Czech Technical University
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HABILITATION THESIS

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Analysis of time discretizations for parabolic problems with application to space discretizations

Abstract

This work summarizes some of the theoretical results of the author in last ten years, where the main area of the research was the numerical analysis for the stable higher order time discretization methods applied on parabolic problems. The main discretization scheme is the time discontinuous Galerkin method in combination with the conforming finite element method or the discontinuous Galerkin method in space. The thesis presents a priori error estimates for nonstationary singularly perturbed convection-diffusion problems, stability results for the problems with the domain evolving in time and a posteriori error estimates based on the equilibrated flux reconstructions. The technique presented for a posteriori analysis in time is applied to purely spatial problem and the quality of the recontruction is investigated with respect to the degree of polynomial approximation.

Keywords

discontinuous Galerkin method, convection-diffusion equation, error analysis, a posteriori analysis, p-robustness, time discontinuous Galerkin, arbitrary Lagrangian-Eulerian description.

Thanks

I would like to thank all of my coworkers, mainly to Monika Balazsová, Vít Dolejší, Miloslav Feistauer, Václav Kučera, Hans-Görg Roos and Filip Roskovec. I would like to thank for the general support to both institutions (MFF UK and FSv ČVUT), where I was employed, and especially to the corresponding departments (Department of Numerical Mathematics and Department of Mathematics).

I would like to thank also to my family and mainly to my wife Zuzana, who tried to help me and support me all the time.

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Chapter 1

Introduction

There is number of areas for application of parabolic problems (mathematics, engineering, physics, biology, chemistry, economy, sociology, etc.). These problems are often discretized in space variables and the resulting large system of *stiff* ordinary differential equations (ODEs) needs to be solved by a suitable method. Backward differentiation formulae (BDF) were often considered as the method of the first choice for *stiff* problems, see e.g. [26], since they are robust and quite cheap. Nevertheless, BDF methods suffer from number of disadvantages. Namely, the order of convergence is limited by order 6, BDF are A-stable only to the order 2 and the robustness (area of stability) of the method decreases with the increasing order. Moreover, these methods are multi-step methods and suffers from usual disadvantages of multi-step methods in general, e.g. the necessity to define artificial starting values and stability issues connected with the step-size adaptation.

On the other hand, certain implicit Runge-Kutta methods and Galerkin time discretizations do not suffer these disadvantages. These methods are A-stable one-step methods of arbitrary order, for the overview about these methods see e.g. [29] and [30] and the citations therein. The main disadvantage of these methods that prevented the use of them in past years was their expensiveness, where the computational costs significantly increase with the order of the method. In comparison, BDF methods remain at the same cost independently of the order. Fortunately, the increase in computational power and advancements in numerical linear algebra in last two decades enabled practical applications of implicit RK or Galerkin methods. This makes implicit Runge-Kutta and Galerkin methods competitive with more traditional approaches like BDF.

This thesis presents some results achieved by the author and his coworkers in last 10 years about theoretical (numerical) analysis of Galerkin time discretizations for unsteady convection-diffusion problems. The main part of the thesis consists from 5 papers [6], [17], [34], [47] and [48] published in impact journals and presented here as Chapters 3–7. Each of these papers is presented in the same form as it is published. Therefore, all of these papers have their own individual style, page numbering, notation and references.

Chapter 3 and Chapter 4 study unsteady singularly perturbed convection-diffusion problems. The convection-diffusion problems appear in many practical applications, especially as a simplified model to Navier-Stokes equations. This problem represents a serious challenge to discretize, whenever the diffusion term is small in comparison to the other terms or data. Such a situation represents the transition state between parabolic and hyperbolic problems, where sharp boundary layers often appear. Usual finite element or finite difference discretizations fail in this situation, since they lead to the solution with highly oscillatory behavior around these layers that pollutes the solution not only in the vicinity of the layer, but at the all computational domain. The overview of discretization techniques and their analysis for linear singularly perturbed problems can be found in [41]. The analysis of unsteady linear singularly perturbed problems can be found in e.g. [1] and [16]. The application to unsteady nonlinear problems can be found in [22]. For the analysis of Runge-Kutta methods applied to hyperbolic problems see [51].

Chapter 5 is devoted to the higher order analysis of unsteady convection-diffusion problems in time dependent domains, where the domain change is driven by a given smooth mapping. There are number of approaches dealing with time dependent domain problems, e.g. the fictitious domain method or the immersed boundary method. Another popular approach is Arbitrary Lagrangian-Eulerian (ALE) method based on one-to-one ALE mapping between the current evolving domain and the fixed reference domain. ALE method was analyzed mainly for the lower (first or second) order time discretization methods in combination with the classical conforming finite element method, see e.g. [23] and [25]. Analysis of higher order discretizations based on the discontinuous Galerkin method can be found in [8], [9] and [44].

Chapter 6 studies a posteriori error estimates for nonlinear parabolic problems. The aim of this chapter is to derive a posteriori error estimates that are cheap in comparison with the original discrete problem, fully computable, reliable and locally efficient. There are number of results devoted to a posteriori error estimates for parabolic problems. Most of these results assume lower (first or second) order time discretizations, see e.g. [27] or [40]. The aposteriori analysis of linear parabolic problems discretized by higher order methods in time based on the discontinuous Galerkin method can be found in [3], [20] and [42]. Nonlinear parabolic problems and higher order time discretizations are addressed in [36], where the upper bound consists from a dual norm and therefore it is not directly computable. For a general overview on a posteriori error concepts see e.g. [45].

Chapter 7 apply the reconstruction principle developed for the time discretization in [17] to the space discretization. Moreover, the efficiency of the derived a posteriori error estimate is studied with respect to the polynomial degree in one dimension. The topic of polynomial robustness (or polynomial dependence of the estimates) is important for the save application of a posteriori error bounds in *hp*-adaptive strategies with high polynomial degrees and it started to be very popular in the community of a posteriori error analysis in recent years. The first results for residual based estimates can be found in [37]. Very important results showing complete polynomial independence of equilibrated reconstructions are in [10]. The results from [10] are applied to large number of numerical methods in [21]. Paper [20] shows a complete polynomial independence of efficiency estimates for the discontinuous Galerkin time discretization for parabolic problems.

A general overview chapter precedes these main chapters. This overview contains a brief description of Chapters 3–7. Moreover, it contains a general description of several concepts for discretizations as well as the corresponding numerical analysis. The notation in this chapter is unified for convenience of the reader and is chosen as close as possible to the notation used in following chapters. The full explanation of the ideas and the full description of the concepts from the original papers can be rather long and technical in many situations. Therefore, the precision of the formulations is not always perfect in this overview, e.g. mean values, penalization parameters, reconstructions, etc., are defined only inside of the computational domain. The complete precise formulations can be found in the original papers or in Chapters 3–7.

Chapter 2

Overview

2.1 Notation

Here, we summarize a basic notation for the upcoming discretizations.

2.1.1 Space discretization notation

Let us assume a bounded polygonal domain $\Omega \subset \mathbb{R}^d$ with Lipschitz continuous boundary. We assume a partition of this domain into closed subsets K with mutually disjoint interiors and covering $\overline{\Omega}$, often called elements. For simplicity, we assume that elements K are simplices and that the partition is conforming, i.e. that the neigbouring elements share the entire edge or face depending on the dimension d. To simplify further notation, we call these boundary objects of co-dimension 1 edges regardless of the dimension d and denote them e.

We assume patches of elements ω_a denoting the patch consisting of the elements containing the common vertex a and ω_K denoting the patch consisting of the elements surrounding K and K itself.

We assume that the elements are shaped regular, i.e. the ratio of the diameters of the inscribed and circumscribed ball is bounded. We denote the local mesh-size $h_K = \mathrm{diam}(K)$ and the global mesh-size $h = \max_K h_K$. Finally, we assume that the mesh is locally quasi-uniform, i.e. the ratio $h_K/h_{K'}$ is bounded for neighnouring elements K and K'.

Moreover, we denote unit normals on edge e as n. The direction of the normals is arbitrary but fixed for the inner edges and outward for the boundary edges.

For piece-wise discontinuous function v, we need to define one-sided values on the edges

$$v_L(x) = \lim_{\epsilon \to 0+} v(x - \epsilon n), \qquad v_R(x) = \lim_{\epsilon \to 0+} v(x + \epsilon n) \tag{2.1}$$

depending on the orientation of n, jumps and mean values

$$[v] = v_L - v_R, \qquad \langle v \rangle = \frac{v_L + v_R}{2}. \tag{2.2}$$

We denote by $(.,.)_M$ and $\|.\|_M$ $L^2(M)$-scalar product and norm, respectively. Typically, we apply this notation with $M = K$ or $M = e$. The global $L^2(\Omega)$-scalar product and norm are denoted by $(.,.)$ and $\|.\|$, respectively. We denote the sum over all elements K or over all edges e of the mesh by $\sum_K$ or $\sum_e$, respectively.

2.1.2 Time discretization notation

Let us assume time interval $I = (0, T)$, where $T > 0$. We assume time partition of $\overline{I}$ by partition nodes $0 = t_0 < t_1 < \ldots < t_r = T$. Although the papers discussed often assume a general time partition, we assume here for simplicity that the partition is equidistant, i.e. $t_m = m\tau$, where τ is a global step-size. We denote local time subintervals $I_m = (t_{m-1}, t_m)$.

Combining the space and time discretization, we denote by $(.,.)_{M,m}$ and $\|.\|_{M,m}$ $L^2(M \times I_m)$-scalar product and norm, respectively. We denote the sum over all elements of the mesh and all the time subintervals by $\sum_{K,m}$.

For any function $f(t)$ defined in $\overline{I}$ we denote one sided nodal values $f(t_m\pm) = f_\pm^m$, where the subscript $\pm$ can be omitted for continuous functions, and we denote the corresponding jump in time as $\{v\}_m = v_+^m - v_-^m$. The time derivative of function $f(t)$ is denoted as $f'(t)$.

2.2 One-step higher order time discretizations

Here, we present some classical one-step discretization techniques. For the overview see e.g. [29] and [30].

2.2.1 One-step discretizations

Let us consider ordinary differential equation (ODE)

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad t \in (0, T), \\ y(0) &= \alpha. \end{aligned} \tag{2.3}$$

Let us denote the approximate solution $\{Y^m\}_{m=0}^r$ such that $y(t_m) \approx Y^m$. We can define three classes of one-step methods.

Runge–Kutta methods: Let $a_{i,j}$, b_i, c_i, $i, j = 1, \ldots, q+1$ be suitable coefficients. Then we call the sequence Y^m satisfying $Y^0 = \alpha$

$$\begin{aligned} g_i^m &= Y^{m-1} + \tau \sum_{j=1}^{q+1} a_{i,j} f(t_{m-1} + \tau c_j, g_j^m), \quad \forall i = 1, \ldots, q+1, \\ Y^m &= Y^{m-1} + \tau \sum_{i=1}^{q+1} b_i f(t_{m-1} + \tau c_i, g_i^m) \end{aligned} \tag{2.4}$$

the Runge-Kutta (RK) solution of (2.3). The auxiliary values g_i^m called *inner stages* represent the approximation of the exact solution in $t_{m-1} + \tau c_i$.

Collocation methods: Let c_i, $i = 1, \ldots, q+1$ be suitable coefficients. Let $Y^0 = \alpha$. In every step we construct polynomial p of degree at most $q+1$ such that

$$\begin{aligned} p(t_{m-1}) &= Y^{m-1}, \\ p'(t_{m-1} + \tau c_i) &= f(t_{m-1} + \tau c_i, p(t_{m-1} + \tau c_i)), \quad \forall i = 1, \ldots, q+1. \end{aligned} \tag{2.5}$$

Then we put $Y^m = p(t_m)$. We call the resulting sequence the collocation solution of (2.3). The points $t_{m-1} + \tau c_i$ are called collocation points. The method produces a piecewise polynomial function that satisfies the original equation (2.3) in these collocation points only.

Continuous and discontinuous Galerkin method: Let us define function spaces

$$X^\tau = \{v \in L^2(0, T) : v|_{I_m} \in P^q(I_m)\}, \tag{2.6}$$

$$Y^\tau = \{v \in C(0, T) : v|_{I_m} \in P^{q+1}(I_m), v(0) = \alpha\}, \tag{2.7}$$

where P^q and P^{q+1} are spaces of polynomials of degree q and $q+1$, respectively. It should be pointed out that both these spaces have the same dimension. We call $u \in Y^\tau$ the continuous Galerkin solution of (2.3) if

$$\int_{I_m} u'(t)v(t)\mathrm{d}t = \int_{I_m} f(t,u(t))v(t)\mathrm{d}t, \quad \forall v \in X^\tau. \tag{2.8}$$

We call $u \in X^\tau$ the discontinuous Galerkin solution of (2.3) if $u_-^0 = \alpha$ and

$$\int_{I_m} u'(t)v(t)\mathrm{d}t + \{u\}_{m-1}v_+^{m-1} = \int_{I_m} f(t,u(t))v(t)\mathrm{d}t, \quad \forall v \in X^\tau. \tag{2.9}$$

For comparison with previous methods we focus mainly on endpoints of intervals: $u_-^m = Y^m \approx y(t_m)$.

The integrals in the definition of continuous and discontinuous Galerkin method are often approximated by quadratures. Suitable quadratures are Gauss or right Radau quadratures on $q+1$ quadrature nodes, respectively, since they approximate all linear terms involved in the integrals exactly. We refer to these Galerkin methods approximated by Gauss or Radau quadrature as to quadrature variants.

2.2.2 Mutual connection between Runge-Kutta methods and Galerkin methods

It is very useful in the numerical analysis to understand the mutual connections among Runge-Kutta methods, collocation methods and Galerkin methods. This connection can be described by following lemmae.

Lemma 2.2.1 *Let the RK coefficients be chosen in the following way*

$$a_{i,j} = \int_0^{c_i} \ell_j(t)\mathrm{d}t, \quad i,j = 1,\dots,q+1, \tag{2.10}$$

$$b_i = \int_0^1 \ell_i(t)\mathrm{d}t, \quad i = 1,\dots,q+1, \tag{2.11}$$

where ℓ_i is the Lagrange interpolation basis function

$$\ell_i(t) = \prod_{j\neq i} \frac{t-c_j}{c_i-c_j}. \tag{2.12}$$

Then the values g_i^m, $i = 1,\dots,q+1$ and Y^m produced by such a RK method are equal to the values $p(t_{m-1}+\tau c_i)$, $i = 1,\dots,q+1$ and Y^m produced by the collocation method with the same coefficients c_i.

The proof can be found in [28] or [50].

Lemma 2.2.2 *Let $p \in P^{q+1}$ be the collocation polynomial on I_m associated to the collocation method with coefficients c_i chosen as Gauss quadrature nodes on $(0,1)$, $u \in P^{q+1}$ be the quadrature variant of continuous Galerkin solution on I_m. Then*

$$p(t) = u(t). \tag{2.13}$$

Lemma 2.2.3 *Let $p \in P^{q+1}$ be the collocation polynomial on I_m associated to the collocation method with coefficients c_i chosen as right Radau quadrature nodes, $u \in P^q$ be the quadrature variant of discontinuous Galerkin solution on I_m and $r_m \in P^{q+1}$ satisfy $r_m(t_{m-1}) = 1$, $r_m(t_m) = 0$ and $r_m \perp P^{q-1}$ on I_m. Then*

$$p(t) = u(t) - \{u\}_{m-1}r_m(t). \tag{2.14}$$

The proof for continuous Galerkin version can be found directly in [31]. The proof for discontinuous Galerkin version can be made similarly, see e.g. [49].

Summarizing these results, it is possible to realize that both Galerkin methods (up to corresponding quadrature and mild reconstruction (2.14) in case of discontinuous version) are special variants of the collocation methods and the collocation methods are special variants of the implicit Runge-Kutta methods. This can be exploited in the numerical analysis by application of the knowledge from one area to another area, especially by using the results about very well understood Runge-Kutta methods for the analysis of the Galerkin methods. The variants of Runge-Kutta methods corresponding to continuous and discontinuous Galerkin method are well known Kuntzmann-Butcher method (also known as Gauss-Legendre method) and Radau IIA method, respectively. For more details see [35] and [18], respectively.

2.3 Discontinuous Galerkin space discretization

Although most of the papers in this thesis are devoted to the time discretization techniques and their analysis, the space discretization is often made with the aid of the discontinuous Galerkin method. We shall briefly describe the discontinuous Galerkin method on simplified example of the Poisson equation

$$-\Delta u = f, \quad \text{in } \Omega. \tag{2.15}$$

We assume for simplicity the homogeneous Dirichlet boundary conditions. The other possibilities can be found in [15].

We apply the notation from Section 2.1.1. The difference between the classical finite element method and the discontinuous Galerkin method is in application of the discontinuous finite element space

$$X_h = \{v \in L^2(\Omega) : v|_K \in P^p(K)\}. \tag{2.16}$$

Since $X_h \not\subset H^1_0(\Omega)$, we could not apply the week formulation of problem (2.15) directly. In fact, we enhance the classical week formulation with additional terms. Among many variants of the discontinuous Galerkin method, one of the most popular approaches is the interior penalty method

$$\begin{aligned}(-\Delta u, v) \approx A_h(u,v) = &\sum_K (\nabla u, \nabla v)_K - \sum_e (\langle \nabla u\rangle \cdot n, [v])_e \\ &-\theta \sum_e (\langle \nabla v\rangle \cdot n, [u])_e + \sum_e (\alpha[u],[v])_e,\end{aligned} \tag{2.17}$$

where the choice of the parameter $\theta = 1, 0, -1$ corresponds to the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variant. The parameter α is usually chosen as

$$\alpha = \frac{C_W}{h_e}, \tag{2.18}$$

where h_e is some intermediate value between h_K and $h_{K'}$ for neigbouring elements K and K' sharing the edge e. The constant $C_W > 0$ needs to be chosen large enough to guarantee the positivity of $A_h(.,.)$ on X_h. The detailed information about the suitable choice of the constant C_W can be found in [15].

The resulting discrete formulation of problem (2.15) is: find $u_h \in X_h$ such that

$$A_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in X_h. \tag{2.19}$$

The corresponding error analysis can be found in [15].

2.4 Analysis of discontinuous Galerkin time discretization

In this section is described the most common approach to the derivation of a priori error estimates for the discontinuous Galerkin time discretization of parabolic problems. For simplicity, let us assume the heat equation

$$\begin{aligned} u' - \Delta u &= f, \quad \text{in } \Omega \times (0,T) \\ u(0) &= u^0, \quad \text{in } \Omega \end{aligned} \tag{2.20}$$

with homogeneous Dirichlet boundary condition.

We apply the notation from Section 2.1.1 and Section 2.1.2. We discretize this problem in space by the classical finite element method with the finite element space

$$X_h = \{v \in H_0^1(\Omega) : v|_K \in P^p(K)\}. \tag{2.21}$$

The resulting semidiscrete problem assumes the solution $u_h \in C^1(0,T,X_h)$ such that

$$\begin{aligned} (u_h', v) + (\nabla u_h, \nabla v) &= (f, v), \quad \forall v \in X_h \\ (u_h(0), v) &= (u^0, v), \quad \forall v \in X_h. \end{aligned} \tag{2.22}$$

The semidiscrete problem (2.22) represents the system of ODEs that can be solved by the discontinuous Galerkin method. Similarly as in Section 2.2.1, we define the fully discrete space

$$X_h^\tau = \{v \in L^2(0,T,X_h) : v|_{I_m} \in P^q(I_m, X_h)\}. \tag{2.23}$$

Then the fully discrete solution $U \in X_h^\tau$ satisfies

$$\begin{aligned} \int_{I_m} (U', v) + (\nabla U, \nabla v)\mathrm{d}t + (\{U\}_{m-1}, v_+^{m-1}) &= \int_{I_m} (f, v)\mathrm{d}t, \quad \forall v \in X_h^\tau, \\ (U_-^0, v) &= (u^0, v), \quad \forall v \in X_h. \end{aligned} \tag{2.24}$$

We may apply the technique of the error analysis described in [43]. Typically, we are interested in upper bounds of the error $e = U - u$ and most often in the nodes of the time partition t_m, i.e. $e_-^m = U_-^m - u(t_m)$. The error analysis most often consists from construction of suitable projection π on X_h^τ and dividing the error into projection part of the error $\eta = \pi u - u$, i.e. the error of the projection of the exact solution, and the rest of the error $\xi = U - \pi u \in X_h^\tau$. We gain the error equation by integrating relation (2.20) in weak form over I_m and subtracting this relation from (2.24). After dividing the error into ξ and η we gain for any $v \in X_h^\tau$

$$\begin{aligned} \int_{I_m} (\xi', v) + (\nabla \xi, \nabla v)\mathrm{d}t + (\{\xi\}_{m-1}, v_+^{m-1}) = &- \int_{I_m} (\nabla \eta, \nabla v)\mathrm{d}t \\ &- \left(\int_{I_m} (\eta', v)\mathrm{d}t + (\{\eta\}_{m-1}, v_+^{m-1}) \right). \end{aligned} \tag{2.25}$$

The most usual projection $\pi : L^2(0,T,L^2(\Omega)) \to X_h^\tau$ is defined as

$$\begin{aligned} \int_{I_m} (\pi u - u, v t^j)\mathrm{d}t &= 0, \quad \forall v \in X_h,\ j \le q-1, \\ ((\pi u)_-^m, v) &= (u(t_m), v), \quad \forall v \in X_h. \end{aligned} \tag{2.26}$$

Advantage of this projection is that the terms on the second row of (2.25) vanish for any $v \in X_h^\tau$. Setting $v = 2\xi$ we gain

$$2\int_{I_m} (\xi', \xi)\mathrm{d}t + 2(\{\xi\}_{m-1}, \xi_+^{m-1}) = \|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 + \|\{\xi\}_{m-1}\|^2, \qquad (2.27)$$

cf [19]. Using (2.27) together with Cauchy inequality gives the error estimate for $\|\xi_-^m\|$ in terms of η

$$\|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 + \|\{\xi\}_{m-1}\|^2 + \int_{I_m} \|\nabla\xi\|^2\mathrm{d}t \le \int_{I_m} \|\nabla\eta\|^2\mathrm{d}t. \qquad (2.28)$$

The estimate

$$\int_{I_m} \|\nabla\eta\|^2 \le C\tau(h^{2p} + \tau^{2q+2}), \qquad (2.29)$$

where the constant C depends on the corresponding derivatives of the exact solution u, are most often derived by the standard scaling argument using Bramble-Hilbert trick applied for Bochner spaces, see e.g. [46]. Since η_-^m is the error of L^2-orthogonal projection of u^m that satisfies $\|\eta_-^m\| \le Ch^{p+1}$, we gain from (2.28) and (2.29) the final desired estimate

$$\|e_-^m\| = \|U_-^m - u^m\| \le \|\xi_-^m\| + \|\eta_-^m\| \le C(h^p + \tau^{q+1}). \qquad (2.30)$$

This estimate is usually considered optimal with respect to the polynomial degree in time, but suboptimal with respect to the polynomial degree in space, since h^{p+1} is usually expected for the finite element error in L^2-norm. The improvement to h^{p+1} can be found in [46]. Moreover, the basic theory of Runge-Kutta methods suggests that the nodal errors should converge with the rate τ^{2q+1} instead of τ^{q+1}. This faster convergence in a finite element setting is usually described as nodal *superconvergence*. Unfortunately, these faster rates appear only exceptionally for parabolic problems when certain compatibility conditions are met, cf. [3]. This *order reduction* phenomenon is analyzed in [11]. See also [24], where the investigation of convergence rate τ^{q+2} for $q \ge 1$ is presented.

2.5 A posteriori error estimates

Let us consider the Poisson problem

$$-\Delta u = f, \quad \text{in } \Omega, \qquad (2.31)$$

where we assume for simplicity the homogeneous boundary condition. The resulting weak solution of problem (2.31) satisfies $u \in H_0^1(\Omega)$. Moreover, it is possible to find out that

$$\nabla u \in H(\mathrm{div}, \Omega) = \{w \in L^2(\Omega)^d : \nabla \cdot w \in L^2(\Omega)\} \qquad (2.32)$$

whenever the right-hand side $f \in L^2(\Omega)$.

Denoting

$$X_h = \{v \in H_0^1(\Omega) : v|_K \in P^p(K)\} \qquad (2.33)$$

the finite element space, we can define the finite element solution $u_h \in X_h$ satisfying

$$(\nabla u_h, \nabla v) = (f, v), \quad \forall v \in X_h. \qquad (2.34)$$

In comparison with a priori analysis, where the convergence of the error with respect to the discretization data is studied and the error bound typically depends on the high derivatives of the unknown exact solution, a posteriori error analysis provides the error bounds depending on the discrete solution itself. There are many techniques for a posteriori error estimates, for overview see e.g. [45].

The goal of this section is to briefly describe the upper bound construction to the error $u_h - u$ by the so called *equilibrated flux reconstruction* technique. The resulting a posteriori error estimate can be viewed as a generalization of the hyper-circle theorem, cf. [39].

Theorem 2.5.1 (Hyper-circle) *Let $u \in H^1_0(\Omega)$ be the exact solution of problem (2.31), $\sigma \in H(\mathrm{div}, \Omega)$ satisfies $f + \nabla \cdot \sigma = 0$ and $v \in H^1_0(\Omega)$ be arbitrary. Then*

$$\|\nabla u - \nabla v\|^2 + \|\sigma - \nabla u\|^2 = \|\nabla v - \sigma\|^2. \tag{2.35}$$

When such a σ is available, then the estimate can be achieved be setting $v = u_h$ and omitting the term $\|\sigma - \nabla u\|^2$, i.e. $\|\nabla u - \nabla u_h\| \leq \|\nabla u_h - \sigma\|$.

Unfortunately, it is not easy to find a suitable $\sigma \in H(\mathrm{div}, \Omega)$ satisfying $f + \nabla \cdot \sigma = 0$ globally. Here, we describe the construction of $\sigma \approx \sigma_h \in H(\mathrm{div}, \Omega)$ that satisfies the relation $f + \nabla \cdot \sigma_h = 0$ in a weaker sense. Let us denote the local Raviart-Thomas space on element K as $\mathrm{RT}(K) = xP^p(K) + (P^p(K))^d$. This space is the usual finite element approximation space to $H(\mathrm{div}, \Omega)$ in the mixed finite element method. For the overview on the mixed finite element method and corresponding polynomial approximations see e.g. [7]. We construct the extension of Raviart-Thomas space to patches ω_a for given vertex a

$$W(\omega_a) = \{w \in H(\mathrm{div}, \omega_a) : w|_K \in \mathrm{RT}(K),\, w|_{\partial\omega_a} \cdot n = 0\}. \tag{2.36}$$

Denoting the space $P^p_*(\omega_a)$ as the space of piece-wise polynomial functions with zero mean value, we can formulate the local *patch-wise* mixed finite element problem: find $\sigma_a \in W(\omega_a)$ and $r_a \in P^p_*(\omega_a)$ such that

$$\begin{aligned} (\sigma_a, v)_{\omega_a} - (r_a, \nabla \cdot v)_{\omega_a} &= (\psi_a \nabla u_h, v)_{\omega_a}, \quad \forall v \in W(\omega_a), \\ (\nabla \cdot \sigma_a, \varphi)_{\omega_a} &= (\nabla \psi_a \cdot \nabla u_h - \psi_a f, \varphi)_{\omega_a}, \quad \forall \varphi \in P^p_*(\omega_a), \end{aligned} \tag{2.37}$$

where ψ_a is the hat function associated with the vertex a and serves as the discrete decomposition of the unity. The final reconstruction σ_h is the sum of all the local contributions σ_a, i.e. $\sigma_h = \sum_a \sigma_a$. Since each of the local contributions $\sigma_a \in H(\mathrm{div}, \Omega)$ if prolongated by zero outside of the patch ω_a then also the complete reconstruction satisfies $\sigma_h \in H(\mathrm{div}, \Omega)$. Moreover, it is possible to show that

$$(f + \nabla \cdot \sigma_h, 1)_K = 0. \tag{2.38}$$

The property (2.38) is usually called the flux equilibration property.

We can derive the error bound using the reconstruction σ_h. Let us assume $v \in H^1_0(\Omega)$. Then

$$(f, v) - (\nabla u_h, \nabla v) = (f + \nabla \cdot \sigma_h, v) + (\sigma_h - \nabla u_h, \nabla v). \tag{2.39}$$

Estimating these terms individually and using the flux equilibration property (2.38) we get

$$(f, v) - (\nabla u_h, \nabla v) \leq \sum_K (C_P h_K \|f + \nabla \cdot \sigma_h\|_K + \|\sigma_h - \nabla u_h\|_K) \|\nabla v\|_K, \tag{2.40}$$

where C_P is the known constant form the Poincare inequality, cf. [38]. Since

$$\|\nabla u - \nabla u_h\| = \sup_{v \in H^1_0(\Omega)} \frac{(\nabla u - \nabla u_h, \nabla v)}{\|\nabla v\|} = \sup_{v \in H^1_0(\Omega)} \frac{(f, v) - (\nabla u_h, \nabla v)}{\|\nabla v\|}, \tag{2.41}$$

we can conclude that

$$\|\nabla u - \nabla u_h\|^2 \leq \sum_K (C_P h_K \|f + \nabla \cdot \sigma_h\|_K + \|\sigma_h - \nabla u_h\|_K)^2. \tag{2.42}$$

The estimate (2.42) is a guaranteed upper bound and the right-hand side contains only the terms that are fully computable from the discrete solution u_h. Since the construction of σ_h is based on the local problems only, cf. (2.37), the evaluation of this reconstruction σ_h as well as the evaluation of the estimator itself is essentially computationally cheaper than the original problem (2.34).

It is possible to provide the efficiency estimates, i.e. the opposite bounds, for the individual local estimators. These estimates are traditionally done under the assumption that the right-hand side f is a piece-wise polynomial, otherwise the additional oscillation term appears in the estimates. Denoting by $\lesssim$ the inequality up to some fixed constant that does not depend on the exact solution u nor the discrete solution u_h nor the mesh-size h, it is possible to derive following local efficiency estimates

$$\begin{aligned} h_K \|f + \nabla \cdot \sigma_h\|_K &\lesssim \|\nabla u - \nabla u_h\|_{\omega_K}, \\ \|\sigma_h - \nabla u_h\|_K &\lesssim \|\nabla u - \nabla u_h\|_{\omega_K}, \end{aligned} \tag{2.43}$$

see e.g. [21]. The proofs are quite technical and therefore they are skipped in this overview. These estimates (2.43) show that large local estimators correspond to large local contributions to the complete error. This property is important for the identifications of the source of the error in possible adaptive strategies. Unfortunately, the locality grows from elements K to patches ω_K.

2.6 Overview of Chapter 3: Linear unsteady singularly perturbed convection-diffusion problems

Chapter 3 is based on the paper *An optimal uniform a priori error estimate for an unsteady singularly perturbed problem* published in International Journal of Numerical Analysis and Modeling in 2014, [48].

The paper deals with the numerical analysis of unsteady singularly perturbed convection-diffusion problems on a square $\Omega = (0,1)^2$

$$u' - \varepsilon\Delta u + b \cdot \nabla u + cu = f \quad \text{in } \Omega \times (0,T) \tag{2.44}$$

with homogeneous Dirichlet boundary condition and corresponding initial condition. The paper presents an optimal a priori error estimate for any general mesh-adapted space discretization and discontinuous Galerkin time discretization.

The paper [48] assumes a singularly perturbed case, where the diffusion coefficient ε is small in comparison to the rest of the data. The goal of the paper is to derive error estimates for mesh-adapted spatial methods in combination with the discontinuous Galerkin method in time that are independent of ε.

2.6.1 Discretization

A possible remedy comes from two different sources: high adaptation of the meshes around the layers (Shishkin meshes, Bakhvalov meshes, etc.) and stabilizations of the method (SUPG, local projection stabilization, etc.), see e.g. [41]. Both approaches are very often used together. The paper assumes the layer-adaptated

S-type meshes in combination with any consistent discretization method either stabilized or not.

The construction of the mesh in each direction is similar. Therefore we describe them in x direction only. Let us assume increasing and differentiable generating function ϕ satisfying $\phi(0) = 0$ and $\phi(1/2) = \ln(N)$, where $N + 1$ is number of discretization nodes in x direction including boundaries. Then the partition nodes x_i can be defined by

$$x_i = \frac{2i}{N}\left(1 - \frac{\sigma\varepsilon}{\beta_1}\phi\left(\frac{1}{2}\right)\right), \quad \forall i = 0, \dots, N/2 \tag{2.45}$$

$$x_i = 1 - \frac{\sigma\varepsilon}{\beta_1}\phi\left(\frac{N-i}{N}\right), \quad \forall i = N/2, \dots, N, \tag{2.46}$$

where $b = (\beta_1, \beta_2)$ and $\sigma \geq 5/2$. For instance, classical Shishkin mesh coresponds to the choice $\phi(s) = 2\ln(N)s$ and the choice $\phi(s) = -\ln(1 - 2s(1 - N^{-1}))$ coresponds to Bakhvalov-type mesh. Such an approach leads to the ε-uniform spatial error estimates even with respect to resulting norms of the exact solution. For the detail see e.g. [41].

The discretization in space is made with the aid of conforming bilinear space V_N, bilinear form $a_{st}(.,.)$ representing the discretization of the spatial terms from (2.44) and corresponding right-hand side f_{st}

$$(u', v) + a_{st}(u, v) = (f_{st}, v), \quad \forall v \in V_N. \tag{2.47}$$

The time discretization is made using the discontinuous Galerkin discretization described in Section 2.2.1, i.e. discrete solution $U \in V_N^\tau$ satisfies

$$\int_{I_m} (U', v) + a_{st}(U, v)\mathrm{d}t + (\{U\}_{m-1}, v_+^{m-1}) = \int_{I_m} (f_{st}, v)\mathrm{d}t \quad \forall v \in V_N^\tau, \tag{2.48}$$

where

$$V_N^\tau = \{v \in L^2(0, T, V_N) : v|_{I_m} \in P^q(I_m, V_N)\}. \tag{2.49}$$

Let us denote right Radau quadrature on $q + 1$ quadrature nodes

$$\int_{I_m} f(t)\mathrm{d}t \approx Q^m[f]. \tag{2.50}$$

Assuming for simplicity that f or f_{st}, respectively, is a polynomial in time of the same degree as the discrete solution we can replace all the integrals in (2.48) by the right Radau quadratures, since all the terms in (2.48) are linear, i.e.

$$Q^m[(U', v)] + Q^m[a_{st}(U, v)] + (\{U\}_{m-1}, v_+^{m-1}) = Q^m[(f_{st}, v)], \quad \forall v \in V_N^\tau. \tag{2.51}$$

2.6.2 Error analysis

We may apply a similar technique of the proof as in Section 2.4. We design a suitable projection π and divide the error into projection part $\eta = \pi u - u$ and $\xi = U - \pi u \in V_N^\tau$. Then the error equation is as follows

$$\begin{aligned} Q^m[(\xi', v)] + Q^m[a_{st}(\xi, v)] + (\{\xi\}_{m-1}, v_+^{m-1}) = &-Q^m[a_{st}(\eta, v)] \\ &- Q^m[(\eta', v)] - (\{\eta\}_{m-1}, v_+^{m-1}) \end{aligned} \tag{2.52}$$

There are two sources of difficulties in the error analysis comparing with the analysis presented in Section 2.4. The first difficulty is that it is not possible to provide ellipticity and continuity estimates of $a_{st}(.,.)$ in any norm in such a way that the constants in these estimates would be independent of ε and N. The second difficulty is that L^2-orthogonal projection on V_N that is essentially involved in the definition of space-time projection π in Section 2.4 is very unsuitable for deriving accurate error estimates with respect to space variables for this specific problem, see e.g. [32], where suboptimal error analysis is presented due to this fact.

To overcome these difficulties, the projection π is designed differently to respect the Runge-Kutta nature of the discontinuous Galerkin method in time, cf. Section 2.2.2, and with the aid of classical Ritz projection in space, namely $\pi = P^\tau R_N$, where P^τ is the Lagrange interpolation operator on right Radau quadrature nodes and $R_N : H_0^1(\Omega) \to V_N$ is the Ritz projection satisfying

$$a_{st}(R_N u - u, v) = 0, \quad \forall v \in V_N. \tag{2.53}$$

Then it is possible to show that

$$\sup_{I_m} \|\pi u - u\| \leq C(\tau^{q+1} + g(N)), \tag{2.54}$$

where the constant C is completely independent of ε even with respect to the derivatives of the exact solution u and the term $g(N)$ depends on the choice of the mesh adaptation and the stabilization, e.g. $g(N) = N^{-2}\ln^2(N)$ for the Shishkin mesh or $g(N) = N^{-2}$ for the Bakhvalov mesh when the classical bilinear finite element method without any stabilization is used.

The advantage of this projection π described above is that the energy term $Q^m[a_{st}(\eta, v)]$ vanishes in (2.52) and it is only necessary to deal with the terms on the second row of the right hand side of (2.52). Following estimate is derived for these terms in the paper [48] or in Chapter 3

$$Q^m[(\eta', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) \leq \tau C(\tau^{q+1} + g(N)) \sup_{I_m} \|v\|. \tag{2.55}$$

2.6.3 Estimates inside of intervals I_m

Since the estimate (2.55) contains the supremum over I_m, we need to handle this supremum term which represents a significant difficulty in comparison with the basic approach described in Section 2.4, where only the nodal values need to be handled. Following paper [2], it can be shown that

$$2Q^m[(\xi', \tilde{\xi})] + 2(\{\xi\}_{m-1}, \tilde{\xi}_+^{m-1}) = \|\xi_-^m\|^2 + \frac{1}{\tau} Q^m[\|\tilde{\xi}\|^2], \tag{2.56}$$

where

$$\tilde{\xi} = P^\tau \left(\frac{\tau \xi(t)}{t - t_{m-1}} \right) \in V_N^\tau. \tag{2.57}$$

Moreover, it is possible to show that

$$0 \leq Q^m[a_{st}(\xi, \xi)] \leq Q^m[a_{st}(\xi, \tilde{\xi})]. \tag{2.58}$$

Since the norms

$$\frac{1}{\tau} Q^m[\|\tilde{\xi}\|^2], \qquad \sup_{I_m} \|\xi\|^2, \qquad \sup_{I_m} \|\tilde{\xi}\|^2 \tag{2.59}$$

are equivalent, we can apply these relations to derive the error estimate. The details are shown in the paper [48] or in Chapter 3.

The interesting question arise: Why the choice of the test function $\tilde{\xi}$ defined in (2.57) gives such a nice result (2.56)? The answer can be found in the connection between Runge-Kutta methods and discontinuous Galerkin methods described in Section 2.2.2 and in the classical analysis for the error estimates of the inner stages of RK. For the overview see e.g. [30], where the results from the original paper [13] are presented.

2.7 Overview of Chapter 4: Semilinear unsteady singularly perturbed convection-diffusion problems

Chapter 4 is based on the paper *A priori diffusion-uniform error estimates for nonlinear singularly perturbed problems: BDF2, midpoint and time DG* published in Mathematical Modelling and Numerical Analysis in 2017, [34].

The paper deals with the numerical analysis of unsteady singularly perturbed semilinear convection-diffusion problems

$$u' - \varepsilon\Delta u + \nabla \cdot f(u) = g \quad \text{in } \Omega \times (0,T) \tag{2.60}$$

with homogeneous Dirichlet boundary condition and corresponding initial condition. The paper presents a priori error estimates for discontinuous Galerkin space discretization in combination with either the second order backward differentiation formula (BDF2) or the midpoint rule or the discontinuous Galerkin method in time.

Once again, we are mostly interested in the singularly perturbed situation, where the parameter ε is small. Since the problem (2.60) is nonlinear, it represents even more difficult challenge then the linear problem from Section 2.6 respectively from [48].

2.7.1 Discretization

We can apply the same notation as in Section 2.1.1 and Section 2.1.2. The space discretization is made with the aid of the discontinuous Galerkin method. The diffusion term $-\Delta u$ is discretized by SIPG formulation described in Section 2.3. The discretization of the convective term $\nabla \cdot f(u)$ is made similarly as in the finite volume method

$$(\nabla \cdot f(u), v) \approx b_h(u,v) = -\sum_K (f(u), \nabla v)_K + \sum_e (H(u_L, u_R, n), [v])_e, \tag{2.61}$$

where the flux $f(u) \cdot n$ is approximated on the edge e by the value $H(u_L, u_R, n)$ called *numerical flux*. We assume that the numerical flux can be arbitrary function satisfying following assumptions

- $H(u,v,n)$ is Lipschitz continuous, i.e.

$$|H(u,v,n) - H(\bar{u},\bar{v},n)| \le C(|u-\bar{u}| + |v-\bar{v}|), \tag{2.62}$$

- $H(u,v,n)$ is consistent, i.e.

$$H(u,u,n) = f(u) \cdot n, \tag{2.63}$$

- $H(u,v,n)$ is conservative, i.e.

$$H(u,v,n) = -H(v,u,-n), \tag{2.64}$$

- $H(u,v,n)$ is E-flux, i.e.

$$H(u,v,n) - f(q)\cdot n \geq 0, \quad \forall q \ \text{ between } \ u,v. \tag{2.65}$$

We shall point out that every monotone numerical flux is E-flux.

The semidiscrete formulation of problem (2.60) is

$$(u_h', v) + \varepsilon A_h(u_h, v) + b_h(u_h, v) = (g, v), \quad \forall v \in X_h. \tag{2.66}$$

This problem is discretized in time by either BDF2

$$\begin{aligned}\left(\frac{3}{2}U^m - 2U^{m-1} + \frac{1}{2}U^{m-2}, v\right) + \tau\varepsilon A_h(U^m, v) + \tau b_h(U^m, v) \\ = \tau(g, v) \quad \forall v \in X_h,\end{aligned} \tag{2.67}$$

where the starting value U^1 is obtained by the backward Euler method, or by the midpoint rule

$$\begin{aligned}(U^m - U^{m-1}, v_h) + \frac{\tau}{2}\varepsilon A_h(U^m + U^{m-1}, v_h) + \tau b_h\left(\frac{U^m + U^{m-1}}{2}\right) \\ = \tau(g(t_{m-1} + \tau/2), v_h), \quad \forall v_h \in X_h\end{aligned} \tag{2.68}$$

or by the quadrature version of the discontinuous Galerkin method

$$\begin{aligned}\int_{I_m} (U', v) + \varepsilon A_h(U, v)\mathrm{d}t + Q^m[b_h(U, v)] + (\{U\}_{m-1}, v_+^{m-1}) = Q^m[(g, v)], \\ \forall v_h \in X_h^\tau,\end{aligned} \tag{2.69}$$

where

$$X_h^\tau = \{v \in L^2(0, T, X_h) : v|_{I_m} \in P^q(I_m, X_h)\} \tag{2.70}$$

and the right Radau quadrature $Q^m[.]$ is defined in Section 2.6.1.

2.7.2 Error analysis

The error analysis follows the idea from the paper [33] following the results from the paper [51]. The complete description of the idea is quite long and very technical. Here, we summarize the most important steps.

The resulting nonlinear form $b_h(.,.)$ with Lipschitz continuous, consistent and conservative numerical fluxes with the E-flux property satisfies following important estimate

$$b_h(v_h, v_h - \Pi u) - b_h(u, v_h - \Pi u) \leq C\left(1 + \frac{\|v_h - u\|^2_{L^\infty(\Omega)}}{h^2}\right)(h^{2p+1} + \|v_h - \Pi u\|^2), \tag{2.71}$$

where Π is L^2-orthogonal projection on X_h, u is any sufficiently regular function and $v_h \in X_h$, cf. [33]. Difficulties come from the term $\|v_h - u\|^2_{L^\infty(\Omega)}/h^2$, where v_h is typically chosen as the discrete solution U or some term directly derived from U . If it is possible to estimate a priori the error as $\|U - u\|_{L^\infty(\Omega)} = O(h)$, then standard application of the technique will give the desired error estimate that is usually much smaller then the considered bound $O(h)$, typically it is $\|U - u\|^2 \leq C(h^{2p+1} + \tau^{2q+2})$, where $q = 1$ for BDF2 and the midpoint rule and q is the degree of the polynomial approximation in time for the discontinuous Galerkin method. Unfortunately, it is not easy to prove the error bound $O(h)$ a priori, since the error is the object of investigation and is unknown.

This problem is solved by the continuous mathematical induction, cf [33]. Let us assume that the discretization parameters h, p, τ and s are chosen in such a way that

$$\|U-u\|^2 \leq C(h^{2p+1}+\tau^{2q+2}) \quad \Longrightarrow \quad \|U-u\|_{L^\infty(\Omega)} \leq \frac{h}{2}. \tag{2.72}$$

If the error is represented by a continuous function and if the error is at some point $t = t_*$ sufficiently small, e.g. $\|U-u\|_{L^\infty(\Omega)} = h/2$, then it takes some time $\delta > 0$ to grow the error over the bound h. Then it is possible to avoid the term $\|U-u\|^2_{L^\infty(\Omega)}/h^2$ on interval $[t_*, t_*+\delta]$ in the estimate (2.71) and it is possible to derive the desired error estimate $\|U-u\|^2 \leq C(h^{2p+1}+\tau^{2q+2})$ on $[t_*, t_*+\delta]$ by rather standard technique, where the constant is independent of ε. Moreover, it is possible to see that (2.72) implies $\|U-u\|_{L^\infty(\Omega)} \leq \frac{h}{2}$ at a new time $t = t_*+\delta$. Since the continuity of the error holds on the bounded interval $[0, T]$, i.e. on a compact set, there exists a minimal finite δ necessary for such a grow and we can deplete the set $[0, T]$ in a finite number of steps. It should be pointed out that the starting error in the initial condition is inherently small.

Alternatively, the concept of the continuous mathematical induction can be replaced by the argument that the error under the assumption (2.72) cannot hit the value $\|U-u\|_{L^\infty(\Omega)} = h$ and assuming the error evolves continuously and is started from the small initial condition error it is not possible to grow over h and therefore the square of the error behaves as $O(h^{2p+1}+\tau^{2q+2})$.

2.7.3 Discrete solution continuation

Since we assume that the exact solution u is continuous in time, the aim of this section is to describe how to reconstruct the discrete solution that is defined nodal-wise as U^m in the case of BDF2 and the midpoint rule and interval-wise (element-wise) as $U|_{I_m}$ in the case of the time discontinuous Galerkin, as a continuous function $U(t)$ that corresponds to the error in the nodal point, i.e. $U(t_m) = U^m$ or $U(t_m) = U^m_-$.

The idea of the construction of the nodal-wise defined solution as a continuous function can be found in [33], where the backward Euler method is discussed. Let us assume that the continuation is well defined on the interval $[0, t_{m-1}]$ and the goal is to define the continuation on the next time interval $(t_{m-1}, t_m]$. Then the value of $U(t_{m-1}+s)$, where $s \in (0, \tau]$, is defined as the discrete solution for the given method by replacing the step-size τ by the new step-size s. Still, it remains to prove a number of technical results that imply that the resulting continuation $U(t_{m-1}+s)$ exists uniquely for arbitrary $s \in (t_{m-1}, t_m]$ and that the resulting function $U(t)$ is really continuous. These results are described in detail in the paper [34] or Chapter 4. It should be pointed out that the BDF2 analysis also applies the stability theory for the multistep methods with non-equidistant time steps, see e.g. [30].

The time discontinuous Galerkin discretization is more complicated, since the solution at the final time of each interval depends on the corresponding inner stages, see Section 2.2.2, and the continuation should respect this fact. Let us assume that the continuation is constructed to the time level $y = t_{m-1}$. The approach from [34] defines the continuation on $(t_{m-1}, t_m]$ as a set of functions on I_m

$$\{U_y\}_{y\in(t_{m-1},t_m]} \subset X_h^\tau. \tag{2.73}$$

Denoting $s \in (0, \tau]$ such that $y = t_{m-1}+s$ and denoting Radau quadrature rescaled from interval I_m to the new interval (t_{m-1}, y) as $Q^m_s[.]$, then each function U_y of the continuation is defined on I_m as

$$\int_{t_{m-1}}^{y} (U_y', v) + \varepsilon A_h(U_y, v)\mathrm{d}t + Q_s^m[b_h(U_y, v)] + (\{U\}_{m-1}, v_+^{m-1}) = Q_s^m[(f, v)], \quad \forall v_h \in X_h^\tau. \tag{2.74}$$

The resulting *continuity* is described by the relations

$$\sup_{(t_{m-1},\min(y,\bar{y}))} \|U_y - U_{\bar{y}}\| \to 0, \text{ as } |y - \bar{y}| \to 0, \tag{2.75}$$

$$\sup_{(t_{m-1},y)} \|U_y - U_-^{m-1}\| \to 0, \text{ as } y \to t_{m-1}+ . \tag{2.76}$$

The proof of this continuity with respect to y is very technical and the details are presented in the paper [34] or Chapter 4.

2.8 Overview of Chapter 5: Nonlinear unsteady convection-diffusion problems in time-dependent domains

Chapter 5 is based on the paper *Stability of the ALE space-time discontinuous Galerkin method for nonlinear convection-diffusion problems in time-dependent domains* published in Mathematical Modelling and Numerical Analysis in 2018, [6].

The paper deals with the numerical analysis of unsteady nonlinear convection-diffusion problems

$$u' - \nabla \cdot (\beta(u)\nabla u) + \nabla \cdot f(u) = g \tag{2.77}$$

with Dirichlet boundary conditions and corresponding initial condition. The nonlinearity in the diffusion term described by the function $\beta(u)$ is considered bounded and Lipschitz continuous, i.e.

$$\beta : \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 \le \beta_1 < \infty, \tag{2.78}$$

$$|\beta(u) - \beta(v)| \le C|u - v|. \tag{2.79}$$

The paper [6] does not assume the singularly perturbed case, where $\beta_0 \to 0$, but the dependence of the derived results on β_0 is tracked for further investigations.

In comparison with previous sections, the problem (2.77) is not considered in a fixed space-time cylinder $\Omega \times (0,T)$, but in an *evolving* space-time cylinder $\Omega_t \times (0,T)$, where the space domain Ω_t depends smoothly on time t. The goal of the paper [6] is to present a stability bound for discontinuous Galerkin space-time discretization.

2.8.1 Arbitrary Lagrangian-Eulerian description

The evolution of the domain Ω_t is described by a one-to-one mapping $\mathcal{A}_t : \Omega_{\mathrm{ref}} \to \Omega_t$ which maps the point $X \in \Omega_{\mathrm{ref}}$ onto the point $x \in \Omega_t$, i.e. $x = \mathcal{A}_t(X) \in \Omega_t$. Collecting these mappings for $t \in [0,T]$ we get the so called ALE mapping $\mathcal{A}$. Although such a mapping is assumed individually for each time interval I_m in the paper [6], we consider here only a single ALE mapping over all time interval $(0,T)$. We also assume that the evolution of the domain as well as the ALE mapping $\mathcal{A}$ is independent of the solution u of problem (2.77). We assume that the evolution of the domain is smooth and that the ALE mapping $\mathcal{A}$ and its inverse $\mathcal{A}^{-1}$satisfies

$$\mathcal{A} \in W^{1,\infty}(0,T,W^{1,\infty}(\Omega_{\mathrm{ref}})), \tag{2.80}$$

$$\mathcal{A}^{-1} \in W^{1,\infty}(0,T,W^{1,\infty}(\Omega_t)).$$

The important concept in the ALE description is the ALE derivative. The ALE derivative D_t of function $f(x,t)$ is defined as the time derivative of the reference function $\tilde{f}(X,t) = f(\mathcal{A}_t(X),t)$, where $x = \mathcal{A}_t(X)$. By the chain rule we gain

$$D_t f(x,t) = \frac{\partial}{\partial t}\tilde{f}(X,t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathcal{A}_t(X)),t) = \nabla f(x,t)\cdot \frac{\partial}{\partial t}\mathcal{A}_t(X) + f'(x,t). \tag{2.81}$$

Denoting the mesh velocity $z(x,t) = \tilde{z}(X,t)$, where $\tilde{z}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X)$, we can rewrite (2.81) as

$$D_t f = z\cdot\nabla f + f'. \tag{2.82}$$

The interpretation of the ALE derivative is the derivative along the ALE curve, where the ALE curve is defined as the evolution of the single point $X \in \Omega_{\mathrm{ref}}$.

Using the ALE derivative, we can reformulate the original problem (2.77) into equivalent problem

$$D_t u - \nabla\cdot(\beta(u)\nabla u) + \nabla\cdot f(u) - z\cdot\nabla u = g. \tag{2.83}$$

2.8.2 Discretization

The aim of this section is to discretize the problem (2.83) by the space-time discontinuous Galerkin method. We apply the notation from Section 2.3 and Section 2.4 on fixed space-time cylinder. Let us assume the discontinuous finite element space

$$\tilde{X}_h = \{\tilde{v}\in L^2(\Omega_{\mathrm{ref}}) : \tilde{v}|_{\tilde{K}} \in P^p(\tilde{K})\} \tag{2.84}$$

on Ω_{ref}. We can define fully discrete space

$$\tilde{X}_h^\tau = \{\tilde{v}\in L^2(0,T,\tilde{X}_h) : v|_{I_m} \in P^q(I_m,\tilde{X}_h)\} \tag{2.85}$$

on the fixed (reference) space-time cylinder. Finally, the fully discrete space X_h^τ on the evolving space-time cylinder is defined as

$$X_h^\tau = \{v : v\circ\mathcal{A} \in \tilde{X}_h^\tau\}. \tag{2.86}$$

Applying the space and time discontinuous Galerkin technique described in Section 2.3 and Section 2.2.1, we arrive to the discrete formulation of problem (2.77)

$$\begin{aligned}\int_{I_m} (D_t U, v)_t + A_h(U,v,t) + b_h(U,v,t) - (z\cdot\nabla U, v)_t \mathrm{d}t \\ +(\{U\}_{m-1}, v_+^{m-1})_{t_{m-1}} = \int_{I_m} \ell(v,t)\mathrm{d}t, \quad \forall v\in X_h^\tau,\end{aligned} \tag{2.87}$$

where $(.,.)_t$ denotes the L^2-scalar product on Ω_t. The detailed description of the forms $A_h(.,.,t)$, $b_h(.,.,t)$ and $\ell(.,t)$ can be found in the paper [6] or in Chapter 5.

2.8.3 Stability analysis

The goal of this section is to derive the stability estimate, i.e. the estimate that bounds the discrete solution $U \in X_h^\tau$ in $L^\infty(L^2)$-norm by the data of the problem in suitable norms, i.e. by the initial and boundary conditions and by the right-hand side g. Setting $v = U$ in (2.87) gives after some manipulations

$$\begin{aligned}\|U_-^m\|_{t_m}^2 - \|U_-^{m-1}\|_{t_{m-1}}^2 + \|\{U\}_{m-1}\|_{t_{m-1}}^2 + \int_{I_m} A_h(U,U,t)\mathrm{d}t \\ \le R_t + C\int_{I_m} \|U\|_t^2 \mathrm{d}t,\end{aligned} \tag{2.88}$$

where $\|.\|_t$ denotes the L^2-norm on Ω_t and the term R_t consists of the norms of the boundary condition and the right-hand side. The main difficulty lies in the estimate of the $L^2(L^2)$-norm of the discrete solution U on the right-hand side of (2.88). Since the discrete solution is from the finite dimensional space, it is possible to show that the norms

$$\int_{I_m} \|U\|_t^2 \mathrm{d}t \quad \text{and} \quad \tau \sup_{I_m} \|U\|_t^2 \tag{2.89}$$

are equivalent. For piece-wise constant or piece-wise linear time approximations, i.e. $q = 0, 1$, it is possible to deal with the supremum term directly, since the supremum over I_m is gained only at the endpoints of the interval I_m, see [5]. The polynomial approximations of higher degree need to be treated more carefully.

2.8.4 Discrete characteristic function

Denoting $y \in [t_{m-1}, t_m]$ such that

$$\|U(y)\|_y^2 = \sup_{I_m} \|U\|_t^2, \tag{2.90}$$

the ideal choice of the test function in (2.87) is $v = U\chi_{(0,y)}$, where $\chi_{(0,y)}$ is the characteristic function of interval $(0, y)$. The applications of this test function in (2.87) leads after some manipulations to

$$\begin{aligned} \|U(y)\|_y^2 - \|U_-^{m-1}\|_{t_{m-1}}^2 + \|\{U\}_{m-1}\|_{t_{m-1}}^2 + \int_{t_{m-1}}^{y} A_h(U, U, t)\mathrm{d}t \\ \leq R_t + C \int_{t_{m-1}}^{y} \|U\|_t^2 \mathrm{d}t. \end{aligned} \tag{2.91}$$

Then the proof of the stability can be finished by Gronwall lemma.

Unfortunately, this choice of the test function is not possible, since $U\chi_{(0,y)} \notin X_h^\tau$, and it is necessary to construct a discrete approximation of $U\chi_{(0,y)}$ in the space X_h^τ. In the paper [6], the approximation $U\chi_{(0,y)} \approx U_y \in X_h^\tau$ is made with the aid of the *discrete characteristic function* described in [12] for fixed domains. Denoting the corresponding function $\tilde{U} \in \tilde{X}_h^\tau$ to the original function $U \in X_h^\tau$, we can define the discrete characteristic function $\tilde{U}_y \in \tilde{X}_h^\tau$ on the fixed space-time cylinder by

$$\begin{aligned} \int_{I_m} (\tilde{U}_y, v)_{\mathrm{ref}} \mathrm{d}t = \int_{t_{m-1}}^{y} (\tilde{U}, v)_{\mathrm{ref}}, \quad \forall v \in P^{q-1}(I_m, \tilde{X}_h), \\ (\tilde{U}_y)_+^{m-1} = (\tilde{U})_+^{m-1}. \end{aligned} \tag{2.92}$$

Then the final discrete characteristic function U_y is defined as the transformation of $\tilde{U}_y$ back to the evolving domain, i.e. $U_y(x, t) = \tilde{U}_y(\mathcal{A}_t^{-1}(x), t) \in X_h^\tau$.

The main properties of this discrete characteristic function $U_y \in X_h^\tau$ is that it behaves similarly as the true characteristic function $U\chi_{(0,y)}$ when applied on the term that corresponds to the discrete time derivative, i.e.

$$\begin{aligned} 2\int_{I_m} (D_t U, U_y)_t \mathrm{d}t + 2(\{U\}_{m-1}, (U_y)_+^{m-1})_{t_{m-1}} \geq \|U(y)\|_y^2 - \|U_-^{m-1}\|_{t_{m-1}}^2 \\ -C\int_{I_m} \|U\|_t^2 \mathrm{d}t. \end{aligned} \tag{2.93}$$

The application of U_y on all of the other terms in (2.87) is treated with the aid of the following continuity property of $U \to U_y$ proved in the paper [6] or in Chapter

5

$$\int_{I_m} \|U_y\|_t^2 \mathrm{d}t \le C \int_{I_m} \|U\|_t^2 \mathrm{d}t, \tag{2.94}$$
$$\int_{I_m} A_h(U_y, U_y, t)\mathrm{d}t \le C \int_{I_m} A_h(U, U, t)\mathrm{d}t.$$

2.9 Overview of Chapter 6: A posteriori error estimates for nonlinear parabolic problems

Chapter 6 is based on the paper *A posteriori error estimates for higher order space-time Galerkin discretizations of nonlinear parabolic problems* published in SIAM Journal on Numerical Analysis in 2021, [17].

The paper deals with the numerical analysis of unsteady singularly perturbed nonlinear convection-diffusion problems

$$u' - \nabla \cdot \sigma(u, \nabla u) + c(u) = 0 \quad \text{in } \Omega \times (0, T) \tag{2.95}$$

with homogeneous Dirichlet boundary condition and corresponding initial condition u^0. The nonlinearity is supposed to be monotone and continuous.

The paper [17] assumes either conforming or nonconforming Galerkin discretizations in space or time resulting in four different types of discretizations. The goal of the paper [17] is to present a unified a posteriori error analysis based on the equilibrated flux reconstructions for all these Galerkin discretizations.

To simplify forthcoming explanations, we only consider the heat equation instead of (2.95), i.e. $\sigma(u, \nabla u) = \nabla u$ and $c(u) = -f$, and the discontiunous Galerkin time discretization in combination with the classical finite element method.

2.9.1 Continuous problem and its discretization

Let us denote spaces

$$\begin{aligned} X &= L^2(0, T, H_0^1(\Omega)), \\ Y &= \{v \in X : v' \in L^2(0, T, L^2(\Omega))\} \subset C([0, T], L^2(\Omega)), \\ Y^0 &= \{v \in Y : v(0) = u^0\}. \end{aligned} \tag{2.96}$$

Then the weak solution satisfies $u \in Y^0$ and

$$\int_0^T (u', v) + (\nabla u, \nabla v)\mathrm{d}t = \int_0^T (f, v)\mathrm{d}t, \quad \forall v \in X. \tag{2.97}$$

We define X_h^τ in the same way as in Section 2.4. It shall be pointed out that X_h^τ is a very natural approximation space to the space X, but not to the spaces Y or Y^0, since $X_h^\tau \not\subset Y$. The fully discrete solution $U \in X_h^\tau$ satisfies

$$\int_{I_m} (U', v) + (\nabla U, \nabla v)\mathrm{d}t + (\{U\}_{m-1}, v_+^{m-1}) = \int_{I_m} (f, v)\mathrm{d}t, \quad \forall v \in X_h^\tau, \tag{2.98}$$

where $U_-^0 = u^0$.

2.9.2 Discrete solution reconstruction

Similarly in Section 2.5, we need to reconstruct the discrete solution. The spatial reconstruction $\sigma_h^\tau \in L^2(0, T, H(\mathrm{div}, \Omega))$ can be obtained in a similar way as described in Section 2.5 and the precise description can be found in the paper [17] or in Chapter 6.

It remains to reconstruct the discrete solution in time. The exact solution u belongs to Y^0. It is possible to see that any function $v \in X_h^\tau$ belongs to the space Y^0 if and only if v is continuous in time and satisfies the initial condition, i.e. $v(0) = u^0$. Then the reconstruction $R_h^\tau \in Y^0$ can be obtained directly from the discrete solution U

$$R_h^\tau(x,t) = U(x,t) - \{U\}_{m-1}(x) r_m(t), \quad t \in I_m,\ x \in \Omega, \tag{2.99}$$

where r_m are Radau polynomials defined in Lemma 2.2.3, i.e. $r_m \in P^{q+1}(I_m)$, $r_m(t_m) = 0$, $r_m(t_{m-1}) = 1$ and $r_m \perp P^{q-1}(I_m)$. In fact, the reconstruction (2.99) is identical to the reconstruction (2.14).

The resulting reconstruction R_h^τ satisfies $R_h^\tau \in Y^0$ and together with the spatial reconstruction σ_h^τ also satisfies the space-time version of the equilibration property (2.38), i.e.

$$(f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau, 1)_{K,m} = 0 \tag{2.100}$$

The details of the proof are presented in the paper [17].

2.9.3 Error measure

Inspired by the work [14], we design the error measure $\mathrm{Res}(U)$ as the dual norm of residual. Since the method (2.98) is nonconforming and the formulations for the exact and the discrete solutions differ, we design a common formulation for both these solutions.

Since the space $X_h^\tau \not\subset Y^0$, we design a new space

$$Y^\tau = \{v \in X : v'|_{I_m} \in L^2(I_m, L^2(\Omega))\}. \tag{2.101}$$

The space Y^τ can be considered as the broken Sobolev space with respect to time. Then this space satisfies $Y^0 \subset Y \subset Y^\tau \subset X$ and also $X_h^\tau \subset Y^\tau$. We can exploit these properties to define the extended formulation that covers the formulation for the exact solution (2.95) as well as the formulation for the discrete solution (2.98): find $u \in Y^\tau$ such that

$$\int_{I_m} (u', v) + (\nabla u, \nabla v) \mathrm{d}t + (\{u\}_{m-1}, v_+^{m-1}) = \int_{I_m} (f, v) \mathrm{d}t, \quad \forall v \in Y^\tau. \tag{2.102}$$

It shall be pointed out that the formulation (2.102) has a unique solution in Y^τ and this solution is the exact solution u of the original problem (2.97).

Then the error measure is defined as a dual norm of residual with respect to the extended formulation (2.102)

$$\mathrm{Res}(U) = \sup_{v \in Y^\tau} \frac{1}{\|v\|_{Y^\tau}} \sum_{K,m} (f - U', v)_{K,m} - (\nabla U, \nabla v)_{K,m} - (\{U\}_{m-1}, v_+^{m-1})_K, \tag{2.103}$$

where the norm $\|.\|_{Y^\tau}$ is designed locally and similarly as in [14]

$$\|v\|_{Y^\tau}^2 = \sum_{K,m} \|v\|_{Y^\tau,K,m}^2, \quad \text{where} \quad \|v\|_{Y^\tau,K,m}^2 = \frac{1}{d_{K,m}^2} h_K^2 \|\nabla v\|_{K,m}^2 + \tau^2 \|v'\|_{K,m}^2. \tag{2.104}$$

The norm $\|.\|_{Y^\tau}$ in [17] contains a user dependent local parameter $d_{K,m}$. To simplify the forthcoming exposition, we assume here $d_{K,m} = 1$.

2.9.4 Error estimate

The upper bound can be derived similarly as in Section 2.5. Let us assume $v \in Y^\tau$. Then

$$\sum_{K,m} (f - U', v)_{K,m} - (\nabla U, \nabla v)_{K,m} - (\{U\}_{m-1}, v_+^{m-1})_K \tag{2.105}$$
$$= \sum_{K,m} (f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau, v)_{K,m} + \sum_{K,m} (\sigma_h^\tau - \nabla U, \nabla v)_{K,m}$$
$$+ \sum_{K,m} ((R_h^\tau)' - U', v)_{K,m} - (\{U\}_{m-1}, v_+^{m-1})_K.$$

Estimation of these terms individually with the aid of (2.100) leads to

$$(f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau, v)_{K,m} \le C_P \|f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau\|_{K,m} \|v\|_{Y^\tau,K,m}, \tag{2.106}$$
$$(\sigma_h^\tau - \nabla U, \nabla v)_{K,m} \le \|\sigma_h^\tau - \nabla U\|_{K,m} \|\nabla v\|_{K,m},$$
$$((R_h^\tau)' - U', v)_{K,m} - (\{U\}_{m-1}, v_+^{m-1})_K \le \|R_h^\tau - U\|_{K,m} \|v'\|_{K,m},$$

where C_P is again the constant from Poincare inequality, cf. [38]. Application of these estimates together and denoting the individual local estimators from (2.106) as

$$\eta_{R,K,m} = C_P \|f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau\|_{K,m}, \tag{2.107}$$
$$\eta_{S,K,m} = \frac{1}{h_K} \|\sigma_h^\tau - \nabla U\|_{K,m},$$
$$\eta_{T,K,m} = \frac{1}{\tau} \|R_h^\tau - U\|_{K,m}$$

gives a posteriori error estimate

$$\mathrm{Res}(U)^2 \le \eta^2 = \sum_{K,m} \left(\eta_{R,K,m} + (\eta_{S,K,m}^2 + \eta_{T,K,m}^2)^{1/2} \right)^2. \tag{2.108}$$

2.9.5 Efficiency estimates

Similarly as in Section 2.5, we can derive local efficiency estimates for the individual error estimators $\eta_{R,K,m}$, $\eta_{S,K,m}$ and $\eta_{T,K,m}$. We assume traditionally that f is a piece-wise polynomial function. Again, we denote by $\lesssim$ the inequality up to constant independent of the exact solution u, the discrete solution U, mesh-size h and step-size τ.

To be able to provide the efficiency estimates locally, we need to define a local version of the error norm $\mathrm{Res}(U)$. Since the error norm is dual norm of residual of the extended formulation, i.e. certain supremum term over all functions $v \in Y^\tau$, see (2.103), we define local versions of the error norm as

$$\mathrm{Res}_{M,m}(U) = \sup_{v \in Y_{M,m}^\tau} \frac{1}{\|v\|_{Y_{M,m}^\tau}} \sum_{K,m} (f - U', v)_{K,m} \tag{2.109}$$
$$- (\nabla U, \nabla v)_{K,m} - (\{U\}_{m-1}, v_+^{m-1})_K,$$

where $Y_{M,m}^\tau \subset Y^\tau$ is a space consisting from functions supported by $\overline{M \times I_m}$, where M is some collection of elements K.

The efficiency estimates for $\eta_{R,K,m}$ and $\eta_{S,K,m}$ can be derived by generalizing the stationary technique, see [21] and [45]. The proof of the efficiency estimate for

$\eta_{T,K,m}$ is made more directly with the aid of a suitable test function, for the details see [17]. The resulting efficiency estimates are following

$$\begin{aligned}\eta_{R,K,m} &= C_P\|f-(R_h^\tau)'+\nabla\cdot\sigma_h^\tau\|_{K,m} \lesssim \mathrm{Res}_{\omega_K,m}(U),\\ \eta_{S,K,m} &= \frac{1}{h_K}\|\sigma_h^\tau-\nabla U\|_{K,m} \lesssim \mathrm{Res}_{\omega_K,m}(U),\\ \eta_{T,K,m} &= \frac{1}{\tau}\|R_h^\tau-U\|_{K,m} \lesssim \mathrm{Res}_{K,m}(U).\end{aligned} \tag{2.110}$$

Since

$$\sum_{K,m}\mathrm{Res}_{K,m}(U) \le \sum_{K,m}\mathrm{Res}_{\omega_K,m}(U) \lesssim \mathrm{Res}(U), \tag{2.111}$$

we can derive from (2.110) the global efficiency estimate

$$\eta = \sum_{K,m}\left(\eta_{R,K,m}+(\eta_{S,K,m}^2+\eta_{T,K,m}^2)^{1/2}\right) \lesssim \mathrm{Res}(U). \tag{2.112}$$

2.10 Overview of Chapter 7: Polynomial robustness of efficiency estimates

Chapter 7 is based on the paper *On polynomial robustness of flux reconstructions* published in Appl. Math. in 2020, [47].

The paper deals with the numerical analysis of convection-diffusion-reaction problems

$$-\Delta u + b\cdot\nabla u + cu = f \quad \text{in } \Omega \tag{2.113}$$

with homogeneous Dirichlet boundary condition. The problem is discretized by the standard finite element method. A posteriori error estimate, where the flux reconstructions are designed element-wise, is derived. The main result of the paper show that the efficiency constant of the flux reconstruction in 1D ($d = 1$) depends on the discretization polynomial degree as $p^{1/2}$ at most. The main contribution behind this paper lies in the application of the reconstruction developed for the time discretization in [17] for the space discretization as well.

To simplify forthcoming explanations, we only consider the Poisson equation instead of (2.113), i.e. $b = 0$ and $c = 0$.

2.10.1 Discretization and upper bound

We descretize the problem (2.113) by the standard finite element method. We can apply the notation from Section 2.1.1. The finite element space is defined as

$$X_h = \{v \in H_0^1(\Omega) : v|_K \in P^p(K)\} \tag{2.114}$$

and we can formulate the discrete problem: find $u_h \in X_h$ such that

$$(\nabla u_h, \nabla v) = (f, v), \quad \forall v_h \in X_h. \tag{2.115}$$

In contrary to Section 2.5, we compose the reconstruction $\sigma_h \in H(\mathrm{div},\Omega)$ from the local element-wise information, i.e. we define $\sigma_h|_K \in \mathrm{RT}(K)$ such that

$$\begin{aligned}\sigma_h|_e\cdot n &= \langle\nabla u_h\rangle|_e\cdot n,\\ (\sigma_h, w)_K &= (\nabla u_h, w)_K, \quad \forall w \in P^{p-1}(K)^d.\end{aligned} \tag{2.116}$$

The resulting global function σ_h is in $H(\mathrm{div}, \Omega)$, since the normal component of σ_h is continuous across the edges. Moreover, σ_h is equilibrated in generalized sense

$$(f + \nabla \cdot \sigma_h, v) = 0, \quad \forall v \in X_h. \tag{2.117}$$

The advantage of the reconstruction defined by (2.116) in comparison with the reconstruction defined in Section 2.5 is its simplicity that enables to evaluate the reconstruction directly without solving an artificial mixed finite element problem on patches ω_a. It shall be pointed out that the relations from (2.116) correspond to the classical (natural) degrees of freedom for $\mathrm{RT}(K)$, see e.g. [7].

Instead of Poincare inequality applied in Section 2.5, we need a more accurate estimate

$$\inf_{v_h \in X_h} \|v - v_h\|_K \leq C_{Fl} \frac{h_K}{p} \|\nabla v\|_K \tag{2.118}$$

that holds for any function $v \in H_0^1(K)$, see e.g. [4]. The constant C_{Fl} is unknown in general, but it can be determined in some special cases. E.g., it is possible to take

$$C_{Fl} = \frac{p}{\sqrt{(2p+3)(2p-1)}} \tag{2.119}$$

in 1D ($d = 1$), see [47] or Chapter 7.

Denoting local estimators

$$\begin{aligned} \eta_{R,K} &= C_{Fl} \frac{h_K}{p} \|f + \nabla \cdot \sigma_h\|_K, \\ \eta_{F,K} &= \|\sigma_h - \nabla u_h\|_K \end{aligned} \tag{2.120}$$

and applying (2.117) together with (2.118) imply a posteriori error estimate

$$\|\nabla u - \nabla u_h\|^2 = \mathrm{Res}(u_h)^2 \leq \eta^2 = \sum_K (\eta_{R,K} + \eta_{F,K})^2. \tag{2.121}$$

The idea of the proof is similar as in Section 2.5.

2.10.2 Efficiency

We derive local efficiency estimates for the individual error estimators $\eta_{R,K}$ and $\eta_{F,K}$ in 1D ($d = 1$). We traditionally assume that $f \in X_h$, similarly as in Section 2.5. Again, we denote by $\lesssim$ the inequality up to constant independent of the exact solution u, the discrete solution u_h and mesh-size h. Since we are also interested in the polynomial dependence of this constant, we assume that this constant is also independent of the discretization polynomial degree p and denote this polynomial dependence separately.

Similarly as in [17], we define local errors

$$\mathrm{Res}_M(u_h) = \sup_{v \in H_0^1(\Omega),\, \mathrm{supp}(v) \subset M} \frac{(f, v) - (\nabla u_h, \nabla v)}{\|\nabla v\|}, \tag{2.122}$$

where M is some collection of elements K. Similarly as in [17], it is possible to show that

$$\sum_K \mathrm{Res}_K(u_h) \leq \sum_K \mathrm{Res}_{\omega_K}(u_h) \lesssim \mathrm{Res}(u_h). \tag{2.123}$$

Moreover, it is possible to see that the reconstruction defined by (2.116) in 1D can be equivalently rewritten on element $K = [a, b]$

$$\sigma_h|_K = \nabla u_h + (\langle \nabla u_h \rangle(a) - \nabla u_h(a))r_a + (\langle \nabla u_h \rangle(b) - \nabla u_h(b))r_b, \qquad (2.124)$$

where the values of $\nabla u_h(a)$ and $\nabla u_h(b)$ are taken from inside of K and $r_a, r_b \in P^{p+1}$ are Radau orthogonal polynomials on K oriented either to the left endpoint a or the right endpoint b. Comparing with the reconstruction (2.14), we can see that the reconstruction of σ_h is defined according to the similar principle, but assumes the jump term on both sides of the interval K.

The efficiency of $\eta_{F,K}$ can be proved by similar argument as in the proof of efficiency of $\eta_{T,K,m}$ in [17]. The advantage of the directness of the proof enables to track the dependence on the polynomial degree

$$\eta_{F,K} = \|\sigma_h - \nabla u_h\|_K \lesssim p^{1/2} \operatorname{Res}_{\omega_K}(u_h). \qquad (2.125)$$

The proof is rather technical and therefore it is skipped here. The details can be found in [47] or in Chapter 7. This estimate can be applied for the proof of the efficiency of $\eta_{R,K}$, where it is possible to show that the polynomial dependence is the same as in $\eta_{F,K}$

$$\eta_{R,K} = C_{Fl} \frac{h_K}{p} \|f + \nabla \cdot \sigma_h\|_K \lesssim p^{1/2} \operatorname{Res}_{\omega_K}(u_h). \qquad (2.126)$$

Again, the proof is quite technical and the details can be found in [47] or in Chapter 7.

The estimate of $\eta_{R,K}$ is quite interesting, since usually the authors in the literature are focused in the efficiency of $\eta_{F,K}$ only. The problem with traditional concept of the term $\eta_{R,K}$ is that only standard Poincare inequality (or a similar inequality like Friedrichs inequality etc.) is applied. This enables to determine the constant C_{Fl} as Poincare constant C_P that is known in standard situations, e.g. on convex domains. On the other hand, classical Poincare inequality only contains the term $C_P h_K$ instead of $C_{Fl} \frac{h_K}{p}$. Avoiding the $1/p$ term seems to lead to suboptimal efficiency analysis with respect to the polynomial degree p.

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Chapter 3

Linear unsteady singularly perturbed convection-diffusion problems

Chapter 4

Semilinear unsteady singularly perturbed convection-diffusion problems

Chapter 5

Nonlinear unsteady convection-diffusion problems in time-dependent domains

Chapter 6

A posteriori error estimates for nonlinear parabolic problems

Chapter 7

Polynomial robustness of efficiency estimates