The Complexity of Fair Division of Indivisible Items with Externalities

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Abstract

We study the computational complexity of fairly allocating a set of indivisible items under externalities. In this recently-proposed setting, in addition to the utility the agent gets from their bundle, they also receive utility from items allocated to other agents. We focus on the extended definitions of envy-freeness up to one item (EF1) and of envy-freeness up to any item (EFX), and we provide the landscape of their complexity for several different scenarios. We prove that it is NP-complete to decide whether there exists an EFX allocation, even when there are only three agents, or even when there are only six different values for the items. We complement these negative results by showing that when both the number of agents and the number of different values for items are bounded by a parameter the problem becomes fixed-parameter tractable. Furthermore, we prove that two-valued and binary-valued instances are equivalent and that EFX and EF1 allocations coincide for this class of instances. Finally, motivated from real-life scenarios, we focus on a class of structured valuation functions, which we term agent/item-correlated. We prove their equivalence to the "standard" setting without externalities. Therefore, all previous results for EF1 and EFX apply immediately for these valuations.

1 Introduction

The *allocation* of a set of *indivisible* resources, e.g., objects, tasks, responsibilities, in a *fair* manner is a question that has received a lot of attention through history. In the last decades though, economists, mathematicians, and computer scientists have systematically started studying the problem with the aim of providing *formal fairness guarantees* (Lipton et al. 2004; Bouveret and Lang 2008; Budish 2011; Caragiannis et al. 2019); for an excellent recent survey on the topic see (Amanatidis et al. 2023). However, despite the significant efforts on this quest, the nature of the problem, i.e., the indivisibility, has not allowed yet for a universally adopted solution concept.

Typically, an instance of the fair division problem consists of a set of indivisible items, and a set of agents each of whom has their own *valuation function*. The task is to *partition* the items into *bundles* and allocate each bundle to an agent such that from the point of view of *every* agent this allocation is "fair". Here, the "fair" part is a mathematical criterion that has to be satisfied by the valuation function of every agent.

Traditionally, in the majority of previous works, the mathematical criterion of fairness for each agent depends *only* on a pairwise comparison between *bundles*. Put simply, each agent cares, i.e., derives value, *only* about the bundle they receive, and they compare it against the bundle of any other agent. However, in many real-life situations this assumption is not sufficient due to inherent underlying *externalities*.

Consider for example the scenario where there is a set of admin tasks that have to be assigned to the faculty members of a CS department. There could exist certain tasks such that some faculty members are objectively better qualified for them – and would even enjoy doing them – while other faculty members are not that suited for them. Here, *every* faculty member evaluates the allocation as a *whole*, since they are affected, either positively or negatively, by the quality of completion of (almost) all tasks.

As a different example, assume that the agents are a priori partitioned into two teams, Team A and Team B, that compete against each other and consider a specific agent in Team A. Then, for any resource the agent considers as *good*, they will get positive value if it is allocated to Team A – maybe the value is discounted compared to the value the agent would get if they got the item – while they get zero, or even negative, value if the resource is allocated to Team B. At the same time, for any resource/task that the agent considers that will decrease the efficiency of the team, i.e. they view it as a *chore*, they would get negative value if it is allocated to some other agent from Team A.

Motivated by real-life scenarios like the two above, Aziz et al. (2023b) recently proposed a new model suitable to capture the situations where externalities occur; interestingly, for *divisible* items, the first models that incorporate externalities were proposed many years ago (Brânzei, Procaccia, and Zhang 2013; Li, Zhang, and Zhang 2015). The foundational principle of their model is that the agents have additive valuations over the items where, for every item a, agent i derives value $V_i(j,a)$ if agent j gets the item; here j can be equal to i. This way, every agent evaluates the *entire* allocation and not just their bundle. Furthermore, in view of the more general valuation functions Aziz et al. appropriately extended the most prominent fairness concepts for indivisible items: *envy-freenes* (EF), *envy-freeness up to one item*

(EF1) (Lipton et al. 2004; Budish 2011), and *envy-freeness up to any item* (EFX) (Caragiannis et al. 2019). Intuitively, they are defined as follows.

An allocation is EF1 if for agent i that prefers the allocation where they swap bundles with agent j, there exists one item in either of the bundles of agents i and j (depending on whether it is a good or a chore), such that by removing it agent i does not longer prefers the allocation with the bundles swapped. An allocation is EFX if instead of removing *some* item from the bundles of agents i and j in order to eliminate agent's i preference towards the allocation with swapped bundles, it suffices to remove *any* item from the same bundles that *strictly* decreases the envy of agent i towards the allocation.

In contrast to the basic setting without externalities, where EF1 allocations always exist and EFX allocations are guaranteed to exist for a few settings, Aziz et al. (2023b) showed that things become significantly more complicated in the presence of externalities. While it is currently unknown, and a major open problem, whether in the basic setting EFX allocations always exist[1], Aziz et al. (2023b) show that there exist instances with externalities without any EFX allocation! However, for those instances where existence of an EFX allocation is not guaranteed, Aziz et al. do not provide any results for the associated computational problem, i.e., decide whether a fair allocation exists or not. We resolve this open problem and we deep dive into the uncharted waters of the computational complexity of fair division with externalities.

Our Contribution

We begin our study of the complexity of fair division with externalities by proving that it is intractable to decide whether a given instance admits an EFX allocation, even for very restricted settings. Firstly, we show that it is NP-complete to solve the problem, even when there are only three agents. This paints a clear dichotomy between tractable and intractable cases, as Aziz et al. (2023b) showed that for two agents an EFX allocation can always be found in polynomial time. This result also shows that fair division with externalities is significantly harder compared to the standard setting without externalities, where pseudo-polynomial algorithms are known, for instance with three (Chaudhury, Garg, and Mehlhorn 2020) and partly with four agents (Berger et al. 2022; Ghosal et al. 2023).

Next, we restrict the problem at a different dimension and we turn our attention to instances with valuations that use only a small number of values. We prove that the problem remains NP-complete even if the domain of the valuation function consists of 6 different values. It is also worth mentioning that in our hardness constructions we do not exploit the presence of chores, as is common in standard fair division settings (Hosseini, Mammadov, and Was 2023).

In light of our hardness lower bounds, we use the framework of parameterized complexity (Niedermeier 2006; Downey and Fellows 2013; Cygan et al. 2015) to reveal at least some tractable fragments of the problem. It should be pointed out that this framework has become *de facto* standard approach when dealing with NP-hard problems in AI, ML, and computer science in general (Kronegger et al. 2014; Igarashi, Bredereck, and Elkind 2017; Bredereck et al. 2019; Ganian and Korchemna 2021; Deligkas et al. 2021; Blažej et al. 2023). Roughly speaking, in this framework, we study the complexity of a problem not merely with respect to the input size n, but also assuming additional information about the instance captured in the so-called *parameter*.[2]

We start our algorithmic journey with the combined parameter: the number of *item types* and the number of agents. Intuitively, two items are of the same type if all agents value them the same if they are allocated to a distinct agent j; this parameter was recently initiated by Gorantla, Marwaha, and Velusamy (2023) for goods and by Aziz et al. (2023a) for chores. As our results indicate, the combination of these two parameters is necessary in order to achieve fixed parameter tractability; our first negative result holds just for three agents, and our second result produces an instance with just three different item types. Hence, our algorithm is the best possible one could hope for, and actually it is capable of finding also EF and EF1 allocations, if they exist.

Moreover, our algorithm serves as the foundation for our second positive result, which is an efficient procedure deciding the existence of an EFX/EF1/EF allocation for the combined parameter the number of agents and the number of different values in agents' preferences. The latter parameter naturally captures widely studied binary valuations (Barman, Krishnamurthy, and Vaish 2018; Freeman et al. 2019; Halpern et al. 2020; Babaioff, Ezra, and Feige 2021; Suksompong and Teh 2022), bi-valued valuations (Ebadian, Peters, and Shah 2022; Garg, Murhekar, and Qin 2022), and was previously used by Amanatidis et al. (2021) and Garg and Murhekar (2023).

Next, we move to instances with structured valuations. Following the approach of Aziz et al. (2023b), we start with binary valuations, i.e., they have $\{0, 1\}$ as domain. First, we show that instances where every agent uses only two different values (which can be different for every agent) are, in fact, equivalent to binary valuations. Additionally, and more importantly, we show that for binary valuations EFX and EF1 allocations coincide. Thus, using the existential result of Aziz et al., we establish the existence of EFX allocations for three agents with binary valuations and no chores.

Finally, we introduce and study a different class of structured valuations which we term *agent/item-correlated* valuations. Intuitively, under agent-correlated valuations, an agent $i \in N$ receives utility $v_{i,a}$ if an item a is given to her, and $\tau_{i,j}$ fraction of $v_{i,a}$ if the item is allocated to agent j. Item-correlated valuations are similar; however, the fractional coefficient depends on the item which is allocated not the agent who gets it. We show that instances with agent/item-correlated valuations can be turned into equivalent instances of fair division without externalities with the same sets of agents and items. To conclude, we show how the agent- and item-correlated preferences capture many real-life scenarios such as team preferences (Igarashi et al. 2023).

[1]The problem is open only for goods or only for chores. Recently, Hosseini et al. (2023) resolved the problem for mixed items.

[2]We provide formal definitions in Preliminaries.

All omitted proofs can be found in the full version of the paper (Deligkas et al. 2023).

2 Preliminaries

We will follow the model and the notation of Aziz et al. (2023b). There is a set of indivisible items $A = \{a_1, a_2, \dots, a_m\}$ and a set of agents $N = \{1, 2, \dots, n\}$. An allocation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ is a partition of the items into n possibly empty sets, i.e., $\pi_i \cap \pi_j = \emptyset$ for every $i \neq j$ and $\bigcup_{i \in N} \pi_i = A$, where set π_i is allocated to agent i. Let Π denote the set of all allocations. For any item $a \in A$, denote $\pi(a)$ the agent who receives item a in allocation π.

We assume that the agents have valuation functions with *additive externalities*. More formally, every agent i has a value $V_i(j, a)$ for every item $a \in A$ and every agent $j \in N$; put simply, agent i gets value $V_i(j, a)$ if item a is allocated to agent j. The value of agent i from allocation π is $V_i(\pi) = \sum_{a \in A} V_i(\pi(a), a)$.

Let $a \in A$ be an item. We define the *item-type* as a vector $\big(V_1(1, a), \dots, V_1(|N|, a), V_2(1, a), \dots, V_{|N|}(|N|, a)\big)$. That is, two items are of the same item-type if the associated vectors are the same; intuitively, these items are "indistinguishable" from the point of view of every agent. By Υ we denote the number of different item-types in an instance.

We will focus on *envy-freeness* and its relaxations, EF1 and EFX, in the presence of externalities. Since now every agent evaluates the *whole allocation* and not just their bundle like in the no-externalities case, the idea of *swapping bundles* needs to be deployed. We use $\pi^{i \leftrightarrow j}$ to denote the new allocation in which agents i and j swap their bundles in π while the bundles of the other agents remaining the same.

Definition 1 (EF (Velez 2016))**.** *An allocation π is envy-free (EF), if for every pair of agents $i, j \in N$ it holds that $V_i(\pi) \geq V_i(\pi^{i \leftrightarrow j})$.*

Definition 2 (EF1 (Aziz et al. 2023b))**.** *An allocation π is envy-free up to one item (EF1), if for every pair of agents $i, j \in N$ there exists an item $a \in A$ and an allocation λ such that: (i) $\lambda_\ell = \pi_\ell \setminus \{a\}$, for all $\ell \in N$; and (ii) $V_i(\lambda) \geq V_i(\lambda^{i \leftrightarrow j})$.*

Definition 3 (EFX (Aziz et al. 2023b))**.** *An allocation π is envy-free up to any item (EFX), if for every pair of agents $i, j \in N$, if $V_i(\pi) < V_i(\pi^{i \leftrightarrow j})$, then for any item $a \in A$ and allocation λ with the properties*

1. *$\lambda_\ell = \pi_\ell \setminus \{a\}$, for all $\ell \in N$;*
2. *$V_i(\lambda) - V_i(\lambda^{i \leftrightarrow j}) > V_i(\pi) - V_i(\pi^{i \leftrightarrow j})$,*

it holds that $V_i(\lambda) \geq V_i(\lambda^{i \leftrightarrow j})$.

Observe that the second property of Definition 3 above implies that we *have to* remove an item from $\pi_i \cup \pi_j$ with *strictly* non-zero value for agent i; depending from which bundle we remove the item, it can be either a good, or a chore. This definition is equivalent to EFX in the absence of externalities, when we have to remove items of non-zero value from a bundle an agent envies. A different, more constrained version, termed EFX_0, requires that envy should be eliminated by removing any item, even if the agent has zero value for it (Plaut and Roughgarden 2020).

For allocation π and two agents $i, j \in N$, we say that i *envies* j or alternatively that there is *envy from* i *towards* j if $V_i(\pi^{i \leftrightarrow j}) > V_i(\pi)$. We will use the following simple observation in our hardness proofs.

Observation 1. *If an agent i does not envy agent j, then the pair i, j satisfies Definition 3 for every item $a \in A$.*

Finally, we are ready to define the computational problems we will study.

Definition 4. *Let $\phi \in \{$EF, EF1, EFX$\}$. An instance $\mathcal{I} = (N, A, V)$ of $\phi-$*FAIR DIVISION WITH EXTERNALITIES *consists of a set of items A and a set of agents N with valuation functions V with additive externalities. The task is to decide whether there exists an allocation that is fair with respect to solution concept ϕ.*

Normalized Valuations. Valuations with externalities allow us to consider only non-negative values in the valuations, i.e. we can "normalize" them as follows. Let $i \in N$ be an agent. For each item $a \in A$ we compute $x_{i,a} := \min_{j \in N} V_i(j, a)$ and we set $V_i(j, a) \leftarrow V_i(j, a) - x_{i,a}$.

Proposition 1. *Let $\mathcal{I}$ be an instance of $\phi-$*FAIR DIVISION WITH EXTERNALITIES. *Then, we can get an instance $\mathcal{I}'$ with normalized valuations such that any solution for instance $\mathcal{I}'$ corresponds to a solution for $\mathcal{I}$.*

Chores. While in the standard setting the definition of chores is straightforward, in the presence of externalities they can be defined in more than one way; Aziz et al. (2023b) defined them informally. Below, we define *strong-chores* and *weak-chores*. Intuitively, a strong-chore is an item that an agent does not want to have at all; this resembles the "standard" chore-definition. On the other hand, an item is a weak-chore, if the agent does not mind having it, but there exist some other agents that it would be better for him if they get it; so, weak-chores capture positive externalities. A strong-chore is a weak-chore, but not vice versa. Hence, negative results with respect to weak-chores carry over to strong-chores.

Definition 5. *An item a is a* strong-chore *for agent i if $V_i(i, a) \leq V_i(j, a)$* for all *$j \neq i$, where for at least one j the inequality is strict. An item a is a* weak-chore *for agent i if there exists an agent j such that $V_i(i, a) < V_i(j, a)$.*

Parameterized Complexity. An instance of a parameterized problem $Q \subseteq \Sigma \times \mathbb{N}$, where Σ is fixed and finite alphabet, is a pair (I, k), where I is an input of the problem and k is *parameter*. The ultimate goal of parameterized algorithmics is to confine the exponential explosion in the running time of an algorithm for some NP-hard problem to the parameter and not to the instance size. In this line of research, the best possible outcome is the so-called *fixed-parameter* algorithm with running time $f(k) \cdot |I|^{\mathcal{O}(1)}$ for any computable function f. That is, for every fixed value of the parameter, we have a polynomial time algorithm where, moreover, the degree of the polynomial is independent of the parameter. For a more comprehensive introduction to parameterized complexity, we refer the interested reader to the monograph of Cygan et al. (2015).

3 General Valuations

In this section, we focus on the case of general valuations for the agents. Our first negative result shows that it is intractable to decide whether there is an EFX allocation even when there are three agents and there are no chores.

Theorem 2. EFX-FAIR DIVISION WITH EXTERNALITIES *is* NP*-complete, even if there are three agents and there are no weak-chores.*

Proof sketch. Firstly, it is not hard to see that we can verify in polynomial time whether an allocation is EFX, since for any agent we can simply calculate whether they envy an allocation where we swap two bundles and whether this can be eliminated by removing each item that satisfies the properties form Definition 3.

Next, we prove hardness by providing a polynomial reduction from the NP-hard EQUAL-CARDINALITY-PARTITION problem (Garey and Johnson 1979) in which, given a sequence $S = (s_1, s_2, \dots, s_{2n})$ of $2n$ integers for some $n \in \mathbb{N}$, the goal is to find a subset $I \subseteq [2n]$ of size n such that $\sum_{i\in I} s_i = \sum_{i\in[2n]\setminus I} s_i$. Let $s_{\min}$ and $s_{\max}$ be the minimum and the maximum integers in S respectively, and let $M = (s_{\max} - s_{\min}) \cdot n^2$. We can assume that $M > 0$, otherwise all the numbers in S are equal and hence any subset of size n forms a solution.

We construct a new instance $S' = (s'_1, s'_2, \dots, s'_{2n})$ of EQUAL-CARDINALITY-PARTITION by shifting all the numbers in S by a constant: $s'_i = M + s_i - s_{\min}$. Then for any $I \subseteq [2n]$ of size n it holds that $\sum_{i\in I} s'_i = \sum_{i\in[2n]\setminus I} s'_i$ if and only if $\sum_{i\in I} s_i = \sum_{i\in[2n]\setminus I} s_i$. Let us now denote $B = \frac{1}{2}\sum_{i\in[2n]}(s_i - s_{\min})$, then EQUAL-CARDINALITY-PARTITION asks to find $I \subseteq [2n]$ of size n such that

$$\sum_{i\in I} s'_i = \frac{1}{2}\sum_{i\in[2n]} s'_i = \frac{1}{2}\sum_{i\in[2n]}(M + s_i - s_{\min}) = Mn + B.$$

In addition, note that

$$\begin{aligned} B =& \frac{1}{2}\sum_{i\in[2n]}(s_i - s_{\min}) \le \frac{1}{2}\sum_{i\in[2n]}(s_{\max} - s_{\min}) = \\ &= n \cdot (s_{\max} - s_{\min}) < n^2 \cdot (s_{\max} - s_{\min}) = M, \end{aligned}$$

so $B < M$. By construction, $M \le s'_i \le M + s_{\max} - s_{\min}$ for every $i \in [2n]$. Therefore, for any set $I \subseteq [2n]$ we have $M \cdot |I| \le \sum_{i\in I} s'_i \le |I| \cdot (M + s_{\max} - s_{\min}) < M \cdot (|I| + 1)$.

We now create an equivalent instance of EFX-FAIR DIVISION WITH EXTERNALITIES with three agents and no chores, where the set $A = \{a_i \mid i \in [2n+2]\}$ consists of $2n+2$ items, first $2n$ associated with integers in S and two auxiliary items. The valuations are defined as follows:

- $V_i(i, a_j) = s'_j$ for $i \in [3]$ and $j \in [2n]$;
- $V_1(1, a_{2n+1}) = V_2(2, a_{2n+2}) = Mn + B$;
- $V_1(1, a_{2n+2}) = V_2(2, a_{2n+1}) = 1$;
- $V_1(2, a_{2n+2}) = V_2(1, a_{2n+1}) = -M^2$;
- $V_3(3, a_{2n+1}) = V_3(3, a_{2n+2}) = \frac{Mn+B}{2}$;
- all remaining values are zeros.

We start by showing how to obtain an EFX allocation π from any $I \subseteq [2n]$ such that $\sum_{i\in I} s'_i = \sum_{i\in[2n]\setminus I} s'_i$. We set $\pi = (\{a_i \mid i \in I\}, \{a_i \mid i \in [2n] \setminus I\}, \{a_{2n+1}, a_{2n+2}\})$.

First, observe that $V_1(\pi) = V_2(\pi) = V_1(\pi^{1\leftrightarrow 2}) = V_2(\pi^{1\leftrightarrow 2}) = V_3(\pi) = V_3(\pi^{1\leftrightarrow 3}) = V_3(\pi^{2\leftrightarrow 3}) = Mn + B$ and $V_1(\pi^{1\leftrightarrow 3}) = V_2(\pi^{2\leftrightarrow 3}) = Mn + B + 1$. Therefore, the only envy is from the agents 1 and 2 towards the agent 3. Since there are no chores, removing items from π_1 or π_2 does not decrease the envy. Moreover, removing a_{2n+1} from π_3 eliminates the envy. Indeed, if λ is such that $\lambda_\ell = \pi_\ell \setminus a_{2n+1}$, then $V_1(\lambda) = V_2(\lambda) = Mn + B$, while $V_1(\lambda^{1\leftrightarrow 3}) = 1$ and $V_2(\lambda^{2\leftrightarrow 3}) = Mn + B$. Similarly, removing a_{2n+2} from π_3 eliminates the envy. Hence, π is EFX.

On the other hand, assume that $\pi = (\pi_1, \pi_2, \pi_3)$ is EFX. For every $j \in [3]$, let $I_j = \{i \mid a_i \in \pi_j\} \cap [2n]$. We will show that I_1 is a solution to EQUAL-CARDINALITY-PARTITION instance S. For this, we will distinguish all possibilities depending on where a_{2n+1} and a_{2n+2} belong and show that the only possible case is $\{a_{2n+1}, a_{2n+2}\} \subseteq \pi_3$, since in rest of the cases, the assignment π cannot be EFX. ☐

Next we restrict the problem at a different dimension and we constrain the valuation function. As our next theorem shows, the problem remains hard even if we severely limit the different values in the valuation functions of the agents and the number of different item types.

Theorem 3. EFX-FAIR DIVISION WITH EXTERNALITIES *is* NP*-complete even if the valuation function uses only* 6 *different values,* 3 *item types, and there are no weak-chores.*

Proof sketch. We prove the statement by providing a polynomial reduction from MIN BISECTION problem on cubic graphs (Bui et al. 1987), i.e., the graphs where every vertex has degree precisely three. In MIN BISECTION we are given a graph $G = (V, E)$ on $2n$ vertices and an integer k and the question is whether there exists a partition (X, Y) of V such that $|X| = |Y| = n$ and there are at most k edges in E with one endpoint in X and the other endpoint in Y. We can assume that no partition of V into two equal parts has only at most $k - 1$ edges across.

Note that if $G = (V, E)$ is a cubic graph with $2n$ vertices and (X, Y) is a partition of V into two subsets of size n such that there are precisely k edges with one endpoint in X and the other in Y, then there are $\frac{3n-k}{2}$ edges with both endpoints in X and $\frac{3n-k}{2}$ edges with both endpoints in Y. We are now ready to describe our reduction to EFX-FAIR DIVISION WITH EXTERNALITIES.

- The set of agents is $N = \{1, \dots, 3n\}$, where each agent $i \in [3n]$ is associated with an edge $e_i \in E$;
- The set of items A is split into three sets (item types):
  - set X of $\frac{3n+k}{2}$ items;
  - set Y of $\frac{3n-k}{2}$ items;
  - set Z of $3n - k$ items.

Intuitively, we want to link any potential solution (X', Y') of MIN BISECTION to the following allocation π of items. Agent i, associated with edge e_i, receives some item from Y only if e_i has both endpoints in Y'. Otherwise, i receives some item from X. In addition, i receives an item from Z if and only if e_i does not belong to the cut (X', Y'). To make such an allocation EFX, we define the valuations as follows.

- For all items $x \in X$, all $i \in [3n]$, $V_i(i, x) = 10n^2$.
- For all items $x \in X$, all $i, j \in N$ such that e_i and e_j do not share an endpoint $V_i(j, x) = 10n^2 - 4n$.
- For all items $y \in Y$, all $i \in [3n]$, $V_i(i, y) = 5n^2$.
- For all items $y \in Y$, all $i, j \in N$ such that e_i and e_j do not share an endpoint $V_i(j, y) = 5n^2 - 4n$.
- For all items $z \in Z$ and all $i \in [3n]$ we have $V_i(i, z) = 1$.
- All the remaining values are zero.

Note that the valuation function only uses six values $0, 1, 5n^2 - 4n, 5n^2, 10n^2 - 4n, 10n^2$ and 3 different item-types.

First, let's assume (X', Y') is a partition of V into equal size parts such that there are exactly k edges with one endpoint in X' and one endpoint in Y'. Let π be the allocation described above. To see that π is EFX, we first consider the case when edges $e_i, e_j \in E$ do not share an endpoint. If in addition the agents i and j receive equal numbers of items from Z, we have $V_i(\pi) = V_i(\pi^{i\leftrightarrow j})$. Indeed, i values items from X and Y by exactly $4n$ more on itself than on j, and since both i and j have exactly one such item, the value does not change after swap. In particular, there is no envy already. Similarly, if one of the agents, say i, is not assigned any item from Z, the envy $V_i(\pi^{i\leftrightarrow j}) - V_i(\pi)$ is at most one. So, by removing any item that decreases the envy, we eliminate it.

It remains to consider the case when the edges e_i and e_j share an endpoint. Again, there are two possibilities. If π_i and π_j contain equal numbers of items from Z, then since e_i and e_j share a vertex, it is straightforward to see that the unique item in $\pi_i \setminus Z$ and the unique item in $\pi_i \setminus Z$ are either both from X or both from Y. Hence $V_i(\pi) = V_i(\pi^{i\leftrightarrow j}) = V_j(\pi) = V_j(\pi^{i\leftrightarrow j})$ and there is no envy.

If only π_j contains an item from Z, then $\pi_i = \{x\}$ for some item $x \in X$. Since all the items are goods, removing x from π makes π_i empty and clearly removes envy from j towards i if there was any envy to begin with. For agent i, if $\pi_j = \{y, z\}$ for some $y \in Y$ and $z \in Z$, then $V_i(\pi^{i\leftrightarrow j}) = V_i(\pi) - 10n^2 + 5n^2 + 1 < 0$, so there is no envy from i towards j. Finally, if $\pi_j = \{x', z\}$ for some $x' \in X$ and $z \in Z$, then $V_i(\pi^{i\leftrightarrow j}) - V_i(\pi) = 1$. Hence, if removal of some item decreases the envy, it completely eliminates it.

For another direction, let π be an EFX assignment. We observe that $|\pi_i \cap (X \cup Y)| = 1$ for all $i \in N$: otherwise there would be two agents i and j such that $|\pi_i \cap (X \cup Y)| = 0$ and $|\pi_j \cap (X \cup Y)| \geq 2$. Then removal of any item from $\pi_j \cap (X \cup Y)$ would decrease, but not eliminate the envy from i towards j. By similar arguments, there are precisely k agents without any item in Z, while all the other agents receive exactly one item from Z. Next, we consider the subset E_I of edges e_i such that $Y \cap \pi_i \neq \emptyset$ and show that if $e_i \in E_I$ and $e_j \notin E_I$ share an endpoint, then $\pi_j \cap Z = \emptyset$. In particular, there are at most k edges that touch E_I, and we use them as a base to obtain our solution to MIN BISECTION. □

Our negative results strongly indicate that in order to derive some positive results for general valuations, we *have to* further restrict ourselves. Interestingly, we show that if we combine the two parameters for which the problem is intractable, when we consider them independently, the problem becomes fixed-parameter tractable. Hence, we provide a dichotomy with respect to this combination of parameters.

We start our journey for the first algorithmic results with instances, where the number of agents and the number of item types is bounded.

Theorem 4. *The ϕ-FAIR DIVISION WITH EXTERNALITIES problem, where $\phi \in \{EF, EF1, EFX\}$, is fixed-parameter tractable when parameterized by the number of different item types Υ and the number of agents $|N|$ combined.*

Proof sketch. As the first step of our algorithm, we partition the items according to their types $T = \{T_1, \dots, T_\Upsilon\}$ and compute the size $n_{T_1}, \dots, n_{T_\Upsilon}$ of each partition. For the rest of this proof, we will use the notion of *bundle-types*. The bundle-type is defined by the subsets of different item types that has at least one representative in the bundle. It is easy to see that there are at most $2^{\mathcal{O}(\Upsilon)}$ bundle-types in total.

Now, we guess for each agent its bundle-type. There are $2^{\Upsilon^{\mathcal{O}(|N|)}}$ such guesses and for each guess, we construct an ILP that verifies whether the guess satisfies the given notion of fairness; in what follows, we assume envy-freeness (EF), but later we show how to tweak the construction to handle also the other notions. We denote by $B(i)$ the set of item types present in the agent's $i \in N$ bundle according to our guess. In addition, we extend the definition of valuation to types and use $V_i(j, t)$ to denote how agent $i \in N$ values the item of type $t \in T$ assigned to agent $j \in N$.

Our ILP contains $\mathcal{O}(|N| \cdot \Upsilon)$ variables $x_{t,i}$ representing the number of items of type t assigned to the agent i. The constraints are then as follows. The first set of $\mathcal{O}(\Upsilon)$ constraints of type $\sum_{i \in N} x_{t,i} = n_t$, where $t \in T$, ensures that all items are allocated. The second set of constraints of size $\mathcal{O}(|N| \cdot \Upsilon)$ secures that the items allocated to each agent correspond to the guessed bundle. Finally, the third set of $\mathcal{O}(|N|^2)$ constraints is to verify that the outcome is envy-free – this can be done by definition of EF.

For EF1 and EFX, we can determine using the guessed bundle-type for every pair of agents $i, j \in N$ the item that decreases envy the most and the item that decreases envy the least (but still by a positive value), respectively. The item to be removed can then easily be incorporated into the third type of constraints.

It is well known that ILPs with parameter-many variables can be solved by a fixed-parameter algorithm (Lenstra Jr. 1983; Kannan 1987; Frank and Tardos 1987), and the theorem follows. □

Theorem 4 above can be used to almost immediately give us the following corollary. The key ingredient here is to show that whenever the number of different values and the number of agents is bounded, so is the number of different item-types.

Corollary 5. *The ϕ-FAIR DIVISION WITH EXTERNALITIES problem, where $\phi \in \{EF, EF1, EFX\}$, is fixed-parameter tractable when parameterized by the number of* *agents $|N|$ and the number of different values d in agents' preferences.*

We conclude this section with a property that could be of independent interest. Specifically, we show that there are instances where no allocation maximizing Nash social welfare is EFX, even if there are 3 agents, 4 items of the same type, and binary valuations. This contrasts the setting of fair division without externalities and additive preferences, as shown by Amanatidis et al. (2021).

Proposition 6. *Let $\mathcal{I}$ be an instance of Fair Division with Externalities and π^* be an allocation maximizing Nash social welfare. Then π^* is not necessarily EFX.*

Proof sketch. Let $\mathcal{I}$ be an instance with $A = \{a_1, a_2, a_3, a_4\}$, $|N| \geq 3$, the valuation function for agent $i \in \{2, \dots, |N|\}$ and every item $a \in A$ be defined as

$$V_i(j, a) = \begin{cases} 1 & \text{if } i = j \text{ or } j = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and for every item $a \in A$ let $V_1(1, a) = 1$ and 0 otherwise.

First, observe that the items are, from the perspective of the agents, indistinguishable, and the only thing that matters for utilities is the number of items allocated to each agent. Therefore, in the rest of this proof, we assume two allocations with the same number of items allocated to the same agents as equivalent. Based on this, we can easily compute the Nash social welfare for an allocation π using the function

$$\mathrm{NW}(\pi) = |\pi_1| \cdot |\pi_2| \cdot \prod_{i \in \{3, \dots, |N|\}} (|\pi_2| + |\pi_i|),$$

subject to the two constraints: $\sum_{i \in N} \pi_i = 4$ and $\forall i \in N \colon \pi_i \in \{0, 1, 2, 3, 4\}$. This clearly attains its global maximum in $|\pi_1| = 1$ and $|\pi_2| = 3$. It can be shown either analytically or by a simple argumentation: there has to be at least one item allocated to agent 1 and at least one item allocated to agent 2. Moreover, agents $3 \dots N$ do not care whether an item is allocated directly to them or to the agent 2, while agent 2 only benefits from items allocated to him. Therefore, the only allocation π^* maximizing Nash social welfare gives one item to agent 1 and three items to agent 2. On the other hand, this allocation is clearly not EFX as the agent 1 benefits from swapping bundle with agent 2 even if we remove any item from π_2^* – as stated before, the items are anyway indistinguishable. $\square$

4 Binary Valuations

In this section, we study the special case of FAIR DIVISION WITH EXTERNALITIES when all the valuations are binary, i.e., all V_i have domain $\{0, 1\}$. We would like to stress here that, in the setting without externalities, the domain for binary valuations is usually $\{-1, 0, 1\}$; $\{0, 1\}$ for goods and $\{-1, 0\}$ for chores (Aziz et al. 2023a). However, in the presence of externalities, there can exist chores even without negative values, so Aziz et al. (2023b) defined binary valuations only using the domain $\{0, 1\}$.

In our first result, we show that, in fact, binary valuations allow us to capture any scenario where every agent has at most two different values.

Proposition 7. *Let $\mathcal{I} = (N, A, V)$ be an instance of $\phi-$*FAIR DIVISION WITH EXTERNALITIES*, where $\phi \in \{$EF, EF1, EFX$\}$. Assume that for every agent i, there exist two numbers x_i and y_i such that $V_i(j, a) \in \{x_i, y_i\}$ for every agent j and item a. Then $\mathcal{I}$ can be transformed in linear time in the equivalent instance of $\phi-$*FAIR DIVISION WITH EXTERNALITIES *with the same sets of agents and objects and binary valuations.*

Next, we show that the notions of EFX and EF1 allocations coincide for binary valuations.

Proposition 8. *Let $\mathcal{I}$ be an instance of Fair Division with Externalities with binary valuations and let π be some allocation of items. Then π is EFX if and only if π is EF1.*

Proof. Observe that every EFX allocation is EF1 by definition, even for general valuations. For another direction, assume that π is EF1 allocation. If agent i envies j, there must be some item a such that removal a from π eliminates the envy. Since the valuations are binary, removal of any item can decrease the envy by at most one, from which we conclude that $V_i(\pi) - V_i(\pi^{i \leftrightarrow j}) = 1$. But then the removal of any other item, decreasing $V_i(\pi) - V_i(\pi^{i \leftrightarrow j})$, eliminates the envy as well. Hence, π is EFX allocation. $\square$

Aziz et al. (2023a) showed that for instances with three-agents, no chores, and binary valuations, an EF1 allocation always exists and can be found in polynomial time. Therefore, by Proposition 8, we obtain the same guarantee also for EFX.

Theorem 9. *Every instance of Fair Division with Externalities with three agents, binary valuations, and no weak-chores, admits an EFX allocation which can be computed in polynomial time.*

5 Correlated Valuations

One of the special cases of valuations with externalities are so-called *agent-correlated* valuations, where an agent $i \in N$ receives for an item $a \in A$:

- the best value $v_{i,a}$ if a is allocated to i and
- some part $(1 - \tau_{i,j})v_{i,a}$ of the best value, $\tau_{i,j} > 0$, if a is allocated to another agent j.

One can imagine that the coefficient $\tau_{i,j}$ indicates the degree of friendship between i and j. We extend this model by adding item-correlations, represented by coefficients $\mu_{i,a}$ for each agent i and item a, so the valuations have the following form: $V = \{V_i(j, a) : i, j \in N, a \in A\}$ such that for every pair of agents $i, j \in N$ and item $a \in A$,

$$V_i(j, a) = \begin{cases} v_{i,a} & \text{if } i = j, \\ (1 - \tau_{i,j}\mu_{i,a}) \cdot v_{i,a} & \text{otherwise,} \end{cases}$$

for some $v_{i,a}$ and $\mu_{i,a}$, where $\tau_{i,j} > 0$. We call such valuations *agent-item-correlated*. Intuitively, $\mu_{i,a}$ shows how important it is for the agent i that they and no one else receives the item a. Surprisingly, it turns out that agent-correlated valuations and even their item-correlated generalizations can be reduced to the valuations without externalities and, therefore, we can use classic algorithms and guarantees from the theory of fair division without externalities (Lipton et al. 2004; Caragiannis et al. 2019; Aziz et al. 2021). In particular, we get that in these settings EF1 allocations always exist and can be found in polynomial time.

Theorem 10. *Let $\phi \in \{EF, EF1, EFX\}$. Any instance $\mathcal{I}$ of $\phi-$*FAIR DIVISION WITH EXTERNALITIES *with agent-item-correlated valuations can be transformed in linear time into the equivalent instance $\mathcal{I}'$ of $\phi-$*FAIR DIVISION *(with no externalities) with the same sets of items and agents.*

Proof. Let $\mathcal{I} = (N, A, V)$, we construct the instance $\mathcal{I}' = (N, A, U)$ of $\phi-$FAIR DIVISION with $U_i(a) = \mu_{i,a}v_{i,a}$ for every agent $i \in N$ and every item $a \in A$. To see the equivalence of $\mathcal{I}$ and $\mathcal{I}'$, fix arbitrary allocation of items π. For every pair of agents i and j, we have $V_i(\pi^{i\leftrightarrow j}) - V_i(\pi) = \sum_{a\in\pi_j}(V_i(i,a) - V_i(j,a)) + \sum_{a\in\pi_i}(V_i(j,a) - V_i(i,a)) = \sum_{a\in\pi_j}\tau_{i,j}\mu_{i,a}v_{i,a} - \sum_{a\in\pi_i}\tau_{i,j}\mu_{i,a}v_{i,a}$, or in terms of U it is $V_i(\pi^{i\leftrightarrow j}) - V_i(\pi) = \tau_{i,j}(U_i(\pi^{i\leftrightarrow j}) - U_i(\pi))$. Since $\tau_{i,j} > 0$, i envies j in $\mathcal{I}$ if and only if i envies j in $\mathcal{I}'$. In particular, the allocation π is envy-free for $\mathcal{I}'$ if and only if it is envy-free for $\mathcal{I}$. The same holds for any allocation obtained from π by removing one item. Therefore, $\mathcal{I}$ and $\mathcal{I}'$ are equivalent as instances of EF or EF1.

For EFX, it remains to show that whenever i envies j, the removal of any item from π decreases the envy in $\mathcal{I}$ if and only if it decreases the envy in $\mathcal{I}'$. Let λ be the allocation obtained from π by removing some item a_0 that decreases envy of i towards j in $\mathcal{I}$, then we have:

$$\begin{aligned}
&V_i(\lambda^{i\leftrightarrow j}) - V_i(\lambda) < V_i(\pi^{i\leftrightarrow j}) - V_i(\pi) \iff \\
&\quad \tau_{i,j}(U_i(\lambda^{i\leftrightarrow j}) - U_i(\lambda)) < \tau_{i,j}(U_i(\pi^{i\leftrightarrow j}) - U_i(\pi)) \\
&\quad\quad \iff U_i(\lambda^{i\leftrightarrow j}) - U_i(\lambda) < \tau_{i,j}U_i(\pi^{i\leftrightarrow j}) - U_i(\pi),
\end{aligned}$$

since $\tau_{i,j} > 0$, which concludes the proof. □

Examples of Agent-Item-Correlated Valuations

For one real-life scenario, imagine that our agents are partitioned into t teams $T_1, \ldots, T_t$. Let $c < 1$ be some constant. The valuation of every agent is defined as follows: for an item $a \in A$, $V_i(i,a) = v_{i,a}$. Moreover, $V_i(j,a) = c \cdot v_{i,a}$ if $j \neq i$ is a teammate of i and $V_i(j,a) = 0$ otherwise. We call such valuations *team-based*.

Therefore, in team-based valuations, agents always want to receive items themselves, but otherwise they prefer items to be given to their teammates. This situation can be captured by setting $\mu_{i,a} = 1$ along with $\tau_{i,j} = 1 - c$ if i and j are in the same team and $\tau_{i,j} = 1$ otherwise.

Corollary 11. *For any instance $\mathcal{I}$ of Fair Division with Externalities with team-based valuations, an EF1 allocation always exists and can be found in polynomial time.*

For another example of agent-item-correlated valuations, assume that the agents form a graph $G = (N, E)$, which we will call a network. For each item a and agent i, if i receives a, this also contributes to the values of any other agent j, and the contribution depends on the distance $d_{i,j}$ between i and j in the network G, namely, $V_i(j,a) = (1 - d_{i,j}\mu_{i,a}) \cdot v_{i,a}$. We call such valuations *network-based*.

Corollary 12. *For any instance $\mathcal{I}$ of Fair Division with Externalities with network-based valuations, an EF1 allocation always exists and can be found in polynomial time.*

For instance, network-based valuations would capture the following scenario: a fixed amount of new transport stops (which will be items) should be added, and there are few possible locations (corresponding to agents) for them. The goal is to ensure that from every location one can reach stops that are not too far. To bring $\mu_{i,a}$ into play, imagine that some stops are more important to have nearby then the others. For instance, it is not crucial to have a railway connection in the vicinity, but highly recommended to ensure that there are underground stations close enough.

6 Conclusions

In this work, we continue the line of research on fair division of indivisible items with externalities initiated by Aziz et al. (2023b). In contrast to previous work, we study the problem from the perspective of computational complexity. To this end, we provide strong intractability results for various restrictions of the problem. On the other hand, we provide several fixed-parameter algorithms that, together with previously mentioned hardness, paint a complete complexity picture of fair division with externalities with respect to its natural parameters. Later, we additionally focus on restricted valuations, providing many properties that lead to previously unknown existence guarantees.

Our algorithmic results leave open a very intriguing question. What is the complexity of deciding whether an EF (or even EF1/EFX) allocation exists when $|N| > |A|$ and we parameterize by the number of items? In the absence of externalities the answer to this is trivial; there is no EF allocation! With externalities though, there is a very easy XP algorithm, but for fixed-parameter tractability the problem becomes very thought provoking.

It is very common in the fair division literature to combine *fairness* of the outcome with its *efficiency*. Arguably, the most widely studied efficiency notions in the context of fair division are Pareto optimality (PO) and social welfare. In our last result, we show that these two notions on their own are easy to compute even under the presence of externalities. Therefore, it is natural to ask for a complexity picture of different combinations of fairness and efficiency.

Proposition 13. *Given an instance $\mathcal{I}$ of Fair Division with Externalities, there is a polynomial-time algorithm that finds an allocation which is Pareto optimal and maximizes utilitarian social welfare.*

Nevertheless, the most appealing question for future research is the (non-)existence of EF1 allocations under binary or general valuations, even if we have only three agents. We conjecture that for binary valuations, EF1 allocation always exists – and, therefore, due to our results, also EFX allocations always exist in this setting. However, despite many attempts, the proof seems highly non-trivial as already with the no-chores assumption, Aziz et al. (2023a) used many branching rules, ramified case distinction, and even computer program to verify some branches. For general valuations, we are more skeptical.

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