb{C}$ and the **full even weighted scalar product** are defined allowing W^e to be of the form (2.28) in defining relations (1.41), (1.43) and (2.24).

The vector space \mathcal{F}_M^e of complex functions $f : F_M^e \rightarrow \mathbb{C}$ is equipped with the full even weighted scalar product and the orthogonality relations of full even orbit functions in the Hilbert space \mathcal{F}_M^e are for $b, b' \in \Lambda_M^e$ of the form

$$\langle \Xi_b^e, \Xi_{b'}^e \rangle_{\mathcal{F}_M^e} = \det C |W^e| M^{n_1+n_2} h_M^{e\vee}(b) \delta_{b,b'}. \quad (2.34)$$

The **forward full even Fourier–Weyl transform** calculates for any $f \in \mathcal{F}_M^e$ its even spectral transform $\widehat{f} : \Lambda_M^e \rightarrow \mathbb{C}$ by prescribing for any $b \in \Lambda_M^e$ the value

$$\widehat{f}(b) = (\det C |W^e| M^{n_1+n_2} h_M^{e\vee}(b))^{-1} \sum_{a \in F_M^e} \varepsilon^e(a) f(a) \overline{\Xi_b^e(a)}. \quad (2.35)$$

Due to the orthogonality relations (2.34), the **backward full even Fourier–Weyl transform** returns the original function $f \in \mathcal{F}_M^e$,

$$f(a) = \sum_{b \in \Lambda_M^e} \widehat{f}(b) \Xi_b^e(a), \quad a \in F_M^e. \quad (2.36)$$

Chapter 3

(Anti)symmetric Trigonometric Transforms

The chapter collects the results concerning the development of the multivariate symmetric and antisymmetric trigonometric transforms. The first contribution of the author to this field is achieved in paper [A5], where the two-dimensional exponential and cosine transforms and related interpolation problems are presented. The two-dimensional sine transforms are further generalized and exemplified in paper [43]. The paper [46] investigated three-dimensional multivariate exponential analogues of the E -functions. Four novel types of multivariate symmetric and antisymmetric discrete cosine transforms and the corresponding interpolation and cubature formulas are developed in [A6]. The cubature formulas in context of the Weyl orbit functions and their corresponding discrete transforms are developed in [33, 45]. In the following sections, the crucial notions from the representative papers [A5, A6] are concisely summarized.

3.1 (Anti)symmetric Exponential Transforms

For any point $a \in \mathbb{R}^n$ with coordinates (a_1, \dots, a_n) in a basis e_1, \dots, e_n , that is orthonormal with respect to the scalar product $\langle \cdot, \cdot \rangle$, the **action of the permutation group** S_n on \mathbb{R}^n is given for $s \in S_n$ as

$$s(a_1, \dots, a_n) = (a_{s(1)}, \dots, a_{s(n)}).$$

The cardinality of the stabilizer $\text{Stab}_{S_n}(a)$ of the permutation action is denoted by

$$H_a = |\text{Stab}_{S_n}(a)|. \tag{3.1}$$

The **cubic lattice** \mathcal{T}_n is the integer span of the set of the basis e_1, \dots, e_n ,

$$\mathcal{T}_n = \mathbb{Z}e_1 + \dots + \mathbb{Z}e_n.$$

The **affine symmetric group** S_n^{aff} is defined as a semidirect product of group of translations \mathcal{T}_n by cubic lattice vectors and the permutation group S_n ,

$$S_n^{\text{aff}} = \mathcal{T}_n \rtimes S_n.$$

Defining the **affine domain** $F(S_n^{\text{aff}})$ by

$$F(S_n^{\text{aff}}) = \{a \in \mathbb{R}^n \mid 1 > a_1 > \cdots > a_n > 0\},$$

its closure $F(\tilde{S}_n^{\text{aff}})$,

$$F(\tilde{S}_n^{\text{aff}}) = \{a \in \mathbb{R}^n \mid 1 \geq a_1 \geq \cdots \geq a_n \geq 0\},$$

contains exactly one point from each S_n^{aff} -orbit.

The **antisymmetric exponential function**, introduced in [58], is defined for any label $b \in \mathbb{R}^n$ with coordinates (b_1, \dots, b_n) and a point $a \in \mathbb{R}^n$ via signed symmetrization of exponential functions over the permutation group S_n ,

$$E_b^-(a) = \sum_{s \in S_n} \text{sgn}(s) e^{2\pi i \langle sb, a \rangle},$$

and the **symmetric exponential function** is defined via symmetrization of exponential functions,

$$E_b^+(a) = \sum_{s \in S_n} e^{2\pi i \langle sb, a \rangle}.$$

For discretized labels $b \in \mathcal{T}_n$, the periodicity and symmetries of the antisymmetric and symmetric exponential functions are for $s \in S_n$ and $r \in \mathcal{T}_n$ of the form

$$E_b^+(sa + r) = E_b^+(a), \quad E_b^-(sa + r) = \text{sgn}(s)E_b^-(a).$$

The periodicity and symmetries of the (anti)symmetric exponential and trigonometric functions allow to restrict these functions to the closure of the fundamental domain $F(S_n^{\text{aff}})$. The **set of points** \hat{F}_N^n for the antisymmetric exponential functions is of the form

$$\hat{F}_N^n = \left\{ \frac{a}{N} \in \mathbb{R}^n \mid a_1 > \cdots > a_n, a_1, \dots, a_n \in \{1, 2, \dots, N\} \right\}$$

and the **set of labels** \hat{D}_N^n is of the form

$$\hat{D}_N^n = \{b \in \mathbb{R}^n \mid b_1 > \cdots > b_n, b_1, \dots, b_n \in \{1, 2, \dots, N\}\}.$$

The vector space $\hat{\mathcal{F}}_N^n$ of complex functions $f : \hat{F}_N^n \rightarrow \mathbb{C}$ is equipped with a scalar product of the following form for any $f, g \in \hat{\mathcal{F}}_N^n$,

$$\langle f, g \rangle_{\hat{\mathcal{F}}_N^n} = \sum_{a \in \hat{F}_N^n} f(a) \overline{g(a)}. \quad (3.2)$$

The orthogonality relations of the antisymmetric exponential functions in the Hilbert space $\hat{\mathcal{F}}_N^n$ are summarized in Proposition 1 in [58]. The orthogonality relations are for any labels $b, b' \in \hat{D}_N^n$ of the form

$$\langle E_b^-, E_{b'}^- \rangle_{\hat{\mathcal{F}}_N^n} = N^n \delta_{b, b'}.$$

The **forward antisymmetric exponential Fourier transform** calculates for any $f \in \hat{\mathcal{F}}_N^n$ its spectral transform $\hat{f} : \hat{D}_N^n \rightarrow \mathbb{C}$ by prescribing for any $b \in \hat{D}_N^n$ the value

$$\hat{f}(b) = N^{-n} \sum_{a \in \hat{F}_N^n} f(a) \overline{E_b^-(a)}. \quad (3.3)$$

Due to the orthogonality relations, the **backward antisymmetric exponential Fourier transform** returns the original function $f \in \widehat{\mathcal{F}}_N^n$,

$$f(a) = \sum_{b \in \widehat{D}_N^n} \widehat{f}(b) E_b^-(a), \quad a \in \widehat{F}_N^n. \quad (3.4)$$

The corresponding Plancherel formula is of the form

$$\sum_{a \in \widehat{F}_N^n} |f(a)|^2 = N^n \sum_{b \in \widehat{D}_N^n} |\widehat{f}(b)|^2.$$

The **set of points** \check{F}_N^n for the symmetric exponential functions is of the form

$$\check{F}_N^n = \left\{ \frac{a}{N} \in \mathbb{R}^n \mid a_1 \geq \cdots \geq a_n, a_1, \dots, a_n \in \{1, 2, \dots, N\} \right\}$$

and the **set of labels** \check{D}_N^n is of the form

$$\check{D}_N^n = \{b \in \mathbb{R}^n \mid b_1 \geq \cdots \geq b_n, b_1, \dots, b_n \in \{1, 2, \dots, N\}\}.$$

The vector space $\check{\mathcal{F}}_N^n$ of complex functions $f : \check{F}_N^n \rightarrow \mathbb{C}$ is equipped with a scalar product of the following form for any $f, g \in \check{\mathcal{F}}_N^n$,

$$\langle f, g \rangle_{\check{\mathcal{F}}_N^n} = \sum_{a \in \check{F}_N^n} H_a^{-1} f(a) \overline{g(a)}. \quad (3.5)$$

The orthogonality relations of the symmetric exponential functions in the Hilbert space $\check{\mathcal{F}}_N^n$ are summarized in Proposition 2 in [58]. The orthogonality relations are for any labels $b, b' \in \check{D}_N^n$ of the form

$$\langle E_b^+, E_{b'}^+ \rangle_{\check{\mathcal{F}}_N^n} = N^n H_b \delta_{b, b'}.$$

The **forward symmetric exponential Fourier transform** calculates for any $f \in \check{\mathcal{F}}_N^n$ its spectral transform $\widehat{f} : \check{D}_N^n \rightarrow \mathbb{C}$ by prescribing for any $b \in \check{D}_N^n$ the value

$$\widehat{f}(b) = N^{-n} H_b^{-1} \sum_{a \in \check{F}_N^n} H_a^{-1} f(a) \overline{E_b^+(a)}. \quad (3.6)$$

Due to the orthogonality relations, the **backward symmetric exponential Fourier transform** returns the original function $f \in \check{\mathcal{F}}_N^n$,

$$f(a) = \sum_{b \in \check{D}_N^n} \widehat{f}(b) E_b^+(a), \quad a \in \check{F}_N^n. \quad (3.7)$$

The corresponding Plancherel formula is of the form

$$\sum_{a \in \check{F}_N^n} H_a^{-1} |f(a)|^2 = N^n H_b \sum_{b \in \check{D}_N^n} |\widehat{f}(b)|^2.$$

3.2 (Anti)symmetric Cosine Transforms

The **antisymmetric cosine function** and **antisymmetric sine function**, introduced in [59], are defined for any label $b \in \mathbb{R}^n$ and a point $a \in \mathbb{R}^n$ via signed symmetrization of products of cosine and sine functions over the permutation group S_n , respectively,

$$\cos_b^-(a) = \sum_{s \in S_n} \operatorname{sgn}(s) \prod_{i=1}^n \cos(\pi b_{s(i)} a_i), \quad \sin_b^-(a) = \sum_{s \in S_n} \operatorname{sgn}(s) \prod_{i=1}^n \sin(\pi b_{s(i)} a_i),$$

and the **symmetric cosine function** and **symmetric sine function** is defined via symmetrization,

$$\cos_b^+(a) = \sum_{s \in S_n} \prod_{i=1}^n \cos(\pi b_{s(i)} a_i), \quad \sin_b^+(a) = \sum_{s \in S_n} \prod_{i=1}^n \sin(\pi b_{s(i)} a_i).$$

For discretized labels $b \in \mathcal{T}_n$, the periodicity and symmetries of the antisymmetric and symmetric trigonometric functions are for $s \in S_n$ and $r \in \mathcal{T}_n$ of the form

$$\begin{aligned} \cos_b^+(sa + r) &= \cos_b^+(a), & \cos_b^-(sa + r) &= \operatorname{sgn}(s) \cos_b^-(a), \\ \sin_b^+(sa + r) &= \sin_b^+(a), & \sin_b^-(sa + r) &= \operatorname{sgn}(s) \sin_b^-(a). \end{aligned}$$

The periodicity and symmetries of the (anti)symmetric trigonometric functions allow to restrict these functions to the closure of the fundamental domain $F(S_n^{\text{aff}})$ and the discrete calculus is thus performed on set of points inside the closure of the fundamental domain $F(S_n^{\text{aff}})$.

The symmetric and antisymmetric discrete cosine transforms of types I–IV are deduced in [59]. The four novel types V–VIII of (anti)symmetric discrete cosine transforms and the corresponding interpolation problem are developed in [A6]. The **set of labels** D_N^+ , corresponding to the symmetric cosine functions, is defined as

$$D_N^+ = \{b \in \mathbb{R}^n \mid b_1 \geq \dots \geq b_n, b_1, \dots, b_n \in \{0, 1, \dots, N-1\}\}.$$

The **sets of points** $F_N^{\text{V},+}$ and $F_N^{\text{VII},+}$ are given as

$$F_N^{\text{V},+} = F_N^{\text{VII},+} = \left\{ \left(\frac{2r_1}{2N-1}, \dots, \frac{2r_n}{2N-1} \right) \mid (r_1, \dots, r_n) \in D_N^+ \right\},$$

and the **sets of points** $F_N^{\text{VI},+}$ and $F_N^{\text{VIII},+}$ are defined as

$$\begin{aligned} F_N^{\text{VI},+} &= \left\{ \left(\frac{2(r_1 + \frac{1}{2})}{2N-1}, \dots, \frac{2(r_n + \frac{1}{2})}{2N-1} \right) \mid (r_1, \dots, r_n) \in D_N^+ \right\}, \\ F_N^{\text{VIII},+} &= \left\{ \left(\frac{2(r_1 + \frac{1}{2})}{2N+1}, \dots, \frac{2(r_n + \frac{1}{2})}{2N+1} \right) \mid (r_1, \dots, r_n) \in D_N^+ \right\}. \end{aligned}$$

To any point $(b_1, \dots, b_n) \in D_N^+$, the following two values are assigned

$$d_b = c_{b_1} \dots c_{b_n}, \tag{3.8}$$

$$\tilde{d}_b = c_{b_1+1} \dots c_{b_n+1}, \tag{3.9}$$

with the symbols c_{b_i} determined by

$$c_b = \begin{cases} \frac{1}{2} & \text{if } b = 0 \text{ or } b = N, \\ 1 & \text{otherwise.} \end{cases} \quad (3.10)$$

To any point $a \in F_N^{V,+}$, which is labeled by the point $(r_1, \dots, r_n) \in D_N^+$, is assigned the value

$$\varepsilon_a = c_{r_1} \dots c_{r_n}, \quad (3.11)$$

and to any point $a \in F_N^{VI,+}$ is assigned the value

$$\tilde{\varepsilon}_a = c_{r_1+1} \dots c_{r_n+1}. \quad (3.12)$$

The vector spaces $\mathcal{F}_N^{V,+} = \mathcal{F}_N^{VII,+}$ of real-valued functions $f : F_N^{V,+} \rightarrow \mathbb{R}$, $f : F_N^{VII,+} \rightarrow \mathbb{R}$ are endowed with the scalar product of any two functions $f, g : F_N^{V,+} \rightarrow \mathbb{R}$, defined by

$$\langle f, g \rangle_{F_N^{V,+}} = \langle f, g \rangle_{F_N^{VII,+}} = \sum_{a \in F_N^{V,+}} \varepsilon_a H_a^{-1} f(a) g(a),$$

and the vector spaces $\mathcal{F}_N^{VI,+}$, $\mathcal{F}_N^{VIII,+}$ of real-valued functions $f : F_N^{VI,+} \rightarrow \mathbb{R}$ and functions $f : F_N^{VIII,+} \rightarrow \mathbb{R}$ are endowed with the scalar products

$$\langle f, g \rangle_{F_N^{VI,+}} = \sum_{a \in F_N^{VI,+}} \tilde{\varepsilon}_a H_a^{-1} f(a) g(a), \quad \langle f, g \rangle_{F_N^{VIII,+}} = \sum_{a \in F_N^{VIII,+}} H_a^{-1} f(a) g(a).$$

The four types of orthogonality relations are for any $b, b' \in D_N^+$ and $\varrho = (\frac{1}{2}, \dots, \frac{1}{2})$ of the form

$$\begin{aligned} \langle \cos_b^+, \cos_{b'}^+ \rangle_{F_N^{V,+}} &= \frac{H_b}{d_b} \left(\frac{2N-1}{4} \right)^n \delta_{b,b'}, \\ \langle \cos_b^+, \cos_{b'}^+ \rangle_{F_N^{VI,+}} &= \frac{H_b}{d_b} \left(\frac{2N-1}{4} \right)^n \delta_{b,b'}, \\ \langle \cos_{b+\varrho}^+, \cos_{b'+\varrho}^+ \rangle_{F_N^{VII,+}} &= \frac{H_b}{\tilde{d}_b} \left(\frac{2N-1}{4} \right)^n \delta_{b,b'}, \\ \langle \cos_{b+\varrho}^+, \cos_{b'+\varrho}^+ \rangle_{F_N^{VIII,+}} &= H_b \left(\frac{2N+1}{4} \right)^n \delta_{b,b'}. \end{aligned} \quad (3.13)$$

The four types of **forward symmetric cosine Fourier transform** calculate for any $f_V \in \mathcal{F}_N^{V,+}, \dots, f_{VIII} \in \mathcal{F}_N^{VIII,+}$ their spectral transforms $\hat{f}_V, \dots, \hat{f}_{VIII} : D_N^+ \rightarrow \mathbb{R}$ by prescribing for any $b \in D_N^+$ the values

$$\begin{aligned} \hat{f}_V(b) &= \frac{d_b}{H_b} \left(\frac{4}{2N-1} \right)^n \sum_{a \in F_N^{V,+}} \varepsilon_a H_a^{-1} f(a) \cos_b^+(a), \\ \hat{f}_{VI}(b) &= \frac{d_b}{H_b} \left(\frac{4}{2N-1} \right)^n \sum_{a \in F_N^{VI,+}} \tilde{\varepsilon}_a H_a^{-1} f(a) \cos_b^+(a), \\ \hat{f}_{VII}(b) &= \frac{\tilde{d}_b}{H_b} \left(\frac{4}{2N-1} \right)^n \sum_{a \in F_N^{VII,+}} \varepsilon_a H_a^{-1} f(a) \cos_{b+\varrho}^+(a), \\ \hat{f}_{VIII}(b) &= \frac{1}{H_b} \left(\frac{4}{2N+1} \right)^n \sum_{a \in F_N^{VIII,+}} H_a^{-1} f(a) \cos_{b+\varrho}^+(a). \end{aligned} \quad (3.14)$$

3.2. (ANTI)SYMMETRIC COSINE TRANSFORMS

Due to the orthogonality relations, the **backward symmetric cosine Fourier transforms** return the original functions $f_V \in \mathcal{F}_N^{V,+}, \dots, f_{VIII} \in \mathcal{F}_N^{VIII,+}$,

$$\begin{aligned} f_V(a) &= \sum_{b \in D_N^+} \widehat{f}_V(b) \cos_b^+(a), \\ f_{VI}(a) &= \sum_{b \in D_N^+} \widehat{f}_{VI}(b) \cos_b^+(a), \\ f_{VII}(a) &= \sum_{b \in D_N^+} \widehat{f}_{VII}(b) \cos_{b+\varrho}^+(a), \\ f_{VIII}(a) &= \sum_{b \in D_N^+} \widehat{f}_{VIII}(b) \cos_{b+\varrho}^+(a). \end{aligned}$$

The **set of labels** D_N^- , corresponding to the antisymmetric cosine functions, is defined as

$$D_N^- = \{b \in \mathbb{R}^n \mid b_1 > \dots > b_n, b_1, \dots, b_n \in \{0, 1, \dots, N-1\}\}.$$

The **sets of points** $F_N^{V,-}$ and $F_N^{VII,-}$ are given as

$$F_N^{V,-} = F_N^{VII,-} = \left\{ \left(\frac{2r_1}{2N-1}, \dots, \frac{2r_n}{2N-1} \right) \mid (r_1, \dots, r_n) \in D_N^- \right\},$$

and the **sets of points** $F_N^{VI,-}$ and $F_N^{VIII,-}$ are defined as

$$\begin{aligned} F_N^{VI,-} &= \left\{ \left(\frac{2(r_1 + \frac{1}{2})}{2N-1}, \dots, \frac{2(r_n + \frac{1}{2})}{2N-1} \right) \mid (r_1, \dots, r_n) \in D_N^- \right\}, \\ F_N^{VIII,-} &= \left\{ \left(\frac{2(r_1 + \frac{1}{2})}{2N+1}, \dots, \frac{2(r_n + \frac{1}{2})}{2N+1} \right) \mid (r_1, \dots, r_n) \in D_N^- \right\}. \end{aligned}$$

The vector spaces $\mathcal{F}_N^{V,-} = \mathcal{F}_N^{VII,-}$ of real-valued functions $f : F_N^{V,-} \rightarrow \mathbb{R}$, $f : F_N^{VII,-} \rightarrow \mathbb{R}$ are endowed with the scalar product of any two functions $f, g : F_N^{V,-} \rightarrow \mathbb{R}$, defined by

$$\langle f, g \rangle_{F_N^{V,-}} = \langle f, g \rangle_{F_N^{VII,-}} = \sum_{a \in F_N^{V,-}} \varepsilon_a f(a) g(a),$$

and the vector spaces $\mathcal{F}_N^{VI,-}$, $\mathcal{F}_N^{VIII,-}$ of real-valued functions $f : F_N^{VI,-} \rightarrow \mathbb{R}$ and functions $f : F_N^{VIII,-} \rightarrow \mathbb{R}$ are endowed with the scalar products

$$\langle f, g \rangle_{F_N^{VI,-}} = \sum_{a \in F_N^{VI,-}} \widetilde{\varepsilon}_a f(a) g(a), \quad \langle f, g \rangle_{F_N^{VIII,-}} = \sum_{a \in F_N^{VIII,-}} f(a) g(a).$$

The four types of orthogonality relations are for any $b, b' \in D_N^-$ and $\varrho = (\frac{1}{2}, \dots, \frac{1}{2})$ of the form

$$\begin{aligned} \langle \cos_b^-, \cos_{b'}^- \rangle_{F_N^{V,-}} &= \frac{1}{d_b} \left(\frac{2N-1}{4} \right)^n \delta_{b,b'}, \\ \langle \cos_b^-, \cos_{b'}^- \rangle_{F_N^{VI,-}} &= \frac{1}{d_b} \left(\frac{2N-1}{4} \right)^n \delta_{b,b'}, \\ \langle \cos_{b+\varrho}^-, \cos_{b'+\varrho}^- \rangle_{F_N^{VII,-}} &= \frac{1}{d_b} \left(\frac{2N-1}{4} \right)^n \delta_{b,b'}, \\ \langle \cos_{b+\varrho}^-, \cos_{b'+\varrho}^- \rangle_{F_N^{VIII,-}} &= \left(\frac{2N+1}{4} \right)^n \delta_{b,b'}. \end{aligned} \tag{3.15}$$

The four types of **forward antisymmetric cosine Fourier transform** calculate for any $f_V \in \mathcal{F}_N^{V,-}, \dots, f_{VIII} \in \mathcal{F}_N^{VIII,-}$ their spectral transforms $\widehat{f}_V, \dots, \widehat{f}_{VIII} : D_N^- \rightarrow \mathbb{R}$ by prescribing for any $b \in D_N^-$ the values

$$\begin{aligned}
 \widehat{f}_V(b) &= d_b \left(\frac{4}{2N-1} \right)^n \sum_{a \in F_N^{V,-}} \varepsilon_a f(a) \cos_b^-(a), \\
 \widehat{f}_{VI}(b) &= d_b \left(\frac{4}{2N-1} \right)^n \sum_{a \in F_N^{VI,-}} \widetilde{\varepsilon}_a f(a) \cos_b^-(a), \\
 \widehat{f}_{VII}(b) &= \widetilde{d}_b \left(\frac{4}{2N-1} \right)^n \sum_{a \in F_N^{VII,-}} \varepsilon_a f(a) \cos_{b+\varrho}^-(a), \\
 \widehat{f}_{VIII}(b) &= \left(\frac{4}{2N+1} \right)^n \sum_{a \in F_N^{VIII,-}} f(a) \cos_{b+\varrho}^-(a).
 \end{aligned} \tag{3.16}$$

Due to the orthogonality relations, the **backward antisymmetric cosine Fourier transforms** return the original functions $f_V \in \mathcal{F}_N^{V,-}, \dots, f_{VIII} \in \mathcal{F}_N^{VIII,-}$,

$$\begin{aligned}
 f_V(a) &= \sum_{b \in D_N^-} \widehat{f}_V(b) \cos_b^-(a), \\
 f_{VI}(a) &= \sum_{b \in D_N^-} \widehat{f}_{VI}(b) \cos_b^-(a), \\
 f_{VII}(a) &= \sum_{b \in D_N^-} \widehat{f}_{VII}(b) \cos_{b+\varrho}^-(a), \\
 f_{VIII}(a) &= \sum_{b \in D_N^-} \widehat{f}_{VIII}(b) \cos_{b+\varrho}^-(a).
 \end{aligned}$$

3.3 2D and 3D Interpolation

A two-dimensional interpolation problem for the antisymmetric exponential functions is formulated in [A5] on a symmetrically placed triangle $K_{[a,a+1]}^-$ in \mathbb{R}^2 ,

$$K_{[a,a+1]}^- = \{(x, y) \in [a, a+1] \times [a, a+1] \mid x > y\}, \tag{3.17}$$

with $a \in \mathbb{R}$. For any parameter $b \in [0, 1]$, a symmetrically placed **point set** $L_{a,b,N}^-$,

$$L_{a,b,N}^- = \{(x_m, y_n) \mid m > n, m, n \in \{0, \dots, N-1\}\},$$

has its points determined by

$$(x_m, y_n) = \left(a + \frac{m+b}{N}, a + \frac{n+b}{N} \right). \tag{3.18}$$

For a given function $f : K_{[a,a+1]}^- \rightarrow \mathbb{C}$ and the set of points $L_{a,b,N}^- \subset K_{[a,a+1]}^-$, with $N = 2M + 1$ or $N = 2M$, the **antisymmetric trigonometric interpolating function**

$\psi_N^- : K_{[a,a+1]}^- \rightarrow \mathbb{C}$ is defined via relation

$$\psi_N^-(x, y) = \sum_{\substack{k, l = -M \\ k > l}}^M c_{kl}^- E_{(k,l)}^-(x, y), \quad x, y \in \mathbb{R}, \quad (3.19)$$

and is required to coincide with f on the point set $L_{a,b,N}^-$,

$$\psi_N^-(x_m, y_n) = f(x_m, y_n), \quad m > n, \quad m, n = 0 \dots N - 1. \quad (3.20)$$

For $N = 2M$, further $2M$ conditions are assumed,

$$\begin{aligned} c_{l,-M}^- &= -e^{2\pi i(Na+b)} c_{M,l}^-, \quad l = -M + 1, \dots, M - 1, \\ c_{M,-M}^- &= 0. \end{aligned} \quad (3.21)$$

As shown in Proposition 3.1 in [A5], there exists a unique antisymmetric interpolating function (3.19) satisfying (3.20). The coefficients c_{kl}^- are given for $N = 2M + 1$ by

$$c_{kl}^- = \frac{1}{N^2} \sum_{\substack{m, n=0 \\ m > n}}^{N-1} f(x_m, y_n) \overline{E_{(k,l)}^-(x_m, y_n)} \quad (3.22)$$

and for $N = 2M$ assuming (3.21) by

$$c_{kl}^- = \frac{g_{k,M} g_{l,M}}{N^2} \sum_{\substack{m, n=0 \\ m > n}}^{N-1} f(x_m, y_n) \overline{E_{(k,l)}^-(x_m, y_n)}, \quad (3.23)$$

where the symbols $g_{k,M}$ are given by

$$g_{k,M} = \begin{cases} \frac{1}{2} & \text{if } k = -M, M, \\ 1 & \text{otherwise.} \end{cases} \quad (3.24)$$

The relation between the forward antisymmetric exponential Fourier transform (3.3) and the interpolation coefficients (3.22) and (3.23) is detailed in equations (45) and (46) in [A5]. The antisymmetric interpolating functions (3.19) are for a specific model function together with the interpolating grids depicted in Figure 4 in [A5]. The convergences of interpolation errors are demonstrated in Table I in [A5].

A two-dimensional interpolation problem for the symmetric exponential functions is formulated on a symmetrically placed triangle $K_{[a,a+1]}^+$ in \mathbb{R}^2 ,

$$K_{[a,a+1]}^+ = \{(x, y) \in [a, a + 1] \times [a, a + 1] \mid x \geq y\}, \quad (3.25)$$

with $a \in \mathbb{R}$. For any parameter $b \in [0, 1]$, a symmetrically placed **point set** $L_{a,b,N}^+$ is defined using the points (3.18) as

$$L_{a,b,N}^+ = \{(x_m, y_n) \mid m \geq n, \quad m, n \in \{0, \dots, N - 1\}\}.$$

For a given function $f : K_{[a,a+1]}^+ \rightarrow \mathbb{C}$ and the set of points $L_{a,b,N}^+ \subset K_{[a,a+1]}^+$, with either $N = 2M + 1$ or $N = 2M$, the **symmetric trigonometric interpolating function** $\psi_N^+ : K_{[a,a+1]}^+ \rightarrow \mathbb{C}$ is defined via relation

$$\psi_N^+(x, y) = \sum_{\substack{k,l=-M \\ k \geq l}}^M c_{kl}^+ E_{(k,l)}^+(x, y), \quad x, y \in \mathbb{R}, \quad (3.26)$$

and is required to coincide with f on the point set $L_{a,b,N}^+$,

$$\psi_N^+(x_m, y_n) = f(x_m, y_n), \quad m \geq n, \quad m, n = 0, \dots, N-1. \quad (3.27)$$

For $N = 2M$, further $2M + 1$ conditions are assumed,

$$c_{l,-M}^+ = -e^{2\pi i(Na+b)} c_{M,l}^+, \quad l = -M, \dots, M. \quad (3.28)$$

As shown in Proposition 3.2 in [A5], there exists a unique symmetric interpolating function (3.26) satisfying (3.27). The coefficients c_{kl}^+ are given for $N = 2M + 1$ by

$$c_{kl}^+ = \frac{1}{H_{(k,l)} N^2} \sum_{\substack{m,n=0 \\ m \geq n}}^{N-1} H_{(m,n)}^{-1} f(x_m, y_n) \overline{E_{(k,l)}^+(x_m, y_n)} \quad (3.29)$$

and for $N = 2M$ assuming (3.28) by

$$c_{kl}^+ = \frac{g_{k,M} g_{l,M}}{H_{(k,l)} N^2} \sum_{\substack{m,n=0 \\ m \geq n}}^{N-1} H_{(m,n)}^{-1} f(x_m, y_n) \overline{E_{(k,l)}^+(x_m, y_n)}, \quad (3.30)$$

where the size of the stabilizer $H_{(k,l)}$, defined by (3.1), is explicitly given by

$$H_{(k,l)} = \begin{cases} 2 & \text{if } k = l, \\ 1 & \text{otherwise.} \end{cases}$$

The relation between the forward symmetric exponential Fourier transform (3.6) and the interpolation coefficients (3.29) and (3.30) is detailed in equations (60) and (61) in [A5]. The symmetric interpolating functions (3.26) are for a specific model function together with the interpolating grids depicted in Figure 5 in [A5]. The convergences of interpolation errors are demonstrated in Table I in [A5].

A general interpolation problem for the antisymmetric cosine functions is formulated for a given real-valued function $f : F(\tilde{S}_n^{\text{aff}}) \rightarrow \mathbb{R}$ and the four point sets $F_N^{\text{V},-} = F_N^{\text{VII},-}$, $F_N^{\text{VI},-}$ and $F_N^{\text{VIII},-}$. The four types of **antisymmetric cosine interpolating polynomials** of f

are expressed as finite sums of antisymmetric multivariate cosine functions for $x \in F(\tilde{S}_n^{\text{aff}})$,

$$\begin{aligned}
 \psi_N^{\text{V},-}(x) &= \sum_{k \in D_N^-} \hat{f}_{\text{V}}(b) \cos_k^-(x), \\
 \psi_N^{\text{VI},-}(x) &= \sum_{k \in D_N^-} \hat{f}_{\text{VI}}(b) \cos_{k+\varrho}^-(x), \\
 \psi_N^{\text{VII},-}(x) &= \sum_{k \in D_N^-} \hat{f}_{\text{VII}}(b) \cos_k^-(x), \\
 \psi_N^{\text{VIII},-}(x) &= \sum_{k \in D_N^-} \hat{f}_{\text{VIII}}(b) \cos_{k+\varrho}^-(x).
 \end{aligned} \tag{3.31}$$

Due to the orthogonality relations (3.15), the forward Fourier transforms $\hat{f}_{\text{V}}, \dots, \hat{f}_{\text{VIII}}$ of the form (3.16) uniquely determine the interpolation polynomials (3.31) which coincide with the interpolating function on the corresponding point sets,

$$\begin{aligned}
 \psi_N^{\text{V},-}(a) &= f(a), & a \in F_N^{\text{V},-}, \\
 \psi_N^{\text{VI},-}(a) &= f(a), & a \in F_N^{\text{VI},-}, \\
 \psi_N^{\text{VII},-}(a) &= f(a), & a \in F_N^{\text{VII},-}, \\
 \psi_N^{\text{VIII},-}(a) &= f(a), & a \in F_N^{\text{VIII},-}.
 \end{aligned}$$

The graph cuts of 3D antisymmetric cosine interpolating functions (3.31) of types V and VII are for a specific model function depicted in Figures 4 and 5 in [A6], respectively. The convergences of interpolation errors are demonstrated in Table I in [A6].

A general interpolation problem for the symmetric cosine functions is formulated for a given real-valued function $f : F(\tilde{S}_n^{\text{aff}}) \rightarrow \mathbb{R}$ and the four point sets $F_N^{\text{V},+} = F_N^{\text{VII},+}$, $F_N^{\text{VI},+}$ and $F_N^{\text{VIII},+}$. The four types of **symmetric cosine interpolating polynomials** of f are expressed as finite sums of symmetric multivariate cosine functions for $x \in F(\tilde{S}_n^{\text{aff}})$,

$$\begin{aligned}
 \psi_N^{\text{V},+}(x) &= \sum_{k \in D_N^+} \hat{f}_{\text{V}}(b) \cos_k^+(x), \\
 \psi_N^{\text{VI},+}(x) &= \sum_{k \in D_N^+} \hat{f}_{\text{VI}}(b) \cos_{k+\varrho}^+(x), \\
 \psi_N^{\text{VII},+}(x) &= \sum_{k \in D_N^+} \hat{f}_{\text{VII}}(b) \cos_k^+(x), \\
 \psi_N^{\text{VIII},+}(x) &= \sum_{k \in D_N^+} \hat{f}_{\text{VIII}}(b) \cos_{k+\varrho}^+(x).
 \end{aligned} \tag{3.32}$$

Due to the orthogonality relations (3.13), the forward Fourier transforms $\hat{f}_{\text{V}}, \dots, \hat{f}_{\text{VIII}}$ of the form (3.14) uniquely determine the interpolation polynomials (3.32) that coincide with the interpolating function on the corresponding point sets,

$$\begin{aligned}
 \psi_N^{\text{V},+}(a) &= f(a), & a \in F_N^{\text{V},+}, \\
 \psi_N^{\text{VI},+}(a) &= f(a), & a \in F_N^{\text{VI},+}, \\
 \psi_N^{\text{VII},+}(a) &= f(a), & a \in F_N^{\text{VII},+}, \\
 \psi_N^{\text{VIII},+}(a) &= f(a), & a \in F_N^{\text{VIII},+}.
 \end{aligned} \tag{3.33}$$

The graph cuts of 3D symmetric cosine interpolating functions (3.32) of types V and VII are for a specific model function depicted in Figures 6 and 7 in [A6], respectively. The convergences of interpolation errors are demonstrated in Table I in [A6].

3.4 Cosine Cubature Formulas

Defining the vectors ϱ_1 and ϱ_2 by

$$\varrho_1 = (n-1, n-2, \dots, 0), \quad (3.34)$$

$$\varrho_2 = \left(n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2} \right), \quad (3.35)$$

and denoting the following n symmetric cosine functions by X_1, X_2, \dots, X_n ,

$$\begin{aligned} X_1(a) &= \cos_{(1,0,\dots,0)}^+(a), \\ X_2(a) &= \cos_{(1,1,0,\dots,0)}^+(a), \\ X_3(a) &= \cos_{(1,1,1,0,\dots,0)}^+(a), \\ &\vdots \\ X_n(a) &= \cos_{(1,1,\dots,1)}^+(a), \end{aligned}$$

the **four kinds of multivariate cosine polynomials** $\mathcal{P}_k^{I,+}$, $\mathcal{P}_k^{I,-}$, $\mathcal{P}_k^{III,+}$, $\mathcal{P}_k^{III,-}$ of n variables X_1, \dots, X_n are determined,

$$\begin{aligned} \mathcal{P}_k^{I,+}(X_1(x), \dots, X_n(x)) &= \cos_k^+(x), & \mathcal{P}_k^{I,-}(X_1(x), \dots, X_n(x)) &= \frac{\cos_{k+\varrho_1}^-(x)}{\cos_{\varrho_1}^-(x)}, \\ \mathcal{P}_k^{III,+}(X_1(x), \dots, X_n(x)) &= \frac{\cos_{k+\varrho}^+(x)}{\cos_{\varrho}^+(x)}, & \mathcal{P}_k^{III,-}(X_1(x), \dots, X_n(x)) &= \frac{\cos_{k+\varrho_2}^-(x)}{\cos_{\varrho_2}^-(x)}. \end{aligned}$$

The recurrence formulas and lower-dimensional examples of these polynomials are found in Section 4 in [A6].

The **transform** $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined for $x \in F(\tilde{S}_n^{\text{aff}})$ by

$$\varphi(x) = (X_1(x), \dots, X_n(x)), \quad (3.36)$$

maps the fundamental domain $F(\tilde{S}_n^{\text{aff}})$ onto the domain $\mathfrak{F}(\tilde{S}_n^{\text{aff}})$,

$$\mathfrak{F}(\tilde{S}_n^{\text{aff}}) = \left\{ \varphi(x) \mid x \in F(\tilde{S}_n^{\text{aff}}) \right\}.$$

The 3D integration domain $\mathfrak{F}(\tilde{S}_3^{\text{aff}})$ is plotted in Figure 8 in [A6].

The square of the Jacobian J of the φ -transform is expressed as a polynomial $p^{I,+}$ in variables in X_1, \dots, X_n ,

$$J(x)^2 = p^{I,+}(X_1(x), \dots, X_n(x)),$$

and the **function** \mathcal{J} is given as

$$\mathcal{J}(X_1, \dots, X_n) = \sqrt{p^{I,+}(X_1, \dots, X_n)}, \quad (X_1, \dots, X_n) \in \mathfrak{F}(\tilde{S}_n^{\text{aff}}). \quad (3.37)$$

3.4. COSINE CUBATURE FORMULAS

The following products of (anti)symmetric cosine functions are expressed as **polynomials** $\mathcal{J}^{I,-}$, $\mathcal{J}^{III,+}$, and $\mathcal{J}^{III,-}$ in variables X_1, \dots, X_n ,

$$\mathcal{J}^{I,-}(X_1(x), \dots, X_n(x)) = \cos_{\varrho_1}^-(x) \cos_{\varrho_1}^-(x), \quad (3.38)$$

$$\mathcal{J}^{III,+}(X_1(x), \dots, X_n(x)) = \cos_{\varrho}^+(x) \cos_{\varrho}^+(x), \quad (3.39)$$

$$\mathcal{J}^{III,-}(X_1(x), \dots, X_n(x)) = \cos_{\varrho_2}^-(x) \cos_{\varrho_2}^-(x), \quad (3.40)$$

and the four **weight functions** $w^{I,\pm}, w^{III,\pm}$ are determined by

$$\begin{aligned} w^{I,+}(X) &= \frac{1}{\mathcal{J}(X)}, \\ w^{I,-}(X) &= \frac{\mathcal{J}^{I,-}(X)}{\mathcal{J}(X)}, \\ w^{III,+}(X) &= \frac{\mathcal{J}^{III,+}(X)}{\mathcal{J}(X)}, \\ w^{III,-}(X) &= \frac{\mathcal{J}^{III,-}(X)}{\mathcal{J}(X)}. \end{aligned}$$

The 3D examples of the polynomials $\mathcal{J}^{I,-}$, $\mathcal{J}^{III,\pm}$ and the function J are calculated in Example 4.2 in [A6]. The symbols $\mathcal{H}_Y, \mathcal{E}_Y$, and $\tilde{\mathcal{E}}_Y$ are for $\varphi(a) = Y$ defined by the relations,

$$\mathcal{H}_Y = H_a, \quad \mathcal{E}_Y = \varepsilon_a, \quad \tilde{\mathcal{E}}_Y = \tilde{\varepsilon}_a. \quad (3.41)$$

The **cubature point sets** $\mathfrak{F}_N^{V,\pm}, \dots, \mathfrak{F}_N^{VIII,\pm}$ are φ -transforms of the point sets for the (anti)symmetric cosine functions, $F_N^{I,\pm}, \dots, F_N^{VIII,\pm}$,

$$\begin{aligned} \mathfrak{F}_N^{V,\pm} &= \mathfrak{F}_N^{VII,\pm} = \left\{ \varphi(a) \mid a \in F_N^{V,\pm} \right\}, \\ \mathfrak{F}_N^{VI,\pm} &= \left\{ \varphi(a) \mid a \in F_N^{VI,\pm} \right\}, \\ \mathfrak{F}_N^{VIII,\pm} &= \left\{ \varphi(a) \mid a \in F_N^{VIII,\pm} \right\}. \end{aligned} \quad (3.42)$$

The four resulting **antisymmetric cosine cubature formulas**, corresponding to the antisymmetric cosine polynomials $\mathcal{P}_k^{I,-}, \mathcal{P}_k^{III,-}$, are of the form

$$\begin{aligned} \int_{\mathfrak{F}(\tilde{S}_n^{\text{aff}})} f(Y) \omega^{I,-}(Y) dY &= \left(\frac{2}{2N-1} \right)^n \sum_{Y \in \mathfrak{F}_N^{V,-}} \mathcal{E}_Y f(Y) \mathcal{J}^{I,-}(Y), \\ \int_{\mathfrak{F}(\tilde{S}_n^{\text{aff}})} f(Y) \omega^{I,-}(Y) dY &= \left(\frac{2}{2N-1} \right)^n \sum_{Y \in \mathfrak{F}_N^{VI,-}} \tilde{\mathcal{E}}_Y f(Y) \mathcal{J}^{I,-}(Y), \\ \int_{\mathfrak{F}(\tilde{S}_n^{\text{aff}})} f(Y) \omega^{III,-}(Y) dY &= \left(\frac{2}{2N-1} \right)^n \sum_{Y \in \mathfrak{F}_N^{VII,-}} \mathcal{E}_Y f(Y) \mathcal{J}^{III,-}(Y), \\ \int_{\mathfrak{F}(\tilde{S}_n^{\text{aff}})} f(Y) \omega^{III,-}(Y) dY &= \left(\frac{2}{2N+1} \right)^n \sum_{Y \in \mathfrak{F}_N^{VIII,-}} f(Y) \mathcal{J}^{III,-}(Y), \end{aligned} \quad (3.43)$$

and the four **symmetric cosine cubature formulas**, corresponding to the symmetric cosine polynomials $\mathcal{P}_k^{I,+}$, $\mathcal{P}_k^{III,+}$, are of the form

$$\begin{aligned}
 \int_{\tilde{\mathfrak{S}}(\tilde{S}_n^{\text{aff}})} f(Y) \omega^{I,+}(Y) dY &= \left(\frac{2}{2N-1} \right)^n \sum_{Y \in \tilde{\mathfrak{S}}_N^{V,+}} \mathcal{E}_Y \mathcal{H}_Y^{-1} f(Y), \\
 \int_{\tilde{\mathfrak{S}}(\tilde{S}_n^{\text{aff}})} f(Y) \omega^{I,+}(Y) dY &= \left(\frac{2}{2N-1} \right)^n \sum_{Y \in \tilde{\mathfrak{S}}_N^{VI,+}} \tilde{\mathcal{E}}_Y \mathcal{H}_Y^{-1} f(Y), \\
 \int_{\tilde{\mathfrak{S}}(\tilde{S}_n^{\text{aff}})} f(Y) \omega^{III,+}(Y) dY &= \left(\frac{2}{2N-1} \right)^n \sum_{Y \in \tilde{\mathfrak{S}}_N^{VII,+}} \mathcal{E}_Y \mathcal{H}_Y^{-1} f(Y) \mathcal{J}^{III,+}(Y), \\
 \int_{\tilde{\mathfrak{S}}(\tilde{S}_n^{\text{aff}})} f(Y) \omega^{III,+}(Y) dY &= \left(\frac{2}{2N+1} \right)^n \sum_{Y \in \tilde{\mathfrak{S}}_N^{VIII,+}} \mathcal{H}_Y^{-1} f(Y) \mathcal{J}^{III,+}(Y).
 \end{aligned} \tag{3.44}$$

Chapter 4

Modified Multiplication and Honeycomb Transforms

The chapter collects the results concerning the development of the connection between the Fourier–Weyl transforms and conformal field theory along with the application to solid state physics. The first contribution of the author to this field is demonstrated in paper [A7], where the multiplication formulas and their modification, together with their Galois symmetry, are presented. Direct link between the Kac–Peterson matrices from conformal field theory and the weight lattice Fourier–Weyl transforms is found in [A8]. The generalization of the discrete Fourier–Weyl and Hartley–Weyl transforms to honeycomb lattice is presented in [A9]. In order to demonstrate further application of the multiplication formulas and Fourier–Weyl transforms in solid state physics, the transversal vibration models of 2D lattices with von Neumann and Dirichlet boundary conditions are exemplified. In the following sections, the crucial notions from the papers [A7–A9] are outlined and the A_2 weight lattice vibration model detailed. Explicit solutions of the models, including propagation of transversal waves in the mechanical graphene model, are deduced.

4.1 Modified Multiplication and Kac–Walton formula

Products of two types of orbit functions φ_λ^σ and $\varphi_{\lambda'}^{\sigma'}$ are decomposed into the sums of orbit functions,

$$\varphi_\lambda^\sigma \varphi_{\lambda'}^{\sigma'} = \sum_{w \in W} \sigma'(w) \varphi_{\lambda+w\lambda'}^{\sigma \cdot \sigma'}, \quad (4.1)$$

and products of orbit functions $\varphi_\lambda^\sigma(a)$ and $\varphi_{\lambda'}^{\sigma'}(a')$ are decomposed as

$$\varphi_\lambda^\sigma(a) \varphi_{\lambda'}^{\sigma'}(a') = \sum_{w \in W} \sigma'(w) \varphi_{\lambda+w\lambda'}^{\sigma \cdot \sigma'}(a + wa'). \quad (4.2)$$

Besides the modified multiplication, these general **product-to-sum decomposition formulas** (4.1) and (4.2) are crucial for vibrations models with von Neumann and Dirichlet boundary conditions.

Using the notation (2.3) and (2.10), which is consistent with [A7], the product decomposition formulas (4.1) of the C - and S -functions are further expressed in the form,

$$\begin{aligned}
 \Phi_\lambda(a) \Phi_\mu(a) &= \sum_{\nu \in P_+} \langle C|CC \rangle_{\lambda,\mu}^\nu \Phi_\nu(a), \\
 \Phi_\lambda(a) \varphi_{\tilde{\mu}}(a) &= \sum_{\tilde{\nu} \in P_{++}} \langle S|CS \rangle_{\lambda,\tilde{\mu}}^{\tilde{\nu}} \varphi_{\tilde{\nu}}(a), \\
 \varphi_{\tilde{\lambda}}(a) \varphi_{\tilde{\mu}}(a) &= \sum_{\nu \in P_+} \langle C|SS \rangle_{\tilde{\lambda},\tilde{\mu}}^\nu \Phi_\nu(a),
 \end{aligned} \tag{4.3}$$

for all dominant weights $\lambda, \mu \in P_+$, and all strictly dominant weights $\tilde{\lambda}, \tilde{\mu} \in P_{++}$.

For the weights from the finite set of labels $\lambda, \mu \in P_+^M$, and $\tilde{\lambda}, \tilde{\mu} \in P_{++}^M$ and the points from the refined dual weight point sets $a \in F_M$ and $\tilde{a} \in \tilde{F}_M$, the **modified multiplication** product decomposition formulas are of the form

$$\begin{aligned}
 \Phi_\lambda(a) \Phi_\mu(a) &= \sum_{\nu \in P_+^M} {}_M \langle C|CC \rangle_{\lambda,\mu}^\nu \Phi_\nu(a), \\
 \Phi_\lambda(\tilde{a}) \varphi_{\tilde{\mu}}(\tilde{a}) &= \sum_{\tilde{\nu} \in P_{++}^M} {}_M \langle S|CS \rangle_{\lambda,\tilde{\mu}}^{\tilde{\nu}} \varphi_{\tilde{\nu}}(\tilde{a}), \\
 \varphi_{\tilde{\lambda}}(\tilde{a}) \varphi_{\tilde{\mu}}(\tilde{a}) &= \sum_{\nu \in P_+^M} {}_M \langle C|SS \rangle_{\tilde{\lambda},\tilde{\mu}}^\nu \Phi_\nu(\tilde{a}).
 \end{aligned} \tag{4.4}$$

Due to the label affine Weyl symmetries (2.6), the relations between the decomposition coefficients form the **dual weight Kac–Walton formulas**,

$$\begin{aligned}
 {}_M \langle C|CC \rangle_{\lambda,\mu}^\nu &= \sum_{w \in \widehat{W}_M^{\text{aff}}} \langle C|CC \rangle_{\lambda,\mu}^{w\nu}, \\
 {}_M \langle S|CS \rangle_{\lambda,\tilde{\mu}}^{\tilde{\nu}} &= \sum_{w \in \widehat{W}_M^{\text{aff}}} \det \widehat{\psi}(w) \langle S|CS \rangle_{\lambda,\tilde{\mu}}^{w\tilde{\nu}}, \\
 {}_M \langle C|SS \rangle_{\tilde{\lambda},\tilde{\mu}}^\nu &= \sum_{w \in \widehat{W}_M^{\text{aff}}} \langle C|SS \rangle_{\tilde{\lambda},\tilde{\mu}}^{w\nu}.
 \end{aligned} \tag{4.5}$$

Examples demonstrating the decomposition formulas are contained in Section 3 in [A7] and illustrated in Figure 2.

For a given $M \in \mathbb{N}$, there exists a minimal number $N \in \mathbb{N}$ such that for all $\lambda \in \Lambda_M$ and $a \in F_M$ it holds that $\langle N\lambda, a \rangle \in \mathbb{Z}$. For any number $l \in \mathbb{N}$, such that $\gcd(l, N) = 1$, and any $\lambda \in P_+^M$ there exist an element $\widehat{w}_l[\lambda] \in \widehat{W}_M^{\text{aff}}$ and $t_l[\lambda] \in P_+^M$, such that $\widehat{w}_l[\lambda](l\lambda) \in P_+^M$. Also for any $\tilde{\lambda} \in P_{++}^M$ there exist an element $\widehat{w}_l[\tilde{\lambda}] \in \widehat{W}_M^{\text{aff}}$ and $t_l[\tilde{\lambda}] \in P_{++}^M$, such that $\widehat{w}_l[\tilde{\lambda}](l\tilde{\lambda}) \in P_{++}^M$. The resulting elements $t_l[\lambda] \in P_+^M$ and $t_l[\tilde{\lambda}] \in P_{++}^M$ determine the **Galois transformation** $t_l : P_+^M \rightarrow P_+^M$ and its restriction $t_l : P_{++}^M \rightarrow P_{++}^M$ of the label sets P_+^M and P_{++}^M , respectively,

$$\begin{aligned}
 t_l[\lambda] &= \widehat{w}_l[\lambda](l\lambda), \\
 t_l[\tilde{\lambda}] &= \widehat{w}_l[\tilde{\lambda}](l\tilde{\lambda}).
 \end{aligned} \tag{4.6}$$

Linearly extending the Galois transform of the C –functions and S –functions,

$$\begin{aligned} t_l(\Phi_\lambda) &= \Phi_{l\lambda}, & \lambda &\in P_+^M, \\ t_l(\varphi_{\tilde{\lambda}}) &= \varphi_{l\tilde{\lambda}}, & \tilde{\lambda} &\in P_{++}^M, \end{aligned} \tag{4.7}$$

yields the Galois transform of the Hilbert spaces $\mathcal{F}_M^1(0, 0)$ and $\mathcal{F}_M^{\sigma^e}(0, 0)$. The Galois transforms of the finite weights sets are illustrated in Figure 3 in [A7].

Denoting the sign change by

$$\hat{\varepsilon}_\ell[\tilde{\lambda}] = \det \hat{\psi}(\hat{w}_\ell[\tilde{\lambda}]),$$

the discrete orthogonality relations (2.15) of the C –functions and S –functions produce the **Galois symmetries of the modified multiplication coefficients** (4.4),

$$\begin{aligned} {}_M\langle C|CC \rangle_{t_\ell[\tilde{\lambda}], t_\ell[\tilde{\mu}]}^{t_\ell[\nu]} &= {}_M\langle C|CC \rangle_{\tilde{\lambda}, \tilde{\mu}}^\nu, \\ \hat{\varepsilon}_\ell[\tilde{\nu}] \hat{\varepsilon}_\ell[\tilde{\mu}] {}_M\langle S|CS \rangle_{t_\ell[\tilde{\lambda}], t_\ell[\tilde{\mu}]}^{t_\ell[\tilde{\nu}]} &= {}_M\langle S|CS \rangle_{\tilde{\lambda}, \tilde{\mu}}^{\tilde{\nu}}, \\ \hat{\varepsilon}_\ell[\tilde{\lambda}] \hat{\varepsilon}_\ell[\tilde{\mu}] {}_M\langle C|SS \rangle_{t_\ell[\tilde{\lambda}], t_\ell[\tilde{\mu}]}^{t_\ell[\nu]} &= {}_M\langle C|SS \rangle_{\tilde{\lambda}, \tilde{\mu}}^\nu. \end{aligned} \tag{4.8}$$

4.2 Generalized Kac–Peterson Matrices

Besides the argument symmetry (2.4) of the four types of orbit functions φ_b^σ , valid for any labels $b \in P$, a different type of label symmetry is induced by restricting the points to the refined weight lattice. For a point $a \in \frac{1}{M}P$, $M \in \mathbb{N}$ together with any $w^{\text{aff}} \in W^{\text{aff}}$ and $b \in \mathbb{R}^n$, the label symmetry of orbit functions is of the form

$$\varphi_{Mw^{\text{aff}}(\frac{b}{M})}^\sigma(a) = \gamma_0^\sigma(w^{\text{aff}})\varphi_b^\sigma(a) \tag{4.9}$$

and the orbit functions φ_b^σ are zero on the boundary $MH^\sigma(0)$,

$$\varphi_b^\sigma(a) = 0, \quad b \in MH^\sigma(0). \tag{4.10}$$

Discrete values of both points $a \in \frac{1}{M}P$ and labels $b \in P$ of the orbit functions $\varphi_b^\sigma(a)$ are due to the argument symmetries restricted to the **set of points** $F_{P,M}^\sigma$,

$$F_{P,M}^\sigma = \frac{1}{M}P \cap F^\sigma(0), \tag{4.11}$$

and due to the label symmetries restricted to the **set of labels** Λ_M^σ ,

$$\Lambda_{P,M}^\sigma = P \cap MF^\sigma(0). \tag{4.12}$$

Relation (38) in [A8] states that the cardinalities of the sets of labels and the sets of points coincide for each case,

$$|\Lambda_{P,M}^\sigma| = |F_{P,M}^\sigma|.$$

The explicit form of the point and label sets, crucial for theoretical implications as well applications, is determined via the symbols (2.12) and the set of points $F_{P,M}^\sigma$ is calculated explicitly as

$$F_{P,M}^\sigma = \left\{ \sum_{i=1}^n \frac{u_i^{\sigma,0}}{M} \omega_i \mid \sum_{i=0}^n u_i^{\sigma,0} m_i = M \right\}, \quad (4.13)$$

and the set of labels $\Lambda_{P,M}^\sigma$ as

$$\Lambda_{P,M}^\sigma = \left\{ \sum_{i=1}^n u_i^{\sigma,0} \omega_i \mid \sum_{i=0}^n u_i^{\sigma,0} m_i = M \right\}.$$

Explicit counting formulas for cardinalities of the sets $F_{P,M}^\sigma$ and $\Lambda_{P,M}^\sigma$, which do not coincide with previously calculated cardinalities of $F_M^\sigma(\varrho, \varrho^\vee)$, are covered in Theorem 4.1 in [A8].

The vector space $\mathcal{F}_{P,M}^\sigma$ of complex functions $f : F_{P,M}^\sigma \rightarrow \mathbb{C}$ is equipped with a scalar product containing the weight function (1.28). This **weight lattice weighted scalar product** is of the following form for any $f, g \in \mathcal{F}_{P,M}^\sigma$,

$$\langle f, g \rangle_{\mathcal{F}_{P,M}^\sigma} = \sum_{a \in F_{P,M}^\sigma} \varepsilon(a) f(a) \overline{g(a)}. \quad (4.14)$$

The orthogonality relations of weight lattice discretized orbit functions in the Hilbert space $\mathcal{F}_{P,M}^\sigma$ are summarized in Theorem 4.5 in [A8]. Using the numbers (1.12) and functions (1.28), the orthogonality relations are for any labels $b, b' \in \Lambda_{P,M}^\sigma$ of the form

$$\langle \varphi_b^\sigma, \varphi_{b'}^\sigma \rangle_{\mathcal{F}_{P,M}^\sigma} = d |W| M^n h_M(b) \delta_{b,b'}. \quad (4.15)$$

The **forward weight lattice Fourier–Weyl transform** calculates for any function $f \in \mathcal{F}_{P,M}^\sigma$ its spectral transform $\widehat{f} : \Lambda_{P,M}^\sigma \rightarrow \mathbb{C}$ by prescribing for any $b \in \Lambda_{P,M}^\sigma$ the value

$$\widehat{f}(b) = (d |W| M^n h_M(b))^{-1} \sum_{a \in F_{P,M}^\sigma} \varepsilon(a) f(a) \overline{\varphi_b^\sigma(a)}. \quad (4.16)$$

Due to the orthogonality relations (4.15), the **backward weight lattice Fourier–Weyl transform** returns the original function $f \in \mathcal{F}_{P,M}^\sigma$,

$$f(a) = \sum_{b \in \Lambda_{P,M}^\sigma} \widehat{f}(b) \varphi_b^\sigma(a), \quad a \in F_{P,M}^\sigma. \quad (4.17)$$

The corresponding Plancherel formula is of the form

$$\sum_{a \in F_{P,M}^\sigma} \varepsilon(a) |f(a)|^2 = d |W| M^n \sum_{b \in \Lambda_{P,M}^\sigma} h_M(b) |\widehat{f}(b)|^2.$$

The symmetric **generalized Kac–Peterson matrices** $S_{\lambda,\mu}^\sigma$, $\lambda, \mu \in \Lambda_{P,k+q^\sigma}^\sigma$, determined by their entries

$$S_{\lambda,\mu}^\sigma = \frac{i^{\lfloor \frac{n!}{2} \rfloor} \varphi_\lambda^\sigma \left(\frac{-\mu}{k+q^\sigma} \right)}{\sqrt{d(k+q^\sigma)^n h_{k+q^\sigma}(\lambda) h_{k+q^\sigma}(\mu)}}, \quad (4.18)$$

are unitary due to the orthogonality relations (4.15). Note that the relations $M = k + q^\sigma$, depending on the comarks (1.2), are substituted for the number M into (4.15). The matrices $S_{\lambda,\mu}^\sigma$ coincide with the standard Kac–Peterson matrices [51].

4.3 Honeycomb Transforms

A specific subtractive construction of the honeycomb lattice in terms of the invariant root and weight lattices of the root system A_2 is considered in [A9]. The key notion is that the weight lattice P of A_2 is disjointly decomposed into three congruence classes [68] as

$$P = Q \cup \{\omega_1 + Q\} \cup \{\omega_2 + Q\}. \quad (4.19)$$

The **extended affine Weyl group** of A_2 extends the Weyl group W by shifts by vectors from the weight lattice P ,

$$W_P^{\text{aff}} = P \rtimes W.$$

The fundamental domain (1.27) of A_2 is denoted for the purposes of this section as in [A9] by F_Q . The **fundamental domain** F_P of the action of W_P^{aff} on \mathbb{R}^2 is a subset of F_Q in the form of a kite,

$$F_P = \{x_1\omega_1 + x_2\omega_2 \in F_Q \mid (2x_1 + x_2 < 1, x_1 + 2x_2 < 1) \vee (2x_1 + x_2 = 1, x_1 \geq x_2)\}.$$

The point set $F_{P,M}$ from [A9] corresponds to the set $F_{P,M}^1$ in (4.11) and the **point set** $F_{Q,M}$ is the intersection of the fundamental domain F_Q with the root lattice,

$$F_{P,M} = \frac{1}{M}P \cap F_Q, \quad (4.20)$$

$$F_{Q,M} = \frac{1}{M}Q \cap F_Q. \quad (4.21)$$

Interiors of the point sets $F_{P,M}$ and $F_{Q,M}$ contain the grid points from the interior of F ,

$$\tilde{F}_{P,M} = \frac{1}{M}P \cap \text{int}(F_Q), \quad (4.22)$$

$$\tilde{F}_{Q,M} = \frac{1}{M}Q \cap \text{int}(F_Q). \quad (4.23)$$

The **honeycomb point set** H_M is obtained from the point set $F_{P,M}$ by subtraction of $F_{Q,M}$,

$$H_M = F_{P,M} \setminus F_{Q,M}. \quad (4.24)$$

The **interior honeycomb point set** $\tilde{H}_M \subset H_M$ contains only the points of H_M belonging to the interior of F_Q ,

$$\tilde{H}_M = \tilde{F}_{P,M} \setminus \tilde{F}_{Q,M}. \quad (4.25)$$

The counting formulas for the numbers of points in the honeycomb point sets H_M and \tilde{H}_M are contained in Propositions 3.1 and 3.2 in [A9], respectively. The honeycomb point set H_6 is depicted in Figure 2 in [A9].

The weight set $\Lambda_{Q,M}$ from [A9] corresponds to the set $\Lambda_{P,M}^1$ in (4.12) and the **weight set** $\Lambda_{P,M}$ is intersection of the lattice P with the magnified fundamental domain MF_P ,

$$\Lambda_{Q,M} = P \cap MF_Q, \quad (4.26)$$

$$\Lambda_{P,M} = P \cap MF_P. \quad (4.27)$$

The weight set $\Lambda_{Q,M}$ is of the following explicit form,

$$\Lambda_{Q,M} = \{ \lambda_1 \omega_1 + \lambda_2 \omega_2 \mid \lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}^{\geq 0}, \lambda_0 + \lambda_1 + \lambda_2 = M \},$$

and the points from $\Lambda_{Q,M}$ are described by their **Kac coordinates** [52] as

$$\lambda = [\lambda_0, \lambda_1, \lambda_2] \in \Lambda_{Q,M}. \quad (4.28)$$

Interiors $\tilde{\Lambda}_{Q,M}$ and $\tilde{\Lambda}_{P,M}$ of the weight sets $\Lambda_{Q,M}$ and $\Lambda_{P,M}$ contain points from the interior of the magnified fundamental domain MF_Q ,

$$\tilde{\Lambda}_{Q,M} = P \cap \text{int}(MF_Q), \quad (4.29)$$

$$\tilde{\Lambda}_{P,M} = P \cap MF_P \cap \text{int}(MF_Q). \quad (4.30)$$

The action of the group Γ_M on a weight $[\lambda_0, \lambda_1, \lambda_2] \in \Lambda_{Q,M}$ is the cyclic permutation group action on the coordinates $[\lambda_0, \lambda_1, \lambda_2]$,

$$\gamma_0[\lambda_0, \lambda_1, \lambda_2] = [\lambda_0, \lambda_1, \lambda_2], \quad \gamma_1[\lambda_0, \lambda_1, \lambda_2] = [\lambda_2, \lambda_0, \lambda_1], \quad \gamma_2[\lambda_0, \lambda_1, \lambda_2] = [\lambda_1, \lambda_2, \lambda_0].$$

The **honeycomb weight set** L_M is given explicitly as,

$$L_M = \{ [\lambda_0, \lambda_1, \lambda_2] \in \Lambda_{Q,M} \mid (\lambda_0 > \lambda_1, \lambda_0 > \lambda_2) \vee (\lambda_0 = \lambda_1 > \lambda_2) \},$$

and the **interior honeycomb weight set** \tilde{L}_M is given as

$$\tilde{L}_M = \left\{ [\lambda_0, \lambda_1, \lambda_2] \in \tilde{\Lambda}_{Q,M} \mid (\lambda_0 > \lambda_1, \lambda_0 > \lambda_2) \vee (\lambda_0 = \lambda_1 > \lambda_2) \right\}.$$

Propositions 3.3 and 3.4 in [A9] relate the numbers of points and weights in the honeycomb sets as

$$\begin{aligned} |L_M| &= \frac{1}{2} |H_M|, \\ |\tilde{L}_M| &= \frac{1}{2} |\tilde{H}_M|. \end{aligned}$$

The honeycomb weight sets L_6 and L_7 are depicted in Figure 3 in [A9].

The **extended C-functions** are for a fixed $M \in \mathbb{N}$ labeled by $\lambda \in L_M$ and introduced by

$$\begin{aligned} \Phi_\lambda^+(x) &= \mu_\lambda^{+,0} \Phi_\lambda(x) + \mu_\lambda^{+,1} \Phi_{\gamma_1 \lambda}(x) + \mu_\lambda^{+,2} \Phi_{\gamma_2 \lambda}(x), \\ \Phi_\lambda^-(x) &= \mu_\lambda^{-,0} \Phi_\lambda(x) + \mu_\lambda^{-,1} \Phi_{\gamma_1 \lambda}(x) + \mu_\lambda^{-,2} \Phi_{\gamma_2 \lambda}(x), \end{aligned} \quad (4.31)$$

where $\mu_\lambda^{\pm,0}, \mu_\lambda^{\pm,1}, \mu_\lambda^{\pm,2} \in \mathbb{C}$ denote for each $\lambda \in L_M$ six **extension coefficients**. For a fixed $M > 3$, the **extended S-functions** φ_λ^\pm labeled by $\lambda \in \tilde{L}_M$ are introduced by

$$\begin{aligned} \varphi_\lambda^+(x) &= \mu_\lambda^{+,0} \varphi_\lambda(x) + \mu_\lambda^{+,1} \varphi_{\gamma_1 \lambda}(x) + \mu_\lambda^{+,2} \varphi_{\gamma_2 \lambda}(x), \\ \varphi_\lambda^-(x) &= \mu_\lambda^{-,0} \varphi_\lambda(x) + \mu_\lambda^{-,1} \varphi_{\gamma_1 \lambda}(x) + \mu_\lambda^{-,2} \varphi_{\gamma_2 \lambda}(x). \end{aligned} \quad (4.32)$$

Two discrete **normalization functions** $\mu^+, \mu^- : L_M \rightarrow \mathbb{R}$ are for any $\lambda \in L_M$ given by

$$\mu^\pm(\lambda) = |\mu_\lambda^{\pm,0}|^2 + |\mu_\lambda^{\pm,1}|^2 + |\mu_\lambda^{\pm,2}|^2 - \text{Re} \left(\mu_\lambda^{\pm,0} \overline{\mu_\lambda^{\pm,1}} + \mu_\lambda^{\pm,0} \overline{\mu_\lambda^{\pm,2}} + \mu_\lambda^{\pm,1} \overline{\mu_\lambda^{\pm,2}} \right), \quad (4.33)$$

and an **intertwining function** $\beta : L_M \rightarrow \mathbb{C}$ is introduced as

$$\begin{aligned} \beta(\lambda) = & 2 \left(\mu_\lambda^{+,0} \overline{\mu_\lambda^{-,0}} + \mu_\lambda^{+,1} \overline{\mu_\lambda^{-,1}} + \mu_\lambda^{+,2} \overline{\mu_\lambda^{-,2}} \right) - \mu_\lambda^{+,0} \left(\overline{\mu_\lambda^{-,1}} + \overline{\mu_\lambda^{-,2}} \right) \\ & - \mu_\lambda^{+,1} \left(\overline{\mu_\lambda^{-,0}} + \overline{\mu_\lambda^{-,2}} \right) - \mu_\lambda^{+,2} \left(\overline{\mu_\lambda^{-,0}} + \overline{\mu_\lambda^{-,1}} \right). \end{aligned} \quad (4.34)$$

The Φ_λ^\pm -functions (4.31), for which both normalization functions are positive and the intertwining functions vanishes,

$$\mu^\pm(\lambda) > 0, \quad \beta(\lambda) = 0, \quad \lambda \in L_M, \quad (4.35)$$

form the **honeycomb C -functions** Ch_λ^\pm . The φ_λ^\pm -functions, which satisfy

$$\mu^\pm(\lambda) > 0, \quad \beta(\lambda) = 0, \quad \lambda \in \tilde{L}_M, \quad (4.36)$$

constitute the **honeycomb S -functions** Sh_λ^\pm .

The vector space \mathcal{H}_M of complex functions $f : H_M \rightarrow \mathbb{C}$ is equipped with a scalar product containing the weight function (1.28). This **honeycomb scalar product** is of the following form for any $f, g \in \mathcal{H}_M$,

$$\langle f, g \rangle_{H_M} = \sum_{a \in H_M} \varepsilon(a) f(a) \overline{g(a)}. \quad (4.37)$$

The orthogonality relations of honeycomb C -functions in the Hilbert space \mathcal{H}_M are summarized in Theorem 5.1 in [A9]. Using the functions (1.28), the orthogonality relations are for any labels $\lambda, \lambda' \in L_M$ of the form

$$\langle \text{Ch}_\lambda^\pm, \text{Ch}_{\lambda'}^\pm \rangle_{H_M} = 12M^2 h_M(\lambda) \mu^\pm(\lambda) \delta_{\lambda\lambda'}, \quad (4.38)$$

$$\langle \text{Ch}_\lambda^+, \text{Ch}_{\lambda'}^- \rangle_{H_M} = 0. \quad (4.39)$$

The **forward honeycomb Fourier–Weyl C -transform** calculates for any $f \in \mathcal{H}_M$ its spectral transforms $\hat{f}^\pm : L_M \rightarrow \mathbb{C}$ by prescribing for any $\lambda \in L_M$ the value

$$\hat{f}^\pm(\lambda) = \frac{\langle f, \text{Ch}_\lambda^\pm \rangle_{H_M}}{\langle \text{Ch}_\lambda^\pm, \text{Ch}_\lambda^\pm \rangle_{H_M}} = (12M^2 h_M(\lambda) \mu^\pm(\lambda))^{-1} \sum_{a \in H_M} \varepsilon(a) f(a) \overline{\text{Ch}_\lambda^\pm(a)}. \quad (4.40)$$

The **backward honeycomb Fourier–Weyl C -transform** returns the original function $f \in \mathcal{H}_M$,

$$f(a) = \sum_{\lambda \in L_M} \left(\hat{f}^+(\lambda) \text{Ch}_\lambda^+(x) + \hat{f}^-(\lambda) \text{Ch}_\lambda^-(x) \right), \quad a \in H_M. \quad (4.41)$$

The corresponding Plancherel formula is of the form

$$\sum_{a \in H_M} \varepsilon(a) |f(a)|^2 = 12M^2 \sum_{\lambda \in L_M} h_M(\lambda) \left(\mu^+(\lambda) \left| \hat{f}^+(\lambda) \right|^2 + \mu^-(\lambda) \left| \hat{f}^-(\lambda) \right|^2 \right).$$

The orthogonality relations of the honeycomb S -functions over \tilde{H}_M , labeled by weights from \tilde{L}_M , are summarized in Theorem 5.3 in [A9]. The honeycomb Hartley C - and S -functions, Cah_λ^\pm and Sah_λ^\pm are specific modifications functions of the honeycomb functions (4.31) and (4.32) containing the Hartley orbit functions (2.2). The orthogonality relations of honeycomb Hartley C - and S -functions are summarized in Theorems 5.2 and 5.3 in [A9], respectively. The forward and backward honeycomb Fourier–Weyl and Hartley–Weyl transforms are of similar form.

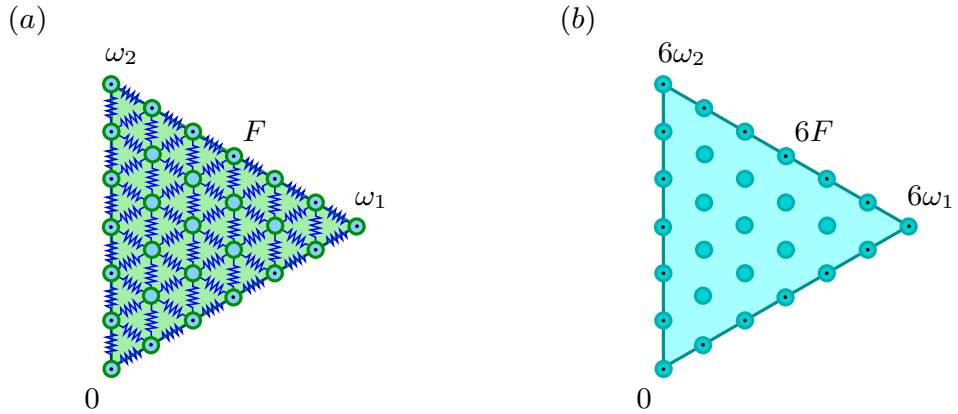


Figure 4.1: (a) The fundamental domain F of A_2 is depicted as the green triangle with vertices ω_1 and ω_2 containing 28 points of the point set $F_{P,6}$. Omitting the dotted points on the boundary of the triangle F , the 10 points of $\tilde{F}_{P,6}$ are obtained. The blue zig-zag lines linking the dots represent the springs of the vibration model. (b) The magnified fundamental domain $6F$ of A_2 is depicted as the cyan triangle with vertices $6\omega_1$ and $6\omega_2$ containing 28 points of the weight set $\Lambda_{Q,6}$. Omitting the dotted weights on the boundary of the triangle $6F$, the 10 points of $\tilde{\Lambda}_{Q,6}$ are obtained. Each weight labels a mode of the transversal vibration model.

4.4 Transversal Vibration Models

The **transversal A_2 weight lattice vibration model** is for $M = 6$ depicted in Figure 4.1. The dots of the point set $F_{P,M}$ depict the points of masses m and the equilibrium distance between the two nearest points is denoted by R_0 . The zig-zag blue lines linking the dots represent the springs of spring constants κ and natural lengths l_0 . The parameter $\eta = l_0/R_0$, $\eta < 1$ measures the level of stretching of the system. Transverse displacement scalar function is denoted as $\psi(a) \equiv \psi(a, t)$, $a \in F_{P,M}$, where t represents time.

The **linearized equation of motion** for transversal displacement of any general point

$$a = a_1\omega_1 + a_2\omega_2 = (a_1, a_2) \in F_{P,M}$$

is of the form

$$m\ddot{\psi}(a) = \kappa(1 - \eta) \left[\psi\left(a + \frac{(1,0)}{M}\right) + \psi\left(a + \frac{(0,-1)}{M}\right) + \psi\left(a + \frac{(-1,1)}{M}\right) \right] + \kappa(1 - \eta) \left[\psi\left(a + \frac{(0,1)}{M}\right) + \psi\left(a + \frac{(-1,0)}{M}\right) + \psi\left(a + \frac{(1,-1)}{M}\right) - 6\psi(a) \right].$$

Assuming a solution of the mode form

$$\psi(a, t) = X(a) \cos(\omega t + \varphi),$$

the equation of motion simplifies as

$$\left(6 - \frac{\omega^2 m}{\kappa(1 - \eta)}\right) X(a) = X\left(a + \frac{(1,0)}{M}\right) + X\left(a + \frac{(0,-1)}{M}\right) + X\left(a + \frac{(-1,1)}{M}\right) + X\left(a + \frac{(0,1)}{M}\right) + X\left(a + \frac{(-1,0)}{M}\right) + X\left(a + \frac{(1,-1)}{M}\right). \quad (4.42)$$

For the case of the root system A_2 and its Weyl group W , the C - and S -functions, labeled by a weight $\lambda = b_1\omega_1 + b_2\omega_2$ and evaluated at a point $a = (x_1, x_2)$, are of the explicit form [A9],

$$\begin{aligned}\Phi_\lambda(a) &= e^{\frac{2}{3}\pi i((2b_1+b_2)x_1+(b_1+2b_2)x_2)} + e^{\frac{2}{3}\pi i(-b_1+b_2)x_1+(b_1+2b_2)x_2} + e^{\frac{2}{3}\pi i((-b_1-2b_2)x_1+(b_1-b_2)x_2)} \\ &\quad + e^{\frac{2}{3}\pi i((-b_1-2b_2)x_1+(-2b_1-b_2)x_2)} + e^{\frac{2}{3}\pi i((-b_1+b_2)x_1+(-2b_1-b_2)x_2)} + e^{\frac{2}{3}\pi i((2b_1+b_2)x_1+(b_1-b_2)x_2)}, \\ \varphi_\lambda(a) &= e^{\frac{2}{3}\pi i((2b_1+b_2)x_1+(b_1+2b_2)x_2)} - e^{\frac{2}{3}\pi i(-b_1+b_2)x_1+(b_1+2b_2)x_2} + e^{\frac{2}{3}\pi i((-b_1-2b_2)x_1+(b_1-b_2)x_2)} \\ &\quad - e^{\frac{2}{3}\pi i((-b_1-2b_2)x_1+(-2b_1-b_2)x_2)} + e^{\frac{2}{3}\pi i((-b_1+b_2)x_1+(-2b_1-b_2)x_2)} - e^{\frac{2}{3}\pi i((2b_1+b_2)x_1+(b_1-b_2)x_2)}.\end{aligned}$$

For the orbit functions $\Phi_\lambda\left(\frac{(1,0)}{M}\right)$ and $\Phi_\lambda\left(\frac{(0,1)}{M}\right)$ multiplied by $\Phi_\lambda(a)$ and $\varphi_\lambda(a)$ specializes the product-to-sum formula (4.2) as

$$\begin{aligned}\Phi_\lambda\left(\frac{(1,0)}{M}\right)\Phi_\lambda(a) &= 2\Phi_\lambda\left(a + \frac{(1,0)}{M}\right) + 2\Phi_\lambda\left(a + \frac{(0,-1)}{M}\right) + 2\Phi_\lambda\left(a + \frac{(-1,1)}{M}\right), \\ \Phi_\lambda\left(\frac{(0,1)}{M}\right)\Phi_\lambda(a) &= 2\Phi_\lambda\left(a + \frac{(0,1)}{M}\right) + 2\Phi_\lambda\left(a + \frac{(-1,0)}{M}\right) + 2\Phi_\lambda\left(a + \frac{(1,-1)}{M}\right),\end{aligned}\tag{4.43}$$

and

$$\begin{aligned}\Phi_\lambda\left(\frac{(1,0)}{M}\right)\varphi_\lambda(a) &= 2\varphi_\lambda\left(a + \frac{(1,0)}{M}\right) + 2\varphi_\lambda\left(a + \frac{(0,-1)}{M}\right) + 2\varphi_\lambda\left(a + \frac{(-1,1)}{M}\right), \\ \Phi_\lambda\left(\frac{(0,1)}{M}\right)\varphi_\lambda(a) &= 2\varphi_\lambda\left(a + \frac{(0,1)}{M}\right) + 2\varphi_\lambda\left(a + \frac{(-1,0)}{M}\right) + 2\varphi_\lambda\left(a + \frac{(1,-1)}{M}\right).\end{aligned}\tag{4.44}$$

Summing the relations in (4.43) and in (4.44) and comparing them to (4.42) yields the dispersion relation

$$\omega_\lambda = \sqrt{\frac{\kappa(1-\eta)}{m} \left(6 - \frac{1}{2}\Phi_\lambda\left(\frac{(1,0)}{M}\right) - \frac{1}{2}\Phi_\lambda\left(\frac{(0,1)}{M}\right)\right)}$$

and the corresponding solutions satisfying von Neumann boundary conditions

$$\mathcal{X}_\lambda(a) = \Phi_\lambda(a), \quad a \in F_{P,M}, \quad \lambda \in \Lambda_{Q,M},$$

as well as Dirichlet boundary conditions,

$$\tilde{\mathcal{X}}_\lambda(\tilde{a}) = \varphi_\lambda(\tilde{a}), \quad \tilde{a} \in \tilde{F}_{P,M}, \quad \tilde{\lambda} \in \tilde{\Lambda}_{Q,M}.$$

In order to obtain real-valued solutions, the corresponding Hartley orbit functions (2.2) are taken as the mechanical modes,

$$\begin{aligned}X_\lambda(a) &= \zeta_\lambda^1(a), \quad a \in F_{P,M}, \quad \lambda \in \Lambda_{Q,M}, \\ \tilde{X}_\lambda(a) &= \zeta_\lambda^{\sigma^e}(a), \quad a \in \tilde{F}_{P,M}, \quad \lambda \in \tilde{\Lambda}_{Q,M}.\end{aligned}$$

Several lower transversal Hartley modes of the A_2 weight lattice vibration model are for von Neumann and Dirichet boundary conditions depicted in Figures 4.2 and 4.3.

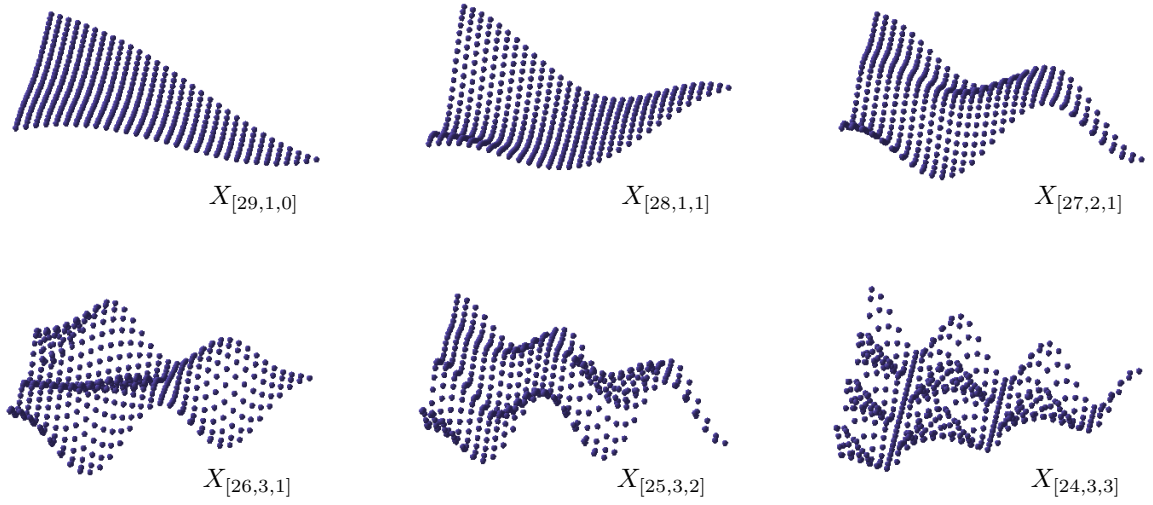


Figure 4.2: Lower transversal Hartley modes of the A_2 vibration model X_λ , $\lambda \in \Lambda_{Q,30}$ satisfying von Neumann boundary conditions. The full set of transversal modes of $F_{P,30}$ contains $|\Lambda_{Q,30}| = 496$ elements.

Spectral analysis of any fixed initial positions

$$\psi(a, 0) = \psi_0(a), \quad (4.45)$$

and any initial velocities

$$\dot{\psi}(a, 0) = V_0(a), \quad (4.46)$$

produces via the Hartley version of the forward Fourier–Weyl C –transform (2.16) from [40] the spectral functions $\widehat{\psi}_0^C, \widehat{V}_0^C$,

$$\begin{aligned} \widehat{\psi}_0^C(\lambda) &= (18M^2 h_M(\lambda))^{-1} \sum_{a \in F_{P,M}} \varepsilon(a) \psi_0(a) X_\lambda(a), \\ \widehat{V}_0^C(\lambda) &= (18M^2 h_M(\lambda))^{-1} \sum_{a \in F_{P,M}} \varepsilon(a) V_0(a) X_\lambda(a) \end{aligned}$$

and via the S –transform the spectral functions $\widehat{\psi}_0^S, \widehat{V}_0^S$,

$$\begin{aligned} \widehat{\psi}_0^S(\tilde{\lambda}) &= (3M^2)^{-1} \sum_{\tilde{a} \in \tilde{F}_{P,M}} \psi_0(\tilde{a}) \tilde{X}_{\tilde{\lambda}}(\tilde{a}), \\ \widehat{V}_0^S(\tilde{\lambda}) &= (3M^2)^{-1} \sum_{\tilde{a} \in \tilde{F}_{P,M}} V_0(\tilde{a}) \tilde{X}_{\tilde{\lambda}}(\tilde{a}). \end{aligned}$$

The unique explicit form of the solution, determined by the initial conditions (4.45) and (4.46), is given for the Dirichlet boundary conditions as

$$\tilde{\psi}(\tilde{a}, t) = \sum_{\tilde{\lambda} \in \tilde{\Lambda}_{Q,M}} \left(\widehat{\psi}_0^S(\tilde{\lambda}) \cos(\omega_{\tilde{\lambda}} t) + \frac{\widehat{V}_0^S(\tilde{\lambda})}{\omega_{\tilde{\lambda}}} \sin(\omega_{\tilde{\lambda}} t) \right) \tilde{X}_{\tilde{\lambda}}(\tilde{a}).$$

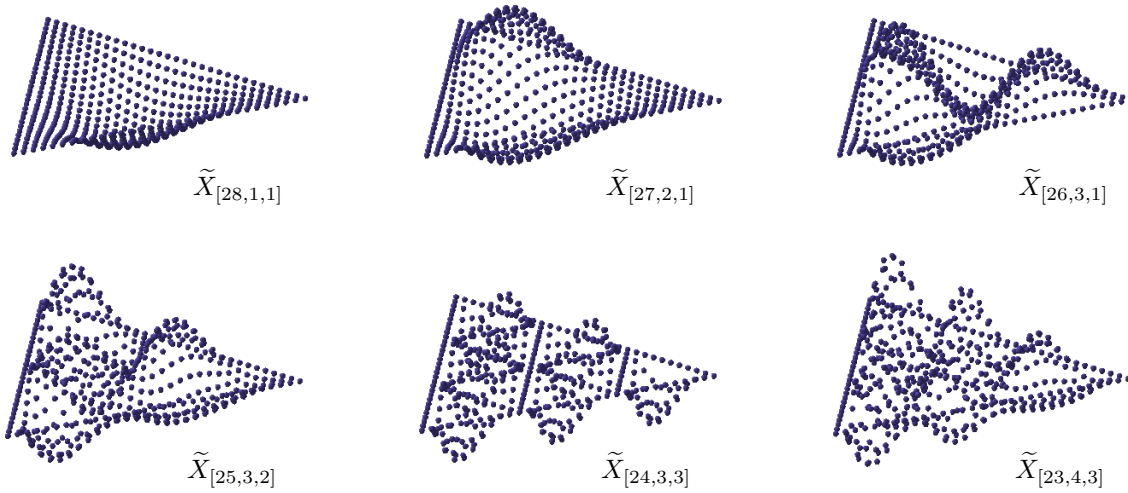


Figure 4.3: Lower transversal Hartley modes of the A_2 vibration model $\tilde{X}_{\tilde{\lambda}}$, $\tilde{\lambda} \in \tilde{\Lambda}_{Q,30}$ satisfying Dirichlet boundary conditions. The full set of transversal modes of $\tilde{F}_{P,30}$ contains $|\tilde{\Lambda}_{Q,30}| = 406$ elements.

For the von Neumann boundary conditions, the additional requirements $\widehat{\psi}_0^C([M, 0, 0]) = 0$ and $\widehat{V}_0^C([M, 0, 0]) = 0$ eliminate the translation mode and the resulting solution is of the form

$$\psi(a, t) = \sum_{\lambda \in \Lambda_{Q,M} \setminus [M,0,0]} \left(\widehat{\psi}_0^C(\lambda) \cos(\omega_\lambda t) + \frac{\widehat{V}_0^C(\lambda)}{\omega_\lambda} \sin(\omega_\lambda t) \right) X_\lambda(a).$$

The **transversal A_2 armchair mechanical graphene vibration model** is for the case and $M = 6$ depicted in Figure 2 from [A9]. The dots of the point set H_M depict the points of masses m and the equilibrium distance between the two nearest points is denoted by R_0 . The green lines linking the honeycomb dots represent the springs of spring constants κ and natural lengths l_0 . The parameter $\eta = l_0/R_0$, $\eta < 1$ measures the level of stretching of the system. Transverse displacement scalar function is denoted as $\psi(a) \equiv \psi(a, t)$, $a \in F_{P,M}$, where t represents time.

The extension coefficients $\mu_\lambda^{\pm,k}$ given by

$$\begin{aligned} \mu_\lambda^{\pm,0} &= \operatorname{Re} \left\{ (3 + \sqrt{3}i) \Phi_\lambda \left(\frac{\omega_1}{M} \right) \right\}, \\ \mu_\lambda^{\pm,1} &= 0, \\ \mu_\lambda^{\pm,2} &= \operatorname{Re} \left\{ (3 - \sqrt{3}i) \Phi_\lambda \left(\frac{\omega_1}{M} \right) \right\} \pm 3 \left| \Phi_\lambda \left(\frac{\omega_1}{M} \right) \right|, \end{aligned} \tag{4.47}$$

lead to the normalization functions (4.33) of the following form [A9],

$$\mu^\pm(\lambda) = 9 \left| \Phi_\lambda \left(\frac{\omega_1}{M} \right) \right| \left(2 \left| \Phi_\lambda \left(\frac{\omega_1}{M} \right) \right| \pm \operatorname{Re} \left\{ (1 - \sqrt{3}i) \Phi_\lambda \left(\frac{\omega_1}{M} \right) \right\} \right). \tag{4.48}$$

The extension coefficients (4.47) determine honeycomb Hartley C - and S -functions of type II in [A9], which are denoted by $\operatorname{Cah}_\lambda^{\pm,\text{II}}$, $\lambda \in L_M$ and $\operatorname{Sah}_\lambda^{\pm,\text{II}}$, $\lambda \in \tilde{L}_M$. Type II honeycomb orbit functions represent transversal eigenvibrations of the model subjected to

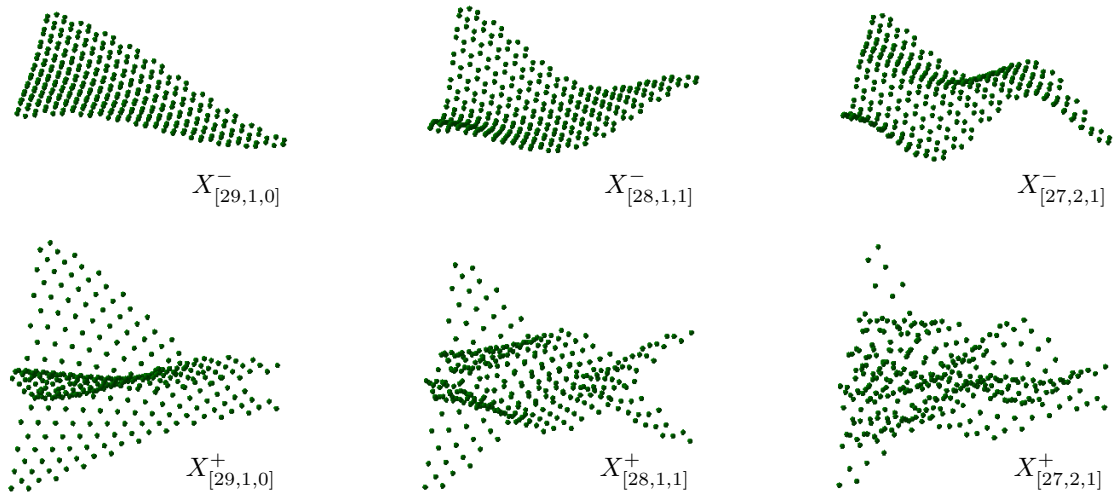


Figure 4.4: Lower transversal Hartley modes of the A_2 armchair mechanical graphene vibration model X_λ^\pm , $\lambda \in L_{30}$ satisfying von Neumann boundary conditions. The full set of transversal modes of H_{30} contains $2|L_{30}| = 330$ elements.

discretized von Neumann boundary conditions

$$X_\lambda^\pm(a) = \text{Cah}_\lambda^{\pm, \text{II}}(a), \quad a \in H_M, \quad \lambda \in L_M,$$

as well as Dirichlet boundary conditions,

$$\tilde{X}_\lambda^\pm(\tilde{a}) = \text{Sah}_\lambda^{\pm, \text{II}}(\tilde{a}), \quad \tilde{a} \in \tilde{H}_M, \quad \tilde{\lambda} \in \tilde{L}_M.$$

Several lower transversal Hartley modes of the transversal A_2 armchair mechanical graphene vibration model are for von Neumann and Dirichet boundary conditions depicted in Figures 4.4 and 4.5.

The eigenfrequencies ω_λ^\pm corresponding to the modes X_λ^\pm , $\lambda \in L_M$ and \tilde{X}_λ^\pm , $\lambda \in \tilde{L}_M$ are given as

$$\omega_\lambda^\pm = \sqrt{\frac{\kappa(1-\eta)}{m} \left(3 \pm \frac{1}{2} \left| \Phi_\lambda \left(\frac{\omega_1}{M} \right) \right| \right)}.$$

Spectral analysis of any fixed initial positions

$$\psi(a, 0) = \psi_0(a), \tag{4.49}$$

and any initial velocities

$$\dot{\psi}(a, 0) = V_0(a), \tag{4.50}$$

yields via the Hartley version of the forward honeycomb Fourier–Weyl C –transform (4.40) of type II from [A9] the spectral functions $\hat{\psi}_0^{C, \pm}$, $\hat{V}_0^{C, \pm}$,

$$\begin{aligned} \hat{\psi}_0^{C, \pm}(\lambda) &= (12M^2 h_M(\lambda) \mu^\pm(\lambda))^{-1} \sum_{a \in H_M} \varepsilon(a) \psi_0(a) X_\lambda^\pm(a), \\ \hat{V}_0^{C, \pm}(\lambda) &= (12M^2 h_M(\lambda) \mu^\pm(\lambda))^{-1} \sum_{a \in H_M} \varepsilon(a) V_0(a) X_\lambda^\pm(a) \end{aligned}$$

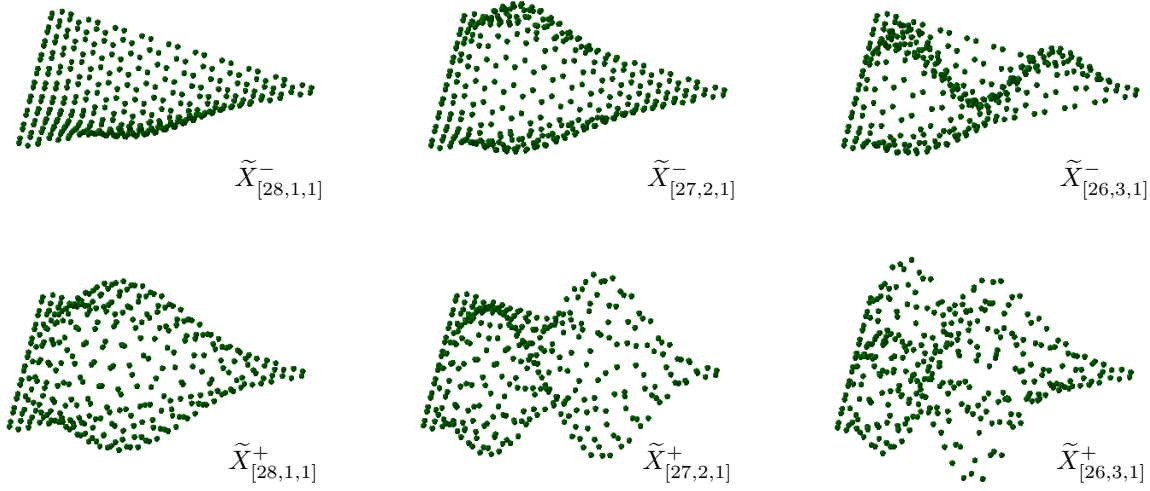


Figure 4.5: Lower transversal Hartley modes of the A_2 armchair mechanical graphene vibration model $\tilde{X}_{\tilde{\lambda}}^{\pm}$, $\tilde{\lambda} \in \tilde{L}_{30}$ satisfying Dirichlet boundary conditions. The full set of transversal modes of \tilde{H}_{30} contains $2|\tilde{L}_{30}| = 270$ elements.

and via the S -transform the spectral functions $\hat{\psi}_0^{S,\pm}$, $\hat{V}_0^{S,\pm}$,

$$\begin{aligned}\hat{\psi}_0^{S,\pm}(\tilde{\lambda}) &= (2M^2\mu^{\pm}(\tilde{\lambda}))^{-1} \sum_{\tilde{a} \in \tilde{H}_M} \psi_0(\tilde{a}) \tilde{X}_{\tilde{\lambda}}^{\pm}(\tilde{a}), \\ \hat{V}_0^{S,\pm}(\tilde{\lambda}) &= (2M^2\mu^{\pm}(\tilde{\lambda}))^{-1} \sum_{\tilde{a} \in \tilde{H}_M} V_0(\tilde{a}) \tilde{X}_{\tilde{\lambda}}^{\pm}(\tilde{a}).\end{aligned}$$

The unique explicit form of the solution, determined by the initial conditions (4.49) and (4.50), is given for the Dirichlet boundary conditions as

$$\begin{aligned}\tilde{\psi}(\tilde{a}, t) &= \sum_{\tilde{\lambda} \in \tilde{L}_M} \left(\hat{\psi}_0^{S,+}(\tilde{\lambda}) \cos(\omega_{\tilde{\lambda}}^+ t) + \frac{\hat{V}_0^{S,+}(\tilde{\lambda})}{\omega_{\tilde{\lambda}}^+} \sin(\omega_{\tilde{\lambda}}^+ t) \right) \tilde{X}_{\tilde{\lambda}}^+(\tilde{a}) \\ &+ \sum_{\tilde{\lambda} \in \tilde{L}_M} \left(\hat{\psi}_0^{S,-}(\tilde{\lambda}) \cos(\omega_{\tilde{\lambda}}^- t) + \frac{\hat{V}_0^{S,-}(\tilde{\lambda})}{\omega_{\tilde{\lambda}}^-} \sin(\omega_{\tilde{\lambda}}^- t) \right) \tilde{X}_{\tilde{\lambda}}^-(\tilde{a}).\end{aligned}$$

For the von Neumann boundary conditions, the additional requirements $\hat{\psi}_0^{C,-}([M, 0, 0]) = 0$ and $\hat{V}_0^{C,-}([M, 0, 0]) = 0$ eliminate the translation mode and the resulting solution is of the form

$$\begin{aligned}\psi(a, t) &= \sum_{\lambda \in L_M} \left(\hat{\psi}_0^{C,+}(\lambda) \cos(\omega_{\lambda}^+ t) + \frac{\hat{V}_0^{C,+}(\lambda)}{\omega_{\lambda}^+} \sin(\omega_{\lambda}^+ t) \right) X_{\lambda}^+(a) \\ &+ \sum_{\lambda \in L_M \setminus [M, 0, 0]} \left(\hat{\psi}_0^{C,-}(\lambda) \cos(\omega_{\lambda}^- t) + \frac{\hat{V}_0^{C,-}(\lambda)}{\omega_{\lambda}^-} \sin(\omega_{\lambda}^- t) \right) X_{\lambda}^-(a).\end{aligned}$$

Conclusion

The results of the selected articles included in the thesis [A1–A9] contribute to the theory and applications of the multidimensional discrete Fourier–Weyl transforms of special functions related to Weyl groups. The developed discrete Fourier–Weyl transforms and related Fourier methods are relevant in their own right in the field of mathematical physics. The theory of the Fourier–Weyl transforms is linked to notions from conformal field theory and generates solutions of vibrations models with boundary conditions in solid state physics. Other potential applications comprise various methods which already benefit from multi-dimensional concatenation of Cartesian products of univariate trigonometric transforms. These methods include numerical solutions of differential equations, solutions of difference equations, numerical integration, spectral and interpolation methods. Moreover, together with the segments pertinent to numerical methods, the presented results contribute to the theory of Chebyshev-like orthogonal polynomials and associated Chebyshev methods.

The discrete Fourier–Weyl transforms on finite fragments of the Weyl group invariant lattices are explicitly described in general forms [A1–A6, A8, A9]. Boundary layouts of the point sets are for each root system dictated by the admissible shifts of the weight and dual weight lattices and by the action of the sign homomorphisms on the generating reflections of the affine Weyl group. Besides the honeycomb lattice case, the point sets underlying in the discrete Fourier–Weyl transforms considered in the thesis are taken as finite subsets of the weight lattices, dual weight lattice and their shifted versions. A mathematical exposition of the root lattice transforms requires several additional notions from the theory of Weyl groups and specifications of the fundamental domains of the extended affine Weyl groups [41]. The root lattice discrete transforms induce jointly with the weight lattice transforms fundamentally novel options for transforms on composed grids. The presented honeycomb lattice case, generated as subtraction of weight and root lattices of the root system A_2 [A9], represents this approach for 2D lattices. The form of the shifted versions of the root lattice transforms and their compositions with the shifted weight lattice transforms deserves further study.

The completeness of the discretely orthogonal sets of the Weyl orbit functions in the finite-dimensional Hilbert spaces is guaranteed by coinciding cardinalities of the point and label sets [A1–A3, A8, A9]. A general algorithm for deriving the specific counting formulas for cardinalities of the point and label sets, which correspond to the dimensions of the functional Hilbert spaces, is developed in [A1]. Special cases of the presented counting formulas [23–25] are related to numbers of elements of finite order in the underlying Lie group [81, 82] as well as to the description of the Voronoi and Delaunay cells of the root

lattices [83]. The algorithm is altered for evaluation of explicit counting formulas for the shifted point sets [A2], the even point sets [A3], the weight lattice point sets [A8] and the honeycomb point sets [A9]. Further generalization of the algorithm for deriving counting formulas from [A1] is applied for calculation of affine fusion tadpoles in conformal field theory [107]. The orthogonality relations and the discrete Fourier–Weyl and Hartley–Weyl transforms crucially depend on orders of the stabilizers in the affine and dual affine Weyl groups. The calculation procedures for orders of the stabilizers, specified for the standard stabilizers in [A1] and the even stabilizers in [A3], permit evaluation and subsequent implementation of the stabilizer functions ε and h_M^\vee for any case.

The six cases of the even Weyl orbit functions exhibit non-standard behavior on the boundaries of the even Weyl group fundamental domains [31, 34, 40]. The even Fourier–Weyl transforms are developed for the standard even Weyl group on the fundamental simplices extended by the reflections of their interiors [A3]. The boundary behavior of the general even orbit functions translates into their intricate boundary points layout [40]. Application of the even Fourier–Weyl transforms in image processing is achieved in [12] via establishing the corresponding convolution theorems [76]. Applicability of the even transforms to conformal field theory is yet to be investigated. A physical realization of the transversal vibration models satisfying the even boundary conditions also deserves further study. Shifted even weight and dual weight lattice transforms, even root lattice transforms, even honeycomb transforms together with their Hartley–Weyl versions have not yet been described. The presented approach of constructing the semisimple even and full even transforms [A4] permit straightforward generalization to all six types of E –functions. Special rich outcome of the even Fourier–Weyl and Hartley–Weyl transforms is expected for direct products of different types of even functions.

The multivariate real-valued discrete Hartley–Weyl transforms have potential applications in fields which utilize the standard versions of the Hartley transform such as signal processing [87, 94], geophysics [66], measurement [104], pattern recognition [3] and optics [72, 112]. Each standard and even Fourier–Weyl and Hartley–Weyl transform is connected to an interpolation problem of multivariate functions sampled over the underlying set of points. The successful interpolation tests are performed for the 2D and 3D symmetric and antisymmetric trigonometric functions in [43, A5, A6] and for the Hartley honeycomb transforms in [A9]. Other fruitful tests are accomplished for the 2D even transforms in [31], 2D transforms of algebras A_2 , C_2 and G_2 in [1, 90–92], 3D alternating transforms in [46], 3D transforms of algebras B_3 and C_3 in [32, 35]. Common feature of the interpolation tests is excellent interpolation and convergence behavior. The Gibbs phenomenon for univariate Fourier expansions conveys specific ringing of Fourier expansions at the points of discontinuities of the model function [39]. The Gibbs phenomenon for bivariate cases is also noted in [37, 38, 106]. The antisymmetric and periodic extensions of a given model function, continuous on the fundamental domain, generate in general border discontinuities and Gibbs ringing of the interpolations. The Gibbs phenomenon is avoided for the cases with continuous periodic and symmetric extension [A3].

Ratios of the antisymmetric and symmetric cosine functions as well as ratios of the Weyl

orbit functions are expressed in the form of multivariate Chebyshev polynomials [45, A6]. The antisymmetric and symmetric cosine polynomials form via their link to polynomials associated to root systems [33] special cases of Heckman–Opdam [86] and Macdonald polynomials [73]. Relations among the special functions associated to the Weyl orbit functions and the Jacobi polynomials are detailed in [33]. These multivariate polynomials inherit the discrete orthogonality properties of trigonometric and orbit functions [18, 22]. Structural characteristics of the ratios related to the antisymmetric and symmetric sine polynomials deserve further study. The Lebesgue constant estimates of the polynomial cubature and interpolation formulas together with the convergence of the polynomials series pose open problems. The cubature rules from [78, 79, A6] indicate that the shifted lattices transforms possess high potential to generate cubature formulas of Gaussian type. Versions of the Clenshaw–Curtis methods of numerical integration [14, 102], developed for the C_2 and A_2 root systems in [33, 97], also need to be further investigated to cover more cases. Hyperinterpolation methods [9, 75, 101, 102] which directly apply the standard polynomial cubature rules, pose for the presented cubature rules open problems.

The existence and explicit forms of generating functions for the related Weyl orbit functions polynomials, developed in [18–20, 100], further enhance the applicability of the presented polynomial Chebyshev methods. The polynomial generating functions are closely related to the character generating functions from Lie theory [55, 89]. The generating functions form an efficient tool for investigating symmetries and parity relations of the generated orthogonal polynomials. The recurrence relations algorithms for the calculation of the bivariate polynomials [69] are superseded by explicit evaluation formulas derived from the generating functions [18]. The explicit evaluation formulas [18] for the polynomials represent practical tool for efficient computer implementation and handling of the generated polynomials. The potential applications of the presented cubature rules [A6] include calculations in laser optics [10], stochastic dynamics [109], fluid flows [16], magnetostatic modeling [110], electromagnetic wave propagation [99], micromagnetic simulations [13], liquid crystal colloids [105] and quantum dynamics [67].

The discretized versions of product-to-sum decomposition formulas, which are summarized for any type of orbit function in [40] and for semisimple even functions in [A4], lead to the dual weight lattice generalization of the Kac–Walton formula [A7]. The properties of the unitary and symmetric Kac–Peterson matrices together with the affine fusion rules and Kac–Walton formulas [26] from conformal field theory motivated the development of the weight lattice discretization of Weyl orbit functions in [A8]. The common argument and label symmetries of the weight lattice transforms, dictated by the affine Weyl groups, yield four types of unitary and symmetric generalizations of the Kac–Peterson matrices. The forms and physical significance of the generalized Kac–Walton formulas and Kac–Peterson matrices for all ten types of weight lattice discretized Weyl orbit functions need to be further investigated. On the other hand, the Fourier–Weyl transforms on the root, shifted and subtracted lattices potentially positively influence advance of conformal field theory. Implications of the weight-lattice discretization for the interpolation methods, the corresponding orthogonal polynomials, cubature rules and polynomials series expansions deserve further

study. Another crucial application of the product-to-sum decomposition formulas is found in description of mechanical vibration models.

The Weyl orbit functions represent solutions of the mechanical vibration models constrained by Dirichlet, von Neumann or mixed boundary conditions on the fundamental domain of the affine Weyl group. The discrete Fourier–Weyl and Hartley–Weyl transforms provide spectral analysis of given initial conditions and determine the explicit solutions. A significant advantage of the current symmetry approach for the honeycomb lattice [A9] lies in the form of the resulting functions. Each solution is determined by a single Hartley honeycomb orbit function, whereas the standard approach yields two different functional descriptions, one for each congruence class of the honeycomb lattice [17, 54, 96]. Permitting an efficient interpolation [A9], the honeycomb Fourier–Weyl and Hartley–Weyl C – and S –transforms thus represent suitable generalizations of the standard discrete cosine and sine transforms to the honeycomb lattice. Contrary to the present armchair boundary triangular honeycomb dot, the zig-zag boundary triangular dot is not a union of two Weyl group invariant lattices and the standard procedures for calculation of solutions [17, 96] cannot be used for this case. On the other hand, a modified approach from [A9] potentially also leads to unique novel solutions and a dispersion relation for the zig-zag honeycomb triangular dot.

With continued research, the significance of the Fourier–Weyl transforms together with the related Fourier and Chebyshev methods in mathematical physics steadily increases. Besides the developed specific transforms, achieved links connecting the transforms to conformal field theory and solid state physics constitute a consequential segment of the field. Since Weyl orbit functions represent generalizations of the trigonometric functions related to fundamental symmetries of nature, their further applications, appreciation and augmentation of value are expected.

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Included Publications

List of Included Articles

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