Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Targeted merging of customers‘ opinions based on trust

Cílené slučování názorů zákazníků na základě důvěry

Master’s Thesis

Author: **Bc. Jurij Ružejnikov**

Supervisor: **doc. Ing. Tatiana V. Guy, Ph.D.**

Consultant: **Ing. Marko Ruman**

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Student: Bc. Jurij Ružejnikov

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Název práce (česky): Cílené slučování názorů zákazníků na základě důvěry

Název práce (anglicky): Targeted merging of customers' opinions based on trust

Jazyk práce: angličtina

Pokyny pro vypracování:

1. Prohlubte si znalost problematiky míchání názorů a stav výzkumu v této oblasti.
2. Navažte na úlohu pravděpodobnostního modelování formování názoru (viz. výzkumný úkol [2]) a navrhněte modifikaci algoritmu, která zohledňuje (subjektivní) stupeň důvěry v poskytovanou informaci.
3. Ilustrujte řešení úlohy míchání názorů na příkladu volby vhodného produktu na základě posudků jiných uživatelů/zákazníků.
4. Implementujte algoritmus z bodu 2 v prostředí Matlab (nebo Python) a aplikujte na simulovaná nebo reálná data.
5. Analyzujte získané výsledky.

Doporučená literatura:

1. Richa Punam Bedi (2021) Trust and Distrust based Crossdomain Recommender System, Applied Artificial Intelligence, 35:4, 326-351,

2. Jurij Ružejnikov: Pravděpodobnostní modelování formování názoru, VU, FJFI, CVUT, 2022.

3. A. Quinn, M. Kárný, T. V. Guy, Optimal design of priors constrained by external predictors. International Journal of Approximate Reasoning 84, 2017, 150-158

Jméno a pracoviště vedoucí diplomové práce:

Ing. Tatiana V. Guy, Ph.D.
Ústav teorie informace a automatizace
Pod Vodárenskou věží 4
182 00, Praha 8

Jméno a pracoviště konzultanta:

Ing. Marko Ruman
Ústav teorie informace a automatizace
Pod Vodárenskou věží 4
182 00, Praha 8

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V Praze dne 31.10.2022

ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE
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Author's declaration:

I declare that this Master's Thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, January 8, 2024 Bc. Jurij Ružejnikov

Název práce:

Cílené slučování názorů zákazníků na základě důvěry

Autor: Bc. Jurij Ružejnikov

Program: Aplikované matematicko-stochastické metody

Druh práce: Diplomová práce

Vedoucí práce: doc. Ing. Tatiana V. Guy, Ph.D.
ÚTIA AVČR, v.v.i, Pod Vodárenskou věží 4, 182 00 Praha 8

Konzultant: Ing. Marko Ruman,
ÚTIA AVČR, v.v.i, Pod Vodárenskou věží 4, 182 00 Praha 8

Abstrakt: V této práci zkoumáme, jak racionální agent formuje svůj názor na základě předchozích znalostí, dostupných informací a názorů jiných agentů. Navrhneme metodu účelného míchání názorů agenta a názorů expertů. Popisujeme agentův názor a názory expertů ve formě distribucí. Formlujeme formování názoru jako rozhodovací problém a řešíme ho s pomocí Plně Pravděpodobnostního Návrhu (PPN). Demonstrujeme řešení na základě simulovaných dat o vlastnostech značek mobilních telefonů. Navrhovaná metodologie je ověřena na příkladu volby mobilní značky na základě názorů expertů zohledňující důvěru v expertův názor.

Klíčová slova: Plně pravděpodobnostní návrh, Reprezentace názoru, Rozhodování, Slučování názorů, Spolehlivost informačního zdroje

Title:

Targeted merging of customers‘ opinions based on trust

Author: Bc. Jurij Ružejnikov

Abstract: In this thesis, we investigate how a rational agent forms their opinion based on prior knowledge, available information, and the opinions of other agents. We propose methodology of how to purposefully merge agent's opinion and expert opinions. We describe the agent's opinion and the opinions of experts in the form of distributions. Formulating opinion formation as a decision-making task and solve it using Fully Probabilistic Design (FPD). To demonstrate our approach, we apply the solution on simulated data describing features of mobile phone brands. Methodology is verified on a testbed example of choosing a mobile phone brand based on expert opinions while taking into account agent's trust in experts.

Key words: Decision-making, Fully probabilistic design, Knowledge representation, Opinion dynamics, Opinion merging, Trust

Contents

Conventions and notation

$\mathbb{R}$ set of real numbers;

$\mathbb{N}$ set of natural numbers;

$\mathbb{N}_0$ set of natural numbers including 0;

$X^{n+1}, n \in \mathbb{N}_0$ discrete random variable (or discrete multivariate random variable);

$x \in \mathbb{X}^{n+1}$ realisation of a discrete multivariate random variable;

$P(X)$ probability distribution of random variable X;

$P(X, Y)$ joint probability distribution of random variables X a Y;

$P(X|Y)$ conditional probability distribution of X conditioned by Y;

$\theta \in \mathbf{\Theta}$ parameter of a model, where $\mathbf{\Theta}$ is a set of parameters;

$\mathcal{D}(P||Q)$ Kullback-Leibler divergence of distributions P and Q;

I subscript indicating that a function is related to agent I, so-called *primary modeller*;

E subscript indicating that a function is related to an external source of information, for instance, an expert.

Introduction

In the modern, data-driven age, vast data is continuously being gathered from various sources. These sources may vary in reliability. For instance, consider two temperature sensors that produce readings equally likely to be close to the actual temperature, however their reliability may vary depending on their position or age. Similarly, the reliability of information from two databases may vary, mainly if these databases cover different or intersecting domains; what is crucial information in one may be a mere detail in the other. Another example is individual experts within the same field, such as managers, medical doctors, mechanics, etc., possess varying amounts of experience and knowledge.

This thesis focuses on the merging of opinions from diverse human perspectives.

The advent of social networks has brought about an overwhelming influx of data that significantly influences human reasoning [1]. The high diversity of information sources and the high variability of information reliability made opinion formation even more difficult. This problem comprises many challenging tasks, among them: extracting opinions from data [2–5], establishing a new opinion based on available information and knowledge [6,7], setting trust in a particular source of information [8–11]. It is important to stress that opinion of an individual is often formed by experience undertaken but also reflect preferences/priorities of the individual. Therefore, the mathematical formulation of opinion formation should reflect that and be solved, respectively.
This work attempts to formulate and solve the task of establishing opinions based on the data provided by external experts and trust in the expert's opinion. We attempt to solve this problem from a single agent (called *primary modeller*) perspective while respecting their prior assumptions and preferences.

One of the first approaches to solving the problem of merging opinions, by combining posterior estimates of beliefs of experts, was introduced by DeGroot [12]. This approach was inspired by applied work on the Delphi method for combining expert opinions [13].

DeGroot's work resulted in a large amount of literature on consensus modelling and opinion dynamics. This includes linear pooling of opinions [14], where the newly acquired opinion, in the form of a distribution, is a weighted sum of distributions representing the opinions of experts. Another method, called logarithmic opinion pooling, uses the weighted geometric mean of expert opinions [15]. An approach based on DeGroot's opinion pooling can be found in [16], where the proposed method models voting outcomes in the USA Senate.

Other approaches include consensual modelling [17], which uses a social trust network for solving group decision-making problem. An example in [6] is similar to the example introduced in this thesis; the opinion merging is in the form of a consensus on the opinions of individual experts, with the goal of solving a decision-making task. Specifically, a person seeking advice on buying a new computer asks friends for their opinions.

Another approach, logic-based merging, aims to combine information from different sources based on propositional bases [18]. Merging opinions by social sampling of posteriors in [19] introduces a way to update an opinion with information from external sources, task similar to one solved in this thesis. In another approach from the opinion dynamics field [20], a Hegselmann-Krause model of opinion formation [21] is implemented. This model represents expert opinions as a time-dependent weighted average of opinions, where each opinion is modelled as a point in a metric space.

Opinion formation can also be achieved by a Bayesian update of the distribution defining the primary modeller's opinion from data of expert opinions [22]. This idea, together with the notion of using a divergence minimisation framework [23] and Fully Probabilistic Design [24] to solve the problem of merging opinion-defining distributions, will be explored in this work.

The data do not represent an objective truth about the topic in question but rather the subjective opinions of individual experts. The update provides a modified opinion for the primary modeller that incorporates the opinions of experts. The solution also integrates the primary modeller's trust in the experts' opinions.

Opinion is quantified as a parameterised distribution of a random variable representing the subject of an opinion. The distribution on the parameter serves as a prior distribution of the parameter, and there is a hyper-prior for the prior distribution.

The work employs the selection of a mobile phone brand as a testbed decision-making problem to demonstrate the utilisation of the resulting merged opinion and applies the theory to this example.

The layout of the work is as follows.

Chapter 1 outlines the necessary theory for opinion merging and illustrates its application through a brief example.

Chapter 2 adapts the theory from Chapter 1 to the specific context of merging opinions about mobile phone brands.

Chapter 3 provides insight into specific aspects related to the merging of opinions about mobile brands.

Chapter 4 presents the data related to mobile phone brands and discusses the results of eight experiments conducted on the dataset.

Conclusion summarises the findings of the experiments and suggests potential avenues for future research.

Chapter 1

Preliminaries

This chapter outlines the necessary theory and describes the considered application example.

1.1 Opinion merging task

Assume two independent modeller's each has an opinion about some random variable, denoted X. Let one of the modellers, say ***primary modeller***, aim to update his opinion based on

- data observed
- opinion provided by another modeller, the ***expert*** (external opinion).

In this section, we will dive into the process of how this new opinion is formed.

Definition 1.1 (Identifiability of probability family)**.**
Let $\{P(X|\theta)|\theta \in \mathbf{\Theta}\}$ be a probability family. It is ***identifiable*** if for each $\theta \in \mathbf{\Theta}$ there is only one probability $P(X|\theta)$, that

$$\theta_1 \neq \theta_2 \implies P(X|\theta_1) \neq P(X|\theta_2).$$

From now on, all parametric distributions in this thesis are chosen from the identifiable families.

Definition 1.2 (Primary modeller's preference)**.**
Let (X, θ) be a set composed of discrete random variable X and parameter $\theta \in \mathbf{\Theta}$. ***Primary modeller's preference*** is defined as joint distribution $P_I(X, \theta)$ on set (X, θ). It represents primary modeller's uncertain opinion about random variable X and parameter θ.

Using the chain rule, [25], $P_I(X, \theta)$ can be decomposed as follows

$$P_I(X, \theta) = P_I(X|\theta) P_I(\theta), \tag{1.1}$$

where individual parts are defined as follows.

Definition 1.3 (Primary modeller's opinion).
Primary modeller's opinion about discrete random variable X is modelled by distribution $P_I(X|\theta)$.

Definition 1.4 (Primary modeller's prior belief).
Let $P_I(X|\theta)$ be a parametric distribution representing the primary modeller's opinion identified by parameter $\theta \in \boldsymbol{\Theta}$. The primary modeller is uncertain of their opinion $P_I(X|\theta)$ and therefore provides ***prior belief*** about parameter θ, denoted as $P_I(\theta)$.

Now, let us assume that an expert provides an external opinion about the same discrete random variable X.

Definition 1.5 (External opinion).
External opinion about discrete random variable X, is modelled by a distribution $P_E(X)$.

From now on, for the sake of simplicity, we sometimes omit the argument, that is,
$P_E(X) = P_E$.

Our further aim is to incorporate external opinion P_E into primary modeller's preference $P_I(X, \theta)$. We achieve this by transitioning from prior distribution $P_I(\theta)$ to distribution $A(\theta|P_E) \in \mathcal{A}$, representing prior distribution after incorporation of the external opinion P_E. Where $\mathcal{A}$ is set of prior distributions conditionally dependent on external opinion P_E.

In other words, primary modeller's prior belief $P_I(\theta)$ (Definition 1.4), after merging of $P_I(X|\theta)$ with $P_E(X)$, becomes $A(\theta|P_E)$.

To summarise our formulation of opinion merging task is defined by:

- Discrete random variable X.
- Parameter $\theta \in \boldsymbol{\Theta}$.
- Updated prior distribution $A(\theta|P_E) \in \mathcal{A}$.

The updated prior distribution $A(\theta|P_E) \in \mathcal{A}$ is assumed to be unknown to primary modeller. Therefore it is modelled via the hyper-prior $S(A|P_E)$ (for detail see [24]).

Next, we define the result of merging primary modeller's opinion $P_I(X|\theta)$ with external opinion P_E as a joint distribution on triple (X, θ, A) adopting a fully Bayesian hierarchical framework.

Consider a Bayesian parametric model of an unknown quantity, X. Primary modeller adopts the parametric distribution, $F_I(X|\theta)$, which is known up to the value of parameter $\theta \in \boldsymbol{\Theta}$. In considered Bayesian modelling, primary modeller also specifies a prior distribution $F_I(\theta)$, which quantifies their beliefs about $\theta \in \boldsymbol{\Theta}$ before X is observed. Further this prior is assumed to be unknown to primary modeller. Therefore, it is modelled hierarchically, via a hyper-prior, $S(A|F_E)$.

Next, (X, θ, A) be a triple composed of discrete random variable X, parameter $\theta \in \boldsymbol{\Theta}$, a prior distribution $A(\theta|P_E) \in \mathcal{A}$, and $P_I(A|P_E)$ be a hyper-prior of $A(\theta|P_E) \in \mathcal{A}$. The result of merging primary modeller's opinion $P_I(X|\theta)$ with external opinion $P_E(X)$ is given by $P_I(X, \theta, A|P_E)$.

By using the chain rule [25] the opinion update can be decomposed:

$$P_I(X, \theta, A|P_E) = P_I(X|\theta, A, P_E)P_I(\theta|A, P_E)P_I(A|P_E). \tag{1.2}$$

Next, we define the following well-grounded assumptions [24].

Assumption 1.6.

- $P_I(X|\theta, A, P_E) = P_E(X)$.
  The unknown model of X is conditioned by external opinion P_E and fully exploits it and therefore coincides with it.
- $P_I(\theta|A, P_E) = A(\theta|P_E)$.
  Known external opinion P_E is sufficient to model the prior belief about unknown parameter θ.
- $P_I(A|P_E) = S(A|P_E)$.
  Following merging with P_E, the rule on how to select $A(\theta|P_E) \in \mathcal{A}$ is modelled by $S(A|P_E)$.

Under Assumption 1.6, (1.2) becomes

$$P_I(X, \theta, A|P_E) = P_E(X)A(\theta|P_E)S(A|P_E). \tag{1.3}$$

Similarly to distribution $P_I(X, \theta, A|P_E)$, $P_I(X, \theta, A|P_I)$ can describe primary modeller's opinion on (X, θ, A) before merging with the external opinion.

Using the chain rule [25], distribution $P_I(X, \theta, A|P_I)$ can be decomposed:

$$P_I(X, \theta, A|P_I) = P_I(X|\theta, A, P_I)P_I(\theta|A, P_I)P_I(A|P_I). \tag{1.4}$$

We can treat the representation of primary modeller's opinion (1.4) as a special case of distribution $P_I(X, \theta, A|P_E)$, without merging with external opinion $P_E(X)$ (Definition 1.5).

Similarly to Assumptions 1.6, $P_I(X|\theta, A, P_I) = P_I(X|\theta)$ as the primary modeller's opinion about X is $P_I(X|\theta)$ (Definition 1.3). $P_I(\theta|A, P_I) = P_I(\theta)$, primary modeller's prior distribution $P_I(\theta)$ (Definition 1.4) is independent of opinion P_I (Definition 1.5). $P_I(A|P_I) = S_I(A|P_I)$, the hyper-prior of $A(\theta|P_I) \in \mathcal{A}$ is labelled as $S_I(A|P_I)$ supplied by the primary modeller.

Inserting the Assumptions described in the previous paragraph into (1.4), we receive the primary modeller's opinion on (X, θ, A):

$$P_I(X, \theta, A|P_I) = P_I(X|\theta)P_I(\theta)S_I(A|P_I). \tag{1.5}$$

The primary modeller's hyper-prior $S_I(A|P_I)$ can be interpreted as a strategy for selecting an appropriate $A(\theta|P_I) \in \mathcal{A}$, that transforms the opinion merging task to a decision-making task.

Definition 1.7 (Primary modeller's strategy)**.**
Let there be a set of prior distributions $A(\theta|P_E) \in \mathcal{A}$. We define hyper-prior distribution $S_I(A|P_E)$ as ***primary modeller's strategy*** of selecting an $A(\theta|P_E) \in \mathcal{A}$, conditioned by external opinion $P_E(X)$ (Definition 1.5).

To solve the decision-making task, we search for distribution $P_I(X, \theta, A|P_E)$ (1.3) that is closest to $P_I(X, \theta, A|P_I)$, (1.5), in the sense of Kullback–Leibler divergence [26].

Definition 1.8.
Let (X, θ, A) be a triple composed of discrete random variable X, parameter $\theta \in \mathbf{\Theta}$ and a prior distribution $A(\theta|P_E)$. Next let us have primary modeller's opinion $P_I(X, \theta, A|P_I)$ (1.5) and distribution $P_I(X, \theta, A|P_E)$ (1.3). We search for a distribution $P^O(X, \theta, A|P_E)$ that meets the following condition:

$$P^O(X, \theta, A|P_E) = \text{argmin}_{P_I(X,\theta,A|P_E)} \mathcal{D}(P_I(X, \theta, A|P_E)||P_I(X, \theta, A|P_I)), \tag{1.6}$$

where $\mathcal{D}(\cdot||\cdot)$ is Kullback–Leibler divergence [26].

We derive a specific form of the distribution $P^O(X, \theta, A|P_E)$ in Definition 1.8 by applying Theorem 1 in [24] (pp: 153-154).

Theorem 1.9.
Let there be discrete random variable X, parameter $\theta \in \mathbf{\Theta}$, prior distribution $A(\theta|P_E)$, external opinion $P_E(X)$ (Definition 1.5), primary modeller's strategy $S_I(A|P_E)$ (Definition 1.7), primary modeller's prior belief $P_I(\theta)$ (Definition 1.4) and primary modeller's opinion $P_I(X|\theta)$ (Definition 1.3). Then the specific form of $P^O(X, \theta, A|P_E)$ (Definition 1.8) is as follows:

$$P^O(X, \theta, A|P_E) = P_E(X)A(\theta|P_E)S^O(A|P_E), \tag{1.7}$$

$$S^O(A|P_E) \propto S_I(A|P_E) \exp\left(-\mathcal{D}(A||\hat{A})\right), \tag{1.8}$$

$$\hat{A}(\theta|P_E) \propto P_I(\theta) \exp\left(\sum_x ln(P_I(x|\theta))P_E(x)\right). \tag{1.9}$$

The updated prior after incorporating external opinion $P_E(X)$ is defined by $A^O(\theta|P_E) \equiv \hat{A}(\theta|P_E)$ and $S^O(A|P_E)$ is primary modeller's strategy modified by the external opinion $P_E(X)$.

Proof. See [24] (pp. 153-154). □

Note 1. The definition of updated prior, $A^O(\theta|P_E) \equiv \hat{A}(\theta|P_E)$ is justified as follows: Distribution $\hat{A}(\theta|P_E)$ (1.9) modifies primary modeller's prior belief, $P_I(\theta)$ (Definition 1.4), with external opinion $P_E(X)$. This makes it a suitable choice as the prior which incorporates external opinion $P_E(X)$.

Definition 1.10 (The primary modeller's updated opinion)**.**
Let there be primary modeller's opinion $P_I(X|\theta)$ (Definition 1.3) identified by $\theta \in \mathbf{\Theta}$, updated prior $A^O(\theta|P_E)$ (Theorem 1.9) and primary modeller's optimal strategy $S^O(A|P_E)$. We define ***primary modeller's updated opinion*** by the following distribution:

$$P_I^O(X) = \int_{\mathbf{\Theta}\times\mathcal{A}} P_I(X|\theta)A^O(\theta|P_E)S^O(A|P_E)d\theta dA. \tag{1.10}$$

1.2 Example

Imagine a person (referred to as 'primary modeller' in the thesis; see Section 1.1) looking to purchase a mobile phone. Primary modeller has opinion about different mobile phone brands, which can be viewed as realisations of discrete random variable X. These opinions are represented by $P_I(X|\theta)$, where $\theta \in \mathbf{\Theta}$ (Definition 1.3).

The primary modeller's opinion is inherently uncertain, and he expresses this uncertainty through a prior distribution, $P_I(\theta)$ (Definition 1.4). The primary modeller's preference can be expressed as $P_I(X,\theta) = P_I(X|\theta)P_I(\theta)$ (Definition 1.2).

Now, consider the presence of an external expert opinion, denoted $P_E(X)$ (Definition 1.5), which also pertains to the same mobile phone brands. External opinion may come from primary modeller's friends, colleagues, online reviews, and other sources. Primary modeller's objective is to incorporate external opinion $P_E(X)$ into his preference $P_I(X,\theta)$. This is achieved by transitioning from the prior distribution $P_I(\theta)$ to a new prior, denoted as $A(\theta|P_E)$, which incorporates the external opinion $P_E(X)$. This transition involves defining distribution $P_I(X,\theta,A|P_E)$ (1.3) and distribution $P_I(X,\theta,A|P_I)$ (1.5). We formulate the problem as decision making task and find a distribution $P^O(X,\theta,A|P_E)$ (Definition 1.8) that is closest to $P_I(X,\theta,A|P_I)$, solution of which is specified in Theorem 1.9. The primary modeller offers a strategy for selecting $A(\theta|P_E)$ in the form of a hyper-prior $S_I(A|P_E)$ (Definition 1.7). The result is the updated prior, denoted as $A^O(\theta|P_E)$ (Theorem 1.9). Finally, we derive the updated opinion of the primary modeller's $P_I^O(X)$, (1.10).

Chapter 2

Merging opinions: Application to the selection of mobile phone brands

Let us have discrete multivariate random variable, $X = (\mathcal{B}, F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}})$, $n \in \mathbb{N}$, where $\mathcal{B}$ is a brand of a mobile phone and $F_{j,\mathcal{B}}$, $j \in \{1, \ldots, n\}$ is the j -th feature of brand $\mathcal{B}$.

The realisations of the components of random variable X are as follows

- $b \in \{1, \ldots, m\}$, $m \in \mathbb{N}$ is the realisation of variable $\mathcal{B}$, representing a particular mobile brand and m is total number of considered mobile brands.

- $f_{j,b} \in \{1, \ldots, k\}$, $k \in \mathbb{N}$ is a realisation of variable $F_{j,\mathcal{B}}$, $j \in \{1, \ldots, n\}$, representing a particular feature of mobile brand b. For instance, feature can be quality of mobile phone display, camera, sound, network coverage or battery life etc. In further text will be referred to as ***feature score***.

2.1 The primary modeller's opinion

Primary modeller's opinion $P_I(X|\theta)$ (Definition 1.4) is specified for the given task by

$$P_I(X|\theta) = P_I(\mathcal{B}, F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}}|\theta). \tag{2.1}$$

Using the chain rule [25], model (2.1) can be factorised as follows:

$$P_I(\mathcal{B}, F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}}|\theta) = P_I(\mathcal{B}|\theta) P_I(F_{1,\mathcal{B}}|\mathcal{B}, \ldots, F_{n,\mathcal{B}}, \theta) \cdots P_I(F_{n,\mathcal{B}}|\mathcal{B}, \theta). \tag{2.2}$$

We assume that features $F_{j,\mathcal{B}}$ are conditionally independent [1]:

$$P_I(\mathcal{B}, F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}}|\theta) = P_I(\mathcal{B}|\theta) P_I(F_{1,\mathcal{B}}|\mathcal{B}, \theta) \cdots P_I(F_{n,\mathcal{B}}|\mathcal{B}, \theta). \tag{2.3}$$

$P_I(\mathcal{B}|\theta)$ reflects primary modeller's preference for a specific mobile brand, meaning his overall opinion about brand $\mathcal{B}$, while $P_I(F_{j,\mathcal{B}}|\mathcal{B}, \theta)$ represents the primary modeller's opinion regarding the j-th feature for brand $\mathcal{B}$.

[1] In practice, the fulfilment of the independence assumption for features $F_{j,\mathcal{B}}$ can be reasonably ensured by the selection of features.

The expression (2.3) can be rewritten as follows

$$P_I(\mathcal{B}, F_{1,\mathcal{B}}, \dots, F_{n,\mathcal{B}}|\theta) = \underbrace{P_I(\mathcal{B}|\theta)}_{\theta_{\mathcal{B}}} \underbrace{P_I(F_{1,\mathcal{B}}|\mathcal{B},\theta)}_{\theta_{F_{1,\mathcal{B}}}} \cdots \underbrace{P_I(F_{n,\mathcal{B}}|\mathcal{B},\theta)}_{\theta_{F_{n,\mathcal{B}}}}, \tag{2.4}$$

where $\theta = (\theta_{\mathcal{B}}, \theta_{F_{1,\mathcal{B}}}, \dots, \theta_{F_{n,\mathcal{B}}})$ is an unknown parameter, and $\theta_{F_{j,\mathcal{B}=b}} = (\theta_{f_{j,b}=1}, \dots \theta_{f_{j,b}=k})$, $k \in \mathbb{N}$ and $\theta_{f_{j,b}=k} \in \langle 0, 1\rangle$.

Parameters $\theta_{\mathcal{B}}$ and $\theta_{f_{j,b}=k}$ represents relative frequency of selection of brands $\mathcal{B}$ and feature scores $f_{j,b}$.

2.2 The primary modeller's prior belief

We model primary modeller's prior belief $P_I(\theta)$ (Definition 1.4) by

$$\begin{aligned} P_I(\theta) = P_I(\theta_{\mathcal{B}}, \theta_{F_{1,\mathcal{B}}}, \dots, \theta_{F_{n,\mathcal{B}}}) &= P_I(\theta_{\mathcal{B}}) P_I(\theta_{F_{1,\mathcal{B}}}) \cdots P_I(\theta_{F_{n,\mathcal{B}}}) \\ &\propto \prod_b \prod_{f_{1,b},\dots,f_{n,b}} \theta_b^{n_b-1} \theta_{f_{1,b}}^{n_{f_{1,b}}-1} \cdots \theta_{f_{n,b}}^{n_{f_{n,b}}-1}. \end{aligned} \tag{2.5}$$

$P_I(\theta_{\mathcal{B}})$ and $P_I(\theta_{F_{j,\mathcal{B}}})$ in (2.5) were chosen as Dirichlet distributions [27]. The Dirichlet model was chosen because the mean of individual parameters are equal to relative frequencies of selecting a certain feature scores, the parameters $n_b > 0$ and $n_{f_{j,b}} > 0$, are therefore weights regulating primary modeller's preference/opinion over each element of X.

The elements of the second product in (2.5), go through all values $f_{j,b} \in \{1, \dots, k\}$ for each $j \in \{1, \dots, n\}$ of $b \in \{1, \dots, m\}$, assigning the weight $n_{f_{j,b}}$ to $\theta_{f_{j,b}}^{n_{f_{j,b}}-1}$, where $\theta_{f_{j,b}} = P_I(F_{j,\mathcal{B}=b} = f_{j,b}|\mathcal{B} = b, \theta)$, see (2.4).

2.3 The external opinion

Let us have $l \in \mathbb{N}$ independent experts each provides external opinion $P_{E_i}(X)$, $i \in \{1, \dots, l\}$ about modelled X.

We specify the external opinion (Definition 1.5) for i-th expert E_i in the following form:

$$\begin{aligned} P_{E_i}(X) &= P_{E_i}(\mathcal{B}, F_{1,\mathcal{B}}, \dots, F_{n,\mathcal{B}}) \\ &= P_{E_i}(\mathcal{B}) P_{E_i}(F_{1,\mathcal{B}}|F_{2,\mathcal{B}}, \dots, F_{n,\mathcal{B}}, \mathcal{B}) \cdots P_{E_i}(F_{n,\mathcal{B}}|\mathcal{B}) \\ &= P_{E_i}(\mathcal{B}) P_{E_i}(F_{1,\mathcal{B}}|\mathcal{B}) \cdots P_{E_i}(F_{n,\mathcal{B}}|\mathcal{B}), n \in \mathbb{N}. \end{aligned} \tag{2.6}$$

Here we used the chain rule [25] (in the second equality) and the assumption of conditional independence of features $F_{j,b}$.

$P_{E_i}(\mathcal{B})$ in (2.6) represents a probability of selecting a certain mobile brand and $P_{E_i}(F_{j,\mathcal{B}}|\mathcal{B})$ represents E_i's, opinion on the j-th feature.

In our experiments, external opinion $P_{E_i}(X)$ is degenerate and $P_{E_i}(X = x) = 1$ for $x = (b_{E_i}, f_{E_i,1,b_{E_i}}, \ldots, f_{E_i,n,b_{E_i}})$, where b_{E_i} is a mobile brand selected by expert E_i, $f_{E_i,j,b_{E_i}} \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n\}$ are the feature scores assigned to the mobile brand b_{E_i} by expert E_i.

Note 2.
Distribution $P_{E_i}(\mathcal{B})$ represents a probability of selecting a certain mobile brand $\mathcal{B}$ by expert E_i, not expert's preference of brand $\mathcal{B}$ as opposed to similar specification of $P_I(\mathcal{B}|\theta)$ described under (2.3). In our experiments, described in Section 4, the preference is not subject to change by the expert's opinion, in other words, the primary modeller's preference of brand $\mathcal{B}$ is influenced only by the expert's opinion about features $F_{j,b}$.

2.4 The primary modeller's strategy

In this thesis, we assume that primary modeller's strategy $S_I(A|P_{E_i})$ (Definition: 1.7) is not subject to change by external opinion P_{E_i}, therefore, we define $S^O(A|P_{E_i}) \equiv S_I(A|P_{E_i})$.

Primary modeller's strategy $S_I(A|P_{E_i})$ is specified as follows:

$$S_I(A|P_{E_i}) \propto \prod_b \prod_{f_{1,b}, f_{2,b}, \ldots, f_{n,b}} \theta_{f_{1,b}}^{s_{f_{1,b}}-1} \cdots \theta_{f_{n,b}}^{s_{f_{n,b}}-1}. \tag{2.7}$$

$S_I(A|P_{E_i})$ is chosen to be a product of Dirichlet distributions.

There is no explicit dependence on P_{E_i} on the right side of $S_I(A|P_{E_i})$, but it expresses that the weights $s_{f_{j,b}}$ are chosen with respect to different experts E_i separately.

Parameters $s_{f_{j,b}} > 0$ are the primary modeller's hyperparameters, representing factors such as trust in expert's opinion and certainty in their own opinion opinion.

Weights, denoted as $s_{f_{j,b}}$, in the product in (2.7), are assigned in a manner similar to that described in the previous subsection.

2.5 The updated prior

In this section we will derive a solution for the decision-making task defined in Definition 1.8 by specifying individual parts of Theorem 1.9.

In other words we merge primary modeller's opinion $P_I(X|\theta)$ with external opinion P_{E_i} by updating primary modeller's prior belief $P_I(\theta)$ specified in (2.1) to prior $\hat{A}(\theta|P_{E_i})$ defined in 1.9. This is done one expert at a time, which is expressed by conditional dependence of $\hat{A}(\theta|P_{E_i})$ on opinion of i-th expert P_{E_i}.

Theorem 2.1.
Let there be updated prior $\hat{A}(\theta|P_{E_i})$ defined in Theorem 1.9, where P_{E_i} represents opinion of i-th expert. It's specific form is:

$$A^O(\theta|P_{E_i}) \equiv \hat{A}(\theta|P_{E_i}) = \frac{1}{\mathcal{N}_{\hat{A}(\theta|P_{E_i})}} \prod_b \prod_{f_{1,b},f_{2,b},\dots,f_{n,b}} \theta_b^{n_b-1} \theta_{f_{1,b}}^{V_{f_{1,b}}-1} \cdots \theta_{f_{n,b}}^{V_{f_{n,b}}-1}, \tag{2.8}$$

where $V_{f_{j,b}}$ are ***updated weights*** $n_{f_{j,b}}$ representing influence of i-th expert opinion on primary modeller's prior $P_I(\theta)$ defined as:

$$V_{f_{j,b}} = n_{f_{j,b}} + s_{f_{1,b}} + \delta(b, b_{E_i})\delta(f_{j,b}, f_{E_i,j,b_{E_i}}) - 1, \tag{2.9}$$

where $i \in \{1, \dots, l\}$ is i-th expert, $b \in \{1, \dots, m\}$ is b-th brand, $b_{E_i} \in \{1, \dots, m\}$ is brand selected by i-th expert, $f_{j,b} \in \{1, \dots, k\}$ is feature score,
$f_{E_i,j,b_{E_i}} \in \{1, \dots, k\}$ - score selected by i-th expert for brand b_{E_i}. The delta function $\delta(.)$ updates the weights $n_{f_{j,b}}$ by increment of $+1$ for scores selected by the i-th expert, $f_{E_i,j,b_{E_i}}$. This update represents the transfer of expert's opinion into opinion of the primary modeller.

Proof. Can be found in Appendix A. □

2.6 Specifying the primary modeller's updated opinion

The final form of primary modeller's updated opinion $P_I^O(X)$ (Definition 1.10) expressed as a probability of selection specific scores f_{j,b_I}, $j \in \{1, \dots, n\}$ are specific feature scores chosen by the primary modeller is as follows:

Theorem 2.2.
Let there be primary modeller's updated opinion $P_I^O(X)$ (Definition 1.10). It's specific form is:

$$\begin{aligned} P_I^O(X = x) &= P_I^O(\mathcal{B} = b_I, F_{1,b_I} = f_{1,b_I}, \dots, F_{n,b_I} = f_{n,b_I}) \\ &= \frac{n_{b_I}}{\sum_b n_b} \prod_j \frac{V_{f_{j,b_I}}}{\sum_{f_{j,b_I}} V_{f_{j,b_I}}}, \end{aligned} \tag{2.10}$$

where $V_{f_{j,b}}$ are ***updated weights*** defined in (2.9), n_{b_I} is primary modeller's weight regulating preference of $P_I(\mathcal{B})$.

Proof. Can be found in Appendix B. □

2.7 Primary modeller's preference

In this section, we calculate primary modeller's preference on brands $P_I(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}})$. This distribution is based on the primary modeller's preference about mobile brands $P_I(\mathcal{B})$ and primary modeller's opinion $P_I(F_{j,\mathcal{B}}|\theta)$, about the j-th feature of the mobile brand $\mathcal{B}$. We calculate this preference both before and after merging it with external opinion $P_{E_i}(X)$ for the purpose of comparing the results in a later chapter.

Note 3.
The intuitive meaning of primary modeller's preference $P_I(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}})$ is similar to primary modeller's preference $P_I(\mathcal{B}|\theta)$ defined in Section 2.1. It expresses preference of brand $\mathcal{B}$ based on features $F_{j,\mathcal{B}}$ as opposed to $P_I(\mathcal{B}|\theta)$ which expresses overall preference of brand $\mathcal{B}$.

2.7.1 Primary modeller's prior preference

In this section, we will define primary modeller's prior preference $P_I^\beta(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}})$ that is before merging with external opinion of the i-th expert $P_{E_i}(X)$ (2.6) of the expert E_i, $i \in \{1, \ldots, l\}$. Where index β indicates state before the opinion merging.

We need to specify the primary modeller's opinion $P^\beta(X)$, which is defined in a similar manner as the primary modeller's updated opinion $P_I^O(X)$ (Definition 1.10). There is no update of weights n_b and $n_{f_{j,b}}$. The weights represent primary modeller's initial preference/opinion over each element of X (as was described in Section 2.2).

$$P_I^\beta(X = x) = \frac{n_{b_I}}{\sum_b n_b} \prod_j \frac{n_{f_{j,b_I}}}{\sum_{f_{j,b_I}} n_{f_{j,b_I}}}. \tag{2.11}$$

Theorem 2.3 (Primary modeller's prior preference)**.**
Let us have the primary modeller's opinion $P_I^\beta(X)$ (2.11).
Primary modeller's prior preference on mobile brands, that is, before the merging of opinions reads:

$$P_I^\beta(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}}) = \frac{P_I^\beta(F_{1,\mathcal{B}}|\mathcal{B}) \cdots P_I^\beta(F_{n,\mathcal{B}}|\mathcal{B}) P_I^\beta(\mathcal{B})}{\sum_b P_I^\beta(F_{1,\mathcal{B}=b}|\mathcal{B} = b) \cdots P_I^\beta(F_{n,\mathcal{B}=b}|\mathcal{B} = b) P_I^\beta(\mathcal{B} = b)}, \tag{2.12}$$

Distribution (2.12) represents preferences of the primary modeller, conditioned by features $F_{j,\mathcal{B}}$.

Proof. Using Bayes rule [25] and the chain rule [25], with the assumption of conditional independence of features $F_{j,\mathcal{B}}$ and specifying the normalisation constant.

$$\begin{aligned} P_I^\beta(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}}) &\propto P_I^\beta(F_{1,\mathcal{B}} \ldots, F_{n,\mathcal{B}}|\mathcal{B}) P_I^\beta(\mathcal{B}) \\ &= \frac{P_I^\beta(F_{1,\mathcal{B}}|\mathcal{B}) \cdots P_I^\beta(F_{n,\mathcal{B}}|\mathcal{B}) P_I^\beta(\mathcal{B})}{\sum_b P_I^\beta(F_{1,\mathcal{B}=b}|\mathcal{B} = b) \cdots P_I^\beta(F_{n,\mathcal{B}=b}|\mathcal{B} = b) P_I^\beta(\mathcal{B} = b)}, \end{aligned} \tag{2.13}$$

□

Distribution (2.12) depends on the specific realisations of the features $F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}}$, which are necessary to be specified for extraction of primary modeller's preference for a specific brand. For that reason we will chose the scores as follows.

We assume that the primary modeller would choose a score $f_{j,b}$ which has the highest probability $P_I^{\beta}(F_{j,b} = f_{j,b}|\mathcal{B} = b)$:

$$f_{j,b}^{\beta} \equiv \underset{f_{j,b}}{argmax}(P_I^{\beta}(F_{j,\mathcal{B}=b} = f_{j,b}|\mathcal{B} = b)), \tag{2.14}$$

for each $j \in \{1, \ldots, n\}$, $b \in \{1, \ldots, m\}$.

2.7.2 Primary modeller's posterior preference

In this section, we will describe updated opinion of the primary modeller $P_I^O(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}})$ after merging with external opinion $P_{E_i}(X)$ (2.6) of expert E_i.

Theorem 2.4 (Primary modeller's posterior preference)**.**
Let us have the primary modeller's opinion after the merging of opinions $P_I^O(X)$ specified in (2.10). Primary modeller's updated opinion on brands after merging with the i-th expert's opinion $P_{E_i}(X)$ reads:

$$P_I^O(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}}) = \frac{P_I^O(F_{1,\mathcal{B}}|\mathcal{B}) \cdots P_I^O(F_{n,\mathcal{B}}|\mathcal{B}) P_I^O(\mathcal{B})}{\sum_b P_I^O(F_{1,\mathcal{B}=b}|\mathcal{B} = b) \cdots P_I^O(F_{n,\mathcal{B}=b}|\mathcal{B} = b) P_I^O(\mathcal{B} = b)}, \tag{2.15}$$

Proof. Is the same as proof of Theorem 2.3 and can be accomplished by changing index β to O. □

Similarly to Section 2.7.1, distribution (2.15) depends on the specific realisations of feature scores $F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}}$. Which need to be selected for extraction of primary modeller's preference of brand $\mathcal{B}$. The realisations of the feature scores in which the distribution $P_I^O(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}})$ (2.15) is evaluated are chosen as follows.

Similarly to Section 2.7.1, we assume that the primary modeller would choose a score $f_{j,b}$ with the highest probability $P_I^O(F_{j,b}|\mathcal{B})$ of score selection:

$$f_{j,b}^{O} \equiv \underset{f_{j,b}}{argmax}(P_I^{O}(F_{j,\mathcal{B}=b} = f_{j,b}|\mathcal{B} = b)), \tag{2.16}$$

for each $j \in \{1, \ldots, n\}$, $b \in \{1, \ldots, m\}$.

Chapter 3

Trust and certainty

In this chapter, we discuss the configuration of the primary modeller's weights, denoted as $n_{f_j,b}$, and the adjustments made to primary modeller's updated weights $V_{f_{j,b}}$ as indicated in (2.9). These adjustments encompass trust in opinion of i-th expert t_{E_i} and primary modeller's certainty in their own opinion, denoted as $c_{I,j,b}$.

Furthermore, the preference sub-setting process is established, which governs the selection of the final scores $f^O_{j,b}$ (2.16) as inputs for primary modeller's preference $P^O_I(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}})$, as shown in (2.15).

3.1 Primary modeller's trust in experts' opinion

Trust in the opinion provided by the expert P_{E_i} (defined in 1.5) is $t_{E_i} \in \langle 0, 1\rangle$. The lower value of t_{E_i} indicates a low trust in the opinion of experts E_i. Higher indicates otherwise.

The trust values t_{E_i} are integrated into the opinion merging process using primary modeller's strategy $S_I(A|P_{E_i})$ specified in (2.7) through weight $s_{f_{j,b}}$.

The trust value t_{E_i} can encapsulates various aspects of the relationship between primary modeller and the expert E_i, such as friendship, prior knowledge of the expert's performance, or other subjective criteria set by primary modeller.

It is important to note that the process of determining trust value t_{E_i} is not elaborated on in this work. For the purpose of experimentation, trust values t_{E_i} are predefined to specific values to illustrate the potential integration of trust t_{E_i} into the opinion merging process.

Extraction of trust value can be achieved for example by similarity (dissimilarity measures) Such solution can be found in [28–30].

3.2 Primary modeller's certainty

The certainty of primary modeller in their opinion $P_I(X)$, as defined in Definition 1.3, is represented by $c_{I,j,b} \in \langle 0, 1\rangle$. A lower value of $c_{I,j,b}$ indicates diminished certainty in their opinion $P_I(X)$, while a higher value signifies the opposite.

This opinion certainty, denoted as $c_{I,j,b}$, governs the impact of the expert's opinion P_{E_i} on primary modeller's opinion $P_I(X)$. A high certainty value, $c_{I,j,b}$, corresponds to a reduced influence of the expert opinion P_{E_i} on primary modeller's opinion, suggesting a lower receptiveness to expert advice. On the contrary, a low certainty value, $c_{I,j,b}$, implies greater receptiveness to expert advice.

The opinion certainty values $c_{I,j,b}$, provided by primary modeller, are integrated into the opinion merging process through primary modeller's strategy $S_I(A|P_{E_i})$ (2.7).

Note 4.
Inclusion of trust t_{E_i} and opinion certainty $c_{I,j,b}$ in $s_{f_{j,b}}$ is carried out through a linear combination: $s_{f_{j,b}} = t_{E_i} + c_{I,j,b}$.

3.3 The preference sub-setting

The preference sub-setting process dictates how primary modeller chooses scores $f_{j,b}$ that form primary modeller's preference on mobile brands before (2.12) and after (2.15) the merging.

The process of preference sub-setting is accomplished using primary modeller's score preference defined as $r_{I,j} \in \{1, 2, \ldots, k\}$, representing the preference for the score of a feature $F_{j,b}$.

This can be intuitively described as primary modeller's inclination toward a specific range of scores $f_{j,b}$ for a feature j across all mobile phone brands $b \in \mathcal{B}$.

The score preference $r_{I,j}$ represents the range for selecting scores $f^{\beta}_{j,b}$ in (2.14) and $f^{O}_{j,b}$ in (2.16) to $f^{\beta}_{j,b} \geq r_{I,j}$ and $f^{O}_{j,b} \geq r_{I,j}$.

Example in terms of the mobile phone selection task:
Let preference score $r_{I,j} = 4$ for for quality of mobile phone display, the preferred feature quality score are in Table 3.1.

Feature scores	1	2	3	4	5	6

Table 3.1: Preferred feature scores for quality of mobile phone display based on preference score highlighted in green.

Consequently, this influences selection of the probabilities $P^{\beta}_I(F_{j,b}|\mathcal{B})$ and $P^{O}_I(F_{j,b}|\mathcal{B})$ entering primary modeller's prior (2.12) and posterior preference (2.15). These probabilities may assume a low value if the primary modeller's initial opinion P_I combined with expert's opinion P_{E_i} is outside the selection range defined by the score preference $r_{I,j}$.

Chapter 4

Experiments

To showcase functionality of the proposed opinion merging framework for the task of choosing a mobile phone based on quality of it's features, the following experimental setup on simulated data is considered.

The simulated data were setup based on data collected from human participants in a survey about quality of features of several mobile brands. Showcase of proposed solution on simulated data was deemed to provide more insight into correctness of the proposed solution.

Let there be a person (primary modeller) who is interested in buying a mobile phone. He chooses to be advised by their friends, colleagues and information on the internet. Primary modeller has a prior preference about existing mobile brands and chooses from 3 mobile brands each having 3 features. External information is in form of a score for each feature, the score scale is from 1 to 6, where 1 means the worst and 6 means the best.

To summarise we have:

- 10 experts for each brand.
- 3 mobile brands - Samsung, Xiaomi, Apple.
- 3 features - Price, Battery life, Camera quality.
- 6 score values.

For this particular experiment setup we consider the following initialisation:

- The preference score $r_{I,n} = 4$ for all features.
- The primary modeller's brand preference $P_I(\mathcal{B})$ follows a uniform distribution.
- Brands $b \in \mathcal{B}$.
- The primary modeller's certainty $c_{I,j,b} \in \langle 0, 1 \rangle$.
- The primary modeller's trust $t_{E_i} \in \langle 0, 1 \rangle$ is established in the experimental setup.

The primary modeller's brand preference $P_I(\mathcal{B})$ is configured to ensure that the preference $P_I(\mathcal{B}|F_{1,\mathcal{B}}, \ldots, F_{n,\mathcal{B}})$ is not influenced by the primary modeller's overall preference of brand. That is, only the preference based on features is considered.

The experts E_i scores are set so that the solution to the selection process can be easily verified.

The expert score tables are in Appendix C.

In the following five examples, the primary modeller's initial chosen scores for each brand is presented in Table 4.1.

	Price	Battery life	Camera quality
Samsung	5	4	5
iPhone	6	5	6
Xiaomi	4	4	3

Table 4.1: The primary modeller's initial chosen scores for each brand.

Primary modeller's initial weights $n_{f_{j,b}}$ described in 2.2 are set to 1 for each score described in Table 4.1.

The primary modeller's prior preference on brands is given by the ordering:

$$\textbf{iPhone} > \textbf{Samsung} > \textbf{Xiaomi.}$$

4.1 Opinion Merging and Preference Subsetting

This section illustrates the influence of expert opinion on the primary modeller's preference.

In the following three experiments, the opinions are merged without considering primary modeller's trust in expert opinions and without considering certainty of the modeller into his opinion. These factors will be investigated in the subsequent sections.

4.1.1 Experiment 1

The experts are configured in the following order:

$$\textbf{Samsung} > \textbf{iPhone} > \textbf{Xiaomi}.$$

In this experiment the primary modeller's opinion is modified by the experts' opinions described in tables in Appendix C.

The aim of this experiment is to test if this ordering is shifted towards Samsung in the final posterior preference (2.15) on brands $b \in \mathcal{B}$.

Experts slightly prefer Samsung over iPhone, which is a setup for the later demonstration of the trust value t_{E_i}, see Section 4.2.

The result of final preference on mobile brands is in Figure 4.1.

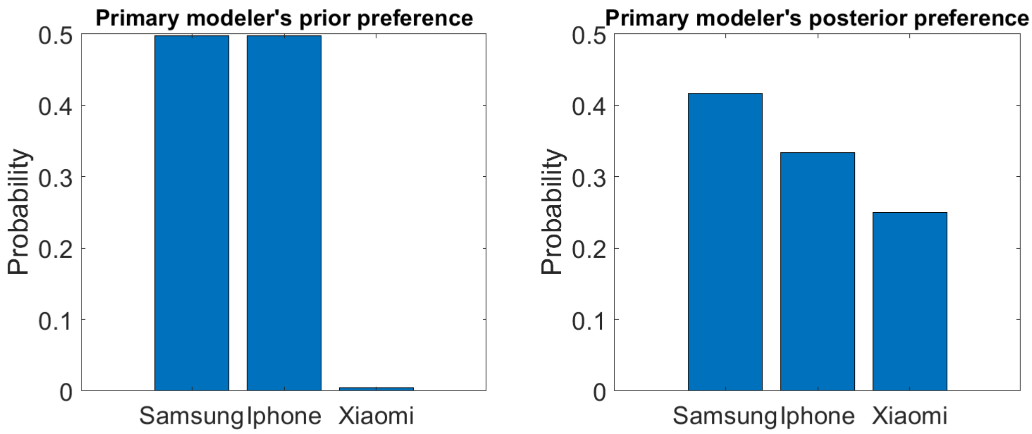

Figure 4.1: Primary modeller's prior and posterior preference.

The left Figure in Figure 4.1 will remains the same for the following experiments as the primary modeller's initial opinion described in Table 4.1 remains unchanged for these experiments and therefore is not a part of the resulting figures. In the following text, the figures that express primary modeller's prior preference are going to be supplied only if primary modeller's prior preference is changed.

The results suggest that the experts influenced the primary modeller's to favour the Samsung brand. Additionally, the primary modeller's probability of choosing the Xiaomi brand prior to opinion merging is low due to the low score assigned to camera quality for the Xiaomi brand. Consequently, this leads to a low probability entering the posterior preference (2.12), significantly reducing the probability of selecting the Xiaomi brand.

4.1.2 Experiment 2

The experts are configured to prefer the brands in the following order:

$$\textbf{Samsung} \approx \textbf{iPhone} \approx \textbf{Xiaomi},$$

The scores are setup to similar **high** values.

The expected result is that the primary modeller's preference is equalised across the brands (in the sense of being almost uniform), but is shifted towards the primary modeller's initial opinion.

The result of final preference on mobile brands is in Figure 4.2, primary modeller's prior preference is left Figure in 4.1.

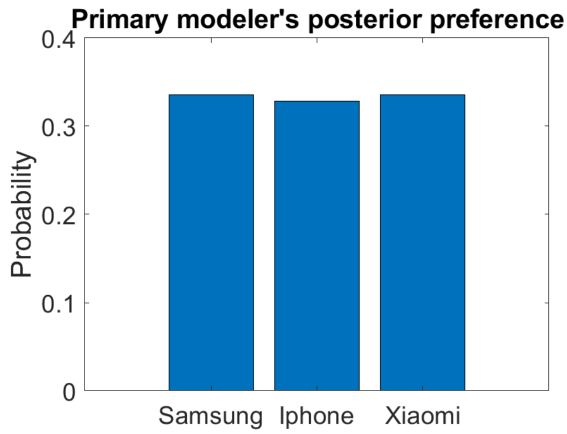

Figure 4.2: Primary modeller's posterior preference.

Comment on this result are given at the end of this section.

4.1.3 Experiment 3

The experts are configured to prefer the brands in the following order:

$$\textbf{Samsung} \approx \textbf{iPhone} \approx \textbf{Xiaomi},$$

The scores are set to similar **low** values.

The expected result is similar to that of Experiment 2. The primary modeller's preference is equalised across the brands (in the sense of being almost uniform), but is shifted towards the primary modeller's initial opinion.

The result of final preference on mobile brands is in Figure 4.3, primary modeller's prior preference is left Figure in Figure 4.1.

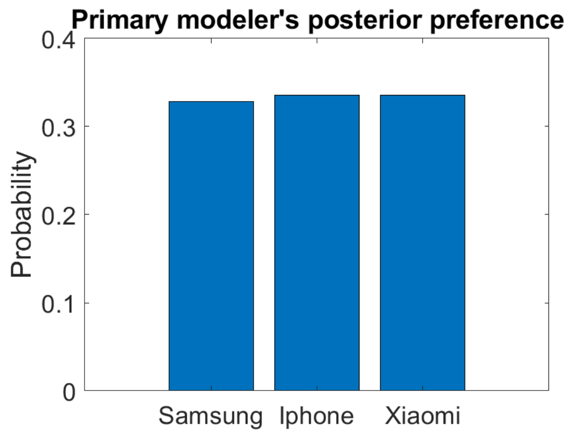

Figure 4.3: Primary modeller's posterior preference.

The results of the last two experiments show only a minor deviation from the uniform distribution. However, the prior preference of the primary modeller was expected to have a stronger influence, pushing the preference towards the original primary modeller's ordering:

$$\textbf{iPhone} > \textbf{Samsung} > \textbf{Xiaomi}.$$

The result is attributed to the fact that each expert's contribution to the updated weight $V_{f_{j,b}}$ has a magnitude of $+1$ (this contribution is expressed by delta function in (2.9)), and the primary modeller's weights $n_{f_{j,b}}$ are set to one and remain so during the mixing of opinions with each expert. The cumulative effect of the expert's contributions diminishes the impact of the primary modeller's opinion.

To address this, the magnitude of the primary modeller's weights $n_{f_{j,b}}$ needs to be adjusted to ensure that the primary modeller's opinion is not diminished. Further elaboration on this adjustment will be provided in Section 4.2.

4.2 Inclusion of trust

In this section, we examine the integration of trust t_{E_i} for each expert's E_i opinion. The issue of diminished initial opinion of the primary modeller described in Section 4.1, at the end of Experiment 3 still remains.

For the following experiments, the primary modeller's opinion will be de-emphasised. This approach allows for a more precise demonstration of trust inclusion, free from any bias introduced by the primary modeller's opinion.

For simplicity, the setup of the following two experiments is the same as in the first experiment from Section 4.1.

4.2.1 Experiment 1

We start by configuring low trust values t_{E_i} for experts reacting to the Samsung brand and high trust values t_{E_i} for experts reacting to the iPhone and Xiaomi. The expected result is that the preferred brand should be the iPhone, since it has similar score values provided by experts E_i as Samsung, with the iPhone being slightly less favoured.

The result of final preference on mobile brands is in Figure 4.4, primary modeller's prior preference is left Figure in 4.1.

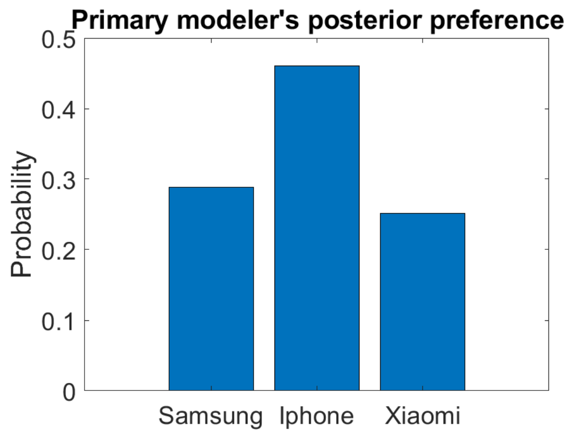

Figure 4.4: Primary modeller's posterior preference.

The result of this experiment complies with the described expectation.

4.2.2 Experiment 2

Setting the trust t_{E_i} for experts reacting to the Samsung and iPhone brands to low values and to high values for Xiaomi. The preferred brand should be Xiaomi.

The result of final preference on mobile brands is in Figure 4.5, primary modeller's prior preference is left Figure in Figure 4.1.

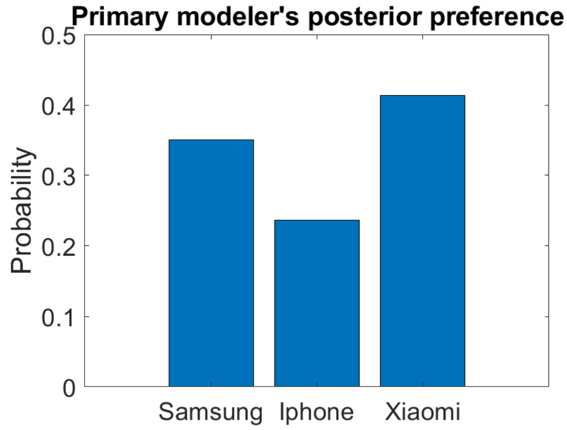

Figure 4.5: Primary modeller's posterior preference.

Both experiments complied with the described expected results. The trust process works as expected.

4.3 Inclusion of certainty

This section aims to provide a solution to the problem of setting the primary modeller's initial weights $n_{f_{1,b}}$ so that the primary modeller's opinion is not diminished during the opinion merging; as was described in the first three examples in Section 4.1.

To better demonstrate the proposed solution, trust values $t_{E_i} \in \langle 0, 1 \rangle$ are intentionally excluded. When the trust value is not set, it is equivalent to setting the trust t_{E_i} to the maximum value of 1.

For simplicity, the setup of the following three examples is the same as in the first experiment from Section 4.1, apart from the setup of the primary modeller's opinion and certainty.

4.3.1 Experiment 1

This experiment illustrates the impact of the certainty $c_{I,j,b}$ on the final ordering of brands. Setting the maximum certainty of opinion $c_{I,j,b} = 1$ in the low scores of the primary modeller for the Samsung brand should lead to the preference for the iPhone as the main brand, with Xiaomi being the second most preferred brand.

The primary modeller's opinion is configured to prefer the brands in the following order:

$$\textbf{iPhone} > \textbf{Xiaomi} > \textbf{Samsung}.$$

The primary modeller's opinion is presented in Table 4.2.

The primary modeller's certainty $c_{I,j,b}$ for each brand is in Table 4.3.

	Feature 1	Feature 2	Feature 3
Samsung	3	4	1
iPhone	6	5	6
Xiaomi	4	4	3

Table 4.2: Primary modeller's opinion for each brand.

	Samsung	iPhone	Xiaomi
	100%	40%	10%

Table 4.3: Primary modeller's opinion certainty for each brand.

Low values of certainty $c_{I,j,b}$ for the iPhone and Xiaomi result in a greater influence of the expert's opinion on the opinion of the primary modeller. This can be intuitively interpreted as the primary modeller being more receptive to the advice of experts E_i for iPhone and Samsung.

The result of final preference on mobile brands is in Figure 4.6.

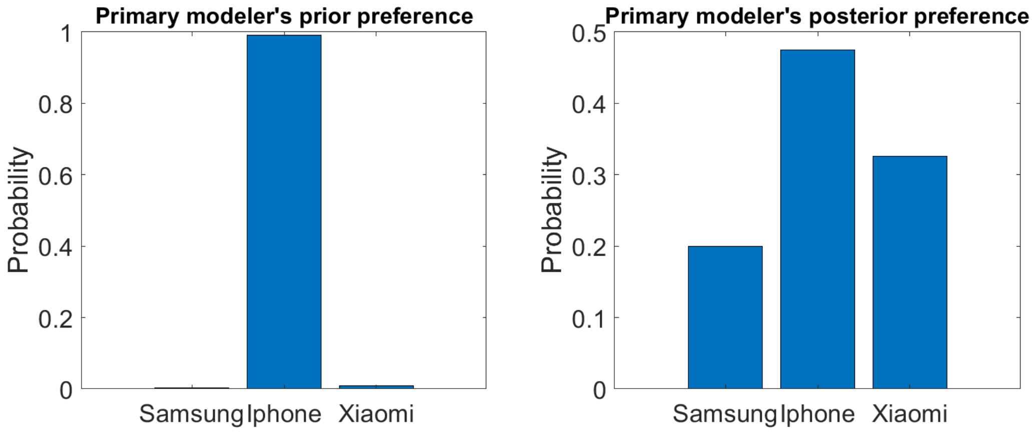

Figure 4.6: Primary modeller's prior and posterior preference.

The result of this experiment complies with the described expectation.

4.3.2 Experiment 2

In this experiment, the maximum certainty $c_{I,j,b}$=1 in the primary modeller's low scores for the Samsung and iPhone brand should lead to a highest preference for the Xiaomi brand.

The primary modeller's opinion is configured to prefer the brands in the following order:

$$\textbf{Xiaomi} > \textbf{iPhone} > \textbf{Samsung}.$$

The primary modeller's opinion is presented in Table 4.4.

	Feature 1	Feature 2	Feature 3
Samsung	3	1	1
iPhone	2	3	1
Xiaomi	6	5	5

Table 4.4: Primary modeller's opinion for each brand.

The primary modeller's certainty $c_{I,j,b}$ for each brand is in Table 4.5.

	Samsung	iPhone	Xiaomi
	20%	20%	100%

Table 4.5: Primary modeller's opinion certainty for each brand.

Similarly to the previous experiment, low values of certainty $c_{I,j,b}$ for Samsung and iPhone result in a higher influence of the expert's opinion on the opinion of the primary modeller. This can be intuitively interpreted as the primary modeller being more receptive to the advice of the experts E_i.

The result of final preference on mobile brands is in Figure 4.7.

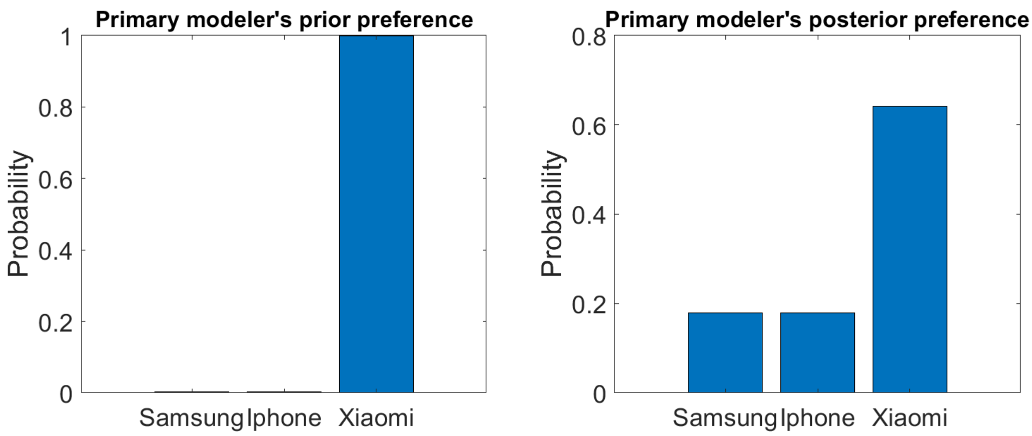

Figure 4.7: Primary modeller's prior and posterior preference.

The result of this experiment complies with the described expectation.

4.3.3 Experiment 3

In this experiment, the maximum certainty $c_{I,j,b} = 1$ in the primary modeller's lowest scores for Samsung brand should lead to highest preference for the iPhone brand.

The primary modeller's opinion is configured to the following order:

$$\textbf{iPhone} > \textbf{Xiaomi} > \textbf{Samsung}.$$

	Feature 1	Feature 2	Feature 3
Samsung	1	1	1
iPhone	6	5	6
Xiaomi	4	4	3

Table 4.6: Primary modeller's opinion for each brand.

The primary modeller's opinion is presented in Table 4.6.

Additionally, to simulate an issue with proposed solution to decision-making task (described below), all expert's opinions for Samsung are deliberately set to the highest scores.

The primary modeller's certainty $c_{I,j,b}$ for each brand is in Table 4.7.

	Samsung	iPhone	Xiaomi
	100%	0%	0%

Table 4.7: Primary modeller's opinion certainty for each brand.

When certainty values $c_{I,j,b}$ for iPhone and Xiaomi are set to zero. This can be intuitively interpreted as the primary modeller completely replacing his own opinion with the opinions of experts E_i.

The result of final preference on mobile brands is in Figure 4.8.

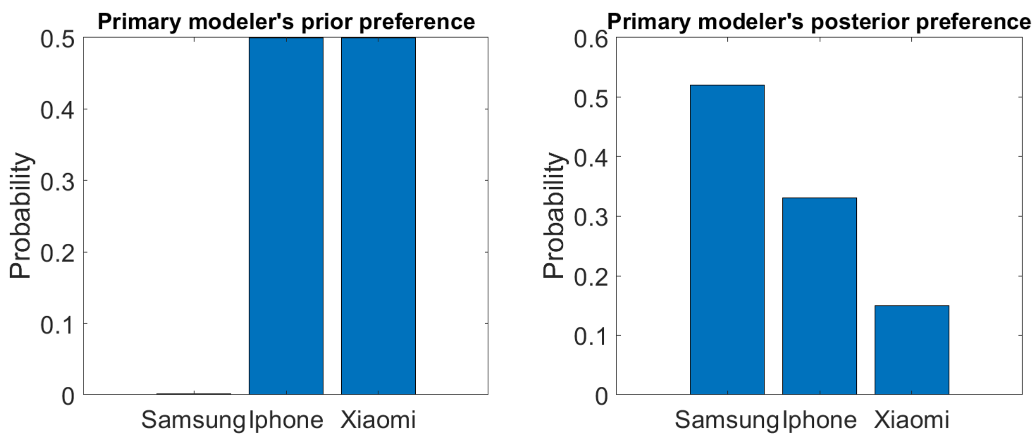

Figure 4.8: Primary modeller's prior and posterior preference.

In this scenario, an unexpected outcome occurs due to the way each expert's E_i contribution to the updated weight $V_{f_{j,b}}$ and the primary modeller's contribution are configured.

In particular:

1. The contribution of each expert to the updated weight $V_{f_{i,b}}$ has a magnitude of $+1$.
2. The primary modeller's contribution to the updated weight $V_{f_{i,b}}$ is $+1$ for each expert, given that certainty $c_{I,j,b}$ is set to the maximum value of 1 (this is due to how the update of weights is setup, see Section 3.2, Note 4).

As a result, the updated weight becomes $V_{f_{i,Samsung}} = 10$ for all $i \in \{1, 2, 3\}$. Consequently, this leads to the probabilities entering the final posterior distribution on the brands (2.15): $P_I^0(F_{i,Samsung}|Samsung)$ for all $i \in \{1, 2, 3\}$, to remain higher than for the other brands, resulting in preference of the Samsung brand, contrary to the expected result.

This experiment represents a case, where all experts coincide in their opinion. This does not happen in practice.

Conclusion

This work formulates and solves the task of establishing a new opinion based on the opinions provided by external experts, user's prior opinion and user's trust in experts' opinions. We have solved this problem from a single agent perspective while respecting their prior assumptions and preferences. The work also considers the user's certainty in their own opinion. The work models opinions in probabilistic way and formulates opinion merging task as a decision making problem.

Selecting mobile phone brand with desired features served as a testbed example to validate the obtained solution. The targeted solution is presented in form of posterior distribution on mobile brands after merging the user's and experts' opinions. This distribution is based on the user's prior preference about brands, the user's opinion about the individual features of the mobile brands and their importance. The data were collected on human participants. As there is no algorithm for "true" merging for comparison, we proposed a series of experiments on simulated data (which were constructed based on real data) that can be judged by logic and intuition. The obtained results are sound and self-explainable.

Future research directions can answer the following questions:

- How to learn trust in an expert's opinion based on previously observed data, external knowledge and other preferences?
- How to model dynamics of trust (i.e. it's change with enriching experience or based on changing user preferences)?
- How does confidence in the provided information influence opinion formation?

Our long-term vision is to combine learning of preferences and opinion merging to get a reliable human-centric support of user interaction within social or customer networks.

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Appendicies

Appendix A

Proof of Theorem 2.1.

Let us substitute $P_I(X)$, (2.4), $P_I(\theta)$, (2.5) and $P_{E_i}(X)$, (2.6) into $\hat{A}(\theta|P_{E_i})$ (1.9).
Substituting $P_I(\theta)$ into (1.9) gives:

$$
\begin{aligned}
\hat{A}(\theta|P_{E_i}) &\propto P_I(\theta)\exp\left[\sum_{x\in X}\ln(P_I(x|\theta))P_{E_i}(x)\right] \\
&\propto P_I(\theta_B,\theta_{F_{1,\mathcal{B}}},\dots\theta_{F_{n,\mathcal{B}}})\exp\left[\sum_{x\in X}\ln(P_I(x|\theta))P_{E_i}(x)\right] \\
&\propto P_I(\theta_B)P_I(\theta_{F_{1,\mathcal{B}}})\cdots P_I(\theta_{F_{n,\mathcal{B}}})\exp\left[\sum_{x\in X}\ln(P_I(x|\theta))P_{E_i}(x)\right] \\
&\propto \prod_b \prod_{f_{1,b},\dots,f_{n,b}} \theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}-1}\cdots\theta_{f_{n,b}}^{n_{f_{n,b}}-1}\exp\left[\sum_{x\in X}\ln(P_I(x|\theta))P_{E_i}(x)\right].
\end{aligned}
\tag{4.1}
$$

Substituting (2.4) into (1.9) and using the notation from (2.4) gives:

$$
\begin{aligned}
\hat{A}(\theta|P_{E_i}) \propto \prod_b \prod_{f_{1,b},\dots,f_{n,b}} &\theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}-1}\cdots\theta_{f_{n,b}}^{n_{f_{n,b}}-1} \\
&\times\exp\left[\sum_b\sum_{f_{1,b},\dots,f_{n,b}}\ln(\theta_b\theta_{f_{1,b}}\cdots\theta_{f_{n,b}})P_{E_i}(x)\right].
\end{aligned}
\tag{4.2}
$$

As mentioned in Note 2, expert E_i, $i\in\{1,\dots,l\}$ provides, among other, $P_{E_i}(\mathcal{B})$. This distribution does not represent the expert's preference for brand $\mathcal{B}$. Therefore, can be omitted during the substitution of $P_{E_i}(X)$ into (1.9), then (4.2) reads

$$
\begin{aligned}
\hat{A}(\theta|P_{E_i}) \propto \prod_b \prod_{f_{1,b},\ldots,f_{n,b}} & \theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}-1}\cdots\theta_{f_{n,b}}^{n_{f_{n,b}}-1} \\
& \times \exp\Bigg[\sum_b \sum_{f_{1,b},\ldots,f_{n,b}} \ln(\theta_b\theta_{f_{1,b}}\cdots\theta_{f_{n,b}}) \\
& \times P_{E_i}(F_{1,\mathcal{B}=b}=f_{1,b}|\mathcal{B}=b)\cdots P_{E_i}(F_{n,\mathcal{B}=b}=f_{n,b}|\mathcal{B}=b)\Bigg].
\end{aligned} \tag{4.3}
$$

As noted, external opinion $P_{E_i}(X)$ degenerates.
Consequently, the summation inside the exponential term in (4.3) has only one nonzero term when $x = (b_{E_i}, f_{E_i,1,b_{E_i}}, \ldots, f_{E_i,n,b_{E_i}})$. Then (4.3) reads

$$
\hat{A}(\theta|P_{E_i}) \propto \prod_b \prod_{f_{1,b},\ldots,f_{n,b}} \theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}-1}\cdots\theta_{f_{n,b}}^{n_{f_{n,b}}-1}\theta_{f_{E_i,1,b_{E_i}}}\cdots\theta_{f_{E_i,n,b_{E_i}}}. \tag{4.4}
$$

Next, we will make the following adjustment in (4.4):

$$
\begin{aligned}
& \prod_b \prod_{f_{1,b},\ldots,f_{n,b}} \theta_{f_{E_i,1,b_{E_i}},b_{E_i}}\cdots\theta_{f_{E_i,n,b_{E_i}},b_{E_i}} \\
&= \prod_b \prod_{f_{1,b},\ldots,f_{n,b}} \theta_{f_{1,b}}^{\delta(b,b_{E_i})\delta(f_{1,b},f_{E_i,1,b_{E_i}})}\cdots\theta_{f_{n,b}}^{\delta(b,b_{E_i})\delta(f_{n,b},f_{E_i,n,b_{E_i}})},
\end{aligned} \tag{4.5}
$$

and substitute into (4.4)

$$
\begin{aligned}
\hat{A}(\theta|P_{E_i}) & \propto \prod_b \prod_{f_{1,b},\ldots,f_{n,b}} \theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}-1}\cdots\theta_{f_{n,b}}^{n_{f_{n,b}}-1} \\
& \times \theta_{f_{1,b}}^{\delta(b,b_{E_i})\delta(f_{1,b},f_{E_i,1,b_{E_i}})}\cdots\theta_{f_{n,b}}^{\delta(b,b_{E_i})\delta(f_{n,b},f_{E_i,n,b_{E_i}})} \\
& \propto \prod_b \prod_{f_{1,b},\ldots,f_{n,b}} \theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}+\delta(b,b_{E_i})\delta(f_{1,b},f_{E_i,1,b_{E_i}})-1}\cdots \\
& \times \theta_{f_{n,b}}^{n_{f_{n,b}}+\delta(b,b_{E_i})\delta(f_{n,b},f_{E_i,n,b_{E_i}})-1}.
\end{aligned} \tag{4.6}
$$

Next, we insert primary modeller's strategy $S_I(A|P_{E_i})$ specified in (2.7), since it serves as a hyper-prior for the distribution $\hat{A}(\theta|P_{E_i})$

$$
\begin{aligned}
\hat{A}(\theta|P_{E_i}) \propto \prod_b \prod_{f_{1,b},\dots,f_{n,b}} & \theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}+\delta(b,b_{E_i})\delta(f_{1,b},f_{E_i,1,b_{E_i}})-1}\dots \\
& \times \theta_{f_{n,b}}^{n_{f_{n,b}}+\delta(b,b_{E_i})\delta(f_{n,b},f_{E_i,n,b_{E_i}})-1} S_I(A|P_{E_i}) \\
& \propto \prod_b \prod_{f_{1,b},\dots,f_{n,b}} \theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}+\delta(b,b_{E_i})\delta(f_{1,b},f_{E_i,1,b_{E_i}})-1}\dots \\
& \times \theta_{f_{n,b}}^{n_{f_{n,b}}+\delta(b,b_{E_i})\delta(f_{n,b},f_{E_i,n,b_{E_i}})-1} \theta_{f_{1,b}}^{s_{f_{1,b}}-1}\theta_{f_{2,b}}^{s_{f_{2,b}}-1}\dots\theta_{f_{n,b}}^{s_{f_{n,b}}-1} \\
& \propto \prod_b \prod_{f_{1,b},\dots,f_{n,b}} \theta_b^{n_b-1}\theta_{f_{1,b}}^{n_{f_{1,b}}+s_{f_{1,b}}-1+\delta(b,b_{E_i})\delta(f_{1,b},f_{E_i,1,b_{E_i}})-1}\dots \\
& \times \theta_{f_{n,b}}^{n_{f_{n,b}}+s_{f_{n,b}}-1+\delta(b,b_{E_i})\delta(f_{n,b},f_{E_i,n,b_{E_i}})-1}
\end{aligned}
\tag{4.7}
$$

We introduce the following definition of the updated weights.

$$
V_{f_{j,b}} = n_{f_{j,b}} + s_{f_{1,b}} + \delta(b, b_{E_i})\delta(f_{j,b}, f_{E_i,j,b_{E_i}}) - 1, \tag{4.8}
$$

where $i \in \{1,\dots,l\}$ is i-th expert, $b \in \{1,\dots,m\}$ is b-th brand, $b_{E_i} \in \{1,\dots,m\}$ is brand selected by i-th expert, $f_{j,b} \in \{1,\dots,k\}$ is feature score,
$f_{E_i,j,b_{E_i}} \in \{1,\dots,k\}$ - score selected by i-th expert for brand b_{E_i}. Then (4.7) reads

$$
\hat{A}(\theta|P_{E_i}) \propto \prod_b \prod_{f_{1,b},f_{2,b},\dots,f_{n,b}} \theta_b^{n_b-1}\theta_{f_{1,b}}^{V_{f_{1,b}}-1}\dots\theta_{f_{n,b}}^{V_{f_{n,b}}-1}. \tag{4.9}
$$

To obtain the final form of updated prior (1.9), we need to calculate the normalisation constant. This is done by noticing that the final distribution is a product of the Dirichlet distributions [27] i.e.

$$
\begin{aligned}
\hat{A}(\theta|P_{E_i}) &= \hat{A}(\theta_{\mathcal{B}}\theta_{F_{1,\mathcal{B}}}\dots\theta_{F_{n,\mathcal{B}}}|P_{E_i}) \\
&= \hat{A}(\theta_{\mathcal{B}}|P_{E_i})\hat{A}(\theta_{F_{1,\mathcal{B}}}|P_{E_i})\cdots\hat{A}(\theta_{F_{n,\mathcal{B}}}|P_{E_i}) \\
&\propto \prod_b \theta_b^{n_b-1}\prod_j\prod_{f_{j,b}}\theta_{f_{j,b},b}^{V_{f_{j,b}}-1},
\end{aligned}
\tag{4.10}
$$

Therefore, the normalisation constant is a product of normalisation constants of Dirichlet distributions $\hat{A}(\theta_{\mathcal{B}}|P_{E_i})$ and $\hat{A}(\theta_{F_{j,\mathcal{B}},\mathcal{B}}|P_{E_i})$, i.e. :

$$
\mathcal{N}_{\hat{A}(\theta|P_{E_i})} = \prod_b \frac{\Gamma(n_b)}{\Gamma(\sum_b n_b)} \prod_j \frac{\prod_{f_{j,b}}\Gamma(V_{f_{j,b}})}{\Gamma(\sum_{f_{j,b}} V_{f_{j,b}})}, \tag{4.11}
$$

where $j \in \{1,\dots,n\}$, $b \in \{1,\dots,m\}$ and Γ is the Gamma function [31].

After merging with the expert's opinion $P_{E_i}(X)$ (2.6), the final form of the updated prior $A^O(\theta|P_{E_i})$ (Theorem 1.9) is :

$$A^O(\theta|P_{E_i}) \equiv \hat{A}(\theta|P_{E_i}) = \frac{1}{\mathcal{N}_{\hat{A}(\theta|P_{E_i})}} \prod_b \prod_{f_{1,b},f_{2,b},\ldots,f_{n,b}} \theta_b^{n_b-1} \theta_{f_{1,b}}^{V_{f_{1,b}}-1} \cdots \theta_{f_{n,b}}^{V_{f_{n,b}}-1}. \tag{4.12}$$

Appendix B

Proof of Theorem 2.2.

Primary modeller's updated opinion can be written as:

$$P_I^O(X) = \int_{\mathbf{\Theta}} P_I^O(X,\theta)d\theta = \int_{\mathbf{\Theta}} P_I(X|\theta)A^O(\theta|P_{E_i})d\theta. \tag{4.13}$$

In the second equality, $A^O(\theta|P_{E_i})$ is the updated prior (Definition 1.9) specified in (4.12).

Next, we substitute primary modeller's opinion (2.4) and updated prior $A^O(\theta|P_{E_i})$ (4.12) into (4.13).

Substituting $P_I(X|\theta)$ (2.4) into (4.13) gives

$$\begin{aligned}
P_I^O(X=x) &= \int_{\mathbf{\Theta}} P_I(X=x|\theta)A^O(\theta|P_{E_i})d\theta \\
&= \int_{\mathbf{\Theta}} P_I(\mathcal{B}=b_I|\theta)P_I(F_{1,\mathcal{B}}=f_{1,b_I}|\mathcal{B},\theta)\cdots\times P_I(F_{n,\mathcal{B}}=f_{n,b_I}|\mathcal{B},\theta)A^O(\theta|P_{E_i})d\theta \\
&= \frac{1}{\mathcal{N}_{\hat{A}(\theta|P_{E_i})}} \int_{\mathbf{\Theta}} \theta_{b_I}\theta_{f_{1,b_I}}\cdots\theta_{f_{n,b_I}} \prod_b \prod_{f_{1,b},f_{2,b},\ldots,f_{n,b}} \theta_b^{n_b-1}\theta_{f_{1,b}}^{V_{f_{1,b}}-1}\cdots\theta_{f_{n,b}}^{V_{f_{n,b}}-1} d\theta \\
&= \frac{1}{\mathcal{N}_{\hat{A}(\theta|P_{E_i})}} \int_{\mathbf{\Theta}} \prod_b \prod_{f_{1,b},f_{2,b},\ldots,f_{n,b}} \theta_b^{n_b+\delta(b,b_I)-1}\theta_{f_{1,b}}^{V_{f_{1,b}}+\delta(b,b_I)\delta(f_{1,b},f_{1,b_I})-1}\cdots \\
&\times \theta_{f_{n,b}}^{V_{f_{n,b}}+\delta(b,b_I)\delta(f_{n,b},f_{n,b_I})-1} d\theta \\
&= \frac{1}{\mathcal{N}_{\hat{A}(\theta|P_{E_i})}} \int_{\mathbf{\Theta}} \prod_b \theta_b^{n_b+\delta(b,b_I)-1} \prod_j \prod_{f_{j,b}} \theta_{f_{j,b},b}^{V_{f_{j,b}}+\delta(b,b_I)\delta(f_{j,b},f_{j,b_I})-1} d\theta.
\end{aligned} \tag{4.14}$$

Similarly to the calculation in (4.10), the integral is a product of Dirichlet distributions [27]

$$P_I^O(X) = \frac{1}{\mathcal{N}_{\hat{A}(\theta|P_{E_i})}} \prod_b \frac{\Gamma(n_b + \delta(b, b_I))}{\Gamma(\sum_b (n_b + \delta(b, b_I)))} \prod_j \frac{\prod_{f_{j,b}} \Gamma(V_{f_{j,b}} + \delta(b, b_I)\delta(f_{j,b}, f_{j,b_I}))}{\Gamma(\sum_{f_{j,b}} (V_{f_{j,b}} + \delta(b, b_I)\delta(f_{j,b}, f_{j,b_I})))}$$
$$= \frac{\frac{n_{b_I}}{\sum_b n_b} \prod_b \frac{\Gamma(n_b)}{\Gamma(\sum_b n_b)} \prod_j \frac{V_{f_{j,b_I}}}{\sum_{f_{j,b_I}} V_{f_{j,b_I}}} \frac{\prod_{f_{j,b}} \Gamma(V_{f_{j,b}})}{\Gamma(\sum_{f_{j,b}} V_{f_{j,b}})}}{\prod_b \frac{\Gamma(n_b)}{\Gamma(\sum_b n_b)} \prod_j \frac{\prod_{f_{j,b}} \Gamma(V_{f_{j,b}})}{\Gamma(\sum_{f_{j,b}} V_{f_{j,b}})}}$$
$$= \frac{n_{b_I}}{\sum_b n_b} \prod_j \frac{V_{f_{j,b_I}}}{\sum_{f_{j,b_I}} V_{f_{j,b_I}}}, \tag{4.15}$$

where in the second equality, we have inserted the normalisation constant (4.11) and used a property of the gamma function $\Gamma(z+1) = z\Gamma(z)$, where $z > 0$ [31].

The final form of the primary modeller's updated opinion $P_I^O(X)$ (Definition 1.10) is:

$$\begin{aligned} P_I^O(X = x) &= P_I^O(\mathcal{B} = b_I, F_{1,b_I} = f_{1,b_I}, \ldots, F_{n,b_I} = f_{n,b_I}) \\ &= P_I^O(\mathcal{B}) P_I(F_{1,\mathcal{B}}|\mathcal{B}) P_I^O(F_{2,\mathcal{B}}|, \mathcal{B}) \cdots P_I^O(F_{n,\mathcal{B}}|\mathcal{B}) \\ &= \frac{n_{b_I}}{\sum_b n_b} \prod_j \frac{V_{f_{j,b_I}}}{\sum_{f_{j,b_I}} V_{f_{j,b_I}}}, \end{aligned} \tag{4.16}$$

where we used the chain rule [25] in the second equality and the assumption that the features $F_{j,b}$ are conditionally independent.

Appendix C

Tables containing simulated expert opinions.

Experts	Price	Battery life	Camera quality
Samsung expert 1	4	3	4
Samsung expert 2	5	3	5
Samsung expert 3	5	6	5
Samsung expert 4	6	5	3
Samsung expert 5	6	6	6
Samsung expert 6	5	6	5
Samsung expert 7	6	6	5
Samsung expert 8	6	3	4
Samsung expert 9	4	5	4
Samsung expert 10	6	4	3

Table 4.8: Samsung experts' simulated opinions.

Experts	Price	Battery life	Camera quality
iPhone expert 1	5	5	5
iPhone expert 2	5	6	5
iPhone expert 3	3	4	4
iPhone expert 4	3	4	5
iPhone expert 5	4	5	5
iPhone expert 6	5	6	4
iPhone expert 7	6	6	6
iPhone expert 8	5	6	6
iPhone expert 9	4	3	4
iPhone expert 10	4	6	3

Table 4.9: Iphone experts' simulated opinions.

Experts	Price	Battery life	Camera quality
Xiaomi expert 1	3	4	3
Xiaomi expert 2	3	3	4
Xiaomi expert 3	4	3	4
Xiaomi expert 4	3	3	3
Xiaomi expert 5	5	4	3
Xiaomi expert 6	3	5	5
Xiaomi expert 7	4	5	6
Xiaomi expert 8	4	3	2
Xiaomi expert 9	3	4	3
Xiaomi expert 10	4	3	4

Table 4.10: Xiaomi experts' simulated opinions.