České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

# Metamateriály na zakřivených varietách

# Metamaterials on curved manifolds

Master Thesis

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*Studijní program:* Matematická fyzika

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*Pokyny pro vypracování:*

1) Laplace-Beltramiho operátor ve Fermiho souřadnicích na varietách.
2) Realizace křivých metamaterálů coby neeliptických samosdružených operátorů.
3) Lokalizace esenciálního spektra pro kritické a nekritické kontrasty.
4) Spektrálně teoretický přístup k efektu neviditelnosti.

*Doporučená literatura:*

[1] J. Behrndt and D. Krejcirik, An indefinite Laplacian on a rectangle, J. Anal. Math. 134 (2018), 501-522.

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# PROHLÁŠENÍ

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**Metamateriály na zakřivených varietách**

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*Název práce:*

**Metamateriály na zakřivených varietách**

*Autor:* Bc. Tomáš Faikl

*Studijní program:* Matematická fyzika

*Druh práce:* Diplomová práce

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*Abstrakt:* Cílem práce je prozkoumání asymptotického chování vlastních hodnot indefinitního laplaciánu na obdélníku vnořeného do riemannovských variet s konstantní křivostí. Hlavní motivací je efekt neviditelnosti v metamateriálech se zápornou permitivitou či permeabilitou, odpovídající operátorově-teoretický popis pomocí existence esenciálního spektra a výsledky studenta z jeho bakalářské práce o vlivu křivosti ambientní variety na spektrální vlastnosti. Práce dává charakterizaci esenciálního spektra operátoru na zakřivené riemannovské varietě.

*Klíčová slova:* indefinitní Laplacián, metamateriál, spektrální analýza, varieta, esenciální spektrum

*Title:*

**Metamaterials on curved manifolds**

*Author:* Bc. Tomáš Faikl

*Abstract:* The aim of the thesis is to investigate the asymptotic behavior of eigenvalues of the indefinite Laplacian on a rectangle embedded in Riemannian manifold with constant curvature. The main motivation is the effect of invisibility in metamaterials with negative permittivity or permeability, the corresponding operator-theoretic description by means of the existence of an essential spectrum, and results from the author's bachelor's thesis on the effect of curvature of the ambient manifold on spectral properties. The project gives a characterization of the essential spectrum of the operator on the curved Riemannian manifold.

*Key words:* indefinite Laplacian, metamaterial, spectral analysis, manifold, essential spectrum

# Contents

# Introduction

In this thesis, we concern ourselves with an indefinite Laplacian on a bounded rectangle on constantly curved surfaces containing a material-metamaterial interface. The indefinite Laplacian frequently occur in mathematical models of metamaterials characterized by negative permitivity and/or permeability. These metamaterials can lead to negative refractive index and interesting optical effects such as *metamaterial cloaking* and superlensing.

It is known from work of other authors (presented later in this thesis) that the operator on a rectangle in flat underlying space $\mathbb{R}^n$, $n \geq 2$, has non-empty essential spectrum and contains zero exactly when a parameter called contrast is "critical" — that is an unusual effect on bounded domain caused by a domain transmission condition on the interface.

We want to explore the effects of curvature on the spectrum of the indefinite Laplacian. Mathematically, the operator considered does not possess ellipticity, nor is semi-bounded and so, standard form-theoretic methods theory do not apply directly.

The thesis is organized as follows. First, we will sketch the problem and provide physical motivation in terms of quasi-static approximation to Maxwell equations and special *metamaterials* with negative permitivity and/or permeability. We then proceed to give a brief overview on the available literature for various cases and generalizations of the problem.

We proceed to give an overview of some main areas of mathematical theory used in the thesis, such as the notion of exponential maps and curvature of Riemannian manifolds and later also Hilbert spaces of square-integrable functions defined on Riemannian manifolds.

After introducing the necessary concepts, we define formal geometrical setting of the problem as a tubular neighbourhood on a constantly-curved Riemannian manifold in terms of exponential map and normal coordinates and finally provide definition of the operator as an indefinite Laplacian on the curved bounded tubular neighbourhood, given in terms of Laplace-Beltrami operator. As was proven in author's previous work, it can be given as an essentially self-adjoint Dirichlet realization.

Finally, we present original results and approaches to spectral analysis of the indefinite Laplacian. By constructing singular Weyl sequences, we obtain that zero is in the essential spectrum whenever a certain parameter called *contrast* is "critical". Then, we refine the argument in case of zero curvature to show that zero is the only point of the essential spectrum and is empty when the contrast is non-critical. The asymptotic analysis of the characteristic equation for eigenvalues in the curved case is much more involved due to presence of associated Legendre functions. We were able to obtain conclusive results only for positive curvature.

The last chapter provides a rather general and elegant approach using forms. We were able to prove emptiness of essential spectrum in non-critical contrast case regardless of curvature and also sketch possible generalizations regarding non-constant curvature.

# Chapter 1

# Motivation

## 1.1 Physical motivation

We will entertain the quasi-static approximation to Maxwell equations for a scalar electric potential. In this framework, the electric and magnetic fields are no longer dependent on the counterpart's field time derivatives. This way, the problems for electric and magnetic field separate. In the following, we will work only with the electric field

$$\operatorname{div} \vec{D} = \rho, \qquad \operatorname{rot} \vec{E} = 0. \tag{1.1}$$

These equations are the Gauss and Faraday law without presence of magnetic field. From differential identity $\operatorname{rot} \operatorname{grad} = 0$ we can see that the field $\vec{E} = -\operatorname{grad} V$ can be described by a potential. Combining with relation for homogeneous material $\vec{D} = \epsilon \vec{E}$, we obtain for the potential $V$

$$-\operatorname{div}(\epsilon \operatorname{grad} V) = \rho. \tag{1.2}$$

In this thesis, we will examine this operator on the left-hand side of (1.2) (as a special case) in a bounded rectangular domain $\Omega \subset \mathbb{R}^2$ in the Hilbert space of $L^2(\Omega)$

$$\begin{aligned} \tilde{A} &= -\operatorname{div} \epsilon \operatorname{grad}, \\ \operatorname{dom} \tilde{A} \ &:= \{\psi \in L^2(\Omega) \mid \tilde{A}\psi \in L^2(\Omega), \psi \,|_{\partial\Omega} = 0\}, \\ \epsilon(x) &:= \begin{cases} \epsilon_+, & x \in \Omega_+, \\ -\epsilon_-, & x \in \Omega_-, \end{cases} \end{aligned} \tag{1.3}$$

for $\Omega = \Omega_+ \cup \Omega_-$, $\Omega_\pm$ rectangular, and constant $\epsilon_\pm > 0$ — the jump signifying the transition between a metamaterial with negative permitivity in $\Omega_-$ and a classical material with positive permitivity in $\Omega_+$. The interface between them will be denoted as $\Gamma$. The operators will be properly defined in the next chapter.

The main focus is to explore the operator on a submanifold $\Omega$ of a two-dimensional Riemannian manifold with a constant Gaussian curvature, importantly in the case of critical contrast $\kappa := \frac{\epsilon_+}{\epsilon_-} = 1$. The geometry model used here is due to Krejčiřík and Siegl [29], although used for a different operator setting.

Detailed self-adjoint realisation in [6] is examined in the case of a particular rectangle in $\mathbb{R}^2$. Complementary results for smooth $\Omega$ and $\Omega_\pm \in \mathbb{R}^n$ with smooth interface $\Gamma$ is discussed and solved in [12] and some applicable results from the references below. In two dimensions, 0 is in the essential spectrum whenever $\kappa = 1$. Note that the problem does not directly lead to an elliptic or semibounded operator and hence is outside the standard frameworks. It was found that the functions in domain of a self-adjoint realisation of $\tilde{A}$ do not belong to any local Sobolev space $H^s$, $s > 0$ in the case of a symmetric flat rectangle in $\mathbb{R}^2$. The case of a rectangle in higher dimensions was examined in [26].

The main motivation of this work is the metamaterial cloaking phenomenon, for mathematical-oriented survey of recent progress, see [37] and [20] for general information on metamaterials.

## 1.2 State of research

One of the first conducted research of mathematical properties of operators appearing in the material-metamaterial interface problems was published in 1999 [9]. There, the authors consider analysis of the problem

$$\begin{gathered} \operatorname{dom} A = \left\{ u \in H_0^1(\Omega) : \operatorname{div}(\epsilon \nabla u) \in L^2(\Omega) \right\}, \\ Au = -\operatorname{div}(\epsilon \nabla u), \quad \forall u \in \operatorname{dom} A, \qquad \epsilon(x) := \begin{cases} \epsilon_+, & x \in \Omega_+, \\ -\epsilon_-, & x \in \Omega_-, \end{cases} \end{gathered} \tag{1.4}$$

for $\epsilon_\pm > 0$, $\epsilon_+ \neq \epsilon_-$ such that $\Omega = \Omega_+ \cup \Omega_-$, boundary $\Sigma$ of $\Omega$ sufficiently regular, boundary $\Gamma_-$ of $\Omega_-$ Lipschitz continuous and $\Sigma \cap \Gamma_- = \emptyset$. For this non-critical contrast, the resulting operator is self-adjoint, has a compact resolvent and its eigenvalues are accumulating to $\pm\infty$. It is also of interest that when interface $\Gamma$ is not smooth, for example when there is a right-angle *corner* on $\Gamma$, then the results extend to values of contrast $\kappa := \frac{\epsilon_+}{\epsilon_-}$ which do not belong to some interval containing the critical contrast of 1. For values of contrast $\kappa$ inside the critical interval, the operator $A$ is not self-adjoint.

Regarding the mathematical justification of negative permitivity and/or permeability appearing in Maxwell equations, these parameters are negative only effectively — they are negative in a sense of homogenisation, i.e. only when electromagnetic waves have wavelength much greater than typical distance of the metamaterial structure. See references [10, 11, 19, 30, 31] for splitring resonators and bulk dielectric inclusions to achieve effectively negative parameters also near resonant effects in the media in both $\Omega \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$.

In [14], the authors explore well-posedness of system

$$\operatorname{div}\left(\frac{1}{\epsilon}\nabla u\right) + \omega^2 \mu u = f \quad \text{in } \Omega \tag{1.5}$$

with Dirichlet boundary conditions in $H^1(\Omega)$ for the case of sign-changing permitivity $\epsilon$ on an interface and source $f \in L^2(\Omega)$. The problem was reformulated in a variational approach and allowed to tackle non-constant permitivities $\epsilon_\pm$ and Lipschitz-regular interface $\Gamma$. In a following paper [8], the authors applied the framework of $\mathbb{T}$-coercivity to achieve better results. In the following paper [7], the framework was improved to prove results about well-posedness based only on the localisation of values of $\epsilon$ to the neighbourhood of interface $\Gamma$. A large quantity of examples was provided for domains in both $\mathbb{R}^2$ and $\mathbb{R}^3$. Finally, similar results were derived for the full non-scalar Maxwell problem in [15] in a time-harmonic case. Important observations are that the time-harmonic Maxwell problem can be fully solved in terms of scalar problems which highlights importance of studying the scalar problems introduced so far.

The situation in $\Omega \subset \mathbb{R}$ is covered in [27, 25, 18]. For general domains with smooth interface $\Gamma$ (this is the case of this thesis for zero Gaussian curvature of the Riemannian manifold), see [12].

## 1.3 Anomalous localized resonance

Regarding the invisibility cloaking phenomena, a framework used for mathematically rigorous description is that of Anomalous Localized Resonance (ALR). When the cloaking phenomenon occurs, it is usually spoken of as Cloaking due to Anomalous Localized Resonance (CALR) [35].

Following [34], we say an inhomogeneous body exhibits ALR if as the loss (imaginary part of permitivity/permeability) goes to zero (or for static problems, as the system of equations lose ellipticity) the field magnitude diverges to infinity throughout a specific region with sharp boundaries not defined by any discontinuities in the moduli, but the field converges to a smooth field outside that region. A region where the field diverges will be called a region of local resonance. For equivalent conditions for ALR to happen, see [38].

For quasi-static results, consult [33] where they concern themselves with quasi-static approximation. They prove that CALR can, at least approximately, occur in the case of quasi-static approximation in $\mathbb{R}^2$ and even in $\mathbb{R}^3$ for some simple setups.

In paper [4], they consider the dielectric problem above with a source term $\alpha f$, proportional to $f$. The domain is the typical annulus and permitivity with loss going to zero. Cloaking of the source is achieved in a region external to the metamaterial. The cloaking issue is directly linked to the existence of ALR. For a fixed dipolar source within a critical distance of the metamaterial, the total electrical power absorbed would become infinite as $\delta \to 0$, which is unphysical. The anomalously resonant fields interact with the source which results in the source having to do a large amount of work to maintain its amplitude; in fact, an infinite amount of work in the limit $\delta \to 0$. Therefore, it makes sense to normalize the source term (by adjusting $\alpha$, letting it depend on $\delta$) so the source supplies power at a constant rate independent of $\delta$. Then outside the region where ALR occurs the field tends to zero as $\delta \to 0$: the source becomes cloaked.

The ALR in $\Omega \subset \mathbb{R}^2$ is usually achieved in an annulus setting with piece-wise constant $\epsilon$,$\mu$. It was proved, that in $\mathbb{R}^3$, the ALR does not occur in the same setup — it does occur, however, when $\epsilon$ and $\mu$ are carefully chosen tensors.

An important connection between the ALR procedure for loss going to zero, $\epsilon = (-1 - \mathrm{i}\delta)1_{\Omega_-} + 1_{\Omega_+}$, $\delta = \Im\epsilon \to 0$, and properties of the limit operator with $\epsilon = -1_{\Omega_-} + 1_{\Omega_+}$ for smooth interface $\Gamma$ is presented in [12].

# Chapter 2

# Mathematical prerequisites

## 2.1 Notation

For a complex number $x \in \mathbb{C}$, we denote its real and imaginary parts as

$$x = \Re x + \mathrm{i}\Im x \tag{2.1}$$

for imaginary unit $\mathrm{i}^2 = -1$.

**Definition 2.1.** Let $C_0^\infty(U)$ denote the space of infinitely differentiable functions $\phi : U \to \mathbb{R}$, with compact support in $U$. We will sometimes call a function $\phi$ belonging to $C_0^\infty(U)$ a test function.

## 2.2 Asymptotics

For references on asymptotics, see [39].

**Definition 2.2.** Let $f : X \to \mathbb{C}$ be a real or complex function, $g : X \to \mathbb{R}$ a real function, $g \geq 0$, both defined on an unbounded subset $X$ of positive real numbers. We say that

$$f(x) = \mathcal{O}\big(g(x)\big) \quad \text{as } x \to +\infty \tag{2.2}$$

if there exists $M > 0$ and $x_0 \in \mathbb{R}$ such that

$$\big|f(x)\big| \leq Mg(x), \quad \forall x \geq x_0. \tag{2.3}$$

**Definition 2.3.** Let $f$ be a real or complex function, $g$ a real function, $g \geq 0$, both defined on an unbounded subset or positive real numbers. We say that

$$f(x) = o\big(g(x)\big) \quad \text{as } x \to +\infty \tag{2.4}$$

if for every $\epsilon > 0$ there exists $x_0 \in \mathbb{R}$ such that

$$\big|f(x)\big| \leq \epsilon g(x), \quad \forall x \geq x_0. \tag{2.5}$$

## 2.3 Geometry

We will assume familiarity with basic geometric concepts. For more details, see for example [32, 41]. We will use the Einstein summation convention unless stated otherwise.

**Definition 2.4.** Let $M$ be a manifold, and let $\mathcal{X}(M)$ denote the space of smooth vector fields on $M$ or equivalently, sections of the tangent bundle $TM$. A linear (or affine) *connection* on $M$ is a map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M), \tag{2.6}$$

written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties for $X_1, X_2, Y_1, Y_2 \in \mathcal{X}(M)$:

$$\begin{aligned} \nabla_{fX_1+gX_2} Y &= f\nabla_{X_1} Y + g\nabla_{X_2} Y \quad \text{for } f, g \in C^\infty(M), \\ \nabla_X (aY_1 + bY_2) &= a\nabla_X Y_1 + b\nabla_X Y_2 \quad \text{for } a, b \in \mathbb{R}, \\ \nabla_X (fY) &= f\nabla_X Y + (Xf)Y \quad \text{for } f \in C^\infty(M). \end{aligned} \tag{2.7}$$

**Note 2.5.** For a linear connection $\nabla$, we can introduce *Christoffel symbols* $\Gamma^k_{ij}$ of $\nabla$ given, in a local coordinate chart $E_i = \frac{\partial}{\partial x^i}$, by

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k. \tag{2.8}$$

Then, for $X, Y \in \mathcal{X}(M)$, given in a local frame by $X = X^i E_i$, $Y = Y^i E_i$,

$$\nabla_X Y = \left( XY^k + X^i Y^j \Gamma^k_{ij} \right) E_k. \tag{2.9}$$

**Definition 2.6.** Let $M$ be a manifold with linear connection $\nabla$ and let $\gamma : I \to M$ be a smooth curve in $M$. Curve $\gamma$ is called a geodesic with respect to $\nabla$ if

$$\nabla_{\dot\gamma} \dot\gamma = 0 \quad \text{on } \gamma. \tag{2.10}$$

**Theorem 2.7** (Existence and Uniqueness of Geodesics)**.** *Let $M$ be a manifold with a linear connection. For any $p \in M$, any $V \in T_pM$, and any $t_0 \in \mathbb{R}$, there exist an open interval $I \subset \mathbb{R}$ containing $t_0$ and a geodesic $\gamma : I \to M$ satisfying $\gamma(t_0) = p$, $\dot\gamma(t_0) = V$. Any two such geodesics agree on their common domain.*

**Definition 2.8.** A Riemannian metric on a smooth manifold $M$ is a 2-tensor field $g$ on $M$ that is symmetric, $g(X, Y) = g(Y, X)$, and positive definite. The manifold $M$ together with a given Riemannian metric is called a Riemannian manifold $(M, g)$.

**Lemma 2.9.** *There is a unique connection $\nabla$ on a Riemannian manifold $(M, g)$ that is compatible with $g$:*

$$\nabla_X \, g(Y, Z) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right) \tag{2.11}$$

*and is symmetric. This connection is called* Levi-Civita connection *and it is assumed to be the default connection when mentioning Riemannian manifolds from now on.*

Theorem 2.7 implicitly defines a map from the tangent bundle to the set of geodesics in $M$, or a map from (a subset of) the tangent bundle to $M$ itself, by sending the vector $V$ to the point obtained by following $\gamma_V$ for time 1. In text below, we will assume that $\gamma_V$ is maximal, i.e. it cannot be non-trivially extended to a larger domain.

**Definition 2.10.** Let $(M, g)$ be a Riemannian manifold. *Exponential map* is a map

$$\exp : \mathcal{E} \to M, \quad \exp V := \gamma_V(1), \tag{2.12}$$

where

$$\mathcal{E} := \left\{ V \in TM : \exists\, \gamma_V : I \supset [0,1] \to M \right\}. \tag{2.13}$$

The *restricted exponential map* is given by

$$\exp_p := \exp|_{\mathcal{E}_p}, \quad \mathcal{E}_p := \mathcal{E} \cap T_pM. \tag{2.14}$$

**Definition 2.11.** If $(M, g)$ is a Riemannian manifold and $\gamma : I \to M$ is a unit-speed curve (parametrized by arc length), we define *geodesic curvature* of $\gamma$ as the function $\kappa : I \to \mathbb{R}$ given by

$$\kappa(t) = \left| \nabla_{\dot\gamma} \dot\gamma(t) \right|. \tag{2.15}$$

In further definitions regarding curvature, we will assume that $(\tilde M, \tilde g)$, $(M, g)$ are Riemannian manifolds, $M \subset \tilde M$, $\iota : M \to \tilde M$ is injective immersion and $g = \iota^* \tilde g$ for pull-back $\iota^*$ and $\tilde M$ is called an ambient space. Then, for each $p \in M$, we have orthogonal direct sum

$$T_p\tilde M = T_pM \oplus \left(T_pM\right)^{\perp}, \tag{2.16}$$

where $\left(T_pM\right)^{\perp} =: N_pM$ is the normal space at point $p$ and $NM := \coprod_{p \in M} N_pM$ is the normal bundle. Notation $\mathcal{N}(M)$ is used for sections on the normal bundle.

If $X, Y \in \mathcal{X}(M)$ are vector fields in $M$, we can extend them to vector fields in $\tilde M$ and then

$$\tilde\nabla_X Y = \left(\tilde\nabla_X Y\right)^{\top} + \left(\tilde\nabla_X Y\right)^{\perp}, \tag{2.17}$$

where $\tilde\nabla$ is the Levi-Civita connection on $\tilde M$.

**Definition 2.12.** *Second fundamental form* of $M$ is defined as the map $\mathrm{I\!I} : \mathcal{X} \times \mathcal{X} \to \mathcal{N}(M)$ given by

$$\mathrm{I\!I}(X, Y) := \left(\tilde\nabla_X Y\right)^{\perp}, \tag{2.18}$$

with $X$ and $Y$ extended to $\tilde M$ arbitrarily.

Consider the special case of $\tilde M = \mathbb{R}^{n+1}$, $\dim M = n$. Then at each point of $M$, there are exactly two unit normal vectors. If $M$ is orientable, the orientation can be used to select the normal vector, otherwise we restrict ourselves to subset of $M$ so that it is orientable. The resulting vector field $N$ is smooth section of $NM$.

**Definition 2.13.** The *scalar second fundamental form* $h : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$ is the symmetric 2-tensor field on $M$ defined by

$$h(X, Y) = g(\mathrm{I\!I}(X, Y), N). \tag{2.19}$$

Since $NM = \operatorname{span} N$, we have

$$\mathrm{I\!I}(X, Y) = h(X, Y)N. \tag{2.20}$$

and local coordinates,

$$h = h_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j, \quad h_{ij} = h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right). \tag{2.21}$$

**Definition 2.14.** By lifting one index of $h$, we get the 1,1-tensor field *shape operator* $s : \mathcal{X}(M)^* \times \mathcal{X}(M) \to C^\infty(M)$ by

$$s = h^i_{\ j} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j, \quad h^i_{\ j} := g^{ik} h_{kj}. \tag{2.22}$$

This tensor field can be identified with endomorphism

$$\hat{s} : \mathcal{X}(M) \to \mathcal{X}(M), \quad s(\omega, X) =: \omega(\hat{s}X) \tag{2.23}$$

for $\omega \in \mathcal{X}^*(M)$, $X \in \mathcal{X}(M)$. Then,

$$g\left(\hat{s}X, Y\right) = h\left(X, Y\right) \tag{2.24}$$

for all $X, Y \in \mathcal{X}(M)$.

**Lemma 2.15.** *As $h$ is symmetric, $\hat{s}$ is self-adjoint and we have for the real eigenvalues*

$$\sigma(\hat{s}) =: \{\kappa_1, \ldots, \kappa_n\}. \tag{2.25}$$

**Definition 2.16.** *Gaussian curvature* $K : M \to \mathbb{R}$ is defined as

$$K := \det \hat{s} = \kappa_1 \cdots \kappa_n. \tag{2.26}$$

**Theorem 2.17** (Gauss's Theorema Egregium)**.** *Let $M \subset \mathbb{R}^3$ be a 2-dimensional Riemannian submanifold. Then Gaussian curvature $K$ is an isometry invariant of $(M, g)$. By suitable definition of Gaussian curvature of general $(M, g)$, it can be shown that it does not depend on particular immersion of $M$ into $\tilde{M}$ and is an intrinsic property of $(M, g)$.*

## 2.4 Analysis

For literature on functional-analysis and spectral parts of this thesis, we recommend literature [17, 16, 2]. For Sobolev spaces on manifolds, see [23], for complex analysis references, see [40].

**Theorem 2.18** (Identity theorem)**.** *Suppose $\Omega \subset \mathbb{C}$ is a connected open subset, $f$ analytic in $\Omega$ and*

$$Z(f) := \left\{a \in \Omega : f(a) = 0\right\}. \tag{2.27}$$

*Then either $Z(f) = \Omega$, or $Z(f)$ has no limit point in $\Omega$. In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that*

$$f(z) = (z - a)^m g(z), \quad z \in \Omega \tag{2.28}$$

*where $g$ is analytic in $\Omega$ and $g(a) \neq 0$. Furthermore, $Z(f)$ is at most countable.*

**Definition 2.19.** Suppose $u, v \in L^1_{\text{loc}}$ are locally integrable functions and $\alpha$ is a multiindex. We say that $v$ is the $\alpha^{\text{th}}$ weak derivative of $u$, written

$$D^\alpha u = v, \tag{2.29}$$

provided

$$\int_U u D^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_U v\phi \,\mathrm{d}x \tag{2.30}$$

for all test functions $\phi \in C_0^\infty(U)$.

**Definition 2.20.** The Sobolev spaces $W^{k,p}(U)$ consists of all locally integrable functions $u : U \to \mathbb{R}$ such that for each multiindex $\alpha$ with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$. We identify functions which agree almost everywhere.

The space $W^{k,p}(U)$ is equipped with a norm

$$||u||_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^{|p} \, \mathrm{d}x\right)^{1/p}, & 1 \leq p < \infty, \\ \sum_{|\alpha|\leq k} \text{ess sup}_U |D^\alpha u|, & p = \infty, \end{cases} \tag{2.31}$$

making it into a Banach space. We write $W^{k,2}(U) = H^k(U)$ for the Hilbert spaces when $p = 2$.

**Note 2.21.** An alternative definition of Sobolev spaces for $p \geq 1$ is as a completion of

$$\left\{u \in C^\infty(U) : ||u||_{W^{k,p}} < \infty\right\} \tag{2.32}$$

for the norm $|| \cdot ||_{W^{k,p}}$ given above.

**Definition 2.22.** We denote by $W_0^{k,p}$ the closure of $C_0^\infty(U)$ in $W^{k,p}(U)$.

**Remark 2.23.** On a Riemannian manifold $(M, g)$ with dimension $n$, Lebesgue integral with measure $\mathrm{d}\nu(g)$ can be introduced. Informally, in local coordinates,

$$\mathrm{d}\nu(g) = \sqrt{|\det g|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{2.33}$$

For formally correct introduction of Lebesgue measure, see the reference.

**Definition 2.24.** Let $(M, g)$ be a smooth Riemannian manifold. For $k$ integer and $u : M \to \mathbb{R}$ smooth, we denote by $\nabla^k$ the $k^{\text{th}}$ covariant derivative of $u$ and $|\nabla^k u|$ is the norm of $\nabla^k u$ defined in a local chart by

$$\left|\nabla^k u\right| = g^{i_1 j_1} \cdots g^{i_k j_k} \left(\nabla^k u\right)_{i_1 \dots i_k} \left(\nabla^k u\right)_{j_1 \dots j_k}, \tag{2.34}$$

where $g^{ij} = (g^{-1})_{ij}$, $(\nabla^k u)_{i_1 \dots i_k} = \nabla_{\frac{\partial}{\partial x^{i_1}}} \dots \nabla_{\frac{\partial}{\partial x^{i_k}}} u$ and the $\nabla$ is the natural extension of the linear connection to general tensor fields given in [23, 32].

Note that $(\nabla u)_i = \partial_i u$, while $\left(\nabla^2 u\right)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u$. Given integer $k$ and $p \geq 1$, set

$$\mathcal{C}^{k,p}(M) := \left\{u \in C^\infty(M) : \forall j = 0, \dots, k, \int_M |\nabla^j u|^p \, \mathrm{d}\nu(g) < +\infty\right\}. \tag{2.35}$$

For $u \in \mathcal{C}_k^p(M)$, set

$$||u||_{W^{k,p}} = \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p \, \mathrm{d}\nu(g)\right)^{1/p}. \tag{2.36}$$

The Sobolev space $W^{k,p}(M)$ is then defined as the completion of $\mathcal{C}^{k,p}(M)$ with respect to $||\cdot||_{W^{k,p}}$.

And finally, we will assume familiarity with theory of linear unbounded operators on Hilbert spaces, for example notions of closure of an operator, its adjoint, symmetric and self-adjoint operators and their spectral theory. See [28, 13] for references.

**Definition 2.25.** Let $H$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then the set of all $\lambda \in \mathbb{C}$ such that $\lambda$ is an eigenvalue of finite multiplicity and it is an isolated point of the spectrum $\sigma(H)$ is called the *discrete spectrum* and is usually denoted $\sigma_d(H)$. The complement

$$\sigma_{\text{ess}}(H) := \sigma(H) \setminus \sigma_d(H) \tag{2.37}$$

is called the *essential spectrum* of $H$.

**Lemma 2.26** ([24]). *Let $H$ be an operator on a Hilbert space $\mathcal{H}$. The essential spectrum of a self-adjoint operator $H$ is then*

$$\sigma_{ess}(H) = \left\{ \lambda \in \mathbb{C} : \exists \text{ non-compact } \{\psi_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} H,\ \|\psi_n\| = 1,\ \lim_{n\to\infty} \|H\psi_n - \lambda\psi_n\| = 0 \right\}.$$

*Non-compact sequences are those that contain no converging subsequence. The sequences $\{\psi_n\}_{n\in\mathbb{N}}$ with such property are called* singular.

**Note 2.27.** For spectrum $\sigma(H)$, there is

$$\sigma(H) = \left\{ \lambda \in \mathbb{C} : \exists\ \{\psi_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} H,\ \|\psi_n\| = 1,\ \lim_{n\to\infty} \|H\psi_n - \lambda\psi_n\| = 0 \right\}. \tag{2.38}$$

**Definition 2.28.** An operator $A$ is *essentially self-adjoint* if it is symmetric and its closure $\overline{A}$ is self-adjoint.

**Theorem 2.29** ([13]). *Let $H$ be a symmetric operator on a Hilbert space $\mathcal{H}$ with domain $L$, and let $\{f_n\}_{n=i}^{\infty}$ be a complete orthonormal set in $\mathcal{H}$. If each $f_n$ lies in $L$ and there exist $\lambda_n \in \mathbb{R}$ such that $Hf_n = \lambda_n f_n$ for every $n$, then $H$ is essentially self-adjoint. Moreover, the spectrum of $H$ is the closure in $\mathbb{R}$ of the set of all $\lambda_n$.*

We will not use the following theorem directly, but we leave it here to compare to theorems in the last chapter. The following version holds for real Hilbert spaces, see [28, §6.2.1] for complex Hilbert spaces — it is usually called the first representation theorem.

**Theorem 2.30** (Lax-Milgram, [17]). *Let $\mathcal{H}$ be a real Hilbert space. Assume that*

$$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \tag{2.39}$$

*is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that it is bounded*

$$|a(u,v)| \le \alpha ||u|| ||v||, \quad u, v \in \mathcal{H}, \tag{2.40}$$

*and* coercive

$$a(u,u) \ge \beta ||u||^2, \quad u \in \mathcal{H}. \tag{2.41}$$

*Finally, let $f : \mathcal{H} \to \mathbb{R}$ be a bounded linear functional on $\mathcal{H}$. Then there exists a unique element $u \in \mathcal{H}$ such that*

$$a(u,v) = f(v), \quad \forall v \in \mathcal{H} \tag{2.42}$$

*and by Riesz theorem, there exists a unique $\eta \in \mathcal{H}$ such that*

$$a(u,v) = (\eta, v), \quad \forall v \in \mathcal{H}. \tag{2.43}$$

# Chapter 3

# Geometrical and functional problem setting

## 3.1 Geometrical setting

We will use the same geometrical setting as in author's previous work [18] heavily inspired by [29]. In this section, we will make frequent use of Sobolev spaces on manifolds [23] and Fermi coordinates [22] in tubular neighbourhoods [21]. See references for details.

Let us define a rectangular domain $\Omega_0 \subset \mathbb{R}^2$,

$$\Omega_0 = (-b, a) \times (0, c) \equiv J_1 \times J_2 \tag{3.1}$$

and denote $\Omega_{0+} = (0, a) \times J_2$, $\Omega_{0-} = (-b, 0) \times J_2$, $\Gamma_0 = \{0\} \times J_2$ and overall, we have a disjoint union $\Omega_0 = \Omega_{0-} \cup \Gamma_0 \cup \Omega_{0+}$. Let the metamaterial interface be $\mathcal{C} = \{0\} \times (0, c)$. The metamaterial is located in $\Omega_{0-} = (-b, 0) \times J_2$ and material with positive permitivity is located in $\Omega_{0+} = (0, a) \times J_2$.

Consider a two-dimensional Riemannian manifold $\mathcal{M}$ and assume that its Gaussian curvature $K$ is continuous (which holds if $\mathcal{M}$ is $C^3$-smooth or is embedded into $\mathbb{R}^3$). Additionally, let $\Gamma : J_2 \to \mathcal{M}$ be a $C^2$ parametrized by arc length. This curve $\Gamma$ will serve as a metamaterial interface. Let us introduce a tubular neighbourhood $\Omega$ of curve $\Gamma$. In case of $a = b$, $\Omega$ can be seen as a set of points on $\mathcal{M}$ with geodesic distance less than $a$ from $\Gamma$. Define a map $\mathcal{L} : \Omega_0 \to \mathcal{M}$ as

$$\mathcal{L}(x_1, x_2) := \exp_{\Gamma(x_2)} \left( x_1 N(x_2) \right), \quad (x_1, x_2) \in \Omega_0 \tag{3.2}$$

where $\exp_q$ is the exponential map of $\mathcal{M}$ at point $q \in \mathcal{M}$ and $N(x_2) \in \mathrm{T}_{\Gamma(x_2)}\mathcal{M}$ is a normal vector to curve $\Gamma$ in $x_2 \in J_2$, an element of tangent space to manifold $\mathcal{M}$. The coordinates are chosen so that $\mathcal{L}(\Gamma_0) = \Gamma$. Finally, denote

$$\Omega := \mathcal{L}(\Omega_0), \quad \Omega_- := \mathcal{L}(\Omega_{0+}), \quad \Omega_+ := \mathcal{L}(\Omega_{0-}). \tag{3.3}$$

In the following text, $\mathcal{L} : \Omega_0 \to \Omega$ will be assumed to be a diffeomorphism. Set $\Omega$ can be parametrized via the geodesic parallel coordinates $(x_1, x_2)$. It follows that

$$g = \begin{pmatrix} 1 & 0 \\ 0 & f^2 \end{pmatrix} \tag{3.4}$$

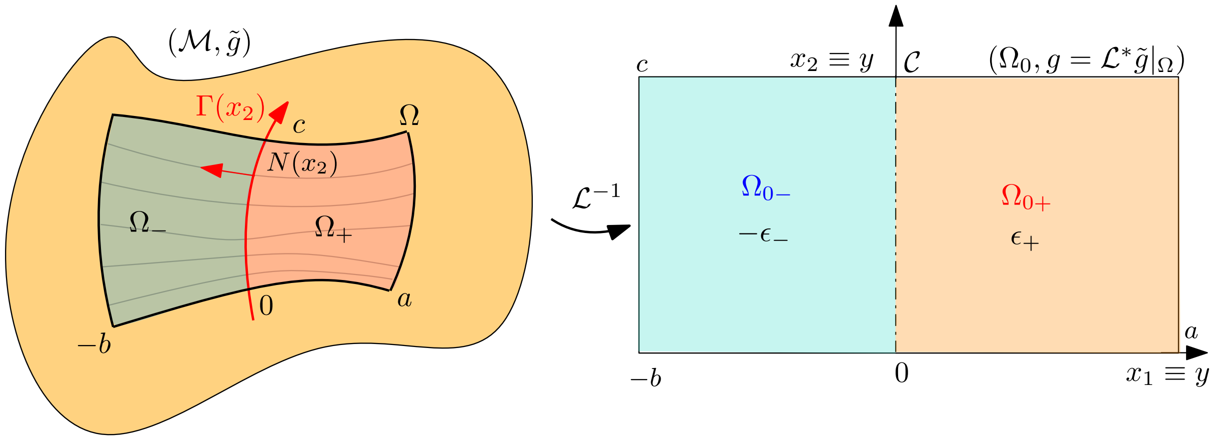



Figure 3.1: Curve $\Gamma$ on a Riemannian manifold and its tubular neighbourhood $\Omega$. At every point of $\Gamma$, there exists a geodesic perpendicular to $\Gamma$ which is used to construct a rectangle on the manifold. The tubular neighbourhood $\Omega$ is diffeomorphic to a rectangle $\Omega_0$ in Fermi coordinates $x_1, x_2$ (on the right) with induced metric $g = \operatorname{diag}(1, f^2)$.

where $f$ is continuous and has continuous partial derivatives $\partial_1 f$, $\partial_1^2 f$ satisfying the Jacobi equation

$$\partial_1^2 f + K f = 0 \quad \wedge \quad \begin{cases} f(0, \cdot) = 1 \\ \partial_1 f(0, \cdot) = -\kappa, \end{cases} \tag{3.5}$$

where $K$ is Gaussian curvature at a point with local coordinates $(x_1, x_2)$ and $\kappa$ is the geodesic curvature of $\Gamma$.

The solutions for constant Gaussian curvatures are

$$f(x_1, x_2) = \begin{cases} \cos(\sqrt{K} x_1) - \frac{\kappa(x_2)}{\sqrt{K}} \sin(\sqrt{K} x_1) & \text{if } K > 0, \\ 1 - \kappa(x_2) \cdot x_1 & \text{if } K = 0, \\ \cosh(\sqrt{|K|} x_1) - \frac{\kappa(x_2)}{\sqrt{|K|}} \sinh(\sqrt{|K|} x_1) & \text{if } K < 0, \end{cases} \tag{3.6}$$

and from now on, we will assume that the geodesic curvature of $\Gamma$ is identically zero, i.e.

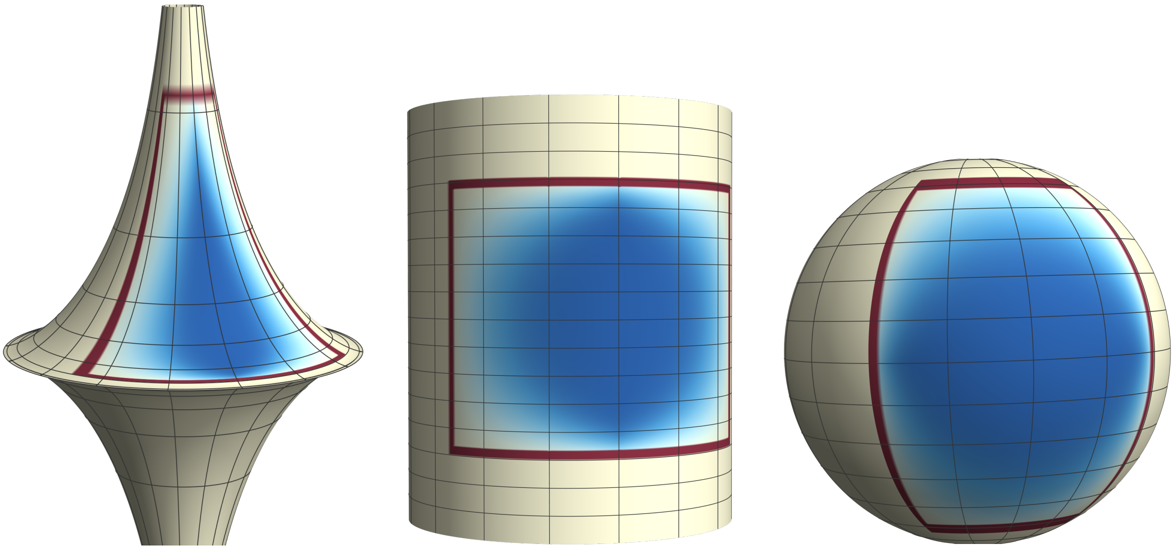
$$\boxed{\text{curve } \Gamma \text{ is a geodesic.}} \tag{3.7}$$

A manifold $(\mathcal{M}, g)$ with arbitrary $K \in \mathbb{R}$ is diffeomorphic to one with $K \in \{-1, 0, 1\}$. Hence, up to diffeomorphism, we can setup our problem in a $L^2(\Omega_0, g)$ space with measure $\mathrm{d}\nu_K(g)$ given by

$$\mathrm{d}\nu_K := \begin{cases} \cos(x_1)\, \mathrm{d}x_1\, \mathrm{d}x_2, & \text{if } K = +1, \\ \mathrm{d}x_1\, \mathrm{d}x_2, & \text{if } K = 0, \\ \cosh(x_1)\, \mathrm{d}x_1\, \mathrm{d}x_2, & \text{if } K = -1, \end{cases} \tag{3.8}$$

and in order for $g$ to be a positive-definite metric, we have to restrict the dimensions of the rectangle in case $K = +1$ to

$$\boxed{K = +1 \implies a, b \in \left(0, \frac{\pi}{2}\right).} \tag{3.9}$$



(a) A *pseudosphere*, $K = -1$. (b) A *cylinder*, $K = 0$. (c) A *sphere*, $K = +1$.

Figure 3.2: Rectangles as defined by construction (3.2) depicted on various manifolds with constant curvature. The boundary of the rectangle is red and the inside is blue. In fact, the blue color represents values of eigenfunction corresponding to mode $m = 1$.

The assumption of $\mathcal{L}$ being a diffemorphism also means that geodesics on $\Omega$ do not intersect as then $\mathcal{L}$ would not be bijective. This assumption can be weakened by only requiring $f, f^{-1} \in L^\infty(\Omega_0, g)$.

Define piecewise-constant permitivity $\epsilon : \Omega_0 \to \mathbb{R}$ using

$$\epsilon(x_1, x_2) := \begin{cases} \epsilon_+, & (x_1, x_2) \in \Omega_+, \\ -\epsilon_-, & (x_1, x_2) \in \Omega_- \end{cases} \tag{3.10}$$

for constants $\epsilon_\pm > 0$. The *contrast* $\kappa \in \mathbb{R}$, $\kappa > 0$ is then defined as

$$\kappa := \frac{\epsilon_+}{\epsilon_-}. \tag{3.11}$$

In the text of the thesis, we will sometimes omit the index of $\Omega_0$ and write only $\Omega$ and we will often use notation for the coordinates on $\Omega_0$ as $x_1 \equiv x$, $x_2 \equiv y$.

## 3.2 Indefinite Laplacian on manifolds

In local coordinates on a manifold $(\Omega_0, g)$ we have the following identities for grad $\psi$ and div $X$

$$(\operatorname{grad} \psi)^i = (\mathrm{d}\psi)^i = g^{ij}\, \partial_j\, \psi, \tag{3.12}$$

$$\operatorname{div} X = \frac{1}{\sqrt{|g|}}\, \partial_i(\sqrt{|g|} X^i). \tag{3.13}$$

Differential expression $-\operatorname{div}(\epsilon \operatorname{grad})$ can be written as

$$-\operatorname{div}(\epsilon \operatorname{grad} \psi) = -\frac{1}{f}\, \partial_1 \left(\epsilon f \frac{\partial \psi}{\partial x^1}\right) - \frac{1}{f}\, \partial_2 \left(\epsilon \frac{1}{f} \frac{\partial \psi}{\partial x^2}\right), \tag{3.14}$$

and for piecewise constant $\epsilon$, it can be given as

$$-\operatorname{div}\left(\epsilon \operatorname{grad}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\right)=\begin{pmatrix}-\epsilon_+\Delta_g\psi_+\\ \epsilon_-\Delta_g\psi_-\end{pmatrix} \tag{3.15}$$

for $\psi \in H^2(\Omega_+, g) \oplus H^2(\Omega_-, g)$ and $\Delta_g$ is a Laplace-Beltrami operator.

Let us summarise definition of the Dirichlet realisation of differential expression $-\operatorname{div}(\epsilon \operatorname{grad})$ on constantly-curved manifold from [18] with permitivity $\epsilon$ given as a piecewise constant function (3.10).

For a Riemannian manifold $(\mathcal{M}, g)$ with constant Gaussian curvature, define an operator $\dot{A}_K : \operatorname{dom}\dot{A}_K \to L^2(\Omega)$ as in [18] with $\operatorname{dom}\dot{A}_K$ considered as a subset $\operatorname{dom}\dot{A}_K \subset L^2(\Omega)$

$$\operatorname{dom}\dot{A}_K := \left\{ \psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \in H^2(\Omega_+, g)\oplus H^2(\Omega_-, g) \;\middle|\; \begin{array}{l} \psi_\pm\,|_{\partial\Omega_0}=0,\\ \psi_+(0,\cdot)=\psi_-(0,\cdot)\\ \epsilon_+\,\partial_1\,\psi_+(0,\cdot) = -\epsilon_-\,\partial_1\,\psi_-'(0,\cdot)\end{array} \right\} \tag{3.16a}$$

$$\dot{A}_K = \begin{pmatrix}\epsilon_+\\ -\epsilon_-\end{pmatrix}\cdot\begin{cases} -\frac{1}{\cos^2(\sqrt{K}x_1)}\,\partial_2^2 - \partial_1^2 + \sqrt{K}\tan(\sqrt{K}x_1)\,\partial_1, & \text{if } K>0,\\ -\partial_2^2-\partial_1^2, & \text{if } K=0,\\ -\frac{1}{\cosh^2(\sqrt{-K}x_1)}\,\partial_2^2 - \partial_1^2 - \sqrt{-K}\tanh(\sqrt{-K}x_1)\,\partial_1, & \text{if } K<0.\end{cases} \tag{3.16b}$$

The operator can be written in a unified fashion using expression valid for $K \in \mathbb{R}$ as

$$\dot{A}_K = \begin{pmatrix}\epsilon_+\\ -\epsilon_-\end{pmatrix}\left(-\frac{1}{\cos^2(\sqrt{K}x_1)}\,\partial_2^2 - \partial_1^2 + \sqrt{K}\tan(\sqrt{K}x_1)\,\partial_1\right). \tag{3.17}$$

The operator $\dot{A}_K$ is essentially self-adjoint for any $K \in \mathbb{R}$ using construction of eigenvectors of $\dot{A}$ which form an orthonormal basis of $L^2(\Omega)$. Denote its self-adjoint closure as $A := \overline{\dot{A}}$.

## 3.3 Known basic properties of the operator

The following observations have been proved in author's previous work [18] along with explicit formulas for eigenvalues and eigenfunctions. The eigenfunctions are in $C^\infty(\overline{\Omega_\pm})$. See sections 4.2 and 4.3 for more details or the original work.

**Theorem 3.1.** *The operator $\dot{A}$ is symmetric and essentially self-adjoint. The spectrum of $A := \overline{\dot{A}}$ is a closure of the set of all eigenvalues of $\dot{A}$.*

*Proof.* The proof relies on separation of variables and unitary transform of the separated self-adjoint operator acting on $L^2\left((-b,a), g|_{(-b,a)}\right)$ into a Laplacian in one dimensional space $L^2\left((-b,a), \mathrm{id}_{(-b,a)}\right)$ without any effect of curvature plus a potential describing the effects of curvature. The potential is attractive in case of positive curvature, repulsive for negative curvature. Then it concludes by finding an orthonormal base of eigenfunctions so that Theorem 2.29 gives the essential self-adjointness. ☐

**Remark 3.2.** The construction in the proof can be generalized to those Riemannian manifolds such that, in normal coordinates, $g(x, y) = \operatorname{diag}(1, f^2(x))$. The unitary transformation

$$\begin{aligned} U : L^2(J_1, \mathrm{d}x) &\to L^2(J_1, \mathrm{d}\nu_K) \\ (U\psi)(x) &:= f^{-1/2}(x)\psi(x) \end{aligned} \tag{3.18}$$

and then the potential in [18] is given as

$$V_K^m(x) = \frac{f''(x)}{2f(x)} - \frac{f'(x)^2}{4f(x)^2} + \frac{m^2}{f(x)^2}. \tag{3.19}$$

It should be noted that the Jacobi equation connecting the metric and curvature is now more complicated.

**Theorem 3.3.** *Let $(\Omega_0, g)$ be a Riemannian manifold with constant Gaussian curvature $K \neq 0$ and $\dot{A}$ the operator (3.16). Then, there exists a homothetic transformation $\tau : \Omega_0 \to \tilde{\Omega}_0$ of domain $\Omega_0 = (-b, a) \times (0, c)$ onto $\tilde{\Omega}_0 = |K|^{\frac{1}{2}}\Omega_0$ so that*

$$\dot{A}_K\psi|_{(x_1,x_2)} = \left(|K|\tilde{\dot{A}}_{(\operatorname{sgn} K)}\tilde{\psi}\right) \circ \tau|_{(x_1,x_2)}. \tag{3.20}$$

*with tildes denoting object on $\tilde{\Omega}_0$. The spectrum of the operator on the original, respectively transformed domain, satisfies*

$$\sigma\left(A_K, a, b, c\right) = |K|\sigma\left(A_{\operatorname{sgn} K}, \sqrt{K}a, \sqrt{K}b, \sqrt{K}c\right). \tag{3.21}$$

**Proposition 3.4.** *In case of $K = 0$, we have*

$$\frac{\epsilon_+}{\epsilon_-} \neq 1 \implies \sigma_{\text{ess}}(A) = \emptyset. \tag{3.22}$$

*In a non-critical case, we have for $K \in \mathbb{R}$ and special case of $a = b$ that $0$ is an eigenvalue of infinite multiplicity and hence*

$$0 \in \sigma_{\text{ess}}(A). \tag{3.23}$$

Aim of this work is to give a more complete characterization of the essential spectrum.

# Chapter 4

# Spectral analysis

## 4.1 Construction of singular sequences

In this section, we will show that for the critical contrast $\frac{\epsilon_+}{\epsilon_-} =: \kappa = 1$, the operator $A$ defined on a manifold with arbitrary constant curvature always contains zero in the essential spectrum.

**Proposition 4.1.** $\epsilon_+ = \epsilon_- \implies 0 \in \sigma_{\text{ess}}(A)$

*Proof.* We can easily see, that the function $(x_1, x_2) \mapsto \sin\left(\frac{n\pi}{c} x_2\right)$ satisfies Dirichlet boundary conditions on boundaries perpendicular to the $x_2$ axis. For this reason, we will now try to find suitable singular sequences in the form $(x_1, x_2) = (x, y) \mapsto g(x) \sin\left(\frac{n\pi}{c} y\right)$, basically making a separation of variables.

To construct our singular sequences, we will utilize equation

$$A\psi(x, y) = 0. \tag{4.1}$$

Consider ansatz $\psi(x, y) = f(x) \sin\left(\frac{n\pi}{c} y\right)$. Then the solution for $f$ of

$$A\psi(x, y) = \left(-f''(x) + \sqrt{K}\tan(\sqrt{K}x) f'(x) + \frac{(\frac{n\pi}{c})^2}{\cos^2(\sqrt{K}x)} f(x)\right) \sin\left(\frac{n\pi}{c} y\right) = 0 \tag{4.2}$$

is in a general form of

$$f(x) = C_1 \cosh\left(\frac{n\pi}{c\sqrt{K}} \operatorname{arctanh} \sin\left(\sqrt{K}x\right)\right) + C_2 \sinh\left(\frac{n\pi}{c\sqrt{K}} \operatorname{arctanh} \sin\left(\sqrt{K}x\right)\right), \tag{4.3}$$

for all $K \in \mathbb{R}\setminus\{0\}$ where $C_1, C_2 \in \mathbb{C}$ are constants. We are not going to construct eigenvectors of $A$ corresponding to eigenvalue $\lambda = 0$ as it is not an eigenvalue in general (only for $a = b$). Instead, we will construct approximations. From these solutions, by a choice $C_1 = 1$ and $C_2 = -1$, we define functions $f_n^{(K)} : (0, a) \to \mathbb{R}$ using

$$f_n^{(K)}(x) := \exp\left(-\frac{n\pi}{c\sqrt{K}} \operatorname{arctanh} \sin(\sqrt{K}x)\right) \tag{4.4}$$

for each $n \in \mathbb{N}$ and fixed $K \in \mathbb{R} \setminus \{0\}$. For case $K = 0$ it is defined via $f_n^{(0)}(x) := \exp\left(-\frac{n\pi}{c} x\right)$. But this case is already present in a limit sense $\lim_{K\to 0} f_n^{(K)}(x) = f_n^{(0)}(x)$, as can be seen from

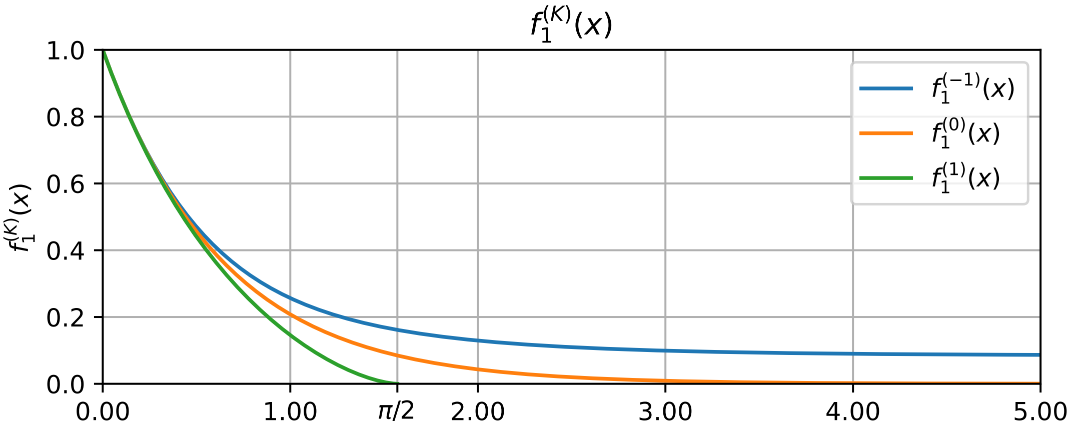



Figure 4.1: Functions $f_n^{(K)}$ for $n = 1$ given in (4.4). We have $f_n^{(K)}(x) = (f_1^{(K)}(x))^{\frac{n\pi}{c}}$ For positive curvature, we have $a, b \in (0, \frac{\pi}{2})$ so we plot only those valid values. For negative curvature, the function does have a non-zero limit as $x \to \infty$.

Taylor expansion in $K$. When the curvature $K$ is obvious from context, we will *denote the function only by* $f_n$. These functions with slight modifications will be our candidates for a singular sequence for zero. Mainly, we need to modify them to satisfy the interface and Dirichlet boundary conditions.

To that end, we will assume $a \leq b$ (for $a = b$, $\lambda = 0$ is an infinitely degenerate eigenvalue, as proven in bachelor's thesis [18, Section 4.8]) and fix two parameters $a_1, a_2 \in \mathbb{R}$, $0 < a_1 < a_2 < a$, $a_2 < b$ and define a smooth cut-off function $\chi : (0, a) \to \mathbb{R}$ with properties

$$\chi(x) := \begin{cases} 1, & x \in (0, a_1), \\ 0, & (a_2, a) \end{cases} \tag{4.5}$$

and with values elsewhere given such that $\chi \in C^\infty((0, a))$. Define sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} A$ for operator $A$ as

$$\varphi_n(x, y) := \begin{cases} f_n(|x|)\chi(|x|)\sqrt{\frac{2}{c}} \sin\left(n\frac{\pi}{c}y\right), & x \in (-a, a), \\ 0, & x \in (-b, -a). \end{cases} \tag{4.6}$$

Indeed, it clearly satisfies Dirichlet boundary conditions and also the interface conditions. The "continuity" at zero is obvious and relation $\varphi_n'(0+, y) = -\varphi_n'(0-, y)$ for all $n \in \mathbb{N}$ and $y \in (0, c)$ follows from the fact that $\varphi_n$ is even on a neighbourhood of $\mathcal{C}$.

Before we consider the three basic cases $K \in \{-1, 0, 1\}$ and prove that $\{\varphi_n\}_{n\in\mathbb{N}}$ is in fact a singular sequence for $A$ and $\lambda = 0$, let us reason about our choice for $f_n^{(K)}$.

For next step, the following observation is crucial. Consider a second-order differential operator $T\phi = \sum_{i=0}^{2} c_i \phi^{(i)}$, $c_i$ sufficiently smooth functions. Then for $\phi = f\eta$, $f$ satisfying $Tf = 0$ and $\eta$ sufficiently smooth, we get

$$T\phi = T(f\eta) = (Tf)\eta + c_2(2f'\eta' + f\eta'') + c_1 f\eta' = c_2(2f'\eta' + f\eta'') + c_1 f\eta' \tag{4.7}$$

and the expression does not depend on $\eta$, only on its derivatives. In our setting, this means that we will need to integrate $A\varphi_n$ only over interval $(a_1, a_2)$ instead of $(0, a_2)$ which will be crucial. We note that $\lambda = 0$ is not an eigenvalue for $a \neq b$, ie. it does not belong to discrete spectrum $\sigma \setminus \sigma_{\text{ess}}$, and sequences $(\varphi_n)_{n\in\mathbb{N}}$ for all three cases are therefore singular.

**Case** $K=0:$ The functions $f_n$ reduce to $f_n(x)=\exp\left(-\frac{n\pi}{c}|x|\right)$. We will estimate

$$\|\varphi_n\|^2 = 2\int_0^a (f_n\chi)^2 = 2\int_0^{a_2} (f_n\chi)^2 \geq 2\int_0^{a_1} \exp\left(-2\frac{n\pi}{c}x\right)\mathrm{d}x = \frac{c}{n\pi}\left(1-\mathrm{e}^{-2\frac{n\pi}{c}a_1}\right) \tag{4.8}$$

and evaluate the following expression using (4.7) and $\chi\in C^\infty((0,a))$:

$$\begin{aligned}\|A\varphi_n\|^2 &= \int_{-b}^a \mathrm{e}^{-\frac{2n\pi}{c}|x|}\left(-\frac{2n\pi}{c}\operatorname{sgn}(x)\chi'(|x|)+\chi''(|x|)\right)^2\mathrm{d}x \leq 2C_n\int_{a_1}^{a_2}\mathrm{e}^{-\frac{2n\pi}{c}x}\,\mathrm{d}x\\ &= \frac{C_nc}{n\pi}\left(\mathrm{e}^{-\frac{2n\pi}{c}a_1}-\mathrm{e}^{-\frac{2n\pi}{c}a_2}\right) = \frac{C_nc}{n\pi}\mathrm{e}^{-\frac{2n\pi}{c}a_1}\left(1-\mathrm{e}^{-(a_2-a_1)\frac{2n\pi}{c}}\right),\end{aligned} \tag{4.9}$$

where we estimated the first bracket using a degree two polynomial $C_n$ in $n$. Hence, $\frac{\varphi_n}{\|\varphi_n\|}$ is a singular sequence as the expression $\lim_{n\to\infty}\frac{1}{\|\varphi_n\|}\|A\varphi_n\|=0$ is zero.

**Case** $K=+1:$ The functions $f_n$ reduce to $f_n(x)=\left(\frac{1+\sin|x|}{1-\sin|x|}\right)^{-\frac{n\pi}{2c}}$ using a known relation for $\operatorname{arctanh}(y)=\frac{1}{2}\ln(\frac{1+y}{1-y})$. Additionally, we have $f_n'(x)=-\frac{n\pi}{c}\frac{\operatorname{sgn}x}{\cos x}f_n(x)$. The following estimates are valid because of convexity of $\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}$, as the second derivative is non-negative on $[0,\frac{\pi}{2}]$:

$$1-x\leq\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\leq 1-\frac{2}{\pi}x,\quad x\in\left(0,\frac{\pi}{2}\right). \tag{4.10}$$

The lower bound is a tangent at 0 to the convex function and the upper bound is a secant to the function crossing points $x=0$ and $x=\frac{\pi}{2}$. Choose $a_1<a_2<1$. Then we estimate

$$\|\varphi_n\|^2 = 2\int_0^a (f_n\chi)^2\,\mathrm{d}\nu_{+1} \geq 2\cos(a_2)\int_0^{a_1}(1-x)^{\frac{2n\pi}{c}}\,\mathrm{d}x = \frac{\cos(a_2)}{\frac{2n\pi}{c}+1}\left(1-(1-a_1)^{\frac{2n\pi}{c}+1}\right), \tag{4.11}$$

where $\mathrm{d}\nu_{+1}=\cos(x)\,\mathrm{d}x$ is measure on the rectangle with curvature $K=+1$. Continue to give an upper bound for our expression of interest:

$$\begin{aligned}\|A\varphi_n\|^2 &= \int_{-b}^a f_n^2(x)\left(-\frac{2n\pi}{c}\frac{\operatorname{sgn}x}{\cos x}\chi'(|x|)+\chi''(|x|)-\tan(x)\,\chi'(|x|)\right)^2\mathrm{d}\nu_{+1}\\ &\leq 2C_n\int_{a_1}^{a_2}\left(1-\frac{2}{\pi}x\right)^{\frac{2n\pi}{c}}\mathrm{d}x = \frac{\pi C_n}{\frac{2n\pi}{c}+1}\left(\left(1-\frac{2}{\pi}a_1\right)^{\frac{2n\pi}{c}+1}-\left(1-\frac{2}{\pi}a_2\right)^{\frac{2n\pi}{c}+1}\right)\\ &= \frac{\pi C_n}{\frac{2n\pi}{c}+1}\left(1-\frac{2}{\pi}a_1\right)^{\frac{2n\pi}{c}+1}\left(1-\left(\frac{1-\frac{2}{\pi}a_2}{1-\frac{2}{\pi}a_1}\right)^{\frac{2n\pi}{c}+1}\right),\end{aligned} \tag{4.12}$$

where we employed boundedness of $\frac{1}{\cos(x)}$ a $\tan(x)$ on $(0,a)\subsetneq(0,\frac{\pi}{2})$ and $C_n$ is again a polynomial in $n$. Same as before, $\frac{\varphi_n}{\|\varphi_n\|}$ is a singular sequence because the expression $\lim_{n\to\infty}\frac{1}{\|\varphi_n\|}\|A\varphi_n\|=0$ is zero.

**Case** $K=-1$**:** Now we have $f_n(|x|)=\exp(-\frac{n\pi}{c}\arctan\sinh|x|)$ and $f_n'(|x|)=-\frac{n\pi}{c}\frac{\operatorname{sgn}x}{\cosh(x)}f_n(|x|)$. The following estimates are, again, valid because of convexity and the fact that $\lim_{x\to\infty}\mathrm{e}^{-\arctan\sinh x}=\mathrm{e}^{-\frac{\pi}{2}}<\frac{1}{2}$:

$$1-x\leq\exp(-\arctan\sinh x)\leq\tilde{f}(x),\quad x\in(0,\infty), \tag{4.13}$$

where function $\tilde{f} : (0,\infty) \to \mathbb{R}$ is given by

$$\tilde{f}(x) := \begin{cases} 1 - \frac{x}{2}, & x \in (0,1), \\ \frac{1}{2}, & x \in (1,\infty). \end{cases} \tag{4.14}$$

There would be three cases, based on a value of $a$, to discuss: $a_1 < a_2 \leq 1$, $a_1 < 1 < a_2$ and $1 \leq a_1 < a_2$. But we can choose values $a_1$, $a_2$ to satisfy the first case for any given $a \in \mathbb{R}$ so there is no need to discuss the other cases. Let us therefore choose $0 < a_1 < a_2 < 1$ and as a consequence $\forall n \in \mathbb{N}, \forall x \in (a_1, a_2) : f_n(x) \leq \left(1 - \frac{x}{2}\right)^{\frac{n\pi}{c}}$. Even with measure $\mathrm{d}\nu_{-1} = \cosh(x)\,\mathrm{d}x$, the expression $\|B_k \varphi_n\|$ will behave very similarly as before because we have already examined a similar upper bound on $f_n$ and also functions $\frac{1}{\cosh x}$ and $\tanh x$ are bounded on $(0,a) \subsetneq (0,\infty)$. Concluding, we have found a singular sequence for $K = -1$. $\square$

**Remark 4.2.** It is possible that a similar construction can be given also for non-constant curvatures with metric $g(x,y) = \operatorname{diag}(1, f(x))$. The ordinary differential equation

$$A_K \left[\psi(x) \sin\left(\frac{m\pi}{c} y\right)\right] = 0 \tag{4.15}$$

has solutions given as

$$\psi(x) := C_1 \exp\left(-\frac{m\pi}{c} \int \frac{1}{f(x)}\,\mathrm{d}x\right) + C_2 \exp\left(\frac{m\pi}{c} \int \frac{1}{f(x)}\,\mathrm{d}x\right). \tag{4.16}$$

for arbitrary constants $C_1$, $C_2$ as can be found by reducing the problem to a system of first order ODEs. For a choice of $C_1 = 1$, $C_2 = 0$, we have

$$\psi(x) = \exp\left(-\frac{m\pi}{c} \int \frac{1}{f(x)}\,\mathrm{d}x\right). \tag{4.17}$$

We have not explored if these functions (for arbitrary $f(x)$) lead to singular sequences due to time concerns. Also, due to usage of cut-off functions, the geometry of the domain could be much richer and the results of this section would still apply.

## 4.2 Accumulation points of the spectrum in zero curvature case

In this section, we will give a full proof of characterization of essential spectrum $\sigma_{\text{ess}}(A_0)$ depending on values $\epsilon_+$ and $\epsilon_-$ for zero Gaussian curvature $K = 0$. The proof presented in [18, Proposition 3.2] was incomplete.

Spectrum of the operator $A_0$ can be described as a closure of the point spectrum of $\dot{A}_0$

$$\sigma(A_0) = \overline{\sigma_\infty \cup \sigma_0},$$

where we define

$$\sigma_\infty = \bigcup_{n=1}^{+\infty} \bigcup_{m=-\infty}^{+\infty} \{\lambda_{n,m}\},$$

and $\left(\lambda_{n,m}\right)_{m\in\mathbb{Z}}$ is for every fixed $n \in \mathbb{N}$ an increasing sequence of roots of equation

$$\frac{\tan(a\sqrt{\frac{\lambda}{\epsilon_+} - (\frac{n\pi}{c})^2})}{\epsilon_+\sqrt{\frac{\lambda}{\epsilon_+} - (\frac{n\pi}{c})^2}} = \frac{\tanh(b\sqrt{\frac{\lambda}{\epsilon_-} + (\frac{n\pi}{c})^2})}{\epsilon_-\sqrt{\frac{\lambda}{\epsilon_-} + (\frac{n\pi}{c})^2}} \tag{4.18}$$

for $\lambda \in \mathbb{R} \setminus \left\{-\epsilon_- \left(\frac{n\pi}{c}\right)^2, \epsilon_+ \left(\frac{n\pi}{c}\right)^2\right\}$. We adopt convention with possibly negative terms under the square roots. Put $\lambda_{n,0}$ if $\lambda = 0$ is a solution, otherwise leave index 0 undefined. In that case, we define $\lambda_{n,\pm 1}$ as the smallest positive, respectively the biggest negative, solution. For each of these eigenvalues $\lambda_{n,m}$, there exists a corresponding eigenvector $f_{n,m}(x,y) = \mathcal{N}_{n,m} \cdot \chi_n(y)\psi_{n,m}(x)$ with $\chi_n(y) = \sqrt{\frac{2}{c}} \sin(\frac{n\pi}{c} y)$,

$$\psi_{n,m}(x) = \begin{cases} \sinh(\sqrt{\frac{\lambda_{n,m}}{\epsilon_-} + (\frac{n\pi}{c})^2}\ b) \sin(\sqrt{\frac{\lambda_{n,m}}{\epsilon_+} - (\frac{n\pi}{c})^2}\ (a-x)), & x > 0, \\ \sin(\sqrt{\frac{\lambda_{n,m}}{\epsilon_+} - (\frac{n\pi}{c})^2}\ a) \sinh(\sqrt{\frac{\lambda_{n,m}}{\epsilon_-} + (\frac{n\pi}{c})^2}\ (b+x)), & x < 0 \end{cases} \tag{4.19}$$

and $\mathcal{N}_{n,m}$ is a normalisation constant.

In the interval $\lambda \in \left(-\epsilon_-(\frac{n\pi}{c})^2, \epsilon_+(\frac{n\pi}{c})^2\right)$, the equation becomes

$$\frac{\tanh(a\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon_+}})}{\epsilon_+\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon_+}}} = \frac{\tanh(b\sqrt{\frac{\lambda}{\epsilon_-} + (\frac{n\pi}{c})^2})}{\epsilon_-\sqrt{\frac{\lambda}{\epsilon_-} + (\frac{n\pi}{c})^2}}. \tag{4.20}$$

The second spectral subset $\sigma_0 \subset \left\{-\epsilon_- \left(\frac{n\pi}{c}\right)^2, \epsilon_+ \left(\frac{n\pi}{c}\right)^2\right\}$ contains at most two eigenvalues for the single $n \in \mathbb{N}$ (if it exists) such that a certain equation is satisfied [18] — in dependence on the setting of parameters $a, b, c$ it contains 0, 1 or both points. If these values exist, then there is exactly one unique eigenvector for each one. From the perspective of essential spectrum, this set is not interesting.

**Lemma 4.3.** *Let $\epsilon_- = \epsilon_+ =: \epsilon > 0$ and choose a fixed $n \in \mathbb{N}$. Then the equation (4.18) has exactly one solution $\lambda$ in interval $\left(-\epsilon \left(\frac{n\pi}{c}\right)^2, \epsilon \left(\frac{n\pi}{c}\right)^2\right)$. If $a = b$, $b < a$, $b > a$, then $\lambda = 0$, $\lambda < 0$, $\lambda > 0$, respectively.*

*Proof.* For each $n \in \mathbb{N}$ define a function $G_n : \left(-\epsilon_-(\frac{n\pi}{c})^2,\ \epsilon_+(\frac{n\pi}{c})^2\right) \to \mathbb{R}$ as a difference of reciprocals of left and right-hand sides of equation (4.18):

$$G_n(\lambda) := \frac{\epsilon\sqrt{\frac{\lambda}{\epsilon} + (\frac{n\pi}{c})^2}}{\tanh(b\sqrt{\frac{\lambda}{\epsilon} + (\frac{n\pi}{c})^2})} - \frac{\epsilon\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}}}{\tanh(a\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}})}. \tag{4.21}$$

After rearranging derivative $G_n'$ into (similar rearrangements as in article [6]):

$$G_n'(\lambda) = \frac{\sinh\left(2a\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}}\right) - 2a\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}}}{4\sinh\left(a\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}}\right)^2 \sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}}} + \frac{\sinh\left(2b\sqrt{(\frac{n\pi}{c})^2 + \frac{\lambda}{\epsilon}}\right) - 2b\sqrt{(\frac{n\pi}{c})^2 + \frac{\lambda}{\epsilon}}}{4\sinh\left(b\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}}\right)^2 \sqrt{(\frac{n\pi}{c})^2 + \frac{\lambda}{\epsilon}}}, \tag{4.22}$$

we readily obtain, using an identity $\sinh x > x$ valid for all $x > 0$, statement $G_n'(\lambda) > 0$ valid on the whole domain of $G_n$.

Limit $\lim_{\lambda \to \epsilon_+(\frac{n\pi}{c})^2 -} G_n(\lambda)$ is positive and a similar limit on the other end of domain $\lim_{\lambda \to -\epsilon_-(\frac{n\pi}{c})^2 +} G_n(\lambda)$ is negative. From this fact, and from continuity of $G_n$, it follows that there exists exactly one

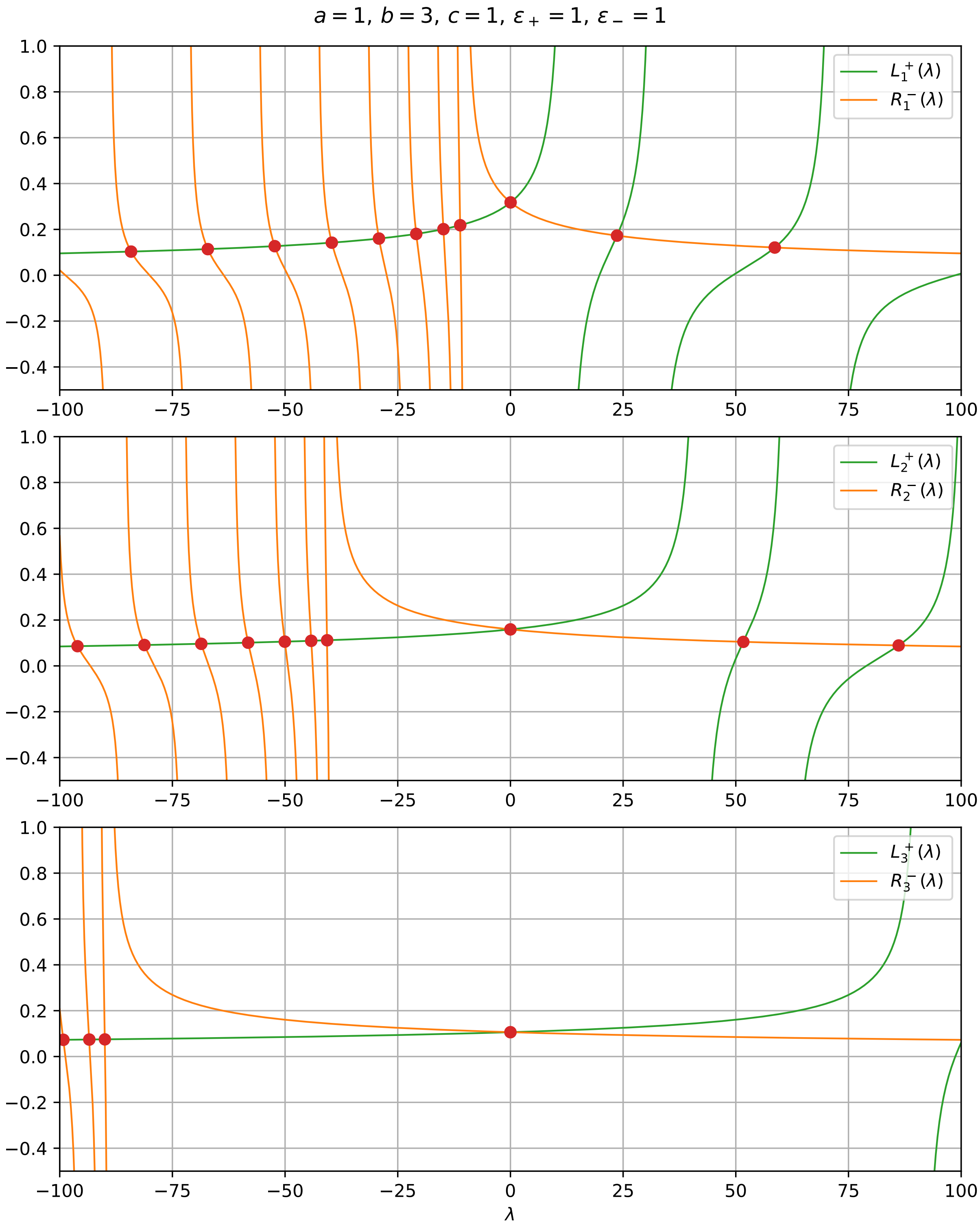



Figure 4.2: Left-hand $L_n^+(\lambda)$ and right-hand $R_n^-(\lambda)$ sides of the characteristic equation (4.18) for progressing $n = 1, 2, 3$ for critical contrast. The red dots are the roots of the equation.

root of characteristic equation (4.18) in the domain of $G_n$. It is also easy to determine the sign of the root — using the value

$$G_n(0) = \epsilon \frac{n\pi}{c} \left( \frac{1}{\tanh\left(b\frac{n\pi}{c}\right)} - \frac{1}{\tanh\left(a\frac{n\pi}{c}\right)} \right) \tag{4.23}$$

and a sign of the root is determined by a sign of $G_n(0)$. This is because intersection of graph $G_n$ with axis $\lambda = 0$ is exactly one, see above. □

**Proposition 4.4.** *When $\epsilon_+ \neq \epsilon_-$, the essential spectrum $\sigma_{\mathrm{ess}}(A_0) = \varnothing$ is empty.*

*Proof.* We prove that if the essential spectrum is nonempty, i.e. there is $\Lambda \in \sigma_{\mathrm{ess}}(A_0)$, then necessarily $\epsilon_+ = \epsilon_-$. Let $\Lambda \in \sigma_{\mathrm{ess}}(A_0)$ be a finite accumulation point of spectrum $\sigma(A_0)$. Then there exists some sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty} \subset \sigma(A_0)$ which converges to $\Lambda$, i.e. $\lim_{n\to+\infty} \lambda_n = \Lambda$. The case for $\Lambda$ being an infinitely degenerate eigenvalue is present by a choice of a constant sequence $\lambda_n = \Lambda$. Without loss of generality, assume $b > a$. As the limit value is finite, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, the value $\lambda_n$ lies in an interval $(0, \epsilon_+(\frac{n\pi}{c})^2)$. Hence, the notation $\lambda_n = \lambda_{n,m=1}$ is founded for all such $n$. We will rearrange characteristic equation (4.20) to form

$$\frac{\tanh(a\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda_n}{\epsilon_+}})}{\tanh(b\sqrt{(\frac{n\pi}{c})^2 + \frac{\lambda_n}{\epsilon_-}})} = \frac{\epsilon_+}{\epsilon_-}\sqrt{\frac{(\frac{n\pi}{c})^2 - \frac{\lambda_n}{\epsilon_+}}{(\frac{n\pi}{c})^2 + \frac{\lambda_n}{\epsilon_-}}} \tag{4.24}$$

and take limit $n \to +\infty$ ($\lambda_n \to \Lambda < +\infty$) on both sides of the equation. This reduces to the necessary condition

$$1 = \frac{\epsilon_+}{\epsilon_-}. \tag{4.25}$$

□

**Proposition 4.5.** $\sigma_{\mathrm{ess}}(A_0) = \{0\} \iff \epsilon_+ = \epsilon_-$.

*Proof.* The other implication $\epsilon_+ = \epsilon_- \implies \Lambda = 0$ is harder to prove. We start by putting $\epsilon := \epsilon_+ = \epsilon_-$, fixing arbitrary $n \in \mathbb{N}$ and denoting $\lambda_n$ the root of the characteristic equation

$$\frac{\tanh(a\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}})}{\sqrt{(\frac{n\pi}{c})^2 - \frac{\lambda}{\epsilon}}} = \frac{\tanh(b\sqrt{\frac{\lambda}{\epsilon} + (\frac{n\pi}{c})^2})}{\sqrt{\frac{\lambda}{\epsilon} + (\frac{n\pi}{c})^2}}. \tag{4.26}$$

lying in $\left(0, \epsilon\left(\frac{n\pi}{c}\right)^2\right)$. According to Lemma 4.3, such a root exists and is unique. Let us define a sequence

$$\alpha_n := \frac{\lambda_n}{\epsilon}\left(\frac{c}{n\pi}\right)^2 \in (0,1) \tag{4.27}$$

for all $n \in \mathbb{N}$. Using this sequence, we will prove $\lambda_n \to 0$.

**Step 1** Rewrite equation (4.26) as

$$a\frac{\tanh(a\frac{n\pi}{c}\sqrt{1-\alpha_n})}{a\frac{n\pi}{c}\sqrt{1-\alpha_n}} = b\frac{\tanh(b\frac{n\pi}{c}\sqrt{1+\alpha_n})}{b\frac{n\pi}{c}\sqrt{1+\alpha_n}}. \tag{4.28}$$

Since the function $x \mapsto \frac{\tanh(x)}{x}$ converges to 0 as $x \to +\infty$ and the sequence $\{\sqrt{1+\alpha_n}\}_n \in (1, \sqrt{2})$, the right hand side converges to 0 as $n \to +\infty$. In other words,

$$\lim_{n\to\infty} a\frac{\tanh(a\frac{n\pi}{c}\sqrt{1-\alpha_n})}{a\frac{n\pi}{c}\sqrt{1-\alpha_n}} = 0. \tag{4.29}$$

Then, as a result of $x \mapsto \frac{\tanh(x)}{x}$ being is strictly positive for all positive $x$, the following holds:

$$\lim_{n\to\infty} n\sqrt{1-\alpha_n} = +\infty. \tag{4.30}$$

**Step 2** Let us again rearrange (4.26) as

$$\frac{\tanh\left(a\frac{n\pi}{c}\sqrt{1-\alpha_n}\right)}{\tanh\left(b\frac{n\pi}{c}\sqrt{1+\alpha_n}\right)} = \frac{\sqrt{1-\alpha_n}}{\sqrt{1+\alpha_n}}. \tag{4.31}$$

From previous step (4.30), the left-hand side converges to 1 as $n \to +\infty$. Thus,

$$\lim_{n\to+\infty} \sqrt{\frac{1-\alpha_n}{1+\alpha_n}} = 1. \tag{4.32}$$

As function $x \mapsto \frac{1-x}{1+x}$ is strictly less than 1 for all $x \in (0,1)$ and $\lim_{x\to 0} \frac{1-x}{1+x} = 1$, it follows that

$$\lim_{n\to\infty} \alpha_n = 0. \tag{4.33}$$

**Step 3** For the last time, let us rearrange and expand hyperbolic functions from (4.26) into

$$\begin{aligned} 1 - 2\frac{\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}}{1+\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}} &= \left(1 - 2\frac{\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}{1+\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}\right)\frac{\sqrt{1-\alpha_n}}{\sqrt{1+\alpha_n}} \\ &= \left(1 - 2\frac{\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}{1+\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}\right)\left(1 - \alpha_n + \mathcal{O}(\alpha_n^2)\right). \end{aligned} \tag{4.34}$$

Expanding the right-hand side and rearranging terms,

$$-2\frac{\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}}{1+\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}} = -\alpha_n - 2\frac{\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}{1+\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}} + o(\alpha_n). \tag{4.35}$$

Now, multiplying the equation by a factor $\frac{1}{2\alpha_n}$, bringing the exponentials to the right-hand side and applying $\lim_{n\to\infty}$ to both sides, we obtain

$$\begin{aligned} \frac{1}{2} &= \lim_{n\to\infty} \frac{1}{\alpha_n}\left(\frac{\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}}{1+\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}} - \frac{\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}{1+\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}\right) \\ &= \lim_{n\to\infty} \frac{\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}} - \mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}{\alpha_n} \frac{\frac{\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}}{1+\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}} - \frac{\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}{1+\mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}}{\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}\left(1 - \mathrm{e}^{-2\frac{n\pi}{c}\left(b\sqrt{1+\alpha_n} - a\sqrt{1-\alpha_n}\right)}\right)} \\ &= \lim_{n\to\infty} \frac{\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}} - \mathrm{e}^{-2b\frac{n\pi}{c}\sqrt{1+\alpha_n}}}{\alpha_n} \\ &= \lim_{n\to\infty} \frac{\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}}{\alpha_n} \\ &= \lim_{n\to\infty} \frac{n^2\,\mathrm{e}^{-2a\frac{n\pi}{c}\sqrt{1-\alpha_n}}}{n^2\alpha_n} = \lim_{n\to\infty} \frac{\mathrm{e}^{-n\left(\frac{2a\pi}{c}\sqrt{1-\alpha_n} - 2\frac{\ln n}{n}\right)}}{n^2\alpha_n} \end{aligned} \tag{4.36}$$

where third and fourth equations hold because $b > a$. As the limit is finite and the numerator in the last term goes to zero, necessarily

$$0 = \lim_{n\to\infty} n^2 \alpha_n = \left(\frac{c}{\pi}\right)^2 \lim_{n\to\infty} \lambda_n. \tag{4.37}$$

Thus, we have proven that $\lambda = 0$ is the only accumulation point and that $\{0\} = \sigma_{\mathrm{ess}}(A_0)$. $\square$

**Corollary 4.6.** *For $\epsilon_+ = \epsilon_-$, the rate of convergence of eigenvalues of $A_0$ to $0$ is $\min_{m\in\mathbb{Z}} |\lambda_{n,m}| = o\left(\mathrm{e}^{-\frac{n\pi}{c}\min\{a,b\}}\right)$.*

*Proof.* In addition to previous proposition, we can establish a rate of convergence. By using a similar trick as in proof of the proposition in the last equation, expand

$$\frac{1}{2} = \left(\frac{\pi}{c}\right)^2 \lim_{n\to\infty} \frac{\mathrm{e}^{-n\left(\frac{2a\pi}{c}\sqrt{1-\alpha_n}-2\frac{\ln n}{n}\right)}}{\lambda_n} \frac{\mathrm{e}^{a\frac{n\pi}{c}}}{\mathrm{e}^{a\frac{n\pi}{c}}} = \lim_{n\to\infty} \frac{\mathrm{e}^{-n\left(\frac{2a\pi}{c}\sqrt{1-\alpha_n}-\frac{a\pi}{c}-2\frac{\ln n}{n}\right)}}{\lambda_n \, \mathrm{e}^{a\frac{n\pi}{c}}} \tag{4.38}$$

and by the same argument as before, we obtain

$$\lambda_n = o\left(\mathrm{e}^{-a\frac{n\pi}{c}}\right). \tag{4.39}$$

$\square$

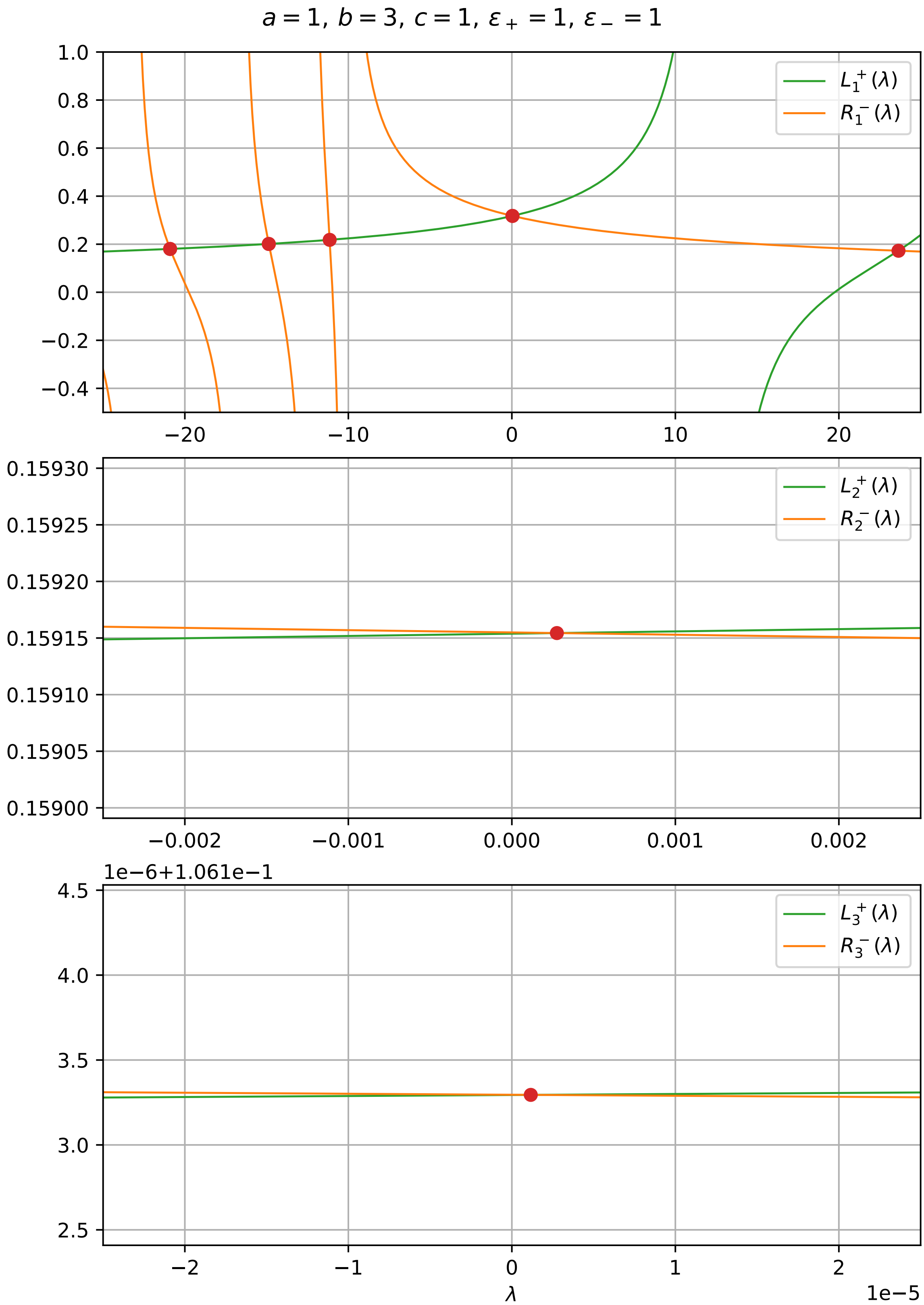



Figure 4.3: Left-hand $L_n^+(\lambda)$ and right-hand $R_n^-(\lambda)$ sides of the characteristic equation (4.18) for progressing $n = 1, 2, 3$ for critical contrast. The red dots are the roots of the equation. We can notice the exponential convergence to zero in $n$ as in Corollary 4.6.

## 4.3 Accumulation points of the spectrum in curved cases

Fix $m \in \mathbb{N}$. According to previous work [18, Section 4.7], the spectrum of operator $A_K$ (denoted as $B_K$ in previous work) contains eigenvalues being the solutions $\lambda_m$ to the characteristic equation

$$\begin{vmatrix} \psi_1^+(a) & \psi_2^+(a) & 0 & 0 \\ 0 & 0 & \psi_1^-(-b) & \psi_2^-(-b) \\ \psi_1^+(0) & \psi_2^+(0) & -\psi_1^-(0) & -\psi_2^-(0) \\ \epsilon_+\psi_1^{+\prime}(0) & \epsilon_+\psi_2^{+\prime}(0) & \epsilon_-\psi_1^{-\prime}(0) & \epsilon_-\psi_2^{-\prime}(0) \end{vmatrix} = 0, \tag{4.40}$$

where the $\psi_\iota^+ : [0,a] \to \mathbb{R}$ and $\psi_\iota^- : [-b,0] \to \mathbb{R}$ are defined as in Table 4.1 and are implicitly dependent on $m$ via $\mu$ and $\nu$. In case there are multiple solutions of (4.40) for a fixed $m \in \mathbb{N}$, we denote the solutions as $\lambda_{m,k}$, $k \in \mathbb{Z}$. This is well-defined notation as self-adjoint operators in separable Hilbert spaces have at most countable point spectrum.

| | $\psi_1^\pm(x)$ | $\psi_2^\pm(x)$ | $\mu$ | $\nu_\pm$ |
|---|---|---|---|---|
| $K = +1$ | $P^\mu_{\nu\pm}(\sin x)$ | $Q^\mu_{\nu\pm}(\sin x)$ | $\frac{m\pi}{c}$ | $\frac{1}{2}\left(\sqrt{1 \pm \frac{4\lambda_{m,k}}{\epsilon_\pm}} - 1\right)$ |
| $K = -1$ | $\frac{P^\mu_{\nu\pm}(\tanh x)}{\sqrt{\cosh x}}$ | $\frac{Q^\mu_{\nu\pm}(\tanh x)}{\sqrt{\cosh x}}$ | $-\frac{1}{2} + \mathrm{i}\frac{m\pi}{c}$ | $\frac{1}{2}\sqrt{1 \mp \frac{4\lambda_{m,k}}{\epsilon_\pm}}$ |

Table 4.1: Choice of eigenfunctions on $\Omega_+$ and $\Omega_-$ for fixed $m \in \mathbb{N}$. Eigenvalue $\lambda_{m,k}$ is a solution to (4.40). $P^\mu_\nu$ and $Q^\mu_\nu$ are associated linearly independent Legendre function of first and second kind, respectively.

Finally, the spectrum of $B_K$ is

$$\sigma(B_K) = \bigcup_{m=1}^{\infty} \bigcup_{k \in \mathbb{Z}} \{\lambda_{m,k}\} \tag{4.41}$$

with $\lambda_{m,k}$ being solutions of (4.40), for fixed $m \in \mathbb{N}$ sorted in an increasing manner as $\lambda_{m,k} < \lambda_{m,k+1}$ for all $k \in \mathbb{Z}$. The resulting eigenfunctions are of form $\psi_{m,k}(x,y) = \phi_{m,k}(x)\sin\left(\frac{m\pi}{c}\right)$ for

$$\phi_{m,k}(x) := \begin{cases} C_1^+\psi_1^+(x) + C_2^+\psi_2^+(x), & x \geq 0, \\ C_1^-\psi_1^-(x) + C_2^-\psi_2^-(x), & x \leq 0, \end{cases} \tag{4.42}$$

for constants $C_1^\pm$, $C_2^\pm$ determined up to the same multiplicative factor using procedure in the reference.

There are multiple conventions of defining associated Legendre functions $P^\mu_\nu(x)$, $Q^\mu_\nu(x)$. Primarily, they are defined as two independent solutions $y \equiv y(x)$ to the second-order linear differential equation called associated Legendre equation with singularities in $x = \pm 1$:

$$\left(1 - x^2\right) y'' - 2xy' + \left(\nu(\nu+1) - \frac{\mu^2}{1 - x^2}\right) y = 0. \tag{4.43}$$

For both cases of curvature, we will employ the functions with parameter range $-1 < x < 1$. In this case, the functions are often referred to as Ferrer's functions. We will use the following conventions [5, 36]. For $\mu$, $\nu \in \mathbb{C}$ (for integer parameters, the functions are defined via limiting procedure),

$$\begin{aligned} P_\nu^\mu(z) &= \frac{1}{\Gamma(1-\mu)}\left(\frac{1+z}{1-z}\right)^{\frac{\mu}{2}} {}_2F_1\left(-\nu,\nu+1;1-\mu;\frac{1-z}{2}\right) \\ Q_\nu^\mu(z) &= \frac{\pi}{2}\frac{1}{\sin(\pi\mu)}\left[\frac{\cos(\pi\mu)}{\Gamma(1-\mu)}\left(\frac{1+z}{1-z}\right)^{\frac{\mu}{2}} {}_2F_1\left(-\nu,\nu+1;1-\mu;\frac{1-z}{2}\right)\right. \\ &\qquad \left. -\frac{\Gamma(1+\nu+\mu)}{\Gamma(1+\nu-\mu)}\left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}}\frac{1}{\Gamma(1+\mu)} {}_2F_1\left(-\nu,\nu+1;1+\mu;\frac{1-z}{2}\right)\right] \end{aligned} \tag{4.44}$$

with ${}_2F_1(a,b;c;z)$ being hypergeometric function, $\Gamma$ the gamma function, $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. The function $\frac{1}{\Gamma(z)}$ is considered as an entire function with zeros in negative integers $z$. The regularized hypergeometric function

$${}_2\tilde{F}_1(a,b;c;z_0) := \frac{1}{\Gamma(c)}\,{}_2F_1(a,b;c;z_0) \tag{4.45}$$

for fixed $|z_0| < 1$ is an entire analytic function [5] in the complex space $\mathbb{C}^3 \ni (a,b,c)$ even for negative-integer $c$. We will utilize these functions for $z \in \mathbb{R}$, $\mu, \nu \in \mathbb{C}$ as seen from Table 4.1. For positive curvature, the parameters $a, b, c$ are real, argument $z$ is $|z| < 1$ and the Legendre functions are real-valued.

Hypergeometric function ${}_2F_1(a,b;c;z)$, $a,b,c \in \mathbb{C}$ with $-c \notin \mathbb{N}$ is defined for $z \in \mathbb{C}$, $|z| \leq 1$[1] in terms of a power series [5]. The series converges absolutely for $|z| < 1$ and diverges for $|z| > 1$,

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{4.46}$$

with $(z)_n$ being a Pochhammer symbol for rising factorial defined for complex $z$ in general as $(z)_n = \frac{\Gamma(z+n)}{\Gamma(n)}$. For positive integer $n$, $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$. From elementary ratio test, the hypergeometric function is continuous in each argument on any compact set in $\mathbb{C}\times\mathbb{C}\times\left(\mathbb{C}\setminus\mathbb{Z}\right) \ni (a,b,c)$ such that neighbourhood of $\mathbb{Z}$ is not present. To include the integers, a regularization multiplier is needed, most commonly $\frac{1}{\Gamma(c)}\,{}_2F_1(a,b;c;z) =: {}_2\tilde{F}_1(a,b;c;z)$. See [1] for limit as $c \to -n$, $n \in \mathbb{N}$. Notice that ${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)$ for all values of arguments.

A natural question to arise is whether these two solutions (4.44) are always linearly independent.

**Remark 4.7** (On linearly independent solutions to Legendre equation)**.** In order to have a two-dimensional solution space of Legendre equation, the two solutions $P_\nu^\mu$ and $Q_\nu^\mu$ have to be linearly independent. As can be seen from the Wronskian

$$W\left\{P_\nu^\mu(z), Q_\nu^\mu(z)\right\} = \frac{\mathrm{e}^{\mathrm{i}\mu\pi}\,2^{2\mu}\Gamma\left(1+\frac{\mu}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)}{(1-z^2)\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)}, \tag{4.47}$$

the functions are linearly dependent when $\mu - \nu = 1, 2, 3, \ldots$. As an example of this behaviour, in the case of $\mu$, $\nu \in \mathbb{N}$, the Legendre function $P_\nu^\mu(z)$ formally given in (4.44) is [5]

- a polynomial in $z$ for $\mu \leq \nu$

[1]For other values of $z$, hypergeometric function is defined using analytic continuation.

- identically zero for $\mu > \nu$.

The case $\mu > \nu$ is of interest due to limit $m \to \infty$ of characteristic equation (4.40) with $\nu$ bounded corresponding to occurrence of essential spectrum. Another expression for solutions of Legendre equation (4.43) are thus needed. We summarize results for all values of parameters $\mu$, $\nu$ below as they do not appear frequently in mathematical physics literature and are important for this section.

This behaviour is described in [5, 2.2] via phenomenon called degeneration. The technical description does not concern this thesis but the implications are useful. First, let us reduce the problem to the hypergeometric equation. By substitution $y(x) = \left(1-x^2\right)^{\frac{\mu}{2}} v(\zeta)$, $\zeta(x) = \frac{1-x}{2}$, Legendre equation (4.43) is converted to hypergeometric equation

$$\zeta(1-\zeta)\frac{\mathrm{d}^2 v}{\mathrm{d}\zeta^2} + \left[c - (a+b+1)\,\zeta\right]\frac{\mathrm{d}v}{\mathrm{d}\zeta} - abv = 0 \tag{4.48}$$

with $a = \mu - \nu$, $b = \mu + \nu + 1$ and $c = \mu + 1$. According to a general theory of hypergeometric function, a degenerate case occurs when at least one of the numbers $a$, $b$, $c-a$, $c-b$ is an integer. This translates to $\mu \pm \nu$, or $\nu$ being integer. For all these cases, linearly independent solutions to the hypergeometric equation are given in reference. The case number corresponds to case in aforementioned reference:

- $\nu \in \mathbb{Z}$:
  - ⋆ $\mu \in \mathbb{Z} \iff \mu \pm \nu \in \mathbb{Z}$: case 19,
  - ⋆ $\mu \notin \mathbb{Z}$: case 4
- $\nu \notin \mathbb{Z}$:
  - ⋆ $\mu \pm \nu \notin \mathbb{Z}$: non-degenerate case (4.44)
  - ⋆ $\mu \pm \nu \in \mathbb{Z}$, $\mu \mp \nu \notin \mathbb{Z}$, respectively: case 2,
  - ⋆ $\mu \pm \nu \in \mathbb{Z} \iff \mu$, $\nu$ are half-integers: case 9.

According to the analysis in all the degenerate cases above, two linearly independent solutions to (4.48) are given using the following Kummer solutions

$$v_1(\zeta) = (1-\zeta)^{c-a-b}\,{}_2F_1\left(c-a, c-b; c; \zeta\right), \quad v_5(\zeta) = \zeta^{1-c}\,{}_2F_1\left(a+1-c, b+1-c; 2-c; \zeta\right). \tag{4.49}$$

One of these solutions will always be a polynomial in $\zeta$ for the degenerate case — see the reference for details. For example, for $\mu, \nu \in \mathbb{Z}$ positive integers, we have $n = l = \nu$, $m = \mu - \nu - 1$ in notation of case 19.

Overall, we obtain (up to a factor of $2^\mu$) two linearly independent solutions to (4.43) as

$$\begin{aligned} P_\nu^\mu(x) &:= \left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}}\,{}_2F_1\left(-\nu, \nu+1; 1+\mu; \frac{1-x}{2}\right), \\ Q_\nu^\mu(x) &:= \left(\frac{1-x}{1+x}\right)^{-\frac{\mu}{2}}\,{}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right). \end{aligned} \tag{4.50}$$

We are interested in finding a set of linearly independent solutions to Legendre and so we concern ourselves only with the case of $\mu - \nu$ being a positive integer — as in other cases, $P_\mu^\nu$, $Q_\nu^\mu$ are already linearly independent.

To demonstrate the need to distinguish this degenerate case, have a look at coefficients $\tilde{\gamma}_m$ of Lemma 4.12. For $\mu - \nu \in \mathbb{Z}$, $\tilde{\gamma}_m = 0$ and hence we would think that $\lambda_m$ corresponding to those $\nu_m = \nu$ would be a root of the characteristic equation (4.53). But we have to remember the requirement $\mu - \nu \notin \mathbb{Z}$ on validity of (4.44) as in that case, the Legendre functions $P_\nu^\mu$ and $Q_\nu^\mu$ as given by (4.53) are not independent solutions.

**Note 4.8** (Singularities of ${}_2F_1\,(a, b; 1-\mu; z)$ in $\mu > 0$)**.** In case of our setup with positive curvature, we encounter problems in associated Legendre functions due to the fact that the sequence $\mu_m = \frac{m\pi}{c}$ can be found infinitely often near an integer[2] as $m \to \infty$ for any value of $\frac{\pi}{c}$. The problem stems from presence of Pochhamer symbol $(1-\mu)_n$ in denominator as in that case, one of the factors will be arbitrarily near zero. It is apriori difficult to obtain boundedness in parameter $c = 1 - \mu_m$ to the hypergeometric function. This is firstly due to the negative $c$ asymptotics being sparse in literature and secondly because of the special care needed to handle negative-integer singularities in the hypergeometric series. The expansion $m \to +\infty$ ($|\mu| \to +\infty$) will be of use in further analysis to determine the essential spectrum.

**Note 4.9** (Concerning literature on asymptotics for $K = +1$)**.** Most asymptotic results [1, 5, 43] for ${}_2F_1\,(a, b; c; z)$ for large $|c|$ are valid only for $|\arg(c)| < \pi - \epsilon$, $\epsilon > 0$ or for restraining conditions on $z$ which make it inapplicable in our case which is $z \in (0, 1)$. The situation is similar in case of the Legendre functions for large $-\mu$ near integers. Asymptotics of [36] are rather complicated and the error bounds are not explicitly calculated for our case. In fact, asymptotic formula for Legendre functions in [1] for $|\mu| \to \infty$ is the same as the one we will later derive, although is not clear under which conditions do the asymptotics hold, and we could not find the original result in literature. Nevertheless, we obtain the same formula for the asymptotics under clear conditions in proof of Lemma 4.12. Apart from this case, the weakest conditions on validity of certain asymptotic expansions (with rigorously stated conditions under which they hold) we were able to find are the following [43]. For the case that $c$ is not near a negative integer, either

$$
\begin{aligned}
&a = -m, \ \text{or} \ b = -n, \ \ n, m \in \mathbb{N}, \\
&\Re z < \frac{1}{2} \ \text{and} \ \forall n \in \mathbb{N} \ |c + n| \geq \delta > 0, \\
&\Re z = \frac{1}{2} \ \text{and} \ |\arg c| \leq \pi - \epsilon, \epsilon > 0
\end{aligned}
\tag{4.51}
$$

and more complicated asymptotics for the case when $c$ is near a negative integer

$$
\begin{aligned}
&\Re z < \frac{1}{2} \ \text{and} \ c = -n + \epsilon, \ \ 0 < \epsilon = o(1), \\
&\Re z = \frac{1}{2} \ \text{and} \ \ \arg(-c) = \epsilon, \ \ 0 < \epsilon = o(1).
\end{aligned}
\tag{4.52}
$$

Overall, these asymptotics cannot be used due to equidistribution theorem sketched above.

**Lemma 4.10.** *Equation for eigenvalues* (4.40) *for fixed $m$ and $\mu - \nu \notin \mathbb{N}$ is equivalent to*

$$
\epsilon_+ \alpha_m(-b)\beta_m(a) + \epsilon_- \alpha_m(a)\beta_m(-b) = 0,
\tag{4.53}
$$

[2] This is known as equidistribution theorem [42, Theorem 2.1], stating that the sequence $\{\alpha n \mod 1\}_{n\in\mathbb{N}}$ is equidistributed (hence dense) in $\mathbb{R}/\mathbb{Z}$ for irrational $\alpha$. For rational values of $\alpha$, the sequence is trivially 0 infinitely often.

*with $\alpha_m, \beta_m, \theta_\mu : [-b, a] \to \mathbb{C}$ defined below for $K \in \{+1, -1\}$:*

$$
\begin{aligned}
\gamma_m &= \frac{\pi\Gamma(1+\mu+\nu)}{2\sin(\pi\mu)\Gamma(1-\mu+\nu)},\\
\frac{\alpha_m(x)}{\gamma_m} &= {}_2\tilde{F}_1\left(-\nu, 1+\nu; 1-\mu; \frac{1}{2}\right) {}_2\tilde{F}_1\left(-\nu, 1+\nu; 1+\mu; \zeta(x)\right)\theta_\mu(x) -\\
&\quad - {}_2\tilde{F}_1\left(-\nu, 1+\nu; 1-\mu; \zeta(z)\right) {}_2\tilde{F}_1\left(-\nu, 1+\nu; 1+\mu; \frac{1}{2}\right)\theta_\mu(x)^{-1},\\
\frac{\beta_m(x)}{\gamma_m} &= {}_2\tilde{F}_1\left(-\nu, 1+\nu; 1-\mu; \zeta(z)\right)\theta_\mu(x)^{-1}\cdot\\
&\quad \left[\frac{\nu(\nu+1)}{2}\,{}_2\tilde{F}_1\left(1-\nu, 2+\nu; 2+\mu; \frac{1}{2}\right) - \mu\,{}_2\tilde{F}_1\left(-\nu, 1+\nu; 1+\mu; \frac{1}{2}\right)\right] -\\
&\quad - {}_2\tilde{F}_1\left(-\nu, 1+\nu; 1+\mu; \zeta(z)\right)\theta_\mu(x)\cdot\\
&\quad \left[\frac{\nu(\nu+1)}{2}\,{}_2\tilde{F}_1\left(1-\nu, 2+\nu; 2-\mu; \frac{1}{2}\right) + \mu\,{}_2\tilde{F}_1\left(-\nu, 1+\nu; 1-\mu; \frac{1}{2}\right)\right],\\
\zeta(x) &= \begin{cases} \frac{1-\sin x}{2}, & K = 1,\\ \frac{1-\tanh x}{2}, & K = -1,\end{cases}\\
\theta_\mu(x) &= \left(\zeta(x)\zeta(-x)^{-1}\right)^{\frac{\mu}{2}} = \begin{cases} \left(\frac{1-\sin x}{1+\sin x}\right)^{\frac{\mu}{2}}, & K = 1,\\ \exp(-\mu x), & K = -1,\end{cases}
\end{aligned}
\tag{4.54}
$$

*with convention $\nu = \nu_{\operatorname{sgn} x}$, i.e. $\nu_+$ for $x > 0$ and $\nu_-$ for $x < 0$ and $\nu$, $\mu$ dependent on $\lambda_m$ as in Table 4.1.*

*Proof.* By introducing contrast $\kappa = \frac{\epsilon_+}{\epsilon_-}$ and using simple algebraic relation

$$
\begin{aligned}
0 = -\det\begin{pmatrix} a & b & 0 & 0\\ 0 & 0 & c & d\\ e & f & -g & -h\\ \kappa i & \kappa j & k & l\end{pmatrix} &= \kappa(bi-aj)(dg-ch) + (be-af)(dk-cl)\\
&= \kappa\alpha_m(-b)\beta_m(a) + \alpha_m(a)\beta_m(-b)
\end{aligned}
\tag{4.55}
$$

we achieve more compact notation by introducing, for positive curvature, $\alpha_m, \beta_m : (-\frac{\pi}{2}, \frac{\pi}{2}) \supset [-b, a] \to \mathbb{R}$,

$$
\begin{aligned}
\alpha_m(x) &:= P_\nu^\mu(\sin x)\,Q_\nu^\mu(0) - P_\nu^\mu(0)Q_\nu^\mu(\sin x),\\
\beta_m(x) &:= \left.\frac{\partial P_\nu^\mu(\sin x)}{\partial x}\right|_{x=0} Q_\nu^\mu(\sin x) - P_\nu^\mu(\sin x)\left.\frac{\partial Q_\nu^\mu(\sin x)}{\partial x}\right|_{x=0}.
\end{aligned}
\tag{4.56}
$$

For negative curvature, the modification is obvious. After substitution from (4.44) and factoring, terms of $Q_\nu^\mu$ containing $\cos(\pi\mu)$ cancel out. In order to handle both cases of curvature in the same manner in further analysis, multiply the equation (4.55) by $\sqrt{\cosh x}$ and appropriately redefine $\alpha_\mu$ and $\beta_\mu$. This leads to the quantities given in the statement of this lemma. ☐

**Lemma 4.11.** *Fix $m \in \mathbb{N}$. Then the only accumulation points of roots $(\lambda_{m,k})_{k\in\mathbb{Z}}$ of the characteristic equation (4.53) for $m$ are $\pm\infty$, or $\omega_\pm$ and the latter is only possible if (4.40) holds for $\lambda_m = \omega_\pm$,*

$$
\omega_\pm = \begin{cases} \mp\frac{\epsilon_\pm}{4}, & K = 1,\\ \pm\frac{\epsilon_\pm}{4}, & K = -1.\end{cases}
\tag{4.57}
$$

*Proof.* The regularized hypergeometric function ${}_2\tilde{F}_1\left(a,b;c;z\right)$ for fixed $|z|<1$ is an entire analytic functions in parameters $a,b,c\in\mathbb{C}$. Denote the left-hand side of (4.53) as $f(\lambda)$ with $\mu$ constant and $\nu$ dependent on $\lambda$. Choose a suitable complex square root such that the branch cut does not intersect the real line. This can be done by a choice defining $\sqrt{0}=0$ and $\sqrt{z}=\mathrm{e}^{\log(z)/2}$, $\log z := \operatorname{Log}|z|+\mathrm{i}\theta$, for $z=|z|\,\mathrm{e}^{\mathrm{i}\theta}$, $|z|>0$, $\theta\in[-\frac{\pi}{4},\frac{3}{2}\pi)$ and Log $z$ the real logarithm. In this notation, the square-root of positive real numbers stays the same, for negative real numbers

$$\sqrt{-x}=\mathrm{i}\sqrt{x},\quad x>0, \tag{4.58}$$

and thus it is holomorphic on $\mathbb{C}\setminus[0,-\mathrm{i}\infty)$ and continuous as a function $\mathbb{R}\to\mathbb{C}$.

Remember that self-adjoint operators have only real eigenvalues. From Theorem 2.18 applied on function $f$ on domain $\mathbb{C}\setminus[0,-\mathrm{i}\infty)$, either limit points of $(\lambda_{m,k})_k$ lie in $\left(\{\infty\}\cup[0,-\mathrm{i}\infty)\right)\cap\mathbb{R}^*$, or $f(\lambda)=0$ on $\mathbb{C}\setminus[0,-\mathrm{i}\infty)$. So either

- there are infinite limit point(s) of $(\lambda_{m,k})_k$ or
- there are limit point(s) $\omega_\pm := \pm\frac{\epsilon_\pm}{4}$ for $K=1$ (or $\omega_\pm := \mp\frac{\epsilon_\pm}{4}$ for $K=-1$),

or $f(\lambda)=0$ for all $\lambda\in\mathbb{R}\setminus\left\{\pm\frac{\epsilon_\pm}{4}\right\}$.

The second point corresponds to $\sqrt{z}$, $z=0$ appearing in $\nu_\pm(\lambda)$. At the same time, the set of all eigenvalues — the point spectrum — of a self-adjoint operators in separable Hilbert spaces is at most countable. As the characteristic equation (4.40) is an equation whose solutions are eigenvalues of a certain self-adjoint operator in $L^2\left((-b,a),\mathrm{d}\nu_K\right)$ (created by separation of variables from $A_K$) [18], we arrive at a contradiction with $f(\lambda)=0$ on $\mathbb{C}\setminus[0,-\mathrm{i}\infty)$. Hence, the limit points belong to $\{\pm\infty,\omega_\pm\}$.

Assume that the second option holds. As $\lambda\mapsto f(\lambda)$ is continuous on $\mathbb{R}$, necessarily $f(\omega_\pm)=0$. ☐

**Lemma 4.12.** *For such $m$ that $\mu_m-\nu_m^\pm\notin\mathbb{N}$, the following hold in notation of Lemma 4.10. Additionally, asymptotics for $|\mu|\to\infty$ hold if $|\nu_{\pm,m}|$ are bounded in $m$:*

$$\tilde{\gamma}_m=\frac{\sin\left(\pi(\mu-\nu)\right)}{2\pi}\frac{\Gamma\left(\mu-\nu\right)\Gamma\left(\mu+\nu+1\right)}{\Gamma\left(\mu^2\right)}=\mu\frac{\sin\left(\pi(\mu-\nu)\right)}{2\pi}\left(1+\mathcal{O}\left(\frac{1}{|\mu|}\right)\right), \tag{4.59}$$

$$\begin{aligned}
\alpha_m(x)&=\frac{\tilde{\gamma}_m}{\mu}\left[\begin{matrix}{}_2F_1\left(-\nu,\nu+1;\mu;\tfrac{1}{2}\right)\,{}_2F_1\left(-\nu,\nu+1;\mu+1;\zeta(x)\right)\theta_\mu(x)\\ -\dfrac{{}_2F_1\left(-\nu,\nu+1;\mu;\zeta(-x)\right)\,{}_2F_1\left(-\nu,\nu+1;\mu+1;\tfrac{1}{2}\right)}{\theta_\mu(x)}\end{matrix}\right],\\
&=\frac{\tilde{\gamma}_m}{\mu}\left(\theta_\mu(x)-\theta_\mu(x)^{-1}\right)\left(1+o(1)\right)
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
\beta_m(x)&=\tilde{\gamma}_m\left[\begin{matrix}\left(\frac{\nu(\nu+1)}{2\mu^2}\,{}_2F_1\left(1-\nu,2+\nu;\mu+1;\tfrac{1}{2}\right)-{}_2F_1\left(-\nu,\nu+1;\mu;\tfrac{1}{2}\right)\right){}_2F_1\left(-\nu,\nu+1;\mu+1;\zeta(x)\right)\theta_\mu(x)\\ +\dfrac{{}_2F_1(-\nu,\nu+1;\mu;\zeta(-x))\left(\frac{\nu(\nu+1)}{2\mu(\mu+1)}\,{}_2F_1\left(1-\nu,\nu+2;\mu+2;\tfrac{1}{2}\right)-{}_2F_1\left(-\nu,\nu+1;\mu+1;\tfrac{1}{2}\right)\right)}{\theta_\mu(x)}\end{matrix}\right]\\
&=\tilde{\gamma}_m\left(-\theta_\mu(x)-\theta_\mu(x)^{-1}\right)\left(1+o(1)\right)
\end{aligned}$$

*Proof.* First, we ought to solve the singularity problems for positive curvature described in Note 4.8. In case of negative curvature, the asymptotic analysis can be done directly, but we would like to maintain unified notation for both curvatures. The transformation described below can be done regardless of sign of curvature.

We will make use of the procedure in proof of [43] and derive improved conditions under which the asymptotics in the near-integer case hold, primarily for $|z| < 1$. We will employ an alternative form of the hypergeometric function given by relations between Kummer's solution to hypergeometric equation. This relation is useful due to transformation of parameter $c \mapsto -c+\kappa$, $\kappa \in \mathbb{N}$. In particular, we will use [5, section 2.9, equation (35)] and [43, equation (25)], valid when the $\Gamma$ factors in nominator are finite and for all $z$ for which the involved series converge,

$$\begin{aligned} {}_2F_1(a,b;c;z) &= k_1(a,b,c)\,{}_2F_1(a,b;a+b-c+1;1-z)+ \\ &\quad + k_2(a,b,c,z)\,{}_2F_1(1-a,1-b;2-c;z), \\ k_1(a,b,c) &:= \frac{\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(a+b-c+1)\,\Gamma(1-c)}, \\ k_2(a,b,c,z) &:= -\Gamma(c-1)\frac{\Gamma(a-c+1)\,\Gamma(b-c+1)}{\Gamma(a)\Gamma(b)\Gamma(1-c)} z^{1-c}(1-z)^{c-b-a}. \end{aligned} \tag{4.61}$$

For regularized hypergeometric functions appearing in Legendre functions,

$$\begin{aligned} \frac{{}_2F_1(a,b;c;z)}{\Gamma(c)} &= \tilde{k}_1(a,b,c)\frac{{}_2F_1(a,b;a+b-c+1;1-z)}{\Gamma(a+b-c+1)}+ \\ &\quad + \tilde{k}_2(a,b,c,z)\frac{{}_2F_1(1-a,1-b;2-c;z)}{\Gamma(2-c)}, \end{aligned} \tag{4.62}$$

the coefficients $\tilde{k}_1$, $\tilde{k}_2$ are obtained by employing definition of $\Gamma(z)$ and Euler reflection identity,

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{C}, \tag{4.63}$$

$$\Gamma(1-z)\Gamma(z)\sin\pi z = \pi, \quad z \in \mathbb{C} \tag{4.64}$$

(the left-hand side of reflection identity is considered as an entire function valid even for negative integers),

$$\begin{aligned} \tilde{k}_1(a,b,c) &:= \frac{\Gamma(a-c+1)\,\Gamma(b-c+1)}{\Gamma(c)\,\Gamma(1-c)} = \Gamma(a-c+1)\,\Gamma(b-c+1)\,\frac{\sin(\pi c)}{\pi} \\ \tilde{k}_2(a,b,c,z) &:= \frac{\Gamma(c-1)\,\Gamma(2-c)}{\Gamma(c)\,\Gamma(1-c)}\frac{\Gamma(a-c+1)\,\Gamma(b-c+1)}{\Gamma(a)\,\Gamma(b)} z^{1-c}(1-z)^{c-b-a} \\ &= -\frac{\Gamma(a-c+1)\,\Gamma(b-c+1)}{\Gamma(a)\,\Gamma(b)} z^{1-c}(1-z)^{c-b-a}. \end{aligned} \tag{4.65}$$

Overall, denoting ${}_2\tilde{F}_1(a,b;c;z) = \frac{1}{\Gamma(c)}\,{}_2F_1(a,b;c;z)$, we obtain sought relation

$$\begin{aligned} \frac{{}_2\tilde{F}_1(a,b;c;z)}{\Gamma(a-c+1)\,\Gamma(b-c+1)} &= \frac{\sin(\pi c)}{\pi}\,{}_2\tilde{F}_1(a,b;a+b-c+1;1-z) \\ &\quad - \frac{(1-z)^{c-b-a}\,z^{1-c}}{\Gamma(a)\,\Gamma(b)}\,{}_2\tilde{F}_1(1-a,1-b;2-c;z) \end{aligned} \tag{4.66}$$

valid for all $a$, $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $c \in \mathbb{C}$ such that $a-c+1, b-c+1 \notin \mathbb{Z}_{\leq 0}$, $|z| < 1$.

Substituting parameters appearing in eigenvalue equation given in Lemma 4.10, in cases where parameter $c$ is negative for large enough $\mu$, we obtain for $t \in \{0,1\}$:

$$\begin{aligned} \frac{{}_2\tilde{F}_1(-\nu+t,\nu+1+t;1+t-\mu;z)}{\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)} &= \frac{(-1)^t\sin(\pi\mu)}{\pi}\,{}_2\tilde{F}_1(-\nu+t,\nu+t+1;\mu+t;1-z)+ \\ + \frac{\left(\frac{1-z}{z}\right)^{-\mu}}{(1-z)^{2t}}\frac{\sin(\pi\nu)}{\pi\left[-\nu(\nu+1)\right]^t}\,{}_2\tilde{F}_1(-\nu-t,\nu-t+1;\mu+1-t;z)&. \end{aligned} \tag{4.67}$$

Turning our attention back to functions $\alpha_m$, $\beta_m$ and substituting relations (4.67), we obtain

$$\frac{\pi\alpha_m(x)/\gamma_m}{\Gamma(\mu-\nu)\,\Gamma(1+\mu+\nu)\sin(\pi\mu)} = \left[\begin{array}{c} {}_2\tilde{F}_1\left(-\nu,\nu+1;\mu;\frac{1}{2}\right)\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\zeta(x)\right)\theta_\mu(x) \\ -\dfrac{{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu;\zeta(-x)\right)\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\frac{1}{2}\right)}{\theta_\mu(x)} \end{array}\right] \tag{4.68}$$

$$\frac{\pi\beta_m(x)/\gamma_m}{\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)\sin(\pi\mu)} =$$
$$= \left[\begin{array}{c} \left(\frac{\nu(\nu+1)}{2}\,{}_2\tilde{F}_1\left(1-\nu,2+\nu;\mu+1;\frac{1}{2}\right)-\mu\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu;\frac{1}{2}\right)\right){}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\zeta(x)\right)\theta_\mu(x) \\ +\dfrac{{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu;\zeta(-x)\right)\left(\frac{\nu(\nu+1)}{2}\,{}_2\tilde{F}_1\left(1-\nu,\nu+2;\mu+2;\frac{1}{2}\right)-\mu\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\frac{1}{2}\right)\right)}{\theta_\mu(x)} \end{array}\right]. \tag{4.69}$$

The terms in $\alpha_m$ containing $\sin(\pi\nu)$ have canceled out. The terms in $\beta_m$ containing $\sin(\pi\nu)$ can be canceled out using identity for contiguous functions:

$${}_2\tilde{F}_1\left(-\nu-1,\nu;\mu;\frac{1}{2}\right)+\frac{\nu(\nu+1)}{4}\,{}_2\tilde{F}_1\left(1-\nu,\nu+2;\mu+2;\frac{1}{2}\right)-\mu\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\frac{1}{2}\right)=0. \tag{4.70}$$

The factors from both $\alpha_m(x)$ and $\beta_m(x)$ can be simplified in relation to $\gamma_m$ from Lemma 4.10 as

$$\gamma_m\frac{\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)\sin(\pi\mu)}{\pi}=\frac{\Gamma(\mu-\nu)^2\,\Gamma(\mu+\nu+1)^2\sin\left(\pi(\mu-\nu)\right)}{2\pi}. \tag{4.71}$$

From $\Gamma(z)=\sqrt{\frac{2\pi}{z}}\left(\frac{z}{\mathrm{e}}\right)^z\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right)$ as $z\to\infty$ and $|\arg z|<\pi$, we obtain

$$\frac{\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)}{\Gamma(\mu)^2}=\frac{\left(\frac{\mu}{\mathrm{e}}\right)^{\mu-\nu}\left(\frac{\mu}{\mathrm{e}}\right)^{\mu+\nu}(\mu+\nu)}{\left(\frac{\mu}{\mathrm{e}}\right)^{2\mu}}\left(1+\mathcal{O}\left(\frac{1}{\mu}\right)\right)=\mu\left(1+\mathcal{O}\left(\frac{1}{\mu}\right)\right). \tag{4.72}$$

All hypergeometric functions are approaching unity, ${}_2F_1(a,b;c;z)=1+\frac{abz}{c}+\mathcal{O}\left(\frac{1}{|c|^2}\right)$ for $\left|\arg(c)\right|<\pi$, $|c|\to\infty$ and $|z|<1$ [5]. To secure the asymptotic relations, notice that for $A\equiv A_m$, $B\equiv B_m$, $\lim_{m\to\infty}A=\lim_{m\to\infty}B=1$,

$$\begin{aligned} A\theta_\mu(x)\pm B\theta_\mu(x)^{-1} &= A\left(\theta_\mu(x)\pm\theta_\mu(x)^{-1}\right)\left(\frac{1\pm\frac{B}{A}\theta_\mu(x)^{-2}}{1\pm\theta_\mu(x)^{-2}}-1+1\right) \\ &= A\left(\theta_\mu(x)\pm\theta_\mu(x)^{-1}\right)\left(\frac{\pm\theta_\mu(x)^{-2}\left(\frac{B}{A}-1\right)}{1\pm\theta_\mu(x)^{-2}}+1\right) \\ &= \left(\theta_\mu(x)\pm\theta_\mu(x)^{-1}\right)\left(1+o(1)\right) \end{aligned} \tag{4.73}$$

as $\frac{\theta_\mu(x)^{-2}}{1\pm\theta_\mu(x)^{-2}}$ is bounded in $m$ for $x\neq 0$ regardless of sign of curvature. □

**Lemma 4.13.** *Let $(\lambda_m)_{m\in\mathbb{N}}$ be solutions to characteristic equation (4.40) for given $m \in \mathbb{N}$. In notation of Table 4.1, define an indication of degeneracy using $s_\pm$ as*

$$s_m^\pm = \begin{cases} 1, & \text{if } \mu_m - \nu_{\pm,m} \in \mathbb{N}, \\ \sin\Big(\pi\left(\mu_m - \nu_{\pm,m}\right)\Big), & \text{else.} \end{cases} \tag{4.74}$$

*Assume that both sequences $(\nu_{\pm,m})_{m\in\mathbb{N}}$ are bounded. Then,*

$$\lim_{m\to\infty} s_m^+ s_m^- \,\xi(\mu_m) = 0, \tag{4.75}$$

*where $\xi : \mathbb{R}_+ \to \mathbb{R}$ for $\theta_\mu(x)$ defined in Lemma 4.10:*

$$\xi(\mu) = \epsilon_+ \Big(\theta_\mu(a) + \theta_\mu(a)^{-1}\Big) \Big(\theta_\mu(b) - \theta_\mu(b)^{-1}\Big) - \epsilon_- \Big(\theta_\mu(a) - \theta_\mu(a)^{-1}\Big) \Big(\theta_\mu(b) + \theta_\mu(b)^{-1}\Big). \tag{4.76}$$

*Proof.*
**Non-degenerate case:** For those $m \in \mathbb{N}$ such that $\big(\mu_m, \nu_{\pm,m}\big)$ is non-degenerate, we have the following for the characteristic equation (4.53):

$$\sin\Big(\pi\left(\mu_m - \nu_+(m)\right)\Big) \sin\Big(\pi\left(\mu_m - \nu_-(m)\right)\Big)\, \xi(\mu_m)\, \big(1 + o(1)\big) = 0 \tag{4.77}$$

due to Lemma 4.12 by using the asymptotics together with identity $\theta_\mu(-x) = \theta_\mu(x)^{-1}$ and dividing by $\mu$.

**Degenerate case:** Concerning those $m \in \mathbb{N}$ such that both $\mu_m - \nu_{\pm,m} \in \mathbb{N}$ (in notation of Table 4.1), i.e. $m$-th eigenvector is given using (4.50) in both $\Omega_\pm$. By analogous process to Lemma 4.12, we arrive at formulas that are the same as in Lemma 4.10, except now $\gamma_m = 1$ and ${}_2\tilde{F}_1\left(a, b; c; z\right)$ is replaced by ${}_2F_1\left(a, b; c; z\right)$ and hence,

$$\xi(\mu_m)\, \big(1 + o(1)\big) = 0. \tag{4.78}$$

**Semi-degenerate case:** For those $m \in \mathbb{N}$ that $\big(\mu_m, \nu_+(m)\big)$ is degenerate and $(\mu_m, \nu_-(m))$ is non-degenerate, eigenvectors are then given using (4.50) on $\Omega_+$ and as non-degenerate (4.44) on $\Omega_-$. In resulting characteristic equation, functions $\alpha_m(x)$, $\beta_m(x)$ for $x < 0$ are given by Lemma 4.12 and for $x > 0$ by previous "degenerate" paragraph. Conclusion of this lemma then follows. □

**Lemma 4.14.** *Let $(\mu_n)_{n\in\mathbb{N}}, (\nu_{+,n})_{n\in\mathbb{N}}, (\nu_{-,n})_{n\in\mathbb{N}}$ be real sequences such that $\mu_n := xn$ for $x \in \mathbb{R}$, $x > 0$ and that both $(\nu_{\pm,n})_{n\in\mathbb{N}}$ converge to finite limits. Define sequences $\left(s_n^\pm\right)_{n\in\mathbb{N}}$ using*

$$s_n^\pm = \begin{cases} 1, & \text{if } \mu_n - \nu_{\pm,n} \in \mathbb{N}, \\ \sin\Big(\pi\left(\mu_m - \nu_{\pm,m}\right)\Big), & \text{else.} \end{cases} \tag{4.79}$$

*Then the sequence $(s_n)_{n\in\mathbb{N}}$ given by*

$$s_n := s_n^+ s_n^- \tag{4.80}$$

*does not have a limit or the limit is not 0.*

*Proof.* We will prove the result by constructing a subsequence $(s_{n_l})_{l\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ that does not converge to zero. Define sets

$$\begin{gathered}\mathcal{N}_D^{\pm} := \left\{n \in \mathbb{N} : \mu_n - \nu_{\pm,n} \in \mathbb{N}\right\}, \\ \mathcal{N}_D := \mathcal{N}_D^{+} \cap \mathcal{N}_D^{-}, \quad \mathcal{N} := \mathbb{N} \setminus \left(\mathcal{N}_D^{+} \cup \mathcal{N}_D^{-}\right).\end{gathered} \tag{4.81}$$

Based on the cardinality of the defined sets, we will proceed to prove the lemma on a case-by-case basis as seen in Table 4.2.

| case | $\lvert\mathcal{N}_D^{+}\rvert$ | $\lvert\mathcal{N}_D^{-}\rvert$ | $\lvert\mathcal{N}_D\rvert$ | $\lvert\mathcal{N}\rvert$ |
|---|---|---|---|---|
| 1 | disc. | disc. | disc. | $\infty$ |
| 2 | $\infty$ | $\infty$ | $\infty$ | $\infty$/disc. |
| 3a | $\infty$ | disc. | disc. | $\infty$ |
| 3b | $\infty$ | $\infty$ | disc. | $\infty$ |
| 4a | $\infty$ | disc. | disc. | disc. |
| 4b | $\infty$ | $\infty$ | disc. | disc. |

Table 4.2: All possible combinations of cardinality of sets defined in (4.81) up to an exchange of roles of $\mathcal{N}_D^{+}$ and $\mathcal{N}_D^{-}$. Label "disc." signifies that the set is discrete, $\infty$ infinite.

Let $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ be two sequences for which there exists an infinite set $N \subset \mathbb{N}$ such that $\exists K > 0,\ \forall n \in N,\ |A_n| > K \wedge |B_n| > K$. Then $|A_n B_n| > K^2$ for all infinitely many $n \in \mathbb{N}$ and so sequence $(A_n B_n)_{n\in\mathbb{N}}$ does not converge to 0. To prove the lemma, we will take $A_n = s_n^{+}$, $B_n = s_n^{-}$.

Before attempting to solve the cases for all values of $x$, entertain possibility of $x = k$ or $x = \frac{k}{2}$ for integer $k \in \mathbb{N}$. In those cases,

$$\begin{aligned}\sin\left(\pi k n - \pi\nu_{\pm,n}\right) &= (-1)^{kn+1}\sin\left(\pi\nu_{\pm,n}\right), \\ \sin\left(\pi\frac{kn}{2} - \pi\nu_{\pm,n}\right) &= \begin{cases}(-1)^{l+1}\sin\left(\pi\nu_{\pm,n}\right), & kn = 2l \text{ is even,} \\ (-1)^{l}\cos\left(\pi\nu_{\pm,n}\right), & kn = 2l+1 \text{ is odd.}\end{cases}\end{aligned} \tag{4.82}$$

$(s_n)_{n\in\mathbb{N}}$ does not converge to 0 for any value of $\nu_{\pm,n}$ (as for the first case $x = k \in \mathbb{N}$ would imply $\nu \in \mathbb{Z}$ and so $\mu - \nu \in \mathbb{N}$) and we can exclude it from further analysis.

We will proceed by contradiction. Let $s_n^{\pm}$ converge to 0. Then there exist sequences $(k_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ such that $\epsilon_n \xrightarrow{n\to\infty} 0$ and

$$\mu_n\pi - \nu_{\pm,n}\pi = k_n\pi + \epsilon_n. \tag{4.83}$$

By rearranging,

$$\mu_n - k_n = \nu_{\pm,n} + \frac{\epsilon_n}{\pi}, \tag{4.84}$$

it is clear that the left hand side of (4.84) has a *finite* limit as the right hand side has. Introduce $(\kappa_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ such that $k_n = \lfloor\mu_n\rfloor - \kappa_n$ where $\lfloor\mu_n\rfloor$ is the largest integer less than $\mu_n$. After substituting we obtain

$$\left(\mu_n - \lfloor\mu_n\rfloor\right) + \kappa_n = \nu_{\pm,n} + \frac{\epsilon_n}{\pi} \tag{4.85}$$

and as $\{\mu_n\} := \mu_n - \lfloor\mu_n\rfloor \in [0, 1)$ is the fractional part of $\mu_n$ and $\kappa_n \in \mathbb{Z}$, then $\kappa_n = \lfloor\nu_{\pm,n}\rfloor$ and

$$\lim_{\substack{n\in\mathcal{N} \\ n\to\infty}} \{\mu_n\} = \lim_{n\to\infty} \left\{\nu_{\pm,n}\right\}. \tag{4.86}$$

Now, assume that $x \in \mathbb{R} \setminus \mathbb{Q}$. Then it can be seen that a limit of (4.86) cannot exist — set of limit points of sequence $(xn - \lfloor xn \rfloor)_{n\in\mathbb{N}}$ is exactly $[0,1]$ from the equidistribution theorem. Similarly, for the case $x \in \mathbb{Q}$, let $x = \frac{a}{b}$ be an irreducible fraction. Then there are at least two limit points of $\left(xn - \lfloor xn \rfloor\right)$, and that gives us a contradiction. Here, we used that $\mathcal{N} = \mathbb{N} \setminus D$ where $D = \mathcal{N}_D^{+} \cup \mathcal{N}_D$ is discrete for contradiction and thus proving case 1.

Proof for case 2 is trivial as $|\mathcal{N}_D| = \infty$ implies existence of subsequence of $s_n$ which is identically equal to 1 and thus 1 is a limit point of $(s_n)_n$, again arriving at a contradiction.

Proof of 3 is similar to 1, except now the final step leading to contradiction requires more information about behaviour of $s_n^{\pm}$ for $\mathcal{N}_D^{\pm}$ as these sets are no longer discrete. For $\mu_n - \nu_{\pm,n} \in \mathbb{N}$, we obtain

$$\{\mu_n\} = \{\nu_{\pm,n}\}\,, \quad n \in \mathcal{N}_D^{\pm}, \tag{4.87}$$

and so

$$\lim_{\substack{n\in\mathcal{N}_D^{\pm}\\ n\to\infty}} \{\mu_n\} = \lim_{n\to\infty} \{\nu_{\pm,n}\}\,. \tag{4.88}$$

Overall with (4.86), we obtain that $(\{\mu_n\})_{n\in\mathbb{N}}$ has at most two limit points. This is a contradiction as the same argument from case 1 now applies except for the case (4.82) which was already solved. Last case is proven in a very similar way. □

**Lemma 4.15.** *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be real sequences such that $\lim_{n\to\infty}(a_n b_n) = 0$ and $(b_n)_{n\in\mathbb{N}}$ does not have a limit or the limit is non-zero. Then $\liminf_{n\to\infty} |a_n| = 0$.*

*Proof.* Sequence $(b_n)_{n\in\mathbb{N}}$ does not converge to 0, so there exists $K > 0$ such that for infinitely many $n \in \mathbb{N}$ is $|b_n| > K$. From $\lim_{n\to\infty}(a_n b_n) = 0$ follows that for all $\epsilon > 0$ is $\epsilon > |a_n b_n| > K|a_n|$ for infinitely many $n \in \mathbb{N}$. Hence, $(a_n)_{n\in\mathbb{N}}$ has a limit point 0. □

**Proposition 4.16.** *Let $K = +1$ and let $\{\lambda_m\}_{m\in\mathbb{N}}$ be a sequence such that the characteristic equation (4.40) is satisfied for a given pair $(m, \lambda_m)$. Let $\Lambda \in \mathbb{R}$, $\Lambda \neq \omega_\pm$ in notation of Lemma 4.11. Then $\lambda_m \xrightarrow{m\to\infty} \Lambda$, $|\Lambda| < \infty \implies \epsilon_+ = \epsilon_-$.*

*Proof.* We will start by deriving identities valid for both positive and negative curvatures and conclude only for positive curvature. When $\lambda_{m,k} \to \Lambda$, necessarily $m \to \infty$ due to Lemma 4.11.

Given the assumptions, Lemma 4.13 holds. We would like to first show that $\liminf_{n\to\infty} |\xi(\mu_n)| = 0$ regardless of sign of curvature. Considering formula

$$\sin(A + \mathrm{i}B) = \sin(A)\cosh(B) + \mathrm{i}\cos(A)\sinh(B), \tag{4.89}$$

we can see that in order for $\sin(\pi\left(\mu_n - \nu_{\pm,n}\right))$ to converge to zero, we need to have

$$\begin{aligned} \sin\left(\pi\Re\left(\mu_n - \nu_{\pm,n}\right)\right) &\xrightarrow{n\to\infty} 0,\\ \Im\left(\mu_n - \nu_{\pm,n}\right) &\xrightarrow{n\to\infty} 0. \end{aligned} \tag{4.90}$$

In Table 4.3, we will assume the same branch of complex square root as in Lemma 4.11 — for negative curvature, the sine does not approach zero as that would contradict the assumptions of this lemma. For the positive curvature, both $\nu_\pm(\Lambda)$ have to be real. That means that $\Lambda \in \left(-\frac{\epsilon_+}{4}, \frac{\epsilon_-}{4}\right)$ and arguments of both sine functions above are eventually real for all $n > n_0$ for some $n_0 \in \mathbb{N}$ and from Lemmas 4.15 and 4.14 follows that $\liminf_{n\to\infty} |\xi(\mu_n)| = 0$.

Now, we will show that $\epsilon_+ = \epsilon_-$. Define functions $\tilde{\theta}_\mu : [-b, a] \to \mathbb{R}$ for $\mu \in \mathbb{R}$:

$$\tilde{\theta}_\mu(x) := \frac{\theta_\mu(x) - \theta_\mu(x)^{-1}}{\theta_\mu(x) + \theta_\mu(x)^{-1}} = \begin{cases} \frac{\left(\frac{1-\sin(x)}{1+\sin(x)}\right)^\mu - 1}{\left(\frac{1-\sin(x)}{1+\sin(x)}\right)^\mu + 1}, & K = 1, \\ -\tanh(\mu x), & K = -1. \end{cases} \tag{4.91}$$

The limit expression $\liminf_{n\to\infty} |\xi(\mu_n)| = 0$ can be rewritten as

$$0 = \liminf_{\mu\to\infty} \left| \left(\theta_\mu(a) + \theta_\mu(a)^{-1}\right) \left(\epsilon_+ \left(\theta_\mu(b) - \theta_\mu(b)^{-1}\right) - \epsilon_- \tilde{\theta}_\mu(a) \left(\theta_\mu(b) + \theta_\mu(b)^{-1}\right)\right)\right|. \tag{4.92}$$

Consider only the case $K = 1$ for the moment. Observe, that $\frac{1-\sin(x)}{1+\sin(x)} < 1$ for $x \in \left(0, \frac{\pi}{2}\right)$. With that knowledge, $\lim_{\mu\to\infty} \left(\theta_\mu(a) + \theta_\mu(a)^{-1}\right) = \infty$ for $a \in \left(0, \frac{\pi}{2}\right)$ as

$$\theta_\mu(a) + \theta_\mu(a)^{-1} = \left(\frac{1-\sin(a)}{1+\sin(a)}\right)^{-\frac{\mu}{2}} \left(\left(\frac{1-\sin(a)}{1+\sin(a)}\right)^\mu + 1\right). \tag{4.93}$$

Hence,

$$\liminf_{\mu\to\infty} \left|\epsilon_+ \left(\theta_\mu(b) - \theta_\mu(b)^{-1}\right) - \epsilon_- \tilde{\theta}_\mu(a) \left(\theta_\mu(b) + \theta_\mu(b)^{-1}\right)\right| = 0 \tag{4.94}$$

by using $\liminf_{n\to\infty} |A_n B_n| = 0 \wedge \lim_{n\to\infty} A_n = +\infty \implies \liminf_{n\to\infty} |B_n| = 0$. By a similar argument we readily obtain

$$\liminf_{\mu\to\infty} \left|\epsilon_+ \tilde{\theta}_\mu(b) - \epsilon_- \tilde{\theta}_\mu(a)\right| = 0. \tag{4.95}$$

From (4.91) and previous estimate follows that for $x \in \left(0, \frac{\pi}{2}\right)$ is $\tilde{\theta}_\mu(x) \to -1$ as $\mu \to +\infty$. Ultimately, using previous equation (4.95), we obtain

$$\epsilon_+ = \epsilon_-. \tag{4.96}$$

$\square$

**Remark 4.17.** Regarding the case of $K = -1$, we could not obtain such strong results. We have even obtained some contradicting results for a special choice of dimensions of the rectangle $a$, $b$, $c$. As $\theta_\mu(x) = \mathrm{e}^{-\mathrm{i}Mx + x/2}$ for $M = \frac{m\pi}{c}$, we obtain

$$\theta_\mu(x) \pm \theta_\mu(x)^{-1} = \left(\mathrm{e}^{x/2} \pm \mathrm{e}^{-x/2}\right) \cos(Mx) - \mathrm{i} \left(\mathrm{e}^{x/2} \mp \mathrm{e}^{-x/2}\right) \sin(Mx) \tag{4.97}$$

and so

$$\left|\theta_\mu(x) \pm \theta_\mu(x)^{-1}\right|^2 = \left(\mathrm{e}^{x/2} \pm \mathrm{e}^{-x/2}\right)^2 \cos(Mx)^2 + \left(\mathrm{e}^{x/2} \mp \mathrm{e}^{-x/2}\right)^2 \sin(Mx)^2 \tag{4.98}$$

| | $\mu$ | $\nu_\pm$ | $\Im\left(\mu_m - \nu_{\pm,m}\right) \xrightarrow{m\to\infty} 0$ |
|---|---|---|---|
| $K = +1$ | $\frac{m\pi}{c}$ | $\frac{1}{2}\left(\sqrt{1 \pm \frac{4\lambda_m}{\epsilon_\pm}} - 1\right)$ | $\Im\nu_{\pm,m} \xrightarrow{m\to\infty} 0$ |
| $K = -1$ | $-\frac{1}{2} + \mathrm{i}\frac{m\pi}{c}$ | $\frac{1}{2}\sqrt{1 \mp \frac{4\lambda_m}{\epsilon_\pm}}$ | $\Im\left(\mu_m - \nu_{\pm,m}\right) \xrightarrow{m\to\infty} 0$ |

Table 4.3: Values of parameters for positive and negative curvature. In the rightmost column is shown an implication of $\Im\left(\mu_n - \nu_{\pm,n}\right) \xrightarrow{n\to\infty} 0$ for the particular curvature.

which is strictly positive for $x > 0$. From (4.92) and (4.91), we have

$$\liminf_{m\to\infty} \left| \frac{\epsilon_+}{\epsilon_-} - \frac{\tilde{\theta}_\mu(a)}{\tilde{\theta}_\mu(b)} \right| = 0, \tag{4.99}$$

meaning that there exists a subsequence of $\frac{\tilde{\theta}_\mu(a)}{\tilde{\theta}_\mu(b)}$ converging to $\frac{\epsilon_+}{\epsilon_-}$. From the estimates

$$\begin{aligned} \left|\tilde{\theta}_\mu(a)\right| =& \left| \frac{\theta_\mu(a) - \theta_\mu(a)}{\theta_\mu(a) + \theta_\mu(-a)} \right| \\ &\geq \frac{\min_m |\theta_\mu(a) - \theta_\mu(-a)|}{\max_m |\theta_\mu(a) + \theta_\mu(-a)|} \\ &= \frac{e^{a/2} - e^{-a/2}}{e^{a/2} + e^{-a/2}} = \tanh(a/2), \end{aligned} \tag{4.100}$$

and

$$\begin{aligned} \left|\tilde{\theta}_\mu(a)\right| \leq& \frac{\max_m |\theta_\mu(a) - \theta_\mu(-a)|}{\min_m |\theta_\mu(a) + \theta_\mu(-a)|} \\ &= \frac{e^{a/2} + e^{-a/2}}{e^{a/2} - e^{-a/2}} = \frac{1}{\tanh(a/2)}, \end{aligned} \tag{4.101}$$

we obtain

$$\left| \frac{\tilde{\theta}_\mu(a)}{\tilde{\theta}_\mu(b)} \right| \in \left[ \tanh(a/2)\tanh(b/2), \frac{1}{\tanh(a/2)\tanh(b/2)} \right]. \tag{4.102}$$

which is an interval always containing 1. Although, when we rewrite

$$\begin{aligned} 0 =& \liminf_{m\to\infty} \left| \epsilon_+ \tilde{\theta}_\mu(b) - \epsilon_- \tilde{\theta}_\mu(a) \right| \\ =& \liminf_{m\to\infty} \left| \begin{aligned} &\epsilon_+ \left( \frac{\sinh(b)}{\cos(2Mb) + \cosh(b)} - \frac{\mathrm{i}\sin(2Mb)}{\cos(2Mb) + \cosh(b)} \right) \\ &- \epsilon_- \left( \frac{\sinh(a)}{\cos(2Ma) + \cosh(a)} - \frac{\mathrm{i}\sin(2Ma)}{\cos(2Ma) + \cosh(a)} \right) \end{aligned} \right| \end{aligned} \tag{4.103}$$

and in particular, for a choice of $a = k_1 c$, $b = k_2 c$, we obtain $Ma = k_1 m\pi$, $Mb = k_2 m\pi$ and hence

$$\begin{aligned} 0 =& \liminf_{m\to\infty} \left| \epsilon_+ \left( \frac{\sinh(b)}{1 + \cosh(b)} \right) - \epsilon_- \left( \frac{\sinh(a)}{1 + \cosh(a)} \right) \right| \\ =& \liminf_{m\to\infty} \left| \epsilon_+ \tanh\left(\frac{b}{2}\right) - \epsilon_- \tanh\left(\frac{a}{2}\right) \right| \end{aligned} \tag{4.104}$$

and as the expression is independent of $m$, we get the necessary conditions

$$\frac{\epsilon_+}{\epsilon_-} = \frac{\tanh(a/2)}{\tanh(b/2)} \tag{4.105}$$

which is not 1 for $a \neq b$. It is in direct contradiction with the result of singular sequences for $a \neq b$ as from Proposition 4.1 we have $\frac{\epsilon_+}{\epsilon_-} = 1 \implies \sigma_{\mathrm{ess}}(A) \neq \emptyset$ for $K = -1$. We conjecture that the limit equation in the case of $K = -1$ in this section is not correct, although we were not able to locate the erroneous step. This reasoning is based on the simplicity of proof of Proposition 4.1 and its numerical simulation confirming the correctness of the construction. Also, in the following chapter, we will see that the essential spectrum in a case of $\frac{\epsilon_+}{\epsilon_-} \neq 1$ does not depend on values of $a$, $b$, $c$.

# Chapter 5

# Form approach for non-critical contrast

In previous section, we proved that for constant (positive) curvature and contrast $\kappa \neq 1$, the essential spectrum $\sigma_{\text{ess}}(A) = \emptyset$ is empty, although the proof using the characteristic equation was rather tedious. Further, we could not obtain conclusive results for negative curvature.

Now, we would like to provide a more elegant and general self-adjoint operator definition via forms which will allow us to tackle also non-constant curvatures for non-critical contrast $\kappa \neq 1$. In the end, we will show that the two resulting operators from each approach coincide for constant curvature.

In this chapter, we will be using notation for Hilbert space $\mathcal{V}$ defined on $\Omega$ with $f \in \mathcal{V}$ as $f_\pm = f|_{\Omega\pm}$ and if $\mathcal{V} = L^2(\Omega)$, $||f_\pm|| := ||f_\pm||_{L^2(\Omega_\pm)}$ for $f \in L^2(\Omega)$. Scalar product $(\cdot,\cdot)_\mathcal{V}$ will be linear in second argument and that of $L^2(\Omega)$, unless specified otherwise.

The approach used here is similar to the $\mathbb{T}$-coercivity approach used in [8, 7] to provide a well-posedness for a similar problem and much more general domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ for $\kappa$ not belonging to a certain neighbourhood of 1. Choice of plausible contrasts depends only on the smoothness of the boundary between $\Omega_+$ and $\Omega_-$. For example, it was shown that when the curve $\Gamma$ contain a segment with right angle, then $\kappa \notin [\frac{1}{3}, 3]$ provide self-adjointness and compact resolvent. Combining $\mathbb{T}$-coercivity and arguments involving smooth partitions of unity, they derive criteria in terms of quantities ($\epsilon$) in the neighbourhood of the boundary.

Here, we apply a similar approach using cut-off function and a generalized Lax-Milgram theorem. Note that $\mathbb{T}$-coercivity is a weaker case of this generalized Lax-Milgram theorem of [3].

Now, we will state the representation theorems on which this chapter is based on. Recall that continuity of a sequilinear form $a$ means that for some $C > 0$

$$|a(u,v)| \leq C||u||_\mathcal{V}||v||_\mathcal{V}, \quad \forall u,v \in \mathcal{V}. \tag{5.1}$$

**Theorem 5.1** ([3, Thm. 2.1]). *Let $\mathcal{V}$ denote a Hilbert space. Let $a$ be a continuous sesquilinear form on $\mathcal{V} \times \mathcal{V}$. If $a$ satisfies, for some $\Phi_1, \Phi_2 \in \mathcal{L}(\mathcal{V})$,*

$$\begin{aligned} |a(u,u)| + |a(u,\Phi_1 u)| &\geq \alpha||u||^2_\mathcal{V}, \quad \forall u \in \mathcal{V}, \\ |a(u,u)| + |a(\Phi_2 u,u)| &\geq \alpha||u||^2_\mathcal{V}, \quad \forall u \in \mathcal{V}, \end{aligned} \tag{5.2}$$

*then* $\mathcal{A} \in \mathcal{L}(\mathcal{V})$ *defined via*

$$a(u, v) = (u, \mathcal{A}v)_{\mathcal{V}} \tag{5.3}$$

*is a continuous isomorphism from* $\mathcal{V}$ *onto* $\mathcal{V}$*. Moreover,* $\mathcal{A}^{-1}$ *is continuous.*

Now, consider two Hilbert spaces $\mathcal{V}$ and $\mathcal{H}$ such that

$$\begin{aligned} &\mathcal{V} \subset \mathcal{H}, \quad \mathcal{V} \text{ is dense in } \mathcal{H}, \\ &\forall u \in \mathcal{V}: \quad ||u||_{\mathcal{H}} \leq C||u||_{\mathcal{V}}. \end{aligned} \tag{5.4}$$

for some $C > 0$.

**Theorem 5.2** ([3, Thm. 2.2])**.** *Let* $a$ *be a continuous sesquilinear form satisfying* (5.2)*. Let* $\mathcal{H} \supset \mathcal{V}$ *be a Hilbert space and suppose that* (5.4) *holds for the Hilbert spaces* $\mathcal{V}$*,* $\mathcal{H}$*. Further assume that* $\Phi_1$*,* $\Phi_2$ *extend to a continuous linear map also in* $\mathcal{H}$*. Define operator* $S : \operatorname{dom} S \to \mathcal{H}$ *using*

$$\begin{aligned} &\operatorname{dom} S := \left\{ v \in \mathcal{V} : u \mapsto a(u, v) \text{ is continuous on } \mathcal{V} \text{ in the norm of } \mathcal{H} \right\}, \\ &a(u, v) =: (u, Sv)_{\mathcal{H}}, \quad \forall v \in \operatorname{dom} S, \ \forall u \in \mathcal{V}. \end{aligned} \tag{5.5}$$

*Then,*

1. $\operatorname{dom} S$ *is dense in both* $\mathcal{V}$ *and* $\mathcal{H}$*,*
2. $S$ *is closed,*
3. $S$ *is bijective from* $\operatorname{dom} S$ *onto* $\mathcal{H}$ *and* $S^{-1} \in \mathcal{L}(\mathcal{H})$*.*
4. *Let* $b$ *denote the conjugate sesquilinear form of* $a$ *given by*

$$(u, v) \mapsto b(u, v) := \overline{a(v, u)}.$$

*and denote* $\tilde{S}$ *the operator associated to* $b$ *by the same construction — then*

$$S^* = (\tilde{S})^* \text{ and } (\tilde{S})^* = S.$$

Start by defining Sobolev spaces of functions zero on the boundary of $\Omega$ restricted to $\Omega_\pm$ as

$$H^1_{0,\Gamma}(\Omega_\pm) := \left\{ f|_{\Omega_\pm} : f \in H^1_0(\Omega) \right\}. \tag{5.6}$$

Further define an even cut-off function $\xi : \Omega \to \mathbb{R}$, $\xi \in C_0^\infty(\Omega)$ for $x_0, x_1 \in \mathbb{R}$, $x_0 < x_1 < a$, $x_0 < x_1 < b$ such that $\xi(-x, y) = \xi(x, y)$ for all $(x, y) \in \Omega$ and, in addition, satisfying

$$\xi(x, y) = \begin{cases} 1, & x \in (-x_0, x_0), \\ 0, & x \in (-b, a) \setminus (-x_1, x_1). \end{cases} \tag{5.7}$$

In interval $(-x_1, -x_0) \cup (x_0, x_1)$ it is defined such that $\xi \in C_0^\infty(\Omega)$ and $||\xi||_{L^2(\Omega_\pm)} = 1$. Define mirroring operator

$$\begin{aligned} &P_\pm : H^1_{0,\Gamma}(\Omega_\pm) \to H^1_{0,\Gamma}(\Omega_\mp), \\ &(P_\pm u)(x, y) = u(-x, y) \text{ for } x \in (-x_1, x_1), \ y \in (0, c). \end{aligned} \tag{5.8}$$

In further analysis, we use it in conjunction with cut-off $\xi$ identically zero outside of $(-x_1, x_1)$. So formally, we rather use the operator $\xi P_\pm \equiv \xi P_\pm : H^1_0(\Omega_\pm) \to H^1_0(\Omega_\mp)$ which is properly defined on functions over the whole of $\Omega$.

Define transforms $T_\iota : H^1_0(\Omega) \to H^1_0(\Omega)$ for $u \in H^1_0(\Omega)$ using

$$T_1 u := \begin{cases} u_+ & \text{in } \Omega_+ \\ -u_- + 2R_+ u_+ & \text{in } \Omega_- \end{cases}, \qquad T_2 u := \begin{cases} -u_+ + 2R_- u_- & \text{in } \Omega_- \\ u_- & \text{in } \Omega_- \end{cases} \tag{5.9}$$

with $R_\pm : H^1_{0,\Gamma}(\Omega_\pm) \to H^1_{0,\Gamma}(\Omega_\mp)$, $R_\pm = \xi P_\pm$. Operators $T_\iota$ are bounded as operators acting in $H^1_0(\Omega)$ as

$$(T_\iota u)_+|_\Gamma = (T_\iota u)_-|_\Gamma, \quad \text{for } u = (u_+, u_-) \in H^1_{0,\Gamma}(\Omega_+) \oplus H^1_{0,\Gamma}(\Omega_-) \tag{5.10}$$

(in sense of traces) due to $(R_\pm u_\pm)|_\Gamma = u_\pm|_\Gamma$ for the same $u$.

For application of generalized Lax-Milgram lemma [3, Theorem 2.2], let us introduce the sesquilinear form associated to operator $\dot{A}$ defined in (3.16) given as

$$\begin{aligned} \dot{a} : L^2(\Omega) \times \operatorname{dom} \dot{A} \to L^2(\Omega), \\ \dot{a}(u, v) := (u, \dot{A} v). \end{aligned} \tag{5.11}$$

By invoking standard density arguments of test functions $C^\infty_0(\Omega)$ in $L^2(\Omega)$ and $\operatorname{dom} \dot{A}$ and definition of weak derivatives, its domain can be augmented and it is given, for $u$, $v \in H^1_0(\Omega)$, by

$$\begin{aligned} a : H^1_0(\Omega) \times H^1_0(\Omega) \to L^2(\Omega), \\ a(u, v) := (\nabla u, \epsilon \nabla v) = \epsilon_+ \int_{\Omega_+} \overline{\nabla u} \nabla v - \epsilon_- \int_{\Omega_-} \overline{\nabla u} \nabla v \end{aligned} \tag{5.12}$$

and the derivatives are understood in the weak sense.

**Proposition 5.3.** *Let $(\mathcal{M}, g)$ be a Riemannian manifold with constant Gaussian curvature and $\Omega$, $\Gamma$ as introduced in Section 3.1. Then, for contrast $\kappa = \frac{\epsilon_+}{\epsilon_-} \neq 1$, there is a unique self-adjoint operator $\mathcal{A}_F : \operatorname{dom} \mathcal{A}_F \subset L^2(\Omega, g) \to L^2(\Omega, g)$ associated to the sesquilinear form $a$ given in (5.12) by*

$$a(u, v) =: (u, \mathcal{A}_F v)_g, \quad u \in H^1_0(\Omega, g),\ v \in \operatorname{dom} \mathcal{A}_F \subset H^1_0(\Omega, g) \tag{5.13}$$

*with domain*

$$\operatorname{dom} \mathcal{A}_F := \left\{ v \in H^1_0(\Omega, g) : \Delta v_\pm \in L^2(\Omega_\pm, g),\ (\epsilon_+ \partial_x v_+ + \epsilon_- \partial_x v_-)\big|_\Gamma = 0 \right\} \tag{5.14}$$

*and $\mathcal{A}_F$ has a compact resolvent and $0 \notin \sigma(\mathcal{A}_F)$.*

*Proof.* In order to obtain a self-adjoint operator associated to $a(u, v)$, estimate

$$\begin{aligned} \left| a(u, T_1 u) \right| &= \left| \epsilon_+ \int_{\Omega_+} \overline{\nabla u_+} \nabla u_+ - \epsilon_- \int_{\Omega_-} \overline{\nabla u_-} \nabla \left( -u_- + 2\xi P u_+ \right) \right| \\ &= \left| \epsilon_+ ||\nabla u_+||^2 + \epsilon_- ||\nabla u_-||^2 - 2\epsilon_- \int_{\Omega_-} \overline{\nabla u_-} \left( \xi \nabla P u_+ + P u_+ \nabla \xi \right) \right| \\ &\geq \epsilon_- \left[ ||\nabla u_+||^2 \left( \frac{\epsilon_+}{\epsilon_-} - \frac{1}{\delta} \right) + ||\nabla u_-||^2 \left( 1 - \delta - \eta \right) - ||u_+||^2 \frac{||\nabla \xi||_{L^\infty(\Omega_+)}}{\eta} \right] \end{aligned} \tag{5.15}$$

where we used Young inequality for $\delta,\ \eta > 0$ and reversed triangle inequality $|x-y| \geq |x| - |y|$. Second equation was estimated using Cauchy-Schwarz inequality, integral substitution and properties of $\xi$

$$\begin{aligned}\left|\int_{\Omega_-} \overline{\nabla u_-}\,(\xi \nabla P u_+)\right| &\leq ||\nabla u_-||\,||(\xi\nabla P u_+)_-|| \leq ||\nabla u_-||\,||\nabla u_+||,\\ \left|\int_{\Omega_-} \overline{\nabla u_-}\,(P u_+ \nabla \xi)\right| &\leq ||\nabla u_-||\,||(P u_+\nabla \xi)_-|| \leq ||\nabla u_-||\,||u_+||\,||\nabla\xi||^2_{L^\infty(\Omega_+)}.\end{aligned} \tag{5.16}$$

In order to compensate for the last negative term without derivatives of $u$ in (5.15), define a complexified form $b_t : H_0^1(\Omega) \times H_0^1(\Omega) \to L^2(\Omega)$ for $t \in \mathbb{R}$, $t > 0$ as

$$b_t(u,v) := a(u,v) + \mathrm{i}t(u,v). \tag{5.17}$$

for $u, v \in H_0^1(\Omega)$. This sesquilinear form satisfies

$$\begin{aligned}|b_t(u,u)| &\geq t||u||^2,\\ |b_t(u,v)| &\geq |a(u,v)| - t||u||\,||v||\end{aligned} \tag{5.18}$$

and is bounded in $H_0^1(\Omega)$ norm with constant $C_t > 0$ due to Poincaré inequality according to

$$\begin{aligned}|b_t(u,v)| &\leq |a(u,v)| + t||u||\,||v||\\ &\leq \max\{\epsilon_+, \epsilon_-\}||\nabla u||\,||\nabla v|| + t||u||\,||v||\\ &\leq C_t ||u||_{H_0^1(\Omega)} ||v||_{H_0^1(\Omega)}.\end{aligned} \tag{5.19}$$

Combining estimates on $|a(u, T_1 u)|$ with boundedness of $T_1$

$$\begin{aligned}||u||\,||T_1 u|| &= ||u|| \sqrt{||u_+||^2 + \int_{\Omega_-} |-u_- + 2\xi P u_+|^2}\\ &\leq ||u||\,\sqrt{||u_+||^2 + 2(||u_-||^2 + 4||u_+||^2)} \leq ||u||\,3||u|| = 3||u||^2,\end{aligned} \tag{5.20}$$

we obtain for $\beta \in \mathbb{R}$, $\beta > 0$

$$\begin{aligned}|b_t(u,u)| + |b_t(u, \beta T_1 u)| &= |b_t(u,u)| + \beta |b_t(u, T_1 u)|\\ &\geq ||u_+||^2\left(t(1-3\beta) - C_\eta \beta\right) + t||u_-||^2\,(1-3\beta)\\ &\quad + \beta\epsilon_- ||\nabla u_+||^2 \left(\frac{\epsilon_+}{\epsilon_-} - \frac{1}{\delta}\right) + \beta \epsilon_- ||\nabla u_-||^2\,(1 - \delta - \eta)\\ &\geq \alpha ||u||_{H_0^1(\Omega)}\end{aligned} \tag{5.21}$$

where $C_\eta = \epsilon_- ||\nabla \xi||^2_{L^\infty(\Omega_+)} \eta^{-1}$. For a choice of $0 < \beta < \frac{1}{3}$, $0 < \delta < \frac{\epsilon_-}{\epsilon_+} < 1$, there exists $\eta > 0$ and $t > t_0$ sufficiently large such that each term on the right-hand side is strictly positive. From here, it is trivial to provide lower bound in terms of $H_0^1(\Omega)$ norm such that $\alpha > 0$ is strictly positive for $\frac{\epsilon_+}{\epsilon_-} > 1$. By the same computation with $b(u, T_2 u)$, we obtain similar strictly positive bounds for $\frac{\epsilon_+}{\epsilon_-} < 1$.

For $\frac{\epsilon_+}{\epsilon_-} > 1$, set $\mathcal{V} = H_0^1(\Omega)$, $\mathcal{H} = L^2(\Omega)$ and $\Phi_1 = \beta T_1$ in Theorem 5.2. Form $a$ is symmetric and thus the second inequality in (5.2) is also satisfied for $\Phi_2 = \beta T_1$. This implies that operator $\mathcal{B}_t$ associated to form $b_t$ defined on $S \subset L^2(\Omega)$ is a closed isomorphism from a dense subset of

$H_0^1(\Omega)$ to $L^2(\Omega)$. At the same time, $\mathcal{B}_t$ has a bounded inverse and due to compact embedding[1] of $H_0^1(\Omega)$ to $L^2(\Omega)$, its resolvent $R(\lambda, \mathcal{B}_t)$ is compact for $\lambda = 0$ and by the first resolvent identity also for all $\lambda$ in the resolvent set. Hence, the essential spectrum is empty [28, Theorem 6.29]. By setting $\Phi_{1,2} = \beta T_2$, we obtain the same results also for $\frac{\epsilon_+}{\epsilon_-} < 1$.

To determine domain of the operator — by Riesz theorem, stating that every continuous linear functional $u \mapsto \varphi(u)$ on $L^2(\Omega)$ is represented by some $\eta \in L^2(\Omega)$ such that $\varphi(u) = (\eta, u)$, we have

$$\operatorname{dom} \mathcal{B}_t = \left\{ v \in H_0^1(\Omega) : \exists \eta \in L^2(\Omega)\, \forall u \in H_0^1(\Omega), b_t(u,v) = (u, \eta) \right\}. \tag{5.22}$$

Based on definition of weak derivatives, it follows that for $u \in C_0^\infty(\Omega)$ and $v$ from

$$\begin{aligned} \operatorname{dom} \mathcal{B}_t &= \left\{ v \in H_0^1(\Omega) : \nabla \cdot (\epsilon \nabla v) \in L^2(\Omega) \right\} \\ &= \left\{ v \in H_0^1(\Omega) : \Delta v_\pm \in L^2(\Omega_\pm),\ (\epsilon_+ \partial_x v_+ + \epsilon_- \partial_x v_-)\big|_\Gamma = 0 \right\}, \end{aligned} \tag{5.23}$$

we have

$$\begin{aligned} b_t(u,v) = a(u,v) + \mathrm{i}t(u,v) &= \int_\Omega \overline{\nabla u} \epsilon \nabla v + \mathrm{i}t(u,v) \\ &= -\int \overline{u} \nabla \cdot (\epsilon \nabla v) + \mathrm{i}t(u,v) \\ &= (u, -\epsilon \Delta v + \mathrm{i}tv) = (u, \eta) \end{aligned} \tag{5.24}$$

where we have used the definition of weak derivative of $\epsilon \nabla v$ and piece-wise constant $\epsilon$. Now, $\Delta v$ is understood in the weak sense of distributions. From density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, $b_t(u,v) = (u, \eta)$ holds also for $u \in H_0^1(\Omega)$.

To extract information about operator without a complex shift, define for $t \in \mathbb{R}$, $t > 0$,

$$\mathcal{A}_F := \mathcal{B}_t - \mathrm{i}tI. \tag{5.25}$$

From (5.25) we have

$$\mathcal{A}_F^* := \mathcal{B}_t^* + \mathrm{i}tI \tag{5.26}$$

and also $\operatorname{dom} \mathcal{A}_F^* = \operatorname{dom} \mathcal{B}_t^* = \operatorname{dom} \mathcal{B}_t = \operatorname{dom} \mathcal{A}_F$ due to operator $\mathcal{B}_t^*$ being associated to $b_t^*(u,v) = \overline{b_t(v,u)} = a(u,v) - \mathrm{i}t(u,v)$ from symmetricity of form $a$. At the same time, for $u \in H_0^1(\Omega)$, $v \in \operatorname{dom} \mathcal{A}_F^*$,

$$\begin{aligned} (u, \mathcal{A}_F^* v) = (u, \left(\mathcal{B}_t^* + \mathrm{i}t\right) v) = b_t^*(u,v) + \mathrm{i}t(u,v) = \overline{a(v,u)} - \mathrm{i}t(u,v) + \mathrm{i}t(u,v) \\ = a(u,v) = (u, \mathcal{A}_F v) \end{aligned} \tag{5.27}$$

and hence, $\mathcal{A}_F^* = \mathcal{A}_F$ is self-adjoint and independent of $t$.

Considering non-zero curvature, introduce respective $L^2(\Omega, g)$ spaces on Riemannian manifolds with constant Gaussian curvature as before. Then all the estimates hold in the same form as above. This is due to metric $f(x)\,\mathrm{d}x\,\mathrm{d}y$ being mirror-symmetric with respect to $\Gamma$ regardless of the position of $\Gamma$ in $\Omega$. See Remark 5.4. ☐

**Remark 5.4** (Generalization to non-constant curvatures)**.** We can notice that in the proof of Proposition 5.3, the estimates (5.16) are the only place in the whole proof where non-constant

[1] This result is known as Rellich-Kondrakov theorem [23].

curvature could potentially be problematic. Note that Poincaré inequality also holds in $L^2$ spaces on Riemannian manifolds (see reference).

Denote $\mathrm{d}\nu_g = f\,\mathrm{d}x\,\mathrm{d}y$ where $f = \sqrt{|\det g|}$ and assume that $g$ is diagonal. Remember

$$|\nabla\phi|_g^2 := g^{ij}\overline{\nabla_i\phi}\nabla_j\phi. \tag{5.28}$$

In the previous estimates, we have used, for $u = (u_+, u_-) \in L^2(\Omega_+, g) \oplus L^2(\Omega_-, g)$

$$\begin{aligned}
||(\chi\nabla P u_\pm)_\mp||_g^2 &= \int_{\Omega_\mp} |\chi\nabla P u_\pm|^2\,\mathrm{d}\nu_g = \int_{\Omega_\mp} |\chi|^2 \left(g^{ij}\nabla_i P u_\pm \nabla_j P u_\pm\right) f\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\Omega_\pm} |P\chi|^2 \left(\left(P g^{ij}\right)\nabla_i u_\pm \nabla_j u_\pm\right) P f\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_\pm} |\chi|^2 \left(g^{ij}\nabla_i u_\pm \nabla_j u_\pm\right) f\,\mathrm{d}x\,\mathrm{d}y \\
&= ||(\chi\nabla u)_\pm||_g^2 \leq ||\nabla u_\pm||_g^2.
\end{aligned} \tag{5.29}$$

Thus, Proposition 5.3 can be generalized to Riemannian manifolds $(\mathcal{N}, \tilde{g})$ such that $g^{ij}(x,y) = g^{ij}(-x,y)$ in some neighbourhood of $\Gamma$.

Finally, we would like to show that the operator $\mathcal{A}_F$ coincides with $\dot{A}$ and its self-adjoint extension $A$ on an intersection of its domains for a non-critical contrast $\kappa \neq 1$.

**Proposition 5.5.** *The operator $\mathcal{A}_F$ defined in Proposition 5.3 via forms coincides, for $\kappa \neq 1$, with the self-adjoint operator $A \supset \dot{A}$ defined in Section 3.2.*

*Proof.* First, we will prove that $\dot{A} \subset \mathcal{A}_F$. As

$$\operatorname{dom}\dot{A}_K = \left\{ \psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \in H^2(\Omega_+, g) \oplus H^2(\Omega_-, g) \;\middle|\; \begin{array}{l} \psi_\pm\,|_{\partial\Omega_0} = 0, \\ \psi_+(0,\cdot) = \psi_-(0,\cdot) \\ \epsilon_+\,\partial_1\,\psi_+(0,\cdot) = -\epsilon_-\,\partial_1\,\psi_-'(0,\cdot) \end{array} \right\}, \tag{5.30}$$

we have that $\operatorname{dom}\dot{A} \subset \operatorname{dom}\mathcal{A}_F$. Based on the action of the operators, we have

$$(u, \dot{A}v) = a(u,v) = (u, \mathcal{A}_F v), \quad \forall u \in C_0^\infty(\Omega),\; v \in \operatorname{dom}\dot{A} \tag{5.31}$$

and by density of test functions then

$$\dot{A} \subset \mathcal{A}_F. \tag{5.32}$$

Whenever there are two densely-defined symmetric operators $L_1$, $L_2$ on a Hilbert space, then the following holds for their adjoints $L_1^*$, $L_2^*$,

$$L_1 \subset L_2 \implies L_2^* \subset L_1^*. \tag{5.33}$$

Hence, $A \subset \mathcal{A}_F$ as $\mathcal{A}_F^* \subset \dot{A}^*$ and

$$A = \overline{\dot{A}} = \dot{A}^{**} \subset \mathcal{A}_F^{**} = \mathcal{A}_F. \tag{5.34}$$

And finally, we obtain also the second extension

$$\mathcal{A}_F = \mathcal{A}_F^* \subset A^* = A. \tag{5.35}$$

□

**Note 5.6** (Cut-off motivation)**.** We can see that if we would have used no cut-off in (5.9), i.e. $\xi \equiv 1$ constant, the last term in (5.15) with $||u_+||$ would be zero and $\mathbb{T}$-coercivity of form $a$ would be achieved rather quickly. Although first, we have to properly define mirroring operator $P$ globally. It turns out that in simpler framework, we cannot recover proper definition of $\mathcal{A}_F$ via forms for the whole range of contrasts $\kappa \in \mathbb{R}$, $\kappa \neq 1$.

Denote $a_* := \min\{a, b\}$. For $T_1$ and $T_2$, use $R_\pm = P_\pm$, $P_+ : H^1_{0,\Gamma}(\Omega_+) \to H^1_{0,\Gamma}(\Omega_-)$ and $P_- : H^1_{0,\Gamma}(\Omega_-) \to H^1_{0,\Gamma}(\Omega_+)$, respectively, defined by

$$(P_\pm u_\pm)(x, y) := \begin{cases} u_\pm(-x, y) & \text{for } x \in (-a_*, a_*), \\ 0 & \text{for } x \notin (-a_*, a_*). \end{cases} \tag{5.36}$$

In order for $P_\pm$ to be also an operator $H^1_{0,\Gamma}(\Omega_\pm) \to H^1_{0,\Gamma}(\Omega_\mp)$, we require that $b \geq a$, or $a \geq b$, respectively (plus and minus). This leads to

$$\begin{aligned} b \geq a &\implies \text{form } a(u, T_1 u) \text{ is coercive for } \kappa > 1, \\ a \geq b &\implies \text{form } a(u, T_2 u) \text{ is coercive for } \kappa < 1. \end{aligned} \tag{5.37}$$

For simplicity, assume $b \geq a$. By using $T_1$ as above, this gives us coercivity for $\kappa > 1$. By defining $T_2$ using different choice of operator $R_- = P$ as

$$(Pu_-)(x, y) = u_-\left(-\frac{b}{a}x, y\right), \tag{5.38}$$

we arrive at $|a(u, u)| + |a(u, T_2 u)| \geq \alpha ||u||_{H^1(\Omega)}$ for $\alpha > 0$, all $u \in H^1_0(\Omega)$ and for $\kappa < \frac{a}{b}$. Overall, combining reflections in $T_2$ and reflections with rescaling in $T_1$, we obtain self-adjoint representations with compact resolvent via form $a(u, v)$ for $\kappa \in (0, \frac{a}{b}) \cup (1, \infty)$. The approach with cut-off functions ensures the same properties also for $\kappa \in (\frac{a}{b}, 1)$.

**Note 5.7.** Let us demonstrate, for curiosity, approach without the cut-off function $\xi$. Now, $T_\iota$ are given using $R_\pm = P_\pm$ as in (5.36). In order for image of $R_\pm$ to be zero at the boundary of $\Omega$, we must have $a \leq b$, or $b \leq a$, respectively. Estimate $|h(\psi, T_1\psi)|$ using

$$\begin{aligned}
\left|h(\psi, \mathbb{T}\psi)\right|^2 &= \left|\epsilon_+\|\nabla\psi\|_+^2 + \epsilon_-\|\nabla\psi\|_-^2 - 2\epsilon_- \int_{\Omega_-} \overline{\nabla\psi_-}\nabla\psi_+^-\right|^2 \\
&= \left(\epsilon_+\|\nabla\psi\|_+^2 + \epsilon_-\|\nabla\psi\|_-^2\right)^2 - 2\left(\epsilon_+\|\nabla\psi\|_+^2 + \epsilon_-\|\nabla\psi\|_-^2\right) 2\epsilon_- \Re\left(\int_{\Omega_-} \overline{\nabla\psi_-}\nabla\psi_+^-\right) \\
&\quad + 4\epsilon_-^2 \left|\int_{\Omega_-} \overline{\nabla\psi_-}\nabla\psi_+^-\right| \\
&\geq \left(\epsilon_+\|\nabla\psi\|_+^2 + \epsilon_-\|\nabla\psi\|_-^2\right)^2 (1-\delta) + 4\epsilon_-^2 \left|\int_{\Omega_-} \overline{\nabla\psi_-}\nabla\psi_+^-\right|^2 \left(1 - \frac{1}{\delta}\right) && (5.39a) \\
&\geq \left(\epsilon_+\|\nabla\psi\|_+^2 + \epsilon_-\|\nabla\psi\|_-^2\right)^2 (1-\delta) - 4\epsilon_-^2 \|\nabla\psi\|_-^2 \|\nabla\psi\|_+^2 \left(\frac{1}{\delta} - 1\right) && (5.39b) \\
&= \left(\epsilon_+^2\|\nabla\psi\|_+^4 + \epsilon_-^2\|\nabla\psi\|_-^4\right)^2 (1-\delta) \\
&\quad + \|\nabla\psi\|_-^2 \|\nabla\psi\|_+^2 \left[2\epsilon_+\epsilon_- (1-\delta) + 4\epsilon_-^2 \left(\frac{1}{\delta} - 1\right)\right] \\
&\geq \|\nabla\psi\|_+^4 \left(\epsilon_+^2 (1-\delta) - \eta\right) + \|\nabla\psi\|_-^4 \left(\epsilon_-^2 (1-\delta) - \eta\right) && (5.39c)
\end{aligned}$$

where we used Young inequality with parameter $\delta > 0$ in equation (5.39a), Cauchy-Schwarz inequality in (5.39b) and chose $\delta \in (0,1)$ so that $\frac{1}{\delta} - 1 > 0$. At the same time, we used a simple integral substitution $\left|\int_{\Omega_-} \overline{\nabla\psi_-} \nabla\psi_+^-\right| \leq \|\nabla\psi\|_-^2 \int_{\Omega_-} \left|\nabla\psi_+^-\right|^2 \leq \|\nabla\psi\|_-^2 \int_{\Omega_+} |\nabla\psi_+|^2 = \|\nabla\psi\|_-^2 \|\nabla\psi\|_+^2$ as when $a \geq b$, integral of non-negative function over a superset is larger than the original integral and when $a < b$, $\psi_+^-(x) = 0$ for $x \in (a,b)$. In (5.39c), we used a simple Young inequality to get the expression with notation

$$\eta = 2\epsilon_-^2 \left(\frac{1}{\delta} - 1\right) - \epsilon_+\epsilon_- (1-\delta)\,. \tag{5.40}$$

We proceed to show that parameter $\delta \in (0,1)$ can be chosen in such a way that both coefficients in (5.39c) in front of $\|\nabla\psi\|_\pm^4$ are positive. The coefficients will be denoted as $\mathcal{K}_\pm$, respectively. Indeed, denote contrast $\kappa := \frac{\epsilon_+}{\epsilon_-}$ and

$$\begin{aligned} \mathcal{K}_+ &= \epsilon_+^2 (1-\delta) - \eta = 2\epsilon_+\epsilon_- (1-\delta)\left(\frac{\kappa+1}{2} - \frac{1}{\kappa\delta}\right), \\ \mathcal{K}_- &= \epsilon_-^2 (1-\delta) - \eta = 2\epsilon_-^2 (1-\delta)\left(\frac{\kappa+1}{2} - \frac{1}{\delta}\right), \end{aligned} \tag{5.41}$$

then $\mathcal{K}_\pm > 0$ for $\delta$ fixed if $\delta > \max\left\{\frac{2}{\kappa(\kappa+1)}, \frac{2}{\kappa+1}\right\}$. Such suitable $\delta \in (0,1)$ can be chosen if $\kappa > 1$ (or $\epsilon_+ > \epsilon_-$). Hence, for such $\delta$ and $\mathcal{K}_\pm > 0$:

$$\begin{aligned} \left|h(\psi, \mathbb{T}\psi)\right| &\geq \sqrt{\mathcal{K}_+\|\nabla\psi\|_+^4 + \mathcal{K}_+\|\nabla\psi\|_+^4} \geq \frac{1}{\sqrt{2}}\left(\sqrt{\mathcal{K}_+}\|\nabla\psi\|_+^2 + \sqrt{\mathcal{K}_-}\|\nabla\psi\|_-^2\right) \\ &\geq \frac{\min\left\{\sqrt{\mathcal{K}_+}, \sqrt{\mathcal{K}_-}\right\}}{\sqrt{2}}\|\nabla\psi\|_\Omega^2 \end{aligned} \tag{5.42}$$

Together with Poincaré inequality for $\psi \in H_0^1(\Omega)$, this gives us the desired result.

# Conclusion

In this thesis, we have explored essential spectrum of a variation of an indefinite Laplacian [6] on a non-smooth domain on constantly-curved surfaces. Note that non-smooth domains are not extensively covered in literature in general. We have not found any effect of the curvature on the essential spectrum.

**Corollary 5.8.** *Let $K$ be a constant Gauss curvature and $A_K$ be operator defined in Section 3.2. Then the following holds for contrast $\kappa = \frac{\epsilon_+}{\epsilon_-}$ :*

1. $\kappa \neq 1 \iff \sigma_{\text{ess}}(A_K) = \varnothing$,
2. $\kappa = 1 \iff 0 \in \sigma_{\text{ess}}(A_K)$.

*Proof.* Corollary is a direct consequence of Proposition 5.3, Proposition 5.5 and Proposition 4.1. □

In the section on singular sequences, we construct Weyl sequences to show presence of zero in the essential spectrum for critical contrast and propose possible generalization to non-constant curvatures. We also refine the argument in case of zero curvature to show that zero is the only point of the essential spectrum and is empty when the contrast is non-critical. We also provide rate of convergence for $\lambda_m \to 0$.

The asymptotic analysis of the characteristic equation for eigenvalues in the curved case is much more involved due to presence of associated Legendre functions. We were able to obtain conclusive results only for positive curvature.

For positive curvature, finite limit points of the spectrum can exist only when $\kappa = 1$, with possible exception of $\omega_\pm \neq 0$, in notation of Lemma 4.11. Although, in the last section we rule out the possibility for $\kappa \neq 1$ with Proposition 5.3, thus $\omega_\pm$ is not a limit point when $\kappa \neq 1$.

Remark 4.17 presents a contradiction with independent construction of singular sequences for negative curvature — we are lead to assume that the results Section 4.3 contains an error in the case of negative curvature, although we were not able to locate the source of the error, despite much effort.

The last chapter provides a rather general and elegant approach to the essential spectrum and operator definition using forms. We were able to prove emptiness of essential spectrum in non-critical contrast case regardless of curvature and also sketch possible generalizations regarding non-constant curvature.

Overall, we presented many new results for non-smooth rectangular domain concerning non-zero curvature and sketched possible generalizations of the used methods to describe also the case of non-constant curvatures and/or different geometries of the domain.

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