Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Mathematics

**Habilitation Thesis**

Zdeněk Mihula

# Reduction Operators, Weighted Inequalities, and Noncompact Sobolev Embeddings

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# Contents

# Chapter 1

# Introduction

Quality of a function can be captured by the fact that the function belongs to a certain function space—a collection of functions sharing some common quality. A function space also often measures the size of its elements by some means. Mappings that take a function from a function space (say $X$) and transform it into another one are usually called operators. Zooming out from individual functions, an operator maps functions from an input function space into some output function space (say $Y$).

Depending on the task at hand, various properties of operators are studied. Two of the most fundamental ones are their boundedness and compactness. Very loosely speaking, boundedness of an operator ensures that the size of outputs in $Y$ is controlled by the size of inputs in $X$. Compactness of operators is a more subtle property; even more loosely speaking, it ensures that from an arbitrary (infinite) collection of inputs with bounded size in $X$, only an "almost finite" number of transformed outputs in $Y$ is significant.

There are an enormous number of various function spaces. A prominent example is Sobolev spaces, which were introduced by Sobolev [102] (cf. [103]) as a tool for studying partial differential equations and their solutions in a systematic way. Partial differential equations and their applications play an important role not only in mathematical physics and mathematical modeling but also in other parts of mathematics, both pure and applied ([14, 43, 97, 76]). So-called Sobolev embeddings and inequalities, the most classical one established by Sobolev and independently by Gagliardo in [42] and by Nirenberg in [84], allow us to deduce some new information about quality of a function from quality of its derivatives. The quality being measured here is integrability of functions and their derivatives. The input function space is a Sobolev space, which measures integrability of derivatives of a function, and the output function space is another function space measuring integrability of the function itself.

An important question in the theory of Sobolev spaces and its applications is the compactness of Sobolev embeddings ([13, 39, 76]). Compactness (in general, not necessarily of Sobolev embeddings) can be studied either from qualitative or quantitative point of views. Qualitative one is whether something is or is not compact. Quantitative one is that compactness or its lack is measured by some quantities. An advantage of the quantitative point of view is that it brings more information that can be exploited further (e.g., [30, 32, 33, 36]). There are various ways of measuring (non)compactness. One is the measure of noncompactness introduced by Kuratowski in [65], and the closely related entropy numbers ([16, 33, 36]). Another possibility is to use various so-called $s$-numbers. An axiomatic approach to $s$-numbers was introduced by Pietsch in [91].

Another prominent class of function spaces is that of rearrangement-invariant function spaces. This class encompasses a large number of function spaces measuring integrability of functions in such a way that the size of functions in these function spaces is invariant with respect to certain rearrangements ([6, 7, 104]). Prototypical examples are widely known Lebesgue spaces—although they are named after H. Lebesgue, they were probably introduced by F. Riesz in [96]—but it became quickly apparent that more sophisticated function spaces were needed (some history can be found, e.g., in [7, 90, 95]). Notable examples of such more sophisticated function spaces are $\Lambda$ spaces, introduced by Lorentz in [74, 75], or Orlicz spaces.

### Thesis outline

This habilitation thesis consists of an introductory text and a collection of six selected research publications. These publications mainly deal with two topics. The first one concerns boundedness of operators between function spaces, and it is divided into two subtopics. The first one concerns boundedness of operators in the class of rearrangement-invariant function spaces and optimality. The second subtopic deals with weighted inequalities. The second topic concerns Sobolev spaces, (non)compactness of Sobolev embeddings and its quantitative aspects.

The thesis is structured as follows.

In Chapter 2, some terminology, notation, and aspects of the theory of rearrangement-invariant function spaces used in the subsequent chapters are introduced and recalled.

Chapter 3 summarizes main results from Papers A and B. Paper A comprehensively studies optimal boundedness properties of quite general Hardy-type operators between rearrangement-invariant function spaces. These operators surprisingly often serve as certain "reduction operators" in the sense that various complicated questions in the framework of rearrangement-invariant function spaces can be reduced to boundedness of such operators. In Paper B, we found reduction operators (one of them being of Hardy type) whose boundedness between a pair of rearrangement-invariant function spaces ensures that every operator satisfying some endpoint estimates, motivated by Gaussian Sobolev inequalities, is also bounded between them.

Chapter 4 summarizes main results from Papers C and D. These papers characterize optimal constants in certain weighted inequalities for integral operators acting on functions of one variable. We obtained a complete explicit characterization of the optimal constant while eliminating some important restrictions from the previous work.

Chapter 5 summarizes main results from Papers E and F. We established new quantitative results for the quality of (non)compactness of three Sobolev embeddings, which are well known to be noncompact—namely for the optimal subcritical Sobolev embedding in Lebesgue spaces, the optimal subcritical Sobolev-Lorentz embedding, and for a certain Sobolev embedding on an infinite strip.

Appendix A contains the selected publications attached to the thesis.

# Chapter 2

# Preliminaries

In this chapter, we will fix some notations and recall some parts of the theory of rearrangement-invariant function spaces, which will be frequently used later. More specific notations and definitions will be introduced where needed.

If not stated otherwise, $(R, \mu)$ is a $\sigma$-finite non-atomic measure space in this thesis. We set

$$\begin{aligned}\mathfrak{M}(R,\mu) &= \{f\colon f \text{ is a } \mu\text{-measurable function on } R \text{ with values in } [-\infty,\infty]\},\\ \mathfrak{M}_0(R,\mu) &= \{f \in \mathfrak{M}(R,\mu)\colon f \text{ is finite } \mu\text{-a.e. in } R\},\end{aligned}$$

and

$$\mathfrak{M}^+(R,\mu) = \{f \in \mathfrak{M}(R,\mu)\colon f \geq 0 \ \mu\text{-a.e. on } R\}.$$

When $R$ is a subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, and $\mu$ is the Lebesgue measure on $R$, we write $R$ instead of $(R, \mu)$, and we will do the same in the notation introduced later.

We will write $A \lesssim B$, where $A$ and $B$ are nonnegative expressions (typically depending on some parameters) if there is a positive constant $c$ independent of all relevant parameters appearing in the expressions such that $A \leq c \cdot B$. When it is not obvious from the context what the relevant parameters are, it will be specified. We will write $A \approx B$ when $A \lesssim B$ and $B \lesssim A$ simultaneously.

The expressions $\frac{1}{\infty}$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, and $0 \cdot \infty$ are to be interpreted as 0. Furthermore, the expression $\frac{1}{0}$ is to be interpreted as $\infty$.

## 2.1 Rearrangements and rearrangement-invariant function spaces

We briefly recall some parts of the theory of rearrangement-invariant function spaces. The interested reader is referred to [7] for more information.

The *nonincreasing rearrangement* $f^*\colon (0,\infty) \to [0,\infty]$ of a function $f \in \mathfrak{M}(R,\mu)$ is defined as

$$f^*(t) = \inf\{\lambda \in (0,\infty) : \mu(\{x \in R\colon |f(x)| > \lambda\}) \leq t\},\ t \in (0,\infty).$$

The nonincreasing rearrangement $f^*$ is a nonincreasing right-continuous function such that $f^*(t) = 0$ for every $t \in [\mu(R),\infty)$.

We say that two functions $f \in M(R,\mu)$ and $g \in M(S,\nu)$, where $(S,\nu)$ is a (possibly different) measure space, are *equimeasurable*, if $f^* = g^*$. This is equivalent to the fact that $\mu(\{x \in R : |f(x)| > \lambda\}) = \mu(\{x \in S : |g(x)| > \lambda\})$ for every $\lambda \in (0,\infty)$. In particular, the functions $f$ and $f^*$ are equimeasurable.

The *maximal nonincreasing rearrangement* $f^{**}\colon (0,\infty) \to [0,\infty]$ of a function $f \in \mathfrak{M}(R,\mu)$ is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s,\ t \in (0,\infty).$$

The maximal nonincreasing rearrangement $f^{**}$ is a nonincreasing continuous function such that $f^* \le f^{**}$.

A functional $\|\cdot\|_{X(R,\mu)}\colon \mathfrak{M}^+(R,\mu) \to [0,\infty]$ is called a *rearrangement-invariant function norm* if, for all $f$, $g$ and $\{f_k\}_{k=1}^\infty$ in $\mathfrak{M}^+(R,\mu)$, and every $\alpha \in [0,\infty)$:

(P1) $\|f\|_{X(R,\mu)} = 0$ if and only if $f = 0$ $\mu$-a.e. in $R$; $\|\alpha f\|_{X(R,\mu)} = \alpha\|f\|_{X(R,\mu)}$; $\|f+g\|_{X(R,\mu)} \le \|f\|_{X(R,\mu)} + \|g\|_{X(R,\mu)}$;

(P2) $\|f\|_{X(R,\mu)} \le \|g\|_{X(R,\mu)}$ if $f \le g$ $\mu$-a.e. in $R$;

(P3) $\|f_k\|_{X(R,\mu)} \nearrow \|f\|_{X(R,\mu)}$ if $f_k \nearrow f$ $\mu$-a.e. in $R$;

(P4) $\|\chi_E\|_{X(R,\mu)} < \infty$ for every $\mu$-measurable $E \subseteq R$ of finite measure;

(P5) for every $\mu$-measurable $E \subseteq R$ of finite measure, there is a constant $C_{E,X} \in (0,\infty)$, possibly depending on $E$ and $\|\cdot\|_{X(R,\mu)}$ but not on $f$, such that $\int_E f(x)\,\mathrm{d}x \le C_{E,X}\|f\|_{X(R,\mu)}$;

(P6) $\|f\|_{X(R,\mu)} = \|g\|_{X(R,\mu)}$ if $f$ and $g$ are equimeasurable.

Extending a rearrangement-invariant function norm $\|\cdot\|_{X(R,\mu)}$ to all functions $f \in \mathfrak{M}(R,\mu)$ by

$$\|f\|_{X(R,\mu)} = \|\,|f|\,\|_{X(R,\mu)},\ f \in \mathfrak{M}(R,\mu),$$

the *rearrangement-invariant function space* $X(R,\mu)$ is defined as

$$X(R,\mu) = \{f \in \mathfrak{M}(R,\mu) : \|f\|_{X(R,\mu)} < \infty\}.$$

The functional $\|\cdot\|_{X(R,\mu)}$ is a norm on $X(R,\mu)$, and $X(R,\mu)$ endowed with it is a Banach space. A function $f \in \mathfrak{M}(R,\mu)$ belongs to $X(R,\mu)$ if and only if $\|f\|_{X(R,\mu)} < \infty$. Every rearrangement-invariant function space $X(R,\mu)$ is contained in $\mathfrak{M}_0(R,\mu)$. In fact, it is contained in $(L^1 + L^\infty)(R,\mu)$.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(R,\mu)}$, the functional $\|\cdot\|_{X'(R,\mu)}$ defined as

$$\|f\|_{X'(R,\mu)} = \sup_{\substack{g \in \mathfrak{M}^+(R,\mu)\\ \|g\|_{X(R,\mu)} \le 1}} \int_R f(x)g(x)\,\mathrm{d}\mu(x),\ f \in \mathfrak{M}^+(R,\mu),$$

is called the *associate function norm*. The associate function norm is also a rearrangement-invariant function norm, and the corresponding rearrangement-invariant function space $X'(R,\mu)$ is called the *associate function space* (of $X(R,\mu)$). The associate function space $X'(R,\mu)$ is (isometrically isomorphic to) a closed norm-fundamental subspace of the dual space of $X(R,\mu)$, but it may be smaller in general (e.g., the associate function space of $L^\infty(R,\mu)$ is $L^1(R,\mu)$).

Rearrangement-invariant function spaces are uniquely determined by their associate function spaces because we always have $(X')'(R,\mu) = X(R,\mu)$. This means that, given a rearrangement-invariant function space $X(R,\mu)$, there is a unique rearrangement-invariant function space $Y(R,\mu)$ whose associate function space is $X(R,\mu)$—namely $Y(R,\mu) = X'(R,\mu)$.

We say that a rearrangement-invariant function space $X(R,\mu)$ is *embedded in* a rearrangement-invariant function space $Y(R,\mu)$, and we write

$$X(R,\mu) \hookrightarrow Y(R,\mu),$$

if there is a constant $C \in (0,\infty)$ such that $\|f\|_{Y(R,\mu)} \leq C\|f\|_{X(R,\mu)}$ for every $f \in \mathfrak{M}(R,\mu)$. In fact, inclusion between rearrangement-invariant function spaces is always continuous because $X(R,\mu) \hookrightarrow Y(R,\mu)$ if and only if $X(R,\mu) \subseteq Y(R,\mu)$.

For every rearrangement-invariant function space $X(R,\mu)$, there is a unique rearrangement-invariant function space $X(0,\mu(R))$ such that $\|f\|_{X(R,\mu)} = \|f^*\|_{X(0,\mu(R))}$. It follows that rearrangement-invariant function spaces over $(R,\mu)$ are completely determined by rearrangement-invariant function spaces over $(0,\mu(R))$. The rearrangement-invariant function space $X(0,\mu(R))$ is called the *representation space* (of $X(R,\mu)$). In particular, when $R = (0,L)$ for $L \in (0,\infty]$ and $\mu$ is the Lebesgue measure, then $X(R,\mu)$ coincides with its representation space.

It is possible to equip *sums and intersections of rearrangement-invariant function spaces* with rearrangement-invariant function norms; hence they are also rearrangement-invariant function spaces. When $\|\cdot\|_{X(R,\mu)}$ and $\|\cdot\|_{Y(R,\mu)}$ are rearrangement-invariant function norms, then so are $\|\cdot\|_{X(R,\mu)\cap Y(R,\mu)}$ and $\|\cdot\|_{(X+Y)(R,\mu)}$ defined as

$$\|f\|_{X(R,\mu)\cap Y(R,\mu)} = \max\{\|f\|_{X(R,\mu)}, \|f\|_{Y(R,\mu)}\},\ f \in \mathfrak{M}^+(R,\mu),$$

and

$$\|f\|_{(X+Y)(R,\mu)} = \inf_{f=g+h}(\|g\|_{X(R,\mu)} + \|h\|_{Y(R,\mu)}),\ f \in \mathfrak{M}^+(R,\mu),$$

where the infimum extends over all possible decompositions $f = g + h$, $g, h \in \mathfrak{M}^+(R,\mu)$. Furthermore, we have (e.g., [28, Lemma 1.12])

$$(X(R,\mu) \cap Y(R,\mu))' = (X' + Y')(R,\mu)$$

and

$$(X + Y)'(R,\mu) = X'(R,\mu) \cap Y'(R,\mu),$$

with equality of norms.

## 2.2 Examples of rearrangement-invariant function spaces

Textbook examples of rearrangement-invariant function spaces are the well-known *Lebesgue spaces* $L^p(R,\mu)$, $p \in [1,\infty]$, defined as

$$L^p(R,\mu) = \{f \in \mathfrak{M}(R,\mu) : \|f\|_{L^p(R,\mu)} < \infty\},$$

where

$$\|f\|_{L^p(R,\mu)} = \begin{cases} \big(\int_R |f(x)|^p \,\mathrm{d}\mu(x)\big)^{\frac{1}{p}} & \text{if } p < \infty, \\ \operatorname{ess\,sup}_{x\in R} |f(x)| & \text{if } p = \infty. \end{cases}$$

When $p < \infty$, the fact that the norm $\|\cdot\|_{L^p(R,\mu)}$ is rearrangement invariant follows from the so-called layer cake representation formula (e.g., [73, Theorem 1.13]), which tells us that

$$\int_R |f(x)|^p \,\mathrm{d}\mu(x) = p\int_0^\infty t^{p-1}\mu(\{x \in R : |f(x)| > t\}) \,\mathrm{d}t = \int_0^{\mu(R)} f^*(t)^p \,\mathrm{d}t.$$

When $p = \infty$, it follows from $\|f\|_{L^\infty(R,\mu)} = \lim_{t\to 0^+} f^*(t)$.

Other important examples are the *Lorentz spaces* $L^{p,q}(R,\mu)$ and *Orlicz spaces* $L^A(R,\mu)$.

For $p, q \in [1,\infty]$, the Lorentz space $L^{p,q}(R,\mu)$ is defined as

$$L^{p,q}(R,\mu) = \{f \in \mathfrak{M}(R,\mu) : \|f\|_{L^{p,q}(R,\mu)} < \infty\},$$

where

$$\|f\|_{L^{p,q}(R,\mu)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\mu(R))}.$$

We have $L^p(R,\mu) = L^{p,p}(R,\mu)$ for every $p \in [1,\infty]$. The functional $\|\cdot\|_{L^{p,q}(R,\mu)}$ is a rearrangement-invariant function norm only when $1 \le q \le p < \infty$ or $p = q = \infty$. However, it is still at least equivalent to a rearrangement-invariant function norm when $1 < p < q \le \infty$—namely to the rearrangement-invariant function norm $\|\cdot\|_{L^{(p,q)}(R,\mu)}$ defined as $\|f\|_{L^{(p,q)}(R,\mu)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t)\|_{L^q(0,\mu(R))}$. Thanks to this, it is usually possible to regard the Lorentz spaces $L^{p,q}(R,\mu)$ as rearrangement-invariant function spaces even when $1 < p < q \le \infty$, as long as exact values of constants are not needed. The Lorentz spaces are increasing in the second parameter in the sense that $L^{p,q_1}(R,\mu) \subsetneq L^{p,q_2}(R,\mu)$ for every $p \in [1,\infty)$ and $1 \le q_1 < q_2 \le \infty$. The Lorentz space $L^{p,\infty}(R,\mu)$ is sometimes called a *weak Lebesgue space*. The interested reader is referred to, e.g., [7, 54, 90] for more information.

Given a Young function $A\colon [0,\infty) \to [0,\infty]$—that is, it is convex, left-continuous, nonconstant, and $A(0) = 0$—the Orlicz space $L^A(R,\mu)$ is defined as

$$L^A(R,\mu) = \{f \in \mathfrak{M}(R,\mu) : \|f\|_{L^A(R,\mu)} < \infty\},$$

where

$$\|f\|_{L^A(R,\mu)} = \inf\left\{\lambda > 0 : \int_R A\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\mu(x) \le 1\right\}.$$

The norm $\|\cdot\|_{L^A(R,\mu)}$ as defined here is usually called the Luxemburg norm. For $p \in [1,\infty)$, the Lebesgue space $L^p(R,\mu)$ is equal to the Orlicz space $L^A(R,\mu)$ defined by the Young function $A(t) = t^p$, $t \in [0, \infty)$. The space $L^\infty(R, \mu)$ coincides with $L^A(R, \mu)$ with $A$ equal to 0 on $[0, 1]$ and to $\infty$ on $(1, \infty)$. Up to equivalence of norms, Orlicz spaces $L^A(R, \mu)$ are determined by the behavior of $A$ near $\infty$ when $\mu(R) < \infty$. When $\mu(R) = \infty$, the behavior of $A$ near 0 is also important. The interested reader can find more information about Orlicz spaces in, e.g., [60, 95].

Some widely used Orlicz spaces, other than Lebesgue spaces, are the following Orlicz spaces of logarithmic and exponential types. Let $\mu(R) < \infty$. The Orlicz space $L^A(R, \mu)$ defined by a Young function $A$ that is equivalent to the function $t^p (\log t)^\alpha$ near $\infty$, where either $p \in (1, \infty)$ and $\alpha \in \mathbb{R}$ or $p = 1$ and $\alpha \geq 0$, is often denoted by $L^p(\log L)^\alpha(R, \mu)$. These spaces satisfy $L^p(\log L)^\alpha(R, \mu) \subsetneq L^p(R, \mu) \subsetneq L^p(\log L)^\beta(R, \mu)$ for every $p \in (1, \infty)$ and $\beta < 0 < \alpha$ (the first inclusion is valid even when $p = 1$). The Orlicz space often denoted by $\exp L^\alpha(R, \mu)$ corresponds to a Young function $A$ equivalent to the function $\exp(t^\alpha)$ near $\infty$, where $\alpha > 0$. These spaces satisfy $L^\infty(R, \mu) \subsetneq \exp L^\alpha(R, \mu) \subsetneq L^q(R, \mu)$ for every $q \in [1, \infty)$. These examples can be generalized to cover more tiers of logarithms/exponentials, or to the case $\mu(R) = \infty$.

There are other important rearrangement-invariant function spaces and classes of function spaces closely related to them. We will meet some later (e.g., Lorentz-Karamata spaces in Chapter 3 or $\Lambda$ and $\Gamma$ spaces in Chapter 4).

# Chapter 3

# Reduction operators

This chapter describes some of the main results from the papers [5, 81]. The common theme of this chapter is the question of reducing complicated problems to simpler ones. The first section, which describes [81], is devoted to the study of optimal behavior of certain operators, usually referred to as (weighted) Hardy operators, on rearrangement-invariant function spaces. The operators studied there are often found to be at the core of such reductions in the setting of rearrangement-invariant function spaces. The second section, which describes [5], deals with one particular reduction—that of reducing the question of whether every operator having certain endpoint behavior is bounded between a given pair of rearrangement-invariant function spaces to the question of whether particular governing operators are bounded.

## 3.1 Weighted Hardy operators on rearrangement-invariant function spaces

For the rest of this section, let $L \in (0, \infty]$. Let $R_{u,v,\nu}$ and $H_{u,v,\nu}$ be Hardy-type operators (formally) defined as, for $f \in \mathfrak{M}(0, L)$,

$$R_{u,v,\nu}f(t) = v(t) \int_0^{\nu(t)} f(s)u(s)\,\mathrm{d}s,\ t \in (0, L),$$

and

$$H_{u,v,\nu}f(t) = u(t) \int_{\nu(t)}^{L} f(s)v(s)\,\mathrm{d}s,\ t \in (0, L).$$

Here $u, v \in \mathfrak{M}^+(0, L)$ are nonincreasing and $\nu$ is an increasing bijection of $(0, L)$ onto itself. The interested reader is referred to [62] for the (pre)history of Hardy-type operators, which in various forms have been an indispensable tool in the mathematical analysis since the early 1900s. In this chapter, we will call both $R_{u,v,\nu}$ and $H_{u,v,\nu}$ Hardy-type operators, but the latter is also frequently called Copson-type operator.

In the paper [81], rearrangement-invariant function norms induced by these operators, optimal rearrangement-invariant function spaces for them and properties of these optimal spaces are studied. We say that a rearrangement-invariant function space $Y$ is the *optimal target space* for an operator $T$ and a rearrangement-invariant function space $X$ if $T$ is bounded from $X$ to $Y$ and $Y$ is the smallest such a rearrangement-invariant function space (in other words, the rearrangement-invariant function norm of $Y$ is the strongest possible)—that is, if $Z$ is a rearrangement-invariant function space such that $T$ is bounded from $X$ to $Z$, then $Y \hookrightarrow Z$. We say that a rearrangement-invariant function space $X$ is the *optimal domain space* for an operator $T$ and a rearrangement-invariant function space $Y$ if $T$ is bounded from $X$ to $Y$ and $X$ is the largest such a rearrangement-invariant function space (in other words, the rearrangement-invariant function norm of $X$ is the weakest possible)—that is, if $Z$ is a rearrangement-invariant function space such that $T$ is bounded from $Z$ to $Y$, then $Z \hookrightarrow X$.

Optimal rearrangement-invariant function spaces for Hardy-type operators can surprisingly often be used for describing optimal rearrangement-invariant function spaces for much more complicated operators. This connection can be easily observed through the following example using the Hardy–Littlewood maximal operator $M$. Thanks to the famous equivalence ([7, Theorem 3.8])

$$C_1 \frac{1}{t} \int_0^t f^*(s)\,\mathrm{d}s \leq (Mf)^*(t) \leq C_2 \frac{1}{t} \int_0^t f^*(s)\,\mathrm{d}s$$

for every $t \in (0,\infty)$ and $f \in \mathfrak{M}(\mathbb{R}^n)$, we immediately see that $M$ is bounded from a rearrangement-invariant function space $X(\mathbb{R}^n)$ to a rearrangement-invariant function space $Y(\mathbb{R}^n)$ if and only if $R_{u,v,\nu}$ with $u \equiv 1$, $v(t) = t^{-1}$, $\nu(t) = t$, and $L = \infty$, is bounded from $X(0,\infty)$ to $Y(0,\infty)$. This means that the optimal rearrangement-invariant function spaces for $M$ are the same as those for $R_{u,v,\nu}$.

There are other important operators of harmonic analysis for which sharp inequalities for their nonincreasing rearrangements in terms of Hardy-type operators are known. For example, the Hilbert transform ([7]; more generally in [21, p. 55]), certain convolution operators ([40, 87]), or the fractional maximal operator and its variants ([24, 35]). In [37], we studied the optimal rearrangement-invariant function spaces for some important operators of harmonic analysis. Even though the number of operators $T$ for which sharp inequalities for their nonincreasing rearrangements are known is limited, what is more often known is at least an upper bound on the nonincreasing rearrangement of a given operator in terms of Hardy-type operators (e.g., [2, 17, 35, 72]). If we find the optimal target space for the governing operator, we find the best possible target rearrangement-invariant function space for the operator $T$ that we can get from the upper bound at our disposal.

Pointwise inequalities for rearrangements are not the only way in which complicated questions can be reduced to simpler ones involving Hardy-type operators. Reductions can sometimes be achieved by the right use of interpolation or by making use of some intrinsic properties of the problem in question. Such approaches have been notably successful in connection with various embeddings of Sobolev-type spaces built upon rearrangement-invariant function spaces into rearrangement-invariant function spaces. In the pioneering paper [58], they proved that the optimal rearrangement-invariant function spaces in Sobolev inequalities of $m$th order on bounded Lipschitz domains in $\mathbb{R}^n$, $1 \leq m < n$, are the same as those for the operator $H_{u,v,\nu}$ with $u \equiv 1$, $v(t) = t^{-1+m/n}$, $\nu(t) = t$, and $L = 1$. Since then, a large number of different Sobolev-type embeddings in rearrangement-invariant function spaces have been reduced (often equivalently, or at least in form of sufficient/necessary conditions) to the boundedness of suitable Hardy-type operators (e.g., [3, 12, 22, 23, 25, 26, 27, 58, 78, 79]).

The paper [81], which generalizes and extends the author's previous results from [80], is aimed to thoroughly and comprehensively address some important properties of the optimal rearrangement-invariant function spaces for the operators $R_{u,v,\nu}$ and $H_{u,v,\nu}$. These properties were already studied before (in particular, see the papers mentioned in the previous paragraph and [37]) but in considerably less generality, usually for particular choices of the functions $u$, $v$, and $\nu$, often scattered and hidden somewhere between the lines with varying degrees of generality. Although some restrictions on the functions $u$, $v$, and $\nu$, apart from their monotonicity, do appear in [81], these restrictions are in general dictated by two things. First, they exclude some pathological cases. Second, they prevent Hardy-type operators with kernels from appearing. Rearrangement-invariant function norms induced by Hardy-type operators with kernels go beyond the scope of the paper and they should be studied in more detail in a future project. For example, rearrangement-invariant function norms induced by Hardy-type operators with kernels appeared in [26] in connection with optimal rearrangement-invariant function spaces for higher order Sobolev embeddings on bounded domains in $\mathbb{R}^n$ with respect to their isoperimetric function, or recently in the author's paper [82] in connection with optimal rearrangement-invariant function spaces for higher order iterations of the Laplace–Beltrami operator in the hyperbolic space.

We now begin describing main results from [81].

We start with a proposition characterizing when the operator $R_{u,v,\nu}$ induces a rearrangement-invariant function norm. It also describes the connection of the induced rearrangement-invariant function norm with the optimal domain space for the operator and a rearrangement-invariant function space. Since it can be easily observed that the operator $H_{u,v,\nu}$ is bounded from a rearrangement-invariant function space $X(0,L)$ to a rearrangement-invariant function space $Y(0,L)$ if and only if the operator $R_{u,v,\nu^{-1}}$, where $\nu^{-1}$ is the inverse function to $\nu$, is bounded from $Y'(0,L)$ to $X'(0,L)$, this proposition also has a "dual version", which characterizes the optimal target space for the operator $H_{u,v,\nu}$ and a rearrangement-invariant function space $X(0,L)$ (see [81, Proposition 3.7]).

**Proposition 3.1** ([81, Proposition 3.1]). *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

- *Let $\nu\colon (0,L) \to (0,L)$ be an increasing bijection.*
- *Let $u\colon (0,L) \to (0,\infty)$ be a nonincreasing function that is integrable on a right-neighborhood of $0$. If $L < \infty$, assume that $u(L^-) > 0$.*
- *Let $v\colon (0,L) \to (0,\infty)$ be measurable.*

*Set*

$$\|f\|_{Y(0,L)} = \|R_{u,v,\nu}(f^*)\|_{X(0,L)},\ f \in \mathfrak{M}^+(0,L),$$

*and*

$$\xi(t) = \begin{cases} v(t)U(\nu(t)),\ t\in(0,L), & \text{if } L<\infty,\\ v(t)U(\nu(t))\chi_{(0,1)}(t) + v(t)\chi_{(1,\infty)}(t),\ t\in(0,\infty), & \text{if } L=\infty, \end{cases}$$

*where*

$$U(t) = \int_0^t u(s)\,\mathrm{d}s,\ \ t \in (0,L). \tag{3.1}$$

*The functional* $\|\cdot\|_{Y(0,L)}$ *is a rearrangement-invariant function norm if and only if* $\xi \in X(0,L)$.

*If* $\xi \in X(0,L)$, *then the rearrangement-invariant function space* $Y(0,L)$ *is the optimal domain space for the operator* $R_{u,v,\nu}$ *and* $X(0,L)$. *If* $\xi \notin X(0,L)$, *then there is no rearrangement-invariant function space* $Z(0,L)$ *such that* $R_{u,v,\nu}\colon Z(0,L) \to X(0,L)$ *is bounded.*

Before proceeding, we recall *slowly varying functions*. We say that a measurable function $b\colon (0,L) \to (0,\infty)$ is slowly varying if for every $\varepsilon > 0$ there are nondecreasing and nonincreasing functions $b_\varepsilon\colon (0,L) \to (0,\infty)$ and $b_{-\varepsilon}\colon (0,L) \to (0,\infty)$, respectively, such that $t^\varepsilon b(t) \approx b_\varepsilon(t)$ and $t^{-\varepsilon} b(t) \approx b_{-\varepsilon}(t)$ on $(0,L)$. Prototypical examples of slowly varying functions (apart from positive constant functions) are functions such as $b(t) = 1 + |\log t|$, $t \in (0,L)$, and its real powers (with possibly different powers near $0$ and near $\infty$ when $L = \infty$).

In the following theorem (and also in the rest of this section), we will sometimes need to impose certain mild conditions on the bijection (temporarily denoted by $\nu$) appearing in the integration range of the Hardy operators.

- We write $\nu \in \underline{D}^0$ if there is $\theta > 1$ such that $\liminf_{t\to 0^+} \frac{\nu(\theta t)}{\nu(t)} > 1$.
- We write $\nu \in \underline{D}^\infty$ if there is $\theta > 1$ such that $\liminf_{t\to \infty} \frac{\nu(\theta t)}{\nu(t)} > 1$.
- We write $\nu \in \overline{D}^0$ if there is $\theta > 1$ such that $\limsup_{t\to 0^+} \frac{\nu(\theta t)}{\nu(t)} < \infty$.
- We write $\nu \in \overline{D}^\infty$ if there is $\theta > 1$ such that $\limsup_{t\to \infty} \frac{\nu(\theta t)}{\nu(t)} < \infty$.

For example, $\nu \in \underline{D}^\infty$ is satisfied if $\nu$ is equivalent to the function $t \mapsto t^\alpha b(t)$ near $\infty$, where $\alpha > 0$ and $b$ is a slowly varying function (cf. [47, Proposition 2.2(iii)]). On the other hand, it is not satisfied for example for $\nu$ equivalent to $\log^\alpha$ near $\infty$, where $\alpha > 0$. Furthermore, some functions will be required to satisfy an averaging condition. We say that a positive a.e. finite measurable function on $(0,L)$ (temporarily denoted by $w$) satisfies the *averaging condition* if

$$\operatorname*{ess\,sup}_{t\in(0,L)} \frac{1}{t w(t)} \int_0^t w(s)\,\mathrm{d}s < \infty. \tag{3.2}$$

Prototypical examples of functions that satisfy the averaging condition are the functions of the form $w(t) = t^{-1+\alpha} b(t)$, $t \in (0,L)$, where $\alpha > 0$ and $b\colon (0,L) \to (0,L)$ is any slowly varying function. On the other hand, a prototypical example of a function $w$ that is integrable over $(0,t)$ for every $t \in (0,L)$ but does not satisfy the averaging condition is the function $w(t) = t^{-1}(1 + |\log t|)^\alpha$, $t \in (0,L)$, with $\alpha < -1$.

Now, suppose that we start with a rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$, and let $\|\cdot\|_{Y_1(0,L)}$ be the rearrangement-invariant function norm induced by the operator $R_{u_1,v_1,\nu_1}$—that is,

$$\|f\|_{Y_1(0,L)} = \|R_{u_1,v_1,\nu_1}(f^*)\|_{X(0,L)},\ \ f \in \mathfrak{M}^+(0,L).$$

Starting now with $Y_1(0,L)$ instead of $X(0,L)$ and considering other functions $u_2$, $v_2$, and $\nu_2$, we consider the rearrangement-invariant function norm $\|\cdot\|_{Y_2(0,L)}$ induced by the operator $R_{u_2,v_2,\nu_2}$—that is,

$$\|f\|_{Y_2(0,L)} = \|R_{u_2,v_2,\nu_2}(f^*)\|_{Y_1(0,L)} = \|R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2}(f^*))^*)\|_{X(0,L)}, \ f \in \mathfrak{M}^+(0,L).$$

The question is: is the iterated rearrangement-invariant function norm $\|\cdot\|_{Y_2(0,L)}$ equivalent to a rearrangement-invariant function norm induced by a noniterated operator $R_{u,v,\nu}$? In other words, are there functions $u$, $v$, and $\nu$ such that

$$\|R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2}(f^*))^*)\|_{X(0,L)} \approx \|R_{u,v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L)?$$

The following theorem deals with this problem. It should be noted that the function $R_{u_2,v_2,\nu_2}(f^*)$ is usually not (equivalent to) a nonincreasing function, and so the outer star cannot be erased just like that.

Such an iteration is not artificial. It first appeared in [26] as an essential tool for establishing sharp iteration principles for various Sobolev embeddings. Loosely speaking, their sharp iteration principles ensure that the optimal rearrangement-invariant target spaces in various Sobolev embeddings of $(k+l)$th order are the same as those obtained by composing the optimal Sobolev embedding of order $k$ with the optimal Sobolev embedding of order $l$. They used it for obtaining optimal higher order Sobolev embeddings from the respective first order ones, where it is possible to exploit suitable isoperimetric inequalities. Another iteration principle appeared in [23], where it was used for establishing optimal higher order Sobolev trace embeddings.

The iteration principle [26, Theorem 9.5] is different from the one presented here (Theorem 3.2). There, the functions $v_1$ and $v_2$ are the same and the other functions are limited to $u_1 = u_2 \equiv 1$, $\nu_1 = \nu_2 = \mathrm{id}$ (and $L$ is finite). Moreover, the iteration principle there in general leads to a Hardy-type operator with a kernel. On the other hand, the iteration principle presented here not only considerably generalizes [23, Theorem 3.4] but also improves it. The iteration principle there is limited to the functions $u_j \equiv 1$, $v_j(t) = t^{\alpha_j - 1}$, $\nu_j(t) = t^{\beta_j}$, $t \in (0,L)$, $L$ finite, $j = 1,2$, with the parameters $\alpha_j, \beta_j \in (0,\infty)$ satisfying

$$\beta_2 + \alpha_2 \geq 1, \ \beta_1 + \alpha_1 \geq 1, \ \beta_1\alpha_2 + \alpha_1 < 1. \tag{3.3}$$

However, the iteration principle presented here can be used for this particular choice of the functions if (see [81, Remark 5.3] for a more general example)

$$\beta_1(\beta_2 + \alpha_2) + \alpha_1 \geq 1, \ \beta_1 + \alpha_1 \geq 1, \ \beta_1\alpha_2 + \alpha_1 < 1. \tag{3.4}$$

Note that (3.3) implies (3.4), but the opposite implication is false. For example, consider $\alpha_1 = \alpha_2 = 1/2$ and $\beta_1 = 3/4$. Then (3.4) is satisfied if and only if $\beta_2 \geq 1/6$, whereas (3.3) is satisfied if and only if $\beta_2 \geq 1/2$.

**Theorem 3.2** ([81, Theorem 5.2]). *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

- *Let $\nu_1, \nu_2 \colon (0,L) \to (0,L)$ be increasing bijections. Assume that $\nu_2 \in \overline{D}^0$. If $L = \infty$, assume that $\nu_2 \in \overline{D}^\infty$.*

- *Let $u_1, u_2\colon (0,L) \to (0,\infty)$ be nonincreasing.*
- *Let $v_1\colon (0,L) \to (0,\infty)$ be a continuous function. Let $v_2\colon (0,L) \to (0,\infty)$ be defined by*
$$\frac{1}{v_2(t)} = \int_0^{\nu_2(t)} \xi(s)\,\mathrm{d}s,\ t \in (0,L),$$
*where $\xi\colon (0,L) \to (0,\infty)$ is a measurable function. Assume that the function $u_1v_2$ satisfies the averaging condition* (3.2)*.*

*Let $v$ be the function defined as*
$$v(t) = u_1(\nu_1(t))v_1(t)\nu_1(t)v_2(\nu_1(t)),\ t \in (0,L).$$
*Set $\nu = \nu_2 \circ \nu_1$ and*
$$\eta(t) = \frac{1}{U_2(t)v(\nu^{-1}(t))},\ t \in (0,L),$$
*where $U_2$ is defined by* (3.1) *with $u$ replaced by $u_2$. Assume that $\eta$ and $\eta/\xi$ are equivalent to nonincreasing functions. Furthermore, assume that there are positive constants $C_1$ and $C_2$ such that*
$$\int_0^t \eta(s)u_2(s)\,\mathrm{d}s \le C_1U_2(t)\eta(t) \quad \textit{for every } t \in (0,L)$$
*and*
$$\frac{1}{t}\int_0^t U_2(\nu(s))v(s)\,\mathrm{d}s \ge C_2U_2(\nu(t))v(t) \quad \textit{for every } t \in (0,L).$$
*Then we have*
$$\|R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2}(f^*))^*)\|_{X(0,L)} \approx \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)}$$
*for every $f \in \mathfrak{M}^+(0,L)$.*

We now turn our attention to the operator $H_{u,v,\nu}$. The question of when the operator $H_{u,v,\nu}$ induces a rearrangement-invariant function norm is considerably more complicated because there is a problem that we inevitably face. The problem is that the functional
$$\mathfrak{M}^+(0,L) \ni f \mapsto \|H_{u,v,\nu}(f^*)\|_{X(0,L)} \tag{3.5}$$
is usually not subadditive and the triangle inequality fails (however, it is subadditive when the functions $u$, $v$, and $\nu$ are related to each other in a specific way (see [81, Proposition 4.1]). If we just replaced the functional (3.5) with $\mathfrak{M}^+(0,L) \ni f \mapsto \|H_{u,v,\nu}f\|_{X(0,L)}$, the resulting functional would be subadditive, but it would not be rearrangement invariant anymore. Nevertheless, both the subadditivity and rearrangement invariance can be saved by taking a supremum over all nonnegative equimeasurable functions with $f$. This is the content of the following proposition, which also characterizes the optimal domain space for the operator $H_{u,v,\nu}$ and a rearrangement-invariant function space (for its "dual version", which describes the optimal target space for the operator $R_{u,v,\nu}$ and a rearrangement-invariant function space, see [81, Proposition 3.8]).

**Proposition 3.3** ([81, Proposition 3.3])**.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

- *Let $\nu\colon (0,L)\to(0,L)$ be an increasing bijection. If $L=\infty$, assume that $\nu\in\underline{D}^\infty$.*
- *Let $u\colon (0,L)\to(0,\infty)$ be nonincreasing.*
- *Let $v\colon (0,L)\to(0,\infty)$ be nonincreasing. If $L<\infty$, assume that $v(L^-)>0$.*

*Set*

$$\|f\|_{Y(0,L)} = \sup_{h\sim f} \|H_{u,v,\nu}h\|_{X(0,L)},\ f\in\mathfrak{M}^+(0,L), \tag{3.6}$$

*where the supremum extends over all $h\in\mathfrak{M}^+(0,L)$ equimeasurable with $f$. The functional $\|\cdot\|_{Y(0,L)}$ is a rearrangement-invariant function norm if and only if*

$$\begin{cases} u(t)\int_{\nu(t)}^L v(s)\,\mathrm{d}s \in X(0,L) & \text{if } L<\infty,\\ u(t)\chi_{(0,\nu^{-1}(1))}(t)\displaystyle\int_{\nu(t)}^1 v(s)\,\mathrm{d}s\in X(0,\infty) \text{ and} & \\ \qquad \limsup\limits_{\tau\to\infty} v(\tau)\|u\chi_{(0,\nu^{-1}(\tau))}\|_{X(0,\infty)}<\infty & \text{if } L=\infty. \end{cases} \tag{3.7}$$

*If (3.7) is satisfied, then the rearrangement-invariant function space $Y(0,L)$ is the optimal domain space for the operator $H_{u,v,\nu}$ and $X(0,L)$. If (3.7) is not satisfied, then there is no rearrangement-invariant function space $Z(0,L)$ such that $H_{u,v,\nu}\colon Z(0,L)\to X(0,L)$ is bounded.*

As with the operator $R_{u,v,\nu}$, there is also a certain iteration principle for the operator $H_{u,v,\nu}$. The interested reader is referred to [81, Proposition 5.4].

In concrete situations, we are often interested in explicit descriptions of optimal rearrangement-invariant function spaces—preferably, in terms of common function spaces. The supremum in (3.6) usually makes an explicit description of the corresponding rearrangement-invariant function space very hard, unless the function norm is somehow simplified first. Ideally, the rearrangement-invariant function norm would be equivalent to the functional (3.5), which is significantly less complicated. While this is not always the case in general, the following result can often be used for the simplification.

**Proposition 3.4** ([81, Proposition 4.2])**.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

- *Let $\nu\colon (0,L)\to(0,L)$ be an increasing bijection. If $L=\infty$, assume that $\nu\in\underline{D}^\infty$.*
- *Let $u\colon (0,L)\to(0,\infty)$ be nonincreasing.*
- *Let $v\colon (0,L)\to(0,\infty)$ be defined by*

$$\frac{1}{v(t)} = \int_0^{\nu^{-1}(t)} \xi(s)\,\mathrm{d}s \quad \text{for every } t\in(0,L), \tag{3.8}$$

*where $\xi\colon (0,L) \to (0,\infty)$ is measurable. If $L<\infty$, assume that $\xi \in L^1(0,L)$. Furthermore, assume that the operator $T_\varphi$ defined as*

$$T_\varphi f(t) = \frac{1}{\varphi(t)} \operatorname*{ess\,sup}_{s\in[t,L)} \varphi(s) f^*(s),\ t\in(0,L),\ f\in\mathfrak{M}(0,L), \tag{3.9}$$

*with $\varphi = u/\xi$ is bounded on $X'(0,L)$.*

*Assume that*

$$\Big\| u(t)\chi_{(0,\nu^{-1}(a))}(t) \int_{\nu(t)}^a v(s)\,\mathrm{d}s \Big\|_{X(0,L)} < \infty,$$

*where $a$ is defined as*

$$a = \begin{cases} L & \text{if } L<\infty,\\ 1 & \text{if } L=\infty.\end{cases}$$

*Let $\|\cdot\|_{Y(0,L)}$ be the functional defined by* (3.6) *and set*

$$\|f\|_{Z(0,L)} = \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L f^*(s)v(t)\Big(\int_0^{\nu^{-1}(t)} T_\varphi g(s)u(s)\,\mathrm{d}s\Big)\,\mathrm{d}t,\ f\in\mathfrak{M}^+(0,L).$$

*The functionals $\|\cdot\|_{Y(0,L)}$ and $\|\cdot\|_{Z(0,L)}$ are rearrangement-invariant function norms. Furthermore, we have*

$$\begin{aligned}\|H_{u,v,\nu}(f^*)\|_{X(0,L)} &\le \sup_{h\sim f}\|H_{u,v,\nu}h\|_{X(0,L)} \le \|f\|_{Z(0,L)}\\ &\le \|T_\varphi\|_{X'(0,L)}\|H_{u,v,\nu}(f^*)\|_{X(0,L)}\end{aligned}$$

*for every $f\in\mathfrak{M}^+(0,L)$. Here $\|T_\varphi\|_{X'(0,L)}$ stands for the operator norm of $T_\varphi$ on $X'(0,L)$. In particular, the rearrangement-invariant function norms $\|\cdot\|_{Y(0,L)}$ and $\|\cdot\|_{Z(0,L)}$ are equivalent; moreover, they are equivalent to the functional* (3.5)*.*

The boundedness of the supremum operator $T_\varphi$ on so-called Lambda spaces, which encompass a large number of common rearrangement-invariant function spaces (see Chapter 4), is characterized by [48, Theorem 3.2]. In concrete situations, $\varphi$ is usually (equivalent to) a quasiconcave function.

We say that a function $\varphi\colon(0,L)\to(0,\infty)$ is *quasiconcave* if it is nondecreasing and the function $(0,L)\ni t\mapsto \varphi(t)/t$ is nonincreasing. A typical example of a function on $(0,L)$ that is equivalent to a quasiconcave function is a function that is equivalent to the function $(0,L)\ni t\mapsto t^\alpha b(t)$, where $b$ is slowly varying and either $\alpha\in(0,1)$, or $\alpha=0$ and $b$ is (equivalent to) a nondecreasing function, or $\alpha=1$ and $b$ is (equivalent to) a nonincreasing function.

It turns out that the question of whether the supremum operator $T_\varphi$ is bounded on a rearrangement-invariant function space is closely connected with the general notion of being an optimal function space and with the notion of being an interpolation space. We start with the former.

**Theorem 3.5** ([81, Theorem 4.6])**.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

- *Let* $\nu\colon (0,L) \to (0,L)$ *be an increasing bijection. If* $L = \infty$, *assume that* $\nu \in \underline{D}^\infty$.
- *Let* $u\colon (0,L) \to (0,\infty)$ *be a nonincreasing function that is integrable on a right-neighborhood of* $0$. *If* $L < \infty$, *assume that* $u(L^-) > 0$.
- *Let* $v\colon (0,L) \to (0,\infty)$ *be a nonincreasing function. If* $L < \infty$, *assume that* $v(L^-) > 0$.

*Let* $\|\cdot\|_{Y(0,L)}$ *be the functional defined by* (3.6). *The following three statements are equivalent.*

(i) *The space* $X(0,L)$ *is the optimal target space for the operator* $H_{u,v,\nu}$ *and some rearrangement-invariant function space.*

(ii) *The space* $X'(0,L)$ *is the optimal domain space for the operator* $R_{u,v,\nu^{-1}}$ *and some rearrangement-invariant function space.*

(iii) *We have that*

$$\|f\|_{X'(0,L)} \approx \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{Y(0,L)}\le 1}} \int_0^L g(t) R_{u,v,\nu^{-1}}(f^*)(t)\,\mathrm{d}t \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

*Finally, assume, in addition, that*

- $v$ *is defined by* (3.8) *with* $\xi$ *satisfying*

$$\frac{u(t)}{U(t)} \int_0^t \xi(s)\,\mathrm{d}s \lesssim \xi(t) \quad \text{for a.e. } t \in (0,L),$$

  *where* $U$ *is defined by* (3.1);
- *the function* $\varphi \circ \nu^{-1}$, *where* $\varphi = u/\xi$, *is equivalent to a quasiconcave function.*

*Then each of the three equivalent statements above implies that*

*(iv) the operator* $T_\varphi$, *defined by* (3.9), *is bounded on* $X'(0,L)$.

For example, when $u(t) = t^{-1+\alpha}$, $v(t) = t^{-1+\beta}$, and $\nu(t) = t^\gamma$, $t \in (0,L)$, the assumptions (including the additional ones) of the preceding theorem are satisfied if $\alpha \in (0,1]$, $\beta \in [0,1)$, $\gamma > 0$ and $1 \le \frac{\alpha}{\gamma} + \beta \le 2$. Furthermore, it is worth noting that, if $X'(0,L)$ is the optimal domain space for $R_{u,v,\nu^{-1}}$ and some rearrangement-invariant function space $Y(0,L)$, then $X'(0,L)$ is actually the optimal domain space for $R_{u,v,\nu^{-1}}$ and its own optimal target space. Similarly, if $X(0,L)$ is the optimal target space for $H_{u,v,\nu}$ and some rearrangement-invariant function space $Y(0,L)$, then $X(0,L)$ is actually the optimal target space for $H_{u,v,\nu}$ and its own optimal domain space (see [81, Remark 4.7]).

The next theorem provides a sufficient and a necessary condition for the boundedness of $T_\varphi$ on a rearrangement-invariant function space in terms of being an interpolation space between suitable endpoint spaces. When $L < \infty$, the theorem actually provides a characterization.

We say that a rearrangement-invariant function space $X(R,\mu)$ is an *interpolation space* between rearrangement-invariant function spaces $X_1(R,\mu)$ and $X_2(R,\mu)$ if the following two conditions are satisfied. First, $X(R,\mu)$ is an intermediate space between $X_1(R,\mu)$ and $X_2(R,\mu)$, that is, $X_1(R,\mu)\cap X_2(R,\mu)\hookrightarrow X(R,\mu)\hookrightarrow (X_1+X_2)(R,\mu)$. Second, if a linear operator $T$ defined on $(X_1+X_2)(R,\mu)$ with values in $(X_1+X_2)(R,\mu)$ is bounded on both $X_1(R,\mu)$ and $X_2(R,\mu)$, then it is also bounded on $X(R,\mu)$. The interested reader can find more information on interpolation spaces in, for example, [7, 8, 15, 61].

**Theorem 3.6** ([81, Theorem 4.11])**.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $\varphi\colon (0,L)\to(0,\infty)$ be a measurable function that is equivalent to a continuous nondecreasing function. Set $\xi = 1/\varphi$. Assume that $\xi$ satisfies the averaging condition* (3.2)*. Consider the following three statements.*

(i) *The operator $T_\varphi$, defined by* (3.9)*, is bounded on $X'(0,L)$.*

(ii) $X(0,L)\in \operatorname{Int}\big(\Lambda^1_\xi(0,L), L^\infty(0,L)\big)$*, where*

$$\Lambda^1_\xi(0,L)=\Big\{f\in\mathfrak{M}(0,L) : \|f\|_{\Lambda^1_\xi(0,L)}=\int_0^L f^*(t)\xi(t)\,\mathrm{d}t<\infty\Big\}.$$

(iii) $X'(0,L)\in \operatorname{Int}\big(L^1(0,L), M_\varphi(0,L)\big)$*, where*

$$M_\varphi(0,L)=\Big\{f\in\mathfrak{M}(0,L) : \|f\|_{M_\varphi(0,L)}=\sup_{t\in(0,L)} f^{**}(t)\varphi(t)<\infty\Big\}.$$

*If $L<\infty$, then the three statements are equivalent to each other. If $L=\infty$, then (i) implies (ii), and (iii) implies (i).*

At the beginning of the proof of the preceding theorem, it is observed that the function $\varphi$ is equivalent to a quasiconcave function. The function spaces $\Lambda^1_\xi(0,L)$ and $M_\varphi(0,L)$ are so-called Lorentz and Marcinkiewicz endpoint spaces (see, e.g., [61] for more information on these spaces). The assumptions of the preceding theorem guarantee that both functionals $\|\cdot\|_{\Lambda^1_\xi(0,L)}$ and $\|\cdot\|_{M_\varphi(0,L)}$ are equivalent to rearrangement-invariant function norms, and so the respective function spaces can be regarded as rearrangement-invariant function spaces.

Now, recall that, loosely speaking, we have seen so far that

- the boundedness of $T_\varphi$ on $X'(0,L)$ for a suitable function $\varphi$ is sufficient for the equivalence of the functionals (3.5) and (3.6) (Proposition 3.4);

- if $X(0,L)$ is the optimal target space for the operator $H_{u,v,\nu}$ and some rearrangement-invariant function space, then $T_\varphi$ with a suitable function $\varphi$ is bounded on $X'(0,L)$ (Theorem 3.5);

- the boundedness of $T_\varphi$ goes hand in hand with the notion of interpolation spaces (Theorem 3.6).

We conclude this section with the following theorem concerning the important case $u\equiv 1$. In this case, we have the following characterization.

**Theorem 3.7** ([81, Theorem 4.16]). *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

- *Let $\nu\colon (0,L) \to (0,L)$ be an increasing bijection. Assume that $\nu^{-1} \in \overline{D}^0$. If $L = \infty$, assume that $\nu^{-1} \in \overline{D}^\infty$ and $\nu \in \underline{D}^\infty$.*
- *Let $v\colon (0,L) \to (0,\infty)$ be defined by (3.8) with $\xi\colon (0,L) \to (0,\infty)$ satisfying the averaging condition (3.2). Assume that $v$, too, satisfies the averaging condition (3.2). Furthermore, assume that the function $\varphi \circ \nu^{-1}$ is equivalent to a quasiconcave function, where $\varphi = 1/\xi$.*

*Let $\|\cdot\|_{Y(0,L)}$ be the functional defined by (3.6) with $u \equiv 1$. The following five statements are equivalent.*

(i) *The operator $T_\varphi$, defined by (3.9), is bounded on $X'(0,L)$.*

(ii) *The functionals (3.5) and (3.6) with $u \equiv 1$ are equivalent.*

(iii) *The space $X(0,L)$ is the optimal target space for the operator $H_{1,v,\nu}$ and some rearrangement-invariant function space.*

(iv) *The space $X'(0,L)$ is the optimal domain space for the operator $R_{1,v,\nu^{-1}}$ and some rearrangement-invariant function space.*

(v) *We have*

$$\|f\|_{X'(0,L)} \approx \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{Y(0,L)}\leq 1}} \int_0^L g(t) R_{1,v,\nu^{-1}}(f^*)(t)\,\mathrm{d}t \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

*Moreover, if $L < \infty$, these five statements are also equivalent to*

(vi) $X(0,L) \in \operatorname{Int}\big(\Lambda^1_\xi(0,L), L^\infty(0,L)\big)$.

## 3.2 Reduction of the Gaussian $K$-inequality

This section describes the main results from [5].

Throughout the section, $(R,\mu)$ and $(S,\nu)$ are two (possibly different) probabilistic non-atomic measure spaces; however, the results and their proofs in [5] could be easily generalized to finite (not necessarily probabilistic) measure spaces.

Given two, say, rearrangement-invariant function spaces $Z_1(R,\mu)$ and $Z_2(R,\mu)$, the corresponding *$K$-functional* is defined, for each $g \in \mathfrak{M}_0(R,\mu)$ and $t > 0$, as

$$K(g,t;Z_1,Z_2) = \inf_{g=g_1+g_2}(\|g_1\|_{Z_1(R,\mu)} + t\|g_2\|_{Z_2(R,\mu)}),$$

where the infimum extends over all possible decompositions $g = g_1 + g_2$, $g_i \in Z_i(R,\mu)$, $i = 1,2$. The interested reader can find more information on the $K$-functional and its importance in the interpolation theory in, for example, [7, 8, 15, 61].

Let $T$ be an operator defined on $(X_1 + X_2)(R, \mu)$ with values in $\mathfrak{M}_0(S, \nu)$, for which we have the following endpoint estimates at our disposal:

$$T\colon X_1(R, \mu) \to Y_1(S, \nu) \qquad \text{and} \qquad T\colon X_2(R, \mu) \to Y_2(S, \nu), \tag{3.10}$$

where $X_j(R, \mu)$ and $Y_j(S, \nu)$, $j = 1, 2$, are (rearrangement-invariant) function spaces. Here $T\colon X_j(R, \mu) \to Y_j(S, \nu)$ stands for the fact that the operator $T$ is bounded from $X_j(R, \mu)$ to $Y_j(S, \nu)$. For a wide class of operators $T$, which includes not only quasi-linear operators ([7, Chapter 3, Definition 5.3]) but also other types of operators (see [15, Section 4.1] for more detail), the endpoint estimates (3.10) imply the following inequality between the corresponding $K$-functionals:

$$K(Tf, t; Y_1, Y_2) \lesssim K(f, t; X_1, X_2) \quad \text{for every } f \in (X_1 + X_2)(R, \mu) \text{ and } t > 0.$$

Such an inequality between $K$-functionals will be referred to as a *$K$-inequality*.

In [5], we considered a $K$-inequality corresponding to a certain choice of the endpoint spaces. The prototypical example of a $K$-inequality studied in [5] is

$$K(Tf, t; L\sqrt{\log L}, e^{L^2}) \lesssim K(f, t; L^1, L^\infty) \tag{3.11}$$

for every $f \in L^1(R, \mu)$ and $t > 0$. We addressed the question of whether there is a reduction operator (or operators) whose boundedness between rearrangement-invariant function spaces $X(0, 1)$ and $Y(0, 1)$ is equivalent to the fact that *every* operator $T$ satisfying the $K$-inequality is bounded from $X(R, \mu)$ to $Y(S, \nu)$. By means of such a reduction operator (or operators), the question of whether every operator satisfying the $K$-inequality is bounded from a rearrangement-invariant function space to another is reduced to a simpler question of boundedness of the reduction operator(s) between the corresponding representation spaces.

The prototypical example (3.11) is motivated by the Gaussian Sobolev inequalities—that is, Sobolev-type inequalities in $\mathbb{R}^n$ endowed with the (standard) Gaussian measure $\gamma_n$ defined as

$$\mathrm{d}\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} \, \mathrm{d}x.$$

In the general setting of rearrangement-invariant function spaces, these inequalities can be stated as

$$\|u - u_{\gamma_n}\|_{Y(\mathbb{R}^n, \gamma_n)} \leq C \|\nabla u\|_{X(\mathbb{R}^n, \gamma_n)} \tag{3.12}$$

for every weakly differentiable function $u$ such that $|\nabla u| \in X(\mathbb{R}^n, \gamma_n)$. Here $u_{\gamma_n}$ is the integral mean of $u$ with respect to the measure $\gamma_n$, and $C$ is a constant that is independent not only of $u$ but also of the dimension $n$. The ongoing interested in such inequalities was raised by the seminal paper [55], which established such an inequality for $X(\mathbb{R}^n, \gamma_n) = L^2(\mathbb{R}^n, \gamma_n)$ and $Y(\mathbb{R}^n, \gamma_n) = L^2 \log L(\mathbb{R}^n, \gamma_n)$, and which pointed out its importance for the study of quantum fields and hypercontractivity semigroups. Since then, such inequalities have been intensively studied in various forms and settings. In particular, in the general setting of rearrangement-invariant function spaces, these inequalities were intensively studied in [22] with focus being on the optimality of function spaces. In that paper, they described the optimal rearrangement-invariant function space $Y(\mathbb{R}^n, \gamma_n)$ in (3.12) for a given rearrangement-invariant function space $X(\mathbb{R}^n, \gamma_n)$ (see also [26]). For the endpoint spaces

$L^1(\mathbb{R}^n, \gamma_n)$ and $L^\infty(\mathbb{R}^n, \gamma_n)$, the optimal form of (3.12) reads as

$$\|u - u_{\gamma_n}\|_{L\sqrt{\log L}(\mathbb{R}^n,\gamma_n)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^n,\gamma_n)}$$

and

$$\|u - u_{\gamma_n}\|_{e^{L^2}(\mathbb{R}^n,\gamma_n)} \leq C\|\nabla u\|_{L^\infty(\mathbb{R}^n,\gamma_n)},$$

which is exactly the endpoint behavior modeled by (3.11). What makes such endpoint behavior rather nonstandard is that $L^p(\mathbb{R}^n, \gamma_n) \subsetneq L\sqrt{\log L}(\mathbb{R}^n, \gamma_n) \subsetneq L^1(\mathbb{R}^n, \gamma_n)$ for every $p \in (1, \infty]$, whereas $L^\infty(\mathbb{R}^n, \gamma_n) \subsetneq e^{L^2}(\mathbb{R}^n, \gamma_n) \subsetneq L^q(\mathbb{R}^n, \gamma_n)$ for every $q \in [1, \infty)$. This means that one endpoint estimate corresponds to slight gain in integrability, whereas the other corresponds to slight loss. Moreover, loosely speaking, both gain and loss are just logarithmic, which entails further difficulties.

A more general version of the $K$-inequality studied in [5] is

$$K(Tf, t; L^{1,1,b_1}, L^{\infty,\infty,b_2}) \lesssim K(f, t; L^1, L^\infty) \tag{3.13}$$

for every $f \in L^1(R, \mu)$ and $t > 0$. The endpoint spaces on the left-hand side are so-called Lorentz-Karamata spaces, which will be introduced soon. In fact, we studied a more general $K$-inequality than (3.13) in the setting of $p$-convex rearrangement-invariant quasi-Banach function spaces; however, we stick with the setting of rearrangement-invariant (Banach) function spaces here (i.e., $p = 1$) in order not to make things needlessly technical.

Given a $\sigma$-finite (non necessarily probabilistic/finite, as is the case in the rest of this section) non-atomic measure space $(S, \nu)$, $p, q \in [1, \infty]$, and a slowly varying function (see page 11) $b\colon (0, \nu(S)) \to (0, \infty)$, the *Lorentz-Karamata space* $L^{p,q,b}(S, \nu)$ is defined as

$$L^{p,q,b}(S, \nu) = \{f \in \mathfrak{M}(S, \nu) : \|f\|_{L^{p,q,b}(S,\nu)} < \infty\},$$

where

$$\|f\|_{L^{p,q,b}(S,\nu)} = \|t^{\frac{1}{p}-\frac{1}{q}} b(t) f^*(t)\|_{L^q(0,\nu(S))}.$$

The interested reader is referred to [83, 89] (cf. [47, 88]) for comprehensive information on this class of function spaces. Lorentz-Karamata spaces are not (equivalent to) rearrangement-invariant function spaces for every $p, q$ and $b$ (to this end, see [89, Theorem 3.33]). In particular, $L^{1,1,b}(S, \nu)$ is (equivalent to) a rearrangement-invariant function space if and only if $b$ is (equivalent to a) nonincreasing (function), whereas $L^{\infty,\infty,b}(S, \nu)$ is equivalent to a rearrangement-invariant function space if and only if $b\chi_{(0,1)} \in L^\infty(0, \nu(S))$.

The class of Lorentz-Karamata spaces contains not only Lebesgue and Lorentz spaces ($b \equiv 1$) but also some Orlicz spaces. In particular, when $\nu(S) = 1$ and $b(t) = (1 - \log t)^\alpha$, $t \in (0, 1)$, then $L^{1,1,b}(S, \nu)$ is equivalent to the Orlicz space $L^1(\log L)^\alpha(S, \nu)$ provided that $\alpha \geq 0$, and $L^{\infty,\infty,b}(S, \nu)$ is equivalent to the Orlicz space $\exp L^{-\frac{1}{\alpha}}(S, \nu)$ provided that $\alpha < 0$ (see [88, Section 8]).

In (3.13), $b_1$ and $b_2$ is a pair of continuous slowly varying functions on $(0, 1)$ satisfying:

*(i)* $b_1$ is nonincreasing and $b_2$ is nondecreasing;

*(ii)* $b_1(t) \approx b_1\left(tb_1(t)b_2(t)^{-1}\right)$ near $0^+$;

*(iii)* $\sup_{0<t<1} b_2(t) \int_t^1 \frac{1}{sb_1(s)}\,\mathrm{d}s < \infty$.

We will write $(b_1, b_2) \in \mathcal{B}$. For example, let $\alpha, \beta \in \mathbb{R}$ and consider

$$b_1(t) = (1-\log t)^{\alpha} \qquad \text{and} \qquad b_2(t) = (1-\log t)^{-\beta},\ t \in (0,1). \tag{3.14}$$

Then $(b_1, b_2) \in \mathcal{B}$ if and only if $\alpha, \beta \geq 0$ and either $\alpha + \beta \geq 1$ and $\beta > 0$ or $\alpha > 1$ and $\beta = 0$. In particular, when $\alpha = \beta = \frac{1}{2}$, we have $L^{1,1,b_1}(S,\nu) = L\sqrt{\log L}(S,\nu)$ and $L^{\infty,\infty,b_2}(S,\nu) = e^{L^2}(S,\nu)$, and so (3.13) reads as (3.11).

Given a pair $(b_1, b_2)$ of slowly varying functions on $(0,1)$, we define the function $\sigma = \sigma(b_1,b_2)\colon [0,1] \to [0,1]$ as the increasing bijection satisfying

$$t = \frac{1}{C}\int_0^{\sigma(t)} \frac{b_1(s)}{b_2(s)}\,\mathrm{d}s \quad \text{for every } t \in [0,1], \tag{3.15}$$

where

$$C = \int_0^1 \frac{b_1(s)}{b_2(s)}\,\mathrm{d}s.$$

We begin with a characterization of the $K$-inequality (3.13) in terms of the validity of a certain inequality involving integrals.

**Theorem 3.8** ([5, Theorem 3.3])**.** *Let $(R,\mu)$ and $(S,\nu)$ be probabilistic non-atomic measure spaces. Let $(b_1,b_2)$ be a pair of continuous slowly varying functions on $(0,1)$. Assume that $b_2$ is nondecreasing. Let $f \in L^1(R,\mu)$ and $g \in L^{1,1,b_1}(S,\nu)$. The inequality*

$$K(g,t;L^{1,1,b_1},L^{\infty,\infty,b_2}) \lesssim K(f,t;L^1,L^{\infty})$$

*is valid for every $t \in (0,\infty)$ with a multiplicative constant independent of $f$ and $g$ if and only if the inequality*

$$\int_0^t g^*(s)b_1(s)\,\mathrm{d}s \lesssim \int_0^t f^*\left(\sigma^{-1}(s)\right)\frac{b_1(s)}{b_2(s)}\,\mathrm{d}s$$

*is valid for every $t \in (0,1)$ with a multiplicative constant independent of $f$ and $g$.*

In view of the preceding theorem, an operator $T$ defined on $L^1(R,\mu)$ with values in $\mathfrak{M}_0(S,\nu)$ satisfies the $K$-inequality (3.13) if and only if

$$\int_0^t (Tf)^*(s)b_1(s)\,\mathrm{d}s \lesssim \int_0^t f^*\left(\sigma^{-1}(s)\right)\frac{b_1(s)}{b_2(s)}\,\mathrm{d}s$$

for every $t \in (0,1)$ and $f \in L^1(R,\mu)$. Motivated by (3.11) and the endpoint behavior of the Gaussian Sobolev inequalities, we call such operators $(b_1,b_2)$-*gaussible*.

The following theorem characterizes those pairs of rearrangement-invariant function spaces between which every $(b_1,b_2)$-gaussible operator is bounded by means of boundedness of certain governing operators. Given a pair $(b_1, b_2)$ of slowly varying functions, the operators $U_{b_1,b_2}$, $T_{b_1,b_2}$, and $S_{b_1}$ are defined as, for every $g \in \mathfrak{M}(0,1)$,

$$U_{b_1,b_2}g(t) = \frac{g^*(\sigma^{-1}(t))}{b_2(t)}, \ t \in (0,1),$$
$$T_{b_1,b_2}g(t) = \sup_{t\leq s<1} \frac{g^*(\sigma(s))}{b_1(\sigma(s))}, \ t \in (0,1),$$

and

$$S_{b_1}g(t) = \int_t^1 \frac{|g(s)|}{sb_1(s)}\,\mathrm{d}s, \ t \in (0,1).$$

Here $\sigma$ is the increasing bijection defined by (3.15).

**Theorem 3.9** ([5, Theorem 3.11]). *Let $(R,\mu)$ and $(S,\nu)$ be probabilistic non-atomic measure spaces. Let $X(R,\mu)$ and $Y(S,\nu)$ be rearrangement-invariant function spaces. Let $(b_1,b_2) \in \mathcal{B}$. The following statements are equivalent.*

(i) *Every $(b_1,b_2)$-gaussible operator $T$ is bounded from $X(R,\mu)$ to $Y(S,\nu)$.*

(ii) *The operators $U_{b_1,b_2}$ and $S_{b_1}$ are bounded from $X(0,1)$ to $Y(0,1)$.*

(iii) *The operator $T_{b_1,b_2}$ is bounded from $Y'(0,1)$ to $X'(0,1)$.*

Note that Theorem 3.9(ii) involves two governing operators. The following theorem shows that, under some extra assumptions on the functions $b_1$ and $b_2$, only one of them is needed.

**Theorem 3.10** ([5, Theorem 3.12]). *Let $(R,\mu)$ and $(S,\nu)$ be probabilistic non-atomic measure spaces. Let $Y(0,1)$ be a rearrangement-invariant function space. Let $(b_1,b_2) \in \mathcal{B}$. Furthermore, assume that the function $b_1/b_2$ is decreasing and that*

$$\lim_{s\to 0^+} b_2(s) \int_s^1 \frac{1}{\tau b_1(\tau)}\,\mathrm{d}\tau \in (0,\infty).$$

*Then we have*

$$\|S_{b_1}(f^*)\|_{Y(0,1)} \lesssim \|U_{b_1,b_2}f\|_{Y(0,1)} \quad \textit{for every } f \in \mathfrak{M}(0,1).$$

*Moreover, if $U_{b_1,b_2}$ is bounded from a rearrangement-invariant function space $X(0,1)$ to $Y(0,1)$, then so is $S_{b_1}$. In particular, the equivalent statements from Theorem 3.9 are also equivalent to:*

(ii') *The operator $U_{b_1,b_2}$ is bounded from $X(0,1)$ to $Y(0,1)$.*

When the functions $b_1$ and $b_2$ are defined by (3.14), the extra assumptions of the preceding theorem are satisfied if and only if either $\alpha \in [0,1)$ and $\beta = 1-\alpha$ or $\alpha > 1$ and $\beta = 0$. In particular, they are satisfied when $\alpha = \beta = \frac{1}{2}$, which corresponds to (3.11).

# Chapter 4

# Weighted inequalities

This chapter describes the main results from the papers [52, 66]. These papers are devoted to characterizing optimal constants, up to equivalences, in certain weighted inequalities involving functions of one variable. The important feature of the results from [52, 66] is that they do not involve various important restrictions from the previous work.

Throughout this chapter, let $a, b \in [-\infty, \infty]$, $a < b$. Recall that all the expressions $\frac{1}{\infty}$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, and $0 \cdot \infty$ are to be interpreted as 0. Furthermore, the expression $\frac{1}{0}$ is to be interpreted as $\infty$.

## 4.1 The embedding $\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$ with no restriction on the weights

In [66], we characterized the optimal (i.e., the least) constant, up to equivalences, $C \in [0, \infty]$ with which the inequality

$$\left(\int_0^b f^*(t)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C \left(\int_0^b \left(\frac{1}{U(t)}\int_0^t f^*(s)u(s)\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{4.1}$$

is satisfied for every $f \in \mathfrak{M}^+(0,b)$. Here $p, q \in (0, \infty)$ and $u, v, w \in \mathfrak{M}^+(0,b)$ are fixed weights that are locally integrable on $[0,b)$, with $u$ being positive a.e. Recall that $u \in \mathfrak{M}^+(0,b)$ is locally integrable on $[0,b)$ if

$$U(t) = \int_0^t u(s)\,\mathrm{d}s < \infty \quad \text{for every } t \in (0,b).$$

This definition is extended to the functions $v$ and $w$ in the obvious way. We also considered the weak variant of the inequality (see (4.5)) corresponding to $p = \infty$.

From the point of view of function spaces, the optimal constant $C$ in (4.1) is equal to the embedding constant of the embedding $\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$, that is,

$$C = \sup_{\|f\|_{\Gamma^p_u(v)} \le 1} \|f\|_{\Lambda^q(w)}. \tag{4.2}$$

Given a $\sigma$-finite non-atomic measure space $(R, \mu)$ with $\mu(R) = b$, the *Lambda space* $\Lambda^q(w)$ is defined as

$$\Lambda^q(w) = \{f \in \mathfrak{M}(R,\mu) : \|f\|_{\Lambda^q(w)} < \infty\},$$

where

$$\|f\|_{\Lambda^q(w)} = \left(\int_0^b f^*(t)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

The (generalized) *Gamma space* $\Gamma^p_u(v)$ is defined as

$$\Gamma^p_u(v) = \{f \in \mathfrak{M}(R,\mu) : \|f\|_{\Gamma^p_u(v)} < \infty\},$$

where

$$\|f\|_{\Gamma^p_u(v)} = \left(\int_0^b \left(\frac{1}{U(t)}\int_0^t f^*(s)u(s)\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

The Lambda spaces are sometimes called *classical Lorentz spaces*. They were introduced by G. G. Lorentz in [75] already in 1951, and they have been widely studied since then. In particular, they encompass a large number of customary function spaces, such as Lebesgue spaces, Lorentz spaces, some Orlicz spaces, or Lorentz–Karamata spaces. The wide-ranging interest in Gamma spaces was sparked by the paper [98], where E. Sawyer established his so-called principle of duality for Lambda spaces in connection with boundedness of some classical operators between Lambda spaces. Nowadays, Gamma spaces are known to play an important role not only in boundedness of operators but also in the interpolation theory, Sobolev embeddings, regularity of so-called very weak solutions to partial differential equations, or in the elasticity theory. The interested reader is referred to the introductory section of [51] and references therein for more information.

In fact, explicit manageable equivalent expressions for the optimal constant in (4.1) for all values of the parameters $p, q \in (0,\infty)$ were already found in [45], where they used the method of discretization (cf. [53, 56]) innovatively followed by so-called antidiscretization. Even though the entire range of the parameters is covered there and the equivalent expressions are explicit and easily manageable, there is still an important restriction in their result. The restriction lies in the fact that $b = \infty$ and, much more importantly, that the weights are assumed to be not only locally integrable on $[0,\infty)$ but also "non-degenerate" in the sense that

$$\int_0^\infty \frac{v(s)}{U(s)^p + U(t)^p}\,\mathrm{d}s < \infty \quad \text{for every } t \in (0,\infty) \tag{4.3}$$

and

$$\int_0^1 \frac{v(s)}{U(s)^p}\,\mathrm{d}s = \int_1^\infty v(s)\,\mathrm{d}s = \infty. \tag{4.4}$$

With use of different techniques, explicit manageable equivalent expressions for the optimal constant in (4.1) were also obtained in [99, 100] and later in [44]. While they do not involve any nondegeneracy restrictions on the weights, the range of the parameters $p$ and $q$ is limited to $p \in (1,\infty)$ and $q = 1$ instead.

The novelty of our results lies in the fact that, while the entire range of the parameters is covered, there are *no extra assumptions on the weights* $u$, $v$, and $w$, except for the natural one that they are locally integrable on $[0,b)$. Although (4.3) can be assumed without loss of generality (see [66, Proof of Theorem 4.1]), (4.4) effectively rules out, apart from the weights on $(0,\infty)$ that are really "degenerate" in some sense, the possibility of using their result in the case $b<\infty$. Since the case $b<\infty$ naturally appears when the function spaces $\Gamma^p_u(v)$ and $\Lambda^q(w)$ are considered over measure spaces of finite measure, this restriction is important. It is worth noting that the obvious idea of replacing (4.4) with

$$\int_0^c \frac{v(s)}{U(s)^p}\,\mathrm{d}s = \int_c^b v(s)\,\mathrm{d}s = \infty \quad \text{for some } c\in(0,b)$$

does not work, because the second integral is typically finite when $b<\infty$. In fact, we came across this restriction, and the need for eliminating it, in the joint paper [20] when we were characterizing some particular examples of compactness results for traces of Sobolev functions onto low-dimensional sets. The interested reader is referred to [66, Remark 4.6] for more information.

In [66], we also used the discretization and antidiscretization technique, but no nondegeneracy restrictions on the weights are imposed, and, at the same time, the entire range of the parameters $p,q\in(0,\infty)$ is covered.

**Theorem 4.1** ([66, Theorem 4.1])**.** *Let $p,q\in(0,\infty)$. Let $u,v,w\in\mathfrak{M}^+(0,b)$ be locally integrable $[0,b)$, with $u$ being positive a.e. in $(0,b)$. Let $C$ be the optimal constant in* (4.1)*, that is, $C$ is defined by* (4.2)*.*

(i) *If $1\le q$ and $p\le q<\infty$, then*

$$C \approx \sup_{0<t<b} \frac{W(t)^{\frac1q}}{\left(V(t)+U(t)^p\int_t^b v(s)U(s)^{-p}\,\mathrm{d}s\right)^{\frac1p}}.$$

(ii) *If $1\le q<p<\infty$, then*

$$\begin{aligned} C \approx{}& \left(\int_0^b \frac{V(t)\int_t^b v(s)U(s)^{-p}\,\mathrm{d}s\, U(t)^{\frac{pq}{p-q}+p-1}u(t)\sup_{\tau\in[t,b)}U(\tau)^{-\frac{pq}{p-q}}W(\tau)^{\frac{p}{p-q}}}{\left(V(t)+U(t)^p\int_t^b v(s)U(s)^{-p}\,\mathrm{d}s\right)^{\frac{q}{p-q}+2}}\,\mathrm{d}t\right)^{\frac{p-q}{pq}} \\ &+ \left(\lim_{t\to0^+}\frac{U^p(t)}{V(t)+U(t)^p\int_t^b v(s)U(s)^{-p}\,\mathrm{d}s}\right)^{\frac1p}\left(\sup_{t\in(0,b)}\frac{W(t)}{U(t)^q}\right)^{\frac1q} \\ &+ \left(\lim_{t\to b^-}\frac{1}{V(t)+U(t)^p\int_t^b v(s)U(s)^{-p}\,\mathrm{d}s}\right)^{\frac1p} W(b)^{\frac1q}. \end{aligned}$$

(iii) *If $p\le q<1$, then*

$$C \approx \sup_{0<t<b} \frac{W(t)^{\frac1q}+U(t)\left(\int_t^b W(s)^{\frac{q}{1-q}}w(s)U(s)^{-\frac{q}{1-q}}\,\mathrm{d}s\right)^{\frac{1-q}{q}}}{\left(V(t)+U(t)^p\int_t^b v(s)U(s)^{-p}\,\mathrm{d}s\right)^{\frac1p}}.$$

(iv) *If $q < 1$ and $q < p < \infty$, then*

$$C \approx \left( \lim_{t\to 0^+} \frac{U(t)^p}{V(t) + U(t)^p \int_t^b v(s)U(s)^{-p}\,\mathrm{d}s} \right)^{\frac{1}{p}} \left( \int_0^b W(t)^{\frac{q}{1-q}} w(t) U(t)^{-\frac{q}{1-q}}\,\mathrm{d}t \right)^{\frac{1-q}{q}}$$
$$+ \left( \lim_{t\to b^-} \frac{1}{V(t) + U(t)^p \int_t^b v(s)U(s)^{-p}\,\mathrm{d}s} \right)^{\frac{1}{p}} \left( \int_0^b W(t)^{\frac{q}{1-q}} w(t)\,\mathrm{d}t \right)^{\frac{1-q}{q}}$$
$$+ \left( \int_0^b \frac{\left( W(t)^{\frac{1}{1-q}} + U(t)^{\frac{q}{1-q}} \int_t^b W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\,\mathrm{d}s \right)^{\frac{p(1-q)}{p-q}}}{\left( V(t) + U(t)^p \int_t^b v(s)U(s)^{-p}\,\mathrm{d}s \right)^{\frac{q}{p-q}+2}} \times V(t)U(t)^{p-1}u(t) \int_t^b v(s)U(s)^{-p}\,\mathrm{d}s\,\mathrm{d}t \right)^{\frac{p-q}{pq}}.$$

*The equivalence constants depend only on the parameters $p$ and $q$. In particular, they are independent of the weights $u$, $v$, and $w$.*

In [66], we also characterized the optimal constant $C \in [0, \infty]$ in the weak variant of (4.1) corresponding to $p = \infty$, that is,

$$\left( \int_0^b f^*(t)^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \leq C \operatorname*{ess\,sup}_{t\in(0,b)} \left( \frac{1}{U(t)} \int_0^t f^*(s)u(s)\,\mathrm{d}s \right) v(t) \tag{4.5}$$

for every $f \in \mathfrak{M}^+(0,b)$, *without any nondegeneracy restrictions on the weights.* As with (4.1), explicit manageable expressions for the optimal constant in (4.5) were already established in [46] for every $q \in (0, \infty)$ but with nondegeneracy restrictions again.

From the point of view of function spaces, the optimal constant in (4.5) is equal to the embedding constant of the embedding $\Gamma_u^\infty(v) \hookrightarrow \Lambda^q(w)$, that is,

$$C = \sup_{\|f\|_{\Gamma_u^\infty(v)} \leq 1} \|f\|_{\Lambda^q(w)}. \tag{4.6}$$

The weak Gamma space $\Gamma_u^\infty(v)$ (cf. [18]) is defined as

$$\Gamma_u^\infty(v) = \{ f \in \mathfrak{M}(R,\mu) : \|f\|_{\Gamma_u^\infty(v)} < \infty \},$$

where

$$\|f\|_{\Gamma_u^\infty(v)} = \operatorname*{ess\,sup}_{t\in(0,b)} \left( \frac{1}{U(t)} \int_0^t f^*(s)u(s)\,\mathrm{d}s \right) v(t).$$

The characterization in the following theorem is slightly less explicit than that in Theorem 4.1 because it is expressed in terms of a so-called representation measure (see (4.8)) of the fundamental function of $\Gamma_u^\infty(v)$, which is defined as

$$\varphi(t) = \operatorname*{ess\,sup}_{\tau\in(0,t)} U(\tau) \operatorname*{ess\,sup}_{s\in(\tau,b)} \frac{v(s)}{U(s)},\ t \in (0,b). \tag{4.7}$$

The reason behind this is the fact that the fundamental function of $\Gamma_u^\infty(v)$ is a supremum, whereas that of $\Gamma_u^p(v)$, $p \in (0,\infty)$, is an integral—namely,

$$\varphi(t) = \int_0^b \min\{U(t)^p, U(s)^p\} \frac{v(s)}{U(s)^p} \,\mathrm{d}s, \ t \in (0,b).$$

However, a representation (4.8) always exists, and it is usually easy to obtain such a representation in an explicit form (see [41, Chapter 2] and [66, Remark 4.4] for more information).

**Theorem 4.2** ([66, Theorem 4.3]). *Let $q \in (0,\infty)$. Let $u, v, w \in \mathfrak{M}^+(0,b)$ be locally integrable on $[0,b)$, with $u$ being positive a.e. in $(0,b)$. Let $C$ be the optimal constant in (4.5), that is, $C$ is defined by (4.6). Let $\varphi$ be the function defined by (4.7). Let $B_1, B_2 \in (0,\infty)$, $\gamma, \delta \in [0,\infty)$, and $\nu$ a nonnegative Borel measure on $(0,b)$ such that*

$$B_1\varphi(t) \le \gamma + \delta U(t) + \int_{(0,b)} \min\{U(t), U(s)\} \,\mathrm{d}\nu(s) \le B_2\varphi(t) \tag{4.8}$$

*for every $t \in (0,b)$.*

(i) *If $1 \le q < \infty$, then*

$$\begin{aligned} C \approx &\left(\lim_{t\to 0^+} \frac{U(t)}{\varphi(t)}\right)\left(\sup_{t\in(0,b)} \frac{W(t)^{\frac{1}{q}}}{U(t)}\right) + \lim_{t\to b^-} \frac{1}{\varphi(t)} W(b)^{\frac{1}{q}} \\ &+ \left(\int_0^b U(t)^q \left(\sup_{\tau\in(t,b)} \frac{W(\tau)}{U(\tau)^q}\right) \varphi(t)^{-(q+2)} u(t) \right. \\ &\quad \left. \times \left(\gamma + \int_{(0,t]} U(s)\,\mathrm{d}\nu(s)\right)\left(\delta + \int_{[t,b)} \mathrm{d}\nu(s)\right) \mathrm{d}t\right)^{\frac{1}{q}}. \end{aligned}$$

(ii) *If $0 < q < 1$, then*

$$\begin{aligned} C \approx &\left(\lim_{t\to 0^+} \frac{U(t)}{\varphi(t)}\right)\left(\int_0^b W(t)^{\frac{q}{1-q}} w(t) U(t)^{-\frac{q}{1-q}} \mathrm{d}t\right)^{\frac{1-q}{q}} + \lim_{t\to b^-} \frac{1}{\varphi(t)} W(b)^{\frac{1}{q}} \\ &+ \left(\int_0^b \xi(t)\varphi(t)^{-(q+2)} u(t) \left(\gamma + \int_{(0,t]} U(s)\,\mathrm{d}\nu(s)\right)\left(\delta + \int_{[t,b)} \mathrm{d}\nu(s)\right) \mathrm{d}t\right)^{\frac{1}{q}}, \end{aligned}$$

*where*

$$\xi(t) = \left(\int_0^b W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}} \min\{U(t)^{\frac{q}{1-q}}, U(s)^{\frac{q}{1-q}}\} \,\mathrm{d}s\right)^{1-q}, \ t \in (0,b).$$

*The equivalence constants depend only on the parameter $q$ and on the constants $B_1$ and $B_2$. In particular, they are independent of the weights $u$, $v$, and $w$.*

## 4.2 Superposition of the weighted Hardy and Copson operators

In [52], we characterized the optimal constant $C \in [0,\infty]$, up to equivalences, in the inequality

$$\left(\int_a^b \left(\int_t^b \left(\int_a^s f(\tau)^p v(\tau)\,\mathrm{d}\tau\right)^{\frac{q}{p}} u(s)\,\mathrm{d}s\right)^{\frac{r}{q}} w(t)\,\mathrm{d}t\right)^{\frac{1}{r}} \le C\int_a^b f(t)\,\mathrm{d}t \tag{4.9}$$

for every $f \in \mathfrak{M}^+(a,b)$, where $u,v,w \in \mathfrak{M}^+(a,b)$ are a.e. positive, $p \in (0,1]$, and $q,r \in (0,\infty)$. The fact that $p \in (0,1]$ is actually no restriction at all, because if $p > 1$, then it is always possible to find a function $f \in L^1(a,b)$ such that $f^p v \notin L^1(a,s)$ for every $s > a$; in other words, the left-hand side is infinite for such a function, whereas the right-hand side is finite.

The inequality (4.9) can be viewed as a weighted inequality for a superposition of two different types of integral operators, namely of Hardy and Copson types. The importance of studying weighted inequalities for compositions of operators of Hardy and Copson types, whether of the same or different types, comes from various sources. Several of them is pointed out in the introductory section of [52] and references therein.

Partial cases of the inequality (usually in its transformed form (4.10) below) were already studied before but under various restrictions, such as on the weights (cf. [49, 50]) or on the parameters (cf. [9, 85, 86]). Similarly to the previous section, the novelty of our result is that the characterization is complete without any unnecessary restrictions, whether on the parameters, on the weights, or on the underlying interval $(a,b)$. We removed those previous restrictions by carefully combining discrete Hardy inequalities with discretization of the inequality done in such a way that no nondegeneracy restrictions on the weights were brought in. At the same time, the discretization method that we used enabled us to completely avoid duality techniques, and so the entire range of the parameters is covered. Moreover, the equivalent expressions for $C$ are explicit and manageable thanks to the antidiscretization done after the discretization part.

Before we state our main result, it is worth noting that the inequality (4.9) can be turned into other important inequalities. For example, by performing the change of variables $\tau \mapsto -\tau$ in the innermost integral and replacing the interval $(a,b)$ with $(-b,-a)$, it is possible to switch the order of the two operators in (4.9)—it then becomes

$$\left(\int_a^b \left(\int_a^t \left(\int_s^b f(\tau)^p v(\tau)\,\mathrm{d}\tau\right)^{\frac{q}{p}} u(s)\,\mathrm{d}s\right)^{\frac{r}{q}} w(t)\,\mathrm{d}t\right)^{\frac{1}{r}} \le C\int_a^b f(t)\,\mathrm{d}t.$$

A second example is

$$\left(\int_a^b \left(\int_t^b \left(\int_a^s f(\tau)\,\mathrm{d}\tau\right)^{q} u(s)\,\mathrm{d}s\right)^{\frac{r}{q}} w(t)\,\mathrm{d}t\right)^{\frac{1}{r}} \le C\left(\int_a^b f(t)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{4.10}$$

with $p \in [1,\infty)$, which can be obtained from (4.9) by replacing (in this order) $f$ with $f^{\frac{1}{p}} v^{-\frac{1}{p}}$, $v$ with $v^{-p}$, $q$ with $qp$, $r$ with $rp$, and $p$ with $1/p$. Therefore, the following theorem can be easily reformulated to provide equivalent expressions also for the optimal constants in these inequalities.

**Theorem 4.3** ([52, Theorem A])**.** *Let $p \in (0,1]$, $q, r \in (0,\infty)$, and $u, v, w \in \mathfrak{M}^+(a,b)$ be a.e. positive. For $t \in (a,b)$, define the function $V_{a,p}$ as*

$$V_{a,p}(t) = \begin{cases} \left( \int_a^t v(s)^{\frac{1}{1-p}} \,\mathrm{d}s \right)^{\frac{1-p}{p}} & \text{if } 0 < p < 1, \\ \operatorname{ess\,sup}_{s\in(a,t)} v(s) & \text{if } p = 1. \end{cases}$$

*The optimal constant $C$ in the inequality* (4.9)*, that is,*

$$C = \sup_{\|f\|_{L^1(a,b)} \le 1,\, f \ge 0} \left( \int_a^b \left( \int_t^b \left( \int_a^s f(\tau)^p v(\tau) \,\mathrm{d}\tau \right)^{\frac{q}{p}} u(s) \,\mathrm{d}s \right)^{\frac{r}{q}} w(t) \,\mathrm{d}t \right)^{\frac{1}{r}},$$

*satisfies*

$$C \approx \begin{cases} C_1 + C_2 & \text{if } r \ge 1 \text{ and } q \ge 1, \\ C_2 + C_3 & \text{if } r \ge 1 \text{ and } q < 1, \\ C_4 + C_5 & \text{if } r < 1 \text{ and } q \ge 1, \\ C_5 + C_6 & \text{if } r < 1 \text{ and } q < 1, \end{cases}$$

*where*

$$C_1 = \sup_{t\in(a,b)} \left( \int_a^t w(s) \,\mathrm{d}s \right)^{\frac{1}{r}} \operatorname*{ess\,sup}_{s\in(t,b)} \left( \int_s^b u(\tau) \,\mathrm{d}\tau \right)^{\frac{1}{q}} V_{a,p}(s)$$

$$C_2 = \sup_{t\in(a,b)} \left( \int_t^b w(s) \left( \int_s^b u(\tau) \,\mathrm{d}\tau \right)^{\frac{r}{q}} \mathrm{d}s \right)^{\frac{1}{r}} V_{a,p}(t)$$

$$C_3 = \sup_{t\in(a,b)} \left( \int_a^t w(s) \,\mathrm{d}s \right)^{\frac{1}{r}} \left( \int_t^b \left( \int_s^b u(\tau) \,\mathrm{d}\tau \right)^{\frac{q}{1-q}} u(s) V_{a,p}(s)^{\frac{q}{1-q}} \,\mathrm{d}s \right)^{\frac{1-q}{q}}$$

$$C_4 = \left( \int_a^b \left( \int_a^t w(s) \,\mathrm{d}s \right)^{\frac{r}{1-r}} w(t) \operatorname*{ess\,sup}_{s\in(t,b)} \left( \int_s^b u(\tau) \,\mathrm{d}\tau \right)^{\frac{r}{q(1-r)}} V_{a,p}(s)^{\frac{r}{1-r}} \,\mathrm{d}t \right)^{\frac{1-r}{r}}$$

$$C_5 = \left( \int_a^b \left( \int_t^b w(s) \left( \int_s^b u(\tau) \,\mathrm{d}\tau \right)^{\frac{r}{q}} \mathrm{d}s \right)^{\frac{r}{1-r}} w(t) \left( \int_t^b u(\tau) \,\mathrm{d}\tau \right)^{\frac{r}{q}} V_{a,p}(t)^{\frac{r}{1-r}} \,\mathrm{d}t \right)^{\frac{1-r}{r}},$$

*and*

$$C_6 = \left( \int_a^b \left( \int_a^t w(s) \,\mathrm{d}s \right)^{\frac{r}{1-r}} w(t) \left( \int_t^b \left( \int_s^b u(\tau) \,\mathrm{d}\tau \right)^{\frac{q}{1-q}} u(s) V_{a,p}(s)^{\frac{q}{1-q}} \,\mathrm{d}s \right)^{\frac{r(1-q)}{q(1-r)}} \mathrm{d}t \right)^{\frac{1-r}{r}}.$$

# Chapter 5

# Quantitative aspects of noncompact Sobolev embeddings

This chapter describes the main results from [38, 68].

In the papers [19, 20], we investigated the question of compactness of Sobolev (trace) embeddings on (regular enough) bounded domains in $\mathbb{R}^d$, $d \geq 2$, in the general framework of rearrangement-invariant function spaces in the situation where the target space is endowed with an (upper) Ahlfors regular measure. The results from these papers considerably extend those from [59], where the measure is the Lebesgue measure, and are complementary to those from [101], where the measure is related to the isoperimetric function of the (possibly irregular) underlying domain. The results in all these papers are qualitative—they address the question of whether the considered Sobolev embedding is compact or not.

Opposite to the aforementioned papers, the papers [38, 68] focus on quantitative aspects of compactness. Whereas there are a large number of quantitative results for compact Sobolev embeddings (e.g., [32, 36, 63, 93, 105]), much less seems to be known about noncompact Sobolev embeddings. In [38, 68], we considered certain noncompact Sobolev embeddings and addressed the question of how much noncompact they are.

Loosely speaking, there are three usual reasons why a Sobolev embedding is noncompact. Namely,

*(i)* the underlying domain is too irregular (e.g., [76, 77]);

*(ii)* the target space is too small (in other words, the norm of the target space is too strong), that is, the target space is optimal or “almost optimal” (e.g., [59, 70]);

*(iii)* the underlying domain is “too unbounded” (e.g., [1]).

The results described in this chapter fall into the categories *(ii)* and *(iii)*. The interested reader is referred to [67] for some quantitative results when the lack of compactness is caused by the (ir)regularity of the underlying domain. The interested reader is also referred to [70], where we qualitatively studied the lack of compactness caused by *(ii)* in the general framework of rearrangement-invariant spaces.

One of the standard ways of measuring (non)compactness of operators between Banach spaces is *the (ball) measure of noncompactness.* The measure of noncompactness $\beta(T)$ of a bounded linear operator $T$ from a Banach space $X$ to a Banach space $Y$ (in short, $T \in B(X,Y)$) is defined as

$$\beta(T) = \lim_{n\to\infty} e_n(T),$$

where $e_n(T)$ is the $n$th *entropy number* of the operator $T \in B(X,Y)$ defined as

$$e_n(T) = \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{n-1} \text{ balls in } Y \text{ with radius } \varepsilon\}.$$

Here $B_X$ denotes the (closed) unit ball of $X$. Since the sequence $\{e_n(T)\}_{n=1}^{\infty}$ is nonincreasing, the limit always exists. Furthermore, it can be easily observed that $0 \leq \beta(T) \leq \|T\|$ (the operator norm of $T \in B(X,Y)$), and that $T$ is compact if and only if $\beta(T) = 0$.

There is also an important axiomatic way of measuring the quality of operators between Banach spaces by means of so-called *s-numbers*, introduced by A. Pietsch in [91]. A (strict) $s$-number is any rule $s$ that assigns to each bounded linear operator $T \in B(X,Y)$ a sequence $\{s_n(T)\}_{n=1}^{\infty}$ of nonnegative numbers having, for every $n \in \mathbb{N}$, the following properties:

(S1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$;

(S2) $s_n(S+T) \leq s_n(S) + \|T\|$ for every $S \in B(X,Y)$;

(S3) $s_n(BTA) \leq \|B\| s_n(T) \|A\|$ for every $A \in B(W,X)$ and $B \in B(Y,Z)$, where $W, Z$ are Banach spaces;

(S4) $s_n(I\colon E \to E) = 1$ for every Banach space $E$ with $\dim E \geq n$;

(S5) $s_n(T) = 0$ if $\operatorname{rank} T < n$.

The $s$-numbers as defined here are sometimes referred to as *strict* $s$-numbers. "Non-strict" $s$-numbers are then define as the strict ones but in the property (S4) $E$ is replaced by $\mathbb{R}^n$ (with the Euclidean norm). Examples of (strict) $s$-numbers are the approximation numbers $a_n$, the isomorphism numbers $i_n$, the Gelfand numbers $c_n$, the Bernstein numbers $b_n$, the Kolmogorov numbers $d_n$, or the Mityagin numbers $m_n$. We will meet the Bernstein numbers, whose definition is given below. The interested reader is referred to [33, 34] for the definitions of the other (strict) $s$-numbers as well as of some "non-strict" ones.

Even though entropy numbers possess similar properties to those of (strict) $s$-number, they are not $s$-numbers (e.g., property (S4) is violated ([33, Chapter 2, Proposition 1.3])). The interested reader is referred to [16, 33, 36] for more information about entropy numbers and $s$-number and how they are connected to spectral properties of differential operators, and to [92] for more information about the history of these quantities.

The *Bernstein numbers* $b_n$ of an operator $T \in B(X,Y)$ are defined as

$$b_n(T) = \sup_{X_n \subseteq X} \inf_{\substack{x\in X_n \\ \|x\|_X = 1}} \|Tx\|_Y,$$

where the supremum extends over all $n$-dimensional subspaces of $X$. From the point of view of the approximation theory, Bernstein numbers are useful for proving lower bounds (e.g., [29, 31, 64, 93]). From the point of view of the operator theory, the Bernstein numbers are related to the concept of strictly singular and finitely strictly singular operators, which are classes of operators that need not be compact but whose behavior is still in some sense better than that of merely bounded operators.

An operator $T \in B(X, Y)$ is said to be *strictly singular* if it is not bounded from below on any (closed) infinite dimensional subspace of $X$. In other words, for every infinite dimensional (closed) subspace $Z$ of $X$,

$$\inf \{\|Tx\|_Y : \|x\|_X = 1, x \in Z\} = 0.$$

If this unboundedness from below is in a sense uniform over finite dimensional subspaces, the operator is called finitely strictly singular. More precisely, an operator $T \in B(X, Y)$ is said to be *finitely strictly singular* if it has the property that given any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that, if $Z$ is a subspace of $X$ with $\dim Z \geq N(\varepsilon)$, then there exists $x \in Z$, $\|x\|_X = 1$, such that $\|Tx\|_Y \leq \varepsilon$. This can be equivalently reformulated in terms of the Bernstein numbers of $T$. The operator $T$ is finitely strictly singular if and only if

$$\lim_{n\to\infty} b_n(T) = 0.$$

The relation between these classes of operators is:

$$T \text{ is compact } \implies T \text{ is finitely strictly singular } \implies T \text{ is strictly singular},$$

with each converse implication being false in general. The interested reader is referred to [4, 71, 94] for more information about (finitely) strictly singular operators.

## 5.1 Quantitative results

In the paper [68], we quantitatively studied two noncompact Sobolev embeddings, namely

$$I\colon V_0^{m,p}(\Omega) \to L^{p^*}(\Omega) \tag{5.1}$$

and

$$I\colon V_0^{m,p}(\Omega) \to L^{p^*,p}(\Omega). \tag{5.2}$$

Here $d, m \in \mathbb{N}$, $1 \leq m < d$, $p \in [1, d/m)$, $p^* = dp/(n - mp)$, $\Omega \subseteq \mathbb{R}^d$ is a bounded domain, and $I$ is the identity operator. The Sobolev space $V_0^{m,p}(\Omega)$ is a Banach space of all $m$times weakly differentiable functions in $\Omega$ whose continuation by 0 outside $\Omega$ is $m$times weakly differentiable function and whose $m$th order weak partial derivatives belong to the Lebesgue space $L^p(\Omega)$. The space $V_0^{m,p}(\Omega)$ is endowed with the norm $\|u\|_{V_0^{m,p}(\Omega)} = \||\nabla^m u|_{\ell_p}\|_{L^p(\Omega)}$, where $\nabla^m u$ is the vector of all $m$th order weak derivatives of $u$.

Both target spaces in (5.1) and (5.2) are in a sense optimal. It is well known that the Lebesgue space $L^{p^*}(\Omega)$ is the optimal target space in (5.1) among all Lebesgue spaces (i.e., the exponent $p^*$ cannot be replaced by a bigger one). However, if more general target function spaces are considered, (5.1) can be improved to (5.2). The latter is indeed an improvement because the Lorentz space $L^{p^*,p}(\Omega)$ is strictly smaller than the Lebesgue space $L^{p^*}(\Omega)$, inasmuch as $1 \le p < p^*$. In fact, the Lorentz space $L^{p^*,p}(\Omega)$ is actually the optimal target space in (5.2) among all rearrangement-invariant function spaces ([58])—that is, if (5.2) is valid with $L^{p^*,p}(\Omega)$ replaced by a rearrangement-invariant funtion space $Y(\Omega)$, then $L^{p^*,p}(\Omega) \hookrightarrow Y(\Omega)$.

It was proved in [10, 57] that the measure of noncompactness of both embeddings (5.1) and (5.2) is the worst possible, that is, it coincides with their norms. From this point of view, the embeddings are *maximally noncompact*. Moreover, even when the Lebesgue space $L^{p^*}(\Omega)$ is enlarged to the weak Lebesgue space $L^{p^*,\infty}(\Omega)$, the Sobolev embedding $I\colon V_0^{m,p}(\Omega) \to L^{p^*,\infty}(\Omega)$ is still maximally noncompact in this sense ([69]). Nevertheless, we proved in [68] that there is a quantitative difference between the noncompactness of (5.1) and that of (5.2). This difference is captured by different behavior of the corresponding Bernstein numbers.

On the one hand, measured by means of the Bernstein numbers, the noncompactness of the "really optimal" Sobolev embedding (5.2) is again the worst possible.

**Theorem 5.1** ([68, Theorem 3.4])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty bounded open set, $m \in \mathbb{N}$, $1 \le m < d$, and $p \in [1, d/m)$. Let $I$ be the identity operator $I\colon V_0^{m,p}(\Omega) \to L^{p^*,p}(\Omega)$, where $p^* = dp/(d-mp)$. Then*

$$b_n(I) = \|I\| \qquad \text{for every } n \in \mathbb{N},$$

*where $\|I\|$ denotes the operator norm. Furthermore, $I$ is not strictly singular.*

On the other hand, when (5.1) is considered instead of (5.2), the corresponding Bernstein numbers decay to 0. Therefore, there is a quantitative difference between the noncompactness of (5.1) and (5.2). In the particular case $m = p = 1$, the upper estimate in the following theorem was proved in [11].

**Theorem 5.2** ([68, Theorem 3.3])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty bounded open set, $m \in \mathbb{N}$, $1 \le m < d$, and $p \in [1, d/m)$. Let $I$ be the identity operator $I\colon V_0^{m,p}(\Omega) \to L^{p^*}(\Omega)$, where $p^* = dp/(d-mp)$. There exists $n_0 \in \mathbb{N}$, depending only on $d$ and $m$, such that*

$$C_1 n^{-\frac{m}{d}} \le b_n(I) \le C_2 n^{-\frac{m}{d}} \qquad \text{for every } n \ge n_0.$$

*Here $C_1$ and $C_2$ are constants depending only on $d$, $m$, and $p$. In particular, $I$ is finitely strictly singular.*

Finally, we turn our attention to [38].

In [38], we quantitatively studied a Sobolev embedding on an infinite strip in $\mathbb{R}^d$, $d \ge 2$. The Sobolev embedding in question is

$$I\colon W_0^1 L^p(\Omega) \to L^p(\Omega), \tag{5.3}$$

where $p \in (1, \infty)$ and

$$\Omega = \mathbb{R}^k \times \prod_{j=1}^{d-k} (q_j, r_j), \tag{5.4}$$

$-\infty < q_j < r_j < \infty$, $k \in \{1, \dots, d-1\}$, is an infinite strip in $\mathbb{R}^d$. The Sobolev space $W_0^1 L^p(\Omega)$ is defined as the closure of smooth functions compactly supported in $\Omega$ with respect to the norm $\|u\|_{W_0^1 L^p(\Omega)} = (\|u\|_{L^p(\Omega)}^p + \||\nabla u|_{\ell_p}\|_{L^p(\Omega)}^p)^{1/p}$.

What attracted our attention to this embedding was that, while (5.3) is an example of a simple Sobolev embedding where the lack of compactness is caused by the unboundedness of the domain, it was open what the exact value of its measure of noncompactness was or what the behavior of its $s$-numbers was. In [38], we were able to establish the exact values of these quantities.

**Theorem 5.3** ([38, Theorem 3.9]). *Let $p \in (1, \infty)$ and $\Omega$ be as in (5.4). We have*

$$s_n(I) = \beta(I) = \left(1 + \left(\frac{2\pi}{p \sin(\frac{\pi}{p})}\right)^p (p-1) \sum_{j=1}^{d-k} \frac{1}{(r_j - q_j)^p}\right)^{-\frac{1}{p}}$$

*for every $n \in \mathbb{N}$ and every strict $s$-number $s$, where $I$ stands for the identity operator $I \colon W_0^{1,p}(\Omega) \to L^p(\Omega)$.*

Although it is not mentioned in the paper [38], the proof of [38, Theorem 3.9] also shows that the Sobolev embedding (5.3) is not strictly singular. The reason is that, for any fix $\varepsilon > 0$, using the construction at the beginning of the proof, we find a system of functions $\{u_j\}_{j=1}^{\infty}$ having mutually disjoint supports and satisfying $\|u_j\|_{L^p(\Omega)} = \|I\|/(1+\varepsilon)$ and $\|u\|_{W_0^{1,p}(\Omega)} = 1$. Then we just need to observe that the identity operator $I \colon W_0^{1,p}(\Omega) \to L^p(\Omega)$ is bounded from below on the infinite dimensional subspace of $W_0^{1,p}(\Omega)$ spanned by the system.

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# Appendix A

# Attached publications

The following publications are included (listed in the order they appear):

[Paper A] Z. Mihula. Optimal behavior of weighted Hardy operators on rearrangement-invariant spaces. *Math. Nachr.*, 296(8):3492–3538, 2023. doi: 10.1002/mana.202200015.

[Paper B] S. Baena-Miret, A. Gogatishvili, Z. Mihula, and L. Pick. Reduction principle for Gaussian $K$-inequality. *J. Math. Anal. Appl.*, 516(2):Paper No. 126522, 23 pp., 2022. doi: 10.1016/j.jmaa.2022.126522.

[Paper C] M. Křepela, Z. Mihula, and H. Turčinová. Discretization and antidiscretization of Lorentz norms with no restrictions on weights. *Rev. Mat. Complut.*, 35(2):615–648, 2022. doi: 10.1007/s13163-021-00399-7.

[Paper D] A. Gogatishvili, Z. Mihula, L. Pick, H. Turčinová, and T. Ünver. Weighted inequalities for a superposition of the Copson operator and the Hardy operator. *J. Fourier Anal. Appl.*, 28(2):Paper No. 24, 24 pp., 2022. doi: 10.1007/s00041-022-09918-6.

[Paper E] J. Lang and Z. Mihula. Different degrees of non-compactness for optimal Sobolev embeddings. *J. Funct. Anal.*, 284(10):Paper No. 109880, 22 pp., 2023. doi: 10.1016/j.jfa.2023.109880.

[Paper F] D. E. Edmunds, J. Lang, and Z. Mihula. Measure of noncompactness of Sobolev embeddings on strip-like domains. *J. Approx. Theory*, 269:Paper No. 105608, 13 pp., 2021. doi: 10.1016/j.jat.2021.105608.

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MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

# Optimal behavior of weighted Hardy operators on rearrangement-invariant spaces

**Zdeněk Mihula**[1,2]

[1]Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Czech Republic

[2]Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Czech Republic

**Correspondence**
Zdeněk Mihula, Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Praha 6, Czech Republic.
Email: mihulzde@fel.cvut.cz

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**Abstract**

The behavior of certain weighted Hardy-type operators on rearrangement-invariant function spaces is thoroughly studied. Emphasis is put on the optimality of the obtained results. First, the optimal rearrangement-invariant function spaces guaranteeing the boundedness of the operators from/to a given rearrangement-invariant function space are described. Second, the optimal rearrangement-invariant function norms being sometimes complicated, the question of whether and how they can be simplified to more manageable expressions is addressed. Next, the relation between optimal rearrangement-invariant function spaces and interpolation spaces is investigated. Last, iterated weighted Hardy-type operators are also studied.

**KEYWORDS**

iterated operators, optimal spaces, rearrangement-invariant spaces, supremum operators, weighted Hardy operators

**MSC (2020)**

46E30, 47G10

## 1 | INTRODUCTION

In this paper, we thoroughly study the behavior of Hardy-type operators $R_{u,v,\nu}$ and $H_{u,v,\nu}$ on rearrangement-invariant function spaces, focusing on the optimality of our results. The Hardy-type operators are defined for measurable functions $g$ on $(0, L)$, $L \in (0, \infty]$, as

$$R_{u,v,\nu}g(t) = v(t)\int_0^{\nu(t)} |g(s)|u(s)\,ds,\ t \in (0, L), \tag{1.1}$$

and

$$H_{u,v,\nu}g(t) = u(t)\int_{\nu(t)}^{L} |g(s)|v(s)\,ds,\ t \in (0, L). \tag{1.2}$$

Here, $u, v$ are nonnegative nonincreasing functions on $(0, L)$ and $\nu$ is an increasing bijection of the interval $(0, L)$ onto itself. Recall that rearrangement-invariant function spaces are, loosely speaking, Banach spaces of functions whose norms are invariant with respect to measure-preserving rearrangements/transformations of functions. Rearrangement-invariant function spaces constitute a broad class of function spaces. Some classical examples of rearrangement-invariant function spaces are Lebesgue spaces, Orlicz spaces, or Lorentz(–Zygmund) spaces to name a few. Precise definitions as well as some preliminary results and notations used in this paper are presented in Section 2.

First, let $T$ be either of the operators and $X(0,L)$ a rearrangement-invariant function space over the interval $(0,L)$. We characterize the optimal domain and the optimal target rearrangement-invariant function space $Y(0,L)$ for $X(0,L)$ and $T$. By that we mean the following. We describe the weakest rearrangement-invariant function norm $\|\cdot\|_{Y(0,L)}$ for which there is a positive constant $C$ such that $\|Tf\|_{X(0,L)} \leq C\|f\|_{Y(0,L)}$ for every $f \in Y(0,L)$. We also describe the strongest rearrangement-invariant function norm $\|\cdot\|_{Y(0,L)}$ for which there is a positive constant $C$ such that $\|Tf\|_{Y(0,L)} \leq C\|f\|_{X(0,L)}$ for every $f \in X(0,L)$. In other words, we characterize the largest and the smallest rearrangement-invariant function space $Y(0,L)$ such that $T$ is bounded from $Y(0,L)$ to $X(0,L)$ and from $X(0,L)$ to $Y(0,L)$, respectively. This is the content of Section 3. As a simple corollary, we also obtain a description of the optimal rearrangement-invariant function spaces for a sum of the two operators, each with possibly different functions $u, v, \nu$. The description is less explicit than it could be if we studied directly the sum, though. Next, in Section 4, we take a close look at how to simplify the description of these optimal rearrangement-invariant function norms and whether it is possible at all. The motivation behind this is simple: The simpler and more manageable description we have at our disposal, the more useful it is. It turns out that this problem is more complex than it may appear at first glance. It leads us to studying a certain supremum operator, and it is closely related to the notion of interpolation spaces. Next, in Section 5, we investigate the optimal behavior of iterated Hardy-type operators—namely, $R_{u_1,v_1,\nu_1} \circ R_{u_2,v_2,\nu_2}$ and $H_{u_1,v_1,\nu_1} \circ H_{u_2,v_2,\nu_2}$—on rearrangement-invariant function spaces. These iterated operators naturally arise when one studies the question of whether iteration of optimal function spaces leads to an optimal function space. Last, in Section 6, we present some concrete examples of optimal rearrangement-invariant function spaces when $X(0,L)$ is a Lorentz–Zygmund space.

In considerably less general settings, the questions mentioned in the preceding paragraph were already studied, see [20–23, 28, 30, 39, 40, 46, 58] and references therein. However, those results are limited to some particular choices of the functions $u$, $v$, and $\nu$—namely, $u \equiv 1$ and $v, \nu$ being power functions for the most part, but see [33, 37]. Moreover, they are also scattered and often hidden somewhere between the lines with varying degrees of generality. The aim of this paper is to thoroughly address the questions in a coherent unified way and in considerable generality. Not only do the results obtained here encompass their already-known particular cases, but they also provide a general theory suitable for various future applications. Some are outlined at the end of this introductory section.

General as the results in this paper are, we do usually impose some mild restrictions on the functions $u$, $v$, $\nu$ so that we can obtain interesting, strong results. However, the imposed assumptions on the functions are actually not too restrictive for the most part and often exclude only cases being in a way pathological. The assumptions also often reflect the very forms of the Hardy-type operators considered here. In particular, the operators do not involve kernels. Indisputably, Hardy-type operators with kernels are of great importance, too. However, they go beyond the scope of this paper, although to investigate thoroughly their behavior on rearrangement-invariant function spaces would be of interest (e.g., see [1, 22]).

Our motivation behind studying the Hardy-type operators $R_{u,v,\nu}$ and $H_{u,v,\nu}$ is the following. Questions involving considerably more complicated operators can sometimes be reduced to questions concerning these Hardy-type operators for suitable choices of $u$, $v$, and $\nu$. In turn, the better we control the Hardy-type operators, the better we control the more complicated ones. Arguably the most straightforwardly, this can be illustrated by the following well-known example, which traces back to the 1930s. Consider the question of establishing the boundedness of the Hardy–Littlewood maximal operator $M$ from a rearrangement-invariant function space $X(\mathbb{R}^n)$ to a rearrangement-invariant function space $Y(\mathbb{R}^n)$. It turns out that $M$ is bounded from $X(\mathbb{R}^n)$ to $Y(\mathbb{R}^n)$ if and only if $R_{u,v,\nu}$ with $u \equiv 1$, $v(t) = t^{-1}$, $\nu(t) = t$, $L = \infty$, is bounded from $X(0,\infty)$ to $Y(0,\infty)$. This is a consequence of the famous equivalence

$$C_1 \frac{1}{t} \int_0^t f^*(s)\,ds \leq (Mf)^*(t) \leq C_2 \frac{1}{t} \int_0^t f^*(s)\,ds \quad \text{for every } t \in (0,\infty).$$

Here $^*$ denotes the nonincreasing rearrangement and $C_1, C_2$ are positive constants depending only on $n$. The upper bound on $(Mf)^*$ was proved by F. Riesz ([54], $n = 1$) and N. Wiener ([61], $n \in \mathbb{N}$). The lower one was proved by C. Hertz ([38], $n = 1$) and by C. Bennett and R. Sharpley ([4], $n \in \mathbb{N}$). There are other important operators of harmonic analysis that sharp inequalities for their nonincreasing rearrangements are known for. For example, the Hilbert transform ([3, Theorem 16.12], [57, Lemma 2.1]), or, more generally, certain singular integral operators with odd kernels [16, p. 55]. It is easy to show that the boundedness of these operators on rearrangement-invariant function spaces is equivalent to the boundedness of a sum of two Hardy-type operators—namely, $R_{u,v,\nu} + H_{u,v,\nu}$. Here $u, v, \nu$ are the same as those for the Hardy–Littlewood maximal function. Other classical operators whose nonincreasing rearrangements are controlled by $R_{u,v,\nu}$ and/or $H_{u,v,\nu}$

for suitable choices of $u, v$, and $\nu$ are certain convolution operators [32, 49] or the fractional maximal operator and its variants [24, 29]. The number of operators that sharp inequalities for their nonincreasing rearrangements are known for is limited. Nevertheless, what is often at our disposal is at least an upper bound on the nonincreasing rearrangement of a given operator. Obviously, the better we control the upper bound, the better we control the given operator. It is also worth noting that inequalities for rearrangements of various maximal operators may actually involve a Hardy-type operator inside a supremum (see [42] and references therein). However, the supremum usually does not cause any trouble (see [30, Lemma 4.10]). For example, consider the fractional maximal operator $M_\gamma$ of order $\gamma \in (0, n)$. It is bounded from $X(\mathbb{R}^n)$ to $Y(\mathbb{R}^n)$ if and only if the supremum operator mapping a measurable function $f$ on $(0, \infty)$ to the function

$$(0,\infty) \ni t \mapsto \sup_{s\in[t,\infty)} R_{u,v,\nu}(f^*)(s)$$

is bounded from $X(0, \infty)$ to $Y(0, \infty)$. Here, $u \equiv 1$, $v(t) = t^{\gamma/n-1}$, and $\nu(t) = t$. This equivalence follows from the sharp inequality for the nonincreasing rearrangement of $M_\gamma$ (see [24, Theorem 1.1] and [42, Example 1]). Importantly, it turns out that the supremum operator is bounded from $X(0, \infty)$ to $Y(0, \infty)$ if and only if the Hardy-type operator $R_{u,v,\nu}$ itself is. This follows from [30, Lemma 4.10] combined with the Hardy–Littlewood inequality (see (2.5)). The interested reader can find more information on boundedness of some classical operators of harmonic analysis on rearrangement-invariant function spaces in [30].

Pointwise inequalities for rearrangements are not the only way to reduce complicated questions to simpler ones involving Hardy-type operators. Reductions are also sometimes achieved with the right use of interpolation or by making use of some intrinsic properties of the problem in question. Such approaches have been notably successful in connection with various embeddings of Sobolev-type spaces built upon rearrangement-invariant function spaces into rearrangement-invariant function spaces. There, interpolation techniques, symmetrization principles, and isoperimetric inequalities have been of great use. For a wide variety of such embeddings, either complete characterizations or at least sufficient and/or necessary conditions have been obtained. See [2, 7, 17, 20–23, 39, 46] for complete characterizations and [18, 19, 23, 47] for sufficient and/or necessary conditions. These so-called reduction principles effectively transform the question of whether a certain Sobolev-type embedding is valid to that of whether a Hardy-type operator is bounded. For example, consider the Sobolev-type embedding

$$W^m X(\Omega) \hookrightarrow Y(\overline{\Omega}, \mu), \tag{1.3}$$

which was thoroughly studied in [23]. Here, $W^m X(\Omega)$ is the $m$-th order Sobolev space built upon a rearrangement-invariant function space $X$ over a bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$, $m < n$, $m \in \mathbb{N}$, and $Y$ is a rearrangement-invariant function space over $\overline{\Omega}$ endowed with a positive $d$-upper Ahlfors measure $\mu$. A $d$-upper Ahlfors measure $\mu$ is a finite Borel measure $\mu$ on $\overline{\Omega}$ satisfying

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{\mu(B_r(x)\cap\overline{\Omega})}{r^d} < \infty \tag{1.4}$$

with $d \in (0, n]$. Here, $B_r(x)$ is the open ball centered at $x$ with radius $r$. It turns out that, when $d \in [n - m, n]$, the question of whether (1.3) is valid leads us to the Hardy-type operator $H_{u_1,v_1,\nu}$ with $u_1 \equiv 1$, $v_1(t) = t^{-1+\frac{m}{n}}$, $\nu(t) = t^{\frac{n}{d}}$, and $L = 1$. When $d \in (0, n - m)$, the question is more complicated and leads us not only to the same Hardy-type operator $H_{u_1,v_1,\nu}$ but also to $R_{u_2,v_2,\nu}$ with $u_2(t) = t^{-\frac{m}{n-d}}$ and $v_2(t) = t^{-1+\frac{m}{n-d}}$.

We conclude this introductory section by briefly mentioning some new applications that general results obtained in this paper could be useful for. For example, we get under control the optimal behavior of upper bounds for nonincreasing rearrangements of various less standard (nonfractional and fractional) maximal operators (see [29, 42]). Furthermore, we get under control the optimal behavior of upper bounds for some operators that play a role in the a.e. convergence of the partial spherical Fourier integrals or in the solvability of the Dirichlet problem for the Laplacian on planar domains. See [12–14] and references therein for more information on such operators. Another possible application is related to traces of Sobolev functions. There are $d$-dimensional sets $\Omega_d \subseteq \mathbb{R}^n$, $d \in (0, n]$, that are "unrecognizable" by $d$-upper Ahlfors measures $\mu$, that is, it may happen that $\mu(\Omega_d) = 0$ for every $d$-upper Ahlfors measure $\mu$. For instance, this is, with probability 1, the case when $\Omega_d$ is a Brownian path in $\mathbb{R}^n$, $n \geq 2$. With probability 1, its Hausdorff dimension is 2 but it is unrecognizable by 2-upper Ahlfors measures. To rectify the situation, more general functions than power functions have to be considered in (1.4). For more information, see [10, 25, 31]. Inevitably, if one is to generalize the results of [23] to cover such exceptional sets, one will need to deal with general enough Hardy-type operators. In particular, one would need to allow $\nu$ to have nonpower growth, as is the case with the Hardy-type operators studied in this paper.

## 2 | PRELIMINARIES

### Conventions and notation

(1) Throughout the paper, $L \in (0, \infty]$.
(2) We adhere to the convention that $\frac{1}{\infty} = 0 \cdot \infty = 0$.
(3) We write $P \lesssim Q$, where $P, Q$ are nonnegative quantities, when there is a positive constant $c$ independent of all appropriate quantities appearing in the expressions $P$ and $Q$ such that $P \leq c \cdot Q$. If not stated explicitly, what "the appropriate quantities appearing in the expressions $P$ and $Q$" are should be obvious from the context. At the few places where it is not obvious, we will explicitly specify what the appropriate quantities are. We also write $P \gtrsim Q$ with the obvious meaning, and $P \approx Q$ when $P \lesssim Q$ and $P \gtrsim Q$ simultaneously.
(4) When $A \subseteq (0, L)$ is a (Lebesgue) measurable set, $|A|$ stands for its Lebesgue measure.
(5) When $u$ is a nonnegative measurable function defined on $(0, L)$, we denote by $U$ the function defined as $U(t) = \int_0^t u(s)\,ds$, $t \in (0, L]$. We say that $u$ is *nondegenerate* if there is $t_0 \in (0, L)$ such that $0 < U(t_0) < \infty$.

We set

$$\mathfrak{M}(0, L) = \{f : f \text{ is a measurable function on } (0, L) \text{ with values in } [-\infty, \infty]\},$$

$$\mathfrak{M}_0(0, L) = \{f \in \mathfrak{M}(0, L) : f \text{ is finite a.e. on } (0, L)\},$$

and

$$\mathfrak{M}^+(0, L) = \{f \in \mathfrak{M}(0, L) : f \geq 0 \text{ a.e. on } (0, L)\}.$$

The *nonincreasing rearrangement* $f^* : (0, \infty) \to [0, \infty]$ of a function $f \in \mathfrak{M}(0, L)$ is defined as

$$f^*(t) = \inf\{\lambda \in (0, \infty) : |\{s \in (0, L) : |f(s)| > \lambda\}| \leq t\}, \ t \in (0, \infty).$$

Note that $f^*(t) = 0$ for every $t \in [L, \infty)$. We say that functions $f, g \in \mathfrak{M}(0, L)$ are *equimeasurable*, and we write $f \sim g$, if $|\{s \in (0, L) : |f(s)| > \lambda\}| = |\{s \in (0, L) : |g(s)| > \lambda\}|$ for every $\lambda \in (0, \infty)$. We always have that $f \sim f^*$. The relation $\sim$ is transitive.

The *maximal nonincreasing rearrangement* $f^{**} : (0, \infty) \to [0, \infty]$ of a function $f \in \mathfrak{M}(0, L)$ is defined as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\,ds, \ t \in (0, \infty).$$

The mapping $f \mapsto f^*$ is monotone in the sense that, for every $f, g \in \mathfrak{M}(0, L)$,

$$|f| \leq |g| \quad \text{a.e. on } (0, L) \quad \Longrightarrow \quad f^* \leq g^* \quad \text{on } (0, \infty).$$

The same implication remains true if $^*$ is replaced by $^{**}$. We have that

$$f^* \leq f^{**} \quad \text{for every } f \in \mathfrak{M}(0, L). \tag{2.1}$$

The operation $f \mapsto f^*$ is neither subadditive nor multiplicative. Although $f \mapsto f^*$ is not subadditive, the following pointwise inequality is valid for every $f, g \in \mathfrak{M}_0(0, L)$ [5, Chapter 2, Proposition 1.7, (1.16)]:

$$(f + g)^*(t) \leq f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right) \quad \text{for every } t \in (0, L). \tag{2.2}$$

Furthermore, the lack of subadditivity of the operation of taking the nonincreasing rearrangement is, up to some extent, compensated by the following fact [5, Chapter 2, (3.10)]. For every $t \in (0,\infty)$ and $f, g \in \mathfrak{M}_0(0,L)$, we have that

$$\int_0^t (f+g)^*(s)\,ds \le \int_0^t f^*(s)\,ds + \int_0^t g^*(s)\,ds. \tag{2.3}$$

In other words, the operation $f \mapsto f^{**}$ is subadditive.

There are a large number of inequalities concerning rearrangements (e.g., [36, 41, Chapter II, Section 2]). We state two of them, which shall prove particularly useful for us. The *Hardy–Littlewood inequality* [5, Chapter 2, Theorem 2.2] tells us that, for every $f, g \in \mathfrak{M}(0,L)$,

$$\int_0^L |f(t)||g(t)|\,dt \le \int_0^L f^*(t)g^*(t)\,dt. \tag{2.4}$$

In particular, by taking $g = \chi_E$ in (2.4), one obtains that

$$\int_E |f(t)|\,dt \le \int_0^{|E|} f^*(t)\,dt \tag{2.5}$$

for every measurable $E \subseteq (0,L)$. The *Hardy lemma* [5, Chapter 2, Proposition 3.6] states that

$$\begin{aligned} &\text{if } f, g \in \mathfrak{M}^+(0,\infty) \text{ are such that } \int_0^t f(s)\,ds \le \int_0^t g(s)\,ds \quad \text{for all } t \in (0,\infty),\\ &\text{then } \int_0^\infty f(t)h(t)\,dt \le \int_0^\infty g(t)h(t)\,dt \quad \text{for every nonincreasing } h \in \mathfrak{M}^+(0,\infty). \end{aligned} \tag{2.6}$$

A functional $\|\cdot\|_{X(0,L)} : \mathfrak{M}^+(0,L) \to [0,\infty]$ is called a *rearrangement-invariant function norm* (on $(0,L)$) if, for all $f$, $g$ and $\{f_k\}_{k=1}^\infty$ in $\mathfrak{M}^+(0,L)$, and every $\lambda \in [0,\infty)$,

- (P1) $\|f\|_{X(0,L)} = 0$ if and only if $f = 0$ a.e. on $(0,L)$; $\|\lambda f\|_{X(0,L)} = \lambda\|f\|_{X(0,L)}$, $\|f+g\|_{X(0,L)} \le \|f\|_{X(0,L)} + \|g\|_{X(0,L)}$;
- (P2) $\|f\|_{X(0,L)} \le \|g\|_{X(0,L)}$ if $f \le g$ a.e. on $(0,L)$;
- (P3) $\|f_k\|_{X(0,L)} \nearrow \|f\|_{X(0,L)}$ if $f_k \nearrow f$ a.e. on $(0,L)$;
- (P4) $\|\chi_E\|_{X(0,L)} < \infty$ for every measurable $E \subseteq (0,L)$ of finite measure;
- (P5) for every measurable $E \subseteq (0,L)$ of finite measure, there is a positive, finite constant $C_{E,X}$, possibly depending on $E$ and $\|\cdot\|_{X(0,L)}$ but not on $f$, such that $\int_E f(t)\,dt \le C_{E,X}\|f\|_{X(0,L)}$;
- (P6) $\|f\|_{X(0,L)} = \|g\|_{X(0,L)}$ whenever $f \sim g$.

The *Hardy–Littlewood–Pólya principle* [5, Chapter 2, Theorem 4.6] asserts that, for every $f, g \in \mathfrak{M}(0,L)$ and every rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$,

$$\text{if } \int_0^t f^*(s)\,ds \le \int_0^t g^*(s)\,ds \text{ for all } t \in (0,L), \text{ then } \|f\|_{X(0,L)} \le \|g\|_{X(0,L)}. \tag{2.7}$$

With every rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$, we associate another functional $\|\cdot\|_{X'(0,L)}$ defined as

$$\|f\|_{X'(0,L)} = \sup_{\substack{g \in \mathfrak{M}^+(0,L)\\ \|g\|_{X(0,L)} \le 1}} \int_0^L f(t)g(t)\,dt,\ f \in \mathfrak{M}^+(0,L). \tag{2.8}$$

The functional $\|\cdot\|_{X'(0,L)}$ is also a rearrangement-invariant function norm [5, Chapter 2, Proposition 4.2], and it is called the *associate function norm* of $\|\cdot\|_{X(0,L)}$. Furthermore, we always have that [5, Chapter 1, Theorem 2.7]

$$\|f\|_{X(0,L)} = \sup_{\substack{g \in \mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)} \le 1}} \int_0^L f(t)g(t)\,dt \quad \text{for every} f \in \mathfrak{M}^+(0,L), \tag{2.9}$$

that is,

$$\| \cdot \|_{(X')'(0,L)} = \| \cdot \|_{X(0,L)}. \tag{2.10}$$

Consequently, statements like "Let $\| \cdot \|_{X(0,L)}$ be *the* rearrangement-invariant function norm whose associate function norm is ..." are well justified. The supremum in (2.9) does not change when the functions involved are replaced with their nonincreasing rearrangements [5, Chapter 2, Proposition 4.2], that is,

$$\|f\|_{X(0,L)} = \sup_{\substack{g \in \mathfrak{M}^+(0,L) \\ \|g\|_{X'(0,L)} \leq 1}} \int_0^L f^*(t) g^*(t)\, dt \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{2.11}$$

Given a rearrangement-invariant function norm $\| \cdot \|_{X(0,L)}$, we extend it from $\mathfrak{M}^+(0,L)$ to $\mathfrak{M}(0,L)$ by $\|f\|_{X(0,L)} = \| \, |f| \, \|_{X(0,L)}$. The extended functional $\| \cdot \|_{X(0,L)}$ restricted to the linear set $X(0,L)$ defined as

$$X(0,L) = \{f \in \mathfrak{M}(0,L) : \|f\|_{X(0,L)} < \infty\}$$

is a norm, provided that we identify any two functions from $\mathfrak{M}(0,L)$ coinciding a.e. on $(0,L)$, as usual. In fact, $X(0,L)$ endowed with the norm $\| \cdot \|_{X(0,L)}$ is a Banach space [5, Chapter 1, Theorem 1.6]. We say that $X(0,L)$ is a *rearrangement-invariant function space*. Note that $f \in \mathfrak{M}(0,L)$ belongs to $X(0,L)$ if and only if $\|f\|_{X(0,L)} < \infty$. We always have that

$$S(0,L) \subseteq X(0,L) \subseteq \mathfrak{M}_0(0,L), \tag{2.12}$$

where $S(0,L)$ denotes the set of all simple functions on $(0,L)$. By a simple function, we mean a (finite) linear combination of characteristic functions of measurable sets having finite measure. Moreover, the second inclusion is continuous if the linear set $\mathfrak{M}_0(0,L)$ is endowed with the (metrizable) topology of convergence in measure on sets of finite measure [5, Chapter 1, Theorem 1.4].

The class of rearrangement-invariant function spaces contains a large number of customary function spaces, such as Lebesgue spaces $L^p$ ($p \in [1,\infty]$), Lorentz spaces $L^{p,q}$ (e.g., [5, pp. 216–220]), Orlicz spaces (e.g., [53]), or Lorentz–Zygmund spaces (e.g., [3, 50]), to name a few. Here, we provide definitions of only those rearrangement-invariant function norms that we shall explicitly need. For $p \in [1,\infty]$, we define the Lebesgue function norm $\| \cdot \|_{L^p(0,L)}$ as

$$\|f\|_{L^p(0,L)} = \begin{cases} \int_0^L f(t)^p \, dt & \text{if } p \in [1,\infty), \\ \operatorname{ess\,sup}_{t \in (0,L)} f(t) & \text{if } p = \infty, \end{cases}$$

$f \in \mathfrak{M}^+(0,L)$. Given a measurable function $v : (0,L) \to (0,\infty)$ such that $V(t) < \infty$ for every $t \in (0,L)$, we define the functional $\| \cdot \|_{\Lambda^1_v(0,L)}$ as

$$\|f\|_{\Lambda^1_v(0,L)} = \int_0^L f^*(s) v(s)\, ds, \ f \in \mathfrak{M}^+(0,L).$$

Here, $V(t) = \int_0^t v(s)\, ds$, $t \in (0,L)$. The functional is equivalent to a rearrangement-invariant function norm if and only if

$$\frac{V(t)}{t} \lesssim \frac{V(s)}{s} \quad \text{for every } 0 < s < t < L. \tag{2.13}$$

This follows from [11, Theorem 2.3] (see also [59, Proposition 1] with regard to local embedding of $\Lambda^1_v(0,L)$ in $L^1(0,L)$). By the fact that it is equivalent to a rearrangement-invariant function norm, we mean that there is a rearrangement-invariant function norm $\| \cdot \|_{X(0,L)}$ on $(0,L)$ such that $\|f\|_{\Lambda^1_v(0,L)} \approx \|f\|_{X(0,L)}$ for every $f \in \mathfrak{M}^+(0,L)$. Hence, we can treat $\Lambda^1_v(0,L)$ as a rearrangement-invariant function space whenever (2.13) is satisfied. Let $\psi : (0,L) \to (0,\infty)$ be a *quasiconcave function*, that is, a nondecreasing function such that the function $(0,L) \ni t \mapsto \frac{\psi(t)}{t}$ is nonincreasing. The functional $\| \cdot \|_{M_\psi(0,L)}$

defined as

$$\|f\|_{M_\psi(0,L)} = \sup_{t\in(0,L)} \psi(t)f^{**}(t),\ t\in(0,L),$$

is a rearrangement-invariant function norm [52, Proposition 7.10.2]. We shall also meet Lorentz–Zygmund spaces. They are defined in Section 6, where they are used.

The rearrangement-invariant function space $X'(0,L)$ built upon the associate function norm $\|\cdot\|_{X'(0,L)}$ of a rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$ is called the *associate function space* of $X(0,L)$. Thanks to (2.10), we have that $(X')'(0,L) = X(0,L)$. Furthermore, one has that

$$\int_0^L |f(t)||g(t)|\,dt \le \|f\|_{X(0,L)}\|g\|_{X'(0,L)} \quad \text{for every} f,g\in\mathfrak{M}(0,L). \tag{2.14}$$

Inequality (2.14) is a Hölder-type inequality, and we shall refer to it as the Hölder inequality.

Let $X(0,L)$ and $Y(0,L)$ be rearrangement-invariant function spaces. We say that $X(0,L)$ is *embedded in* $Y(0,L)$, and we write $X(0,L)\hookrightarrow Y(0,L)$, if there is a positive constant $C$ such that $\|f\|_{Y(0,L)}\le C\|f\|_{X(0,L)}$ for every $f\in\mathfrak{M}(0,L)$. If $X(0,L)\hookrightarrow Y(0,L)$ and $Y(0,L)\hookrightarrow X(0,L)$ simultaneously, we write that $X(0,L) = Y(0,L)$. We have that [5, Chapter 1, Theorem 1.8]

$$X(0,L)\hookrightarrow Y(0,L) \quad \text{if and only if} \quad X(0,L)\subseteq Y(0,L). \tag{2.15}$$

Furthermore,

$$X(0,L)\hookrightarrow Y(0,L) \quad \text{if and only if} \quad Y'(0,L)\hookrightarrow X'(0,L) \tag{2.16}$$

with the same embedding constants.

If $\|\cdot\|_{X(0,L)}$ and $\|\cdot\|_{Y(0,L)}$ are rearrangement-invariant function norms, then so are $\|\cdot\|_{X(0,L)\cap Y(0,L)}$ and $\|\cdot\|_{(X+Y)(0,L)}$ defined as

$$\|f\|_{X(0,L)\cap Y(0,L)} = \max\{\|f\|_{X(0,L)}, \|f\|_{Y(0,L)}\},\ f\in\mathfrak{M}^+(0,L),$$

and

$$\|f\|_{(X+Y)(0,L)} = \inf_{f=g+h}(\|g\|_{X(0,L)} + \|h\|_{Y(0,L)}),\ f\in\mathfrak{M}^+(0,L).$$

Here, the infimum extends over all possible decompositions $f = g+h$, $g,h\in\mathfrak{M}^+(0,L)$. Furthermore, we have that ([43, Theorem 3.1], also [27, Lemma 1.12])

$$(X(0,L)\cap Y(0,L))' = (X'+Y')(0,L) \quad \text{and} \quad (X+Y)'(0,L) = X'(0,L)\cap Y'(0,L) \tag{2.17}$$

with equality of norms. The K-*functional* between $X(0,L)$ and $Y(0,L)$ is, for every $f\in(X+Y)(0,L)$ and $t\in(0,\infty)$, defined as

$$\mathrm{K}(f,t;X,Y) = \inf_{f=g+h}(\|g\|_{X(0,L)} + t\|h\|_{Y(0,L)}).$$

Here, the infimum extends over all possible decompositions $f = g+h$ with $g\in X(0,L)$ and $h\in Y(0,L)$. For every $f\in(X+Y)(0,L)\setminus\{0\}$, $\mathrm{K}(f,\cdot;X,Y)$ is a positive increasing concave function on $(0,\infty)$ [5, Chapter 5, Proposition 1.2].

Let $X_0(0,L)$ and $X_1(0,L)$ be rearrangement-invariant function spaces. We say that a rearrangement-invariant function space $X(0,L)$ is an *intermediate space* between $X_0(0,L)$ and $X_1(0,L)$ if $X_0(0,L)\cap X_1(0,L)\hookrightarrow X(0,L)\hookrightarrow (X_0+X_1)(0,L)$. A linear operator $T$ defined on $(X_0+X_1)(0,L)$ having values in $(X_0+X_1)(0,L)$ is said to be *admissible* for the couple $(X_0(0,L),X_1(0,L))$ if $T$ is bounded on both $X_0(0,L)$ and $X_1(0,L)$. An intermediate space $X(0,L)$ between $X_0(0,L)$ and $X_1(0,L)$ is an *interpolation space* with respect to the couple $(X_0(0,L),X_1(0,L))$ if every admissible operator for the couple is bounded on $X(0,L)$. By [9, Theorem 3], $X(0,L)$ is always an interpolation space with respect to the couple $(L^1(0,L),L^\infty(0,L))$.

We always have that [5, Chapter 2, Theorem 6.6]

$$L^1(0,L) \cap L^\infty(0,L) \hookrightarrow X(0,L) \hookrightarrow L^1(0,L) + L^\infty(0,L).$$

In particular,

$$L^\infty(0,L) \hookrightarrow X(0,L) \hookrightarrow L^1(0,L) \tag{2.18}$$

provided that $L < \infty$.

Let $a > 0$. The *dilation operator* $D_a$ maps a function $f \in \mathfrak{M}(0,L)$ to the function

$$D_a f(t) = \begin{cases} f(\frac{t}{a}), & \text{if } L = \infty, \\ f(\frac{t}{a})\chi_{(0,aL)}(t), & \text{if } L < \infty. \end{cases}$$

Importantly, $D_a$ is bounded on every rearrangement-invariant function space $X(0,L)$ (see [5, Chapter 3, Proposition 5.11]). More precisely, we have that

$$\|D_a f\|_{X(0,L)} \le \max\{1,a\}\|f\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}(0,L). \tag{2.19}$$

## 3 | OPTIMAL REARRANGEMENT-INVARIANT FUNCTION SPACES

In this section, we shall investigate optimal mapping properties of the operators $H_{u,v,\nu}$ and $R_{u,v,\nu}$. Let $T$ be either of them. We say that a rearrangement-invariant function space $Y(0,L)$ is *the optimal domain space* for the operator $T$ and a rearrangement-invariant function space $X(0,L)$ if the following two facts are true. $T : Y(0,L) \to X(0,L)$ is bounded and $Z(0,L) \hookrightarrow Y(0,L)$ whenever $Z(0,L)$ is a rearrangement-invariant function space such that $T : Z(0,L) \to X(0,L)$ is bounded. In other words, $\|\cdot\|_{Y(0,L)}$ is the weakest domain rearrangement-invariant function norm for $T$ and $\|\cdot\|_{X(0,L)}$. We say that a rearrangement-invariant function space $Y(0,L)$ is *the optimal target space* for the operator $T$ and a rearrangement-invariant function space $X(0,L)$ if the following two facts are true. $T : X(0,L) \to Y(0,L)$ is bounded and $Y(0,L) \hookrightarrow Z(0,L)$ whenever $Z(0,L)$ is a rearrangement-invariant function space such that $T : X(0,L) \to Z(0,L)$ is bounded. In other words, $\|\cdot\|_{Y(0,L)}$ is the strongest target rearrangement-invariant function norm for $T$ and $\|\cdot\|_{X(0,L)}$.

### 3.1 | Optimal domain spaces

We start by characterizing when the functional $\mathfrak{M}^+(0,L) \ni f \mapsto \|R_{u,v,\nu}(f^*)\|_{X(0,L)}$ is a rearrangement-invariant function norm. It turns out that it also enables us to characterize optimal domain spaces for $R_{u,v,\nu}$. In the following subsection, we will also use it to characterize optimal target spaces for $H_{u,v,\nu}$.

**Proposition 3.1.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection.*
*(2) Let $u : (0,L) \to (0,\infty)$ be a nondegenerate nonincreasing function. If $L < \infty$, assume that $u(L^-) > 0$.*
*(3) Let $v : (0,L) \to (0,\infty)$ be measurable.*

*Set*

$$\|f\|_{Y(0,L)} = \|R_{u,v,\nu}(f^*)\|_{X(0,L)},\ f \in \mathfrak{M}^+(0,L),$$

*and*

$$\xi(t) = \begin{cases} v(t)U(\nu(t)),\ t \in (0,L), & \text{if } L < \infty, \\ v(t)U(\nu(t))\chi_{(0,1)}(t) + v(t)\chi_{(1,\infty)}(t),\ t \in (0,\infty), & \text{if } L = \infty. \end{cases} \tag{3.1}$$

*The functional $\|\cdot\|_{Y(0,L)}$ is a rearrangement-invariant function norm if and only if $\xi \in X(0,L)$.*

*If $\xi \in X(0,L)$, then the rearrangement-invariant function space $Y(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu}$ and $X(0,L)$. If $\xi \notin X(0,L)$, then there is no rearrangement-invariant function space $Z(0,L)$ such that $R_{u,v,\nu} : Z(0,L) \to X(0,L)$ is bounded.*

*Proof.* We shall show that $\|\cdot\|_{Y(0,L)}$ is a rearrangement-invariant function norm provided that $\xi \in X(0,L)$. Before we do that, note that, since $u$ is positive and nonincreasing, its nondegeneracy implies that $0 < U(t) < \infty$ for every $t \in (0,L] \cap \mathbb{R}$.

*Property* (P1). The positive homogeneity and positive definiteness of $\|\cdot\|_{Y(0,L)}$ can be readily verified. As for the subadditivity of $\|\cdot\|_{Y(0,L)}$, it follows from (2.3) combined with Hardy's lemma (2.6) that

$$\int_0^L (f+g)^*(s)u(s)\chi_{(0,\nu(t))}(s)\,ds \le \int_0^L f^*(s)u(s)\chi_{(0,\nu(t))}(s)\,ds + \int_0^L g^*(s)u(s)\chi_{(0,\nu(t))}(s)\,ds$$

for every $f,g \in \mathfrak{M}^+(0,L)$ and $t \in (0,L)$ thanks to the fact that $u$ is nonincreasing. Since $\|\cdot\|_{X(0,L)}$ is subadditive, it follows that

$$\|f+g\|_{Y(0,L)} \le \|f\|_{Y(0,L)} + \|g\|_{Y(0,L)} \quad \text{for every } f,g \in \mathfrak{M}^+(0,L).$$

*Properties* (P2) *and* (P3). Since $\|\cdot\|_{X(0,L)}$ has these properties, it can be readily verified that $\|\cdot\|_{Y(0,L)}$, too, has them.

*Property* (P4). First, assume that $L<\infty$. Clearly, $\|\chi_{(0,L)}\|_{Y(0,L)} < \infty$ if and only if $v(t)U(\nu(t)) \in X(0,L)$. Since $\|\cdot\|_{Y(0,L)}$ has property (P2), $\|\cdot\|_{Y(0,L)}$ has property (P4) if and only if $v(t)U(\nu(t)) \in X(0,L)$. Second, assume that $L=\infty$. Let $E \subseteq (0,\infty)$ be a set of finite positive measure. Clearly, $\|\chi_E\|_{Y(0,\infty)} < \infty$ if and only if $v(t)U(\nu(t))\chi_{(0,|E|)}(t) + v(t)\chi_{(|E|,\infty)}(t) \in X(0,\infty)$. If $|E| \le 1$, then

$$\begin{aligned}
&\|v(t)U(\nu(t))\chi_{(0,|E|)}(t) + v(t)\chi_{(|E|,\infty)}(t)\|_{X(0,\infty)}\\
&\quad\le \|v(t)U(\nu(t))\chi_{(0,1)}(t)\|_{X(0,\infty)} + \|v(t)\chi_{(|E|,1)}(t)\|_{X(0,\infty)} + \|v(t)\chi_{(1,\infty)}(t)\|_{X(0,\infty)}\\
&\quad\le \|v(t)U(\nu(t))\chi_{(0,1)}(t)\|_{X(0,\infty)} + \frac{1}{U(\nu(|E|))}\|U(\nu(t))v(t)\chi_{(|E|,1)}(t)\|_{X(0,\infty)}\\
&\qquad+ \|v(t)\chi_{(1,\infty)}(t)\|_{X(0,\infty)}\\
&\quad\le \left(1+\frac{1}{U(\nu(|E|))}\right)\|v(t)U(\nu(t))\chi_{(0,1)}(t)\|_{X(0,\infty)} + \|v(t)\chi_{(1,\infty)}(t)\|_{X(0,\infty)}.
\end{aligned}$$

If $E \ge 1$, we can obtain, in a similar way, that

$$\begin{aligned}
&\|v(t)U(\nu(t))\chi_{(0,|E|)}(t) + v(t)\chi_{(|E|,\infty)}(t)\|_{X(0,\infty)}\\
&\quad\le \|v(t)U(\nu(t))\chi_{(0,1)}(t)\|_{X(0,\infty)} + (1+U(\nu(|E|)))\|v(t)\chi_{(1,\infty)}(t)\|_{X(0,\infty)}.
\end{aligned}$$

Either way, we have that $\|\chi_E\|_{Y(0,\infty)} < \infty$ if and only if

$$v(t)U(\nu(t))\chi_{(0,1)}(t) + v(t)\chi_{(1,\infty)}(t) \in X(0,\infty).$$

*Property* (P5). Let $E \subseteq (0,L)$ be a set of finite positive measure. Let $f \in \mathfrak{M}^+(0,L)$. Note that the function $(0,L) \ni t \mapsto \frac{1}{U(\nu(t))}\int_0^{\nu(t)} f^*(s)u(s)\,ds$ is nonincreasing because it is the integral mean of a nonnegative nonincreasing function over the interval $(0,\nu(t))$ with respect to the measure $u(s)\,ds$. Thanks to that and the monotonicity of $u$, we obtain that

$$\begin{aligned}
\left\|v(t)\int_0^{\nu(t)} f^*(s)u(s)\,ds\right\|_{X(0,L)} &\ge \left\|v(t)\chi_{(0,\nu^{-1}(|E|))}(t)\int_0^{\nu(t)} f^*(s)u(s)\,ds\right\|_{X(0,L)}\\
&\ge \left\|v(t)U(\nu(t))\chi_{(0,\nu^{-1}(|E|))}(t)\right\|_{X(0,L)}\frac{1}{U(|E|)}\int_0^{|E|} f^*(s)u(s)\,ds
\end{aligned}$$

$$\geq \left\|v(t)U(\nu(t))\chi_{(0,\nu^{-1}(|E|))}(t)\right\|_{X(0,L)} \frac{u(|E|^-)}{U(|E|)} \int_0^{|E|} f^*(s)\,ds$$

$$\geq \left\|v(t)U(\nu(t))\chi_{(0,\nu^{-1}(|E|))}(t)\right\|_{X(0,L)} \frac{u(|E|^-)}{U(|E|)} \int_E f(s)\,ds.$$

Here, we used (2.5) in the last inequality.

*Property* (P6). Since $f^* = g^*$ when $f, g \in \mathfrak{M}^+(0,L)$ are equimeasurable, this is obvious.

Note that the necessity of $\xi \in X(0,L)$ for $\|\cdot\|_{Y(0,L)}$ to be a rearrangement-invariant function norm was already proved in the paragraph devoted to property (P4).

Assume now that $\xi \in X(0,L)$. Thanks to the Hardy–Littlewood inequality (2.4) and the monotonicity of $u$, we have that

$$\|R_{u,v,\nu} f\|_{X(0,L)} \leq \|R_{u,v,\nu}(f^*)\|_{X(0,L)} = \|f\|_{Y(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Hence, $R_{u,v,\nu} : Y(0,L) \to X(0,L)$ is bounded. Next, if $Z(0,L)$ is a rearrangement-invariant function space such that $R_{u,v,\nu} : Z(0,L) \to X(0,L)$ is bounded, then we have that

$$\|f\|_{Y(0,L)} = \|R_{u,v,\nu}(f^*)\|_{X(0,L)} \lesssim \|f^*\|_{Z(0,L)} = \|f\|_{Z(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L),$$

and so $Z(0,L) \hookrightarrow Y(0,L)$. Finally, note that, if $R_{u,v,\nu} : Z(0,L) \to X(0,L)$ is bounded, then

$$\|\xi\|_{X(0,L)} \approx \|R_{u,v,\nu}(\chi_{(0,a)})\|_{X(0,L)} \lesssim \|\chi_{(0,a)}\|_{Z(0,L)} < \infty.$$

Here,

$$a = \begin{cases} L & \text{if } L < \infty, \\ 1 & \text{if } L = \infty. \end{cases} \tag{3.2}$$

Hence, $\xi \in X(0,L)$. □

*Remark* 3.2. Since $\nu : (0,L) \to (0,L)$ will always be an increasing bijection, let us briefly comment on this assumption. It may seem that the assumption is quite restrictive. However, from the point of view of applications that this paper is motivated by, the assumption is quite natural and not overly restrictive. They tell us that it is reasonable to assume that $\nu$ is an increasing bijection mapping the interval $(0,L)$ onto an interval $(0,\tilde{L})$, and that $L$ is finite if and only if $\tilde{L}$ is. Furthermore, when $L < \infty$, our assumption that $L = \tilde{L}$ only makes some computations easier and is not restrictive. The reason is that, if $\nu : (0,L) \to (0,\tilde{L})$ is an increasing bijection, then $\tilde{\nu}(t) = L\nu(t)/\tilde{L}$ is an increasing bijection of the interval $(0,L)$ onto itself. Although it might be of interest to allow $\nu$ to map an unbounded interval onto a bounded one or vice versa, that would make this paper considerably more technical. Nevertheless, the interested reader should be able to follow proofs presented in this paper and modify them if needed.

We now turn our attention to $H_{u,v,\nu}$. It turns out that the situation becomes significantly more complicated. Notably the fact that, unlike with $R_{u,v,\nu}$, the integration is carried out over intervals away from 0 often causes great difficulties. In particular, the functional $\mathfrak{M}^+(0,L) \ni f \mapsto \|H_{u,v,\nu}(f^*)\|_{X(0,L)}$ is hardly ever subadditive. Instead, in general, we need to consider a more complicated functional (see Proposition 4.1, however). Here and in subsequent sections, we will often need to impose certain mild conditions on $\nu$.

(1) We write $\nu \in \underline{D}^0$ if there is $\theta > 1$ such that $\liminf_{t\to 0^+} \frac{\nu(\theta t)}{\nu(t)} > 1$.
(2) We write $\nu \in \underline{D}^\infty$ if there is $\theta > 1$ such that $\liminf_{t\to\infty} \frac{\nu(\theta t)}{\nu(t)} > 1$.
(3) We write $\nu \in \overline{D}^0$ if there is $\theta > 1$ such that $\limsup_{t\to 0^+} \frac{\nu(\theta t)}{\nu(t)} < \infty$.
(4) We write $\nu \in \overline{D}^\infty$ if there is $\theta > 1$ such that $\limsup_{t\to\infty} \frac{\nu(\theta t)}{\nu(t)} < \infty$.

When we need to emphasize the exact value of $\theta$, we will write $\nu \in \underline{D}^0_\theta$ and so forth.

**Proposition 3.3.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. If $L = \infty$, assume that $\nu \in \underline{D}^{\infty}$.*
*(2) Let $u : (0,L) \to (0,\infty)$ be nonincreasing.*
*(3) Let $v : (0,L) \to (0,\infty)$ be nonincreasing. If $L < \infty$, assume that $v(L^-) > 0$.*

*Set*

$$\|f\|_{Y(0,L)} = \sup_{h\sim f} \|H_{u,v,\nu} h\|_{X(0,L)},\ f \in \mathfrak{M}^+(0,L), \tag{3.3}$$

*where the supremum extends over all $h \in \mathfrak{M}^+(0,L)$ equimeasurable with $f$. The functional $\|\cdot\|_{Y(0,L)}$ is a rearrangement-invariant function norm if and only if*

$$\begin{cases} u(t)\int_{\nu(t)}^{L} v(s)\,ds \in X(0,L) & \text{if } L<\infty, \\ u(t)\chi_{(0,\nu^{-1}(1))}(t)\int_{\nu(t)}^{1} v(s)\,ds \in X(0,\infty) \text{ and} & \\ \ \limsup_{\tau\to\infty} v(\tau)\|u\chi_{(0,\nu^{-1}(\tau))}\|_{X(0,\infty)} < \infty & \text{if } L=\infty. \end{cases} \tag{3.4}$$

*If (3.4) is satisfied, then the rearrangement-invariant function space $Y(0,L)$ is the optimal domain space for the operator $H_{u,v,\nu}$ and $X(0,L)$. If (3.4) is not satisfied, then there is no rearrangement-invariant function space $Z(0,L)$ such that $H_{u,v,\nu} : Z(0,L) \to X(0,L)$ is bounded.*

*Proof.* We shall show that $\|\cdot\|_{Y(0,L)}$ is a rearrangement-invariant function norm provided that (3.4) is satisfied.

*Property* (P2). Let $f, g \in \mathfrak{M}^+(0,L)$ be such that $f \le g$ a.e. Consequently, $f^* \le g^*$. Suppose that $\|f\|_{Y(0,L)} > \|g\|_{Y(0,L)}$. It implies that there is $\widetilde{f} \in \mathfrak{M}^+(0,L)$, $\widetilde{f} \sim f$, such that

$$\sup_{h\sim g} \|H_{u,v,\nu} h\|_{X(0,L)} < \|H_{u,v,\nu}\widetilde{f}\|_{X(0,L)}. \tag{3.5}$$

When $L = \infty$, we may assume that $\lim_{t\to\infty}(\widetilde{f})^*(t) = \lim_{t\to\infty} f^*(t) = 0$, for we would otherwise approximate $\widetilde{f}$ by functions $f_n = \widetilde{f}\chi_{(0,n)}$, $n \in \mathbb{N}$. The monotone convergence theorem and property (P3) of $\|\cdot\|_{X(0,L)}$ would guarantee that the inequality above holds with $\widetilde{f}$ replaced by $f_n$ for $n$ large enough. Thanks to [5, Chapter 2, Corollary 7.6] (also [55, Proposition 3]), there is a measure-preserving transformation $\sigma : (0,L) \to (0,L)$ such that $\widetilde{f} = f^*\circ\sigma$. For the definition of measure-preserving transformations, see [5, Chapter 2, Definition 7.1]. Since $\sigma$ is measure preserving, we have that $(g^*\circ\sigma) \sim g^* \sim g$ [5, Chapter 2, Proposition 7.2]. Consequently,

$$\begin{aligned} \sup_{h\sim g}\left\| u(t)\int_{\nu(t)}^{L} h(s)v(s)\,ds \right\|_{X(0,L)} &\ge \left\| u(t)\int_{\nu(t)}^{L} g^*(\sigma(s))v(s)\,ds \right\|_{X(0,L)} \\ &\ge \left\| u(t)\int_{\nu(t)}^{L} f^*(\sigma(s))v(s)\,ds \right\|_{X(0,L)} \\ &= \left\| u(t)\int_{\nu(t)}^{L} \widetilde{f}(s)v(s)\,ds \right\|_{X(0,L)}. \end{aligned} \tag{3.6}$$

By combining (3.5) and (3.6), we reach a contradiction. Hence, $\|f\|_{Y(0,L)} \le \|g\|_{Y(0,L)}$.

*Property* (P3). Let $f, f_k \in \mathfrak{M}^+(0,L)$, $k \in \mathbb{N}$, be such that $f_k \nearrow f$ a.e. Thanks to property (P2) of $\|\cdot\|_{Y(0,L)}$, the limit $\lim_{k\to\infty}\|f_k\|_{Y(0,L)}$ exists and we clearly have that $\lim_{k\to\infty}\|f_k\|_{Y(0,L)} \le \|f\|_{Y(0,L)}$. The fact that $\lim_{k\to\infty}\|f_k\|_{Y(0,L)} = \|f\|_{Y(0,L)}$ can be proved by contradiction in a similar way to the proof of (P2).

*Property* (P1). The positive homogeneity and positive definiteness of $\|\cdot\|_{Y(0,L)}$ can be readily verified. As for the subadditivity of $\|\cdot\|_{Y(0,L)}$, let $f, g \in \mathfrak{M}^+(0,L)$ be simple functions. Let $h \in \mathfrak{M}^+(0,L)$ be such that $h \sim f + g$. Being equimeasurable with $f+g$, $h$ is a simple function having the same range as $f+g$. Furthermore, it is easy to see that $h$ can be decomposed as $h = h_1 + h_2$, where $h_1, h_2 \in \mathfrak{M}^+(0,L)$ are simple functions such that $h_1 \sim f$ and $h_2 \sim g$. Using the subadditivity of $\|\cdot\|_{X(0,L)}$, we obtain that

$$\begin{aligned}\left\|u(t)\int_{\nu(t)}^{L} h(s)v(s)\,ds\right\|_{X(0,L)} &\le \left\|u(t)\int_{\nu(t)}^{L} h_1(s)v(s)\,ds\right\|_{X(0,L)} + \left\|u(t)\int_{\nu(t)}^{L} h_2(s)v(s)\,ds\right\|_{X(0,L)}\\ &\le \|f\|_{Y(0,L)} + \|g\|_{Y(0,L)}.\end{aligned}$$

Hence, $\|f+g\|_{Y(0,L)} \le \|f\|_{Y(0,L)} + \|g\|_{Y(0,L)}$. When $f, g \in \mathfrak{M}^+(0,L)$ are general functions, we approximate each of them by a nondecreasing sequence of nonnegative, simple functions and use property (P3) of $\|\cdot\|_{Y(0,L)}$ to get $\|f+g\|_{Y(0,L)} \le \|f\|_{Y(0,L)} + \|g\|_{Y(0,L)}$.

*Property* (P4). Assume that $L < \infty$. Since $\|\cdot\|_{Y(0,L)}$ has property (P2), $\|\cdot\|_{Y(0,L)}$ has property (P4) if and only if $\|\chi_{(0,L)}\|_{Y(0,L)} < \infty$. If $h \in \mathfrak{M}^+(0,L)$ is equimeasurable with $\chi_{(0,L)}$, then $h = 1$ a.e. on $(0,L)$; therefore,

$$\|\chi_{(0,L)}\|_{Y(0,L)} = \|H_{u,v,\nu}\chi_{(0,L)}\|_{X(0,L)}.$$

Hence, $\|\cdot\|_{Y(0,L)}$ has property (P4) if and only if $u(t)\int_{\nu(t)}^{L} v(s)\,ds \in X(0,L)$. Assume now that $L = \infty$. Fix $\theta > 1$ such that $\nu \in \underline{D}_\theta^\infty$. Let $E \subseteq (0,\infty)$ be of finite measure. Set $b = \max\left\{1, \nu(1), \frac{\theta|E|}{M-1}\right\}$, where $M = \inf_{t\in[1,\infty)} \frac{\nu(\theta t)}{\nu(t)}$. Note that $M > 1$. Let $h \in \mathfrak{M}^+(0,\infty)$ be equimeasurable with $\chi_E$. It is easy to see that $h = \chi_F$ for some measurable $F \subseteq (0,\infty)$ such that $|F| = |E|$. Thanks to the (outer) regularity of the Lebesgue measure, there is an open set $G \supseteq F$ such that $|G| \le \theta|F|$. Since $G$ is an open set on the real line, there is a countable system of mutually disjoint open intervals $\{(a_k, b_k)\}_k$ such that $G \cap (b,\infty) = \bigcup_k (a_k, b_k)$. We plainly have that $F \subseteq (0,b] \cup (G \cap (b,\infty))$ and $a_k > b$. Furthermore, we have that $b_k - a_k \le \theta|F| \le (M-1)b < (M-1)a_k$, whence

$$\nu^{-1}(b_k) - \nu^{-1}(a_k) < (\theta - 1)\nu^{-1}(a_k). \tag{3.7}$$

We have that

$$\begin{aligned}\left\|u(t)\int_{\nu(t)}^{\infty} \chi_F(s)v(s)\,ds\right\|_{X(0,\infty)} &\le \left\|u(t)\int_{\nu(t)}^{\infty}\left(\chi_{(0,b]}(s) + \sum_k \chi_{(a_k,b_k)}(s)\right)v(s)\,ds\right\|_{X(0,\infty)}\\ &\le \left\|u(t)\chi_{(0,\nu^{-1}(b))}(t)\int_{\nu(t)}^{b} v(s)\,ds\right\|_{X(0,\infty)}\\ &\quad + \sum_k \left\|u(t)\chi_{(0,\nu^{-1}(a_k))}(t)\int_{a_k}^{b_k} v(s)\,ds\right\|_{X(0,\infty)}\\ &\quad + \sum_k \left\|u(t)\chi_{(\nu^{-1}(a_k),\nu^{-1}(b_k))}(t)\int_{\nu(t)}^{b_k} v(s)\,ds\right\|_{X(0,\infty)}.\end{aligned} \tag{3.8}$$

Note that the assumption

$$\left\|u(t)\chi_{(0,\nu^{-1}(1))}(t)\int_{\nu(t)}^{1} v(s)\,ds\right\|_{X(0,\infty)} < \infty \tag{3.9}$$

together with the monotonicity of $u$ and $v$ implies that

$$\|u\chi_{(0,a)}\|_{X(0,\infty)} < \infty \quad \text{for every } a \in (0,\infty). \tag{3.10}$$

Indeed, since $u$ is nonincreasing, it is sufficient to show that $\|u\chi_{(0,\nu^{-1}(\frac{1}{2}))}\|_{X(0,\infty)} < \infty$, which follows from

$$\infty > \left\| u(t)\chi_{(0,\nu^{-1}(1))}(t)\int_{\nu(t)}^{1} v(s)\,ds \right\|_{X(0,\infty)} \geq \left\| u(t)\chi_{(0,\nu^{-1}(\frac{1}{2}))}(t)\int_{\nu(t)}^{1} v(s)\,ds \right\|_{X(0,\infty)} \geq \frac{v(1)}{2}\left\| u\chi_{(0,\nu^{-1}(\frac{1}{2}))}\right\|_{X(0,\infty)}.$$

Furthermore, note that (3.10) guarantees that

$$\limsup_{\tau\to\infty} v(\tau)\|u\chi_{(0,\nu^{-1}(\tau))}\|_{X(0,\infty)} < \infty \tag{3.11}$$

if and only if

$$\sup_{\tau\in[1,\infty)} v(\tau)\|u\chi_{(0,\nu^{-1}(\tau))}\|_{X(0,\infty)} < \infty. \tag{3.12}$$

Now, as for the first term on the right-hand side of (3.8), we have that

$$\begin{aligned} \left\| u(t)\chi_{(0,\nu^{-1}(b))}(t)\int_{\nu(t)}^{b} v(s)\,ds \right\|_{X(0,\infty)} &\leq \left\| u(t)\chi_{(0,\nu^{-1}(1))}(t)\int_{\nu(t)}^{1} v(s)\,ds \right\|_{X(0,\infty)} \\ &\quad + \left\| u(t)\chi_{(0,\nu^{-1}(1))}(t)\int_{1}^{b} v(s)\,ds \right\|_{X(0,\infty)} \\ &\quad + \left\| u(t)\chi_{(\nu^{-1}(1),\nu^{-1}(b))}(t)\int_{\nu(t)}^{b} v(s)\,ds \right\|_{X(0,\infty)} \\ &\leq A < \infty. \end{aligned} \tag{3.13}$$

Here,

$$\begin{aligned} A = {} & \left\| u(t)\chi_{(0,\nu^{-1}(1))}(t)\int_{\nu(t)}^{1} v(s)\,ds \right\|_{X(0,\infty)} + v(1)(b-1)\|u\chi_{(0,\nu^{-1}(1))}\|_{X(0,\infty)} \\ & + v(1)(b-1)\|u\chi_{(0,\nu^{-1}(b)-\nu^{-1}(1))}\|_{X(0,\infty)}. \end{aligned}$$

As for the second term on the right-hand side of (3.8), we have that

$$\begin{aligned} \left\| u(t)\chi_{(0,\nu^{-1}(a_k))}(t)\int_{a_k}^{b_k} v(s)\,ds \right\|_{X(0,\infty)} &\leq v(a_k)(b_k-a_k)\|u\chi_{(0,\nu^{-1}(a_k))}\|_{X(0,\infty)} \\ &\leq B(b_k-a_k), \end{aligned} \tag{3.14}$$

where $B$ is the supremum in (3.12), which is independent of $k$. Next,

$$\begin{aligned} \left\| u(t)\chi_{(\nu^{-1}(a_k),\nu^{-1}(b_k))}(t)\int_{\nu(t)}^{b_k} v(s)\,ds \right\|_{X(0,\infty)} &\leq \int_{a_k}^{b_k} v(s)\,ds\|u\chi_{(\nu^{-1}(a_k),\nu^{-1}(b_k))}\|_{X(0,\infty)} \\ &\leq v(a_k)(b_k-a_k)\|u\chi_{(\nu^{-1}(a_k),\nu^{-1}(b_k))}\|_{X(0,\infty)} \\ &\leq v(a_k)(b_k-a_k)\|u\chi_{(0,(\theta-1)\nu^{-1}(a_k))}\|_{X(0,\infty)} \\ &\leq \lceil\theta-1\rceil v(a_k)(b_k-a_k)\|u\chi_{(0,\nu^{-1}(a_k))}\|_{X(0,\infty)} \\ &\leq \lceil\theta-1\rceil B(b_k-a_k). \end{aligned} \tag{3.15}$$

Here, we used the monotonicity of $u$ and $v$ in the second inequality, (3.7) in the third one, and the monotonicity of $u$ in the fourth one. By combining (3.8) with (3.13), (3.14), and (3.15), we obtain that

$$\left\| u(t)\int_{\nu(t)}^{\infty} h(s)v(s)\,ds \right\|_{X(0,\infty)} \le A + \lceil\theta\rceil B\sum_k (b_k - a_k) \le A + \lceil\theta\rceil\theta B|E| < \infty.$$

Hence, $\|\chi_E\|_{Y(0,L)} < \infty$ provided that (3.9) and (3.11) are satisfied. The necessity of (3.9) is obvious because we have that

$$\left\| u(t)\chi_{(0,\nu^{-1}(1))}(t)\int_{\nu(t)}^{1} v(s)\,ds \right\|_{X(0,\infty)} \le \|\chi_{(0,1)}\|_{Y(0,\infty)}.$$

As for the necessity of (3.11), suppose that $\limsup_{\tau\to\infty} v(\tau)\left\|u\chi_{(0,\nu^{-1}(\tau))}\right\|_{X(0,\infty)} = \infty$. It follows that there is a sequence $\tau_k \nearrow \infty$, $k \to \infty$, such that

$$\lim_{k\to\infty} v(\tau_k)\|u\chi_{(0,\nu^{-1}(\tau_k))}\|_{X(0,\infty)} = \infty.$$

Since $\inf_{t\in[1,\infty)} \frac{\nu(\theta t)}{\nu(t)} > 1$, we can find an $\varepsilon > 0$ such that $\frac{\nu(\theta t)}{\nu(t)} \ge 1+\varepsilon$ for every $t \in [1,\infty)$. Moreover, we may clearly assume that $\tau_k \ge \nu(1)+1$ and $\frac{\tau_k}{\tau_k - 1} \le 1+\varepsilon$. Hence,

$$\nu^{-1}(\tau_k) - \nu^{-1}(\tau_k - 1) \le (\theta - 1)\nu^{-1}(\tau_k - 1) \tag{3.16}$$

inasmuch as $\frac{\nu(\theta\nu^{-1}(\tau_k-1))}{\nu(\nu^{-1}(\tau_k-1))} \ge 1+\varepsilon$. Using (3.16) and the fact that $u$ is nonincreasing, we obtain that

$$\begin{aligned}
\|u\chi_{(0,\nu^{-1}(\tau_k))}\|_{X(0,\infty)} &\le \|u\chi_{(0,\nu^{-1}(\tau_k-1))}\|_{X(0,\infty)} + \|u\chi_{(\nu^{-1}(\tau_k-1),\nu^{-1}(\tau_k))}\|_{X(0,\infty)}\\
&\le \|u\chi_{(0,\nu^{-1}(\tau_k-1))}\|_{X(0,\infty)} + \|u\chi_{(0,\nu^{-1}(\tau_k)-\nu^{-1}(\tau_k-1))}\|_{X(0,\infty)}\\
&\le \|u\chi_{(0,\nu^{-1}(\tau_k-1))}\|_{X(0,\infty)} + \|u\chi_{(0,(\theta-1)\nu^{-1}(\tau_k-1))}\|_{X(0,\infty)}\\
&= \lceil\theta\rceil\|u\chi_{(0,\nu^{-1}(\tau_k-1))}\|_{X(0,\infty)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\chi_{(0,1)}\|_{Y(0,\infty)} &\ge \left\| u(t)\chi_{(0,\nu^{-1}(\tau_k-1))}(t)\int_{\nu(t)}^{\infty} \chi_{(\tau_k-1,\tau_k)}(s)v(s)\,ds \right\|_{X(0,\infty)}\\
&\ge v(\tau_k)\|u\chi_{(0,\nu^{-1}(\tau_k-1))}\|_{X(0,\infty)} \ge \frac{1}{\lceil\theta\rceil} v(\tau_k)\|u\chi_{(0,\nu^{-1}(\tau_k))}\|_{X(0,\infty)},
\end{aligned}$$

which tends to $\infty$ as $k \to \infty$. Hence, $\|\chi_{(0,1)}\|_{Y(0,\infty)} = \infty$, and so $\|\cdot\|_{Y(0,\infty)}$ does not have property (P4).

*Property* (P5). Assume that $L < \infty$. Note that (3.4) together with $v(L^-) > 0$ implies that $\|u\|_{X(0,L)} < \infty$. Let $f \in \mathfrak{M}^+(0,L)$. Since $f^*$ is nonincreasing, we have that $\int_0^L f^*(s)\,ds \le 2\int_0^{\frac{L}{2}} f^*(s)\,ds$. Since the function $(0,L) \ni t \mapsto f^*(L-t)$ is equimeasurable with $f$, we have that

$$\begin{aligned}
\|f\|_{Y(0,L)} &\ge \left\| u(t)\int_{\nu(t)}^{L} f^*(L-s)v(s)\,ds \right\|_{X(0,L)}\\
&\ge v(L^-)\left\| u(t)\chi_{(0,\nu^{-1}(\frac{L}{2}))}(t)\int_{\nu(t)}^{L} f^*(L-s)\,ds \right\|_{X(0,L)}\\
&= v(L^-)\left\| u(t)\chi_{(0,\nu^{-1}(\frac{L}{2}))}(t)\int_{0}^{L-\nu(t)} f^*(s)\,ds \right\|_{X(0,L)}
\end{aligned}$$

$$\geq v(L^-)\|u\chi_{(0,\nu^{-1}(\frac{L}{2}))}\|_{X(0,L)}\int_0^{\frac{L}{2}} f^*(s)\,ds$$
$$\geq \frac{v(L^-)}{2}\|u\chi_{(0,\nu^{-1}(\frac{L}{2}))}\|_{X(0,L)}\int_0^{L} f^*(s)\,ds$$
$$\geq \frac{v(L^-)}{2}\|u\chi_{(0,\nu^{-1}(\frac{L}{2}))}\|_{X(0,L)}\int_0^{L} f(s)\,ds.$$

Here, we used (2.5) in the last inequality. Since $\frac{v(L^-)}{2}\|u\chi_{(0,\nu^{-1}(\frac{L}{2}))}\|_{X(0,L)} \in (0,\infty)$ does not depend on $f$, property (P5) follows. Assume now that $L = \infty$. Recall that (3.10) is satisfied provided that (3.9) is satisfied. Let $f \in \mathfrak{M}^+(0,\infty)$ and $E \subseteq (0,\infty)$ be of finite measure. The function $(0,\infty) \ni t \mapsto f^*(t-|E|)\chi_{(|E|,\infty)}(t)$ is equimeasurable with $f$. By arguing similarly to the case $L < \infty$, we obtain that

$$\|f\|_{Y(0,\infty)} \geq v(2|E|)\|u\chi_{(0,\nu^{-1}(|E|))}\|_{X(0,\infty)}\int_E f(s)\,ds,$$

whence property (P5) follows.

*Property* (P6). Since the relation $\sim$ is transitive, it plainly follows that $\|\cdot\|_{Y(0,L)}$ has property (P6).

Note that the necessity of (3.4) for $\|\cdot\|_{Y(0,L)}$ to be a rearrangement-invariant function norm was already proved in the paragraph devoted to property (P4).

Assume now that (3.4) is satisfied. We plainly have that

$$\|H_{u,v,\nu}f\|_{X(0,L)} \leq \|f\|_{Y(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L),$$

and so $H_{u,v,\nu} : Y(0,L) \to X(0,L)$ is bounded. Next, let $Z(0,L)$ be a rearrangement-invariant function space such that $H_{u,v,\nu} : Z(0,L) \to X(0,L)$ is bounded. For every $f \in \mathfrak{M}^+(0,\infty)$ and each $h \in \mathfrak{M}^+(0,L)$ equimeasurable with $f$, we have that

$$\|H_{u,v,\nu}h\|_{X(0,L)} \lesssim \|h\|_{Z(0,L)} = \|f\|_{Z(0,L)}.$$

Therefore,

$$\|f\|_{Y(0,L)} \lesssim \|f\|_{Z(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Hence, $Z(0,L) \hookrightarrow Y(0,L)$. Finally, we claim that (3.4) needs to be satisfied if there is any rearrangement-invariant function space $Z(0,L)$ such that $H_{u,v,\nu} : Z(0,L) \to X(0,L)$ is bounded. If $L < \infty$, we plainly have that

$$\left\|u(t)\int_{\nu(t)}^{L} v(s)\,ds\right\|_{X(0,L)} = \|H_{u,v,\nu}\chi_{(0,L)}\|_{X(0,L)} \lesssim \|\chi_{(0,L)}\|_{Z(0,L)} < \infty.$$

If $L = \infty$, we can argue as in the paragraph devoted to property (P4) to show that, if (3.4) is not satisfied, then

$$\sup_{h\sim\chi_{(0,1)}} \|H_{u,v,\nu}h\|_{X(0,\infty)} = \infty.$$

However, this implies, thanks to the boundedness of $H_{u,v,\nu} : Z(0,\infty) \to X(0,\infty)$,

$$\infty = \sup_{h\sim\chi_{(0,1)}} \|H_{u,v,\nu}h\|_{X(0,\infty)} \lesssim \|\chi_{(0,1)}\|_{Z(0,\infty)} < \infty,$$

which would be a contradiction. $\square$

*Remark* 3.4.

(1) The assumption $\nu \in \underline{D}^\infty$ is not overly restrictive. For example, it is satisfied whenever $\nu$ is equivalent to $t \mapsto t^\alpha b(t)$ near $\infty$ for some $\alpha > 0$ and a slowly varying function $b$ (cf. [35, Proposition 2.2]). On the other hand, $\nu(t) = \log^\alpha(t)$ near $\infty$, where $\alpha > 0$, is a typical example of a function not satisfying the assumption. The same remark (with the obvious modifications) is true for the assumption $\nu \in \underline{D}^0$, which will appear in Proposition 5.4.
(2) When $u \equiv 1$, (3.11) is equivalent to

$$\limsup_{t\to\infty} v(\nu(t))\|\chi_{(0,t)}\|_{X(0,\infty)} < \infty.$$

(3) The functional (3.3) is quite complicated; however, we shall see in Section 4 that it can often be significantly simplified.
(4) Let $Y_1(0,L)$ and $Y_2(0,L)$ be the optimal domain spaces for $R_{u_1,v_1,\nu_1}$ and $H_{u_2,v_2,\nu_2}$, respectively. Note that $(R_{u_1,v_1,\nu_1} + H_{u_2,v_2,\nu_2}) : Z(0,L) \to X(0,L)$ is bounded if and only if both $R_{u_1,v_1,\nu_1}$ and $H_{u_2,v_2,\nu_2}$ are bounded from $Z(0,L)$ to $X(0,L)$. Consequently, $Y_1(0,L) \cap Y_2(0,L)$ is the optimal domain space for $R_{u_1,v_1,\nu_1} + H_{u_2,v_2,\nu_2}$ and $X(0,L)$.

## 3.2 | Optimal target spaces

We start with an easy but useful observation concerning the Hardy-type operators defined by (1.1) and (1.2). Let $u, v : (0,L) \to (0,\infty)$ be measurable functions. Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. The operators $R_{u,v,\nu}$ and $H_{u,v,\nu^{-1}}$ are in a sense dual to each other. More precisely, by using the Fubini theorem, one can easily verify that

$$\int_0^L f(t) R_{u,v,\nu} g(t)\,dt = \int_0^L g(t) H_{u,v,\nu^{-1}} f(t)\,dt \quad \text{for every } f, g \in \mathfrak{M}^+(0,L). \tag{3.17}$$

Here, $\nu^{-1}$ is the inverse function to $\nu$.

The validity of (3.17) has an unsurprising, well-known consequence, which we state here for future reference (see also Corollary 4.9).

**Proposition 3.5.** *Let $\|\cdot\|_{X(0,L)}$, $\|\cdot\|_{Y(0,L)}$ be rearrangement-invariant function norms.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection.*
*(2) Let $u, v : (0,L) \to (0,\infty)$ be measurable.*

*We have that*

$$\sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu} f\|_{Y(0,L)} = \sup_{\|g\|_{Y'(0,L)}\le 1} \|H_{u,v,\nu^{-1}} g\|_{X'(0,L)}. \tag{3.18}$$

*In particular,*

$$\begin{aligned} R_{u,v,\nu} &: X(0,L) \to Y(0,L) \quad \textit{is bounded if and only if} \\ H_{u,v,\nu-1} &: Y'(0,L) \to X'(0,L) \quad \textit{is bounded.} \end{aligned} \tag{3.19}$$

*Proof.* We have that

$$\begin{aligned} \sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu} f\|_{Y(0,L)} &= \sup_{\|f\|_{X(0,L)}\le 1} \sup_{\|g\|_{Y'(0,L)}\le 1} \int_0^L R_{u,v,\nu} f(t)|g(t)|\,dt \\ &= \sup_{\|f\|_{X(0,L)}\le 1} \sup_{\|g\|_{Y'(0,L)}\le 1} \int_0^L |f(t)| H_{u,v,\nu^{-1}} g(t)\,dt \\ &= \sup_{\|g\|_{Y'(0,L)}\le 1} \|H_{u,v,\nu^{-1}} g\|_{X'(0,L)} \end{aligned}$$

thanks to (2.9), (3.17), and (2.8). □

*Remark* 3.6. Thanks to (3.19) and (2.10), $Y(0,L)$ is the optimal target space for the operator $H_{u,v,\nu}$ and $X(0,L)$ if and only if $Y'(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $X'(0,L)$. Similarly, $Y(0,L)$ is the optimal target space for the operator $R_{u,v,\nu}$ and $X(0,L)$ if and only if $Y'(0,L)$ is the optimal domain space for the operator $H_{u,v,\nu^{-1}}$ and $X'(0,L)$.

As immediate corollaries of Remark 3.6 combined with Proposition 3.1 and Proposition 3.3, we obtain the following descriptions of the optimal target spaces for the operators $H_{u,v,\nu}$ and $R_{u,v,\nu}$, respectively.

**Proposition 3.7.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

- *Let $\nu, u, v$ be as in Proposition 3.1.*

*Assume that $\xi \in X'(0,L)$, where $\xi$ is defined by (3.1) with $\nu$ replaced by $\nu^{-1}$. Let $\|\cdot\|_{Y(0,L)}$ be the rearrangement-invariant function norm whose associate function norm $\|\cdot\|_{Y'(0,L)}$ is defined as*

$$\|f\|_{Y'(0,L)} = \|R_{u,v,\nu^{-1}}(f^*)\|_{X'(0,L)},\ f \in \mathfrak{M}^+(0,L).$$

*The rearrangement-invariant function space $Y(0,L)$ is the optimal target space for the operator $H_{u,v,\nu}$ and $X(0,L)$. Moreover, if $\xi \notin X'(0,L)$, then there is no rearrangement-invariant function space $Z(0,L)$ such that $H_{u,v,\nu} : X(0,L) \to Z(0,L)$ is bounded.*

**Proposition 3.8.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

- *Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. If $L = \infty$, assume that $\nu^{-1} \in \underline{D}^\infty$.*
- *Let $u, v$ be as in Proposition 3.3.*

*Assume that*

$$\begin{cases} u(t)\int_{\nu^{-1}(t)}^{L} v(s)\,ds \in X'(0,L) & \text{if } L < \infty, \\ u(t)\chi_{(0,\nu(1))}(t)\int_{\nu^{-1}(t)}^{1} v(s)\,ds \in X'(0,\infty) \text{ and} & \\ \quad \limsup_{\tau\to\infty} v(\tau)\|u\chi_{(0,\nu(\tau))}\|_{X'(0,\infty)} < \infty & \text{if } L = \infty. \end{cases} \tag{3.20}$$

*Let $\|\cdot\|_{Y(0,L)}$ be the rearrangement-invariant function norm whose associate function norm $\|\cdot\|_{Y'(0,L)}$ is defined as*

$$\|f\|_{Y'(0,L)} = \sup_{h\sim f} \|H_{u,v,\nu^{-1}}h\|_{X'(0,L)},\ f \in \mathfrak{M}^+(0,L). \tag{3.21}$$

*Here, the supremum extends over all $h \in \mathfrak{M}^+(0,L)$ equimeasurable with $f$. The rearrangement-invariant function space $Y(0,L)$ is the optimal target space for the operator $R_{u,v,\nu}$ and $X(0,L)$. Moreover, if (3.20) is not satisfied, then there is no rearrangement-invariant function space $Z(0,L)$ such that $R_{u,v,\nu} : X(0,L) \to Z(0,L)$ is bounded.*

*Remark* 3.9. Let $Y_1(0,L)$ and $Y_2(0,L)$ be the optimal target spaces for $R_{u_1,v_1,\nu_1}$ and $H_{u_2,v_2,\nu_2}$, respectively. Note that $(R_{u_1,v_1,\nu_1} + H_{u_2,v_2,\nu_2}) : X(0,L) \to Z(0,L)$ is bounded if and only if both $R_{u_1,v_1,\nu_1}$ and $H_{u_2,v_2,\nu_2}$ are bounded from $X(0,L)$ to $Z(0,L)$. It follows that $(Y_1 + Y_2)(0,L)$ is the optimal target space for $R_{u_1,v_1,\nu_1} + H_{u_2,v_2,\nu_2}$ and $X(0,L)$.

# 4 | SIMPLIFICATION OF OPTIMAL FUNCTION NORMS AND THEIR CONNECTION WITH INTERPOLATION

## 4.1 | Simplification of optimal function norms

The description of the optimal domain spaces for the operator $H_{u,v,\nu}$ is complicated by the fact that the functional $\mathfrak{M}^+(0,L) \ni f \mapsto \|H_{u,v,\nu}(f^*)\|_{X(0,L)}$ is usually not a rearrangement-invariant function norm. However, it actually is a rearrangement-invariant function norm when $u$, $v$, and $\nu$ are related to each other in such a way that the function

$R_{u,v,\nu^{-1}}(g^*)$ is nonincreasing for every $g \in \mathfrak{M}^+(0,L)$. This fact is the content of the following proposition, in which we omit its obvious consequences for optimal spaces. In the light of Proposition 3.8, the situation is similar for the optimal target spaces for the operator $R_{u,v,\nu}$.

**Proposition 4.1.** *Let* $\|\cdot\|_{X(0,L)}$ *be a rearrangement-invariant function norm.*

*(1) Let* $\nu : (0,L) \to (0,L)$ *be an increasing bijection.*
*(2) Let* $u : (0,L) \to (0,\infty)$ *be a nondegenerate nonincreasing function.*
*(3) Let* $v : (0,L) \to (0,\infty)$ *be defined by*

$$\frac{1}{v(t)} = \int_0^{\nu^{-1}(t)} u(s)\,ds \quad \text{for every } t \in (0,L).$$

*Set*

$$\|f\|_{Y(0,L)} = \|H_{u,v,\nu}(f^*)\|_{X(0,L)}, \ f \in \mathfrak{M}^+(0,L).$$

*The functional* $\|\cdot\|_{Y(0,L)}$ *is a rearrangement-invariant function norm if and only if*

$$\left\| u(t)\chi_{(0,\nu^{-1}(a))}(t) \int_{\nu(t)}^{a} \frac{1}{U(\nu^{-1}(s))}\,ds \right\|_{X(0,L)} < \infty, \tag{4.1}$$

*where a is defined by (3.2).*

*Proof.* We only sketch the proof, which is significantly easier than that of Proposition 3.3. The functional $\|\cdot\|_{Y(0,L)}$ plainly possesses properties (P2), (P3), and (P6). It is easy to see that $\|\cdot\|_{Y(0,L)}$ has property (P4) if and only if (4.1) is satisfied. To this end, note that (4.1) implies (3.10). As for property (P1), only the subadditivity needs a comment. The key observation is that $(0,L) \ni t \mapsto R_{u,v,\nu^{-1}}(h^*)(t)$ is nonincreasing for every $h \in \mathfrak{M}^+(0,L)$ inasmuch as it is the integral mean of a nonnegative nonincreasing function over the interval $(0,\nu^{-1}(t))$ with respect to the measure $u(s)\,ds$. Hence, thanks to (2.11), (3.17), and (2.3) combined with the Hardy lemma (2.6), we have that

$$\begin{aligned}
\|f+g\|_{Y(0,L)} = \|H_{u,v,\nu}((f+g)^*)\|_{X(0,L)} &= \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\le 1}} \int_0^L H_{u,v,\nu}((f+g)^*)(t)h^*(t)\,dt\\
&= \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\le 1}} \int_0^L (f+g)^*(t)R_{u,v,\nu^{-1}}(h^*)(t)\,dt\\
&\le \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\le 1}} \int_0^L f^*(t)R_{u,v,\nu^{-1}}(h^*)(t)\,dt\\
&\quad + \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\le 1}} \int_0^L g^*(t)R_{u,v,\nu^{-1}}(h^*)(t)\,dt\\
&= \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\le 1}} \int_0^L H_{u,v,\nu}(f^*)(t)h^*(t)\,dt\\
&\quad + \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\le 1}} \int_0^L H_{u,v,\nu}(g^*)(t)h^*(t)\,dt\\
&= \|f\|_{Y(0,L)} + \|g\|_{Y(0,L)}
\end{aligned}$$

for every $f, g \in \mathfrak{M}^+(0,L)$. Finally, as for the validity of property (P5), owing to (2.11), (3.17), the monotonicity of $R_{u,v,\nu^{-1}}(g^*)$, and the Hardy–Littlewood inequality (2.4), we have that

$$
\begin{aligned}
\|f\|_{Y(0,L)} \geq \|f\chi_E\|_{Y(0,L)} &= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \int_0^L (f\chi_E)^*(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt \\
&= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \int_0^{|E|} (f\chi_E)^*(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt \\
&\geq \int_0^{|E|} (f\chi_E)^*(t)\,dt \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} R_{u,v,\nu^{-1}}(g^*)(|E|) \\
&\geq R_{u,v,\nu^{-1}}\left(\frac{\chi_{(0,|E|)}}{\|\chi_{(0,|E|)}\|_{X'(0,L)}}\right)(|E|) \int_0^{|E|} (f\chi_E)^*(t)\,dt \\
&\geq \frac{v(|E|)}{\|\chi_{(0,|E|)}\|_{X'(0,L)}} U(\nu^{-1}(|E|)) \int_E f(t)\,dt
\end{aligned}
$$

for every $f \in \mathfrak{M}^+(0,L)$ and $E \subseteq (0,L)$ having finite measure. □

In general, when the functions $u$, $v$, and $\nu$ are not related to each other in the particular way as in Proposition 4.1, we have to live with the complicated functional (3.3). Nevertheless, we shall see that the functional is often equivalent to a significantly more manageable functional (cf. [30, Theorem 4.2]). To this end, we need to introduce a supremum operator. For a fixed measurable function $\varphi : (0,L) \to (0,\infty)$, we define the operator $T_\varphi$ as

$$
T_\varphi f(t) = \frac{1}{\varphi(t)} \operatorname*{ess\,sup}_{s\in[t,L)} \varphi(s) f^*(s),\ t \in (0,L),\ f \in \mathfrak{M}(0,L). \tag{4.2}
$$

Note that $T_\varphi f(t) = \frac{1}{\varphi(t)} \sup_{s\in[t,L)} \varphi(s) f^*(s)$ for every $t \in (0,L)$ provided that $\varphi$ is nondecreasing and/or right-continuous. If $\varphi$ is nonincreasing, we have that $T_\varphi f(t) = \frac{f^*(t)}{\varphi(t)} \varphi(t^+)$ for every $t \in (0,L)$, and so $T_\varphi f = f^*$ possibly up to a countably many points.

**Proposition 4.2.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. If $L = \infty$, assume that $\nu \in \underline{D}^\infty$.*
*(2) Let $u : (0,L) \to (0,\infty)$ be nonincreasing.*
*(3) Let $v : (0,L) \to (0,\infty)$ be defined by*

$$
\frac{1}{v(t)} = \int_0^{\nu^{-1}(t)} \xi(s)\,ds \quad \text{for every } t \in (0,L), \tag{4.3}
$$

*where $\xi : (0,L) \to (0,\infty)$ is a measurable function. If $L < \infty$, assume that $v(L^-) > 0$. Furthermore, assume that the operator $T_\varphi$ defined by (4.2) with $\varphi = u/\xi$ is bounded on $X'(0,L)$.*

*Assume that*

$$
\left\| u(t)\chi_{(0,\nu^{-1}(a))}(t) \int_{\nu(t)}^a v(s)\,ds \right\|_{X(0,L)} < \infty,
$$

*where a is defined by (3.2). Let $\|\cdot\|_{Y(0,L)}$ be the functional defined by (3.3) and set*

$$\|f\|_{Z(0,L)} = \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L f^*(s)v(t)\int_0^{\nu^{-1}(t)} T_\varphi g(s)u(s)\,ds\,dt,\ f\in\mathfrak{M}^+(0,L).$$

*The functionals $\|\cdot\|_{Y(0,L)}$ and $\|\cdot\|_{Z(0,L)}$ are rearrangement-invariant function norms. Furthermore, we have that*

$$\begin{aligned}\|H_{u,v,\nu}(f^*)\|_{X(0,L)} &\le \sup_{h\sim f}\|H_{u,v,\nu}h\|_{X(0,L)} \le \|f\|_{Z(0,L)}\\ &\le \|T_\varphi\|_{X'(0,L)}\|H_{u,v,\nu}(f^*)\|_{X(0,L)}\end{aligned} \tag{4.4}$$

*for every $f\in\mathfrak{M}^+(0,L)$. Here, $\|T_\varphi\|_{X'(0,L)}$ stands for the operator norm of $T_\varphi$ on $X'(0,L)$. In particular, the rearrangement-invariant function norms $\|\cdot\|_{Y(0,L)}$ and $\|\cdot\|_{Z(0,L)}$ are equivalent.*

*Proof.* Since $f\sim f^*$ for every $f\in\mathfrak{M}^+(0,L)$, the first inequality in (4.4) plainly holds. As for the second inequality, note that the function $(0,L)\ni t\mapsto R_{u,v,\nu^{-1}}(T_\varphi g)(t)$ is nonincreasing for every $g\in\mathfrak{M}^+(0,L)$. Indeed, it is the integral mean of the nonincreasing function $(0,L)\ni s\mapsto \operatorname{ess\,sup}_{\tau\in[s,L)}\varphi(\tau)g^*(\tau)$ over the interval $(0,\nu^{-1}(t))$ with respect to the measure $\xi(s)\,ds$. Consequently, for every $f\in\mathfrak{M}^+(0,L)$ and every $h\in\mathfrak{M}^+(0,L)$ equimeasurable with $f$, we have that

$$\begin{aligned}\|H_{u,v,\nu}h\|_{X(0,L)} &= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L h(t)R_{u,v,\nu^{-1}}(g^*)(t)\,dt\\ &\le \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L h(t)R_{u,v,\nu^{-1}}(T_\varphi g)(t)\,dt\\ &\le \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L h^*(t)R_{u,v,\nu^{-1}}(T_\varphi g)(t)\,dt\\ &= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L f^*(t)R_{u,v,\nu^{-1}}(T_\varphi g)(t)\,dt\\ &= \|f\|_{Z(0,L)}.\end{aligned} \tag{4.5}$$

Here, we used (2.11) together with (3.17) in the first equality, the pointwise estimate $g^*(t)\le T_\varphi g(t)$ for a.e. $t\in(0,L)$ in the first inequality, the Hardy–Littlewood inequality (2.4) in the second inequality, and the equimeasurability of $f$ and $h$ in the last inequality. Hence, the second inequality in (4.4) follows from (4.5). As for the third inequality in (4.4), we have that

$$\begin{aligned}\sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L f^*(t)R_{u,v,\nu^{-1}}(T_\varphi g)(t)\,dt &= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L T_\varphi g(t)H_{u,v,\nu}(f^*)(t)\,dt\\ &\le \|H_{u,v,\nu}(f^*)\|_{X(0,L)} \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \|T_\varphi g\|_{X'(0,L)}\\ &= \|T_\varphi\|_{X'(0,L)}\|H_{u,v,\nu}(f^*)\|_{X(0,L)}\end{aligned}$$

for every $f\in\mathfrak{M}^+(0,L)$ thanks to (3.17) and the Hölder inequality (2.14).

Second, we shall prove that the functional $\|\cdot\|_{Y(0,L)}$, defined by (3.3), is a rearrangement-invariant function norm. If $L<\infty$, this follows immediately from Proposition 3.3. If $L=\infty$, owing to Proposition 3.3 again, we only need to verify that (3.11) is satisfied. To this end, it follows from the proof of property (P4) of $\|\cdot\|_{Y(0,L)}$ that, if (3.11) did not hold, then we would have

$$\sup_{h\sim\chi_{(0,1)}}\|H_{u,v,\nu}h\|_{X(0,\infty)}=\infty.$$

However, thanks to (4.4), we have that

$$\begin{aligned}\sup_{h\sim\chi_{(0,1)}}\|H_{u,v,\nu}h\|_{X(0,\infty)}&\approx\|H_{u,v,\nu}\chi_{(0,1)}\|_{X(0,\infty)}\\&=\left\|u(t)\chi_{(0,\nu^{-1}(1))}(t)\int_{\nu(t)}^1 v(s)\,ds\right\|_{X(0,\infty)}<\infty.\end{aligned}$$

Therefore, (3.11) is satisfied.

Finally, now that we know that the functionals $\|\cdot\|_{Y(0,L)}$ and $\|\cdot\|_{Z(0,L)}$ are equivalent and the former is a rearrangement-invariant function norm, it readily follows that $\|\cdot\|_{Z(0,L)}$, too, is a rearrangement-invariant function norm once we observe that $\|\cdot\|_{Z(0,L)}$ is subadditive. The subadditivity follows from

$$\begin{aligned}\|f+g\|_{Z(0,L)}&=\sup_{\substack{h\in\mathfrak{M}^+(0,L)\\\|h\|_{X'(0,L)}\le1}}\int_0^L(f+g)^*(t)R_{u,v,\nu^{-1}}(T_\varphi h)(t)\,dt\\&\le\sup_{\substack{h\in\mathfrak{M}^+(0,L)\\\|h\|_{X'(0,L)}\le1}}\int_0^L f^*(t)R_{u,v,\nu^{-1}}(T_\varphi h)(t)\,dt\\&\quad+\sup_{\substack{h\in\mathfrak{M}^+(0,L)\\\|h\|_{X'(0,L)}\le1}}\int_0^L g^*(t)R_{u,v,\nu^{-1}}(T_\varphi h)(t)\,dt\\&=\|f\|_{Z(0,L)}+\|g\|_{Z(0,L)}\quad\text{for every } f,g\in\mathfrak{M}^+(0,L).\end{aligned}$$

Here, we used (2.3) together with the Hardy lemma (2.6) (recall that the function $R_{u,v,\nu^{-1}}(T_\varphi h)$ is nonincreasing for every $h\in\mathfrak{M}^+(0,L)$). □

*Remark* 4.3.

(i) If $\varphi=u/\xi$ is (equivalent to) a nonincreasing function, $T_\varphi f(t)$ is (equivalent to) $f^*(t)$ for a.e. $t\in(0,L)$; hence $T_\varphi$ is bounded on every rearrangement-invariant function space in this case. Furthermore, when $\varphi=u/\xi$ is nonincreasing, the norm of $T_\varphi$ on every rearrangement-invariant function space is equal to 1; therefore, all the inequalities in (4.4) are actually equalities (cf. Proposition 4.1) in this case.

(ii) The boundedness of $T_\varphi$ on a large number of rearrangement-invariant function spaces is characterized by [34, Theorem 3.2].

By combining Proposition 4.2 and Proposition 3.8, we obtain the following proposition. It tells us that the optimal target space for the operator $R_{u,v,\nu}$ and a rearrangement-invariant function space $X(0,L)$ often has a much more manageable description than that given by Proposition 3.8. This is the case if the supremum operator $T_\varphi$ defined by (4.2) with an appropriate function $\varphi$ is bounded on $X(0,L)$.

**Proposition 4.4.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu:(0,L)\to(0,L)$ be an increasing bijection. If $L=\infty$, assume that $\nu^{-1}\in\underline{D}^\infty$.*
*(2) Let $u:(0,L)\to(0,\infty)$ be nonincreasing.*

*(3) Let $v : (0,L) \to (0,\infty)$ be defined by*

$$\frac{1}{v(t)} = \int_0^{\nu(t)} \xi(s)\,ds \quad \text{for every } t \in (0,L),$$

*where $\xi : (0,L) \to (0,\infty)$ is a measurable function. If $L < \infty$, assume that $v(L^-) > 0$. Furthermore, assume that the operator $T_\varphi$ defined by (4.2) with $\varphi = u/\xi$ is bounded on $X(0,L)$.*

*Assume that*

$$\left\| u(t)\chi_{(0,\nu(a))}(t) \int_{\nu^{-1}(t)}^{a} v(s)\,ds \right\|_{X'(0,L)} < \infty,$$

*where a is defined by (3.2). Let $\|\cdot\|_{Y(0,L)}$ be the rearrangement-invariant function norm whose associate function norm $\|\cdot\|_{Y'(0,L)}$ is defined as*

$$\|f\|_{Y'(0,L)} = \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X(0,L)}\le 1}} \int_0^L f^*(s)v(t)\int_0^{\nu(t)} T_\varphi g(s)u(s)\,ds\,dt,\ f \in \mathfrak{M}^+(0,L).$$

*The rearrangement-invariant function space $Y(0,L)$ is the optimal target space for the operator $R_{u,v,\nu}$ and $X(0,L)$. Moreover,*

$$\|H_{u,v,\nu^{-1}}(f^*)\|_{X'(0,L)} \le \|f\|_{Y'(0,L)} \le \|T_\varphi\|_{X(0,L)}\|H_{u,v,\nu^{-1}}(f^*)\|_{X'(0,L)}$$

*for every $f \in \mathfrak{M}^+(0,L)$. Here, $\|T_\varphi\|_{X(0,L)}$ stands for the operator norm of $T_\varphi$ on $X(0,L)$.*

*Remark* 4.5. Owing to Remark 3.6, Proposition 4.4 can also be used to get a simpler description of optimal domain spaces for the operator $H_{u,v,\nu}$.

A great deal of our effort has been devoted to describing optimal rearrangement-invariant function spaces. A natural, somewhat related question is, can every rearrangement-invariant function space be an optimal space? Suppose that $Z(0,L)$ is the optimal domain space for $H_{u,v,\nu}$ and $X(0,L)$, and denote by $W(0,L)$ the optimal target space for $H_{u,v,\nu}$ and $Z(0,L)$. Owing to the optimality of $W(0,L)$, we immediately see that $W(0,L) \hookrightarrow X(0,L)$. What is not obvious, however, is whether the opposite embedding, too, (is)/(can be) true. This leads us to the following theorem, which shows that the notion of being an optimal function space is related to the question of whether the complicated functional (3.3) can be simplified.

**Theorem 4.6.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. If $L = \infty$, assume that $\nu \in \underline{D}^\infty$.*
*(2) Let $u : (0,L) \to (0,\infty)$ be a nondegenerate nonincreasing function. If $L < \infty$, assume that $u(L^-) > 0$.*
*(3) Let $v : (0,L) \to (0,\infty)$ be a nonincreasing function. If $L < \infty$, assume that $v(L^-) > 0$.*

*Let $\|\cdot\|_{Y(0,L)}$ be the functional defined by (3.3). The following three statements are equivalent.*

*(i) The space $X(0,L)$ is the optimal target space for the operator $H_{u,v,\nu}$ and some rearrangement-invariant function space.*
*(ii) The space $X'(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and some rearrangement-invariant function space.*
*(iii) We have that*

$$\|f\|_{X'(0,L)} \approx \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{Y(0,L)}\le 1}} \int_0^L g(t)R_{u,v,\nu^{-1}}(f^*)(t)\,dt \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{4.6}$$

*Finally, assume, in addition, that*

*(1) $v$ is defined by (4.3) with $\xi$ satisfying*

$$\frac{u(t)}{U(t)} \int_0^t \xi(s)\,ds \lesssim \xi(t) \quad \text{for a.e. } t \in (0, L), \tag{4.7}$$

*(2) the function $\varphi \circ \nu^{-1}$, where $\varphi = u/\xi$, is equivalent to a quasiconcave function.*

*Then, each of the three equivalent statements above implies that*

*(iv) the operator $T_\varphi$, defined by (4.2), is bounded on $X'(0, L)$.*

*Proof.* We start off by observing that each of the three equivalent statements implies that the functional $\|\cdot\|_{Y(0,L)}$ is actually a rearrangement-invariant function norm. Statements (i) and (ii) imply it thanks to Proposition 3.3 and Proposition 3.8, respectively. If we assume (iii), then, in particular, the set $\{g \in \mathfrak{M}^+(0, L) : \|g\|_{Y(0,L)} \le 1\}$ needs to contain a function $g \in \mathfrak{M}^+(0, L)$ not equal to 0 a.e. Thanks to Proposition 3.3 and its proof, $\|g\|_{Y(0,L)} = \infty$ for every $g \in \mathfrak{M}^+(0, L)$ not equal to 0 a.e. if $\varrho$ fails to be a rearrangement-invariant function norm. Hence, $\|\cdot\|_{Y(0,L)}$ is a rearrangement-invariant function norm if (iii) is assumed. Therefore, in all of the cases, we are entitled to denote the corresponding rearrangement-invariant function space over $(0, L)$ by $Y(0, L)$. Moreover, note that (4.6) actually reads as

$$\|f\|_{X'(0,L)} \approx \|R_{u,v,\nu^{-1}}(f^*)\|_{Y'(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0, L). \tag{4.8}$$

Second, statements (i) and (ii) are clearly equivalent to each other owing to Remark 3.6.

Next, the proof of the fact that (ii) implies (iii) is based on the following important observation. If $X'(0, L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and a rearrangement-invariant function space $Z(0, L)$, then, in particular, $R_{u,v,\nu^{-1}} : X'(0, L) \to Z(0, L)$ is bounded. Consequently, by virtue of Proposition 3.8, the rearrangement-invariant function space whose associate function norm is $\|\cdot\|_{Y(0,L)}$ is the optimal target space for the operator $R_{u,v,\nu^{-1}}$ and $X'(0, L)$. By (2.10), this optimal target space is actually the space $Y'(0, L)$. Owing to Proposition 3.1, the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $Y'(0, L)$ exists, and we denote it by $W(0, L)$. Moreover,

$$\|f\|_{W(0,L)} \approx \|R_{u,v,\nu^{-1}}(f^*)\|_{Y'(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0, L). \tag{4.9}$$

The crucial observation is that we have, in fact, that $X'(0, L) = W(0, L)$. The embedding $X'(0, L) \hookrightarrow W(0, L)$ is valid because $R_{u,v,\nu^{-1}} : X'(0, L) \to Y'(0, L)$ is bounded and $W(0, L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $Y'(0, L)$. The validity of the opposite embedding is slightly more complicated. Since $R_{u,v,\nu^{-1}} : X'(0, L) \to Z(0, L)$ is bounded and $Y'(0, L)$ is the optimal target space for the operator $R_{u,v,\nu^{-1}}$ and $X'(0, L)$, we have that $Y'(0, L) \hookrightarrow Z(0, L)$. Consequently, since $R_{u,v,\nu^{-1}} : W(0, L) \to Y'(0, L)$ is bounded, so is $R_{u,v,\nu^{-1}} : W(0, L) \to Z(0, L)$. Using the fact that $X'(0, L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $Z(0, L)$, we obtain that $W(0, L) \hookrightarrow X'(0, L)$. Now that we know that $X'(0, L) = W(0, L)$, (4.8) follows from (4.9).

Next, note that (iii) implies (ii). Indeed, (4.8) coupled with Proposition 3.1 tells us that $X'(0, L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $Y'(0, L)$.

Finally, it only remains to prove that (iii) implies (iv) under the additional assumptions. By (4.8), we have that

$$\begin{aligned}
\|T_\varphi f\|_{X'(0,L)} &\approx \|R_{u,v,\nu^{-1}}((T_\varphi f)^*)\|_{Y'(0,L)} \approx \|R_{u,v,\nu^{-1}}(T_\varphi f)\|_{Y'(0,L)} \\
&= \left\| v(t) \int_0^{\nu^{-1}(t)} \xi(s) \operatorname*{ess\,sup}_{\tau \in [s,L)} \varphi(\tau) f^*(\tau)\,ds \right\|_{Y'(0,L)} \\
&\le \left\| v(t) \int_0^{\nu^{-1}(t)} \xi(s) \operatorname*{ess\,sup}_{\tau \in [s,\nu^{-1}(t))} \varphi(\tau) f^*(\tau)\,ds \right\|_{Y'(0,L)}
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| v(t)\left( \operatorname*{ess\,sup}_{\tau\in[\nu^{-1}(t),L)} \varphi(\tau) f^*(\tau)\right) \int_0^{\nu^{-1}(t)} \xi(s)\,ds \right\|_{Y'(0,L)} \\
&= \left\| v(t) \int_0^{\nu^{-1}(t)} \xi(s) \operatorname*{ess\,sup}_{\tau\in[s,\nu^{-1}(t))} \varphi(\tau) f^*(\tau)\,ds \right\|_{Y'(0,L)} \\
&+ \left\| \operatorname*{ess\,sup}_{\tau\in[\nu^{-1}(t),L)} \varphi(\tau) f^*(\tau) \right\|_{Y'(0,L)}. \qquad (4.10)
\end{aligned}
$$

Here, we used the fact that $T_\varphi f$ is equivalent to a nonincreasing function for every $f \in \mathfrak{M}^+(0,L)$, and the multiplicative constants in this equivalence are independent of $f$. Since $\varphi$ is equivalent to a continuous nondecreasing function and $\xi$ satisfies (4.7), it follows from [34, Theorem 3.2] that

$$
\int_0^{\nu^{-1}(t)} \xi(s) \operatorname*{ess\,sup}_{\tau\in[s,\nu^{-1}(t))} \varphi(\tau) f^*(\tau)\,ds \lesssim \int_0^{\nu^{-1}(t)} f^*(s) u(s)\,ds \quad \text{for every } t\in(0,L).
$$

Hence,

$$
\left\| v(t) \int_0^{\nu^{-1}(t)} \xi(s) \operatorname*{ess\,sup}_{\tau\in[s,\nu^{-1}(t))} \varphi(\tau) f^*(\tau)\,ds \right\|_{Y'(0,L)} \lesssim \|R_{u,v,\nu^{-1}}(f^*)\|_{Y'(0,L)}. \tag{4.11}
$$

Since the function $\varphi\circ\nu^{-1}$ is equivalent to a quasiconcave function, it follows from [30, Lemma 4.10] that

$$
\left\| \operatorname*{ess\,sup}_{\tau\in[\nu^{-1}(t),L)} \varphi(\tau) f^*(\tau) \right\|_{Y'(0,L)} \lesssim \|\varphi(\nu^{-1}(t)) f^*(\nu^{-1}(t))\|_{Y'(0,L)}.
$$

We note that, although [30, Lemma 4.10] deals only with the case $L=\infty$, its proof translates verbatim to the case of $L\in(0,\infty)$. Furthermore, we have that

$$
\begin{aligned}
\|\varphi(\nu^{-1}(t)) f^*(\nu^{-1}(t))\|_{Y'(0,L)} &\lesssim \left\| \frac{U(\nu^{-1}(t))}{\int_0^{\nu^{-1}(t)} \xi(s)\,ds} f^*(\nu^{-1}(t)) \right\|_{Y'(0,L)} \\
&\le \|R_{u,v,\nu^{-1}}(f^*)\|_{Y'(0,L)}. \qquad (4.12)
\end{aligned}
$$

Here, we used the fact that $\xi$ satisfies (4.7) in the first inequality and the monotonicity of $f^*$ in the second one. By combining (4.10) with (4.11) and (4.12) and using (4.8), we obtain that

$$
\|T_\varphi f\|_{X'(0,L)} \lesssim \|R_{u,v,\nu^{-1}}(f^*)\|_{Y'(0,L)} \approx \|f\|_{X'(0,L)} \quad \text{for every } f\in\mathfrak{M}^+(0,L);
$$

hence $T_\varphi$ is bounded on $X'(0,L)$. □

*Remark* 4.7.

(i) If $X'(0,L)$ is the optimal domain space for $R_{u,v,\nu^{-1}}$ and some rearrangement-invariant function space $Y(0,L)$, then $X'(0,L)$ is actually the optimal domain space for $R_{u,v,\nu^{-1}}$ and its own optimal target space. This follows from the following. Thanks to Proposition 3.8 and Proposition 3.1, we are entitled to denote by $Z(0,L)$ the optimal target space for $R_{u,v,\nu^{-1}}$ and $X'(0,L)$ and by $W(0,L)$ the optimal domain space for $R_{u,v,\nu^{-1}}$ and $Z(0,L)$. We need to show that $X'(0,L)=W(0,L)$. On the one hand, since $R_{u,v,\nu^{-1}} : X'(0,L)\to Z(0,L)$ is bounded and $W(0,L)$ is the optimal domain space for $R_{u,v,\nu^{-1}}$ and $Z(0,L)$, we have that $X'(0,L)\hookrightarrow W(0,L)$. On the other hand, since $R_{u,v,\nu^{-1}} : X'(0,L)\to Y(0,L)$

is bounded and $Z(0,L)$ is the optimal target space for $R_{u,v,\nu^{-1}}$ and $X'(0,L)$, we have that $Z(0,L) \hookrightarrow Y(0,L)$. Consequently, $R_{u,v,\nu^{-1}} : W(0,L) \to Y(0,L)$ is bounded. Finally, since $X'(0,L)$ is the optimal domain space for $R_{u,v,\nu^{-1}}$ and $Y(0,L)$, we obtain that $W(0,L) \hookrightarrow X'(0,L)$. Furthermore, by combining this observation with Remark 3.6, we also obtain the following fact. If $X(0,L)$ is the optimal target space for $H_{u,v,\nu}$ and some rearrangement-invariant function space $Y(0,L)$, then $X(0,L)$ is actually the optimal target space for $H_{u,v,\nu}$ and its own optimal domain space.

(ii) If $\xi$ satisfies the averaging condition (4.24), then (4.7) is satisfied for every nonincreasing function $u$ inasmuch as $tu(t) \le U(t)$ for every $t \in (0,L)$.

(iii) When $u(t) = t^{-1+\alpha}$, $v(t) = t^{-1+\beta}$, and $\nu(t) = t^{\gamma}$, $t \in (0,L)$, the additional assumptions of Theorem 4.6 are satisfied if $\alpha \in (0,1]$, $\beta \in [0,1)$, $\gamma > 0$, and $1 \le \frac{\alpha}{\gamma} + \beta \le 2$.

We conclude this subsection with a result that is somewhat unrelated to the rest but of independent interest. It shows that, to verify the boundedness of $H_{u,v,\nu}$ between a pair of rearrangement-invariant function spaces, it is sufficient to verify it on nonincreasing functions. It is an easy consequence of Hardy–Littlewood inequality (2.4) that this is the case for the operator $R_{u,v,\nu}$, provided that $u$ is nonincreasing. However, the validity of such a result for the operator $H_{u,v,\nu}$ is far from being obvious because this time the integration is not carried out over a right-neighborhood of 0. Such a result was first obtained by Cianchi, Pick, and Slavíková in [22, Corollary 9.8] for $u \equiv 1$, $\nu = \mathrm{id}$, and $L < \infty$. Later, Peša generalized their result to cover also the case $L = \infty$ in [51, Theorem 3.10]. In [15], we needed such a result for $\nu(t) = t^{\alpha}$, $\alpha > 0$, and $u \not\equiv 1$, and, while we felt certain that their proofs would carry over to the needed setting, we still had to carefully check them because there is plenty of fine analysis involved. The following proposition extends the result to the generality considered in this paper. It turns out that their proofs can easily be adapted for our setting. Our proof is actually in a way simpler because they considered operators with kernels.

**Proposition 4.8.** *Let $\|\cdot\|_{X(0,L)}$ and $\|\cdot\|_{Y(0,L)}$ be rearrangement-invariant function norms.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. Assume that $\nu^{-1} \in \overline{D}^0_\theta$ for some $\theta > 1$. If $L = \infty$, assume that $\nu^{-1} \in \overline{D}^\infty_\theta$.*
*(2) Let $u, v : (0,L) \to (0,\infty)$ be nonincreasing.*

*The following two statements are equivalent.*

*(i) There is a positive constant $C$ such that*

$$\|H_{u,v,\nu} f\|_{Y(0,L)} \le C\|f\|_{X(0,L)} \tag{4.13}$$

*for every $f \in \mathfrak{M}(0,L)$.*
*(ii) There is a positive constant $C$ such that*

$$\|H_{u,v,\nu}(f^*)\|_{Y(0,L)} \le C\|f\|_{X(0,L)} \tag{4.14}$$

*for every $f \in \mathfrak{M}(0,L)$.*

*Moreover, if (4.14) holds with a constant $C$, then (4.13) holds with the constant $C\frac{\theta}{\theta-1}\sup_{t\in(0,L)}\frac{\nu^{-1}(t)}{\nu^{-1}(\frac{t}{\theta})}$.*

*Proof.* Since (i) plainly implies (ii), we only need to prove that (ii) implies (i). Since the quantities in (4.13) and (4.14) do not change when the function $v$ is redefined on a countable set, we may assume that $v$ is left continuous. Note that $H_{u,v,\nu} f$ is nonincreasing for every $f \in \mathfrak{M}(0,L)$. Hence, thanks to (2.11) and (3.17), in order to prove that (ii) implies (i), we need to show that

$$\sup_{\substack{f\in\mathfrak{M}^+(0,L)\\ \|f\|_{X(0,L)}\le 1}} \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{Y'(0,L)}\le 1}} \int_0^L f(s) R_{u,v,\nu^{-1}}(g^*)(s)\,ds \lesssim \sup_{\substack{f\in\mathfrak{M}^+(0,L)\\ \|f\|_{X(0,L)}\le 1}} \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{Y'(0,L)}\le 1}} \int_0^L f^*(s) R_{u,v,\nu^{-1}}(g^*)(s)\,ds. \tag{4.15}$$

We define the operator $G$ as

$$Gg(t) = \sup_{\tau\in[t,L)} R_{u,v,\nu^{-1}}(g^*)(\tau),\ t \in (0, L),$$

for every $g \in \mathfrak{M}^+(0, L)$. Note that $Gg$ is nonincreasing for every $g \in \mathfrak{M}^+(0, L)$. Fix $g \in \mathfrak{M}^+(0, L)$ such that $|\{t \in (0, L) : g(t) > 0\}| < \infty$, and set

$$E = \{t \in (0, L) : R_{u,v,\nu^{-1}}(g^*)(t) < Gg(t)\}.$$

It can be shown that there is a countable system $\{(a_k, b_k)\}_{k\in\mathcal{I}}$ of mutually disjoint, bounded intervals in $(0, L)$ such that

$$E = \bigcup_{k\in\mathcal{I}} (a_k, b_k), \tag{4.16}$$

$$Gg(t) = R_{u,v,\nu^{-1}}(g^*)(t) \quad \text{if } t \in (0, L) \setminus E, \tag{4.17}$$

$$Gg(t) = R_{u,v,\nu^{-1}}(g^*)(b_k) \quad \text{if } t \in (a_k, b_k) \text{ for } k \in \mathcal{I}. \tag{4.18}$$

This was proved in [22, Proposition 9.3] for $L < \infty$ and in [51, Lemma 3.9] for $L = \infty$. Their proofs are for $u \equiv 1$ and $\nu = \mathrm{id}$, but the fact that $g^*u$ is nonincreasing and $R_{u,v,\nu^{-1}}(g^*)$ is upper semicontinuous remains valid in our situation, and so it can be readily seen that their proofs carry over verbatim to our setting.

Note that $M = \sup_{t\in(0,L)} \frac{\nu^{-1}(t)}{\nu^{-1}(\frac{t}{\theta})} < \infty$. Set $\sigma = \frac{\theta}{\theta-1} \in (1, \infty)$. Since $v$ and $g^*u$ are nonincreasing, we have that, for every $k \in \mathcal{I}$,

$$\begin{aligned}
(b_k - a_k) R_{u,v,\nu^{-1}}(g^*)(b_k) &= \sigma \int_{\frac{a_k+(\sigma-1)b_k}{\sigma}}^{b_k} R_{u,v,\nu^{-1}}(g^*)(b_k)\,dt \\
&= \sigma \int_{\frac{a_k+(\sigma-1)b_k}{\sigma}}^{b_k} \frac{v(b_k)}{\nu^{-1}(b_k)} \nu^{-1}(b_k) \int_0^{\nu^{-1}(b_k)} g^*(s)u(s)\,ds\,dt \\
&\le \sigma \int_{\frac{a_k+(\sigma-1)b_k}{\sigma}}^{b_k} \frac{v(t)}{\nu^{-1}(t)} \nu^{-1}(b_k) \int_0^{\nu^{-1}(t)} g^*(s)u(s)\,ds\,dt \\
&\le \sigma \frac{\nu^{-1}(b_k)}{\nu^{-1}(\frac{a_k+(\sigma-1)b_k}{\sigma})} \int_{\frac{a_k+(\sigma-1)b_k}{\sigma}}^{b_k} R_{u,v,\nu^{-1}}(g^*)(t)\,dt \\
&\le \sigma \frac{\nu^{-1}(b_k)}{\nu^{-1}(\frac{b_k}{\theta})} \int_{\frac{a_k+(\sigma-1)b_k}{\sigma}}^{b_k} R_{u,v,\nu^{-1}}(g^*)(t)\,dt \\
&\le \sigma M \int_{\frac{a_k+(\sigma-1)b_k}{\sigma}}^{b_k} R_{u,v,\nu^{-1}}(g^*)(t)\,dt \\
&\le \sigma M \int_{a_k}^{b_k} R_{u,v,\nu^{-1}}(g^*)(t)\,dt.
\end{aligned} \tag{4.19}$$

Here, we used the fact that $v$ and $(g^*u)^{**}$ are nonincreasing in the first inequality.

Consider the averaging operator $A$ defined as

$$Af = f^*\chi_{(0,L)\setminus E} + \sum_{k\in\mathcal{I}} \left( \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f^*(s)\,ds \right) \chi_{(a_k,b_k)},\ f \in \mathfrak{M}^+(0, L).$$

Note that $Af$ is a nonincreasing function for every $f \in \mathfrak{M}^+(0,L)$. Furthermore, it is known [5, Chapter 2, Theorem 4.8] that

$$\|Af\|_{X(0,L)} \le \|f\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{4.20}$$

For every $f \in \mathfrak{M}^+(0,L)$, we have that

$$\begin{aligned}
\int_0^L f(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt \le \int_0^L f(t) Gg(t)\,dt \le \int_0^L f^*(t) Gg(t)\,dt \\
&= \int_{(0,L)\setminus E} f^*(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt + \sum_{k\in\mathcal{I}} \int_{a_k}^{b_k} f^*(t) R_{u,v,\nu^{-1}}(g^*)(b_k)\,dt \\
&\le \int_0^L f^*(t)\chi_{(0,L)\setminus E}(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt \\
&\quad + \sigma M \sum_{k\in\mathcal{I}} \left(\frac{1}{b_k - a_k}\int_{a_k}^{b_k} f^*(t)\,dt\right)\left(\int_{a_k}^{b_k} R_{u,v,\nu^{-1}}(g^*)(t)\,dt\right) \\
&\le \sigma M \int_0^L Af(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt.
\end{aligned} \tag{4.21}$$

Here, we used the Hardy–Littlewood inequality (2.4), (4.16), (4.17), (4.18), and (4.19). If $L = \infty$ and $g \in \mathfrak{M}^+(0,\infty)$ is positive on a set of infinite measure, we consider $g\chi_{(0,n)} \nearrow g$, $n \to \infty$, and obtain (4.21) even for such functions $g$, thanks to the monotone convergence theorem. Hence, we have proved that

$$\int_0^L f(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt \le \sigma M \int_0^L Af(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt \quad \text{for every } f, g \in \mathfrak{M}^+(0,L). \tag{4.22}$$

By combining (4.20) and (4.22), we obtain that

$$\int_0^L f(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt \le \sigma M \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X(0,L)}\le 1}} \int_0^L h^*(t) R_{u,v,\nu^{-1}}(g^*)(t)\,dt$$

for every $f \in \mathfrak{M}^+(0,L)$, $\|f\|_{X(0,L)} \le 1$, and $g \in \mathfrak{M}^+(0,L)$. Note that here we used the fact that $Af$ is nonincreasing for every $f \in \mathfrak{M}^+(0,L)$. By taking the supremum over all $f, g \in \mathfrak{M}^+(0,L)$ from the closed unit balls of $X(0,L)$ and $Y'(0,L)$, respectively, we obtain (4.15) with the multiplicative constant equal to $\sigma M$. □

Proposition 4.8 together with Proposition 3.5 has the following important corollary. Note that the first equality is just a consequence of the Hardy–Littlewood inequality (2.4) combined with the obvious inequality

$$\sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu}(f^*)\|_{Y(0,L)} \le \sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu} f\|_{Y(0,L)}.$$

**Corollary 4.9.** *Let $\|\cdot\|_{X(0,L)}$ and $\|\cdot\|_{Y(0,L)}$ be rearrangement-invariant function norms.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. Assume that $\nu \in \overline{D}^0$. If $L = \infty$, assume that $\nu \in \overline{D}^\infty$.*
*(2) Let $u, v : (0,L) \to (0,\infty)$ be nonincreasing.*

*We have that*

$$\begin{aligned}
\sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu}(f^*)\|_{Y(0,L)} &= \sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu} f\|_{Y(0,L)} \\
&= \sup_{\|g\|_{Y'(0,L)}\le 1} \|H_{u,v,\nu^{-1}} g\|_{X'(0,L)} \\
&\approx \sup_{\|g\|_{Y'(0,L)}\le 1} \|H_{u,v,\nu^{-1}}(g^*)\|_{X'(0,L)}.
\end{aligned}$$

*Remark* 4.10. The assumption $\nu \in \overline{D}^0$ is not overly restrictive. For example, it is satisfied whenever $\nu$ is equivalent to $t \mapsto t^\alpha \ell_1(t)^{\beta_1} \cdots \ell_k(t)^{\beta_k}$ near 0 for any $\alpha > 0$, $k \in \mathbb{N}_0$, and $\beta_j \in \mathbb{R}$, $j = 1, 2, \dots, k$. Here, the functions $\ell_j$ are $j$-times iterated logarithmic functions defined as

$$\ell_j(t) = \begin{cases} 1 + |\log t| & \text{if } j = 1, \\ 1 + \log \ell_{j-1}(t) & \text{if } j > 1, \end{cases} \tag{4.23}$$

for $t \in (0, L)$. On the other hand, $\nu(t) = \exp(-t^\alpha)$ near 0, where $\alpha < 0$, is a typical example of a function not satisfying the assumption. The same remark (with the obvious modifications) is true for the assumption $\nu \in \overline{D}^\infty$.

## 4.2 | Optimal function norms and their connection with interpolation

We already know that a sufficient condition for simplification of the complicated function norm (3.21) is boundedness of a certain supremum operator. Furthermore, we have also already seen that the supremum operator is often bounded on optimal function spaces. We shall also soon see that the boundedness of the supremum operator goes hand in hand with a certain interpolation property of the rearrangement-invariant function space on which the supremum operator acts. In other words, the question of whether the supremum in the function norm (3.21) can be "simplified," the notion of being an optimal function space, and interpolation are all closely related to each other.

As the following theorem shows, there is a connection between a rearrangement-invariant function space $X(0, L)$ being an interpolation space with respect to a certain pair of endpoint spaces and the boundedness of $T_\varphi$ on the associate space of $X(0, L)$ (cf. [39, Theorem 3.12]). We say that a measurable a.e. positive function on $(0, L)$ satisfies *the averaging condition* (4.24) (cf. [60, Lemma 2.3]) if

$$\operatorname*{ess\,sup}_{t \in (0,L)} \frac{1}{t w(t)} \int_0^t w(s)\, ds < \infty. \tag{4.24}$$

Here, $w$ temporarily denotes the function in question. The value of the essential supremum will be referred to as *the averaging constant* of the function.

**Theorem 4.11.** *Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $\varphi : (0, L) \to (0, \infty)$ be a measurable function that is equivalent to a continuous nondecreasing function. Set $\xi = 1/\varphi$. Assume that $\xi$ satisfies the averaging condition (4.24). Set $\psi(t) = t / \int_0^t \xi(s)\, ds$, $t \in (0, L)$. Consider the following three statements.*

*(i) The operator $T_\varphi$, defined by (4.2), is bounded on $X'(0, L)$.*
*(ii) $X(0, L) \in \operatorname{Int}\left(\Lambda^1_\xi(0, L), L^\infty(0, L)\right)$.*
*(iii) $X'(0, L) \in \operatorname{Int}\left(L^1(0, L), M_\psi(0, L)\right)$.*

*If $L < \infty$, then the three statements are equivalent to each other. If $L = \infty$, then (i) implies (ii), and (iii) implies (i).*

*Proof.* We start off by noting that we may without loss of generality assume that $\varphi$ is continuous and nondecreasing. Furthermore, $(\Lambda^1_\xi)'(0, L) = M_\psi(0, L)$ [52, Theorem 10.4.1] and

$$\psi \approx \varphi \quad \text{on } (0, L), \tag{4.25}$$

thanks to the fact that $\xi$ satisfies the averaging condition (4.24) and is (equivalent to) a nonincreasing function. We shall show that (i) implies (ii), whether $L$ is finite or infinite. First, we observe that $X(0, L)$ is an intermediate space between $\Lambda^1_\xi(0, L)$ and $L^\infty(0, L)$. Set $\Xi_L = \int_0^L \xi(s)\, ds \in (0, \infty]$, and note that $\Xi_L < \infty$ if $L < \infty$. Let $\Xi^{-1} : (0, \Xi_L) \to (0, L)$ be the increasing bijection that is inverse to the function $(0, L) \ni t \mapsto \int_0^t \xi(s)\, ds$. By [56, Lemma 6.8], we have that

$$\mathrm{K}(f, t; \Lambda^1_\xi, L^\infty) \approx \int_0^{\Xi^{-1}(t)} f^*(s) \xi(s)\, ds \quad \text{for every } f \in \mathfrak{M}^+(0, L) \text{ and } t \in (0, \Xi_L). \tag{4.26}$$

Let $a$ be defined by (3.2). The embedding $X(0,L) \hookrightarrow (\Lambda^1_\xi + L^\infty)(0,L)$ follows from

$$\begin{aligned}\|f\|_{(\Lambda^1_\xi+L^\infty)(0,L)} &= \mathrm{K}(f,1;\Lambda^1_\xi,L^\infty) \le \max\left\{1, \frac{1}{\int_0^a \xi(s)\,ds}\right\} \mathrm{K}\left(f, \int_0^a \xi(s)\,ds; \Lambda^1_\xi, L^\infty\right)\\ &\approx \int_0^a f^*(t)\xi(t)\,dt \lesssim \frac{1}{\psi(a)}\int_0^a f^*(t)T_\varphi\chi_{(0,a)}(t)\,dt\\ &\lesssim \|f\|_{X(0,L)}\|T_\varphi\chi_{(0,a)}\|_{X'(0,L)} \lesssim \|f\|_{X(0,L)}.\end{aligned}$$

Here, we used Hölder's inequality (2.14) in the last but one inequality and the boundedness of $T_\varphi$ on $X'(0,L)$ in the last one. We now turn our attention to the embedding $\Lambda^1_\xi(0,L)\cap L^\infty(0,L) \hookrightarrow X(0,L)$. If $L<\infty$, the embedding is plainly true owing to (2.18). If $L=\infty$, it is sufficient to observe that, for every $f\in X'(0,\infty)$, $f^* = g+h$ for some functions $g\in L^1(0,\infty)$ and $h\in M_\psi(0,\infty)$, thanks to (2.16), the fact that $(\Lambda^1_\xi(0,\infty)\cap L^\infty(0,\infty))' = L^1(0,\infty)+M_\psi(0,\infty)$ by (2.17), and (2.15). Set $g = f^*\chi_{(0,1)}$ and $h = f^*\chi_{(1,\infty)}$. Clearly, $g\in L^1(0,\infty)$ thanks to property (P5) of $\|\cdot\|_{X'(0,L)}$. Furthermore,

$$\begin{aligned}\|h\|_{M_\psi(0,\infty)} &\approx \sup_{t\in(0,\infty)} \psi(t)(f^*\chi_{(1,\infty)})^*(t) \lesssim \sup_{t\in(0,\infty)}\psi(t+1)f^*(t+1) = \frac{\psi(1)}{\psi(1)}\sup_{t\in[1,\infty)}\psi(t)f^*(t)\\ &= \psi(1)T_\psi f(1) \approx T_\varphi f(1) < \infty.\end{aligned}$$

Here, we used the fact that $\xi$ satisfies the averaging condition (4.24) in the first equivalence (cf. [48, Lemma 2.1]) and (4.25) in the last one. Note that $T_\varphi f(1)$ is finite owing to (2.12) inasmuch as $T_\varphi f\in X'(0,\infty)$ and it is a nonincreasing function. Hence, $X(0,L)$ is an intermediate space between $\Lambda^1_\xi(0,L)$ and $L^\infty(0,L)$. Next, in order to prove that (i) implies (ii), it remains to show that every admissible operator $S$ for the couple $\left(\Lambda^1_\xi(0,L), L^\infty(0,L)\right)$ is bounded on $X(0,L)$. Let $S$ be such an operator. Since $S$ is linear and bounded on both $\Lambda^1_\xi(0,L)$ and $L^\infty(0,L)$, it follows that (see [5, Chapter 5, Theorem 1.11])

$$\mathrm{K}(Sf,t;\Lambda^1_\xi,L^\infty) \lesssim \mathrm{K}(f,t;\Lambda^1_\xi,L^\infty) \quad \text{for every } f\in(\Lambda^1_\xi+L^\infty)(0,L) \text{ and } t\in(0,L). \tag{4.27}$$

By combining (4.26) and (4.27), we obtain that

$$\int_0^t (Sf)^*(s)\xi(s)\,ds \lesssim \int_0^t f^*(s)\xi(s)\,ds \quad \text{for every } f\in(\Lambda^1_\xi+L^\infty)(0,L) \text{ and } t\in(0,L). \tag{4.28}$$

Since the function $(0,L)\ni t\mapsto \sup_{t\le s<L}\varphi(s)g^*(s)$ is nonincreasing for every $g\in\mathfrak{M}^+(0,L)$, the Hardy lemma (2.6) together with (4.28) implies that

$$\int_0^L (Sf)^*(t)T_\varphi g(t)\,dt \lesssim \int_0^L f^*(t)T_\varphi g(t)\,dt \quad \text{for every } f\in(\Lambda^1_\xi+L^\infty)(0,L) \text{ and } g\in\mathfrak{M}^+(0,L).$$

Therefore,

$$\begin{aligned}\|Sf\|_{X(0,L)} &= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L (Sf)^*(t)g^*(t)\,dt \le \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L (Sf)^*(t)T_\varphi g(t)\,dt\\ &\lesssim \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \int_0^L f^*(t)T_\varphi g(t)\,dt \le \|f\|_{X(0,L)} \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}} \|T_\varphi g\|_{X'(0,L)}\\ &\lesssim \|f\|_{X(0,L)}\end{aligned}$$

for every $f\in X(0,L)$. Here, we used (2.11) in the equality, Hölder's inequality (2.14) in the last but one inequality, and the boundedness of $T_\varphi$ on $X'(0,L)$ in the last one. Hence, $S$ is bounded on $X(0,L)$.

We shall now prove that (iii) implies (i), whether $L$ is finite or infinite. Since $\xi$ is nonincreasing and satisfies the averaging condition (4.24), it follows from [34, Theorem 3.2] (cf. [48, Lemma 3.1]) that $T_\varphi$ is bounded on $L^1(0,L)$. Furthermore, $T_\varphi$ is also bounded on $M_\psi(0,L)$, for

$$\begin{aligned}\|T_\varphi f\|_{M_\psi(0,L)} &= \sup_{t\in(0,L)} (T_\varphi f)^{**}(t)\psi(t) = \sup_{t\in(0,L)} \frac{1}{\int_0^t \xi(s)\,ds}\int_0^t \xi(s) \sup_{\tau\in[s,L)} \varphi(\tau)f(\tau)\,ds \\ &\leq \sup_{t\in(0,L)} \varphi(t)f^*(t) \lesssim \|f\|_{M_\psi(0,L)}.\end{aligned}$$

Here, we used (4.25) and (2.1) in the last inequality. Fix $f \in (L^1 + M_\psi)(0,L)$. We claim that

$$\mathrm{K}(T_\varphi f, t; L^1, M_\psi) \lesssim \mathrm{K}(f, t; L^1, M_\psi) \quad \text{for every } t\in(0,\infty) \tag{4.29}$$

with a multiplicative constant independent of $f$. Let $f = g + h$ with $g \in L^1(0,L)$ and $h \in M_\psi(0,L)$ be a decomposition of $f$. Note that the fact that $\xi$ is nonincreasing and satisfies the averaging condition (4.24) implies that

$$\varphi(s) \lesssim \varphi\left(\frac{s}{2}\right) \quad \text{for every } s\in(0,L).$$

Thanks to this and (2.2), we have that

$$\begin{aligned}T_\varphi f(s) &\leq \frac{1}{\varphi(s)}\left(\sup_{\tau\in[s,L)} \varphi(\tau)g^*\left(\frac{\tau}{2}\right) + \sup_{\tau\in[s,L)} \varphi(\tau)h^*\left(\frac{\tau}{2}\right)\right) \\ &\lesssim T_\varphi g\left(\frac{s}{2}\right) + T_\varphi h\left(\frac{s}{2}\right)\end{aligned} \tag{4.30}$$

for every $s \in (0,L)$. By combining (4.30) and the boundedness of the dilation operator $D_2$ (see (2.19)) with the fact that $T_\varphi$ is bounded on both $L^1(0,L)$ and $M_\psi(0,L)$, we obtain that (cf. [8, p. 497])

$$\begin{aligned}\mathrm{K}(T_\varphi f, t; L^1, M_\psi) &\lesssim \mathrm{K}\left(T_\varphi g\left(\frac{\cdot}{2}\right), t; L^1, M_\psi\right) + \mathrm{K}\left(T_\varphi h\left(\frac{\cdot}{2}\right), t; L^1, M_\psi\right) \\ &\lesssim \|T_\varphi g\|_{L^1(0,L)} + t\|T_\varphi h\|_{M_\psi(0,L)} \lesssim \|g\|_{L^1(0,L)} + t\|h\|_{M_\psi(0,L)}\end{aligned}$$

for every $t \in (0,\infty)$. Here, the multiplicative constants are independent of $f, g, h$, and $t$. Hence (4.29) is true. Now, since we have (4.29) at our disposal, there is a linear operator $S$ bounded on both $L^1(0,L)$ and $M_\psi(0,L)$ with norms that can be bounded from above by a constant independent of $f$ such that $Sf = T_\varphi f$. This follows from [26, Theorem 2]. Owing to (iii), $S$ is also bounded on $X'(0,L)$; moreover, its norm on $X'(0,L)$ can be bounded from above by a constant independent of $f$ [5, Chapter 3, Proposition 1.11]. Therefore,

$$\|T_\varphi f\|_{X'(0,L)} = \|Sf\|_{X'(0,L)} \lesssim \|f\|_{X'(0,L)},$$

in which the multiplicative constant is independent of $f$. Hence, $T_\varphi$ is bounded on $X'(0,L)$.

Finally, if $L < \infty$, then (ii) is equivalent to (iii); hence, the three statements are equivalent to each other in this case. Indeed, since $(\Lambda^1_\xi + L^\infty)(0,L) = \Lambda^1_\xi(0,L)$ and $(L^1 + M_\psi)(0,L) = L^1(0,L)$ owing to (2.18), the equivalence of (ii) and (iii) follows from [44, Corollary 3.6]. Here, we used the fact that both $\Lambda^1_\xi(0,L)$ and $L^1(0,L)$ have absolutely continuous norm (in the sense of [5, Chapter 1, Definition 3.1]). □

## 4.3 | More on the case $u \equiv 1$

The remainder of this section is devoted to the particular but important case $u \equiv 1$. We shall see that the connection between the supremum operator and the various notions that we have met is even tighter in this case.

First, we need to equip ourselves with the following auxiliary result, which generalizes [30, Lemma 4.9] and whose immediate corollary for $u \equiv 1$ is of independent interest.

**Proposition 4.12.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. Assume that $\nu^{-1} \in \overline{D}^0$. If $L = \infty$, assume that $\nu^{-1} \in \overline{D}^\infty$.*
*(2) Let $u : (0,L) \to (0,\infty)$ be nonincreasing.*
*(3) Let $v : (0,L) \to (0,\infty)$ be nonincreasing. Assume that $v$ satisfies the averaging condition (4.24), and denote its averaging constant by $C$.*

*Set $f = \sum_{i=1}^N c_i \chi_{(0,a_i)}$, where $c_i \in (0,\infty)$, $i = 1,\dots,N$, and $0 < a_1 < \dots < a_N < L$. We have that*

$$\|H_{u,v,\nu} f\|_{X(0,L)} \approx \left\| u(t) \sum_{i=1}^N a_i c_i v(a_i) \chi_{(0,\nu^{-1}(a_i))}(t) \right\|_{X(0,L)}, \tag{4.31}$$

*in which the multiplicative constants depend only on $\nu$ and $C$.*

*Proof.* First, observe that $\inf_{t\in(0,L)} \frac{\nu^{-1}(\frac{t}{\theta})}{\nu^{-1}(t)} \in (0,1)$, where $\theta > 1$ is such that $\nu^{-1} \in \overline{D}^0_\theta$ and, if $L = \infty$, also $\nu^{-1} \in \overline{D}^\infty_\theta$. We denote the infimum by $M$.

Second, we have that

$$\begin{aligned}
\left\| u(t) \int_{\nu(t)}^L f(s) v(s)\,ds \right\|_{X(0,L)} &= \left\| u(t) \sum_{i=1}^N c_i \chi_{(0,\nu^{-1}(a_i))}(t) \int_{\nu(t)}^{a_i} v(s)\,ds \right\|_{X(0,L)} \\
&\ge \left\| u(t) \sum_{i=1}^N c_i \chi_{(0,\nu^{-1}(\frac{a_i}{\theta}))}(t) v(a_i)(a_i - \nu(t)) \right\|_{X(0,L)} \\
&\ge \frac{\theta-1}{\theta} \left\| u(t) \sum_{i=1}^N c_i \chi_{(0,\nu^{-1}(\frac{a_i}{\theta}))}(t) v(a_i) a_i \right\|_{X(0,L)} \\
&\ge \frac{\theta-1}{\theta} \left\| u(t) \sum_{i=1}^N c_i \chi_{(0,M\nu^{-1}(a_i))}(t) v(a_i) a_i \right\|_{X(0,L)} \\
&\ge M\frac{\theta-1}{\theta} \left\| u(Mt) \sum_{i=1}^N c_i \chi_{(0,\nu^{-1}(a_i))}(t) v(a_i) a_i \right\|_{X(0,L)} \\
&\ge M\frac{\theta-1}{\theta} \left\| u(t) \sum_{i=1}^N c_i \chi_{(0,\nu^{-1}(a_i))}(t) v(a_i) a_i \right\|_{X(0,L)}
\end{aligned} \tag{4.32}$$

thanks to the fact that $u$ and $v$ are nonincreasing and the boundedness of $D_{\frac{1}{M}}$ (see (2.19)).

Last, using the fact that $v$ satisfies the averaging condition (4.24), we obtain that

$$\begin{aligned}
\left\| u(t) \int_{\nu(t)}^L f(s) v(s)\,ds \right\|_{X(0,L)} &= \left\| u(t) \sum_{i=1}^N c_i \chi_{(0,\nu^{-1}(a_i))}(t) \int_{\nu(t)}^{a_i} v(s)\,ds \right\|_{X(0,L)} \\
&\le C \left\| u(t) \sum_{i=1}^N c_i \chi_{(0,\nu^{-1}(a_i))}(t) a_i v(a_i) \right\|_{X(0,L)}.
\end{aligned} \tag{4.33}$$

By combining (4.33) and (4.32), we obtain (4.31). □

Since every nonnegative, nonincreasing function on $(0,L)$ is the pointwise limit of a nondecreasing sequence of nonnegative, nonincreasing simple functions, Proposition 4.12 with $u \equiv 1$ has the following important corollary.

**Corollary 4.13.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu, v$ be as in Proposition 4.12.*

*Let $f \in \mathfrak{M}^+(0,L)$. There is a nondecreasing sequence $\{f_k\}_{k=1}^\infty$ of nonnegative, nonincreasing simple functions on $(0,L)$ such that*

$$\lim_{k\to\infty} \|H_{1,v,\nu}(f_k)\|_{X(0,L)} \approx \|f^*\|_{X(0,L)} = \|f\|_{X(0,L)}.$$

*Here, the multiplicative constants depend only on $\nu$ and the averaging constant of $v$.*

*Remark* 4.14. The assumption $\nu^{-1} \in \overline{D}^0$ is not overly restrictive. For example, it is satisfied whenever $\nu$ is equivalent to $t \mapsto t^\alpha \ell_1(t)^{\beta_1} \cdots \ell_k(t)^{\beta_k}$ near 0 for any $\alpha > 0, k \in \mathbb{N}_0$ and $\beta_j \in \mathbb{R}$, $j = 1, 2, \dots, k$. Here, the functions $\ell_j$ are iterated logarithmic functions defined by (4.23). In this case, $\nu^{-1}$ is equivalent to $t \mapsto t^{\frac{1}{\alpha}} \ell_1(t)^{-\frac{\beta_1}{\alpha}} \cdots \ell_k(t)^{-\frac{\beta_k}{\alpha}}$ near 0 (cf. [6, Appendix 5]). On the other hand, $\nu(t) = \log^\alpha(\frac{1}{t})$ near 0, where $\alpha < 0$, is a typical example of a function not satisfying the assumption. The same remark (with the obvious modifications) is true for the assumption $\nu^{-1} \in \overline{D}^\infty$.

While Proposition 4.2 provides a sufficient condition for simplification of (3.3), the following proposition provides a necessary one.

**Proposition 4.15.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. Assume that $\nu^{-1} \in \overline{D}^0$. If $L = \infty$, assume that $\nu^{-1} \in \overline{D}^\infty$ and $\nu \in \underline{D}^\infty$.*
*(2) Let $v : (0,L) \to (0,\infty)$ be a nonincreasing function satisfying the averaging condition (4.24).*

*If there is a positive constant $C$ such that*

$$\sup_{h\sim f} \|H_{1,v,\nu} h\|_{X(0,L)} \le C \|H_{1,v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L), \tag{4.34}$$

*then the three equivalent statements from Theorem 4.6 with $u \equiv 1$ are satisfied.*

*Proof.* Let $a$ be defined by (3.2). Since $v$ is integrable over $(0,a)$, for it satisfies the averaging condition (4.24), we have that

$$\left\| \chi_{(0,\nu^{-1}(a))}(t) \int_{\nu(t)}^a v(s)\,ds \right\|_{X(0,L)} \le \int_0^a v(s)\,ds \|\chi_{(0,\nu^{-1}(a))}\|_{X(0,L)} < \infty. \tag{4.35}$$

Furthermore, if $L = \infty$, then $\limsup_{\tau\to\infty} v(\tau) \|\chi_{(0,\nu^{-1}(\tau))}\|_{X(0,\infty)} < \infty$. Indeed, suppose that $\limsup_{\tau\to\infty} v(\tau) \|\chi_{(0,\nu^{-1}(\tau))}\|_{X(0,\infty)} = \infty$. It follows from the proof of Proposition 3.3 that

$$\sup_{h\sim\chi_{(0,1)}} \|H_{1,v,\nu} h\|_{X(0,\infty)} = \infty. \tag{4.36}$$

However, since

$$\sup_{h\sim\chi_{(0,1)}} \|H_{1,v,\nu} h\|_{X(0,\infty)} \approx \|H_{1,v,\nu}\chi_{(0,1)}\|_{X(0,\infty)} = \left\| \chi_{(0,\nu^{-1}(1))}(t) \int_{\nu(t)}^1 v(s)\,ds \right\|_{X(0,\infty)} < \infty,$$

thanks to (4.34) and (4.35), (4.36) is not possible. Hence, Proposition 3.3 guarantees that the optimal domain space for $H_{1,v,\nu}$ and $X(0,L)$ exists. Moreover, if we denote it by $Z(0,L)$, then (4.34) implies that

$$\|f\|_{Z(0,L)} \approx \|H_{1,v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{4.37}$$

Now, we finally turn our attention to proving that (4.34) implies statement (iii) from Theorem 4.6. Let $Y(0,L)$ be the optimal target space for the operator $H_{1,\upsilon,\nu}$ and $Z(0,L)$. Its existence is guaranteed by Proposition 3.7, and we have that

$$\|f\|_{Y'(0,L)} = \|R_{1,\upsilon,\nu^{-1}}(f^*)\|_{Z'(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{4.38}$$

Using the optimality of $Y(0,L)$ combined with the fact that $H_{1,\upsilon,\nu} : Z(0,L) \to X(0,L)$ is bounded, and (4.37), we obtain that

$$\|H_{1,\upsilon,\nu}(f^*)\|_{X(0,L)} \lesssim \|H_{1,\upsilon,\nu}(f^*)\|_{Y(0,L)} \lesssim \|f\|_{Z(0,L)} \approx \|H_{1,\upsilon,\nu}(f^*)\|_{X(0,L)}$$

for every $f \in \mathfrak{M}^+(0,L)$. Hence,

$$\|H_{1,\upsilon,\nu}(f^*)\|_{X(0,L)} \approx \|H_{1,\upsilon,\nu}(f^*)\|_{Y(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

In particular, we have that

$$\|H_{1,\upsilon,\nu}h\|_{X(0,L)} \approx \|H_{1,\upsilon,\nu}h\|_{Y(0,L)} \tag{4.39}$$

for every nonincreasing simple function $h \in \mathfrak{M}^+(0,L)$. By combining (4.39) with Corollary 4.13, we obtain that

$$\|f^*\|_{X(0,L)} \approx \|f^*\|_{Y(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Owing to the rearrangement invariance of both function norms, it follows that $X(0,L) = Y(0,L)$. Hence, (4.6) with $u \equiv 1$ follows from (4.38) combined with (2.8). $\square$

We obtain the final result of this subsection by combining Theorem 4.6, Proposition 4.15, Proposition 4.2, and Theorem 4.11.

**Theorem 4.16.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. Assume that $\nu^{-1} \in \overline{D}^0$. If $L = \infty$, assume that $\nu^{-1} \in \overline{D}^\infty$ and $\nu \in \underline{D}^\infty$.*
*(2) Let $\upsilon : (0,L) \to (0,\infty)$ be defined by (4.3) with $\xi : (0,L) \to (0,\infty)$ satisfying the averaging condition (4.24). Assume that $\upsilon$, too, satisfies the averaging condition (4.24). Furthermore, assume that the function $\varphi \circ \nu^{-1}$ is equivalent to a quasiconcave function, where $\varphi = 1/\xi$.*

*Let $\|\cdot\|_{Y(0,L)}$ be the functional defined by (3.3) with $u \equiv 1$. The following five statements are equivalent.*

*(i) The operator $T_\varphi$, defined by (4.2), is bounded on $X'(0,L)$.*
*(ii) There is a positive constant $C$ such that*

$$\sup_{h\sim f} \|H_{1,\upsilon,\nu}h\|_{X(0,L)} \le C\|H_{1,\upsilon,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

*(iii) The space $X(0,L)$ is the optimal target space for the operator $H_{1,\upsilon,\nu}$ and some rearrangement-invariant function space.*
*(iv) The space $X'(0,L)$ is the optimal domain space for the operator $R_{1,\upsilon,\nu^{-1}}$ and some rearrangement-invariant function space.*
*(v) We have that*

$$\|f\|_{X'(0,L)} \approx \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{Y(0,L)}\le 1}} \int_0^L g(t) R_{1,\upsilon,\nu^{-1}}(f^*)(t)\,dt \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

*If $L < \infty$, these five statements are also equivalent to*

*(vi) $X(0,L) \in \operatorname{Int}\left(\Lambda^1_\xi(0,L), L^\infty(0,L)\right)$.*

*Remark* 4.17.

(1) The assumption that $v$ satisfies the averaging condition (4.24) is natural. It forbids weights $v$ for which the question of whether $X(0,L)$ (or $X'(0,L)$) is the optimal target (or domain) space for $H_{1,v,\nu}$ (or $R_{1,v,\nu^{-1}}$) and some rearrangement-invariant function space cannot be decided by the boundedness of the corresponding supremum operator $T_\varphi$. This can be illustrated by a very simple example. Consider $\nu = \mathrm{id}$ and $\xi \equiv 1$. Since $T_\varphi f = f^*$, $T_\varphi$ is bounded on any $X'(0,L)$. However, $R_{1,v,\nu^{-1}} f(t) = \int_0^t |f(s)|\,ds/t$ clearly need not be bounded from $X'(0,L)$ to $(L^1 + L^\infty)(0,L)$, which is the largest rearrangement-invariant function space. To this end, consider, for example, $X(0,L) = L^\infty(0,\infty)$ (cf. [58, Proposition 4.1]).

(2) When $v(t) = t^{-1+\beta}$ and $\nu(t) = t^\gamma$, $t \in (0,L)$, the assumptions of Theorem 4.16 are satisfied if $\beta \in (0,1)$, $\gamma > 0$, and $1 \le \frac{1}{\gamma} + \beta \le 2$.

## 5 | ITERATION OF OPTIMAL FUNCTION NORMS

This section is devoted to so-called *sharp iteration principles* for the operators $R_{u,v,\nu}$ and $H_{u,v,\nu}$. To illustrate their meaning and importance, suppose that $Y_1(0,L)$ is the optimal target space for $H_{u_1,v_1,\nu_1}$ and a rearrangement-invariant function space $X(0,L)$. Let us now go one step further and suppose that $Y_2(0,L)$ is the optimal target space for $H_{u_2,v_2,\nu_2}$ and $Y_1(0,L)$. In the light of Proposition 3.7, the associate function norm of $\|\cdot\|_{Y_2(0,L)}$ is equal to $\|f\|_{Y_2'(0,L)} = \|R_{u_1,v_1,\nu_1^{-1}}((R_{u_2,v_2,\nu_2^{-1}}(f^*))^*)\|_{X'(0,L)}$. We immediately see that there is an inevitable difficulty that we face if we wish to understand the iterated norm. This difficulty is caused by the fact that the function $R_{u_2,v_2,\nu_2^{-1}}(f^*)$ is hardly ever (equivalent to) a nonincreasing function, unless $u_2$, $v_2$, and $\nu_2$ are related to each other in a very specific way (see Proposition 4.1). Therefore, we cannot just readily "delete" the outer star. Nevertheless, with some substantial effort, we shall be able to equivalently express the iterated norm as a noniterated one under suitable assumptions. The suitable assumptions are such that the iteration does not lead to the presence of kernels, which would go beyond the scope of this paper (see [22, section 8] in that regard). It should be noted that such an iteration is not artificial. For example, it is an essential tool for establishing sharp iteration principles for various Sobolev embeddings. Roughly speaking, they ensure that the optimal rearrangement-invariant target space in a Sobolev embedding of $(k+l)$-th order is the same as that obtained by composing the optimal Sobolev embedding of order $k$ with the optimal Sobolev embedding of order $l$ (see [21, 23, 46] and references therein). Another possible application is description of optimal rearrangement-invariant function norms for compositions of some operators of harmonic analysis (see [30] and references therein for optimal behavior of some classical operators on rearrangement-invariant function spaces). Finally, the motivation behind studying function norms induced by $H_{u_1,v_1,\nu_1} \circ H_{u_1,v_1,\nu_1}$ is similar.

### 5.1 | Iteration principle for $R_{u,v,\nu}$

The following proposition is the first step toward the sharp iteration principle for $R_{u,v,\nu}$.

**Proposition 5.1.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu_1, \nu_2 : (0,L) \to (0,L)$ be increasing bijections. Assume that $\nu_2 \in \overline{D}^0$. If $L = \infty$, assume that $\nu_2 \in \overline{D}^\infty$.*
*(2) Let $u_1, u_2 : (0,L) \to (0,\infty)$ be nonincreasing.*
*(3) Let $v_1 : (0,L) \to (0,\infty)$ be measurable. Let $v_2 : (0,L) \to (0,\infty)$ be a nonincreasing function satisfying the averaging condition (4.24).*

*Set $\nu = \nu_2 \circ \nu_1$ and*

$$v(t) = u_1(\nu_1(t)) v_1(t) \nu_1(t) v_2(\nu_1(t)),\ t \in (0,L). \tag{5.1}$$

*We have that*

$$\|R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2}(f^*))^*)\|_{X(0,L)} \gtrsim \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)}$$

*for every $f \in \mathfrak{M}^+(0,L)$, in which the multiplicative constant depends only on $\nu_2$ and the averaging constant of $v_2$.*

*Proof.* Note that $\inf_{t\in(0,L)} \frac{\nu_2(\frac{t}{\theta})}{\nu_2(t)} > 0$, where $\theta > 1$ is such that $\nu_2 \in \overline{D}^0_\theta$ and, if $L = \infty$, also $\nu_2 \in \overline{D}^\infty_\theta$. Consequently, there is $N \in \mathbb{N}$, such that $\nu_2(t) \le N\nu_2(\frac{t}{\theta})$ for every $t \in (0, L)$. Hence, for every $f \in \mathfrak{M}^+(0, L)$, we have that

$$\int_0^{\nu_2(t)} f^*(s)u_2(s)\,ds \le N \int_0^{\nu_2(\frac{t}{\theta})} f^*(s)u_2(s)\,ds \quad \text{for every } t \in (0, L) \tag{5.2}$$

owing to the fact that $f^*u_2$ is nonincreasing. Thanks to the monotonicity of $u_1$ and $v_2$, the fact that $v_2$ satisfies the averaging condition (4.24) and the inequality (5.2), we have that

$$\begin{aligned}
&\left\| v_1(t)u_1(\nu_1(t))\nu_1(t)v_2(\nu_1(t)) \int_0^{\nu(t)} f^*(s)u_2(s)\,ds \right\|_{X(0,L)} \\
&\lesssim \left\| v_1(t)u_1(\nu_1(t)) \int_0^{\nu_1(t)} v_2(s)\,ds \int_0^{\nu(t)} f^*(s)u_2(s)\,ds \right\|_{X(0,L)} \\
&\approx \left\| v_1(t)u_1(\nu_1(t)) \int_{\frac{\nu_1(t)}{\theta}}^{\nu_1(t)} v_2(s)\,ds \int_0^{\nu(t)} f^*(s)u_2(s)\,ds \right\|_{X(0,L)} \\
&\le \left\| v_1(t) \int_{\frac{\nu_1(t)}{\theta}}^{\nu_1(t)} v_2(s)u_1(s)\,ds \int_0^{\nu(t)} f^*(s)u_2(s)\,ds \right\|_{X(0,L)} \\
&\lesssim \left\| v_1(t) \int_{\frac{\nu_1(t)}{\theta}}^{\nu_1(t)} v_2(s)u_1(s)\,ds \int_0^{\nu_2(\frac{\nu_1(t)}{\theta})} f^*(s)u_2(s)\,ds \right\|_{X(0,L)} \\
&\le \left\| v_1(t) \int_{\frac{\nu_1(t)}{\theta}}^{\nu_1(t)} \left( v_2(s) \int_0^{\nu_2(s)} f^*(\tau)u_2(\tau)\,d\tau \right) u_1(s)\,ds \right\|_{X(0,L)} \\
&\le \left\| v_1(t) \int_0^{\nu_1(t)} \left( v_2(s) \int_0^{\nu_2(s)} f^*(\tau)u_2(\tau)\,d\tau \right) u_1(s)\,ds \right\|_{X(0,L)} \\
&\le \left\| R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2}(f^*))^*) \right\|_{X(0,L)}
\end{aligned}$$

for every $f \in \mathfrak{M}^+(0, L)$. Here, we used the Hardy–Littlewood inequality (2.4) in the last inequality. □

We are now in a position to establish the sharp iteration principle for $R_{u,v,\nu}$.

**Theorem 5.2.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu_1, \nu_2, u_1, u_2$ be as in Proposition 5.1.*
*(2) Let $v_1 : (0, L) \to (0, \infty)$ be a continuous function. Let $v_2 : (0, L) \to (0, \infty)$ be defined by*

$$\frac{1}{v_2(t)} = \int_0^{\nu_2(t)} \xi(s)\,ds,\ t \in (0, L),$$

*where $\xi : (0, L) \to (0, \infty)$ is a measurable function. Assume that the function $u_1v_2$ satisfies the averaging condition (4.24).*

*Let $v$ be the function defined by (5.1). Set $\nu = \nu_2 \circ \nu_1$ and*

$$\eta(t) = \frac{1}{U_2(t)v(\nu^{-1}(t))},\ t \in (0, L).$$

*Assume that $\eta$ and $\eta/\xi$ are equivalent to nonincreasing functions. Furthermore, assume that there are positive constants $C_1$ and $C_2$ such that*

$$\int_0^t \eta(s)u_2(s)\,ds \le C_1 U_2(t)\eta(t) \quad \text{for a.e. } t \in (0,L) \tag{5.3}$$

*and*

$$\frac{1}{t}\int_0^t U_2(\nu(s))v(s)\,ds \ge C_2 U_2(\nu(t))v(t) \quad \text{for a.e. } t \in (0,L). \tag{5.4}$$

*We have that*

$$\|R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2}(f^*))^*)\|_{X(0,L)} \approx \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)}$$

*for every $f \in \mathfrak{M}^+(0,L)$. Here, the multiplicative constants depend only on $\nu_1$, $\nu_2$, $C_1$, $C_2$, the averaging constant of $u_1v_2$ and the multiplicative constants in the equivalences of $\eta$ and $\frac{\eta}{\xi}$ to nonincreasing functions.*

*Proof.* First, note that the fact that $v_2u_1$ satisfies the averaging condition (4.24) together with the monotonicity of $u_1$ implies that $v_2$, too, satisfies the averaging condition (4.24) (with the same multiplicative constant). Hence, we have that

$$\|R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2}(f^*))^*)\|_{X(0,L)} \gtrsim \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L),$$

thanks to Proposition 5.1. Therefore, we only need to prove the opposite inequality.

We may assume that $u_2$ is nondegenerate and $\psi \in X(0,L)$, where $\psi$ is defined as $\psi(t) = v(t)U_2(\nu(t))\chi_{(0,L)}(t) + v(t)\chi_{(L,\infty)}(t)$, $t \in (0,L)$. Indeed, if it is not the case, then $\|R_{u_2,v,\nu}(f^*)\|_{X(0,L)} = \infty$ for every $f \in \mathfrak{M}^+(0,L)$ that is not equivalent to 0 a.e. Proposition 3.1 with $u = u_2$ guarantees that there is a rearrangement-invariant function space $Z(0,L)$ such that

$$\|f\|_{Z(0,L)} = \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Furthermore, by (3.18) and the Hardy–Littlewood inequality (2.4), we have that

$$\sup_{\|g\|_{X'(0,L)}\le 1} \|H_{u_2,v,\nu^{-1}}g\|_{Z'(0,L)} = 1. \tag{5.5}$$

Note that, for every $f \in \mathfrak{M}^+(0,L)$, the function

$$(0,L) \ni t \mapsto v_2(t)\int_0^{\nu_2(t)} \xi(s)u_2(s) \sup_{\tau\in[s,L)} \frac{1}{\xi(\tau)}f^*(\tau)\,ds$$

is nonincreasing. Indeed it is the integral mean of the nonincreasing function $(0,L) \ni s \mapsto u_2(s)\sup_{\tau\in[s,L)} \frac{1}{\xi(\tau)}f^*(\tau)$ over the interval $(0,\nu_2(t))$ with respect to the measure $\xi(s)\,ds$. By (2.9) and (3.17), we have that

$$\begin{aligned}
\|R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2}(f^*))^*)\|_{X(0,L)} &= \sup_{\|g\|_{X'(0,L)}\le 1}\int_0^L (R_{u_2,v_2,\nu_2}(f^*))^*(t)H_{u_1v_1,\nu_1^{-1}}g(t)\,dt \\
&= \sup_{\|g\|_{X'(0,L)}\le 1}\int_0^L \left[v_2(s)\int_0^{\nu_2(s)} u_2(\tau)f^*(\tau)\,d\tau\right]^*(t)H_{u_1,v_1,\nu_1^{-1}}g(t)\,dt \\
&\le \sup_{\|g\|_{X'(0,L)}\le 1}\int_0^L \left[v_2(s)\int_0^{\nu_2(s)} \xi(\tau)u_2(\tau)\sup_{x\in[\tau,L)}\frac{1}{\xi(x)}f^*(x)\,d\tau\right]^*(t)H_{u_1,v_1,\nu_1^{-1}}g(t)\,dt \\
&= \sup_{\|g\|_{X'(0,L)}\le 1}\int_0^L v_2(t)\int_0^{\nu_2(t)} \xi(s)u_2(s)\sup_{\tau\in[s,L)}\frac{1}{\xi(\tau)}f^*(\tau)\,ds\,H_{u_1,v_1,\nu_1^{-1}}g(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|g\|_{X'(0,L)}\le 1} \int_0^L \left(\xi(s) \sup_{\tau\in[s,L)} \frac{1}{\xi(\tau)} f^*(\tau)\right)\left(u_2(s)\int_{\nu_2^{-1}(s)}^L v_2(t)u_1(t)\int_{\nu_1^{-1}(t)}^L g(x)v_1(x)\,dx\,dt\right) ds\\
&\le \left\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\right\|_{Z(0,L)} \sup_{\|g\|_{X'(0,L)}\le 1} \left\|u_2(t)\int_{\nu_2^{-1}(t)}^L v_2(s)u_1(s)\int_{\nu_1^{-1}(s)}^L g(\tau)v_1(\tau)\,d\tau\,ds\right\|_{Z'(0,L)}\\
&= \left\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\right\|_{Z(0,L)} \sup_{\|g\|_{X'(0,L)}\le 1} \left\|u_2(t)\int_{\nu^{-1}(t)}^L g(\tau)v_1(\tau)\int_{\nu_2^{-1}(t)}^{\nu_1(\tau)} v_2(s)u_1(s)\,ds\,d\tau\right\|_{Z'(0,L)}\\
&\lesssim \left\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\right\|_{Z(0,L)} \sup_{\|g\|_{X'(0,L)}\le 1} \left\|u_2(t)\int_{\nu^{-1}(t)}^L g(\tau)v_1(\tau)\nu_1(\tau)u_1(\nu_1(\tau))v_2(\nu_1(\tau))\,d\tau\right\|_{Z'(0,L)}\\
&= \left\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\right\|_{Z(0,L)} \sup_{\|g\|_{X'(0,L)}\le 1} \|H_{u_2,v,\nu^{-1}}g\|_{Z'(0,L)}\\
&= \left\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\right\|_{Z(0,L)},
\end{aligned}
$$

for every $f \in \mathfrak{M}^+(0,L)$. Here, we used Fubini's theorem in the fourth and fifth equalities, the Hölder inequality (2.14) in the second inequality, the fact that $u_1v_2$ satisfies the averaging condition (4.24) in the last inequality, and (5.5) in the last equality. Therefore, the proof will be finished once we show that

$$
\left\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\right\|_{Z(0,L)} \lesssim \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).
$$

Since the function $\frac{\eta}{\xi}$ is equivalent to a nonincreasing function, we have that

$$
\left\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\right\|_{Z(0,L)} \lesssim \left\|\eta(t) \sup_{s\in[t,L)} \frac{1}{\eta(s)} f^*(s)\right\|_{Z(0,L)}
$$

for every $f \in \mathfrak{M}^+(0,L)$. Hence, it is sufficient to show that

$$
\left\|\eta(t) \sup_{s\in[t,L)} \frac{1}{\eta(s)} f^*(s)\right\|_{Z(0,L)} \lesssim \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{5.6}
$$

Note that, for every $f \in \mathfrak{M}^+(0,L)$,

$$
\begin{aligned}
\left\|\eta(t) \sup_{s\in[t,L)} \frac{1}{\eta(s)} f^*(s)\right\|_{Z(0,L)} &\approx \left\|v(t)\int_0^{\nu(t)} u_2(s)\eta(s) \sup_{\tau\in[s,L)} \frac{1}{\eta(\tau)} f^*(\tau)\,ds\right\|_{X(0,L)}\\
&\le \left\|v(t)\int_0^{\nu(t)} u_2(s)\eta(s) \sup_{\tau\in[s,\nu(t))} \frac{1}{\eta(\tau)} f^*(\tau)\,ds\right\|_{X(0,L)}\\
&\quad + \left\|v(t)\left(\sup_{\tau\in[\nu(t),L)} \frac{1}{\eta(\tau)} f^*(\tau)\right)\int_0^{\nu(t)} u_2(s)\eta(s)\,ds\right\|_{X(0,L)},
\end{aligned} \tag{5.7}
$$

inasmuch as $\eta$ is equivalent to a nonincreasing function. Furthermore, since $\eta$ is equivalent to a nonincreasing function and satisfies (5.3), [34, Theorem 3.2] guarantees that

$$
\int_0^{\nu(t)} u_2(s)\eta(s) \sup_{\tau\in[s,\nu(t))} \frac{1}{\eta(\tau)} f^*(\tau)\,ds \lesssim \int_0^{\nu(t)} f^*(s)u_2(s)\,ds
$$

for every $t \in (0, L)$ and every $f \in \mathfrak{M}^+(0, L)$. Here, the multiplicative constant depends only on $C_2$. Hence,

$$\left\| v(t) \int_0^{\nu(t)} u_2(s)\eta(s) \sup_{\tau\in[s,\nu(t))} \frac{1}{\eta(\tau)} f^*(\tau)\,ds \right\|_{X(0,L)} \lesssim \left\| v(t) \int_0^{\nu(t)} f^*(s)u_2(s)\,ds \right\|_{X(0,L)}$$
$$= \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)} \tag{5.8}$$

for every $f \in \mathfrak{M}^+(0, L)$. Furthermore, thanks to the fact that $\eta$ satisfies (5.3) again, we have that

$$\left\| v(t) \left( \sup_{\tau\in[\nu(t),L)} \frac{1}{\eta(\tau)} f^*(\tau) \right) \int_0^{\nu(t)} u_2(s)\eta(s)\,ds \right\|_{X(0,L)} \lesssim \left\| v(t)U_2(\nu(t))\eta(\nu(t)) \sup_{\tau\in[\nu(t),L)} \frac{1}{\eta(\tau)} f^*(\tau) \right\|_{X(0,L)}$$
$$= \left\| \sup_{\tau\in[\nu(t),L)} \frac{1}{\eta(\tau)} f^*(\tau) \right\|_{X(0,L)}$$
$$= \left\| \sup_{\tau\in[t,L)} \frac{1}{\eta(\nu(\tau))} f^*(\nu(\tau)) \right\|_{X(0,L)} \tag{5.9}$$

for every $f \in \mathfrak{M}^+(0, L)$. We claim that

$$\left\| \sup_{\tau\in[t,L)} \frac{1}{\eta(\nu(\tau))} f^*(\nu(\tau)) \right\|_{X(0,L)} \lesssim \|R_{u_2,v,\nu}(f^*)\|_{X(0,L)}. \tag{5.10}$$

Thanks to the Hardy–Littlewood–Pólya principle (2.7), it is sufficient to show that

$$\int_0^t \sup_{\tau\in[s,L)} \frac{1}{\eta(\nu(\tau))} f^*(\nu(\tau))\,ds \lesssim \int_0^t (R_{u_2,v,\nu}(f^*))^*(s)\,ds \quad \text{for every } t \in (0, L). \tag{5.11}$$

To this end, we have that

$$\int_0^t \sup_{\tau\in[s,t)} \frac{1}{\eta(\nu(\tau))} f^*(\nu(\tau))\,ds \lesssim \int_0^t \frac{1}{\eta(\nu(s))} f^*(\nu(s))\,ds = \int_0^t U_2(\nu(s))v(s) f^*(\nu(s))\,ds$$
$$\le \int_0^t R_{u_2,v,\nu}(f^*)(s)\,ds \le \int_0^t (R_{u_2,v,\nu}(f^*))^*(s)\,ds \tag{5.12}$$

for every $t \in (0, L)$. Here, the first inequality follows from [34, Theorem 3.2] (the fact that the function $(0, L) \ni s \mapsto \frac{1}{\eta(\nu(s))} = U_2(\nu(s))v(s)$ is equivalent to a nondecreasing function and satisfies (5.4) was used here), the second inequality follows from the monotonicity of $f^*$, and the last one follows from the Hardy–Littlewood inequality (2.4). Furthermore, owing to (5.4) again, we have that

$$\sup_{\tau\in[t,L)} \frac{1}{\eta(\nu(\tau))} f^*(\nu(\tau)) = \sup_{\tau\in[t,L)} U_2(\nu(\tau))v(\tau) f^*(\nu(\tau)) \lesssim \sup_{\tau\in[t,L)} \left( \frac{1}{\tau} \int_0^\tau U_2(\nu(s))v(s)\,ds \right) f^*(\nu(\tau))$$
$$\le \sup_{\tau\in[t,L)} \frac{1}{\tau} \int_0^\tau U_2(\nu(s))v(s) f^*(\nu(s))\,ds \le \sup_{\tau\in[t,L)} \frac{1}{\tau} \int_0^\tau R_{u_2,v,\nu}(f^*)(s)\,ds$$
$$\le \sup_{\tau\in[t,L)} \frac{1}{\tau} \int_0^\tau (R_{u_2,v,\nu}(f^*))^*(s)\,ds = \frac{1}{t} \int_0^t (R_{u_2,v,\nu}(f^*))^*(s)\,ds. \tag{5.13}$$

Inequality (5.11) now follows from (5.12) and (5.13) inasmuch as

$$\int_0^t \sup_{\tau\in[s,L)} \frac{1}{\eta(\nu(\tau))} f^*(\nu(\tau))\,ds \le \int_0^t \sup_{\tau\in[s,t)} \frac{1}{\eta(\nu(\tau))} f^*(\nu(\tau))\,ds + t \sup_{\tau\in[t,L)} \frac{1}{\eta(\nu(\tau))} f^*(\nu(\tau))$$

for every $t \in (0, L)$.

Finally, by combining (5.7) with (5.8), (5.9), and (5.10), we obtain (5.6). □

*Remark* 5.3. Since Theorem 5.2 has several assumptions, it is instructive to provide a concrete, important example, which is also quite general. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in (0, \infty)$. Set $\nu_j(t) = t^{\alpha_j}$, $u_j(t) = t^{\beta_j - 1} b_j(t)$, and $v_j(t) = t^{\gamma_j - 1} c_j(t)$, $t \in (0, L)$, $j = 1, 2$, where $b_j, c_j$ are continuous slowly varying functions. Set $d = (b_1 \circ \nu_1) \cdot c_1 \cdot (c_2 \circ \nu_1)$ and $\widetilde{d} = (b_1 \circ \nu_1) \cdot c_1$. Assume that $\gamma_2 < 1$, $\beta_1 + \gamma_2 > 1$, and

$$\alpha_1(\beta_1 + \alpha_2\beta_2 + \gamma_2 - 1) + \gamma_1 \geq 1,\ \alpha_1(\beta_1 + \alpha_2\beta_2 - \alpha_2) + \gamma_1 \geq 1,\ \alpha_1(\beta_1 + \gamma_2 - 1) + \gamma_1 < 1.$$

If $\alpha_1(\beta_1 + \alpha_2\beta_2 + \gamma_2 - 1) + \gamma_1 = 1$ or $\alpha_1(\beta_1 + \alpha_2\beta_2 - \alpha_2) + \gamma_1 = 1$, also assume that $d$ or $\widetilde{d}$, respectively, is equivalent to a nondecreasing function. Under these assumptions, we can use Theorem 5.2 to obtain that

$$\left\| v_1(t) \int_0^{t^{\alpha_1}} \left[ v_2(\tau) \int_0^{\tau^{\alpha_2}} f^*(\sigma) u_2(\sigma)\, d\sigma \right]^* (s) u_1(s)\, ds \right\|_{X(0,L)} \approx \left\| t^\delta d(t) \int_0^{t^{\alpha_1\alpha_2}} f^*(s) s^{\beta_2 - 1}\, ds \right\|_{X(0,L)}$$

for every $f \in \mathfrak{M}^+(0, L)$, where $\delta = \alpha_1(\beta_1 + \gamma_2 - 1) + \gamma_1 - 1$.

When $\beta_j = 1$ and $b_j = c_j \equiv 1$, $j = 1, 2$, the assumptions are satisfied provided that

$$\alpha_1(\alpha_2 + \gamma_2) + \gamma_1 \geq 1,\ \alpha_1 + \gamma_1 \geq 1,\ \alpha_1\gamma_2 + \gamma_1 < 1. \tag{5.14}$$

In particular, (5.14) is satisfied if (cf. [21, Theorem 3.4])

$$\alpha_2 + \gamma_2 \geq 1,\ \alpha_1 + \gamma_1 \geq 1,\ \alpha_1\gamma_2 + \gamma_1 < 1.$$

## 5.2 | Iteration principle for $H_{u,v,\nu}$

We conclude this section with a $H_{u,v,\nu}$ counterpart to Theorem 5.2, whose proof is substantially simpler than that of the theorem.

**Proposition 5.4.** *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm.*

*(1) Let $\nu_1, \nu_2 : (0, L) \to (0, \infty)$ be increasing bijections. Assume that $\nu_1 \in \underline{D}^0$. If $L = \infty$, assume that $\nu_1 \in \underline{D}^\infty$.*
*(2) Let $u_1, u_2, v_1, v_2 : (0, L) \to (0, \infty)$ be measurable. Assume that the function $v_1 u_2$ is equivalent to a nonincreasing function and that it satisfies the averaging condition (4.24).*

*Set*

$$v(t) = \nu_2^{-1}(t) v_1(\nu_2^{-1}(t)) u_2(\nu_2^{-1}(t)) v_2(t),\ t \in (0, L),$$

*and $\nu = \nu_2 \circ \nu_1$. We have that*

$$\|H_{u_1,v_1,\nu_1}(H_{u_2,v_2,\nu_2} f)\|_{X(0,L)} \approx \|H_{u_1,v,\nu} f\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0, L). \tag{5.15}$$

*Here, the multiplicative constants depend only on $\nu_1$, the averaging constant of $v_1 u_2$, and the multiplicative constants in the equivalence of $v_1 u_2$ to a nonincreasing function.*

*Finally, assume, in addition, that*

*(1) $u_1$ and $u_2$ are nonincreasing,*
*(2) $v_1$ is defined by*

$$\frac{1}{v_1(t)} = \int_0^{\nu_1^{-1}(t)} \xi(s)\, ds \quad \text{for every } t \in (0, L),$$

*where $\xi : (0, L) \to (0, \infty)$ is a measurable function,*
*(3) the operator $T_\varphi$ defined by (4.2) with $\varphi = u_1/\xi$ is bounded on $X'(0, L)$.*

*Then,*

$$\sup_{\substack{g\sim f\\ g\in\mathfrak{M}^+(0,L)}}\ \sup_{\substack{h\sim H_{u_2,v_2,\nu_2}g\\ h\in\mathfrak{M}^+(0,L)}}\|H_{u_1,v_1,\nu_1}h\|_{X(0,L)}\approx\sup_{\substack{g\sim f\\ g\in\mathfrak{M}^+(0,L)}}\|H_{u_1,v,\nu}g\|_{X(0,L)}\quad\text{for every } f\in\mathfrak{M}^+(0,L).$$

*Here, the multiplicative constants depend only on the norm of $T_\varphi$ on $X'(0,L)$ and the multiplicative constant in (5.15).*

*Proof.* On the one hand, we have that

$$\begin{aligned}
\|H_{u_1,v_1,\nu_1}(H_{u_2,v_2,\nu_2}f)\|_{X(0,L)} &= \left\|u_1(t)\int_{\nu_1(t)}^L\left(u_2(s)\int_{\nu_2(s)}^L f(\tau)v_2(\tau)\,d\tau\right)u_2(s)v_1(s)\,ds\right\|_{X(0,L)}\\
&= \left\|u_1(t)\int_{\nu(t)}^L f(\tau)v_2(\tau)\int_{\nu_1(t)}^{\nu_2^{-1}(\tau)} u_2(s)v_1(s)\,ds\,d\tau\right\|_{X(0,L)}\\
&\le \left\|u_1(t)\int_{\nu(t)}^L f(\tau)v_2(\tau)\int_0^{\nu_2^{-1}(\tau)} u_2(s)v_1(s)\,ds\,d\tau\right\|_{X(0,L)}\\
&\lesssim \left\|u_1(t)\int_{\nu(t)}^L f(\tau)v_2(\tau)\nu_2^{-1}(\tau)u_2(\nu_2^{-1}(\tau))v_1(\nu_2^{-1}(\tau))\,d\tau\right\|_{X(0,L)}\\
&= \|H_{u_1,v,\nu}f\|_{X(0,L)}
\end{aligned}$$

for every $f\in\mathfrak{M}^+(0,L)$, thanks to the fact that $v_1u_2$ satisfies the averaging condition (4.24).

As for the opposite inequality, observe that $M=\inf_{t\in(0,\frac{L}{\theta})}\frac{\nu_1(\theta t)}{\nu_1(t)}>1$, where $\theta>1$ is such that $\nu_1\in\underline{D}_\theta^0$ and, if $L=\infty$, also $\nu_1\in\underline{D}_\theta^\infty$. Set $K=\min\{\frac{1}{\theta},\nu_1^{-1}(\frac{1}{M})\}$. We have that

$$\begin{aligned}
&\|H_{u_1,v_1,\nu_1}(H_{u_2,v_2,\nu_2}f)\|_{X(0,L)}\\
&= \left\|u_1(t)\int_{\nu(t)}^L f(\tau)v_2(\tau)\int_{\nu_1(t)}^{\nu_2^{-1}(\tau)} u_2(s)v_1(s)\,ds\,d\tau\right\|_{X(0,L)}\\
&\ge \left\|\chi_{(0,KL)}(t)u_1(t)\int_{\nu_2(M\nu_1(t))}^L f(\tau)v_2(\tau)\int_{\nu_1(t)}^{\nu_2^{-1}(\tau)} u_2(s)v_1(s)\,ds\,d\tau\right\|_{X(0,L)}\\
&\gtrsim \left\|\chi_{(0,KL)}(t)u_1(t)\int_{\nu_2(M\nu_1(t))}^L f(\tau)v_2(\tau)u_2(\nu_2^{-1}(\tau))v_1(\nu_2^{-1}(\tau))(\nu_2^{-1}(\tau)-\nu_1(t))\,d\tau\right\|_{X(0,L)}\\
&\ge \frac{M-1}{M}\left\|\chi_{(0,KL)}(t)u_1(t)\int_{\nu_2(M\nu_1(t))}^L f(\tau)v(\tau)\,d\tau\right\|_{X(0,L)}\\
&\ge \frac{M-1}{M}\left\|\chi_{(0,KL)}(t)u_1(t)\int_{\nu_2(\nu_1(\theta t))}^L f(\tau)v(\tau)\,d\tau\right\|_{X(0,L)}\\
&\ge \frac{M-1}{M}\left\|\chi_{(0,L)}\left(\frac{t}{K}\right)u_1(t)\int_{\nu_2(\nu_1(\frac{t}{K}))}^L f(\tau)v(\tau)\,d\tau\right\|_{X(0,L)}\\
&\ge \frac{M-1}{M}K\left\|u_1(t)\int_{\nu(t)}^L f(\tau)v(\tau)\,d\tau\right\|_{X(0,L)}
\end{aligned}$$

for every $f \in \mathfrak{M}^+(0, L)$. Here, we used the fact that $v_1 u_2$ is equivalent to a nonincreasing function and the boundedness of the dilation operator $D_{\frac{1}{K}}$ (see (2.19)).

Finally, under the additional assumptions, we have that

$$\sup_{\substack{g\sim f\\ g\in\mathfrak{M}^+(0,L)}} \sup_{\substack{h\sim H_{u_2,v_2,\nu_2}g\\ h\in\mathfrak{M}^+(0,L)}} \|H_{u_1,v_1,\nu_1}h\|_{X(0,L)} \approx \sup_{\substack{g\sim f\\ g\in\mathfrak{M}^+(0,L)}} \|H_{u_1,v_1,\nu_1}(H_{u_2,v_2,\nu_2}g)\|_{X(0,L)}$$

$$\approx \sup_{\substack{g\sim f\\ g\in\mathfrak{M}^+(0,L)}} \|H_{u_1,v,\nu}g\|_{X(0,L)}$$

for every $f \in \mathfrak{M}^+(0, L)$, thanks to (4.4) combined with (5.15). □

*Remark* 5.5. If $T_\varphi$ is not bounded on $X'(0, \infty)$, then, while we still have that

$$\sup_{\substack{g\sim f\\ g\in\mathfrak{M}^+(0,L)}} \sup_{\substack{h\sim H_{u_2,v_2,\nu_2}g\\ h\in\mathfrak{M}^+(0,L)}} \|H_{u_1,v_1,\nu_1}h\|_{X(0,L)} \gtrsim \sup_{\substack{g\sim f\\ g\in\mathfrak{M}^+(0,L)}} \|H_{u_1,v,\nu}g\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0, L),$$

it remains an open question whether the opposite inequality (is)/(can be) valid.

## 6 CONCRETE EXAMPLES OF OPTIMAL FUNCTION SPACES

We conclude this paper with a few concrete examples of optimal function spaces. Let $\gamma \in [0, 1)$ and $\delta > 0$. Set $v(t) = t^{-1+\gamma}$ and $\nu(t) = ct^\delta$, $t \in (0, L)$, where $c = 1$ if $L = \infty$ and $c = L^{1-\delta}$ if $L < \infty$. Throughout this section, $R_{\gamma,\delta}$ and $H_{\gamma,\delta}$ denote $R_{1,v,\nu}$ and $H_{1,v,\nu}$, respectively. In all the examples, the fixed function space is a Lorentz–Zygmund space. The class of Lorentz–Zygmund spaces contains several customary function spaces, for example, the Lebesgue spaces $L^p$, the Lorentz spaces $L^{p,q}$, and some Orlicz spaces—namely, logarithmic and exponential Orlicz spaces.

If $L = \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, we define the broken logarithmic function $\ell_j^{\mathbb{A}}$ as $\ell_j^{\mathbb{A}} = \ell_j^{\alpha_0}\chi_{(0,1]} + \ell_j^{\alpha_\infty}\chi_{(1,\infty)}$, where $\ell_j$ is defined by (4.23). Let $p, q \in [1, \infty]$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$, and $\alpha, \beta \in \mathbb{R}$. If $L = \infty$, the Lorentz–Zygmund space $L^{p,q;\mathbb{A},\mathbb{B}}(0, \infty)$ is defined as the collection of all the functions $f \in \mathfrak{M}(0, \infty)$ for which the functional $\|\cdot\|_{L^{p,q;\mathbb{A},\mathbb{B}}(0,\infty)}$ defined as

$$\|f\|_{L^{p,q;\mathbb{A},\mathbb{B}}(0,\infty)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} \ell_1^{\mathbb{A}}(t) \ell_2^{\mathbb{B}}(t) f^*(t) \right\|_{L^q(0,\infty)} \tag{6.1}$$

is finite. If $L < \infty$, the Lorentz–Zygmund space $L^{p,q;\alpha,\beta}(0, L)$ is defined as the collection of all the functions $f \in \mathfrak{M}(0, L)$ for which the functional $\|\cdot\|_{L^{p,q;\alpha,\beta}(0,L)}$ defined as

$$\|f\|_{L^{p,q;\alpha,\beta}(0,L)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} \ell_1^{\alpha}(t) \ell_2^{\beta}(t) f^*(t) \right\|_{L^q(0,L)} \tag{6.2}$$

is finite. When $\mathbb{B} = (0, 0)$ and $\beta = 0$, we write $L^{p,q;\mathbb{A}}(0, \infty)$ and $L^{p,q;\alpha}(0, L)$ for short, respectively. Similarly, when $\mathbb{A} = \mathbb{B} = (0, 0)$ and $\alpha = \beta = 0$, we write $L^{p,q}(0, \infty)$ and $L^{p,q}(0, L)$ for short, respectively. Note that these are the usual Lorentz spaces. We shall also encounter Lorentz–Zygmund spaces $L^{(p,q;\mathbb{A},\mathbb{B})}(0, \infty)$ and $L^{(p,q;\alpha,\beta)}(0, L)$. In the definitions of these spaces, the nonincreasing rearrangement $f^*$ is replaced by the maximal nonincreasing rearrangement $f^{**}$. At one point, we will need a Lorentz–Zygmund space with three tiers of logarithm, which is defined in the obvious way. For more information on Lorentz–Zygmund spaces, see [50]. In particular, the functional $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm if and only if $p = q = 1$, $\alpha_0 \geq 0$, and $\alpha_\infty \leq 0$, or if $p \in (1, \infty)$ and $q \in [1, \infty]$, or if $p = \infty$, $q \in [1, \infty)$, and $\alpha_0 + 1/q < 0$, or if $p = q = \infty$ and $\alpha_0 \leq 0$. The functional $\|\cdot\|_{L^{p,q;\alpha}(0,L)}$ is equivalent to a rearrangement-invariant function norm if and only if $p = q = 1$, $\alpha \geq 0$, or if $p \in (1, \infty)$ and $q \in [1, \infty]$, or if $p = \infty$, $q \in [1, \infty)$, and $\alpha + 1/q < 0$, or if $p = q = \infty$ and $\alpha \leq 0$. Throughout the rest of this section, we implicitly assume that the parameters $p, q, \mathbb{A}$, and $\alpha$ satisfy one of these conditions.

We omit proofs in this section, which are to some extent straightforward but lengthy and technical. However, the interested reader can find detailed proofs in the author's PhD thesis [45, section 2.2.3]. Some particular examples with detailed proofs can also be found in [21, 30, 46].

## 6.1 | Optimal function spaces for $R_{\gamma,\delta}$

We start with optimal domain spaces for the operator $R_{\gamma,\delta}$.

**Proposition 6.1.** *Let $\gamma \in [0,1)$ and $\delta > 0$.*

*If $L = \infty$, the optimal domain space $X(0,\infty)$ for the operator $R_{\gamma,\delta}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$ satisfies*

$$X(0,\infty) = \begin{cases} L^{1,1;(0,\alpha_\infty+1)}(0,\infty) & \textit{if } p = \frac{1}{1-\gamma},\ q = 1,\ \alpha_0+1<0,\ \alpha_\infty+1<0,\ \gamma\in(0,1); \\ L^{1,1;(\alpha_0+1,\alpha_\infty+1)}(0,\infty) & \textit{if } p = \frac{1}{1-\gamma},\ q = 1,\ \alpha_0+1>0,\ \alpha_\infty+1<0,\ \gamma\in(0,1) \textit{ or} \\ & p=q=1,\ \alpha_0\geq 0,\ \alpha_\infty+1<0,\ \gamma=0; \\ L^{1,1;(0,\alpha_\infty+1),(1,0)}(0,\infty) & \textit{if } p = \frac{1}{1-\gamma},\ q = 1,\ \alpha_0+1=0,\ \alpha_\infty+1<0,\ \gamma\in(0,1); \\ L^{(1,q;\mathbb{A})}(0,\infty) & \textit{if } p = \frac{1}{1-\gamma},\ q \in (1,\infty),\ \alpha_\infty+\frac{1}{q}<0,\ \gamma\in(0,1) \textit{ or} \\ & p = \frac{1}{1-\gamma},\ q=\infty,\ \alpha_\infty\leq 0,\ \gamma\in(0,1); \\ L^{\frac{\delta p}{1+p(\gamma+\delta-1)},q;\mathbb{A}}(0,\infty) & \textit{if } p\in\left(\frac{1}{1-\gamma},\frac{1}{1-\gamma-\delta}\right),\ \gamma+\delta<1 \textit{ or} \\ & p\in\left(\frac{1}{1-\gamma},\infty\right),\ \gamma+\delta\geq 1; \\ L^{\infty,q;\mathbb{A}}(0,\infty) & \textit{if } p=\frac{1}{1-\gamma-\delta},\ \alpha_0+\frac{1}{q}<0,\ \gamma+\delta\leq 1 \textit{ or} \\ & p=\frac{1}{1-\gamma-\delta},\ q=\infty,\ \alpha_0\leq 0,\ \gamma+\delta\leq 1; \\ L^{\frac{\delta}{\gamma+\delta-1},\infty;\mathbb{A}}(0,\infty) & \textit{if } p=q=\infty,\ \alpha_0\leq 0,\ \alpha_\infty\geq 0,\ \gamma+\delta>1. \end{cases}$$

*If $L < \infty$, the optimal domain space $X(0,L)$ for the operator $R_{\gamma,\delta}$ and the space $L^{p,q;\alpha}(0,L)$ satisfies*

$$X(0,L) = \begin{cases} L^{1}(0,L) & \textit{if } p\in\left[1,\frac{1}{1-\gamma}\right),\ \gamma\in(0,1) \textit{ or} \\ & p=\frac{1}{1-\gamma},\ \alpha+\frac{1}{q}<0,\ \gamma\in(0,1) \textit{ or} \\ & p=\frac{1}{1-\gamma},\ q=\infty,\ \alpha\leq 0,\ \gamma\in(0,1); \\ L^{1,1;\alpha+1}(0,L) & \textit{if } p=\frac{1}{1-\gamma},\ q=1,\ \alpha+1>0,\ \gamma\in(0,1) \textit{ or} \\ & p=q=1,\ \alpha\geq 0,\ \gamma=0; \\ L^{1,1;0,1}(0,L) & \textit{if } p=\frac{1}{1-\gamma},\ q=1,\ \alpha+1=0,\ \gamma\in(0,1); \\ L^{(1,q;\alpha)}(0,L) & \textit{if } p=\frac{1}{1-\gamma},\ q\in(1,\infty),\ \alpha+\frac{1}{q}\geq 0,\ \gamma\in(0,1) \textit{ or} \\ & p=\frac{1}{1-\gamma},\ q=\infty,\ \alpha>0,\ \gamma\in(0,1); \\ L^{\frac{\delta p}{1+p(\gamma+\delta-1)},q;\alpha}(0,L) & \textit{if } p\in\left(\frac{1}{1-\gamma},\frac{1}{1-\gamma-\delta}\right),\ \gamma+\delta<1 \textit{ or} \\ & p\in\left(\frac{1}{1-\gamma},\infty\right),\ \gamma+\delta\geq 1; \\ L^{\infty,q;\alpha}(0,L) & \textit{if } p=\frac{1}{1-\gamma-\delta},\ \alpha+\frac{1}{q}<0,\ \gamma+\delta\leq 1 \textit{ or} \\ & p=\frac{1}{1-\gamma-\delta},\ q=\infty,\ \alpha\leq 0,\ \gamma+\delta\leq 1; \\ L^{\frac{\delta}{\gamma+\delta-1},\infty;\alpha}(0,L) & \textit{if } p=q=\infty,\ \alpha\leq 0,\ \gamma+\delta>1. \end{cases}$$

The following proposition describes optimal target spaces for the operator $R_{\gamma,\delta}$.

**Proposition 6.2.** *Let $\gamma \in [0,1)$ and $\delta > 0$.*

*If $L = \infty$, the optimal target space $X(0,\infty)$ for the operator $R_{\gamma,\delta}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$ satisfies*

$$X(0,\infty) = \begin{cases} L^{\frac{1}{1-\gamma},\infty}(0,\infty) & \text{if } p = q = 1,\ \alpha_0 = \alpha_\infty = 0,\ \gamma \in (0,1); \\ L^{1,1;(\alpha_0-1,\alpha_\infty-1)}(0,\infty) & \text{if } p = q = 1,\ \alpha_0 \geq 1,\ \alpha_\infty < 0,\ \gamma = 0; \\ L^{\frac{p}{p(1-\gamma-\delta)+\delta},q;\mathbb{A}}(0,\infty) & \text{if } p \in (1,\infty),\ \gamma+\delta \leq 1,\ \gamma \in (0,1) \text{ or} \\ & p \in (1,\infty),\ \delta < 1,\ \gamma = 0 \text{ or} \\ & p \in (1, \frac{\delta}{\gamma+\delta-1}),\ \gamma+\delta > 1; \\ L^{p,q;\mathbb{A}}(0,\infty) & \text{if } p \in (1,\infty),\ \delta = 1,\ \gamma = 0 \text{ or} \\ & p = \infty,\ q \in [1,\infty),\ \alpha_0 + \frac{1}{q} < 0,\ \delta = 1,\ \gamma = 0 \text{ or} \\ & p = q = \infty,\ \alpha_0 \leq 0,\ \delta = 1,\ \gamma = 0; \\ L^{\frac{1}{1-\gamma-\delta},\infty;\mathbb{A}}(0,\infty) & \text{if } p = q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty \geq 0,\ \gamma+\delta \leq 1,\ \gamma \in (0,1) \text{ or} \\ & p = q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty \geq 0,\ \delta < 1,\ \gamma = 0; \\ L^{\infty,\infty;\mathbb{A}}(0,\infty) & \text{if } p = \frac{\delta}{\gamma+\delta-1},\ q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty \geq 0,\ \gamma+\delta > 1; \\ L^\infty(0,\infty) & \text{if } p = \frac{\delta}{\gamma+\delta-1},\ q \in [1,\infty),\ \alpha_0 = \alpha_\infty = 0,\ \gamma+\delta > 1,\ \gamma \in (0,1). \end{cases}$$

*If $L < \infty$, the optimal target space $X(0,L)$ for the operator $R_{\gamma,\delta}$ and the space $L^{p,q;\alpha}(0,L)$ satisfies*

$$X(0,L) = \begin{cases} L^{\frac{1}{1-\gamma},\infty}(0,L) & \text{if } p = q = 1,\ \alpha = 0,\ \gamma \in (0,1); \\ L^{1,1;\alpha-1}(0,L) & \text{if } p = q = 1,\ \alpha \geq 1,\ \gamma = 0; \\ L^{\frac{p}{p(1-\gamma-\delta)+\delta},q;\alpha}(0,L) & \text{if } p \in (1,\infty),\ \gamma+\delta \leq 1,\ \gamma \in (0,1) \text{ or} \\ & p \in (1,\infty),\ \delta < 1,\ \gamma = 0 \text{ or} \\ & p \in (1, \frac{\delta}{\gamma+\delta-1}),\ \gamma+\delta > 1; \\ L^{p,q;\alpha}(0,L) & \text{if } p \in (1,\infty),\ \delta = 1,\ \gamma = 0 \text{ or} \\ & p = \infty,\ q \in [1,\infty),\ \alpha + \frac{1}{q} < 0,\ \delta = 1,\ \gamma = 0 \text{ or} \\ & p = q = \infty,\ \alpha \leq 0,\ \delta = 1,\ \gamma = 0; \\ L^{\frac{1}{1-\gamma-\delta},\infty;\alpha}(0,L) & \text{if } p = q = \infty,\ \alpha \leq 0,\ \gamma+\delta \leq 1,\ \gamma \in (0,1) \text{ or} \\ & p = q = \infty,\ \alpha \leq 0,\ \delta < 1,\ \gamma = 0; \\ L^{\infty,\infty;\alpha}(0,L) & \text{if } p = \frac{\delta}{\gamma+\delta-1},\ q = \infty,\ \alpha \leq 0,\ \gamma+\delta > 1; \\ L^\infty(0,L) & \text{if } p = \frac{\delta}{\gamma+\delta-1},\ q < \infty,\ \alpha = 0,\ \gamma+\delta > 1,\ \gamma \in (0,1) \text{ or} \\ & p \in (\frac{\delta}{\gamma+\delta-1}, \infty],\ \gamma+\delta > 1. \end{cases}$$

## 6.2 | Optimal function spaces for $H_{\gamma,\delta}$

The following proposition describes optimal domain spaces for the operator $H_{\gamma,\delta}$.

**Proposition 6.3.** *Let $\gamma \in [0,1)$ and $\delta > 0$.*

*If $L = \infty$, the optimal domain space $X(0,\infty)$ for the operator $H_{\gamma,\delta}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$ satisfies*

$$X(0,\infty)=\begin{cases} L^{\frac{1}{\gamma+\delta},1;\mathbb{A}}(0,\infty) & \textit{if } p=q=1,\ \alpha_0\geq 0,\ \alpha_\infty\leq 0,\ \gamma+\delta\leq 1;\\ L^{1,1;\mathbb{A}}(0,\infty) & \textit{if } p=\frac{\delta}{1-\gamma},\ q=1,\ \alpha_0\geq 0,\ \alpha_\infty\leq 0,\ \gamma+\delta>1;\\ L^{1}(0,\infty) & \textit{if } p=\frac{\delta}{1-\gamma},\ \alpha_0=\alpha_\infty=0,\ \gamma+\delta>1,\ \gamma\in(0,1);\\ L^{\frac{p}{p\gamma+\delta},q;\mathbb{A}}(0,\infty) & \textit{if } p\in(1,\infty),\ \gamma+\delta\leq 1 \textit{ or}\\ & p\in(\frac{\delta}{1-\gamma},\infty),\ \gamma+\delta>1;\\ L^{\frac{1}{\gamma},1}(0,\infty) & \textit{if } p=q=\infty,\ \alpha_0=\alpha_\infty=0,\ \gamma\in(0,1);\\ L^{\infty,\infty;(\alpha_0+1,\alpha_\infty+1)}(0,\infty) & \textit{if } p=q=\infty,\ \alpha_0+1\leq 0,\ \alpha_\infty>0,\ \gamma=0. \end{cases}$$

*If $L<\infty$, the optimal domain space $X(0,L)$ for the operator $H_{\gamma,\delta}$ and the space $L^{p,q;\alpha}(0,L)$ satisfies*

$$X(0,L)=\begin{cases} L^{\frac{1}{\gamma+\delta},1;\alpha}(0,L) & \textit{if } p=q=1,\ \alpha\geq 0,\ \gamma+\delta\leq 1;\\ L^{1}(0,L) & \textit{if } p\in[1,\frac{\delta}{1-\gamma}),\ \gamma+\delta>1 \textit{ or}\\ & p=\frac{\delta}{1-\gamma},\ \alpha=0,\ \gamma+\delta>1,\ \gamma\in(0,1);\\ L^{1,1;\alpha}(0,L) & \textit{if } p=\frac{\delta}{1-\gamma},\ q=1,\ \alpha\geq 0,\ \gamma+\delta>1;\\ L^{\frac{p}{p\gamma+\delta},q;\alpha}(0,L) & \textit{if } p\in(1,\infty),\ \gamma+\delta\leq 1 \textit{ or}\\ & p\in(\frac{\delta}{1-\gamma},\infty),\ \gamma+\delta>1;\\ L^{\frac{1}{\gamma},1}(0,L) & \textit{if } p=q=\infty,\ \alpha=0,\ \gamma\in(0,1);\\ L^{\infty,\infty;\alpha+1}(0,L) & \textit{if } p=q=\infty,\ \alpha+1\leq 0,\ \gamma=0. \end{cases}$$

Finally, we end with optimal target spaces for the operator $H_{\gamma,\delta}$.

**Proposition 6.4.** *Let $\gamma\in[0,1)$ and $\delta>0$.*
*If $L=\infty$, the optimal target space $X(0,\infty)$ for the operator $H_{\gamma,\delta}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$ satisfies*

$$X(0,\infty)=\begin{cases} L^{\frac{p\delta}{1-p\gamma},q;\mathbb{A}}(0,\infty) & \textit{if } p=q=1,\ \alpha_0\geq 0,\ \alpha_\infty\leq 0,\ \gamma+\delta\geq 1 \textit{ or}\\ & p\in(1,\frac{1}{\gamma}),\ \gamma+\delta\geq 1 \textit{ or}\\ & p\in(\frac{1}{\gamma+\delta},\frac{1}{\gamma}),\ \gamma+\delta<1;\\ L^{1,1;\mathbb{A}}(0,\infty) & \textit{if } p=\frac{1}{\gamma+\delta},\ q=1,\ \alpha_0\geq 0,\ \alpha_\infty\leq 0,\ \gamma+\delta<1;\\ L^{1,1;(\alpha_0,0)}(0,\infty) & \textit{if } p=\frac{1}{\gamma+\delta},\ q=1,\ \alpha_0\geq 0,\ \alpha_\infty>0,\ \gamma+\delta<1;\\ L^{(1,q;(\alpha_0-1,\alpha_\infty-1))}(0,\infty) & \textit{if } p=\frac{1}{\gamma+\delta},\ q\in(1,\infty],\ \alpha_0>1-\frac{1}{q},\ \alpha_\infty<1-\frac{1}{q},\ \gamma+\delta<1;\\ L^{(1,q;(\alpha_0-1,-\frac{1}{q}),(0,-1))}(0,\infty) & \textit{if } p=\frac{1}{\gamma+\delta},\ q\in(1,\infty],\ \alpha_0>1-\frac{1}{q},\ \alpha_\infty=1-\frac{1}{q},\ \gamma+\delta<1;\\ X_1(0,\infty) & \textit{if } p=\frac{1}{\gamma+\delta},\ q\in(1,\infty],\ \alpha_0>1-\frac{1}{q},\ \alpha_\infty>1-\frac{1}{q},\ \gamma+\delta<1;\\ L^{\infty,q;(\alpha_0-1,\alpha_\infty-1)}(0,\infty) & \textit{if } p=\frac{1}{\gamma},\ \alpha_0<1-\frac{1}{q},\ \alpha_\infty>1-\frac{1}{q},\ \gamma\in(0,1);\\ X_2(0,\infty) & \textit{if } p=\frac{1}{\gamma},\ q\in[1,\infty),\ \alpha_0>1-\frac{1}{q},\ \alpha_\infty>1-\frac{1}{q},\ \gamma\in(0,1) \textit{ or}\\ & p=\frac{1}{\gamma},\ q=1,\ \alpha_0=0,\ \alpha_\infty>0,\ \gamma\in(0,1);\\ L^{\infty,\infty;(0,\alpha_\infty-1)}(0,\infty) & \textit{if } p=\frac{1}{\gamma},\ q=\infty,\ \alpha_0>1,\ \alpha_\infty>1,\ \gamma\in(0,1);\\ L^{\infty,q;(-\frac{1}{q},\alpha_\infty-1),(-1,0)}(0,\infty) & \textit{if } p=\frac{1}{\gamma},\ q\in(1,\infty],\ \alpha_0=1-\frac{1}{q},\ \alpha_\infty>1-\frac{1}{q},\ \gamma\in(0,1);\\ X_3(0,\infty) & \textit{if } p=\frac{1}{\gamma},\ q=1,\ \alpha_0<0,\ \alpha_\infty=0,\ \gamma\in(0,1);\\ L^{\infty}(0,\infty) & \textit{if } p=\frac{1}{\gamma},\ q=1,\ \alpha_0\geq 0,\ \alpha_\infty=0,\ \gamma\in(0,1);\\ L^{\infty,\infty;(\alpha_0-1,\alpha_\infty-1)}(0,\infty) & \textit{if } p=q=\infty,\ \alpha_0\leq 0,\ \alpha_\infty>1,\ \gamma=0. \end{cases}$$

*Here, $X_1(0,\infty)$, $X_2(0,\infty)$, and $X_3(0,\infty)$ are rearrangement-invariant function spaces such that*

$$\|f\|_{X_1(0,\infty)} \approx \|t^{1-\frac{1}{q}}\ell^{\alpha_0-1}(t)f^{**}(t)\|_{L^1(0,1)} + \|f\|_{L^1(0,\infty)},$$

$$\|f\|_{X_2(0,\infty)} \approx \|t^{-\frac{1}{q}}\ell^{\alpha_\infty-1}(t)f^{*}(t)\chi_{(1,\infty)}(t)\|_{L^q(0,\infty)} + \|f\|_{L^\infty(0,\infty)},$$

$$\|f\|_{X_3(0,\infty)} \approx \|t^{-1}\ell^{\alpha_0-1}(t)f^{*}(t)\|_{L^1(0,1)},$$

*for every $f \in \mathfrak{M}(0,\infty)$.*

*If $L < \infty$, the optimal target space $X(0,L)$ for the operator $H_{\gamma,\delta}$ and the space $L^{p,q;\alpha}(0,L)$ satisfies*

$$X(0,L) = \begin{cases} L^{\frac{p\delta}{1-p\gamma},q;\alpha}(0,L) & \text{if } p = q = 1,\ \alpha \ge 0,\ \gamma+\delta \ge 1 \text{ or} \\ & p \in (1,\frac{1}{\gamma}),\ \gamma+\delta \ge 1 \text{ or} \\ & p \in (\frac{1}{\gamma+\delta},\frac{1}{\gamma}),\ \gamma+\delta < 1; \\ L^{1,1;\alpha}(0,L) & \text{if } p = \frac{1}{\gamma+\delta},\ q = 1,\ \alpha \ge 0,\ \gamma+\delta < 1; \\ L^{(1,q;\alpha-1)}(0,L) & \text{if } p = \frac{1}{\gamma+\delta},\ q \in (1,\infty],\ \alpha > 1-\frac{1}{q},\ \gamma+\delta < 1; \\ L^{\infty,q;\alpha-1}(0,L) & \text{if } p = \frac{1}{\gamma},\ \alpha < 1-\frac{1}{q},\ \gamma \in (0,1); \\ L^{\infty,q;-\frac{1}{q},-1}(0,L) & \text{if } p = \frac{1}{\gamma},\ q \in (1,\infty],\ \alpha = 1-\frac{1}{q},\ \gamma \in (0,1); \\ L^{\infty}(0,L) & \text{if } p = \frac{1}{\gamma},\ \alpha > 1-\frac{1}{q},\ \gamma \in (0,1) \text{ or} \\ & p = \frac{1}{\gamma},\ q = 1,\ \alpha \ge 0,\ \gamma \in (0,1) \text{ or} \\ & p > \frac{1}{\gamma},\ \gamma \in (0,1); \\ L^{\infty,\infty;\alpha-1}(0,L) & \text{if } p = q = \infty,\ \alpha \le 0,\ \gamma = 0. \end{cases}$$


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## CONFLICT OF INTEREST STATEMENT

The author declares no potential conflict of interests.

## ORCID

*Zdeněk Mihula* https://orcid.org/0000-0001-6962-7635

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# Reduction principle for Gaussian $K$-inequality [☆]

Sergi Baena-Miret [a], Amiran Gogatishvili [b], Zdeněk Mihula [c,d], Luboš Pick [d,*]

[a] *Department of Mathematics and Computer Science, University of Barcelona, Gran Via de les Corts Catalanes 585, Barcelona, 08007, Spain*
[b] *Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic*
[c] *Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Technická 2, 166 27 Praha 6, Czech Republic*
[d] *Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic*

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We study interpolation properties of operators (not necessarily linear) which satisfy a specific $K$-inequality corresponding to endpoints defined in terms of Orlicz–Karamata spaces modeled upon the example of the Gaussian–Sobolev embedding. We prove a reduction principle for a fairly wide class of such operators.



## 1. Introduction

The principal motivation for our research is to investigate the applicability of interpolation techniques, in particular the $K$-method, to sharp Gaussian–Sobolev embeddings, or, more generally, Boltzman–Sobolev embeddings. Such an approach was successfully applied earlier for example to Euclidean–Sobolev embeddings ([18]), boundary trace embeddings ([7]), or to a wide variety of classical operators of harmonic analysis ([11]). The method can be outlined as follows: we begin with two sharp endpoint estimates from which an inequality between corresponding $K$-functionals is derived (we will refer to this step as a *$K$-inequality*). The $K$-inequality typically gives a pointwise comparison of certain operators involving nonincreasing rear-

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* Corresponding author.
*E-mail addresses:* sergibaena@ub.edu (S. Baena-Miret), gogatish@math.cas.cz (A. Gogatishvili), mihulzde@fel.cvut.cz (Z. Mihula), pick@karlin.mff.cuni.cz (L. Pick).

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rangements of images of an operator to those involving the functions themselves. This inequality is then corroborated using ideas involving some modification of the Hardy–Littlewood–Pólya relation. In case of Sobolev embeddings, some result in the spirit of the DeVore–Scherer theorem is used at the end.

In [18], [7] and [11], this approach worked very well. On the other hand, for example in [8], where sharp Gaussian–Sobolev embeddings were established, interpolation methods were not used. Instead, the optimal embeddings were derived from an appropriate isoperimetric inequality. This step required a symmetrization argument exploiting a general Pólya–Szegő principle on the decrease of rearrangement-invariant norms of the gradient of Sobolev functions in the Gauss space, extending earlier results of [12] and [27]. The proof relied upon the Gaussian isoperimetric inequality by Borell [4] which gives an explicit description of the isoperimetric profile of $\mathbb{R}^n$ endowed with the probability Gaussian measure. A serious disadvantage of this technique consists in the fact that it works only for the first-order embeddings. This shortcoming was later overcome by establishing higher-order results using sharp iteration methods ([9]). Thanks to those results, sharp function spaces appearing in such embeddings are known, at least in the rearrangement-invariant environment.

In the light of the described situation, it would clearly be of interest to investigate the very existence of an operator (or operators) whose boundedness between a given pair of rearrangement-invariant function spaces would guarantee that every operator satisfying the $K$-inequality corresponding to specific pairs of endpoint spaces, modeled upon the example of endpoint spaces appropriate for Gaussian–Sobolev embeddings, is bounded between that pair. This idea is to some extent connected with the classical result of Calderón ([6]). However, in this paper, we are not so much interested in characterizing Calderón couples, but instead we aim at nailing down those pairs of function spaces for which the corresponding $K$-inequality always guarantees the boundedness of operators. Although our research was originally motivated by Gaussian–Sobolev embeddings, operators having similar endpoint behavior appear also in other circumstances, for instance in studying problems appearing in Gaussian harmonic analysis ([28]).

A prototypical example, motivated by the Gaussian–Sobolev embeddings, of endpoint behavior that we have in our mind is that of an operator $T$ satisfying the $K$-inequality

$$K(Tf,t;L\sqrt{\log L},e^{L^2}) \lesssim K(f,t;L^1,L^\infty) \quad \text{for every } t\in(0,1) \tag{1.1}$$

with a multiplicative constant independent of $f$. It will be useful to notice that both of the spaces on the left-hand side are the classical Orlicz spaces of either logarithmic or exponential type, sometimes also called *Zygmund classes*. It is also important to recall that these spaces are neither Lebesgue spaces nor two-parameter Lorentz spaces, which makes their study through interpolation techniques considerably difficult. On the other hand, they are special cases of the Lorentz–Zygmund spaces ([2]), and also of the yet more general Lorentz–Karamata spaces, based on the so-called *slowly varying functions*. These spaces were first introduced in [10] and then treated by many authors (see e.g. [1,13,14,25]). It might be useful to note that, in the notation of Lorentz–Zygmund spaces [2], (1.1) reads as

$$K(Tf,t;L^{1,1;\frac12},L^{\infty,\infty;-\frac12}) \lesssim K(f,t;L^1,L^\infty) \quad \text{for every } t\in(0,1). \tag{1.2}$$

It turns out that the principal property of every operator $T$ satisfying (1.2) is the validity of

$$\int_0^t \frac{(Tf)^*(s)}{\sqrt{\log\frac{e}{s}}}\,ds \lesssim \int_0^t \frac{f^*(s\log\frac{e}{\sqrt{s}})}{\log\frac{e}{s}}\,ds \quad \text{for every } t\in(0,1). \tag{1.3}$$

We shall use a far more general form of this inequality as a point of departure. Namely, for $p\in(0,\infty)$ and for a pair $(\mathscr{b}_1,\mathscr{b}_2)$ of slowly varying functions, we will consider operators $T$ satisfying

$$\int_0^t [(Tf)^*(s)\mathscr{b}_1(s)]^p\,ds \lesssim \int_0^t \left[f^*\big(\sigma^{-1}(s^{1/p})^p\big)\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p ds$$

for every $f \in L^p(R,\mu)$ and $t \in (0,1)$, where

$$\sigma\colon [0,1] \to [0,1]$$

is the increasing, bijective function such that

$$t^p = \frac{1}{C}\int_0^{\sigma(t)^p} \left[\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p ds \quad \text{for every } t \in [0,1]$$

for an appropriate constant $C$. Motivated by the principal inspiration and motivation, we shall call such operators $(p,\mathscr{b}_1,\mathscr{b}_2)$-*gaussible*.

Our main result is Theorem 3.11, complemented with Theorem 3.12, below. It gives several characterizations of boundedness of every $(p,\mathscr{b}_1,\mathscr{b}_2)$-gaussible operator from $X$ to $Y$, where $X, Y$ is a prescribed pair of rearrangement-invariant spaces over two (possibly different) nonatomic measure spaces of measure 1. Of course, our choice of the value 1 is made only for technical convenience, and is immaterial as simple modifications can be used to extend the results to any nonatomic finite measure spaces. For the particular case corresponding to (1.1), Theorems 3.11 and 3.12 yield (among other results) that for a given pair of rearrangement-invariant spaces $X$ and $Y$, the following three statements are equivalent:

(i) every operator $T$ satisfying (1.3) is bounded from $X$ to $Y$,
(ii) every operator $T$ satisfying (1.1) is bounded from $X$ to $Y$,
(iii) the operator $U$ defined by $Uf(s) = f^*(s\log\frac{e}{\sqrt{s}})\sqrt{\log\frac{e}{s}}$ for $s \in (0,1)$ and every suitable $f$ is bounded from $X(0,1)$ to $Y(0,1)$, where $X(0,1)$ and $Y(0,1)$ are the representation spaces of $X$ and $Y$ in the classical Lorentz–Luxemburg sense.

It is worth noticing that the operator $U$ in (iii) is far away from being quasilinear, let alone linear.

To provide the interested reader with some useful information, we shall now describe the motivation and what lies at the root of (1.2) in more detail. The story begins with the seminal paper [16] of L. Gross, who established the first of Gaussian–Sobolev embeddings and also pointed out its importance. In the study of quantum fields and hypercontractivity semigroups, one often needs semigroup estimates, which can be equivalently described in terms of inequalities of Sobolev type in infinitely many variables (see, for instance, [23] and the references therein). In [16], the major problem occurring in attempts to generalize classical Sobolev embeddings to cases of infinitely many variables (recall that the Lebesgue measure does not make sense for infinitely many variables) was solved by replacing the Lebesgue measure by the Gaussian probability measure in $\mathbb{R}^n$, $n \geq 1$, having the density

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{\frac{-|x|^2}{2}}\,dx,$$

and requiring the embedding constants to be independent of the dimension. Since $\gamma_n(\mathbb{R}^n) = 1$ for every $n \in \mathbb{N}$, taking limit as $n \to \infty$ makes sense. It should be mentioned though that another very important question was settled in the same paper, a question concerned with the comparison of integrability of the gradient of a scalar function of several variables with the integrability of the function itself. While in the Euclidean environment there is always a huge gain in integrability, expressible by change of certain power, no such thing is available in the Gaussian setting. Typically, if $\nabla u \in L^p(\mathbb{R}^n, dx)$ for some $p \in [1,n)$, then

$u \in L^{\frac{np}{n-p}}(\mathbb{R}^n, dx)$, in which $dx$ stands for the $n$-dimensional Lebesgue measure, and, of course, $\frac{np}{n-p} > p$. But with $n \to \infty$ one has $\frac{np}{n-p} \to p$, so there is a good chance that the gain will be lost. However, L. Gross discovered that there still is some gain, albeit only of a logarithmic, rather than power, nature. Namely, he proved that if a function $u$ satisfies $\nabla u \in L^2(\mathbb{R}^n, \gamma_n)$ and is suitably normalized (for example when its median is zero), then $u$ itself belongs to a slightly "better" space (better means smaller here), namely $L^2 \log L(\mathbb{R}^n, \gamma_n)$. A more precise formulation of this inequality reads as follows:

$$\|u - u_{\gamma_n}\|_{L^2 \log L(\mathbb{R}^n,\gamma_n)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^n,\gamma_n)},$$

where $C$ depends on neither $u$ nor $n$, and $u_{\gamma_n}$ denotes the integral mean of $u$, that is

$$u_{\gamma_n} = \int_{\mathbb{R}^n} u(x)\, d\gamma_n(x).$$

The discovery of L. Gross paved the way for extensive research of all kinds. His result has been extended, generalized and modified many times, and simple proofs occurred as well as new applications. In [8], a comprehensive treatment of sharp Gaussian–Sobolev embeddings of the form

$$\|u - u_{\gamma_n}\|_{Y(\mathbb{R}^n,\gamma_n)} \leq C\|\nabla u\|_{X(\mathbb{R}^n,\gamma_n)} \tag{1.4}$$

was carried out, in which $X$ and $Y$ are general rearrangement-invariant spaces. The focus has been on the "optimality" of the function spaces involved. One of the most important discoveries of [8] was that the Gaussian–Sobolev embedding can be equivalently described by the action of an operator acting on functions of a single variable, providing thus a considerable simplification of the problem in hand. Namely, it was shown in [8, Theorem 3.1] that the inequality (1.4) is equivalent to the boundedness of the operator $S$ defined as

$$Sg(t) = \int_t^1 \frac{g(s)}{s\sqrt{\log \frac{e}{s}}}\, ds$$

for suitable functions $g\colon (0,1) \to \mathbb{R}$ and every $t \in (0,1)$ from $X(0,1)$ to $Y(0,1)$. This result is usually called a *reduction principle*. The operator $S$ is known to satisfy

$$\begin{aligned} &S\colon L^1(0,1) \to L(\log L)^{\frac{1}{2}}(0,1),\\ &S\colon L^\infty(0,1) \to \exp L^2(0,1), \end{aligned} \tag{1.5}$$

and, interestingly, this "endpoint behavior" is shared also by the operator $U$. We shall, however, prove as a particular case of Theorem 3.12 that $U$ majorizes $S$ in the sense that, for every rearrangement-invariant space $Y(0,1)$,

$$\|Sf^*\|_{Y(0,1)} \lesssim \|Uf\|_{Y(0,1)} \quad \text{for every } f.$$

Moreover, unlike $U$, $S$ is linear. The reduction principle leads to a surprising discovery: while the operator $S$, hence the Gaussian–Sobolev embedding, always provides a gain in integrability for example when $X(\mathbb{R}^n, \gamma_n) = L^p(\mathbb{R}^n, \gamma_n)$ for $p < \infty$, there is actually a *loss* of integrability for example when $X(\mathbb{R}^n, \gamma_n) = L^\infty(\mathbb{R}^n, \gamma_n)$ or $X(\mathbb{R}^n, \gamma_n) = \exp L^\beta(\mathbb{R}^n, \gamma_n)$ for $\beta > 0$, in which $\exp L^\beta(\mathbb{R}^n, \gamma_n)$ is the classical exponential-type Orlicz space. Roughly speaking, the gain in integrability vanishes, and eventually is even turned to a loss when we are near the endpoint $L^\infty$. This is very graphically reflected by the second part of (1.5).

In this paper we focus on operators with endpoint behavior similar to that of (1.1) or (1.2), but for a considerably wider class of function spaces of which the operator governing the Gaussian–Sobolev embeddings is a particular instance. Let us finally add for the sake of completeness that operators of another type of nonstandard behavior were studied by different methods in [15,20]. However, both the motivation of the research and techniques used in those papers were completely different.

## 2. Preliminaries

**Conventions.**

- Throughout this paper, $(R,\mu)$ and $(S,\nu)$ are two (possibly different) probabilistic nonatomic measure spaces. If $(R,\mu)=((0,1),\lambda)$, where $\lambda$ is the 1-dimensional Lebesgue measure on $(0,1)$, we write in short $(0,1)$ instead of $((0,1),\lambda)$.
- We write $P\lesssim Q$, where $P,Q$ are nonnegative quantities, when there is a positive constant $c$ independent of all appropriate quantities appearing in the expressions $P$ and $Q$ such that $P\le c\cdot Q$. If not stated explicitly, what "the appropriate quantities appearing in the expressions $P$ and $Q$" are should be obvious from the context. At the few places where it is not obvious, we will explicitly specify what the appropriate quantities are. We also write $P\gtrsim Q$ with the obvious meaning. Furthermore, we write $P\approx Q$ when $P\lesssim Q$ and $P\gtrsim Q$ simultaneously.
- We adhere to the convention that $\frac{1}{\infty}=0\cdot\infty=0$.

We set

$$\mathcal{M}(R,\mu)=\{f\colon f\text{ is a }\mu\text{-measurable complex-valued function on }R\},$$

and

$$\mathcal{M}_+(R,\mu)=\{f\in\mathcal{M}(R,\mu)\colon f\ge 0\ \mu\text{-a.e.}\}.$$

*Rearrangements and rearrangement-invariant function spaces.* The *nonincreasing rearrangement* $f^*\colon(0,1)\to[0,\infty]$ of a function $f\in\mathcal{M}(R,\mu)$ is defined as

$$f^*(t)=\inf\{\lambda\in(0,\infty)\colon\mu\left(\{x\in R\colon|f(x)|>\lambda\}\le t\}\right),\ t\in(0,1).$$

The *maximal nonincreasing rearrangement* $f^{**}\colon(0,1)\to[0,\infty]$ of a function $f\in\mathcal{M}(R,\mu)$ is defined as

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\,ds,\ t\in(0,1).$$

If there is any possibility of misinterpretation, we use the more explicit notations $f^*_\mu$ and $f^{**}_\mu$ instead of $f^*$ and $f^{**}$, respectively, to stress what measure the rearrangements are taken with respect to. The mapping $f\mapsto f^*$ is monotone in the sense that, for every $f,g\in\mathcal{M}(R,\mu)$,

$$|f|\le|g|\quad\mu\text{-a.e. on }R\quad\Longrightarrow\quad f^*\le g^*\quad\text{on }(0,1);$$

consequently, the same implication remains true if $^*$ is replaced by $^{**}$. We have that $f^*\le f^{**}$ for every $f\in\mathcal{M}(R,\mu)$.

The *Hardy lemma* ([3, Chapter 2, Proposition 3.6]) ensures that, for every $f,g\in\mathcal{M}_+(0,1)$ and every nonincreasing $h\in\mathcal{M}_+(0,1)$,

$$\begin{aligned} &\text{if} \quad \int_0^t f(s)\,ds \le \int_0^t g(s)\,ds \quad \text{for every } t \in (0,1), \\ &\text{then} \quad \int_0^1 f(t)h(t)\,dt \le \int_0^1 g(t)h(t)\,dt. \end{aligned} \tag{2.1}$$

A functional $\|\cdot\|_{X(0,1)} \colon \mathcal{M}_+(0,1) \to [0,\infty]$ is called a *rearrangement-invariant Banach function norm* (on $(0,1)$) if, for all $f$, $g$ and $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}_+(0,1)$, and every $\lambda \in [0,\infty)$:

(P1) $\|f\|_{X(0,1)} = 0$ if and only if $f = 0$ a.e. on $(0,1)$; $\|\lambda f\|_{X(0,1)} = \lambda\|f\|_{X(0,1)}$; $\|f+g\|_{X(0,1)} \le \|f\|_{X(0,1)} + \|g\|_{X(0,1)}$;
(P2) $\|f\|_{X(0,1)} \le \|g\|_{X(0,1)}$ if $f \le g$ a.e. on $(0,1)$;
(P3) $\|f_k\|_{X(0,1)} \nearrow \|f\|_{X(0,1)}$ if $f_k \nearrow f$ a.e. on $(0,1)$;
(P4) $\|1\|_{X(0,1)} < \infty$;
(P5) there is a positive constant $C_X$, possibly depending on $\|\cdot\|_{X(0,1)}$ but not on $f$, such that $\int_0^1 f(t)\,dt \le C_X\|f\|_{X(0,1)}$;
(P6) $\|f\|_{X(0,1)} = \|g\|_{X(0,1)}$ whenever $f^* = g^*$.

The *Hardy–Littlewood–Pólya principle* ([3, Chapter 2, Theorem 4.6]) asserts that, for every $f, g \in \mathcal{M}(0,1)$ and every rearrangement-invariant Banach function norm $\|\cdot\|_{X(0,1)}$,

$$\text{if} \int_0^t f^*(s)\,ds \le \int_0^t g^*(s)\,ds \text{ for every } t \in (0,1), \text{ then } \|f\|_{X(0,1)} \le \|g\|_{X(0,1)}. \tag{2.2}$$

With every rearrangement-invariant Banach function norm $\|\cdot\|_{X(0,1)}$, we associate another functional $\|\cdot\|_{X'(0,1)}$ defined as

$$\|f\|_{X'(0,1)} = \sup_{\substack{g\in\mathcal{M}_+(0,1)\\ \|g\|_{X(0,1)}\le 1}} \int_0^1 f(t)g(t)\,dt, \ \ f \in \mathcal{M}_+(0,1).$$

The functional $\|\cdot\|_{X'(0,1)}$ is also a rearrangement-invariant Banach function norm ([3, Chapter 2, Proposition 4.2]), and it is called the *associate Banach function norm* of $\|\cdot\|_{X(0,1)}$. Furthermore, we always have that ([3, Chapter 1, Theorem 2.7])

$$\|f\|_{X(0,1)} = \sup_{\substack{g\in\mathcal{M}_+(0,1)\\ \|g\|_{X'(0,1)}\le 1}} \int_0^1 f(t)g(t)\,dt \quad \text{for every } f \in \mathcal{M}_+(0,1), \tag{2.3}$$

that is,

$$\|\cdot\|_{(X')'(0,1)} = \|\cdot\|_{X(0,1)}. \tag{2.4}$$

The supremum in (2.3) does not change when the functions involved are replaced with their nonincreasing rearrangements ([3, Chapter 2, Proposition 4.2]), that is,

$$\|f\|_{X(0,1)} = \sup_{\substack{g\in\mathcal{M}_+(0,1)\\ \|g\|_{X'(0,1)}\le 1}} \int_0^1 f^*(t)g^*(t)\,dt \quad \text{for every } f \in \mathcal{M}_+(0,1).$$

Given a rearrangement-invariant Banach function norm $\|\cdot\|_{X(0,1)}$, we define the functional $\|\cdot\|_{X(R,\mu)}$ as

$$\|f\|_{X(R,\mu)} = \|f_\mu^*\|_{X(0,1)} \quad \text{for every } f \in \mathcal{M}(R,\mu). \tag{2.5}$$

Note that $\|f\|_{X(R,\mu)} = \||f|\|_{X(R,\mu)}$. When $(R,\mu) = (0,1)$, (2.5) extends the given rearrangement-invariant Banach function norm to all $f \in \mathcal{M}(0,1)$. The functional $\|\cdot\|_{X(R,\mu)}$ restricted to the linear set $X(R,\mu)$ defined as

$$X(R,\mu) = \{f \in \mathcal{M}(R,\mu) \colon \|f\|_{X(R,\mu)} < \infty\} \tag{2.6}$$

is a norm (provided that we identify any two functions from $\mathcal{M}(R,\mu)$ coinciding $\mu$-a.e. on $R$, as usual). In fact, $X(R,\mu)$ endowed with the norm $\|\cdot\|_{X(R,\mu)}$ is a Banach space ([3, Chapter 1, Theorem 1.6]). We say that $X(R,\mu)$ is a *rearrangement-invariant Banach function space* (an *r.i. Banach function space*). Note that $f \in \mathcal{M}(R,\mu)$ belongs to $X(R,\mu)$ if and only if $\|f\|_{X(R,\mu)} < \infty$.

The rearrangement-invariant Banach function space $X'(R,\mu)$ built upon the associate Banach function norm $\|\cdot\|_{X'(0,1)}$ of $\|\cdot\|_{X(0,1)}$ is called the *associate Banach function space* of $X(R,\mu)$. Thanks to (2.4), we have that $(X')'(R,\mu) = X(R,\mu)$. Furthermore, one has that

$$\int_R |f||g| \, d\mu \le \|f\|_{X(R,\mu)} \|g\|_{X'(R,\mu)} \quad \text{for every } f, g \in \mathcal{M}(R,\mu). \tag{2.7}$$

We shall refer to (2.7) as the Hölder inequality.

A functional $\|\cdot\|_{X(0,1)} \colon \mathcal{M}_+(0,1) \to [0,\infty]$ is called a *rearrangement-invariant quasi-Banach function norm* (on $(0,1)$) if it satisfies all the properties of a rearrangement-invariant Banach function norm but (P1) and (P5), and instead of (P1) it satisfies, for every $f, g \in \mathcal{M}_+(0,1)$ and $\lambda \ge 0$,

(P1') $\|f\|_{X(0,1)} = 0$ if and only if $f = 0$ a.e. on $(0,1)$; $\|\lambda f\|_{X(0,1)} = \lambda \|f\|_{X(0,1)}$; there is a constant $C \ge 1$, such that $\|f + g\|_{X(0,1)} \le C \left( \|f\|_{X(0,1)} + \|g\|_{X(0,1)} \right)$.

Given a rearrangement-invariant quasi-Banach function norm $\|\cdot\|_{X(0,1)}$, the functional defined by (2.5) is a quasinorm on the linear set defined by (2.6). Moreover, $X(R,\mu)$ endowed with the quasinorm $\|\cdot\|_{X(R,\mu)}$ is a quasi-Banach space ([22, Corollary 3.7]), and we called it a *rearrangement-invariant quasi-Banach function space* (an *r.i. quasi-Banach function space*). The rearrangement-invariant (quasi-)Banach function space $X(0,1)$ is called the *representation space* of $X(R,\mu)$.

Statements like, "let $X(R,\mu)$ be a rearrangement-invariant (quasi-)Banach function space", are to be interpreted as "let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant (quasi-)Banach function norm and let $X(R,\mu)$ be the corresponding rearrangement-invariant (quasi-)Banach function space".

Let $X(R,\mu)$ and $Y(R,\mu)$ be rearrangement-invariant (quasi-)Banach function spaces over the same measure space. We say that $X(R,\mu)$ is *embedded in* $Y(R,\mu)$, and we write $X(R,\mu) \hookrightarrow Y(R,\mu)$, if there is a positive constant $C$ such that $\|f\|_{Y(R,\mu)} \le C\|f\|_{X(R,\mu)}$ for every $f \in \mathcal{M}(R,\mu)$. If $X(R,\mu) \hookrightarrow Y(R,\mu)$ and $Y(R,\mu) \hookrightarrow X(R,\mu)$ simultaneously, we write $X(R,\mu) = Y(R,\mu)$. We have that ([3, Chapter 1, Theorem 1.8] and [22, Corollary 3.9])

$$X(R,\mu) \hookrightarrow Y(R,\mu) \quad \text{if and only if} \quad X(R,\mu) \subseteq Y(R,\mu).$$

We say that a rearrangement-invariant quasi-Banach function norm $\|\cdot\|_{X(0,1)}$ is *p-convex*, where $p \in (0,\infty)$, if the functional

$$\|f\|_{X^{\frac{1}{p}}(0,1)} = \|f^{\frac{1}{p}}\|_{X(0,1)}^p, \ f \in \mathcal{M}_+(0,1),$$

is a rearrangement-invariant Banach function norm. The corresponding rearrangement-invariant Banach function space $X^{\frac{1}{p}}(R,\mu)$ is said to be $p$-convex.

The $K$*-functional* for a couple of (quasi-)Banach function spaces $(X_0(R,\mu), X_1(R,\mu))$ is defined, for every $f \in \mathcal{M}(R,\mu)$ and $t \in (0,\infty)$, as

$$K(f,t;X_0,X_1) = \inf\{\|g\|_{X_0} + t\|h\|_{X_1} : f = g + h\},$$

where the infimum is taken over all representations $f = g+h$ with $g \in X_0$ and $h \in X_1$. If $f \notin (X_0+X_1)(R,\mu)$, then the infimum is to be interpreted as $\infty$.

*Orlicz–Karamata spaces.* A measurable function $b\colon (0,1) \to (0,\infty)$ is said to be *slowly varying* if for every $\varepsilon > 0$ there are a nondecreasing function $b_\varepsilon$ and a nonincreasing function $b_{-\varepsilon}$ such that $t^\varepsilon b(t) \approx b_\varepsilon(t)$ and $t^{-\varepsilon} b(t) \approx b_{-\varepsilon}(t)$ on $(0,1)$. A slowly varying function $b$ satisfies

$$0 < \inf_{t\in[a,1)} b(t) \le \sup_{t\in[a,1)} b(t) < \infty \quad \text{for every } a \in (0,1).$$

A positive linear combination of slowly varying functions is a slowly varying function. If $b_1, b_2$ are slowly varying functions, so is $b_1 b_2$. Any real power of a slowly varying function is a slowly varying function. For every $\alpha \in \mathbb{R}\setminus\{0\}$ and a slowly varying function $b$, we have that $\lim_{t\to 0^+} t^\alpha b(t) = \lim_{t\to 0^+} t^\alpha$. Furthermore, if $\alpha > 0$, then

$$\int_0^t s^{-1+\alpha} b(s)\, ds \approx t^\alpha b(t) \quad \text{for every } t \in (0,1).$$

For more details, we refer the reader to [14,25].

The Orlicz–Karamata functional $\|\cdot\|_{L^{p,b}(0,1)}$, where $p \in (0,\infty]$ and $b$ is a slowly varying function, is defined as

$$\|f\|_{L^{p,b}(0,1)} = \|b(t) f^*(t)\|_{L^p(0,1)},\ f \in \mathcal{M}_+(0,1),$$

where $\|\cdot\|_{L^p(0,1)}$ is the Lebesgue quasi-norm on $(0,1)$, that is,

$$\|f\|_{L^p(0,1)} = \begin{cases} \left(\int_0^1 |f(t)|^p\, dt\right)^{\frac{1}{p}} & \text{if } p \in (0,\infty), \\ \operatorname*{ess\,sup}_{t\in(0,1)} |f(t)| & \text{if } p = \infty. \end{cases}$$

The functional $\|\cdot\|_{L^{p,b}(0,1)}$ is a rearrangement-invariant quasi-Banach function norm provided that either $p \in (0,\infty)$ or $p = \infty$ and $b \in L^\infty(0,1)$ ([25, Proposition 3.7]). The corresponding function spaces are called *Orlicz–Karamata spaces*. The Orlicz–Karamata functional $\|\cdot\|_{L^{p,b}(0,1)}$ is equivalent to a rearrangement-invariant Banach function norm if and only if $p = 1$ and $b$ is equivalent to a nonincreasing function, or $p \in (1,\infty)$, or $p = \infty$ and $b \in L^\infty(0,1)$ ([25, Theorem 3.26]). The class of Orlicz–Karamata spaces contains Lebesgue spaces as well as some important Orlicz spaces. If $b \equiv 1$, then $\|\cdot\|_{L^{p,b}(0,1)} = \|\cdot\|_{L^p(0,1)}$ ([3, Chapter 2, Proposition 1.8]). Set $\ell(t) = 1 - \log(t)$, $t \in (0,1)$. If $p \in [1,\infty)$ and $b = \ell^\alpha$, where $\alpha > 0$ if $p = 1$, otherwise $\alpha \in \mathbb{R}$, then $L^{p,b}(R,\mu) = L^p (\log L)^{\alpha p} (R,\mu)$, the Orlicz space induced by a Young function $\Phi$ satisfying, for large values of $t$, $\Phi(t) \approx t^p (\log t)^{\alpha p}$. Furthermore, if $b = \ell^\alpha$, where $\alpha < 0$, then $L^{\infty,b}(R,\mu) = \exp L^{-\frac{1}{\alpha}}(R,\mu)$, the Orlicz space induced by a Young function $\Phi$ satisfying, for large values of $t$, $\Phi(t) \approx \exp(t^{-\frac{1}{\alpha}})$. For more details, we refer the reader to [24, Section 8].

## 3. Main results

In this section we shall state and prove our main results. We begin by introducing a key function.

**Definition 3.1.** Let $p \in (0, \infty)$ and $b_1, b_2$ be slowly varying functions. We define the function

$$\sigma = \sigma(b_1, b_2, p) \colon [0,1] \to [0,1]$$

as the increasing, bijective function satisfying

$$t^p = \frac{1}{C} \int_0^{\sigma(t)^p} \left[b_1(s) b_2(s)^{-1}\right]^p \, ds \quad \text{for every } t \in [0,1], \tag{3.1}$$

where

$$C = \int_0^1 \left[b_1(s) b_2(s)^{-1}\right]^p \, ds \in (0, \infty).$$

**Remark 3.2.** Let $C([0,1])$ denote the space of continuous (real-valued) functions on the interval $[0,1]$, and let $C^1(0,1)$ denote the space of continuously differentiable functions on the interval $(0,1)$. It immediately follows from (3.1) that $\sigma, \sigma^{-1} \in \mathcal{C}([0,1])$, where $\sigma^{-1}$ denotes the inverse function. Furthermore, we have that

$$\sigma^{-1}(t) \approx t b_1(t^p) b_2(t^p)^{-1}$$

on $(0,1)$. If the functions $b_1, b_2$ are continuous, then $\sigma, \sigma^{-1} \in \mathcal{C}^1(0,1)$ and

$$\left(\sigma^{-1}(t^{1/p})^p\right)' \approx b_1(t)^p b_2(t)^{-p}$$

on $(0,1)$. We shall use these properties of $\sigma$ without making any explicit reference to them.

We shall now characterize the $K$-inequality corresponding to the couples $(L^{p,b_1}, L^{\infty,b_2})$ and $(L^p, L^\infty)$ by an inequality for certain integrals.

Note that, since $(R,\mu)$ and $(S,\nu)$ are finite nonatomic measure spaces, we have the embeddings $L^{\infty,b_2}(S,\nu) \hookrightarrow L^{p,b_1}(S,\nu)$ and $L^\infty(R,\mu) \hookrightarrow L^p(R,\mu)$, and so to write $f \in L^p(R,\mu)$ and $g \in L^{p,b_1}(S,\nu)$ is the same as to write $f \in (L^p + L^\infty)(R,\mu)$ and $g \in \left(L^{p,b_1} + L^{\infty,b_2}\right)(S,\nu)$.

**Theorem 3.3.** *Let $p \in (0,\infty)$ and $(b_1, b_2)$ be a pair of continuous slowly varying functions. Assume that $b_2$ is nondecreasing. Let $f \in L^p(R,\mu)$ and $g \in L^{p,b_1}(S,\nu)$. The inequality*

$$K(g, t; L^{p,b_1}, L^{\infty,b_2}) \lesssim K(f, t; L^p, L^\infty) \tag{3.2}$$

*holds for every $t \in (0,1)$ with a multiplicative constant independent of $f$ and $g$ if and only if the inequality*

$$\int_0^t [g^*(s) b_1(s)]^p \, ds \lesssim \int_0^t \left[f^*\left(\sigma^{-1}(s^{1/p})^p\right) b_1(s) b_2(s)^{-1}\right]^p \, ds$$

*holds for every $t \in (0,1)$ with a multiplicative constant independent of $f$ and $g$.*

**Proof.** First, we claim that, for every $g \in L^{p,\mathscr{b}_1}(S,\nu)$ and every $t \in (0,1)$,

$$K(g,t;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2}) \approx \left(\int\limits_0^{\sigma(t)^p} [g^*(s)\mathscr{b}_1(s)]^p\, ds\right)^{\frac{1}{p}} + t\cdot \sup_{\sigma(t)^p\le s<1} g^*(s)\mathscr{b}_2(s), \tag{3.3}$$

in which the multiplicative constants are independent of $g$ and $t$. This result can be derived from a general implicit formula appearing in [21, Theorem 4.1]. Since we need the explicit formula here, we shall prove it in detail for the sake of completeness. Let $g \in L^{p,\mathscr{b}_1}(S,\nu)$ and $t \in (0,1)$. For the sake of brevity, we set

$$I(g)(t) = \left(\int\limits_0^{\sigma(t)^p} [g^*(s)\mathscr{b}_1(s)]^p\, ds\right)^{\frac{1}{p}} + t\cdot \sup_{\sigma(t)^p\le s<1} g^*(s)\mathscr{b}_2(s).$$

Let $g = g_1 + g_2$, where $g_1 \in L^{p,\mathscr{b}_1}(S,\nu)$ and $g_2 \in L^{\infty,\mathscr{b}_2}(S,\nu)$, be a decomposition of $g$. We have that

$$\begin{aligned} I(g)(t) &= \left(\int\limits_0^{\sigma(t)^p} [(g_1+g_2)^*(s)\mathscr{b}_1(s)]^p\, ds\right)^{\frac{1}{p}} + t\cdot \sup_{\sigma(t)^p\le s<1} (g_1+g_2)^*(s)\mathscr{b}_2(s) \\ &\le \left(\int\limits_0^{\sigma(t)^p} [(g_1^*(s/2)+g_2^*(s/2))\mathscr{b}_1(s)]^p\, ds\right)^{\frac{1}{p}} + t\cdot \sup_{\sigma(t)^p\le s<1} [g_1^*(s/2)+g_2^*(s/2)]\,\mathscr{b}_2(s) \\ &\lesssim I(g_1^*(\cdot/2))(t) + I(g_2^*(\cdot/2))(t). \end{aligned} \tag{3.4}$$

As for $I(g_1^*(\cdot/2))(t)$, since

$$g_1^*(s) \approx g_1^*(s)\left(\frac{1}{s\mathscr{b}_1(s)^p}\int\limits_0^s \mathscr{b}_1(\tau)^p\, d\tau\right)^{\frac{1}{p}} \le \frac{1}{s^{\frac{1}{p}}\mathscr{b}_1(s)}\left(\int\limits_0^s [g_1^*(\tau)\mathscr{b}_1(\tau)]^p\, d\tau\right)^{\frac{1}{p}}$$

for every $s \in (0,1)$, we obtain that

$$\begin{aligned} I(g_1^*(\cdot/2))(t) &\lesssim \|g_1\|_{L^{p,\mathscr{b}_1}(S,\nu)} + t\cdot \sup_{\sigma(t)^p\le s<1} g_1^*(s/2)\mathscr{b}_2(s) \\ &\lesssim \|g_1\|_{L^{p,\mathscr{b}_1}(S,\nu)} + t\cdot \sup_{\sigma(t)^p\le s<1} \frac{\left(\int\limits_0^{\frac{s}{2}} [g_1^*(\tau)\mathscr{b}_1(\tau)]^p\, d\tau\right)^{\frac{1}{p}}}{s^{\frac{1}{p}}\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}} \\ &\le \|g_1\|_{L^{p,\mathscr{b}_1}(S,\nu)}\left(1 + t\cdot \sup_{\sigma(t)^p\le s<1} \frac{1}{s^{\frac{1}{p}}\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}}\right) \\ &\approx \|g_1\|_{L^{p,\mathscr{b}_1}(S,\nu)}\left(1 + t\cdot \frac{1}{\sigma(t)\mathscr{b}_1(\sigma(t)^p)\mathscr{b}_2(\sigma(t)^p)^{-1}}\right) \\ &\approx \|g_1\|_{L^{p,\mathscr{b}_1}(S,\nu)}. \end{aligned} \tag{3.5}$$

As for the second term on the right-hand side of (3.4), note that

$$
\begin{aligned}
I(g_2^*(\cdot/2))(t) &\lesssim \|g_2\|_{L^{\infty,\mathfrak{b}_2}(S,\nu)} \left( \int_0^{\sigma(t)^p} \left[\mathfrak{b}_1(s)\mathfrak{b}_2(s)^{-1}\right]^p \, ds \right)^{\frac{1}{p}} + t \cdot \sup_{\sigma(t)^p \le s < 1} g_2^*(s/2)\mathfrak{b}_2(s/2) \\
&\approx t\|g_2\|_{L^{\infty,\mathfrak{b}_2}(S,\nu)} + t \cdot \sup_{\frac{\sigma(t)^p}{2} \le s < \frac{1}{2}} g_2^*(s)\mathfrak{b}_2(s) \\
&\lesssim t \cdot \|g_2\|_{L^{\infty,\mathfrak{b}_2}(S,\nu)},
\end{aligned} \tag{3.6}
$$

in which we used (3.1). Hence, by combining (3.6) and (3.5) together with (3.4), and taking the infimum over all such representations $g = g_1 + g_2$, we obtain that

$$
I(g)(t) \lesssim K(g,t; L^{p,\mathfrak{b}_1}, L^{\infty,\mathfrak{b}_2}). \tag{3.7}
$$

As for the opposite inequality, we may assume that $I(g)(t) < \infty$, for otherwise there is nothing to prove. Define the functions $g_1, g_2 \in \mathcal{M}(S,\nu)$ as

$$
g_1(x) = \max\{|g(x)| - g^*(\sigma(t)^p), 0\} \cdot \operatorname{sgn} g(x), \ \ x \in S,
$$

and

$$
g_2(x) = g(x) - g_1(x) = \min\{|g(x)|, g^*(\sigma(t)^p)\} \cdot \operatorname{sgn} g(x), \ \ x \in S.
$$

Clearly, $g = g_1 + g_2$ and we have that

$$
g_1^*(s) = (g^*(s) - g^*(\sigma(t)^p))\chi_{(0,\sigma(t)^p)}(s) \quad \text{and} \quad g_2^*(s) = \min\{g^*(s), g^*(\sigma(t)^p)\}
$$

for every $s \in (0,1)$. Note that

$$
\|g_1\|_{L^{p,\mathfrak{b}_1}(S,\nu)} = \left( \int_0^{\sigma(t)^p} [(g^*(s) - g^*(\sigma(t)^p))\mathfrak{b}_1(s)]^p \, ds \right)^{\frac{1}{p}} \le \left( \int_0^{\sigma(t)^p} [g^*(s)\mathfrak{b}_1(s)]^p \, ds \right)^{\frac{1}{p}}
$$

and

$$
\|g_2\|_{L^{\infty,\mathfrak{b}_2}(S,\nu)} = g^*(\sigma(t)^p) \sup_{0 < s \le \sigma(t)^p} \mathfrak{b}_2(s) + \sup_{\sigma(t)^p \le s < 1} g^*(s)\mathfrak{b}_2(s) \approx \sup_{\sigma(t)^p \le s < 1} g^*(s)\mathfrak{b}_2(s);
$$

consequently,

$$
\|g_1\|_{L^{p,\mathfrak{b}_1}(S,\nu)} + t\|g_2\|_{L^{\infty,\mathfrak{b}_2}(S,\nu)} \lesssim I(g)(t)
$$

and $g_1 \in L^{p,\mathfrak{b}_1}(S,\nu)$ and $g_2 \in L^{\infty,\mathfrak{b}_2}(S,\nu)$. Hence,

$$
K(g,t; L^{p,\mathfrak{b}_1}, L^{\infty,\mathfrak{b}_2}) \lesssim I(g)(t),
$$

which together with (3.7) establishes our claim (3.3).

Second, since we have that (see [19], also [17, Theorem 4.1])

$$
K(f,t; L^p, L^\infty) \approx \left( \int_0^{t^p} f^*(s)^p \, ds \right)^{\frac{1}{p}},
$$

in which the multiplicative constants depend only on $p$, in view of (3.3), we need to prove that

$$\left(\int_0^{\sigma(t)^p} [g^*(s)\mathscr{b}_1(s)]^p \, ds\right)^{\frac{1}{p}} + t \cdot \sup_{\sigma(t)^p \le s < 1} g^*(s)\mathscr{b}_2(s) \lesssim \left(\int_0^{t^p} f^*(s)^p \, ds\right)^{\frac{1}{p}} \tag{3.8}$$

for every $t \in (0,1)$ if and only if

$$\int_0^t [g^*(s)\mathscr{b}_1(s)]^p \, ds \lesssim \int_0^t \left[f^*\big(\sigma^{-1}(s^{1/p})^p\big)\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p \, ds \quad \text{for every } t \in (0,1). \tag{3.9}$$

We shall observe that

$$\int_0^{\sigma(t)^p} [g^*(s)\mathscr{b}_1(s)]^p \, ds \lesssim \int_0^{t^p} f^*(s)^p \, ds \quad \text{for every } t \in (0,1) \tag{3.10}$$

if and only if

$$t^p \cdot \sup_{\sigma(t)^p \le s < 1} \frac{1}{s\mathscr{b}_1(s)^p\mathscr{b}_2(s)^{-p}} \int_0^s [g^*(\tau)\mathscr{b}_1(\tau)]^p \, d\tau \lesssim \int_0^{t^p} f^*(s)^p \, ds \quad \text{for every } t \in (0,1). \tag{3.11}$$

Note that (3.11) plainly implies (3.10) inasmuch as

$$\sigma(t)\mathscr{b}_1(\sigma(t)^p)\mathscr{b}_2(\sigma(t)^p)^{-1} \approx t \quad \text{for every } t \in (0,1).$$

As for the opposite implication, let $t \in (0,1)$. Since the function $(0,1) \ni t \mapsto (|f|^p)^{**}(t^p)$ is nonincreasing, (3.10) actually implies that

$$\sup_{t \le s < 1} \frac{1}{s^p} \int_0^{\sigma(s)^p} [g^*(\tau)\mathscr{b}_1(\tau)]^p \, d\tau \lesssim \frac{1}{t^p} \int_0^{t^p} f^*(s)^p \, ds. \tag{3.12}$$

Since $\sigma^{-1}$ is an increasing bijection of $[0,1]$ onto itself, by the change of variables $s = \sigma^{-1}(\tilde{s}^{1/p})$, (3.12) is equivalent to

$$\sup_{\sigma(t)^p \le \tilde{s} < 1} \frac{1}{\tilde{s}\mathscr{b}_1(\tilde{s})^p\mathscr{b}_2(\tilde{s})^{-p}} \int_0^{\tilde{s}} [g^*(\tau)\mathscr{b}_1(\tau)]^p \, d\tau \lesssim \frac{1}{t^p} \int_0^{t^p} f^*(s)^p \, ds,$$

whence (3.11) follows. Furthermore, by the change of variables $s = \sigma^{-1}(\tilde{s}^{1/p})^p$, we have that

$$\int_0^{t^p} f^*(s) \, ds \approx \int_0^{\sigma(t)^p} f^*(\sigma^{-1}(\tilde{s}^{1/p})^p)\mathscr{b}_1(\tilde{s})^p\mathscr{b}_2(\tilde{s})^{-p} \, d\tilde{s} \quad \text{for every } t \in (0,1).$$

Hence, since $\sigma$ is a bijection of $[0,1]$ onto itself, (3.9) is equivalent to (3.10).

Finally, the proof will be complete once we show that (3.11) is equivalent to (3.8). Since (3.8) plainly implies (3.10), which is equivalent to (3.11), we only need to observe that (3.11) implies (3.8) (the former actually implies the latter pointwise). To this end, note that

$$\sup_{\sigma(t)^p\le s<1} g^*(s)\mathscr{b}_2(s) \approx \sup_{\sigma(t)^p\le s<1} \frac{g^*(s)}{s^{\frac{1}{p}}\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}}\left(\int_0^s \mathscr{b}_1(\tau)^p\,d\tau\right)^{\frac{1}{p}}$$
$$\le \sup_{\sigma(t)^p\le s<1} \frac{1}{s^{\frac{1}{p}}\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}}\left(\int_0^s \left[g^*(\tau)\mathscr{b}_1(\tau)\right]^p\,d\tau\right)^{\frac{1}{p}}$$

for every $t\in(0,1)$. Hence, if (3.11) is true (consequently, so is (3.10)), then

$$\begin{aligned}&K(g,t;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2})\\ &\lesssim \left(\int_0^{\sigma(t)^p}\left[g^*(s)\mathscr{b}_1(s)\right]^p\,ds\right)^{\frac{1}{p}} + t\cdot \sup_{\sigma(t)^p\le s<1}\frac{1}{s^{\frac{1}{p}}\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}}\left(\int_0^s\left[g^*(\tau)\mathscr{b}_1(\tau)\right]^p\,d\tau\right)^{\frac{1}{p}}\\ &\lesssim \left(\int_0^{t^p} f^*(s)^p\,ds\right)^{\frac{1}{p}}\end{aligned}$$

for every $t\in(0,1)$. $\square$

**Remark 3.4.** If (3.2) is valid for every $t\in(0,1)$, it is actually valid for every $t\in(0,\infty)$ (with a possibly different multiplicative constant). Indeed, owing to the embeddings mentioned before Theorem 3.3, we have that $K(f,t;L^p,L^\infty)\approx K(f,1;L^p,L^\infty)$ and $K(g,t;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2})\approx K(g,1;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2})$ for every $t\in[1,\infty)$, in which the multiplicative constants are independent of $f,g$ and $t$; therefore

$$\begin{aligned}K(g,t;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2}) &\approx K(g,1;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2})\\ &\lesssim K(f,1;L^p,L^\infty)\approx K(f,t;L^p,L^\infty)\end{aligned}$$

for every $t\in[1,\infty)$.

Now we shall introduce a key notion of a gaussible operator.

**Definition 3.5.** Let $p\in(0,\infty)$ and $\mathscr{b}_1,\mathscr{b}_2$ be slowly varying functions. We say that an operator $T$ defined on $L^p(R,\mu)$ having values in $\mathcal{M}(S,\nu)$ is $(p,\mathscr{b}_1,\mathscr{b}_2)$-*gaussible* if

$$\int_0^t \left[(Tf)^*(s)\mathscr{b}_1(s)\right]^p\,ds \lesssim \int_0^t \left[f^*\big(\sigma^{-1}(s^{1/p})^p\big)\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p\,ds$$

for every $f\in L^p(R,\mu)$ and $t\in(0,1)$.

**Remarks 3.6.**

(i) It follows immediately from the definition that a $(p,\mathscr{b}_1,\mathscr{b}_2)$-gaussible operator is bounded from $L^p(R,\mu)$ to $L^{p,\mathscr{b}_1}(S,\nu)$. Indeed, any $(p,\mathscr{b}_1,\mathscr{b}_2)$-gaussible operator $T$ satisfies

$$\|Tf\|_{L^{p,\mathscr{b}_1}(S,\nu)} \lesssim \left(\int_0^1\left[f^*\big(\sigma^{-1}(s^{1/p})^p\big)\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p\,ds\right)^{1/p} \approx \|f\|_{L^p(R,\mu)}.$$

(ii) In view of Theorem 3.3, an operator $T$ defined on $L^p(R,\mu)$ having values in $\mathcal{M}(S,\nu)$ is $(p,\mathfrak{b}_1,\mathfrak{b}_2)$-gaussible if and only if it satisfies

$$K(Tf,t;L^{p,\mathfrak{b}_1},L^{\infty,\mathfrak{b}_2}) \lesssim K(f,t;L^p,L^\infty) \quad \text{for every } f \in L^p(R,\mu) \text{ and } t \in (0,1). \tag{3.13}$$

(iii) The class of operators satisfying the $K$-inequality (3.13) actually coincides with a certain class of operators introduced in [5, Section 4.1]. An operator $T$ defined on $X_0 + X_1$ having values in $Y_0 + Y_1$, where $(X_0, X_1)$ and $(Y_0, Y_1)$ are two pairs of compatible couples of quasi-Banach spaces, belongs to the class $B(X_0, X_1; Y_0, Y_1)$ if there is a constant $C > 0$ such that, for every $f_i \in X_i$, $i = 0, 1$, and every $\varepsilon > 0$, there are $g_i \in Y_i$, $i = 0, 1$, such that

$$T(f_0 + f_1) = g_0 + g_1 \quad \text{and} \quad \|g_i\|_{Y_i} \le C\|f_i\|_{X_i} + \varepsilon,\ i = 0, 1.$$

By [5, Proposition 4.1.3] with some appropriate modifications, an operator $T$ defined on $L^p(R,\mu)$ having values in $\mathcal{M}(S,\nu)$ satisfies the $K$-inequality (3.13) if and only if it belongs to the class $B(L^p(R,\mu), L^\infty(R,\mu); L^{p,\mathfrak{b}_1}(S,\nu), L^{\infty,\mathfrak{b}_2}(S,\nu))$.

Assume that (3.13) holds. Let $f_0 \in L^p(R,\mu)$, $f_1 \in L^\infty(R,\mu)$ and $\varepsilon > 0$ be given. Assume that neither $f_0$ nor $f_1$ is equivalent to the zero function (otherwise the proof is trivial). Thanks to (3.13) with $t = t_0 = \dfrac{\|f_0\|_{L^p(R,\mu)}}{\|f_1\|_{L^\infty(R,\mu)}}$, there are $g_0 \in L^{p,\mathfrak{b}_1}(S,\nu)$ and $g_1 \in L^{\infty,\mathfrak{b}_2}(S,\nu)$ such that $T(f_0 + f_1) = g_0 + g_1$ and

$$\|g_0\|_{L^{p,\mathfrak{b}_1}(S,\nu)} + \frac{\|f_0\|_{L^p(R,\mu)}}{\|f_1\|_{L^\infty(R,\mu)}}\|g_1\|_{L^{\infty,\mathfrak{b}_2}(S,\nu)} \le 2C\|f_0\|_{L^p(R,\mu)} + \min\left\{\frac{\|f_0\|_{L^p(R,\mu)}}{\|f_1\|_{L^\infty(R,\mu)}}, 1\right\}\varepsilon,$$

whence

$$\|g_0\|_{L^{p,\mathfrak{b}_1}(S,\nu)} \le 2C\|f_0\|_{L^p(R,\mu)} + \varepsilon$$

and

$$\|g_1\|_{L^{\infty,\mathfrak{b}_2}(S,\nu)} \le 2C\|f_1\|_{L^\infty(R,\mu)} + \varepsilon.$$

Hence $T \in B(L^p(R,\mu), L^\infty(R,\mu); L^{p,\mathfrak{b}_1}(S,\nu), L^{\infty,\mathfrak{b}_2}(S,\nu))$. Conversely, assume that $T \in B(L^p(R,\mu), L^\infty(R,\mu); L^{p,\mathfrak{b}_1}(S,\nu), L^{\infty,\mathfrak{b}_2}(S,\nu))$. Let $f \in L^p(R,\mu)$ and $t \in (0,\infty)$ be given. Let $f = f_0 + f_1$ be a decomposition of $f$, where $f_0 \in L^p(R,\mu)$, $f_1 \in L^\infty(R,\mu)$. Fix arbitrary $\varepsilon > 0$. There are $g_0 \in L^{p,\mathfrak{b}_1}(S,\nu)$ and $g_1 \in L^{\infty,\mathfrak{b}_2}(S,\nu)$ such that $T(f_0 + f_1) = g_0 + g_1$ and

$$\|g_0\|_{L^{p,\mathfrak{b}_1}(S,\nu)} \le C\|f_0\|_{L^p(R,\mu)} + \varepsilon \quad \text{and} \quad \|g_1\|_{L^{\infty,\mathfrak{b}_2}(S,\nu)} \le C\|f_1\|_{L^\infty(R,\mu)} + \varepsilon,$$

where $C > 0$ is a constant independent of $f_0$, $f_1$, $g_0$, $g_1$, $t$ and $\varepsilon$. Consequently,

$$K(Tf,t;L^{p,\mathfrak{b}_1},L^{\infty,\mathfrak{b}_2}) \le C(\|f_0\|_{L^p(R,\mu)} + t\|f_1\|_{L^\infty(R,\mu)}) + (1+t)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$K(Tf,t;L^{p,\mathfrak{b}_1},L^{\infty,\mathfrak{b}_2}) \le C(\|f_0\|_{L^p(R,\mu)} + t\|f_1\|_{L^\infty(R,\mu)}).$$

By taking the infimum over all decompositions $f = f_0 + f_1$, $f_0 \in L^p(R,\mu)$, $f_1 \in L^\infty(R,\mu)$, we obtain (3.13).

We now specify the class of pairs of slowly varying functions for which we shall later obtain our main result.

**Definition 3.7.** Let $p \in (0, \infty)$. We say that a pair $(\mathscr{b}_1, \mathscr{b}_2)$ of slowly varying functions belongs to the class $\mathcal{B}_p$ if

(a) $\mathscr{b}_1, \mathscr{b}_2 \in \mathcal{C}(0,1)$,
(b) $\mathscr{b}_1$ is nonincreasing and $\mathscr{b}_2$ is nondecreasing,
(c) $\mathscr{b}_1(t) \approx \mathscr{b}_1\left(t\mathscr{b}_1(t)^p\mathscr{b}_2(t)^{-p}\right)$ near $0^+$,
(d) $\sup\limits_{0<t<1} \mathscr{b}_2(t)^p \int\limits_t^1 \frac{ds}{s\mathscr{b}_1(s)^p} < \infty$.

**Remarks 3.8.**

(i) Note that (c) in Definition 3.7 actually implies that

$$\mathscr{b}_1(t) \approx \mathscr{b}_1\big(\sigma^{-1}(t^{1/p})^p\big) \approx \mathscr{b}_1\big(\sigma(t^{1/p})^p\big) \quad \text{for every } t \in (0,1).$$

(ii) Since the function $t \mapsto \mathscr{b}_1(t)^p\mathscr{b}_2(t)^{-p}$, $t \in (0,1)$, is nonincreasing, it follows that

$$t \leq \sigma^{-1}(t^{1/p})^p \quad \text{for every } t \in (0,1). \tag{3.14}$$

Indeed, owing to (3.1), we have that

$$\begin{aligned}
\frac{\sigma^{-1}(t^{1/p})^p}{t} &= \left(\int\limits_0^1 \left[\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p ds\right)^{-1} \frac{1}{t}\int\limits_0^t \left[\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p ds \\
&\geq \left(\int\limits_0^1 \left[\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p ds\right)^{-1} \int\limits_0^1 \left[\mathscr{b}_1(s)\mathscr{b}_2(s)^{-1}\right]^p ds \\
&= 1
\end{aligned}$$

for every $t \in (0,1)$. Moreover, if the function $t \mapsto \mathscr{b}_1(t)^p\mathscr{b}_2(t)^{-p}$, $t \in (0,1)$, is decreasing, then the inequality in (3.14) is strict.

Now we shall introduce three operators, which will play an essential role in what follows.

**Definition 3.9.** Let $p \in (0, \infty)$ and $\mathscr{b}_1, \mathscr{b}_2$ be slowly varying functions. We define the operators $U_{\mathscr{b}_1,\mathscr{b}_2,p}$, $T_{\mathscr{b}_1,\mathscr{b}_2,p}$ and $S_{\mathscr{b}_1,p}$ as, for every $f \in \mathcal{M}(0,1)$,

$$\begin{aligned}
U_{\mathscr{b}_1,\mathscr{b}_2,p}f(t) &= f^*(\sigma^{-1}(t^{1/p})^p)\mathscr{b}_2(t)^{-1}, \ t \in (0,1), \\
T_{\mathscr{b}_1,\mathscr{b}_2,p}f(t) &= \sup_{t \leq s < 1} \frac{f^*(\sigma(s^{1/p})^p)}{\mathscr{b}_1(\sigma(s^{1/p})^p)^p}, \ t \in (0,1),
\end{aligned}$$

and

$$S_{\mathscr{b}_1,p}f(t) = \left( \int_t^1 \frac{|f(s)|^p}{s\mathscr{b}_1(s)^p}\, ds \right)^{\frac{1}{p}}, \ t \in (0,1),$$

where $\sigma$ is the function from Definition 3.1.

**Remarks 3.10.**

(i) Note that the functions $U_{\mathscr{b}_1,\mathscr{b}_2,p}f$, $T_{\mathscr{b}_1,\mathscr{b}_2,p}f$ and $S_{\mathscr{b}_1,p}f$ are nonincreasing for every $f \in \mathcal{M}(0,1)$.

(ii) The operator $U_{\mathscr{b}_1,\mathscr{b}_2,p}$ is plainly $(p,\mathscr{b}_1,\mathscr{b}_2)$-gaussible (with $(R,\mu)=(S,\nu)=(0,1)$). Hence

$$K(U_{\mathscr{b}_1,\mathscr{b}_2,p}f,t;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2}) \lesssim K(f,t;L^p,L^\infty) \quad \text{for every } f \in L^p(R,\mu) \text{ and } t \in (0,1)$$

owing to Remark 3.6(ii). Moreover, although $U_{\mathscr{b}_1,\mathscr{b}_2,p}$ is neither linear nor quasilinear, it is bounded (in the classic sense) from $L^p(0,1)$ to $L^{p,\mathscr{b}_1}(0,1)$ (see Remark 3.6(i)) and from $L^\infty(0,1)$ to $L^{\infty,\mathscr{b}_2}(0,1)$; indeed,

$$\|U_{\mathscr{b}_1,\mathscr{b}_2,p}f\|_{L^{\infty,\mathscr{b}_2}(0,1)} = \sup_{0<t<1} \left( \frac{f^*(\sigma^{-1}(t^{1/p})^p)}{\mathscr{b}_2(t)} \right) \mathscr{b}_2(t) = \|f\|_{L^\infty(0,1)}$$

for every $f \in \mathcal{M}(0,1)$.

Now we are in a position to state and prove our main results.

**Theorem 3.11.** *Let $p \in (0,\infty)$ and $(\mathscr{b}_1,\mathscr{b}_2) \in \mathcal{B}_p$. Let $X(R,\mu)$ and $Y(S,\nu)$ be r.i. quasi-Banach function spaces that are $p$-convex. The following four statements are equivalent.*

(i) *Every $(p,\mathscr{b}_1,\mathscr{b}_2)$-gaussible operator $T$ is bounded from $X(R,\mu)$ to $Y(S,\nu)$.*

(ii) *Every operator $T$ defined on $L^p(R,\mu)$ having values in $\mathcal{M}(S,\nu)$ that satisfies*

$$K(Tf,t;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2}) \lesssim K(f,t;L^p,L^\infty) \quad \textit{for every } f \in L^p(R,\mu) \textit{ and } t \in (0,1)$$

*is bounded from $X(R,\mu)$ to $Y(S,\nu)$.*

(iii) *The operators $U_{\mathscr{b}_1,\mathscr{b}_2,p}$ and $S_{\mathscr{b}_1,p}$ are bounded from $X(0,1)$ to $Y(0,1)$.*

(iv) *The operator $T_{\mathscr{b}_1,\mathscr{b}_2,p}$ is bounded from $\big(Y^{\frac{1}{p}}\big)'(0,1)$ to $\big(X^{\frac{1}{p}}\big)'(0,1)$.*

**Proof.** *(i) and (ii) are equivalent.* This is an immediate consequence of the very definition of $(p,\mathscr{b}_1,\mathscr{b}_2)$-gaussible operators and Theorem 3.3, as was already observed in Remark 3.6(ii).

*(i) implies (iii).* Note that, for every $f \in L^p(R,\mu)$, the function $U_{\mathscr{b}_1,\mathscr{b}_2,p}(f^*)$ is a nonnegative, nonincreasing, finite function on $(0,1)$. By [3, Chapter 2, Corollary 7.8], there is a function $g_f \in \mathcal{M}(S,\nu)$ such that $g_f^* = U_{\mathscr{b}_1,\mathscr{b}_2,p}(f^*)$. The auxiliary operator $T$ defined as $Tf = g_f$, $f \in L^p(R,\mu)$, is plainly $(p,\mathscr{b}_1,\mathscr{b}_2)$-gaussible (note that this does not depend on particular choices of $g_f$). Hence, owing to $(i)$, $T$ is bounded from $X(R,\mu)$ to $Y(S,\nu)$. By [3, Chapter 2, Corollary 7.8] again, for every $h \in X(0,1)$, there is a function $f_h \in X(R,\mu)$ such that $f_h^* = h^*$. Therefore,

$$\|U_{\mathscr{b}_1,\mathscr{b}_2,p}h\|_{Y(0,1)} = \|(Tf_h)^*\|_{Y(0,1)} = \|Tf_h\|_{Y(S,\nu)} \lesssim \|f_h\|_{X(R,\mu)} = \|h^*\|_{X(0,1)} = \|h\|_{X(0,1)}$$

for every $h \in X(0,1)$. Hence $U_{\mathscr{b}_1,\mathscr{b}_2,p}$ is bounded from $X(0,1)$ to $Y(0,1)$.

Next, it is easy to see that $S_{\mathscr{b}_1,p}$ is bounded from $L^p(0,1)$ to $L^{p,\mathscr{b}_1}(0,1)$ and from $L^\infty(0,1)$ to $L^{\infty,\mathscr{b}_2}(0,1)$. Moreover, the operator is quasilinear. It follows that

$$K(S_{\mathscr{b}_1,p}f,t;L^{p,\mathscr{b}_1},L^{\infty,\mathscr{b}_2}) \lesssim K(f,t;L^p,L^\infty) \quad \text{for every } f\in L^p(0,1) \text{ and } t\in(0,1);$$

hence $S_{\mathscr{b}_1,p}$ is a $(p,\mathscr{b}_1,\mathscr{b}_2)$-gaussible operator with respect to $(R,\mu)=(S,\nu)=(0,1)$ (see Remark 3.6(ii)). Arguing along the same lines as for $U_{\mathscr{b}_1,\mathscr{b}_2,p}$, we obtain that $S_{\mathscr{b}_1,p}$ is bounded from $X(0,1)$ to $Y(0,1)$.

*(iii) implies (iv).* Fix $f\in\mathcal{M}(0,1)$. First, note that

$$T_{\mathscr{b}_1,\mathscr{b}_2,p}f(t) \lesssim \frac{f^*(\sigma(t^{1/p})^p)}{\mathscr{b}_1(\sigma(t^{1/p})^p)^p} + T_{\mathscr{b}_1,\mathscr{b}_2,p}f(\sigma^{-1}(t^{1/p})^p) \quad \text{for every } t\in(0,1). \tag{3.15}$$

Indeed, since $t\le\sigma^{-1}(t^{1/p})^p$, we have that

$$\begin{aligned} T_{\mathscr{b}_1,\mathscr{b}_2,p}f(t) &\le \sup_{t\le s\le\sigma^{-1}(t^{1/p})^p} \frac{f^*(\sigma(s^{1/p})^p)}{\mathscr{b}_1(\sigma(s^{1/p})^p)^p} + \sup_{\sigma^{-1}(t^{1/p})^p\le s<1} \frac{f^*(\sigma(s^{1/p})^p)}{\mathscr{b}_1(\sigma(s^{1/p})^p)^p} \\ &\le \frac{f^*(\sigma(t^{1/p})^p)}{\mathscr{b}_1(t)^p} + T_{\mathscr{b}_1,\mathscr{b}_2,p}f(\sigma^{-1}(t^{1/p})^p) \\ &\approx \frac{f^*(\sigma(t^{1/p})^p)}{\mathscr{b}_1(\sigma(t^{1/p})^p)^p} + T_{\mathscr{b}_1,\mathscr{b}_2,p}f(\sigma^{-1}(t^{1/p})^p). \end{aligned}$$

It follows from (3.15) that

$$\begin{aligned} \int_0^1 T_{\mathscr{b}_1,\mathscr{b}_2,p}f(t)g^*(t)\,dt &\lesssim \int_0^1 \left(\frac{f^*(\sigma(t^{1/p})^p)}{\mathscr{b}_1(\sigma(t^{1/p})^p)^p}\right) g^*(t)\,dt + \int_0^1 T_{\mathscr{b}_1,\mathscr{b}_2,p}f(\sigma^{-1}(t^{1/p})^p)g^*(t)\,dt \\ &= \mathcal{I}_1(g)+\mathcal{I}_2(g) \end{aligned} \tag{3.16}$$

for every $g\in\mathcal{M}(0,1)$. As for $\mathcal{I}_1(g)$, by the change of variables $t=\sigma^{-1}(\tilde{t}^{1/p})^p$, Hölder's inequality (2.7), and $(iii)$, we have that

$$\begin{aligned} \mathcal{I}_1(g) &\approx \int_0^1 f^*(\tilde{t})(U_{\mathscr{b}_1,\mathscr{b}_2,p}(|g|^{1/p})(\tilde{t}))^p\,d\tilde{t} \le \|f\|_{(Y^{\frac{1}{p}})'(0,1)} \|U_{\mathscr{b}_1,\mathscr{b}_2,p}(|g|^{1/p})\|^p_{Y(0,1)} \\ &\lesssim \|f\|_{(Y^{\frac{1}{p}})'(0,1)} \||g|^{1/p}\|^p_{X(0,1)} = \|f\|_{(Y^{\frac{1}{p}})'(0,1)} \|g\|_{X^{\frac{1}{p}}(0,1)}. \end{aligned} \tag{3.17}$$

As for $\mathcal{I}_2(g)$, since the function $\frac{1}{\mathscr{b}_1}$ is equivalent to a quasiconcave function on $(0,1)$, it follows from [11, Lemma 4.10] that

$$\int_0^t T_{\mathscr{b}_1,\mathscr{b}_2,p}f(\sigma^{-1}(s^{1/p})^p)\,ds = \int_0^t \sup_{s\le\tau<1}\frac{f^*(\tau)}{\mathscr{b}_1(\tau)^p}\,ds \lesssim \int_0^t \left(\frac{f^*(\tau)}{\mathscr{b}_1(\tau)^p}\right)^*(s)\,ds$$

for every $t\in(0,1)$. Hence, by virtue of Hardy's lemma (2.1),

$$\mathcal{I}_2(g) \lesssim \int_0^1 \left(\frac{f^*(s)}{\mathscr{b}_1(s)^p}\right)^*(t)g^*(t)\,dt. \tag{3.18}$$

Finally, by combining (3.16) with (3.17) and (3.18), and using Hölder's inequality (2.7) and the boundedness of $S_{\mathscr{b}_1,p}$, we obtain

$$\begin{aligned}
\|T_{\theta_1,\theta_2,p}f\|_{(X^{\frac{1}{p}})'(0,1)} &= \sup_{\|g\|_{X^{\frac{1}{p}}(0,1)}\le 1} \int_0^1 T_{\theta_1,\theta_2,p}f(t)g^*(t)\,dt \lesssim \sup_{\|g\|_{X^{\frac{1}{p}}(0,1)}\le 1} (\mathcal{I}_1(g)+\mathcal{I}_2(g)) \\
&\lesssim \|f\|_{(Y^{\frac{1}{p}})'(0,1)} + \sup_{\|g\|_{X^{\frac{1}{p}}(0,1)}\le 1} \int_0^1 \left(\frac{f^*(s)}{\theta_1(s)^p}\right)^*(t)g^*(t)\,dt \\
&= \|f\|_{(Y^{\frac{1}{p}})'(0,1)} + \sup_{\|g\|_{X^{\frac{1}{p}}(0,1)}\le 1} \int_0^1 \frac{f^*(t)}{\theta_1(t)^p}|g(t)|\,dt \\
&\le \|f\|_{(Y^{\frac{1}{p}})'(0,1)} + \sup_{\|g\|_{X^{\frac{1}{p}}(0,1)}\le 1} \int_0^1 \frac{f^{**}(t)}{\theta_1(t)^p}|g(t)|\,dt \\
&= \|f\|_{(Y^{\frac{1}{p}})'(0,1)} + \sup_{\|g\|_{X^{\frac{1}{p}}(0,1)}\le 1} \int_0^1 f^*(t)S_{\theta_1,p}(|g|^{1/p})(t)^p\,dt \\
&\le \|f\|_{(Y^{\frac{1}{p}})'(0,1)} + \|f\|_{(Y^{\frac{1}{p}})'(0,1)} \sup_{\|g\|_{X^{\frac{1}{p}}(0,1)}\le 1} \|S_{\theta_1,p}(|g|^{1/p})\|^p_{Y(0,1)} \\
&\lesssim \|f\|_{(Y^{\frac{1}{p}})'(0,1)} + \|f\|_{(Y^{\frac{1}{p}})'(0,1)} \sup_{\|g\|_{X^{\frac{1}{p}}(0,1)}\le 1} \||g|^{1/p}\|^p_{X(0,1)} \\
&\approx \|f\|_{(Y^{\frac{1}{p}})'(0,1)}.
\end{aligned}$$

Hence $T_{\theta_1,\theta_2,p}$ is bounded from $\left(Y^{\frac{1}{p}}\right)'(0,1)$ to $\left(X^{\frac{1}{p}}\right)'(0,1)$.

*(iv) implies (i).* Since $T$ is $(p,\theta_1,\theta_2)$-gaussible, by virtue of Hardy's lemma (2.1) we have that

$$\begin{aligned}
\int_0^1 (Tf)^*(s)^p g^*(s)\,ds &\le \int_0^1 [(Tf)^*(s)\theta_1(s)]^p \left(\sup_{s\le\tau<1} \frac{g^*(\tau)}{\theta_1(\tau)^p}\right) ds \\
&\lesssim \int_0^1 f^*(\sigma^{-1}(s^{1/p})^p)^p \theta_1(s)^p \theta_2(s)^{-p}\, T_{\theta_1,\theta_2,p}g(\sigma^{-1}(s^{1/p})^p)\,ds \\
&\approx \int_0^1 f^*(s)^p\, T_{\theta_1,\theta_2,p}g(s)\,ds
\end{aligned}$$

for every $g \in \mathcal{M}(0,1)$. Hence, by using Hölder's inequality (2.7) on the right-hand side and $(iv)$,

$$\int_0^1 (Tf)^*(s)^p g^*(s)\,ds \lesssim \||f|^p\|_{X^{\frac{1}{p}}(0,1)} \|T_{\theta_1,\theta_2,p}g\|_{(X^{\frac{1}{p}})'(0,1)} \lesssim \|f\|^p_{X(R,\mu)} \|g\|_{(Y^{\frac{1}{p}})'(0,1)},$$

whence, by taking the supremum over all $g$ from the unit ball of $\left(Y^{\frac{1}{p}}\right)'(0,1)$, we obtain that $T$ is bounded from $X(R,\mu)$ to $Y(S,\nu)$. $\square$

It turns out that the statement Theorem 3.11(iii) is often actually equivalent to (in turn, so are the other three statements):

(iii') The operator $U_{\ell_1,\ell_2,p}$ is bounded from $X(0,1)$ to $Y(0,1)$.

**Theorem 3.12.** *Let $p \in (0,\infty)$ and $Y(0,1)$ be a $p$-convex r.i. quasi-Banach function space. Let $(\ell_1,\ell_2) \in \mathcal{B}_p$. Furthermore, assume that the function $(0,1) \ni \tau \mapsto \ell_1(\tau)\ell_2(\tau)^{-1}$ is strictly decreasing and that*

$$\lim_{s\to 0^+} \ell_2(s)^p \int_s^1 \frac{d\tau}{\tau \ell_1(\tau)^p} \in (0,\infty).$$

*We have that*

$$\|S_{\ell_1,p}(f^*)\|_{Y(0,1)} \lesssim \|U_{\ell_1,\ell_2,p} f\|_{Y(0,1)} \quad \text{for every } f \in \mathcal{M}(0,1). \tag{3.19}$$

*Moreover, let $X(0,1)$ be another r.i. quasi-Banach function space that is $p$-convex. If $U_{\ell_1,\ell_2,p}$ is bounded from $X(0,1)$ to $Y(0,1)$, so is $S_{\ell_1,p}$.*

**Proof.** First, note that $t < \sigma^{-1}(t^{1/p})^p$ for every $t \in (0,1)$ (recall Remark 3.8(ii)).

Next, since $Y^{\frac{1}{p}}(0,1)$ is an r.i. Banach function space, in order to prove (3.19), by virtue of the Hardy–Littlewood–Pólya principle (2.2) it is sufficient to show that

$$\int_0^t \int_s^1 \frac{f^*(\tau)^p}{\tau \ell_1(\tau)^p}\, d\tau\, ds \lesssim \int_0^t f^*(\sigma^{-1}(s^{1/p})^p)^p \ell_2(s)^{-p}\, ds \tag{3.20}$$

for every $t \in (0,1)$ and every $f \in \mathcal{M}(0,1)$ with a multiplicative constant independent of $f$ and $t$. Fix such $f$ and $t$. By Fubini's theorem, the left-hand side of (3.20) is equal to

$$\int_0^t \frac{f^*(s)^p}{\ell_1(s)^p}\, ds + t\int_t^1 \frac{f^*(s)^p}{s\ell_1(s)^p}\, ds, \tag{3.21}$$

and, by the change of variables $\tilde{s} = \sigma^{-1}(s^{1/p})^p$, the right-hand side of (3.20) is equivalent to

$$\int_0^{\sigma^{-1}(t^{1/p})^p} \frac{f^*(\tilde{s})^p}{\ell_1(\tilde{s})^p}\, d\tilde{s} = \int_0^t \frac{f^*(\tilde{s})^p}{\ell_1(\tilde{s})^p}\, d\tilde{s} + \int_t^{\sigma^{-1}(t^{1/p})^p} \frac{f^*(\tilde{s})^p}{\ell_1(\tilde{s})^p}\, d\tilde{s}. \tag{3.22}$$

In the light of (3.21) and (3.22), in order to prove (3.20), it is sufficient to show that

$$t\int_t^1 \frac{f^*(s)^p}{s\ell_1(s)^p}\, ds \lesssim \int_t^{\sigma^{-1}(t^{1/p})^p} \frac{f^*(s)^p}{\ell_1(\sigma(s^{1/p})^p)^p}\, ds \tag{3.23}$$

with a multiplicative constant independent of $f$ and $t$. To this end, owing to the monotone convergence theorem and the fact that every nonnegative, nonincreasing function on $(0,1)$ is the pointwise limit of a nondecreasing sequence of nonincreasing simple functions on $(0,1)$, it is actually sufficient to prove (3.23) for $f^* = \chi_{(0,a)}$, where $a \in (0,1)$. Therefore, (3.23) will follow once we prove that

$$t\int_t^1 \frac{\chi_{(0,a)}(s)}{s\ell_1(s)^p}\, ds \lesssim \int_t^{\sigma^{-1}(t^{1/p})^p} \frac{\chi_{(0,a)}(s)}{\ell_1(\sigma(s^{1/p})^p)^p}\, ds \tag{3.24}$$

for every $a \in (0,1)$ with a multiplicative constant independent of $a$ and $t$. We claim that

$$s \int_s^1 \frac{d\tau}{\tau \mathscr{b}_1(\tau)^p} \lesssim \int_s^{\sigma^{-1}(s^{1/p})^p} \frac{d\tau}{\mathscr{b}_1(\sigma(\tau^{1/p})^p)^p} \quad \text{for every } s \in (0,1). \tag{3.25}$$

Before we set out to prove the claim, we will make three observations. First, since the function $\tau \mapsto \mathscr{b}_1(\tau)^p \mathscr{b}_2(\tau)^{-p}$, $\tau \in (0,1)$, is decreasing, we have that

$$\int_0^1 \mathscr{b}_1(\tau)^p \mathscr{b}_2(\tau)^{-p}\, d\tau > \lim_{s \to 1^-} \mathscr{b}_1(s)^p \mathscr{b}_2(s)^{-p}.$$

Second,

$$\lim_{s \to 0^+} \mathscr{b}_1(s)^p \int_s^1 \frac{d\tau}{\tau \mathscr{b}_1(\tau)^p} = \infty \qquad \text{and} \qquad \lim_{s \to 0^+} \frac{\mathscr{b}_1(s)}{\mathscr{b}_2(s)} = \infty,$$

for

$$\frac{\mathscr{b}_1(s)^p}{\mathscr{b}_2(s)^p} \gtrsim \mathscr{b}_1(s)^p \int_s^1 \frac{d\tau}{\tau \mathscr{b}_1(\tau)^p} \geq \int_s^1 \frac{d\tau}{\tau} \quad \text{for every } s \in (0,1).$$

Third, in order to prove (3.25), it is sufficient to prove that the inequality is valid near $0^+$ and near $1^-$ inasmuch as

$$\sup_{s \in [c,d]} \frac{s \int_s^1 \frac{d\tau}{\tau \mathscr{b}_1(\tau)^p}}{\int_s^{\sigma^{-1}(s^{1/p})^p} \frac{d\tau}{\mathscr{b}_1(\sigma(\tau^{1/p})^p)^p}} < \infty \quad \text{for every } 0 < c < d < 1.$$

Set $M = (\int_0^1 \mathscr{b}_1(\tau)^p \mathscr{b}_2(\tau)^{-p}\, d\tau)^{-1}$. As for the validity near $1^-$, note that, for every $s \in (0,1)$,

$$\frac{s \int_s^1 \frac{d\tau}{\tau \mathscr{b}_1(\tau)^p}}{\int_s^{\sigma^{-1}(s^{1/p})^p} \frac{d\tau}{\mathscr{b}_1(\sigma(\tau^{1/p})^p)^p}} \lesssim \frac{\frac{(1-s)}{\mathscr{b}_1(s)^p}}{\frac{\sigma^{-1}(s^{1/p})^p - s}{\mathscr{b}_1(s)^p}} = \frac{1-s}{\sigma^{-1}(s^{1/p})^p - s}$$

and that both numerator and denominator on the right-hand side go to 0 as $s \to 1^-$. Hence, owing to L'Hôpital's rule,

$$\lim_{s \to 1^-} \frac{1-s}{\sigma^{-1}(s^{1/p})^p - s} = \lim_{s \to 1^-} \frac{-1}{M \mathscr{b}_1(s)^p \mathscr{b}_2(s)^{-p} - 1} = \frac{-1}{M(\lim_{s \to 1^-} \mathscr{b}_1(s)^p \mathscr{b}_2(s)^{-p}) - 1} \in (0, \infty).$$

As for the validity near $0^+$, we use L'Hôpital's rule again to obtain that

$$\begin{aligned} \lim_{s \to 0^+} \frac{s \int_s^1 \frac{d\tau}{\tau \mathscr{b}_1(\tau)^p}}{\int_s^{\sigma^{-1}(s^{1/p})^p} \frac{d\tau}{\mathscr{b}_1(\sigma(\tau^{1/p})^p)^p}} &= \lim_{s \to 0^+} \frac{\int_s^1 \frac{d\tau}{\tau \mathscr{b}_1(\tau)^p} - \frac{1}{\mathscr{b}_1(s)^p}}{\frac{M \mathscr{b}_1(s)^p \mathscr{b}_2(s)^{-p}}{\mathscr{b}_1(s)^p} - \frac{1}{\mathscr{b}_1(\sigma(s^{1/p})^p)^p}} \\ &= \lim_{s \to 0^+} \frac{\mathscr{b}_1(s)^p \int_s^1 \frac{d\tau}{\tau \mathscr{b}_1(\tau)^p} - 1}{M \mathscr{b}_1(s)^p \mathscr{b}_2(s)^{-p} - \frac{\mathscr{b}_1(s)^p}{\mathscr{b}_1(\sigma(s^{1/p})^p)^p}} \end{aligned}$$

$$= \frac{1}{M}\left(\lim_{s\to 0^+} \theta_2(s)^p \int_s^1 \frac{d\tau}{\tau\theta_1(\tau)^p}\right) \in (0,\infty).$$

Therefore, (3.25) is valid. Having (3.25) at our disposal, it is now easy to prove (3.24). If $a \le t$, then (3.24) plainly holds. If $t < a \le \sigma^{-1}(t^{1/p})^p$, then

$$\begin{aligned} t\int_t^1 \frac{\chi_{(0,a)}(s)}{s\theta_1(s)^p}\,ds = t\int_t^a \frac{ds}{s\theta_1(s)^p} &\le \int_t^a \frac{ds}{\theta_1(s)^p} \\ &\approx \int_t^a \frac{ds}{\theta_1(\sigma(s^{1/p})^p)^p} = \int_t^{\sigma^{-1}(t^{1/p})^p} \frac{\chi_{(0,a)}(s)}{\theta_1(\sigma(s^{1/p})^p)^p}\,ds. \end{aligned}$$

If $a > \sigma^{-1}(t^{1/p})^p$, then

$$\begin{aligned} t\int_t^1 \frac{\chi_{(0,a)}(s)}{s\theta_1(s)^p}\,ds \le t\int_t^1 \frac{ds}{s\theta_1(s)^p} &\lesssim \int_t^{\sigma^{-1}(t^{1/p})^p} \frac{ds}{\theta_1(\sigma(s^{1/p})^p)^p} \\ &= \int_t^{\sigma^{-1}(t^{1/p})^p} \frac{\chi_{(0,a)}(s)}{\theta_1(\sigma(s^{1/p})^p)^p}\,ds, \end{aligned}$$

in which the multiplicative constant is that from (3.25). Hence (3.24) is true, which completes the proof of (3.19).

Finally, let $X(0,1)$ be an r.i. quasi-Banach function space that is $p$-convex and assume that $U_{\theta_1,\theta_2,p} : X(0,1) \to Y(0,1)$ is bounded. It follows from [9, Corollary 9.8] (cf. [26, Theorem 1]) that

$$\left\|\int_t^1 \frac{|f(s)|^p}{s\theta_1(s)^p}\,ds\right\|_{Y^{\frac{1}{p}}(0,1)} \lesssim \||f|^p\|_{X^{\frac{1}{p}}(0,1)} \quad \text{for every } f \in \mathcal{M}(0,1)$$

if and only if

$$\left\|\int_t^1 \frac{f^*(s)^p}{s\theta_1(s)^p}\,ds\right\|_{Y^{\frac{1}{p}}(0,1)} \lesssim \||f|^p\|_{X^{\frac{1}{p}}(0,1)} \quad \text{for every } f \in \mathcal{M}(0,1).$$

Owing to this equivalence, $S_{\theta_1,p}\colon X(0,1) \to Y(0,1)$ is bounded if (and only if)

$$\|S_{\theta_1,p}(f^*)\|_{Y(0,1)} \lesssim \|f\|_{X(0,1)} \quad \text{for every } f \in \mathcal{M}(0,1). \tag{3.26}$$

Thanks to (3.19), we have that

$$\|S_{\theta_1,p}(f^*)\|_{Y(0,1)} \lesssim \|U_{\theta_1,\theta_2,p}f\|_{Y(0,1)} \lesssim \|f\|_{X(0,1)} \quad \text{for every } f \in \mathcal{M}(0,1),$$

whence (3.26) follows. □

We shall finish by illustrating our results with a particular example. Recall that the function $\ell\colon (0,1) \to (0,\infty)$ is defined as $\ell(t) = 1 - \log(t)$, $t \in (0,1)$. Set $\theta_1 = \ell^{\alpha}$, $\theta_2 = \ell^{-\beta}$. Let $p \in (0,\infty)$. It is a matter of straightforward computations to check that $(\ell_1, \ell_2) \in \mathcal{B}_p$ if and only if $\alpha, \beta \geq 0$ and either $\alpha + \beta \geq \frac{1}{p}$ and $\beta > 0$ or $\alpha > \frac{1}{p}$ and $\beta = 0$. Moreover, if either $0 \leq \alpha < \frac{1}{p}$ and $\alpha + \beta = \frac{1}{p}$ or $\alpha > \frac{1}{p}$ and $\beta = 0$, then the pair $(\ell_1, \ell_2)$ also satisfies the assumptions of Theorem 3.12. Therefore, by combining Theorems 3.11 and 3.12, we obtain the following important particular example. If $\alpha > \frac{1}{p}$, then $L^{\infty, \ell_2} = L^\infty$, and so this case is not so interesting.

**Theorem 3.13.** *Let $p \in (0, \infty)$ and $0 \leq \alpha < \frac{1}{p}$. Set $\beta = \frac{1}{p} - \alpha$. Let $X(R, \mu)$ and $Y(S, \nu)$ be r.i. quasi-Banach function spaces that are $p$-convex. The following four statements are equivalent.*

(i) *Every $(p, \ell_1, \ell_2)$-gaussible operator $T$ is bounded from $X(R, \mu)$ to $Y(S, \nu)$.*
(ii) *Every operator $T$ defined on $L^p(R, \mu)$ having values in $\mathcal{M}(S, \nu)$ that satisfies*

$$K(Tf, t; L^p(\log L)^\alpha, \exp L^{\frac{1}{\beta}}) \lesssim K(f, t; L^p, L^\infty) \quad \text{for every } f \in L^p(R, \mu) \text{ and } t \in (0, 1)$$

*is bounded from $X(R, \mu)$ to $Y(S, \nu)$.*
(iii) *The operator $U_{\ell^\alpha, \ell^{-\beta}, p}$ is bounded from $X(0, 1)$ to $Y(0, 1)$.*
(iv) *The operator $T_{\ell^\alpha, \ell^{-\beta}, p}$ is bounded from $\left(Y^{\frac{1}{p}}\right)'(0, 1)$ to $\left(X^{\frac{1}{p}}\right)'(0, 1)$.*

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# Discretization and antidiscretization of Lorentz norms with no restrictions on weights

Martin Křepela[1] · Zdeněk Mihula[1,2] · Hana Turčinová[2]

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**Abstract**
We improve the discretization technique for weighted Lorentz norms by eliminating all "non-degeneracy" restrictions on the involved weights. We use the new method to provide equivalent estimates on the optimal constant $C$ such that the inequality

$$\left(\int_0^L (f^*(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^L \left(\int_0^t u(s)\,\mathrm{d}s\right)^{-p}\left(\int_0^t f^*(s)u(s)\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

holds for all relevant measurable functions, where $L \in (0, \infty]$, $p, q \in (0, \infty)$ and $u$, $v$, $w$ are locally integrable weights, $u$ being strictly positive. In the case of weights that would be otherwise excluded by the restrictions, it is shown that additional limit terms naturally appear in the characterizations of the optimal $C$. A weak analogue for $p = \infty$ is also presented.

**Keywords** Rearrangement-invariant spaces · Weights · Discretization · Lorentz spaces · Embeddings

**Mathematics Subject Classification** 46E30 · 46E35

## 1 Introduction

Consider the problem of determining the optimal (i.e., least) constant $C \in [0, \infty]$ such that the inequality

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Extended author information available on the last page of the article

$$\left(\int_0^L (f^*(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \leq C\left(\int_0^L \left(\frac{1}{U(t)}\int_0^t f^*(s)u(s)\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{1.1}$$

is satisfied for all functions $f$ defined on an "appropriate" measure space, where $f^*$ stands for the nonincreasing rearrangement of $f$ and $U(t) = \int_0^t u(s)\,\mathrm{d}s$. The values of the other involved parameters are fixed, namely $L \in (0, \infty]$, $p, q \in (0, \infty)$, and $u$, $v$, $w$ are locally integrable, nonnegative weights on $(0, L)$, $u$ strictly positive.

In other words, this corresponds to the problem of characterizing the embedding $\Gamma_u^p(v) \hookrightarrow \Lambda^q(w)$ (all the involved notation is discussed in Sect. 2 below). This problem has been extensively studied and there are several possible approaches leading to a solution. Let us briefly inspect what is at our disposal.

Gogatishvili and Pick provided in [7] what is currently the most cited solution to the problem. It relies on a method of discretization, in which the integral expressions in (1.1) are reformulated in terms of specific sequences. What was innovative in their paper was the so-called "antidiscretization" part, where the discrete conditions were transformed back into integral ones. The technique from [7] is our point of departure and will be discussed in detail. We note that, in what follows, by "discretization" we will actually refer to the whole process including the antidiscretization part.

Although [7] satisfied the demand for conditions in a form that may be easily verified, there is a catch. Namely, only the case $L = \infty$ is covered and it is assumed there that $v$ is "non-degenerate" with respect to $u$ in the sense that

$$\int_0^\infty \frac{v(s)\,\mathrm{d}s}{U^p(s) + U^p(t)} < \infty \quad \text{for all } t \in (0, \infty), \quad \int_0^1 \frac{v(s)\,\mathrm{d}s}{U^p(s)} = \int_1^\infty v(s)\,\mathrm{d}s = \infty.$$

It turns out that the first of these conditions can be assumed without loss of generality (see the beginning of the proof of Theorem 4.1); therefore, what rules out "degenerated" weights is the condition

$$\int_0^1 \frac{v(s)\,\mathrm{d}s}{U^p(s)} = \int_1^\infty v(s)\,\mathrm{d}s = \infty. \tag{1.2}$$

Unfortunately, this means that, besides leaving out some "degenerated" weights on $(0, \infty)$, the result cannot be used in any direct way (e.g., using the obvious idea of truncating $v$ and $w$) in the case where $L < \infty$. A finite $L$ appears naturally when the considered weighted Lorentz spaces consist of functions defined on a finite measure space. Such a setting is perfectly reasonable and it even becomes inevitable when embeddings of weighted Sobolev-Lorentz spaces on domains into Lorentz $\Lambda$-spaces and/or their compactness are studied, which is an application that we have in mind. More details on this matter are given in Remark 4.6.

A completely different way of approaching the problem (1.1) was found independently by Sinnamon in [11,12]. It is based on reformulating (1.1) as an inequality for quasinconcave functions. His method is actually far simpler than discretization. However, the goal of [11] was to describe the Köthe dual of Lorentz $\Gamma$-spaces; therefore only the case $q = 1$ and $u \equiv 1$ was considered there. It appears that there is no easy way to modify his proof technique to allow other values of $q$. Nevertheless, the result obtained in [11] does not require any additional assumptions on weights and it gives a hint on how the conditions characterizing (1.1) change in the general case. Namely, there appear certain limit terms of the same nature as in other embeddings between Lorentz $\Lambda$ and $\Gamma$-spaces (cf. [2]). Another description of the Köthe dual of Lorentz $\Gamma$-spaces was given by Gogatishvili and Kerman in [6]. Their method is not built on discretization either, and so their result does not require any extra assumptions on weights, but, again, it covers the problem (1.1) only for $q = 1$ and $u \equiv 1$.

In [5], the discretization technique was modified in order to encompass "degenerated" weights as well. Our goal in the present paper is to use this modification to finally provide a complete characterization of the optimal constant $C$ in (1.1) without the restriction (1.2) on the weights $u$ and $v$, and for all positive values of $p$ and $q$, including the "weak-type" modification for $p = \infty$ as well.

It should be noted that inequality (1.1) with a general $u$ in fact follows from the case $u \equiv 1$. Indeed, since $u$ is locally integrable and positive a.e. in $(0, L)$, the function $U$ is absolutely continuous and $U' > 0$ a.e. in $(0, L)$; hence its inverse $U^{-1}$ is also absolutely continuous. Thus, performing the change of variables $t \mapsto U^{-1}(t)$ and considering that, since $U^{-1}$ is strictly increasing, a function $h$ is nonincreasing if and only if $h \circ U^{-1}$ is nonincreasing, we observe that (1.1) holds for all $f \in \mathfrak{M}(0, L)$ if and only if

$$\left(\int_0^{U(L)} (g^*(t))^q w(U^{-1}(t))\, \mathrm{d}U^{-1}(t)\right)^{\frac{1}{q}}$$
$$\le C \left(\int_0^{U(L)} \left(\frac{1}{t}\int_0^t g^*(s)\,\mathrm{d}s\right)^p v(U^{-1}(t))\, \mathrm{d}U^{-1}(t)\right)^{\frac{1}{p}}$$

holds for all $g \in \mathfrak{M}(0, U(L))$ (in here, $(0, L)$ may be, of course, replaced by any nonatomic measure space of measure $L$). Hence, to tackle (1.1) it suffices to consider $u \equiv 1$ and, at the end, perform the suggested change of variables to get the general version. Nevertheless, in the proofs of the main results in Sect. 4, we use the discretization technique directly with the general $u$. There would be little difference if we used $u \equiv 1$, namely only in writing $t$ instead of $U(t)$ in the proofs. By using $U(t)$ we also avoid the need for performing the substitution to obtain the final result.

## 2 Preliminaries

Let us summarize the notation and auxiliary results that we shall use in this paper. Throughout the paper, $L \in (0, \infty]$ is a fixed positive (possibly infinite) number.

**Convention 2.1** We adhere to the following conventions:

(i) If $f$ is a function on $(0, L)$, then $f(0)$ and $f(L)$ stand for $\lim_{t\to 0^+} f(t)$ and $\lim_{t\to L^-} f(t)$, respectively.

(ii) All of the expressions $\frac{1}{\infty}$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ and $0\cdot\infty$ are to be interpreted as 0. The expression $\frac{1}{0}$ is to be interpreted as $\infty$.

Let $(X, \mu)$ be a nonatomic, $\sigma$-finite measure space such that $\mu(X) = L$. By $\mathfrak{M}_\mu(X)$ we denote the set of all $\mu$-measurable (extended) real-valued functions defined on $X$. The symbol $\mathfrak{M}(0, L)$ denotes the set of all Lebesgue-measurable functions on $(0, L)$, and $\mathfrak{M}_+(0, L)$ denotes the set of all $f \in \mathfrak{M}(0, L)$ such that $f \geq 0$ a.e. on $(0, L)$.

We say that a function $v \in \mathfrak{M}_+(0, L)$ is a *weight* on $(0, L)$ if $0 < V(t) < \infty$ for every $t \in (0, L)$, where

$$V(t) = \int_0^t v(s)\,\mathrm{d}s, \quad t \in [0, L].$$

Furthermore, we denote

$$V(a, b) = \int_a^b v(s)\,\mathrm{d}s, \quad 0 \leq a < b \leq L.$$

If $f \in \mathfrak{M}_\mu(X)$, the symbol $f^*$ denotes the *nonincreasing rearrangement* of $f$, that is,

$$f^*(t) = \inf\{\lambda \in [0, \infty) \colon \mu(\{x \in X \colon |f(x)| > \lambda\}) \leq t\},\ t \in (0, L),$$

(for details see [1]). Let $p \in (0, \infty]$ and $v$ be a weight on $(0, L)$. We define the following functionals:

$$\|f\|_{\Lambda^p(v)} = \begin{cases} \left(\int_0^L (f^*(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} & \text{if } p \in (0, \infty), \\ \operatorname{ess\,sup}_{t\in(0,L)} f^*(t)v(t) & \text{if } p = \infty. \end{cases}$$

Let $u$ be an a.e. positive weight on $(0, L)$. Let

$$f_u^{**}(t) = \frac{1}{U(t)} \int_0^t f^*(s)u(s)\,\mathrm{d}s,\ t \in (0, L),$$

be the nonincreasing maximal function of $f$ with respect to $u$ (cf. [7]). We define the functional

$$\|f\|_{\Gamma_u^p(v)} = \|f_u^{**}\|_{\Lambda^p(v)}.$$

Accordingly, we denote

$$\Lambda^p(v) = \left\{f \in \mathfrak{M}_\mu(X) :\ \|f\|_{\Lambda^p(v)} < \infty\right\},$$
$$\Gamma_u^p(v) = \left\{f \in \mathfrak{M}_\mu(X) :\ \|f\|_{\Gamma_u^p(v)} < \infty\right\}.$$

These classes of functions are the usual weighted Lorentz $\Lambda$ and $\Gamma$-spaces (cf. [2,7]).

A function $\varrho\colon (0,L)\to(0,\infty)$ is called *admissible* if it is positive, increasing and continuous on $(0,L)$. If $\varrho$ is admissible, we say that a function $h\colon(0,L)\to[0,\infty)$ is $\varrho$-*quasiconcave*, and we write $h\in Q_\varrho(0,L)$, if $h$ is nondecreasing on $(0,L)$ and the function $\frac{h}{\varrho}$ is nonincreasing on $(0,L)$. Thanks to the monotonicity properties of $\varrho$-quasiconcave functions, it follows that $h\not\equiv 0$ on $(0,L)$ if and only if $h(t)\neq 0$ for each $t\in(0,L)$. Throughout the paper, we implicitly assume that $\varrho$ is an admissible function on $(0,L)$.

If $h\not\equiv 0$ is a $\varrho$-quasiconcave function, so is the function $\frac{\varrho}{h}$. Furthermore, the function $h^p$ is $\varrho^p$-quasiconcave for each $p>0$. A nonnegative linear combination of $\varrho$-quasiconcave functions is also $\varrho$-quasiconcave. If functions $h_1$ and $h_2$ are $\varrho_1$-quasiconcave and $\varrho_2$-quasiconcave, respectively, then $h_1\cdot h_2$ is $(\varrho_1\cdot\varrho_2)$-quasiconcave.

Every $h\in Q_\varrho(0,L)$ has an integral representation with limit terms (see [5, Theorem 2.4.1]). Precisely, there is a nonnegative Borel measure $\nu$ on $(0,L)$ such that

$$\begin{aligned}h(t)\le \lim_{s\to0^+}h(s)+\left(\lim_{s\to L^-}\frac{h(s)}{\varrho(s)}\right)\varrho(t)\\ +\int_{(0,L)}\min\{\varrho(t),\varrho(s)\}\,\mathrm{d}\nu(s)\le 4h(t)\quad\text{for each } t\in(0,L).\end{aligned}\tag{2.1}$$

For more information on $\varrho$-quasiconcave functions, see [5, Chapter 2].

The cornerstone of the discretization technique is the construction of a covering sequence. The properties of such a sequence, as listed below, as well as their proofs can be found in [5, Chapter 3]. For every $h\in Q_\varrho(0,L)$, $h\not\equiv 0$, and each $a>1$, there are numbers $K_-, K^+\in\{\mathbb{Z},\pm\infty\}$ with $-\infty\le K_-\le 0\le K^+\le\infty$, and a sequence $\{x_k\}_{k\in\mathcal{K}_-^+}$, where $\mathcal{K}_-^+=\{k\in\mathbb{Z}\colon K_-\le k\le K^+\}$, with the following properties:

- The sequence $\{x_k\}_{k\in\mathcal{K}_-^+}$ is increasing and $x_k\in(0,L)$ for every $k\in\mathbb{Z}$ such that $K_-+1\le k\le K^+-1$.
- $K^+=\infty$ if and only if

$$\lim_{t\to L^-}h(t)=\infty\quad\text{and}\quad\lim_{t\to L^-}\frac{\varrho(t)}{h(t)}=\infty.\tag{2.2}$$

  If $K^+=\infty$, then $\lim_{k\to\infty}x_k=L$. Otherwise, $x_{K^+}=L$.
- $K_-=-\infty$ if and only if

$$\lim_{t\to0^+}h(t)=0\quad\text{and}\quad\lim_{t\to0^+}\frac{\varrho(t)}{h(t)}=0.\tag{2.3}$$

  If $K_-=-\infty$, then $\lim_{k\to-\infty}x_k=0$. Otherwise, $x_{K_-}=0$.
- For every $k\in\mathbb{Z}$ such that $K_-+2\le k\le K^+-1$, one has

$$ah(x_{k-1})\le h(x_k)\quad\text{and}\quad a\frac{\varrho(x_{k-1})}{h(x_{k-1})}\le\frac{\varrho(x_k)}{h(x_k)}.\tag{2.4}$$

- For every $k \in \mathbb{Z}$ such that $K_- + 2 \le k \le K^+ - 1$, one has

$$\frac{1}{a} h(x_k) \le h(t) \le h(x_k) \quad \text{for each } t \in [x_{k-1}, x_k]$$
or
$$\frac{1}{a} \frac{\varrho(x_k)}{h(x_k)} \le \frac{\varrho(t)}{h(t)} \le \frac{\varrho(x_k)}{h(x_k)} \quad \text{for each } t \in [x_{k-1}, x_k].$$

- If $K^+ < \infty$, then

$$h(x_{K^+-1}) \le h(t) \le a h(x_{K^+-1}) \quad \text{for each } t \in [x_{K^+-1}, L)$$
or
$$\frac{\varrho(x_{K^+-1})}{h(x_{K^+-1})} \le \frac{\varrho(t)}{h(t)} \le a \frac{\varrho(x_{K^+-1})}{h(x_{K^+-1})} \quad \text{for each } t \in [x_{K^+-1}, L).$$

- If $K_- > -\infty$, then

$$\frac{1}{a} h(x_{K_-+1}) \le h(t) \le h(x_{K_-+1}) \quad \text{for each } t \in (0, x_{K_-+1}]$$
or
$$\frac{1}{a} \frac{\varrho(x_{K_-+1})}{h(x_{K_-+1})} \le \frac{\varrho(t)}{h(t)} \le \frac{\varrho(x_{K_-+1})}{h(x_{K_-+1})} \quad \text{for each } t \in (0, x_{K_-+1}].$$

If $\{x_k\}_{k \in \mathcal{K}_-^+}$ satisfies these conditions, it is called a *covering sequence (with respect to* $h, \varrho, a$*)*.

The family of all covering sequences with respect to $h$, $\varrho$, $a$ is denoted by $CS(h, \varrho, a)$ (we omit any reference to the interval $(0, L)$ in this notation because it will always be apparent what the underlying interval is). We also denote

$$\mathcal{K} = \{k \in \mathbb{Z} \colon K_- + 1 \le k \le K^+ - 1\},$$
$$\mathcal{K}^+ = \{k \in \mathbb{Z} \colon K_- + 1 \le k \le K^+\}.$$

The properties of $\{x_k\}_{k \in \mathcal{K}_-^+}$ imply that

$$(0, L) \subseteq \bigcup_{k \in \mathcal{K}^+} (x_{k-1}, x_k] \subseteq (0, L], \tag{2.5}$$

where the first inclusion is strict if and only if $K^+ \ne \infty$.

Furthermore, if $\{x_k\}_{k \in \mathcal{K}_-^+} \in CS(h, \varrho, a)$, then there is a decomposition

$$\mathcal{K}^+ = \mathcal{Z}_1 \cup \mathcal{Z}_2 \quad \text{and} \quad \mathcal{Z}_1 \cap \mathcal{Z}_2 = \emptyset \tag{2.6}$$

such that

$$h(t) \approx h(x_k) \quad \text{for every } t \in [x_{k-1}, x_k] \text{ and each } k \in \mathcal{Z}_1, \tag{2.7}$$

and

$$\frac{\varrho(t)}{h(t)} \approx \frac{\varrho(x_k)}{h(x_k)} \quad \text{for every } t \in [x_{k-1}, x_k] \text{ and each } k \in \mathcal{Z}_2, \tag{2.8}$$

where the equivalence constants depend only on $a$. Moreover, if $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(h, \varrho, a)$, then $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(\frac{\varrho}{h}, \varrho, a)$, and $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(h^p, \varrho^p, a^p)$ for every $p \in (0, \infty)$.

## 3 Discretization of generalized Lorentz norms

This section contains technical lemmas necessary for implementing discretization to solve our main problem. The results below extend their counterparts in [7] by eliminating the "non-degeneracy" assumptions in there. Namely, Lemmas 3.3, 3.4 and 3.5 below correspond, in this sense, to Lemmas 3.7, 3.6 and 3.8 in [7], respectively.

We start with an auxiliary result that is frequently used when one deals with weighted inequalities.

**Lemma 3.1** *Let $L \in (0, \infty]$ and let $v$ be a weight on $(0, L)$. Let $0 \le a < b \le L$. If $\gamma > -1$, then*

$$V^{\gamma+1}(a, b) = (\gamma + 1) \int_a^b \left( \int_a^t v(s)\,\mathrm{d}s \right)^{\gamma} v(t)\,\mathrm{d}t.$$

***Proof*** Assume that $b < L$. Since $v$ is a weight on $(0, L)$, the function $\psi(t) = V(a, t)$, $t \in [a, b]$, is absolutely continuous on $[a, b]$. Hence the claim follows from the change of variables $y = \psi$ in the integral on the right-hand side (e.g., [10, page 156]).

If $b = L$, the claim follows from the part already proved and the monotone convergence theorem. □

The following theorem generalizes [7, Corollary 2.13] by allowing degenerated weights (cf. [5, Lemma 4.1.1]).

**Theorem 3.2** *Let $p \in (0, \infty)$ and $h \in Q_\varrho(0, L)$. Assume that there exist $C_1,\ C_2 \in (0, \infty)$, $\alpha, \beta \in [0, \infty)$ and a nonnegative Borel measure $\nu$ on $(0, L)$ such that*

$$C_1 h(t) \le \alpha + \beta\varrho(t) + \int_{(0,L)} \min\{\varrho(t), \varrho(s)\}\,\mathrm{d}\nu(s) \le C_2 h(t) \quad \textit{for every } t \in (0, L). \tag{3.1}$$

*If $a > 0$ is sufficiently large, $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(h, \varrho, a)$ and $f \in Q_{\varrho^p}(0, L)$, then*

$$\sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)} \approx \alpha^p \lim_{t\to 0^+} \frac{f(t)}{\varrho^p(t)} + \beta^p \lim_{t\to L^-} f(t) + \int_{(0,L)} f(t)\varrho^{1-p}(t) \left(\int_{(0,L)} \min\{\varrho(t), \varrho(s)\}\, \mathrm{d}\nu(s)\right)^{p-1} \mathrm{d}\nu(t). \tag{3.2}$$

*Precisely, it is sufficient if the parameter $a$ satisfies*

$$a^p > 12 \cdot \frac{3^{p+\max\{1,p\}} C_2^p}{\min\{1, p\} C_1^p}.$$

*Moreover, the equivalence constants in (3.2) depend only on $p, a, C_1, C_2$.*

***Proof*** Without loss of generality, we may assume that $h \not\equiv 0$ and $f \not\equiv 0$. Since $h$ and $f$ are in $Q_\varrho(0, L)$ and $Q_{\varrho^p}(0, L)$, respectively, we have $h \neq 0$ and $f \neq 0$ on $(0, L)$.

It can be easily shown that the limit terms on the left-hand side of (3.2) are well defined if they are to appear (recall Convention 2.1(i)). This follows from the fact that $h \in Q_\varrho(0, L)$ and from the properties of $\{x_k\}_{k\in\mathcal{K}_-^+}$ (cf. [5, the proof of Lemma 4.1.1]). Furthermore, the limit terms on the right-hand side of (3.2) are always well defined thanks to $f \in Q_{\varrho^p}(0, L)$ and Convention 2.1(ii).

In order to simplify the notation, we set

$$\psi(t) = \varrho^{1-p}(t) \left(\int_{(0,L)} \min\{\varrho(t), \varrho(s)\}\, \mathrm{d}\nu(s)\right)^{p-1}, \quad t \in (0, L).$$

We begin the proof of (3.2) by showing that

$$\sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)} \gtrsim \alpha^p \lim_{t\to 0^+} \frac{f(t)}{\varrho^p(t)} + \beta^p \lim_{t\to L^-} f(t) + \int_{(0,L)} f(t)\psi(t)\, \mathrm{d}\nu(t). \tag{3.3}$$

We first check that

$$\alpha^p \lim_{t\to 0^+} \frac{f(t)}{\varrho^p(t)} + \beta^p \lim_{t\to L^-} f(t) \lesssim \sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}.$$

If $\alpha = \beta = 0$, the inequality holds trivially. Assume that $\alpha > 0$. We have $\lim_{t\to 0^+} h(t) \gtrsim \alpha > 0$ thanks to (3.1), and thus $K_- > -\infty$ by (2.3). Furthermore, we have

$$\alpha^p \lim_{t\to 0^+} \frac{f(t)}{\varrho^p(t)} \lesssim \left(\lim_{t\to 0^+} h^p(t)\right) \left(\lim_{t\to 0^+} \frac{f(t)}{\varrho^p(t)}\right) = \lim_{t\to 0^+} \frac{f(t)h^p(t)}{\varrho^p(t)} = \left(\frac{fh^p}{\varrho^p}\right)(x_{K_-})$$

$$\leq \sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}.$$

Note that the first equality is indeed valid because both limits are positive. One can similarly prove that $\beta^p \lim_{t\to L^-} f(t) \lesssim \sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}$ owing to (2.2).

Hence, in order to prove (3.3) it remains to show

$$\int_{(0,L)} f(t)\psi(t)\,\mathrm{d}\nu(t) \lesssim \sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}. \tag{3.4}$$

We have

$$\begin{aligned} \widetilde{C}_1 h^p(t) &\leq \alpha^p + \beta^p\varrho^p(t) + \int_{(0,L)} \min\{\varrho^p(t), \varrho^p(s)\}\psi(s)\,\mathrm{d}\nu(s) \\ &\leq \widetilde{C}_2 h^p(t) \quad \text{for every } t\in(0,L), \end{aligned} \tag{3.5}$$

where $\widetilde{C}_1 = \frac{C_1^p}{3^{p+\max\{p,1\}}}$ and $\widetilde{C}_2 = \frac{6C_2^p}{\min\{1,p\}}$, owing to [5, Theorem 2.4.3 and its proof]. Let $\mathcal{K}^+ = \mathcal{Z}_1 \cup \mathcal{Z}_2$ be a decomposition of $\mathcal{K}^+$ from (2.6). Denote by $\widetilde{\nu}$ the extension of $\nu$ to $(0, L]$ by zero. Using (2.7), the fact that $f \in Q_{\varrho^p}(0, L)$, and (3.5), one has

$$\begin{aligned} \sum_{k\in\mathcal{Z}_1} \int_{(x_{k-1},x_k]} f(t)\psi(t)\,\mathrm{d}\widetilde{\nu}(t) &= \sum_{k\in\mathcal{Z}_1} \int_{(x_{k-1},x_k]} \frac{f(t)}{\varrho^p(t)}\varrho^p(t)\psi(t)\,\mathrm{d}\widetilde{\nu}(t) \\ &\leq \sum_{k\in\mathcal{Z}_1} \frac{f(x_{k-1})}{\varrho^p(x_{k-1})} \int_{(x_{k-1},x_k]} \min\{\varrho^p(x_k), \varrho^p(t)\}\psi(t)\,\mathrm{d}\widetilde{\nu}(t) \\ &\lesssim \sum_{k\in\mathcal{Z}_1} \frac{f(x_{k-1})}{\varrho^p(x_{k-1})} h^p(x_k) \approx \sum_{k\in\mathcal{Z}_1} \frac{f(x_{k-1})}{\varrho^p(x_{k-1})} h^p(x_{k-1}) \\ &\leq \sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}. \end{aligned}$$

Here and below, if $x_{k-1} = 0$ or $x_k = L$, the corresponding terms are to be understood as the corresponding limits. Hence

$$\sum_{k\in\mathcal{Z}_1} \int_{(x_{k-1},x_k]} f(t)\psi(t)\,\mathrm{d}\widetilde{\nu}(t) \lesssim \sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}. \tag{3.6}$$

Furthermore, by (2.8), by the fact that $h^p \in Q_{\varrho^p}(0, L)$, by the fact that $f$ is nondecreasing, and by (3.5),

$$\sum_{k\in\mathcal{Z}_2} \int_{(x_{k-1},x_k]} f(t)\psi(t)\,\mathrm{d}\widetilde{\nu}(t) \leq \sum_{k\in\mathcal{Z}_2} f(x_k)\frac{\varrho^p(x_{k-1})}{\varrho^p(x_{k-1})} \int_{(x_{k-1},x_k]} \psi(t)\,\mathrm{d}\widetilde{\nu}(t)$$

$$
\begin{aligned}
&= \sum_{k\in\mathcal{Z}_2} f(x_k)\frac{1}{\varrho^p(x_{k-1})}\int_{(x_{k-1},x_k]}\min\{\varrho^p(x_{k-1}),\varrho^p(t)\}\psi(t)\,\mathrm{d}\widetilde{\nu}(t)\\
&\lesssim \sum_{k\in\mathcal{Z}_2} f(x_k)\frac{h^p(x_{k-1})}{\varrho^p(x_{k-1})} \approx \sum_{k\in\mathcal{Z}_2} f(x_k)\frac{h^p(x_k)}{\varrho^p(x_k)} \le \sum_{k\in\mathcal{K}_-^+}\frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}.
\end{aligned}
$$

Hence

$$
\sum_{k\in\mathcal{Z}_2}\int_{(x_{k-1},x_k]} f(t)\psi(t)\,\mathrm{d}\widetilde{\nu}(t) \lesssim \sum_{k\in\mathcal{K}_-^+}\frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}. \tag{3.7}
$$

Desired inequality (3.4) now follows from (3.6), (3.7) and (2.5).

Our next goal is to prove the estimate

$$
\sum_{k\in\mathcal{K}_-^+}\frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)} \lesssim \alpha^p \lim_{t\to 0^+}\frac{f(t)}{\varrho^p(t)} + \beta^p \lim_{t\to L^-} f(t) + \int_{(0,L)} f(t)\psi(t)\,\mathrm{d}\nu(t). \tag{3.8}
$$

We start by showing that

$$
\int_{(0,L)} f(t)\psi(t)\,\mathrm{d}\nu(t) \gtrsim \sum_{k=K_-+2}^{K^+-2}\frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}. \tag{3.9}
$$

By (2.4), we have

$$
h^p(x_{k-1}) \le \frac{1}{a^p}h^p(x_k) \quad \text{for every } k\in\mathbb{Z},\ K_-+2\le k\le K^+-1
$$

and

$$
\frac{\varrho(x_k)^p}{\varrho(x_{k+1})^p}h^p(x_{k+1}) \le \frac{1}{a^p}h^p(x_k) \quad \text{for every } k\in\mathbb{Z},\ K_-+1\le k\le K^+-2.
$$

By combining these two inequalities we obtain

$$
\widetilde{C}_1 h^p(x_k) - \widetilde{C}_2 h^p(x_{k-1}) - \widetilde{C}_2\frac{\varrho^p(x_k)}{\varrho^p(x_{k+1})}h^p(x_{k+1}) \ge \left(\widetilde{C}_1 - \frac{2\widetilde{C}_2}{a^p}\right)h^p(x_k) \tag{3.10}
$$

for every $k \in \mathbb{Z}$, $K_- + 2 \le k \le K^+ - 2$. Since $f \in Q_{\varrho^p}(0, L)$ and (2.5), we have that

$$
\begin{aligned}
2 \int_{(0,L)} f(t)\psi(t)\,\mathrm{d}\nu(t) &\ge \sum_{k=K_-+2}^{K^+-2} \int_{(x_{k-1},x_{k+1}]} f(t)\psi(t)\,\mathrm{d}\nu(t) \\
&\ge \sum_{k=K_-+2}^{K^+-2} \left( \frac{f(x_k)}{\varrho^p(x_k)} \int_{(x_{k-1},x_k]} \varrho^p(t)\psi(t)\,\mathrm{d}\nu(t) \right. \\
&\quad \left. + \frac{f(x_k)}{\varrho^p(x_k)} \int_{(x_k,x_{k+1}]} \varrho^p(x_k)\psi(t)\,\mathrm{d}\nu(t) \right) \\
&= \sum_{k=K_-+2}^{K^+-2} \frac{f(x_k)}{\varrho^p(x_k)} \left( \int_{(x_{k-1},x_{k+1}]} \min\{\varrho^p(x_k), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t) \right).
\end{aligned}
\tag{3.11}
$$

Using (3.5) and (3.10), we have

$$
\begin{aligned}
&\int_{(x_{k-1},x_{k+1}]} \min\{\varrho^p(x_k), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t) \\
&= \alpha^p + \beta^p \varrho^p(x_k) + \int_{(0,L)} \min\{\varrho^p(x_k), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t) \\
&\quad - \left( \alpha^p + \int_{(0,x_{k-1}]} \min\{\varrho^p(x_{k-1}), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t) \right) \\
&\quad - \frac{\varrho^p(x_k)}{\varrho^p(x_{k+1})} \left( \beta^p \varrho^p(x_{k+1}) + \int_{(x_{k+1},L)} \min\{\varrho^p(x_{k+1}), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t) \right) \\
&\ge \widetilde{C}_1 h^p(x_k) - \widetilde{C}_2 h^p(x_{k-1}) - \frac{\varrho^p(x_k)}{\varrho^p(x_{k+1})} \widetilde{C}_2 h^p(x_{k+1}) \ge \left( \widetilde{C}_1 - \frac{2\widetilde{C}_2}{a^p} \right) h^p(x_k)
\end{aligned}
$$

for each $k \in \mathbb{Z}$, $K_- + 2 \le k \le K^+ - 2$. Note that choosing a sufficiently large parameter $a$ assures that $\widetilde{C}_1 - \frac{2\widetilde{C}_2}{a^p} > 0$. Hence inequality (3.9) follows from the last chain of inequalities and (3.11).

If $K_- = -\infty$ and $K^+ = \infty$ (and so $K_- + 2 = K_-$ and $K^+ - 2 = K^+$), inequality (3.9) clearly implies inequality (3.8). Thus, the proof is finished when $K_- = -\infty$ and $K^+ = \infty$. Now suppose that $K_- > -\infty$ or $K^+ < \infty$. Since $f \in Q_{\varrho^p}(0, L)$, we have

$$
\begin{aligned}
\int_{(0,L)} f(t)\psi(t)\,\mathrm{d}\nu(t) &= \int_{(0,x]} f(t)\psi(t)\,\mathrm{d}\nu(t) + \int_{(x,L)} f(t)\psi(t)\,\mathrm{d}\nu(t) \\
&\ge \frac{f(x)}{\varrho^p(x)} \int_{(0,x]} \min\{\varrho^p(x), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t) \\
&\quad + \frac{f(x)}{\varrho^p(x)} \int_{(x,L)} \min\{\varrho^p(x), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t) \\
&= \frac{f(x)}{\varrho^p(x)} \int_{(0,L)} \min\{\varrho^p(x), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t)
\end{aligned}
$$

for every $x \in (0, L)$, whence

$$\alpha^p \lim_{t\to 0^+} \frac{f(t)}{\varrho^p(t)} + \beta^p \lim_{t\to L^-} f(t) + \int_{(0,L)} f(t)\psi(t)\,\mathrm{d}\nu(t)$$
$$\geq \frac{f(x)}{\varrho^p(x)} \left( \alpha^p + \beta^p \varrho^p(x) + \int_{(0,L)} \min\{\varrho^p(x), \varrho^p(t)\}\psi(t)\,\mathrm{d}\nu(t) \right) \quad (3.12)$$
$$\approx \frac{f(x)h^p(x)}{\varrho^p(x)}.$$

Assume, for example, $K_- > -\infty$ and $K^+ = \infty$ (and so $K^+ - 2 = K^+$ and $\beta = 0$), (3.12) together with (3.9) implies that

$$\alpha^p \lim_{t\to 0^+} \frac{f(t)}{\varrho^p(t)} + \int_{(0,L)} f(t)\psi(t)\,\mathrm{d}\nu(t) \gtrsim \lim_{t\to 0^+} \frac{f(t)h^p(t)}{\varrho^p(t)} + \frac{f(x_{K_-+1})h^p(x_{K_-+1})}{\varrho^p(x_{K_-+1})}$$
$$+ \sum_{k=K_-+2}^{K^+-2} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}$$
$$= \sum_{k\in\mathcal{K}_-^+} \frac{f(x_k)h^p(x_k)}{\varrho^p(x_k)}.$$

The other two cases can be handled similarly. □

**Lemma 3.3** *Let $p \in (0, \infty)$ and $h \in Q_\varrho(0, L)$. Assume that there exist $C_1, C_2 \in (0, \infty)$, $\alpha, \beta \in [0, \infty)$ and a nonnegative Borel measure $\nu$ on $(0, L)$ such that (3.1) holds. Let $a > 108\frac{C_2}{C_1}$ and $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(h, \varrho, a)$. Then for every $f \in \mathfrak{M}_+(0, L)$ we have*

$$\sum_{k\in\mathcal{K}^+} \left( \operatorname*{ess\,sup}_{t\in(x_{k-1},x_k]} \frac{h^{\frac{1}{p}}(t)}{\varrho^{\frac{1}{p}}(t)} f(t) \right)^p \approx \sum_{k\in\mathcal{K}_-^+} h(x_k) \left( \operatorname*{ess\,sup}_{t\in(0,L)} \frac{f(t)}{\varrho^{\frac{1}{p}}(x_k) + \varrho^{\frac{1}{p}}(t)} \right)^p$$
$$\approx \alpha \left( \operatorname*{ess\,sup}_{t\in(0,L)} \frac{f(t)}{\varrho^{\frac{1}{p}}(t)} \right)^p + \beta \left( \operatorname*{ess\,sup}_{t\in(0,L)} f(t) \right)^p \quad (3.13)$$
$$+ \int_{(0,L)} \left( \operatorname*{ess\,sup}_{\tau\in(0,L)} \frac{\varrho^{\frac{1}{p}}(t) f(\tau)}{\varrho^{\frac{1}{p}}(t) + \varrho^{\frac{1}{p}}(\tau)} \right)^p \mathrm{d}\nu(t).$$

*Moreover, the equivalence constants in (3.13) depend only on $p, a, C_1, C_2$.*

***Proof*** We may clearly assume that $h \not\equiv 0$ on $(0, L)$ (recall Convention 2.1(iii)), and so $h \neq 0$ on $(0, L)$. Note that $h^{\frac{1}{p}} \in Q_{\varrho^{\frac{1}{p}}}(0, L)$ and $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(h^{\frac{1}{p}}, \varrho^{\frac{1}{p}}, a^{\frac{1}{p}})$. The first equivalence follows from [5, Theorem 4.2.7 and Remark 4.2.8].

As for the second equivalence, observe that the function $t \mapsto \left( \operatorname{ess\,sup}_{\tau\in(0,L)} \frac{\varrho^{\frac{1}{p}}(t) f(\tau)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(\tau)} \right)^p$ is in $Q_\varrho(0, L)$ (we may assume that $f$ is finite a.e. in

$(0, L)$, otherwise (3.13) holds plainly). Consider Theorem 3.2 with the setting $\widetilde{p} = 1$, $\widetilde{f}(t) = \left(\operatorname*{ess\,sup}_{\tau\in(0,L)} \frac{\varrho^{\frac{1}{p}}(t)f(\tau)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(\tau)}\right)^p$, and $\widetilde{\varrho} = \varrho$, where the symbols with tildes correspond to those from the statement of the theorem. It implies

$$
\begin{aligned}
&\sum_{k\in\mathcal{K}_-^+} h(x_k)\left(\operatorname*{ess\,sup}_{t\in(0,L)} \frac{f(t)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(t)}\right)^p \\
&\approx \alpha \lim_{t\to 0+}\left(\operatorname*{ess\,sup}_{\tau\in(0,L)} \frac{f(\tau)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(\tau)}\right)^p \\
&\quad + \beta \lim_{t\to L^-}\left(\operatorname*{ess\,sup}_{\tau\in(0,L)} \frac{\varrho^{\frac{1}{p}}(t)f(\tau)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(\tau)}\right)^p \\
&\quad + \int_{(0,L)}\left(\operatorname*{ess\,sup}_{\tau\in(0,L)} \frac{\varrho^{\frac{1}{p}}(t)f(\tau)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(\tau)}\right)^p \mathrm{d}\nu(t),
\end{aligned}
$$

which is the second equivalence in (3.13) upon observing that

$$
\lim_{t\to 0+}\left(\operatorname*{ess\,sup}_{\tau\in(0,L)} \frac{f(\tau)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(\tau)}\right)^p \approx \left(\operatorname*{ess\,sup}_{\tau\in(0,L)} \frac{f(\tau)}{\varrho^{\frac{1}{p}}(\tau)}\right)^p
$$

and

$$
\lim_{t\to L^-}\left(\operatorname*{ess\,sup}_{\tau\in(0,L)} \frac{\varrho^{\frac{1}{p}}(t)f(\tau)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(\tau)}\right)^p \approx \left(\operatorname*{ess\,sup}_{\tau\in(0,L)} f(\tau)\right)^p.
$$

□

**Lemma 3.4** *Let $p \in (0,\infty)$ and $h \in Q_\varrho(0, L)$. Assume that there exist $C_1, C_2 \in (0,\infty)$, $\alpha, \beta \in [0,\infty)$ and a nonnegative Borel measure $\nu$ on $(0, L)$ such that (3.1) holds. Let $a > 108\frac{C_2}{C_1}$ and $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(h, \varrho, a)$. Then for every $f \in \mathfrak{M}_+(0, L)$ we have*

$$
\begin{aligned}
\sum_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k} \frac{h^{\frac{1}{p}}(t)}{\varrho^{\frac{1}{p}}(t)} f(t)\,\mathrm{d}t\right)^p &\approx \sum_{k\in\mathcal{K}_-^+} h(x_k)\left(\int_0^L \frac{f(t)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(t)}\,\mathrm{d}t\right)^p \\
&\approx \alpha\left(\int_0^L \frac{f(t)}{\varrho^{\frac{1}{p}}(t)}\,\mathrm{d}t\right)^p + \beta\left(\int_0^L f(t)\,\mathrm{d}t\right)^p \\
&\quad + \int_{(0,L)}\left(\int_0^L \frac{\varrho^{\frac{1}{p}}(t)f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p \mathrm{d}\nu(t).
\end{aligned} \tag{3.14}
$$

*Moreover, the equivalence constants in (3.14) depend only on $p, a, C_1, C_2$.*

***Proof*** We omit the proof because it is similar to the proof of Lemma 3.3. We just note that the first equivalence in (3.14) follows from [5, Theorem 4.2.5 and Remark 4.2.6]. □

**Lemma 3.5** *Let $p \in (0,\infty)$, $\varphi \in Q_\varrho(0,L)$ and $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(\varphi,\varrho,a)$ with $a>1$. For every $f \in \mathfrak{M}_+(0,L)$ we have*

$$\sup_{t\in(0,L)} \varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p \approx \sup_{k\in\mathcal{K}_-^+} \varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p \tag{3.15}$$
$$\approx \sup_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k} f(s)\frac{\varphi^{\frac{1}{p}}(s)}{\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p.$$

*Moreover, the equivalence constants depend only on $p$ and $a$.*

***Proof*** We may clearly assume that $\varphi \not\equiv 0$ on $(0,L)$, and so $\varphi \neq 0$ on $(0,L)$. The second equivalence in (3.15) follows from [5, Theorem 4.2.5 and Remark 4.2.6] (note that $\varphi^{\frac{1}{p}} \in Q_{\varrho^{\frac{1}{p}}}(0,L)$ and $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(\varphi^{\frac{1}{p}},\varrho^{\frac{1}{p}},a^{\frac{1}{p}})$) upon observing that

$$\sup_{k\in\mathcal{K}_-^+} \varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p = \left(\sup_{k\in\mathcal{K}_-^+} \varphi^{\frac{1}{p}}(x_k)\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$\approx \left(\sup_{k\in\mathcal{K}_-^+} \frac{\varphi^{\frac{1}{p}}(x_k)}{\varrho^{\frac{1}{p}}(x_k)}\int_0^L \min\{\varrho^{\frac{1}{p}}(x_k),\varrho^{\frac{1}{p}}(s)\}\,\mathrm{d}\nu(s)\right)^p,$$

where $\mathrm{d}\nu(s) = f(s)\varrho^{-\frac{1}{p}}(s)\,\mathrm{d}s$.

We shall prove the first equivalence in (3.15). Using (2.7) and (2.8), we have

$$\sup_{t\in(0,L)} \varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p = \sup_{k\in\mathcal{K}^+}\sup_{t\in(x_{k-1},x_k]} \varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$\approx \sup_{k\in\mathcal{Z}_1}\sup_{t\in(x_{k-1},x_k]} \varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$+ \sup_{k\in\mathcal{Z}_2}\sup_{t\in(x_{k-1},x_k]} \varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$\approx \sup_{k\in\mathcal{Z}_1} \varphi(x_{k-1})\sup_{t\in(x_{k-1},x_k]}\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$+ \sup_{k\in\mathcal{Z}_2} \frac{\varphi(x_k)}{\varrho(x_k)}\sup_{t\in(x_{k-1},x_k]}\left(\int_0^L \frac{\varrho^{\frac{1}{p}}(t)f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$= \sup_{k\in\mathcal{Z}_1} \varphi(x_{k-1})\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_{k-1})+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p + \sup_{k\in\mathcal{Z}_2} \varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p, \tag{3.16}$$

where $\mathcal{K}^+ = \mathcal{Z}_1 \cup \mathcal{Z}_2$ is a decomposition of $\mathcal{K}^+$ from (2.6). Note that the second equivalence in (3.16) is valid even when $K_- + 1 \in \mathcal{Z}_1$ or $K_+ \in \mathcal{Z}_2$. Indeed, if $K_- + 1 \in \mathcal{Z}_1$ (and so $K_- > -\infty$), we have $\varphi(x_{K_-}) = \lim_{t\to 0^+} \varphi(t) \approx \varphi(x_{K_-+1}) > 0$ thanks to (2.7). Hence,

$$\begin{aligned}&\sup_{t\in(0,x_{K_-+1}]} \varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p\\ &\approx \left(\lim_{t\to 0^+}\varphi(t)\right)\left(\sup_{t\in(0,x_{K_-+1}]}\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p\right)\\ &= \varphi(x_{K_-})\left(\sup_{t\in(0,x_{K_-+1}]}\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p\right).\end{aligned}$$

Analogously, one may show that, if $K_+ \in \mathcal{Z}_2$, then

$$\sup_{t\in(x_{K_+-1},L)} \varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p \approx \varphi(x_{K_+})\sup_{t\in(x_{K_+-1},L)}\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p.$$

Next, one clearly has

$$\begin{aligned}&\sup_{k\in\mathcal{Z}_1} \varphi(x_{k-1})\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_{k-1})+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p + \sup_{k\in\mathcal{Z}_2} \varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p\\ &\lesssim \sup_{k\in\mathcal{K}^+_-} \varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p.\end{aligned} \tag{3.17}$$

For any $k \in \mathcal{Z}_1$ we have

$$\begin{aligned}\varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p &\approx \varphi(x_{k-1})\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p\\ &\le \varphi(x_{k-1})\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_{k-1})+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p.\end{aligned}$$

Hence,

$$\begin{aligned}\sup_{k\in\mathcal{K}^+} \varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p &\lesssim \sup_{k\in\mathcal{Z}_1} \varphi(x_{k-1})\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_{k-1})+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p\\ &+ \sup_{k\in\mathcal{Z}_2} \varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p.\end{aligned} \tag{3.18}$$

If $K_- = -\infty$ (and so $\mathcal{K}^+_- = \mathcal{K}^+$), we obtain

$$\sup_{k\in\mathcal{Z}_1}\varphi(x_{k-1})\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_{k-1})+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p + \sup_{k\in\mathcal{Z}_2}\varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$\approx \sup_{k\in\mathcal{K}_-^+}\varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p \tag{3.19}$$

by combining (3.17) with (3.18).

Now suppose that $K_- > -\infty$. If $K_- + 1 \in \mathcal{Z}_2$, then $\lim_{t\to 0^+}\frac{\varphi(t)}{\varrho(t)} \approx \frac{\varphi(x_{K_-+1})}{\varrho(x_{K_-+1})} \in (0,\infty)$ thanks to (2.8), and so

$$\lim_{t\to 0^+}\varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$\le \left(\lim_{t\to 0^+}\frac{\varphi(t)}{\varrho(t)}\right)\left(\sup_{t\in(0,x_{K_-+1}]}\left(\int_0^L \frac{\varrho^{\frac{1}{p}}(t)f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p\right)$$
$$\approx \frac{\varphi(x_{K_-+1})}{\varrho(x_{K_-+1})}\left(\int_0^L \frac{\varrho^{\frac{1}{p}}(x_{K_-+1})f(s)}{\varrho^{\frac{1}{p}}(x_{K_-+1})+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$\le \sup_{k\in\mathcal{Z}_2}\varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p.$$

If $K_- + 1 \in \mathcal{Z}_1$, we plainly have

$$\lim_{t\to 0^+}\varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p \le \sup_{k\in\mathcal{Z}_1}\varphi(x_{k-1})\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_{k-1})+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p.$$

Hence, whether $K_- + 1 \in \mathcal{Z}_1$ or $K_- + 1 \in \mathcal{Z}_2$, we obtain

$$\lim_{t\to 0^+}\varphi(t)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(t)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p \lesssim \sup_{k\in\mathcal{Z}_1}\varphi(x_{k-1})\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_{k-1})+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p$$
$$+ \sup_{k\in\mathcal{Z}_2}\varphi(x_k)\left(\int_0^L \frac{f(s)}{\varrho^{\frac{1}{p}}(x_k)+\varrho^{\frac{1}{p}}(s)}\,\mathrm{d}s\right)^p.$$

Therefore, combining the last inequality with (3.17) and (3.18), we obtain equivalence (3.19) even when $K_- > -\infty$.

Finally, the first equivalence in (3.15) follows by combining (3.16) with (3.19). □

## 4 Main results

We are finally ready to present our main results. The first one is the desired characterization of (1.1) when all the involved exponents are finite.

**Theorem 4.1** *Let $p, q \in (0, \infty)$. Let $v, w$ be weights on $(0, L)$ and $u$ an a.e. positive weight on $(0, L)$. Set*

$$C = \sup_{\|f\|_{\Gamma_u^p(v)} \leq 1} \|f\|_{\Lambda^q(w)}.$$

(i) *If $1 \leq q$ and $p \leq q < \infty$, then $C \approx A_1$, where*

$$A_1 = \sup_{0<t<L} \frac{W^{\frac{1}{q}}(t)}{\left(V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s\right)^{\frac{1}{p}}}.$$

(ii) *If $1 \leq q < p < \infty$, then $C \approx A_2$, where*

$$\begin{aligned} A_2 = &\left( \int_0^L \frac{V(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s\, U^{\frac{pq}{p-q}+p-1}(t) u(t) \sup_{\tau \in [t, L)} U^{-\frac{pq}{p-q}}(\tau) W^{\frac{p}{p-q}}(\tau)}{\left(V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s\right)^{\frac{q}{p-q}+2}}\, \mathrm{d}t \right)^{\frac{p-q}{pq}} \\ &+ \left( \lim_{t \to 0^+} \frac{U^p(t)}{V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s} \right)^{\frac{1}{p}} \left( \sup_{t \in (0, L)} \frac{W(t)}{U^q(t)} \right)^{\frac{1}{q}} \\ &+ \left( \lim_{t \to L^-} \frac{1}{V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s} \right)^{\frac{1}{p}} W^{\frac{1}{q}}(L). \end{aligned}$$

(iii) *If $p \leq q < 1$, then $C \approx A_3$, where*

$$A_3 = \sup_{0<t<L} \frac{W^{\frac{1}{q}}(t) + U(t) \left( \int_t^L W^{\frac{q}{1-q}}(s) w(s) U^{-\frac{q}{1-q}}(s)\, \mathrm{d}s \right)^{\frac{1-q}{q}}}{\left(V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s\right)^{\frac{1}{p}}}.$$

(iv) *If $q < 1$ and $q < p < \infty$, then $C \approx A_4$, where*

$$\begin{aligned} A_4 = &\left( \lim_{t \to 0^+} \frac{U^p(t)}{V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s} \right)^{\frac{1}{p}} \left( \int_0^L W^{\frac{q}{1-q}}(t) w(t) U^{-\frac{q}{1-q}}(t)\, \mathrm{d}t \right)^{\frac{1-q}{q}} \\ &+ \left( \lim_{t \to L^-} \frac{1}{V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s} \right)^{\frac{1}{p}} \left( \int_0^L W^{\frac{q}{1-q}}(t) w(t)\, \mathrm{d}t \right)^{\frac{1-q}{q}} \\ &+ \left( \int_0^L \frac{\left( W^{\frac{1}{1-q}}(t) + U^{\frac{q}{1-q}}(t) \int_t^L W^{\frac{q}{1-q}}(s) w(s) U^{-\frac{q}{1-q}}(s)\, \mathrm{d}s \right)^{\frac{p(1-q)}{p-q}}}{\left(V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s\right)^{\frac{q}{p-q}+2}} \right. \\ &\left. \times\, V(t) U^{p-1}(t) u(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s\, \mathrm{d}t \right)^{\frac{p-q}{pq}}. \end{aligned}$$

*The equivalence constants depend only on the parameters $p$ and $q$. In particular, they are independent of the weights $u$, $v$ and $w$.*

***Proof*** First of all, note that $U$ is admissible. Furthermore, as a prelude to the proof, let us make the following observation. Suppose that there exists a $t_0 \in (0, L)$ such that $\int_{t_0}^{L} v(s)U^{-p}(s)\,\mathrm{d}s = \infty$. Then the same holds, in fact, for all $t \in (0, L)$ (if $t > t_0$, consider that $v$ is locally integrable and $U$ is admissible; thus $\int_t^L v(s)U^{-p}(s)\,\mathrm{d}s = \infty$ must hold as well). It follows that $\Gamma_u^p(v) = \{0\}$, where "0" is the zero-constant function. Therefore, $C = 0$ and, by Convention 2.1 (ii), the quantities $A_1$–$A_4$ are also equal to zero; hence the theorem holds trivially. Thanks to this observation, we may and will assume in the proof that

$$\int_t^L \frac{v(s)}{U^p(s)}\,\mathrm{d}s < \infty \quad \text{for every } t \in (0, L). \tag{4.1}$$

If $p > q$, set $r = \frac{pq}{p-q}$. For each $f \in \mathfrak{M}_\mu(X)$ there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ of functions from $\mathfrak{M}_+(0, L)$ such that $\int_t^L h_n(s)\,\mathrm{d}s \nearrow f^*(t)$ for a.e. $t \in (0, L)$ as $n \to \infty$. The proof of this statement is analogous to that of [12, Lemma 1.2]. Furthermore, for any $t \in (0, L)$ and every $h \in \mathfrak{M}_+(0, L)$, we have

$$\frac{1}{U(t)}\int_0^t u(y)\int_y^L h(s)\,\mathrm{d}s\,\mathrm{d}y \le 2\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s \le \frac{2}{U(t)}\int_0^t u(y)\int_y^L h(s)\,\mathrm{d}s\,\mathrm{d}y.$$

Hence, by the monotone convergence theorem, we get (Convention 2.1 (ii) is in use here)

$$C \approx \sup_{h\in\mathfrak{M}_+(0,L)} \frac{\left(\int_0^L \left(\int_t^L h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}}{\left(\int_0^L \left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}}. \tag{4.2}$$

Define

$$\varphi(t) = \int_0^L \min\{U(t)^p, U(s)^p\}\frac{v(s)}{U(s)^p}\,\mathrm{d}s,\ t \in (0, L).$$

Note that $\varphi \in Q_{U^p}(0, L)$ (in particular, $\varphi$ is finite on $(0, L)$ by assumption (4.1)). Therefore, for every $a > 1$ there exists a covering sequence $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(\varphi, U^p, a)$. We fix $a > 1$ sufficiently large so that the lemmas and theorems that we are to use below may be applied. An appropriate value of $a$ may be determined by inspecting the further course of the proof in each of the cases (i)–(iv). In any of them, however, the sufficient size of the parameter $a$ depends only on $p$ and $q$.

By Lemma 3.4 with $\widetilde{p} = p$, $\widetilde{h} = \varphi$, $\widetilde{\varrho} = U^p$, $\widetilde{f} = Uh$, $\widetilde{\alpha} = \widetilde{\beta} = 0$, and $\mathrm{d}\widetilde{\nu}(t) = \frac{v}{U^p(t)}\,\mathrm{d}t$ (the parameters with tildes are those from the lemma) we have

$$\int_0^L \left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t \approx \sum_{k\in\mathcal{K}_-^+} \varphi(x_k)\left(\int_0^L \frac{U(t)h(t)}{U(x_k)+U(t)}\,\mathrm{d}t\right)^p$$
$$\approx \sum_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^p \tag{4.3}$$

for each $h \in \mathfrak{M}_+(0,L)$. The equivalence constants depend only on $p$. Clearly,

$$\int_0^L \left(\int_t^L h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t = \sum_{k\in\mathcal{K}^+}\int_{x_{k-1}}^{x_k}\left(\int_t^L h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t$$
$$\approx \sum_{k\in\mathcal{K}^+}\int_{x_{k-1}}^{x_k}\left(\int_t^{x_k} h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t + \sum_{k\in\mathcal{K}}\left(\int_{x_k}^L h(s)\,\mathrm{d}s\right)^q \int_{x_{k-1}}^{x_k} w(t)\,\mathrm{d}t \tag{4.4}$$

for each $h \in \mathfrak{M}_+(0,L)$, and the equivalence constants depend only on $q$.

*Upper bounds*. In this part, we shall prove the upper bounds on $C$. This is equivalent to proving that the upper bounds are upper bounds on the supremum on the right-hand side of (4.2). As for cases (i) and (ii), assume that $1 \le q < \infty$. For every $k \in \mathcal{K}^+$, the weighted Hardy inequality (e.g., [9] and references therein) yields

$$\int_{x_{k-1}}^{x_k}\left(\int_t^{x_k} h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t$$
$$\lesssim \left(\int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^q \sup_{t\in(x_{k-1},x_k]} \varphi^{-\frac{q}{p}}(t)\int_{x_{k-1}}^t w(s)\,\mathrm{d}s. \tag{4.5}$$

*Case* (i). Assume that $1 \le q$, $p \le q$ and $A_1 < \infty$. Then

$$\sum_{k\in\mathcal{K}^+}\int_{x_{k-1}}^{x_k}\left(\int_t^{x_k} h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t \lesssim \left(\sup_{t\in(0,L)} \varphi^{-\frac{q}{p}}(t)W(t)\right)\sum_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k}\varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^q$$
$$\le A_1^q\left(\sum_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k}\varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^p\right)^{\frac{q}{p}} \approx A_1^q\left(\int_0^L\left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{q}{p}}. \tag{4.6}$$

The first inequality in (4.6) follows from (4.5), the second inequality is valid since $p \le q$, and the equivalence is valid thanks to (4.3). Furthermore, using $p \le q$, we get

$$
\begin{aligned}
\sum_{k\in\mathcal{K}} \left(\int_{x_k}^L h(s)\,\mathrm{d}s\right)^q \int_{x_{k-1}}^{x_k} w(t)\,\mathrm{d}t &\le \sum_{k\in\mathcal{K}} \left(\int_{x_k}^L h(s)\,\mathrm{d}s\right)^q \varphi^{\frac{q}{p}}(x_k)\varphi^{-\frac{q}{p}}(x_k)W(x_k) \\
&\le \sup_{k\in\mathcal{K}} \left(\varphi^{-\frac{q}{p}}(x_k)W(x_k)\right)\left(\sum_{k\in\mathcal{K}} \left(\int_{x_k}^L h(s)\,\mathrm{d}s\right)^p \varphi(x_k)\right)^{\frac{q}{p}} \\
&\lesssim A_1^q \left(\sum_{k\in\mathcal{K}} \left(\int_{x_k}^L \frac{U(s)h(s)}{U(x_k)+U(s)}\,\mathrm{d}s\right)^p \varphi(x_k)\right)^{\frac{q}{p}} \\
&\lesssim A_1^q \left(\int_0^L \left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{q}{p}}, \qquad (4.7)
\end{aligned}
$$

where the last inequality follows from (4.3). Note that (4.7) is actually valid for any $q \in (0,\infty)$ such that $p \le q$. By combining (4.4), (4.6), (4.7), and considering (4.2), we obtain the estimate $C \lesssim A_1$ in case (i).

*Case* (ii). Assume that $1 \le q < p < \infty$ and $A_2 < \infty$. Owing to the Hölder inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$, we obtain

$$
\begin{aligned}
&\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^q \sup_{t\in(x_{k-1},x_k]} \varphi^{-\frac{q}{p}}(t)\int_{x_{k-1}}^t w(s)\,\mathrm{d}s \\
&\le \left(\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^p\right)^{\frac{q}{p}} \left(\sum_{k\in\mathcal{K}^+} \sup_{t\in(x_{k-1},x_k]} \varphi^{-\frac{r}{p}}(t)\left(\int_{x_{k-1}}^t w(s)\,\mathrm{d}s\right)^{\frac{r}{q}}\right)^{\frac{p-q}{p}} \\
&\le \left(\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^p\right)^{\frac{q}{p}} \left(\sum_{k\in\mathcal{K}^+} \sup_{t\in(x_{k-1},x_k]} \left(\frac{U^p(t)}{\varphi(t)}\right)^{\frac{r}{p}} U^{-r}(t)W(t)^{\frac{r}{q}}\right)^{\frac{p-q}{p}}. \qquad (4.8)
\end{aligned}
$$

By [5, Theorem 2.4.4], the equivalence

$$
\begin{aligned}
\left(\frac{U^p(t)}{\varphi(t)}\right)^{\frac{r}{p}} \approx \left(\lim_{s\to 0+} \frac{U^p(s)}{\varphi(s)}\right)^{\frac{r}{p}} + \left(\lim_{s\to L-} \frac{1}{\varphi(s)}\right)^{\frac{r}{p}} U^r(t) \\
+ \int_0^L \min\{U^r(t), U^r(s)\} \frac{U^{p-1}(s)u(s)V(s)}{\varphi^{\frac{r}{p}+2}(s)} \int_s^L \frac{v(\tau)}{U^p(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s \qquad (4.9)
\end{aligned}
$$

is valid for every $t \in (0, L)$, and the equivalence constants depend only on $p$ and $q$. Note that (4.9) is actually valid for any $q \in (0,\infty)$ such that $q < p$. Lemma 3.3 with the setting $\widetilde{h} = \left(\frac{U^p}{\varphi}\right)^{\frac{r}{p}}$, $\widetilde{\varrho} = U^r$, $\widetilde{p} = 1$, $\widetilde{f} = W^{\frac{r}{q}}$, together with (4.9) gives

$$
\sum_{k\in\mathcal{K}^+} \sup_{t\in(x_{k-1},x_k]} \left(\frac{U^p(t)}{\varphi(t)}\right)^{\frac{r}{p}} U^{-r}(t)W(t)^{\frac{r}{q}} \approx A_2^r. \qquad (4.10)
$$

By using (4.8), (4.10) and (4.3) we obtain

$$\sum_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k}\varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^q \sup_{t\in(x_{k-1},x_k]}\varphi^{-\frac{q}{p}}(t)\int_{x_{k-1}}^{t}w(s)\,\mathrm{d}s$$
$$\lesssim A_2^q\left(\int_0^L\left(\int_0^L\frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{q}{p}}. \tag{4.11}$$

Next, one has

$$\begin{aligned}
&\sum_{k\in\mathcal{K}}\left(\int_{x_k}^{L}h(s)\,\mathrm{d}s\right)^q\int_{x_{k-1}}^{x_k}w(t)\,\mathrm{d}t\\
&\le\sum_{k\in\mathcal{K}}\left(\int_{x_k}^{L}h(s)\,\mathrm{d}s\right)^q\varphi^{\frac{q}{p}}(x_k)\varphi^{-\frac{q}{p}}(x_k)W(x_k)\\
&\le\left(\sum_{k\in\mathcal{K}}\left(\int_{x_k}^{L}h(s)\,\mathrm{d}s\right)^p\varphi(x_k)\right)^{\frac{q}{p}}\left(\sum_{k\in\mathcal{K}}\varphi^{-\frac{r}{p}}(x_k)W^{\frac{r}{q}}(x_k)\right)^{\frac{q}{r}}\\
&=\left(\sum_{k\in\mathcal{K}}\left(\sum_{\substack{l\in\mathbb{Z}\\ k\le l\le K^+-1}}\int_{x_l}^{x_{l+1}}h(s)\,\mathrm{d}s\right)^p\varphi(x_k)\right)^{\frac{q}{p}}\left(\sum_{k\in\mathcal{K}}\varphi^{-\frac{r}{p}}(x_k)W^{\frac{r}{q}}(x_k)\right)^{\frac{q}{r}}\\
&\approx\left(\sum_{k\in\mathcal{K}}\left(\int_{x_k}^{x_{k+1}}h(s)\,\mathrm{d}s\right)^p\varphi(x_k)\right)^{\frac{q}{p}}\left(\sum_{k\in\mathcal{K}}\varphi^{-\frac{r}{p}}(x_k)W^{\frac{r}{q}}(x_k)\right)^{\frac{q}{r}}\\
&\le\left(\sum_{k\in\mathcal{K}}\left(\int_{x_k}^{x_{k+1}}\varphi^{\frac{1}{p}}(s)h(s)\,\mathrm{d}s\right)^p\right)^{\frac{q}{p}}\left(\sum_{k\in\mathcal{K}}\varphi^{-\frac{r}{p}}(x_k)W^{\frac{r}{q}}(x_k)\right)^{\frac{q}{r}}\\
&\lesssim\left(\int_0^L\left(\int_0^L\frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{q}{p}}\left(\sum_{k\in\mathcal{K}}\varphi^{-\frac{r}{p}}(x_k)W^{\frac{r}{q}}(x_k)\right)^{\frac{q}{r}}.
\end{aligned}\tag{4.12}$$

Here, the Hölder inequality was applied in the second step, the fourth step relies on (2.4) and [5, Lemma 1.3.5], and the last step follows from (4.3). Note that (4.12) is actually valid for all $0<q<p<\infty$, although we are currently assuming $1\le q<p<\infty$. Since

$$W(x_k)\le U^q(x_k)\sup_{t\in(x_k,L)}U^{-q}(t)W(t)$$

holds for each $k \in \mathcal{K}$, by (4.11) and (4.12) we get

$$\sum_{k\in\mathcal{K}} \left(\int_{x_k}^{L} h(s)\,\mathrm{d}s\right)^q \int_{x_{k-1}}^{x_k} w(t)\,\mathrm{d}t \lesssim A_2^q \left(\int_0^L \left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{q}{p}}. \tag{4.13}$$

Using (4.4), (4.5), (4.11), (4.13) and considering (4.2), we obtain the estimate $C \lesssim A_2$ in case (ii).

As for cases (iii) and (iv), assume that $0 < q < 1$. One can easily modify [13, Theorem 3.3] to obtain

$$\begin{aligned}
&\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t \\
&\lesssim \left(\int_{x_{k-1}}^{x_k} h(t)\varphi^{\frac{1}{p}}(t)\,\mathrm{d}t\right)^q \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_k}^{t} w(s)\,\mathrm{d}s\right)^{\frac{q}{1-q}} w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{1-q} \\
&\le \left(\int_{x_{k-1}}^{x_k} h(t)\varphi^{\frac{1}{p}}(t)\,\mathrm{d}t\right)^q \left(\int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}}(t)w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{1-q}
\end{aligned} \tag{4.14}$$

for every $k \in \mathcal{K}^+$, where the constant in "$\lesssim$" depends only on $q$. By Lemma 3.5 with the setting $\widetilde{\varphi} = U^q\varphi^{-\frac{q}{p}}$, $\widetilde{f} = W^{\frac{q}{1-q}}w$, $\widetilde{p} = 1-q$, $\widetilde{\varrho} = U^q$, and by Lemma 3.1 one has

$$\begin{aligned}
&\sup_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}}(t)w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{1-q} \\
&\approx \sup_{t\in(0,L)} \frac{\left(\int_0^L W^{\frac{q}{1-q}}(s)w(s)\min\left\{1, \left(\frac{U(t)}{U(s)}\right)^{\frac{q}{1-q}}\right\}\mathrm{d}s\right)^{1-q}}{\varphi^{\frac{q}{p}}(t)} \approx A_3^q.
\end{aligned} \tag{4.15}$$

*Case* (iii). Assume that $0 < p \le q < 1$ and $A_3 < \infty$. Thanks to (4.3), (4.14) and (4.15), we have

$$\begin{aligned}
&\sum_{k\in\mathcal{K}^+} \int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t \\
&\lesssim \left(\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} h(t)\varphi^{\frac{1}{p}}(t)\,\mathrm{d}t\right)^q\right) \left(\sup_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}}(t)w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{1-q}\right) \\
&\lesssim A_3^q \left(\int_0^L \left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{q}{p}}.
\end{aligned} \tag{4.16}$$

Since $A_1 \le A_3$, it follows from (4.7) that

$$\sum_{k\in\mathcal{K}} \left(\int_{x_k}^{L} h(s)\,\mathrm{d}s\right)^q \int_{x_{k-1}}^{x_k} w(t)\,\mathrm{d}t \lesssim A_3^q \left(\int_0^L \left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{q}{p}}. \tag{4.17}$$

Hence, starting with (4.2) and using (4.4), (4.16) and (4.17), we obtain $C \lesssim A_3$ in case (iii).

*Case* (iv). Assume that $0 < q < 1$, $0 < q < p$ and $A_4 < \infty$. Denote

$$\xi(t) = \int_0^L \min\left\{U^{\frac{q}{1-q}}(s), U^{\frac{q}{1-q}}(t)\right\} W^{\frac{q}{1-q}}(s)w(s)U^{-\frac{q}{1-q}}(s)\,\mathrm{d}s, \quad t \in (0, L). \tag{4.18}$$

By Lemma 3.1, we have

$$\xi(t) \approx W^{\frac{1}{1-q}}(t) + U^{\frac{q}{1-q}}(t) \int_t^L W^{\frac{q}{1-q}}(s)w(s)U^{-\frac{q}{1-q}}(s)\,\mathrm{d}s \quad \text{for every } t \in (0, L). \tag{4.19}$$

Thanks to (4.14), the Hölder inequality with exponents $\frac{p}{q}$ and $\frac{r}{q}$, and (4.3), we obtain

$$\begin{aligned} &\sum_{k\in\mathcal{K}^+} \int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t \\ &\le \left(\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} h(t)\varphi^{\frac{1}{p}}(t)\,\mathrm{d}t\right)^p\right)^{\frac{q}{p}} \left(\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}}(t)w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{r(1-q)}{q}}\right)^{\frac{q}{r}} \\ &\approx \left(\int_0^L \left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{q}{p}} \left(\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}}(t)w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{r(1-q)}{q}}\right)^{\frac{q}{r}}. \end{aligned} \tag{4.20}$$

Furthermore, Lemma 3.4 with $\widetilde{h}(t) = \frac{U(t)^r}{\varphi^{\frac{r}{p}}(t)}$, $\widetilde{\varrho} = U^r$, $\widetilde{p} = \frac{r(1-q)}{q}$ and $\widetilde{f} = W^{\frac{q}{1-q}}w$ gives

$$\left(\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}}(t)w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{r(1-q)}{q}}\right)^{\frac{q}{r}} \approx \left(\sum_{k\in\mathcal{K}^+_-} \frac{\xi^{\frac{r(1-q)}{q}}(x_k)}{\varphi^{\frac{r}{p}}(x_k)}\right)^{\frac{q}{r}}. \tag{4.21}$$

Recall that (4.12) is valid for any $0 < q < p < \infty$ and note that $W^{\frac{r}{q}}(t) \lesssim \xi^{\frac{r(1-q)}{q}}(t)$ for every $t \in (0, L)$. Therefore, (4.4), (4.20), (4.21) and (4.12) yield

$$\left(\int_0^L \left(\int_t^L h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim \left(\sum_{k\in\mathcal{K}_-^+} \frac{\xi^{\frac{r(1-q)}{q}}(x_k)}{\varphi^{\frac{r}{p}}(x_k)}\right)^{\frac{1}{r}} \times \left(\int_0^L \left(\int_0^L \frac{U(s)h(s)}{U(s)+U(t)}\,\mathrm{d}s\right)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{4.22}$$

Combining (4.9) and Lemma 3.4 with $\widetilde{h} = U^r\varphi^{-\frac{r}{p}}$, $\widetilde{\varrho} = U^r$, $\widetilde{p} = \frac{r(1-q)}{q}$, $\widetilde{f} = W^{\frac{q}{1-q}}w$, we obtain

$$\left(\sum_{k\in\mathcal{K}_-^+} \frac{\xi^{\frac{r(1-q)}{q}}(x_k)}{\varphi^{\frac{r}{p}}(x_k)}\right)^{\frac{1}{r}} \approx A_4. \tag{4.23}$$

Hence, the desired upper bound $C \lesssim A_4$ follows from (4.22), (4.23) and (4.2).

*Lower bounds.* Now we shall turn our attention to proving the lower bounds. Suppose that $C < \infty$. Fix an arbitrary $t \in (0, L)$ and choose any function $f \in \mathfrak{M}_\mu(X)$ such that

$$f^* = \frac{\chi_{[0,t)}}{\left(V(t) + U^p(t)\int_t^L v(s)U^{-p}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}}.$$

Such a function indeed exists, see [1, Corollary 7.8, p. 86]. Observe that $\|f\|_{\Gamma_u^p(v)} = 1$, and so

$$\frac{W^{\frac{1}{q}}(t)}{\left(V(t) + U^p(t)\int_t^L v(s)U^{-p}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}} \le C$$

because the left-hand side is equal to $\|f\|_{\Lambda^q(w)}$. Since $t$ was arbitrary, we get the estimate $A_1 \le C$ by taking the supremum over $t \in (0, L)$. Notice that no additional assumptions on $p$ or $q$ were needed. Hence, not only does this complete case (i), but it also shows that $A_1 \le C$ in all cases (i)–(iv). This is a common feature of inequalities of this type.

Let us continue with the other cases. Thanks to (4.2), (4.4) and (4.3), we have

$$\left(\sum_{k\in\mathcal{K}^+} \int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \lesssim C\left(\sum_{k\in\mathcal{K}^+} \left(\int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}}(t)h(t)\,\mathrm{d}t\right)^p\right)^{\frac{1}{p}}$$

(4.24)

for every $h \in \mathfrak{M}_+(0, L)$.

Exploiting the saturation of the Hardy inequality (see [9, Lemma 5.4] and [13, Theorem 3.3]) and (4.24), by the same argument as in [7, pages 340–344] we obtain the following estimates:

- If $1 \le q < \infty$, $p > q$, then

$$\left(\sum_{k\in\mathcal{K}^+}\left(\sup_{t\in(x_{k-1},x_k]}\varphi^{-\frac{q}{p}}(t)\int_{x_{k-1}}^{t}w(s)\,\mathrm{d}s\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \lesssim C; \tag{4.25}$$

- If $0 < q < 1$, $p \le q$, then

$$\sup_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t}w(s)\,\mathrm{d}s\right)^{\frac{q}{1-q}}w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}} \lesssim C; \tag{4.26}$$

- If $0 < q < 1$, $p > q$, then

$$\left(\sum_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t}w(s)\,\mathrm{d}s\right)^{\frac{q}{1-q}}w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{(1-q)r}{q}}\right)^{\frac{1}{r}} \lesssim C. \tag{4.27}$$

*Case* (ii). Assume that $1 \le q < p < \infty$. We have

$$\begin{aligned}
A_2 &\approx \left(\sum_{k\in\mathcal{K}^+}\sup_{t\in(x_{k-1},x_k]}\frac{W^{\frac{r}{q}}(t)}{\varphi^{\frac{r}{p}}(t)}\right)^{\frac{1}{r}}\\
&\lesssim \left(\sum_{k=K_-+2}^{K^+}\frac{W^{\frac{r}{q}}(x_{k-1})}{\varphi^{\frac{r}{p}}(x_{k-1})}\right)^{\frac{1}{r}} + \left(\sum_{k\in\mathcal{K}^+}\sup_{t\in(x_{k-1},x_k]}\frac{\left(\int_{x_{k-1}}^{t}w(s)\,\mathrm{d}s\right)^{\frac{r}{q}}}{\varphi^{\frac{r}{p}}(t)}\right)^{\frac{1}{r}}\\
&= \left(\sum_{k\in\mathcal{K}}\frac{\left(\sum_{l=K_-+1}^{k}\int_{x_{l-1}}^{x_l}w(s)\,\mathrm{d}s\right)^{\frac{r}{q}}}{\varphi^{\frac{r}{p}}(x_k)}\right)^{\frac{1}{r}} + \left(\sum_{k\in\mathcal{K}^+}\left(\sup_{t\in(x_{k-1},x_k]}\frac{\int_{x_{k-1}}^{t}w(s)\,\mathrm{d}s}{\varphi^{\frac{q}{p}}(t)}\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}\\
&\approx \left(\sum_{k\in\mathcal{K}}\left(\frac{\int_{x_{k-1}}^{x_k}w(s)\,\mathrm{d}s}{\varphi^{\frac{q}{p}}(x_k)}\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} + \left(\sum_{k\in\mathcal{K}^+}\left(\sup_{t\in(x_{k-1},x_k]}\frac{\int_{x_{k-1}}^{t}w(s)\,\mathrm{d}s}{\varphi^{\frac{q}{p}}(t)}\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}\\
&\lesssim C.
\end{aligned}$$

Here, the first and last step are based on (4.10) and (4.25), respectively. The fourth step follows from [5, Lemma 1.3.4] combined with (2.4).

*Case* (iii). Let $0 < p \le q < 1$. Owing to (4.15), we have

$$\begin{aligned} A_3 &\approx \sup_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}}(t)w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}} \\ &\approx \sup_{\substack{K_-+2\le k\le K^+\\ k\in\mathbb{Z}}} W(x_{k-1})\left(\int_{x_{k-1}}^{x_k} w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}} \\ &\quad + \sup_{k\in\mathcal{K}^+}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} w(s)\,\mathrm{d}s\right)^{\frac{q}{1-q}} w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}}. \end{aligned} \tag{4.28}$$

As for the first term, one may use exactly the same argument as in [7, page 343] to obtain

$$\begin{aligned} &\sup_{\substack{K_-+2\le k\le K^+\\ k\in\mathbb{Z}}} W(x_{k-1})\left(\int_{x_{k-1}}^{x_k} w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}} \\ &\lesssim \sup_{k\in\mathcal{K}}\varphi^{-\frac{1}{p}}(x_k)W^{\frac{1}{q}}(x_k) \\ &\quad + \sup_{\substack{K_-+2\le k\le K^+\\ k\in\mathbb{Z}}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} w(s)\,\mathrm{d}s\right)^{\frac{q}{1-q}} w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}}. \end{aligned} \tag{4.29}$$

Moreover,

$$\begin{aligned} \sup_{k\in\mathcal{K}}\varphi^{-\frac{1}{p}}(x_k)W^{\frac{1}{q}}(x_k) &\approx \sup_{k\in\mathcal{K}}\varphi^{-\frac{1}{p}}(x_k)\left(\int_{x_{k-1}}^{x_k} w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \\ &\approx \sup_{k\in\mathcal{K}}\varphi^{-\frac{1}{p}}(x_k)\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} w(s)\,\mathrm{d}s\right)^{\frac{q}{1-q}} w(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}} \\ &\le \sup_{k\in\mathcal{K}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} w(s)\,\mathrm{d}s\right)^{\frac{q}{1-q}} w(t)\varphi^{-\frac{q}{p(1-q)}}(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}}, \end{aligned} \tag{4.30}$$

where the first equivalence is valid thanks to [5, Lemma 1.3.4] again, and the second one follows from Lemma 3.1. Hence, combining (4.28), (4.29) and (4.30) with (4.26), we get $A_3 \lesssim C$.

*Case* (iv). Let $0 < q < 1$, $0 < q < p$. Similarly to the previous case, it can be shown (cf. [7, pages 344–346]) that

$$\begin{aligned}
&\left( \sum_{k\in\mathcal{K}^+} \left( \int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}}(t) w(t) \varphi^{-\frac{q}{p(1-q)}}(t)\, \mathrm{d}t \right)^{\frac{(1-q)r}{q}} \right)^{\frac{1}{r}} \\
&\approx \left( \sum_{k=K_-+2}^{K^+} W^r(x_{k-1}) \left( \int_{x_{k-1}}^{x_k} w(t) \varphi^{-\frac{q}{p(1-q)}}(t)\, \mathrm{d}t \right)^{\frac{(1-q)r}{q}} \right)^{\frac{1}{r}} \\
&\quad + \left( \sum_{k\in\mathcal{K}^+} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^{t} w(s)\, \mathrm{d}s \right)^{\frac{q}{1-q}} w(t) \varphi^{-\frac{q}{p(1-q)}}(t)\, \mathrm{d}t \right)^{\frac{(1-q)r}{q}} \right)^{\frac{1}{r}} \\
&\approx \left( \sum_{k\in\mathcal{K}^+} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^{t} w(s)\, \mathrm{d}s \right)^{\frac{q}{1-q}} w(t) \varphi^{-\frac{q}{p(1-q)}}(t)\, \mathrm{d}t \right)^{\frac{(1-q)r}{q}} \right)^{\frac{1}{r}}.
\end{aligned} \tag{4.31}$$

Hence, the desired inequality $A_4 \lesssim C$ follows from (4.31), (4.21), (4.23) and (4.27). □

***Remark 4.2*** Keeping the setting of Theorem 4.1, we make the following remark. If

$$\int_t^L W^{\frac{q}{1-q}}(s) w(s) U^{-\frac{q}{1-q}}(s)\, \mathrm{d}s < \infty \quad \text{for every } t \in (0, L), \tag{4.32}$$

then $A_4$ (and so also $C$) is equivalent to $A_5$, where

$$A_5 = \left( \int_0^L \frac{\left( W^{\frac{1}{1-q}}(t) + U^{\frac{q}{1-q}}(t) \int_t^L W^{\frac{q}{1-q}}(s) w(s) U^{-\frac{q}{1-q}}(s)\, \mathrm{d}s \right)^{\frac{p(1-q)}{p-q}-1} W^{\frac{q}{1-q}}(t) w(t)}{\left( V(t) + U^p(t) \int_t^L v(s) U^{-p}(s)\, \mathrm{d}s \right)^{\frac{q}{p-q}}}\, \mathrm{d}t \right)^{\frac{p-q}{pq}}.$$

We shall prove this assertion. Note that assumption (4.32) implies that the function $\xi$ defined by (4.18) is finite and, moreover, $\xi \in Q_{U^{\frac{q}{1-q}}}(0, L)$. Therefore, there is a covering sequence $\{\widetilde{x}_k\}_{k\in\widetilde{\mathcal{K}}_-^+} \in CS(\xi, U^{\frac{q}{1-q}}, b)$ for each parameter $b > 1$. We find $b$ sufficiently large so that the assumptions of the theorems that we are to use are satisfied. The sufficient size of $b$ depends only on $p$ and $q$. Combining Theorem 3.2 applied to $\widetilde{h} = \xi$, $\widetilde{\varrho} = U^{\frac{q}{1-q}}$, $\mathrm{d}\widetilde{\nu}(t) = W^{\frac{q}{1-q}}(t) w(t) U^{-\frac{q}{1-q}}(t)\, \mathrm{d}t$, $\widetilde{\alpha} = \widetilde{\beta} = 0$, $\widetilde{p} = \frac{r(1-q)}{q}$, $\widetilde{f} = U^r \varphi^{-\frac{r}{p}}$ with (4.19), we obtain

$$\sum_{k\in\widetilde{\mathcal{K}}_-^+} \frac{\xi^{\frac{r(1-q)}{q}}(\widetilde{x}_k)}{\varphi^{\frac{r}{p}}(\widetilde{x}_k)} \approx A_5^r. \tag{4.33}$$

Furthermore, since both functions $\xi^{\frac{r(1-q)}{q}}$ and $U^r\varphi^{-\frac{r}{p}}$ are $U^r$-quasiconcave on $(0, L)$, we have

$$\sum_{k\in\mathcal{K}_-^+} \frac{\xi^{\frac{r(1-q)}{q}}(x_k)}{\varphi^{\frac{r}{p}}(x_k)} \approx \sum_{k\in\widetilde{\mathcal{K}}_-^+} \frac{\xi^{\frac{r(1-q)}{q}}(\widetilde{x}_k)}{\varphi^{\frac{r}{p}}(\widetilde{x}_k)} \tag{4.34}$$

thanks to [5, Lemma 4.2.9], where $\{x_k\}_{k\in\mathcal{K}_-^+}$ is the covering sequence from the proof of Theorem 4.1. Hence the desired equivalence follows from (4.23) combined with (4.33) and (4.34).

Without the additional assumption, $A_4$ and $A_5$ need not, however, be equivalent. In order to see this, note that $\xi \equiv \infty$ if (4.32) is violated. If this is the case, then $A_4 = \infty$ provided that (4.1) is true, but $A_5 = 0$ provided that $\frac{p(1-q)}{p-q} - 1 < 0$ (which is the case when $0 < q < 1 < p < \infty$). It appears that this peculiar detail was overlooked in [7, Theorem 4.2].

The final theorem, which generalizes [8, Theorem 1.8] by allowing degenerated weights, deals with a variant of the main result in the setting $p = \infty$. It provides an equivalent estimate on the optimal constant $C$ in the inequality

$$\left(\int_0^L (f^*(t))^q w(t)\,dt\right)^{\frac{1}{q}} \le C \operatorname*{ess\,sup}_{t\in(0,L)} \left(\frac{1}{U(t)}\int_0^t f^*(s)u(s)\,ds\right) v(t),$$

which is expressed by (4.35) below. We note that the representation (4.37) below is always possible (see Remark 4.4).

**Theorem 4.3** *Let $q \in (0, \infty)$. Let $v, w$ be weights on $(0, L)$ and $u$ an a.e. positive weight on $(0, L)$. Set*

$$C = \sup_{\|f\|_{\Gamma_u^\infty(v)}\le 1} \|f\|_{\Lambda^q(w)}, \tag{4.35}$$

*and*

$$\varphi(t) = \operatorname*{ess\,sup}_{\tau\in(0,t)} U(\tau) \operatorname*{ess\,sup}_{s\in(\tau,L)} \frac{v(s)}{U(s)},\ \ t \in (0, L). \tag{4.36}$$

*Let $B_1, B_2 \in (0, \infty)$, $\gamma, \delta \in [0, \infty)$ and $\nu$ a nonnegative Borel measure on $(0, L)$ such that*

$$\begin{aligned} B_1\varphi(t) &\le \gamma + \delta U(t) + \int_{(0,L)} \min\{U(t), U(s)\}\,d\nu(s) \\ &\le B_2\varphi(t) \quad \text{for every } t \in (0, L). \end{aligned} \tag{4.37}$$

(i) *If* $1 \le q < \infty$, *then* $C \approx A_6$, *where*

$$A_6 = \left(\lim_{t\to 0^+} \frac{U(t)}{\varphi(t)}\right)\left(\sup_{t\in(0,L)} \frac{W^{\frac{1}{q}}(t)}{U(t)}\right) + \lim_{t\to L^-} \frac{1}{\varphi(t)} W^{\frac{1}{q}}(L)$$
$$+ \left(\int_0^L U^q(t)\left(\sup_{\tau\in(t,L)} \frac{W(\tau)}{U^q(\tau)}\right)\varphi^{-(q+2)}(t)u(t)\right.$$
$$\left.\times\left(\gamma + \int_{(0,t]} U(s)\,\mathrm{d}\nu(s)\right)\left(\delta + \int_{[t,L)} \mathrm{d}\nu(s)\right)\mathrm{d}t\right)^{\frac{1}{q}}.$$

(ii) *If* $0 < q < 1$, *then* $C \approx A_7$, *where*

$$A_7 = \left(\lim_{t\to 0^+} \frac{U(t)}{\varphi(t)}\right)\left(\int_0^L W^{\frac{q}{1-q}}(t)w(t)U^{-\frac{q}{1-q}}(t)\,\mathrm{d}t\right)^{\frac{1-q}{q}} + \lim_{t\to L^-} \frac{1}{\varphi(t)} W^{\frac{1}{q}}(L)$$
$$+ \left(\int_0^L \xi(t)\varphi^{-(q+2)}(t)u(t)\left(\gamma + \int_{(0,t]} U(s)\,\mathrm{d}\nu(s)\right)\left(\delta + \int_{[t,L)} \mathrm{d}\nu(s)\right)\mathrm{d}t\right)^{\frac{1}{q}},$$

*where*

$$\xi(t) = \left(\int_0^L W^{\frac{q}{1-q}}(s)w(s)U^{-\frac{q}{1-q}}(s)\min\{U^{\frac{q}{1-q}}(t), U^{\frac{q}{1-q}}(s)\}\,\mathrm{d}s\right)^{1-q}, \ t \in (0, L). \tag{4.38}$$

*The equivalence constants depend only on the parameter $q$ and the constants $B_1$ and $B_2$. In particular, they are independent of the weights $u$, $v$ and $w$.*

***Proof*** We start off with a few useful observations. Note that

$$\operatorname*{ess\,sup}_{t\in(0,L)} \frac{1}{U(t)} \int_0^t f^*(s)u(s)\,\mathrm{d}s\, v(t) = \operatorname*{ess\,sup}_{t\in(0,L)} \frac{1}{U(t)} \int_0^t f^*(s)u(s)\,\mathrm{d}s\, \varphi(t)$$

for every $f \in \mathfrak{M}_\mu(X)$ (see [8, Lemma 1.5]). Coupling this with an argument similar to that leading to (4.2), we obtain

$$C \approx \sup_{h\in\mathfrak{M}_+(0,L)} \frac{\left(\int_0^L \left(\int_t^L h(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}}{\sup_{t\in(0,L)} \varphi(t) \int_0^L \frac{U(s)h(s)}{U(t)+U(s)}\,\mathrm{d}s}. \tag{4.39}$$

Furthermore, without loss of generality, we may assume that

$$\varphi(t) < \infty \quad \text{for every } t \in (0, L), \tag{4.40}$$

for, if this is not the case, then $\varphi$ is actually identically equal to $\infty$ on $(0, L)$, and so it follows that $C = A_6 = A_7 = 0$ (Convention 2.1(ii) is used here once again).

By interchanging the order of the suprema, we get the identity

$$\varphi(t) = \operatorname*{ess\,sup}_{s\in(0,L)} v(s) \min\left\{1, \frac{U(t)}{U(s)}\right\} \quad \text{for every } t \in (0, L).$$

Hence $\varphi \in Q_U(0, L)$.

Since all key ideas were already presented in the proof of Theorem 4.1, we will only outline the proof instead of going into all detail.

Since $\varphi \in Q_U(0, L)$, there is a covering sequence $\{x_k\}_{k\in\mathcal{K}_-^+} \in CS(\varphi, U, a)$ for each $a > 1$. We fix such a sequence for $a > 1$ sufficiently large so that the assumptions of the theorems that we are to use are satisfied. An explicit estimate on $a$ may be obtained by careful examination of each step of the proof. What is important is that it depends only on the parameter $q$ and on the constants $B_1$ and $B_2$.

By Lemma 3.5 applied to $\widetilde{p} = 1$, $\widetilde{\varphi} = \varphi$, $\widetilde{\varrho} = U$ and $\widetilde{f} = Uh$, we have

$$\begin{aligned}
&\sup_{t\in(0,L)} \varphi(t) \int_0^L \frac{U(s)h(s)}{U(t)+U(s)}\,\mathrm{d}s \approx \sup_{k\in\mathcal{K}_-^+} \varphi(x_k) \int_0^L \frac{U(s)h(s)}{U(x_k)+U(s)}\,\mathrm{d}s\\
&\approx \sup_{k\in\mathcal{K}^+} \int_{x_{k-1}}^{x_k} h(s)\varphi(s)\,\mathrm{d}s.
\end{aligned}$$

By discretizing the right-hand side of (4.39) as in (4.4) and arguing as in [8, the proof of Theorem 1.8], one can show that

$$C^q \approx \sum_{k\in\mathcal{K}^+} \sup_{t\in(x_{k-1},x_k]} \frac{W(t)}{\varphi^q(t)} \quad \text{if } 1 \le q < \infty \tag{4.41}$$

and

$$C^q \approx \sum_{k\in\mathcal{K}_-^+} \frac{\xi(x_k)}{\varphi^q(x_k)} \quad \text{if } 0 < q < 1. \tag{4.42}$$

In order to obtain the desired results, we need to anti-discretize the right-hand sides of (4.41) and (4.42).

Thanks to [5, Theorem 2.4.4], using the representation (4.37) of $\varphi$, we have, for every $t \in (0, L)$,

$$\begin{aligned}
\frac{U^q(t)}{\varphi^q(t)} &\approx \left(\lim_{s\to0^+} \frac{U(t)}{\varphi(s)}\right)^q + \left(\lim_{s\to L^-} \frac{1}{\varphi(s)}\right)^p U^q(t)\\
&+ \int_0^L \min\{U^q(t), U^q(s)\}\varphi^{-(q+2)}(s)u(s)\\
&\times \left(\gamma + \int_{(0,s]} U(\tau)\,\mathrm{d}\nu(\tau)\right)\left(\delta + \int_{[s,L)} \mathrm{d}\nu(\tau)\right)\mathrm{d}s.
\end{aligned} \tag{4.43}$$

The equivalence constants in (4.43) depend only on $q$, $A_1$ and $A_2$.

If $1 \le q$, the equivalence $C \approx A_6$ follows from (4.41), (4.43) and Lemma 3.3 applied to $\widetilde{h} = U^q \varphi^{-q}$, $\widetilde{\varrho} = U^q$, $\widetilde{q} = 1$ and $\widetilde{f} = W$.

Assume now that $0 < q < 1$. Observe that

$$\begin{aligned} \xi(t) &\approx W(t) + U^q(t) \left( \int_t^L W^{\frac{q}{1-q}}(s) w(s) U^{-\frac{q}{1-q}}(s)\,\mathrm{d}s \right)^{1-q} \\ &\approx U^q(t) \left( \int_0^L \frac{W^{\frac{q}{1-q}}(s) w(s)}{U^{\frac{q}{1-q}}(t) + U^{\frac{q}{1-q}}(s)}\,\mathrm{d}s \right)^{1-q} \end{aligned} \tag{4.44}$$

for every $t \in (0, L)$ thanks to Lemma 3.1. Furthermore, by the same reasoning it is easy to see that

$$\left( \int_0^L W^{\frac{q}{1-q}}(s) w(s)\,\mathrm{d}s \right)^{1-q} \approx W(L). \tag{4.45}$$

The equivalence $C \approx A_7$ follows from (4.42), (4.44), (4.45) and Lemma 3.4 applied to $\widetilde{h} = U^q \varphi^{-q}$, $\widetilde{\varrho} = U^q$, $\widetilde{q} = 1 - q$, $\widetilde{f} = W^{\frac{q}{1-q}} w$ and $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\nu}$ given by (4.43). □

***Remark 4.4*** Since the function $\varphi$ defined by (4.36) is in $Q_U(0, L)$, there is always a nonnegative Borel measure $\nu$ on $(0, L)$ that represents $\varphi$ as in (4.37) with $B_1 = 1$, $B_2 = 4$, $\gamma = \lim_{t\to 0^+} \varphi(t)$ and $\delta = \lim_{t\to L^-} \frac{\varphi(t)}{U(t)}$ (recall (2.1)).

Whereas the fundamental function of $\Gamma_u^p(v)$ has an integral form if $p \in (0, \infty)$, the fundamental function of $\Gamma_u^\infty(v)$ is given by a supremum. This is the reason why the statement of Theorem 4.3 is somewhat more implicit than that of Theorem 4.1. Nevertheless, under some extra assumptions, which are often satisfied in applications, we can actually represent $\varphi$ in the form of (4.37) quite explicitly.

For example, if $v \in Q_U(0, L)$, $v$ is differentiable on $(0, L)$, and $\frac{v'}{u}$ is nonincreasing and locally absolutely continuous on $(0, L)$, then $\varphi = v$ and (4.37) holds with $B_1 = B_2 = 1$, $\gamma = \lim_{s\to 0^+} v(s)$, $\delta = \lim_{s\to L^-} \frac{v'(s)}{u(s)}$, and $\mathrm{d}\nu(t) = \left(-\frac{v'}{u}\right)'(t)\,\mathrm{d}t$. This can be proved by integrating by parts upon observing that $\lim_{s\to 0^+} U(s)\frac{v'(s)}{u(s)} = 0$. Moreover, under these extra assumptions, we can also take $B_1 = 1$, $B_2 = 2$, $\gamma = \lim_{s\to 0^+} v(s)$, $\delta = \lim_{s\to L^-} \frac{v(s)}{U(s)}$, and $\mathrm{d}\nu(t) = \left(-\frac{v'}{u}\right)'(t)\,\mathrm{d}t$ thanks to the fact that $\lim_{s\to L^-} \frac{v(s)}{U(s)} \ge \lim_{s\to L^-} \frac{v'(s)}{u(s)}$ and $\lim_{s\to L^-} \frac{v(s)}{U(s)} \le \frac{v(t)}{U(t)}$ for every $t \in (0, L)$.

***Remark 4.5*** In the setting of Theorem 4.3, if $0 < q < 1$, there is an alternative, equivalent expression for $A_7$ under the extra assumption

$$\xi(t) < \infty \quad \text{for every } t \in (0, L), \tag{4.46}$$

where $\xi$ is defined by (4.38) (cf. Remark 4.2). Namely, $A_7 \approx A_8$, where

$$A_8 = \left( \int_0^L \varphi(t)^{-q} \xi(t)^{-\frac{q}{1-q}} W^{\frac{q}{1-q}}(t) w(t)\,\mathrm{d}t \right)^{\frac{1}{q}}.$$

Indeed, the additional assumption implies that $\xi \in Q_{U^q}(0, L)$, and the desired equivalence then follows from (4.42) coupled with the fact that $C \approx A_7$, and Theorem 3.2 applied to $\widetilde{h} = \xi^{\frac{1}{1-q}}$, $\widetilde{\varrho} = U^{\frac{q}{1-q}}$, $\widetilde{\alpha} = \widetilde{\beta} = 0$, $\mathrm{d}\widetilde{\nu}(t) = W^{\frac{q}{1-q}}(t)w(t)U^{-\frac{q}{1-q}}(t)\,\mathrm{d}t$, $\widetilde{p} = 1-q$ and $\widetilde{f} = U^q\varphi^{-q}$ coupled with [5, Lemma 4.2.9].

The constants $A_7$ and $A_8$ need not, however, be equivalent if (4.46) is violated. In order to see this, suppose that (4.40) holds but (4.46) does not. Then $\xi \equiv \infty$ on $(0, L)$, and thus $A_7 = \infty$ but $A_8 = 0$. This detail was probably overlooked in [8, Theorem 1.8].

***Remark 4.6*** We conclude this paper by outlining a possible application of our results. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, $m \in \mathbb{N}$, $m < n$, and $d \in (0, n-m)$. Let $\mu$ be a $d$-upper Ahlfors measure on $\overline{\Omega}$, that is, a finite Borel measure on $\overline{\Omega}$ such that

$$\sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \frac{\mu\big(B(x,r)\cap\overline{\Omega}\big)}{r^d} < \infty,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ of radius $r$ centered at $x$. Notable examples of such measures are $d$-dimensional Hausdorff measures on $d$-dimensional sets. If $X$ and $Y$ are rearrangement-invariant function spaces satisfying

$$\left\| \int_{t^{\frac{n}{d}}}^1 f^*(s)s^{-1+\frac{m}{n}}\,\mathrm{d}s \right\|_{Y(0,1)} \lesssim \|f\|_{X(0,1)}$$

and

$$\left\| t^{-\frac{m}{n-d}} \int_0^{t^{\frac{n}{d}}} f^*(s)s^{-1+\frac{m}{n-d}}\,\mathrm{d}s \right\|_{Y(0,1)} \lesssim \|f\|_{X(0,1)}$$

for every $f \in \mathfrak{M}_+(0,1)$, then [4, Theorem 5.1] ensures boundedness of a linear Sobolev trace operator

$$\mathrm{T}\colon W^m X(\Omega) \to Y^{\langle\frac{n-d}{m}\rangle}(\overline{\Omega}, \mu), \tag{4.47}$$

where $W^m X(\Omega)$ is a Sobolev-type space of $m$th order built upon $X(\Omega)$ and $Y^{\langle\frac{n-d}{m}\rangle}(\overline{\Omega}, \mu)$ is the rearrangement-invariant function space whose norm is defined as

$$\|u\|_{Y^{\langle\frac{n-d}{m}\rangle}(\overline{\Omega},\mu)} = \left\| \Big(\big(g_u^{\frac{n-d}{m}}\big)^{**}\Big)^{\frac{m}{n-d}} \right\|_{Y(0,1)}, \quad u \in \mathfrak{M}_\mu(\overline{\Omega}),$$

where $g_u(t) = u^*(\mu(\overline{\Omega})t)$, $t \in (0,1)$.

Since a large number of customary rearrangement-invariant function spaces are instances of Lorentz $\Lambda$-spaces, it is of interest to know how to apply this result of [4] when $Y$ is a Lorentz $\Lambda$-space. Note that despite this assumption the resulting target space in (4.47) need not be equivalent to a $\Lambda$-space (see [14]).

Supposing that $Y = \Lambda^p(v)$, one might ask if the target space in (4.47) may be replaced by $\Lambda^q(w)$. The answer is positive if $Y^{\langle \frac{n-d}{m} \rangle}(\overline{\Omega}, \mu)$ embeds in $\Lambda^q(w)$, that is, if there is a constant $C > 0$ such that

$$\left(\int_0^L (f^*(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \leq C\left(\int_0^L \left(\frac{1}{t}\int_0^t (f^*(s))^{\frac{n-d}{m}}\,\mathrm{d}s\right)^{\frac{pm}{n-d}} v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

holds for every $f \in \mathfrak{M}_+(0, L)$ with $L = \mu(\overline{\Omega})$. This is where the main results of this paper come into play because, by a standard rescaling argument, the optimal constant $C$ in the inequality above satisfies

$$C^{\frac{n-d}{m}} = \sup_{\|f\|_{\Gamma^{\widetilde{p}}(v)} \leq 1} \|f\|_{\Lambda^{\widetilde{q}}(w)},$$

where $\widetilde{p} = \frac{pm}{n-d}$ and $\widetilde{q} = \frac{qm}{n-d}$. Notably, the absence of the "non-degeneracy" restrictions is crucial because $L < \infty$.

Furthermore, the results obtained in this paper could also be used to improve some of the compactness results for the Sobolev trace operator (4.47) obtained in [3, Theorem 5.3].

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## Authors and Affiliations

**Martin Křepela**[1] · **Zdeněk Mihula**[1,2] · **Hana Turčinová**[2]

Zdeněk Mihula
mihulzde@fel.cvut.cz; mihulaz@karlin.mff.cuni.cz

Martin Křepela
martin.krepela@fel.cvut.cz

Hana Turčinová
turcinova@karlin.mff.cuni.cz

1 Faculty of Electrical Engineering, Department of Mathematics, Czech Technical University in Prague, Technická 2, 166 27 Praha 6, Czech Republic

2 Faculty of Mathematics and Physics, Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

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# Weighted Inequalities for a Superposition of the Copson Operator and the Hardy Operator

**Amiran Gogatishvili**[1] · **Zdeněk Mihula**[2,3] · **Luboš Pick**[3] · **Hana Turčinová**[3] · **Tuğçe Ünver**[1,4]

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## Abstract

We study a three-weight inequality for the superposition of the Hardy operator and the Copson operator, namely

$$\left(\int_a^b\left(\int_t^b\left(\int_a^s f(\tau)^p v(\tau)\,d\tau\right)^{\frac{q}{p}} u(s)\,ds\right)^{\frac{r}{q}} w(t)\,dt\right)^{\frac{1}{r}} \le C\int_a^b f(t)\,dt,$$

in which $(a,b)$ is any nontrivial interval, $q,r$ are positive real parameters and $p \in (0,1]$. A simple change of variables can be used to obtain any weighted $L^p$-norm with $p \ge 1$ on the right-hand side. Another simple change of variables can be used to equivalently turn this inequality into the one in which the Hardy and Copson operators swap their positions. We focus on characterizing those triples of weight functions

Dedicated to the 80th anniversary of Professor Stefan Samko.

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Tuğçe Ünver
unver@math.cas.cz ; tugceunver@kku.edu.tr

Extended author information available on the last page of the article

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$(u, v, w)$ for which this inequality holds for all nonnegative measurable functions $f$ with a constant independent of $f$. We use a new type of approach based on an innovative method of discretization which enables us to avoid duality techniques and therefore to remove various restrictions that appear in earlier work. This paper is dedicated to Professor Stefan Samko on the occasion of his 80th birthday.

**Keywords** Weighted Hardy inequality · Superposition of operators · Copson operator · Hardy operator

**Mathematics Subject Classification** 26D10

## 1 Introduction and the Main Result

The main purpose of this paper is to introduce a new line of argument which enables one to obtain a previously unavailable characterization of the validity of certain specific inequalities involving superposition of integral operators of Copson and Hardy type and three weight functions.

More precisely, given $a, b \in [-\infty, \infty]$, $a < b$, and parameters $q, r \in (0, \infty)$ and $p \in (0, 1]$, we characterize all triples $(u, v, w)$ of *weights* (i.e. positive measurable functions) on $(a, b)$ such that there exists a constant $C > 0$ with which the inequality

$$\left(\int_a^b \left(\int_t^b \left(\int_a^s f(\tau)^p v(\tau)\, d\tau\right)^{\frac{q}{p}} u(s)\, ds\right)^{\frac{r}{q}} w(t)\, dt\right)^{\frac{1}{r}} \le C \int_a^b f(t)\, dt \tag{1}$$

holds for every nonnegative measurable function $f$ on $(a, b)$. Let us note that the restriction $p \in (0, 1]$ is natural and does not cause any weakness. Indeed, the inequality is obviously impossible without it as, if $p > 1$, one can always easily construct a function $f$ that makes the integral on the left diverge while keeping the right-hand side finite.

The inequality (1), as a certain "mother figure", immediately paves the way to many other important inequalities. For instance, one can easily swap the order of the two inner integral operators and obtain the inequality

$$\left(\int_a^b \left(\int_a^t \left(\int_s^b f(\tau)^p v(\tau)\, d\tau\right)^{\frac{q}{p}} u(s)\, ds\right)^{\frac{r}{q}} w(t)\, dt\right)^{\frac{1}{r}} \le C \int_a^b f(t)\, dt \tag{2}$$

instead of (1). This is achieved by a simple change of variables $\tau \mapsto -\tau$ in the innermost integral on the left side of (1) and following the forced changes from thereafter. Similarly, one can turn (1) to the inequality

$$\left(\int_a^b \left(\int_t^b \left(\int_a^s f(\tau)\, d\tau\right)^{q} u(s)\, ds\right)^{\frac{r}{q}} w(t)\, dt\right)^{\frac{1}{r}} \le C \left(\int_a^b f(t)^p v(t)\, dt\right)^{\frac{1}{p}} \tag{3}$$

with $p \geq 1$ by performing the replacements (in this order) $f \mapsto f^{\frac{1}{p}} v^{-\frac{1}{p}}$, $v \mapsto v^{-p}$, $q \mapsto qp$, $r \mapsto rp$ and, finally, $p \mapsto \frac{1}{p}$, in (1). Our characterization of (1) thus immediately yields necessary and sufficient conditions for (2), (3), and their various combinations.

The key innovation is contained in methods of proofs which are based on new discretization techniques that require neither duality methods nor nondegeneracy conditions on weights.

In the theory of weighted inequalities, questions involving iterations of operators have recently been constituting the cutting edge. The subject has been rather fashionable for some time, mainly because inequalities involving compositions of operators, on the one hand, have an impressive array of important applications, while, on the other hand, are quite difficult to handle.

There is plenty of motivation for studying weighted inequalities for a composition of operators, and it pours down from various sources, rather different in spirit. A notable one is the theory of Sobolev-type embeddings where, during the last two decades, various forms of the so-called reduction principles have flourished. The reduction principle is a powerful method which establishes an, perhaps somewhat surprising, equivalence between a difficult problem involving differential operators in several variables, such as a Sobolev-type embedding, and a weighted inequality for an integral operator acting on functions defined on an interval. For the first-order embedding, this is usually achieved by an effective use of some sort of the Pólya–Szegő principle, and the resulting operator is then always a weighted Copson operator.

For higher-order embeddings, however, the Pólya–Szegő principle does not work, and one needs some new way of argumentation. Here, once again, approaches vary. For Euclidean–Sobolev embeddings (in which functions are defined on a sufficiently regular subdomain of the Euclidean ambient space $\mathbb{R}^n$ endowed with the Lebesgue measure), an effective use of interpolation theory leads to satisfactory results [26]. However, when the underlying domain is not sufficiently regular, or, for instance, when $\mathbb{R}^n$ is replaced by the Gauss space $(\mathbb{R}^n, \gamma_n)$, which is still $\mathbb{R}^n$, but endowed with the Gauss probability measure

$$\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}\, dx,$$

interpolation methods turn out to be ineffective and have to be replaced by something else. An extremely efficient argument in this matter, based on the isoperimetric inequality combined with an iteration technique, was developed in [8], and later applied again, this time to higher-order trace embeddings, in [7].

Now here comes the interesting part. While for some specific weights (typically for power weights that occur in reduction principles for Euclidean–Sobolev embeddings) the iterated operator is pointwise equivalent to a suitable single Copson operator [26], this is impossible in general (for example for Euclidean–Sobolev embeddings on bad enough domains or for Gaussian–Sobolev embeddings this approach fails [6]). In result, after performing the reduction principle, one has to grapple either with a kernel operator, or with a superposition of two or more operators [8]. Specifically, superposition of integral operators of different type (Hardy vs. Copson) is encountered for example when operators obtained from the reduction principle are applied to one of many operator-induced function spaces. The simplest examples of these are spaces whose norm involves the operation of the maximal nonincreasing rearrangement such as weak spaces, Marcinkiewicz spaces, etc., but there are more sophisticated ones which are also important.

Another variety of applications in a completely different direction can be found in the theory of function spaces and interpolation theory. These shelter, among others, questions concerning sharp embeddings between important structures [19, 33], Köthe duals of function spaces [17, 20, 37], inequalities restricted to cones of functions such as those of monotone or concave functions [15, 16, 24], or inequalities involving bilinear and multilinear operators [3]. The blocking technique appearing in [24] was also independently developed and applied for integral Hardy-type inequalities in [22].

Several results were obtained recently for iterations of operators of identical type, however always under some rather unpleasant restrictions.

One of the earliest treatments of iteration of identical operators was most likely carried out in [4]. The authors consider an $n$-dimensional problem and using radial weights they reduce it to the iterated Hardy inequality and treat some particular cases of parameters by discretization. Later in [13], inequalities involving Hardy–Hardy or Hardy–Copson iteration were fully characterized. Owing to the fact that a reduction technique was used, the characterization obtained is more complicated and nonstandard. In [25] both cases of iterations (Hardy–Hardy and Hardy–Copson) involving a kernel and using a different discretization are considered, in the simplest case of parameters, and characterization is obtained. Recently, iteration of Copson operators was treated in [27], restricted to nondegenerate weights.

Inequalities for superposition of the Copson and Hardy operators were studied in [18]; however, the results obtained there were restricted to nondegenerate weights. Next, in [31], a three-weight inequality was characterized, motivated by a specific inequality in which a weighted norm of a mean value is compared to that of the derivative of a given function. Techniques of proofs in that paper are related to [36]. The result was later revisited several times, see e.g. [3, 32], where also further applications to bilinear operators are pointed out. Particular cases and related topics had been studied earlier, see for instance [11] or [12] for $p = 1$, or [16] for $p = \infty$, or [36] and [17] for special cases of weights. The subject is also intimately related to the new type of spaces governed by operator-induced norms that have been appearing recently in connection with various other tasks, notably from embeddings of Sobolev spaces endowed with slowly-decaying upper Ahlfors measures [9, 10, 37].

Let us recall that discretization techniques have been around for some time. In the late 1980's and early 1990's they proved to be very useful for example in the theory of one-sided operators and ergodic theory, see e.g. [29, 30, 34, 35] and all the huge amount of subsequent work. In the early 2000's, they were used in order to solve some problems in the theory of classical Lorentz spaces that had been open for long time, see [5, 14]. Later various authors spent considerable efforts in order to chip away certain technical obstacles such as nondegeneracy assumptions with varying success, consider e.g. [11] or [28] and the references therein. However, this research is far from being complete.

Let us note that the current paper is closely related to the project [21], in which some of the new discretization methods presented here will be applied to a different problem, namely to an inequality involving the Hardy operator on one side and the Copson operator on the other, cf. [5].

We shall now present our principal result, that is, a complete characterization of (1). We will formulate it in the form of a single theorem. We shall need the following notation. For $a, b \in [-\infty, \infty]$, $a < b$, and $p \in (0, 1]$, let

$$V_p(a,b) := \begin{cases} \left(\int_a^b v^{\frac{1}{1-p}}\right)^{\frac{1-p}{p}} & \text{if } 0 < p < 1, \\ \operatorname*{ess\,sup}_{t\in(a,b)} v(t) & \text{if } p = 1. \end{cases}$$

Our main result is:

**Theorem A** *Let $a, b \in [-\infty, \infty]$, $a < b$, $q, r \in (0, \infty)$, $p \in (0, 1]$, and let $u, v, w$ be weights on $(a, b)$. Then there exists a constant $C > 0$ such that the inequality (1) holds for all nonnegative measurable functions $f$ on $(a, b)$ if and only if one of the following conditions is satisfied:*

(i) $1 \le r$, $1 \le q$,

$$C_1 := \sup_{t\in(a,b)} \left(\int_a^t w(s)\,ds\right)^{\frac{1}{r}} \operatorname*{ess\,sup}_{s\in(t,b)} \left(\int_s^b u(\tau)\,d\tau\right)^{\frac{1}{q}} V_p(a,s) < \infty$$

*and*

$$C_2 := \sup_{t\in(a,b)} \left(\int_t^b w(s)\left(\int_s^b u(\tau)\,d\tau\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}} V_p(a,t) < \infty;$$

(ii) $1 \le r$, $q < 1$, $C_2 < \infty$ *and*

$$C_3 := \sup_{t\in(a,b)} \left(\int_a^t w(s)\,ds\right)^{\frac{1}{r}} \left(\int_t^b \left(\int_s^b u(\tau)\,d\tau\right)^{\frac{q}{1-q}} \times u(s) V_p(a,s)^{\frac{q}{1-q}}\,ds\right)^{\frac{1-q}{q}} < \infty;$$

(iii) $r < 1$, $1 \le q$,

$$C_4 := \left(\int_a^b \left(\int_a^t w(s)\,ds\right)^{\frac{r}{1-r}} w(t) \operatorname*{ess\,sup}_{s\in(t,b)} \left(\int_s^b u(\tau)\,d\tau\right)^{\frac{r}{q(1-r)}} \times V_p(a,s)^{\frac{r}{1-r}}\,dt\right)^{\frac{1-r}{r}} < \infty$$

*and*

$$C_5 := \left( \int_a^b \left( \int_t^b w(s) \left( \int_s^b u(\tau)\, d\tau \right)^{\frac{r}{q}} ds \right)^{\frac{r}{1-r}} w(t) \left( \int_t^b u(\tau)\, d\tau \right)^{\frac{r}{q}} V_p(a,t)^{\frac{r}{1-r}}\, dt \right)^{\frac{1-r}{r}} < \infty;$$

(iv) $r < 1$, $q < 1$, $C_5 < \infty$ *and*

$$C_6 := \left( \int_a^b \left( \int_a^t w(s)\, ds \right)^{\frac{r}{1-r}} w(t) \left( \int_t^b \left( \int_s^b u(\tau)\, d\tau \right)^{\frac{q}{1-q}} u(s) \times V_p(a,s)^{\frac{q}{1-q}}\, ds \right)^{\frac{r(1-q)}{q(1-r)}} dt \right)^{\frac{1-r}{r}} < \infty.$$

*Moreover, the best constant $C$ in the inequality (1) satisfies*

$$C \approx \begin{cases} C_1 + C_2 & \textit{in the case (i)}, \\ C_2 + C_3 & \textit{in the case (ii)}, \\ C_4 + C_5 & \textit{in the case (iii)}, \\ C_5 + C_6 & \textit{in the case (iv)}. \end{cases} \tag{4}$$

The proof is based on a new type of discretization which avoids the use of any kind of duality principle, enabling us thereby to obtain the result in the required generality.

Theorem A is proved in Sect. 3, along with a side theorem which gives another characterization of (1). Key ingredients of the proofs are collected in Sect. 2.

## 2 Background Discretization Results

In this section we shall establish the background discretization material that will be needed in the proof of the main result. We first fix notation and conventions used in this paper. We denote by LHS($*$) and RHS($*$) the left-hand side and right-hand side of the inequality numbered by $*$, respectively. We adhere to the usual convention that $\frac{1}{\infty} = 0 \cdot \infty = \frac{\infty}{\infty} = \frac{0}{0} = 0$. We denote by $\mathcal{M}^+(c,d)$ the set of all nonnegative measurable functions on $(c,d)$. By increasing we mean strictly increasing. Finally, the small letters $i$ and $k$ are always integers, which are reserved for indices. In particular, when we write $N \le k \le M$, in which $N$ and $M$ can be $-\infty$ and $\infty$, respectively, we mean $k \in \mathbb{Z}$, $N \le k \le M$. This convention is accordingly modified for similar inequalities and the index $i$ in the obvious way.

**Definition 1** Let $N \in \mathbb{Z} \cup \{-\infty\}$, $M \in \mathbb{Z} \cup \{+\infty\}$, $N < M$, and $\{a_k\}_{k=N}^M$ be a sequence of positive numbers. We say that $\{a_k\}_{k=N}^M$ is *strongly increasing* if

$$\inf \left\{ \frac{a_{k+1}}{a_k}, \quad N \le k < M \right\} > 1.$$

Our approach is based on a fine discretization of the inequality in question. Before we start doing that, we need some new information of a general kind from the discrete world. The following lemma is contained in the manuscript [21], where it is also proved. However, since the manuscript is not publicly available yet, we include its proof here for the reader's convenience.

**Lemma 2** *Let $s > 0$, $M \in \mathbb{Z} \cup \{+\infty\}$. Assume that $\{a_k\}_{k=-\infty}^M$ and $\{b_k\}_{k=-\infty}^M$ are sequences of nonnegative numbers such that $\{b_k\}_{k=-\infty}^M$ is nondecreasing. Then*

$$\sum_{k=-\infty}^{M} a_k \Bigg(\sum_{i=k}^{M} a_i\Bigg)^s b_k \approx \sum_{k=-\infty}^{M} (b_k - b_{k-1}) \Bigg(\sum_{i=k}^{M} a_i\Bigg)^{s+1} + \Bigg(\sum_{k=-\infty}^{M} a_k\Bigg)^{s+1} \lim_{k\to-\infty} b_k, \tag{5}$$

*in which the multiplicative constants depend only on $s$.*

***Proof*** First, assume that $\lim_{k\to-\infty} b_k > 0$. Thanks to this assumption, we have that

$$\lim_{N\to-\infty} \Bigg(\sum_{k=N}^{M} a_k\Bigg)^{s+1} b_N = \Bigg(\sum_{k=-\infty}^{M} a_k\Bigg)^{s+1} \lim_{k\to-\infty} b_k,$$

whether the series converges or diverges. Let $N \in \mathbb{Z}$, $N < M$. By virtue of Abel's lemma, we have that

$$\sum_{k=N}^{M} c_k b_k = \sum_{k=N+1}^{M} (b_k - b_{k-1}) \sum_{i=k}^{M} c_i + \Bigg(\sum_{k=N}^{M} c_k\Bigg) b_N \tag{6}$$

for every sequence $\{c_k\}_{k=N}^M$ of nonnegative numbers. Set $c_k = a_k \big(\sum_{i=k}^M a_i\big)^s$ for $k \in \mathbb{Z}$, $N \le k \le M$. Applying power rules (cf. e.g. [2, Lemmas 1 and 1′]), we get

$$\begin{aligned}\sum_{k=N}^{M} a_k \Bigg(\sum_{i=k}^{M} a_i\Bigg)^s b_k &= \sum_{k=N+1}^{M} (b_k - b_{k-1}) \sum_{i=k}^{M} a_i \Big(\sum_{j=i}^{M} a_j\Big)^s + \Bigg(\sum_{k=N}^{M} a_k \Big(\sum_{i=k}^{M} a_i\Big)^s\Bigg) b_N \\ &\approx \sum_{k=N+1}^{M} (b_k - b_{k-1}) \Bigg(\sum_{i=k}^{M} a_i\Bigg)^{s+1} + \Bigg(\sum_{k=N}^{M} a_k\Bigg)^{s+1} b_N,\end{aligned}$$

in which the multiplicative constants depend only on $s$. By letting $N$ go to $-\infty$, we obtain (5).

Second, assume that $\lim_{k\to-\infty} b_k = 0$. It follows that $b_k = \sum_{i=-\infty}^{k} (b_i - b_{i-1})$ for every $k \in \mathbb{Z}$, $k \le M$. Therefore, we have that

$$\sum_{k=-\infty}^{M} c_k b_k = \sum_{k=-\infty}^{M} (b_k - b_{k-1}) \sum_{i=k}^{M} c_i$$

for every sequence $\{c_k\}_{k=-\infty}^M$ of nonnegative numbers. By taking $c_k = a_k\big(\sum_{i=k}^M a_i\big)^s$ for $k \in \mathbb{Z}$, $k \le M$, and using the power rules as above, we obtain (5). $\square$

The proof of the following lemma can be found in [23, Proposition 2.1] and [14, Lemmas 3.2–3.4].

**Lemma 3** *Let $N \in \mathbb{Z} \cup \{-\infty\}$, $M \in \mathbb{Z} \cup \{+\infty\}$, $N < M$, $\beta > 0$, and let $\{a_k\}_{k=N}^M$ and $\{\varrho_k\}_{k=N}^M$ be sequences of positive numbers. If $\{\varrho_k\}_{k=N}^M$ is nondecreasing, then*

$$\sup_{N \le k \le M} \varrho_k \sup_{k \le i \le M} a_i = \sup_{N \le k \le M} \varrho_k a_k. \tag{7}$$

*If $\{\varrho_k\}_{k=N}^M$ is strongly increasing, then*

$$\sum_{k=N}^M \varrho_k \left(\sum_{i=k}^M a_i\right)^\beta \approx \sum_{k=N}^M \varrho_k a_k^\beta, \tag{8}$$

$$\sum_{k=N}^M \varrho_k \sup_{k \le i \le M} a_i \approx \sum_{k=N}^M \varrho_k a_k, \tag{9}$$

*and*

$$\sup_{N \le k \le M} \varrho_k \left(\sum_{i=k}^M a_i\right)^\beta \approx \sup_{N \le k \le M} \varrho_k a_k^\beta. \tag{10}$$

*Moreover, the multiplicative constants depend only on* $\inf\left\{\frac{\varrho_{k+1}}{\varrho_k}, \quad N \le k < M\right\}$ *and $\beta$.*

We should note that in [14], (7) is formulated when $\{\varrho_k\}_{k=N}^M$ is a strongly increasing sequence (and $N = -\infty$); however, the result is a consequence of the interchanging suprema and holds true even when $\{\varrho_k\}_{k=N}^M$ is just nondecreasing.

**Definition 4** Let $G$ be a positive continuous increasing function on $(a, b)$ such that $\lim_{t \to a^+} G(t) = 0$. Define

$$M = \inf\{k \in \mathbb{Z} : G(t) \le 2^k \text{ for every } t \in (a, b)\}$$

(if the set is empty, then $M = \infty$). An increasing sequence $\{x_k\}_{k=-\infty}^M \subset (a, b]$ such that $(a, b) \subset \bigcup_{k=-\infty}^M [x_{k-1}, x_k]$ is said to be the *discretizing sequence* of $G$ if it satisfies $G(x_k) = 2^k$ for every $k < M$.

We note that if $\lim_{t \to b^-} G(t) < \infty$, then $M < \infty$ and $x_M = b$, while, if $\lim_{t \to b^-} G(t) = \infty$, then $M = \infty$ and $\lim_{k \to \infty} x_k = b$. Furthermore, if $M < \infty$, then $2^{M-1} \le G(t) \le 2^M$ for every $t \in [x_{M-1}, b)$. Note that the discretizing sequence (as defined above) is unique, and so the definite article is justified. Finally, the Darboux property of continuous functions implies that the discretizing sequence exists for every $G$ as above.

For a locally integrable nonnegative function $w$ on $[a, b)$, we will use the notation

$$W(t) = \int_a^t w(s)\,ds, \quad t \in [a, b].$$

Note that the discretizing sequence for $W$ exists when $w$ is such a function.

Recall that if $M = \infty$, then $M - 1$ is interpreted as $\infty$.

**Lemma 5** *Let $\alpha \geq 0$. Assume that $w$ is a weight on $(a, b)$, $\{x_k\}_{k=-\infty}^{M}$ is the discretizing sequence of $W$ and $h$ is a nonnegative nonincreasing function on $(a, b)$. Then*

$$\int_a^b W(t)^\alpha w(t) h(t)\,dt \approx \sum_{k=-\infty}^{M-1} 2^{k(\alpha+1)} h(x_k) \tag{11}$$

*holds, in which the equivalence constants depend only on $\alpha$.*

***Proof*** The monotonicity of $h$ and properties of the discretizing sequence $\{x_k\}_{k=-\infty}^{M}$ give

$$\begin{aligned}\int_a^b h(t)W(t)^\alpha w(t)\,dt &= \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} h(t)W(t)^\alpha w(t)\,dt \\ &\lesssim \sum_{k=-\infty}^{M-1} h(x_k) \int_{x_k}^{x_{k+1}} d\big[W(t)^{\alpha+1}\big] \\ &\approx \sum_{k=-\infty}^{M-1} 2^{k(\alpha+1)} h(x_k),\end{aligned}$$

and, conversely,

$$\begin{aligned}\int_a^b h(t)W(t)^\alpha w(t)\,dt &\geq \sum_{k=-\infty}^{M-1} \int_{x_{k-1}}^{x_k} h(t)W(t)^\alpha w(t)\,dt \\ &\gtrsim \sum_{k=-\infty}^{M-1} h(x_k) \int_{x_{k-1}}^{x_k} d\big[W(t)^{\alpha+1}\big] \\ &\approx \sum_{k=-\infty}^{M-1} 2^{k(\alpha+1)} h(x_k).\end{aligned}$$

Therefore, the statement follows. □

Having established necessary background material, we can now take the first step towards an effective discretization of the inequality (1).

**Proposition 6** *Let $0 < p \le 1$, $0 < q, r < \infty$ and let $u, v, w$ be weights on $(a, b)$. Assume that $\{x_k\}_{k=-\infty}^{M}$ is the discretizing sequence of $W$. Then there exists a positive constant $C$ such that the inequality (1) holds for all nonnegative measurable $f$ on $(a, b)$ if and only if there exist positive constants $C'$ and $C''$ such that*

$$\left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^{x_{k+1}} \left(\int_{x_k}^{t} f^p(s)v(s)\,ds\right)^{\frac{q}{p}} u(t)\,dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \le C' \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} f(t)\,dt \tag{12}$$

*and*

$$\left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{a}^{x_k} f^p(t)v(t)\,dt\right)^{\frac{r}{p}} \left(\int_{x_k}^{b} u(t)\,dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \le C'' \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} f(t)\,dt \tag{13}$$

*for all nonnegative measurable functions $f$ on $(a, b)$. Moreover, the best constants $C$, $C'$ and $C''$ in (1), (12) and (13), respectively, satisfy $C \approx C' + C''$.*

***Proof*** Applying (11) with $\alpha = 0$ and then using (8), we obtain

$$\begin{aligned}
LHS(1) &\approx \left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^{x_{k+1}} \left(\int_{a}^{t} f^p v\right)^{\frac{q}{p}} u(t)\,dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \\
&\approx \left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^{x_{k+1}} \left(\int_{x_k}^{t} f^p v\right)^{\frac{q}{p}} u(t)\,dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \\
&\quad + \left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{a}^{x_k} f^p v\right)^{\frac{r}{p}} \left(\int_{x_k}^{x_{k+1}} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \\
&\approx \left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^{x_{k+1}} \left(\int_{x_k}^{t} f^p v\right)^{\frac{q}{p}} u(t)\,dt\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \\
&\quad + \left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{a}^{x_k} f^p v\right)^{\frac{r}{p}} \left(\int_{x_k}^{b} u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}.
\end{aligned}$$

In the last equivalence we have used the fact that either $\int_a^{x_{M-1}} f^p v = 0$, where $x_{M-1}$ is to be interpreted as $b$ if $M = \infty$, or there is $N \in \mathbb{Z} \cup \{-\infty\}$, $N \le M - 1$, such that $\int_a^{x_k} f^p v = 0$ for every $k < N$ and $\left\{2^k \left(\int_a^{x_k} f^p v\right)^{\frac{r}{p}}\right\}_{k=N}^{M-1}$ is a strongly increasing sequence (unless $N = M - 1 < \infty$, which is a trivial case). The assertion follows. □

The next step is based on the saturation of Hardy inequalities and embeddings of weighted Lebesgue spaces on the intervals determined by a discretizing sequence.

**Proposition 7** *Let $0 < p \le 1$, $0 < q, r < \infty$ and let $u, v, w$ be weights on $(a, b)$. Assume that $\{x_k\}_{k=-\infty}^{M}$ is the discretizing sequence of $W$. For every $k \in \mathbb{Z}$, $k \le M$, we denote*

$$A_k := \sup_{g\in\mathcal{M}^+(x_{k-1},x_k)} \frac{\left(\int_{x_{k-1}}^{x_k} g(t)^p v(t)\,dt\right)^{\frac{1}{p}}}{\int_{x_{k-1}}^{x_k} g(t)\,dt} \tag{14}$$

*and*

$$B_k := \sup_{h\in\mathcal{M}^+(x_{k-1},x_k)} \frac{\left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} h(s)^p v(s)\,ds\right)^{\frac{q}{p}} u(t)\,dt\right)^{\frac{1}{q}}}{\int_{x_{k-1}}^{x_k} h(t)\,dt}. \tag{15}$$

*Then there exists a positive constant $C$ such that the inequality (1) holds for all non-negative measurable functions $f$ on $(a, b)$ if and only if there exist positive constants $C'$, $C''$ such that the inequalities*

$$\left(\sum_{k=-\infty}^{M-1} 2^k a_k^r B_{k+1}^r\right)^{\frac{1}{r}} \le C' \sum_{k=-\infty}^{M-1} a_k \tag{16}$$

*and*

$$\left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^{b} u(t)\,dt\right)^{\frac{r}{q}} \left(\sum_{j=-\infty}^{k} a_j^p A_j^p\right)^{\frac{r}{p}}\right)^{\frac{1}{r}} \le C'' \sum_{k=-\infty}^{M-1} a_k \tag{17}$$

*hold for every sequence $\{a_k\}_{k=-\infty}^{M-1}$ of nonnegative numbers. Moreover, the best constants $C$, $C'$ and $C''$ in (1), (16) and (17), respectively, satisfy $C \approx C' + C''$.*

***Proof*** Assume that (12) holds. By (15), there exist nonnegative measurable functions $h_k$, $k \le M-1$, on $(a, b)$ such that $\operatorname{supp} h_k \subset [x_k, x_{k+1}]$, $\int_{x_k}^{x_{k+1}} h_k = 1$, and

$$\left(\int_{x_k}^{x_{k+1}} \left(\int_{x_k}^{t} h_k^p v\right)^{\frac{q}{p}} u(t)\,dt\right)^{\frac{1}{q}} \gtrsim B_{k+1}.$$

Thus, given $\{a_m\}_{m=-\infty}^{M-1}$ and inserting $f = \sum_{m=-\infty}^{M-1} a_m h_m$ into (12), we get (16). Conversely, (12) follows at once from (16) on setting $a_k = \int_{x_k}^{x_{k+1}} f$ for $k \in (-\infty, M-1)$.

Similarly, by (14), there exist nonnegative measurable functions $g_k$, $k \le M-1$, on $(a, b)$ such that $\operatorname{supp} g_k \subset [x_{k-1}, x_k]$, $\int_{x_{k-1}}^{x_k} g_k = 1$, and

$$\left(\int_{x_{k-1}}^{x_k} g_k^p v\right)^{\frac{1}{p}} \gtrsim A_k.$$

Thus, given $\{a_m\}_{m=-\infty}^{M-1}$ and inserting $f = \sum_{m=-\infty}^{M-1} a_m g_m$ into (13), (17) follows. Conversely, inserting $a_k = \int_{x_{k-1}}^{x_k} f$ in (17) gives (13).

The assertion now directly follows from Proposition 6. □

## 3 Proofs

We begin this section with a theorem of auxiliary nature, albeit interesting on its own, which yields a discrete characterization of the inequality in question. We will then use it as the last step towards the proof of Theorem A.

**Theorem 8** *Let $0 < p \le 1$, $0 < q, r < \infty$ and let $u, v, w$ be weights on $(a, b)$. Let $\{x_k\}_{k=-\infty}^{M}$ be the discretizing sequence of $W$. Then there exists a constant $C > 0$ such that the inequality (1) holds for all nonnegative measurable functions $f$ on $(a, b)$ if and only if one of the following conditions is satisfied:*

(i) $1 \le r$, $1 \le q$,

$$A_1^* := \sup_{k \le M-1} 2^{\frac{k}{r}} \operatorname*{ess\,sup}_{t \in (x_k, x_{k+1})} \left( \int_t^{x_{k+1}} u(s)\, ds \right)^{\frac{1}{q}} V_p(x_k, t) < \infty$$

*and*

$$B_1^* := \sup_{k \le M-1} \left( \sum_{i=k}^{M-1} 2^i \left( \int_{x_i}^{b} u(t)\, dt \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} V_p(a, x_k) < \infty;$$

(ii) $1 \le r$, $q < 1$, $B_1^* < \infty$ *and*

$$A_2^* := \sup_{k \le M-1} 2^{\frac{k}{r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_t^{x_{k+1}} u(s)\, ds \right)^{\frac{q}{1-q}} u(t) V_p(x_k, t)^{\frac{q}{1-q}}\, dt \right)^{\frac{1-q}{q}} < \infty;$$

(iii) $r < 1$, $1 \le q$,

$$A_3^* := \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t \in (x_k, x_{k+1})} \left( \int_t^{x_{k+1}} u(s)\, ds \right)^{\frac{r}{q(1-r)}} V_p(x_k, t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} < \infty,$$

*and*

$$B_2^* := \left( \sum_{k=-\infty}^{M-1} 2^k \left( \int_{x_k}^{b} u(t)\, dt \right)^{\frac{r}{q}} \left( \sum_{i=k}^{M-1} 2^i \left( \int_{x_i}^{b} u(t)\, dt \right)^{\frac{r}{q}} \right)^{\frac{r}{1-r}} \right.$$
$$\left. \times V_p(a, x_k)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} < \infty;$$

(iv) $r < 1$, $q < 1$, $B_2^* < \infty$ *and*

$$A_4^* := \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_t^{x_{k+1}} u(s)\, ds \right)^{\frac{q}{1-q}} u(t) \right.\right.$$
$$\left.\left. \times V_p(x_k, t)^{\frac{q}{1-q}}\, dt \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}} < \infty.$$

*Moreover, the best constant $C$ in the inequality (1) satisfies*

$$C \approx \begin{cases} A_1^* + B_1^* & \text{in the case (i),} \\ A_2^* + B_1^* & \text{in the case (ii),} \\ A_3^* + B_2^* & \text{in the case (iii),} \\ A_4^* + B_2^* & \text{in the case (iv).} \end{cases}$$

***Proof*** It follows from Proposition 7 that the best constant $C$ in (1) satisfies $C \approx C' + C''$, where $C'$ and $C''$ are the best constants in (16) and (17). Next, we obtain an appropriate characterization of $C'$ by combining a discrete version of the Landau resonance theorem (cf. e.g. [14, Proposition 4.1]) with the classical Hardy inequality. Finally, an appropriate two-sided estimate of $C''$ can be obtained by combining the known characterization of a discrete Hardy inequality (cf. e.g. [1, Theorem 1] or [24, Theorem 9.2])) with the classical duality expression of the norm in a weighted Lebesgue space. □

***Proof of Theorem A*** First of all, note that the optimal constant $C$ in (1) is equal to $\infty$ if there is $t_0 \in (a, b)$ such that $W(t_0) = \infty$, and so is RHS(4); hence the theorem is trivially true in this pathological case. Therefore, we may assume that $W(t) < \infty$ for every $t \in (a, b)$. Let $\{x_k\}_{k=-\infty}^{M}$, where $M \in \mathbb{Z} \cup \{\infty\}$, be the discretizing sequence of $W$.

(i) Let $p \le 1 \le r$, $1 \le q$. We have from Theorem 8(i) that $C \approx A_1^* + B_1^*$. Define

$$\tilde{A}_1 := \sup_{k \le M-1} 2^{\frac{k}{r}} \operatorname*{ess\,sup}_{t \in (x_k, b)} \left( \int_t^b u \right)^{\frac{1}{q}} V_p(a, t).$$

We will first show that $A_1^* + B_1^* \approx \tilde{A}_1 + B_1^*$. Since obviously $A_1^* \le \tilde{A}_1$, it is enough to prove that $\tilde{A}_1 \lesssim A_1^* + B_1^*$. Using (7), we obtain

$$\begin{aligned}
\tilde{A}_1 &= \sup_{k \le M-1} 2^{\frac{k}{r}} \sup_{k \le i \le M-1} \operatorname*{ess\,sup}_{t \in (x_i, x_{i+1})} \left( \int_t^b u \right)^{\frac{1}{q}} V_p(a, t) \\
&= \sup_{k \le M-1} 2^{\frac{k}{r}} \operatorname*{ess\,sup}_{t \in (x_k, x_{k+1})} \left( \int_t^b u \right)^{\frac{1}{q}} V_p(a, t) \\
&\approx \sup_{k \le M-1} 2^{\frac{k}{r}} \operatorname*{ess\,sup}_{t \in (x_k, x_{k+1})} \left( \int_t^{x_{k+1}} u \right)^{\frac{1}{q}} V_p(a, t) + \sup_{k \le M-2} 2^{\frac{k}{r}} \left( \int_{x_{k+1}}^b u \right)^{\frac{1}{q}} V_p(a, x_{k+1}).
\end{aligned}$$

Since

$$V_p(a,t) \approx V_p(a,x_k) + V_p(x_k,t) \quad \text{for every} t \in (x_k, x_{k+1}), \tag{18}$$

we in fact have

$$\begin{aligned}\tilde{A}_1 &\approx A_1^* + \sup_{k\le M-1} 2^{\frac{k}{r}} \left(\int_{x_k}^{x_{k+1}} u\right)^{\frac{1}{q}} V_p(a,x_k) + \sup_{k\le M-2} 2^{\frac{k}{r}} \left(\int_{x_{k+1}}^{b} u\right)^{\frac{1}{q}} V_p(a,x_{k+1})\\ &\lesssim A_1^* + \sup_{k\le M-1} 2^{\frac{k}{r}} \left(\int_{x_k}^{b} u\right)^{\frac{1}{q}} V_p(a,x_k) \le A_1^* + B_1^*,\end{aligned}$$

establishing the claim.

Next, we will show that $C_1 + C_2 \approx \tilde{A}_1 + B_1^*$. Observe first that

$$C_1 = \sup_{k\le M-1} \sup_{t\in(x_k,x_{k+1})} \left(\int_a^t w\right)^{\frac{1}{r}} \operatorname*{ess\,sup}_{s\in(t,b)} \left(\int_s^b u\right)^{\frac{1}{q}} V_p(a,s) \approx \tilde{A}_1.$$

On the other hand, fixing $k \in \mathbb{Z}$, $k < M$, we have that

$$\begin{aligned}\sum_{i=k}^{M-1} 2^i \left(\int_{x_i}^b u\right)^{\frac{r}{q}} &= 2^k \left(\int_{x_k}^b u\right)^{\frac{r}{q}} + \sum_{i=k+1}^{M-1} 2^i \left(\int_{x_i}^b u\right)^{\frac{r}{q}}\\ &\approx 2^k \left(\int_{x_k}^b u\right)^{\frac{r}{q}} + \sum_{i=k+1}^{M-1} \left(\int_{x_{i-1}}^{x_i} w\right)\left(\int_{x_i}^b u\right)^{\frac{r}{q}}\\ &\le 2^k \left(\int_{x_k}^b u\right)^{\frac{r}{q}} + \int_{x_k}^b w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt\end{aligned} \tag{19}$$

with equivalence constants independent of $k$. Hence, in view of (19), we have

$$\begin{aligned}B_1^* &\lesssim \sup_{k\le M-1} \left(\int_{x_k}^b w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}} V_p(a,x_k) + \sup_{k\le M-1} 2^{\frac{k}{r}} \left(\int_{x_k}^b u\right)^{\frac{r}{q}} V_p(a,x_k)\\ &\le \sup_{k\le M-1} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})} \left(\int_t^b w(s)\left(\int_s^b u\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}} V_p(a,t) + \tilde{A}_1\\ &\approx C_2 + C_1.\end{aligned} \tag{20}$$

Thus, we have $\tilde{A}_1 + B_1^* \lesssim C_1 + C_2$.

Conversely,

$$\int_{x_k}^b w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt = \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt \lesssim \sum_{i=k}^{M-1} 2^i \left(\int_{x_i}^b u\right)^{\frac{r}{q}}. \tag{21}$$

Consequently,

$$C_2 \approx \sup_{k\le M-1} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})} \left(\int_t^{x_{k+1}} w(s)\left(\int_s^b u\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}} V_p(a,t)$$
$$+ \sup_{k\le M-2} \left(\int_{x_{k+1}}^b w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}} V_p(a,x_{k+1}).$$

Hence, in view of (21), (7) and the fact that $\{x_k\}_{k=-\infty}^M$ is the discretizing sequence for $W$, we obtain

$$C_2 \lesssim \sup_{k\le M-1} 2^{\frac{k}{r}} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})} \left(\int_t^b u\right)^{\frac{1}{q}} V_p(a,t)$$
$$+ \sup_{k\le M-2} \left(\sum_{i=k+1}^{M-1} 2^i \left(\int_{x_i}^b u\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} V_p(a,x_{k+1})$$
$$\lesssim \tilde{A}_1 + B_1^*. \tag{22}$$

Consequently, we arrive at $C \approx C_1 + C_2$.

(ii) Let $p \le 1 \le r, q < 1$. Using Theorem 8(ii), we have that $C \approx A_2^* + B_1^*$. Let us show that $A_2^* + B_1^* \approx \tilde{A}_2 + B_1^*$, where

$$\tilde{A}_2 := \sup_{k\le M-1} 2^{\frac{k}{r}} \left(\int_{x_k}^b \left(\int_t^b u\right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}} dt\right)^{\frac{1-q}{q}}.$$

It is easy to see that $A_2^* \lesssim \tilde{A}_2$. On the other hand, using (10), we have

$$\tilde{A}_2 = \sup_{k\le M-1} 2^{\frac{k}{r}} \left(\sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} \left(\int_t^b u\right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}} dt\right)^{\frac{1-q}{q}}$$
$$\approx \sup_{k\le M-1} 2^{\frac{k}{r}} \left(\int_{x_k}^{x_{k+1}} \left(\int_t^b u\right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}} dt\right)^{\frac{1-q}{q}}.$$

Furthermore, for each $k \le M-1$, we have that

$$\left(\int_{x_k}^{x_{k+1}} \left(\int_t^b u\right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}} dt\right)^{\frac{1-q}{q}}$$
$$\lesssim \left(\int_{x_k}^{x_{k+1}} \left(\int_t^b u\right)^{\frac{1}{1-q}} d\left[V_p(a,t)^{\frac{q}{1-q}}\right]\right)^{\frac{1-q}{q}}$$
$$+ \lim_{t\to x_k+} \left(\int_t^b u\right)^{\frac{1}{q}} V_p(a,t). \tag{23}$$

Indeed, by integrating by parts, it is clear that (23) holds for each $k \in \mathbb{Z}$, $k < M-1$, whether $M = \infty$ or $M < \infty$. The remaining case when $M < \infty$ and $k = M-1$ requires more explanation. We may assume that $\max\{A_2^*, B_1^*\} < \infty$; consequently

$$\left(\int_{x_{M-1}}^{b}\left(\int_t^b u\right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}}\,dt\right)^{\frac{1-q}{q}} < \infty.$$

Thus, for each $x \in (x_{M-1}, b)$,

$$V_p(a,x)\left(\int_x^b u\right)^{\frac{1}{q}} \lesssim \left(\int_x^b\left(\int_t^b u\right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}}\,dt\right)^{\frac{1-q}{q}}$$

holds, whence we conclude that

$$\lim_{x\to b-} V_p(a,x)\left(\int_x^b u\right)^{\frac{1}{q}} = 0.$$

Hence, (23) holds. Additionally, observe that

$$\lim_{t\to x_k+}\left(\int_t^b u\right)^{\frac{1}{q}} V_p(a,t) \le \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})}\left(\int_t^b u\right)^{\frac{1}{q}} V_p(a,t). \tag{24}$$

Then, in view of (23) and (24),

$$\begin{aligned}
\tilde{A}_2 &\lesssim \sup_{k\le M-1} 2^{\frac{k}{r}}\left(\int_{x_k}^{x_{k+1}}\left(\int_t^b u\right)^{\frac{1}{1-q}} d\big[V_p(a,t)^{\frac{q}{1-q}}\big]\right)^{\frac{1-q}{q}}\\
&\quad + \sup_{k\le M-1} 2^{\frac{k}{r}} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})}\left(\int_t^b u\right)^{\frac{1}{q}} V_p(a,t)\\
&\lesssim \sup_{k\le M-1} 2^{\frac{k}{r}}\left(\int_{x_k}^{x_{k+1}}\left(\int_t^{x_{k+1}} u\right)^{\frac{1}{1-q}} d\big[V_p(a,t)^{\frac{q}{1-q}}\big]\right)^{\frac{1-q}{q}}\\
&\quad + \sup_{k\le M-1} 2^{\frac{k}{r}} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})}\left(\int_t^{x_{k+1}} u\right)^{\frac{1}{q}} V_p(a,t)\\
&\quad + \sup_{k\le M-2} 2^{\frac{k}{r}}\left(\int_{x_{k+1}}^b u\right)^{\frac{1}{q}} V_p(a,x_{k+1})\\
&=: \tilde{A}_{2,1} + \tilde{A}_{2,2} + \tilde{A}_{2,3}.
\end{aligned}$$

We shall now establish appropriate upper estimates for $\tilde{A}_{2,1}$, $\tilde{A}_{2,2}$ and $\tilde{A}_{2,3}$. Note that, (18) yields that

$$\begin{aligned} &\sup_{k\le M-1} 2^{\frac{k}{r}}\left(\int_{x_k}^{x_{k+1}}\left(\int_t^{x_{k+1}} u\right)^{\frac{q}{1-q}} u(t)\, V_p(a,t)^{\frac{q}{1-q}}\, dt\right)^{\frac{1-q}{q}} \\ &\quad\approx A_2^* + \sup_{k\le M-1} 2^{\frac{k}{r}}\left(\int_{x_k}^{x_{k+1}} u\right)^{\frac{1}{q}} V_p(a,x_k) \\ &\quad\le A_2^* + B_1^*. \end{aligned} \tag{25}$$

Since integration by parts gives

$$\tilde{A}_{2,1} \lesssim \sup_{k\le M-1} 2^{\frac{k}{r}}\left(\int_{x_k}^{x_{k+1}}\left(\int_t^{x_{k+1}} u\right)^{\frac{q}{1-q}} u(t)\, V_p(a,t)^{\frac{q}{1-q}}\, dt\right)^{\frac{1-q}{q}},$$

it follows that $\tilde{A}_{2,1} \lesssim A_2^* + B_1^*$. Furthermore, note that

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in(x,y)}\left(\int_t^y u\right)^{\frac{1}{q}} V_p(a,t) &\approx \operatorname*{ess\,sup}_{t\in(x,y)}\left(\int_t^y\left(\int_s^y u\right)^{\frac{q}{1-q}} u(s)\, ds\right)^{\frac{1-q}{q}} V_p(a,t) \\ &\le \left(\int_x^y\left(\int_t^y u\right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}}\, dt\right)^{\frac{1-q}{q}}. \end{aligned} \tag{26}$$

Thus, applying (26) and (25), we obtain that

$$\tilde{A}_{2,2} \lesssim \sup_{k\le M-1} 2^{\frac{k}{r}}\left(\int_{x_k}^{x_{k+1}}\left(\int_t^{x_{k+1}} u\right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}}\, dt\right)^{\frac{1-q}{q}} \lesssim A_2^* + B_1^*.$$

Finally, it is clear that $\tilde{A}_{2,3} \lesssim B_1^*$. Combining these estimates we arrive at $C \approx \tilde{A}_2 + B_1^*$.

Next, we will prove that $\tilde{A}_2 + B_1^* \approx C_2 + C_3$. We have from (20) that $B_1^* \lesssim C_2$. Moreover,

$$C_3 = \sup_{k\le M-1}\ \sup_{t\in(x_k,x_{k+1})}\left(\int_a^t w\right)^{\frac{1}{r}}\left(\int_t^b\left(\int_s^b u\right)^{\frac{q}{1-q}} u(s) V_p(a,s)^{\frac{q}{1-q}}\, ds\right)^{\frac{1-q}{q}} \approx \tilde{A}_2.$$

Therefore, the proof will be complete once we show that $C_2 \lesssim \tilde{A}_2 + B_1^*$.

Applying (26), we plainly have that $\tilde{A}_1 \lesssim \tilde{A}_2$. Finally, using (22), we obtain $C_2 \lesssim \tilde{A}_1 + B_1^* \lesssim \tilde{A}_2 + B_1^*$. Hence the proof is complete in this case.

(iii) Let $p \le 1, r < 1, 1 \le q$, then we have from Theorem 8(iii) that $C \approx A_3^* + B_2^*$. First, we will show that $C \approx \tilde{A}_3 + B_2^*$, where

$$\tilde{A}_3 := \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t\in(x_k,b)} \left( \int_t^b u \right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}}.$$

It is clear that $A_3^* \le \tilde{A}_3$. Moreover, (9) together with (18) yields

$$\begin{aligned}
\tilde{A}_3 &= \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \sup_{k\le i\le M-1} \operatorname*{ess\,sup}_{t\in(x_i,x_{i+1})} \left( \int_t^b u \right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&\approx \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})} \left( \int_t^b u \right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&\approx \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})} \left( \int_t^{x_{k+1}} u \right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&\quad + \left( \sum_{k=-\infty}^{M-2} 2^{\frac{k}{1-r}} \left( \int_{x_{k+1}}^b u \right)^{\frac{r}{q(1-r)}} V_p(a,x_{k+1})^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&\lesssim A_3^* + \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^b u \right)^{\frac{r}{q(1-r)}} V_p(a,x_k)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&\le A_3^* + B_2^*.
\end{aligned}$$

Next, we will show that $\tilde{A}_3 + B_2^* \approx C_4 + C_5$. We will find equivalent formulations for $B_2^*$. Using (5) for $a_k = 2^k \left( \int_{x_k}^b u \right)^{\frac{r}{q}}$, $b_k = V_p(a,x_k)^{\frac{r}{1-r}}$ and $s = \frac{r}{1-r}$, we get

$$\begin{aligned}
B_2^* \approx &\left( \sum_{k=-\infty}^{M-1} \left( \sum_{i=k}^{M-1} 2^i \left( \int_{x_i}^b u \right)^{\frac{r}{q}} \right)^{\frac{1}{1-r}} \left[ V_p(a,x_k)^{\frac{r}{1-r}} - V_p(a,x_{k-1})^{\frac{r}{1-r}} \right] \right)^{\frac{1-r}{r}} \\
&+ \left( \sum_{i=-\infty}^{M-1} 2^i \left( \int_{x_i}^b u \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} \lim_{k\to-\infty} V_p(a,x_k).
\end{aligned}$$

Applying (19) and (11) with $\alpha = 0$, we have that

$$\begin{aligned}
B_2^* \lesssim &\left( \sum_{k=-\infty}^{M-1} \left( \int_{x_k}^b w(t) \left( \int_t^b u \right)^{\frac{r}{q}} dt \right)^{\frac{1}{1-r}} \left[ V_p(a,x_k)^{\frac{r}{1-r}} - V_p(a,x_{k-1})^{\frac{r}{1-r}} \right] \right)^{\frac{1-r}{r}} \\
&+ \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^b u \right)^{\frac{r}{q(1-r)}} V_p(a,x_k)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_a^b w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}} \lim_{k\to-\infty} V_p(a,x_k)\\
\lesssim &\left(\sum_{k=-\infty}^{M-1}\int_{x_k}^b \left(\int_t^b w(s)\left(\int_s^b u\right)^{\frac{r}{q}} ds\right)^{\frac{r}{1-r}} w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt\right.\\
&\left.\times\left[V_p(a,x_k)^{\frac{r}{1-r}} - V_p(a,x_{k-1})^{\frac{r}{1-r}}\right]\right)^{\frac{1-r}{r}}\\
&+\left(\sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})}\left(\int_t^b u\right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}}\right)^{\frac{1-r}{r}}\\
&+\left(\int_a^b \left(\int_t^b w(s)\left(\int_s^b u\right)^{\frac{r}{q}} ds\right)^{\frac{r}{1-r}} w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt\right)^{\frac{1-r}{r}}\\
&\times \lim_{k\to-\infty} V_p(a,x_k).
\end{aligned}
$$

Now, plugging $b_k = V_p(a,x_k)^{\frac{r}{1-r}}$ and

$$
c_k = \int_{x_k}^{x_{k+1}} \left(\int_t^b w(s)\left(\int_s^b u\right)^{\frac{r}{q}} ds\right)^{\frac{r}{1-r}} w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt
$$

into (6), we obtain

$$
\begin{aligned}
B_2^* \lesssim &\left(\sum_{k=-\infty}^{M-1}\int_{x_k}^{x_{k+1}} \left(\int_t^b w(s)\left(\int_s^b u\right)^{\frac{r}{q}} ds\right)^{\frac{r}{1-r}}\right.\\
&\left.\times w(t)\left(\int_t^b u\right)^{\frac{r}{q}} dt\, V_p(a,x_k)^{\frac{r}{1-r}}\right)^{\frac{1-r}{r}}\\
&+\left(\sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t\in(x_k,x_{k+1})}\left(\int_t^b u\right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}}\right)^{\frac{1-r}{r}}\\
\le &\left(\sum_{k=-\infty}^{M-1}\int_{x_k}^{x_{k+1}} \left(\int_t^b w(s)\left(\int_s^b u\right)^{\frac{r}{q}} ds\right)^{\frac{r}{1-r}}\right.\\
&\left.\times w(t)\left(\int_t^b u\right)^{\frac{r}{q}} V_p(a,t)^{\frac{r}{1-r}} dt\right)^{\frac{1-r}{r}} + \tilde{A}_3\\
\approx &\; C_5 + C_4.
\end{aligned}
\tag{27}
$$

In the last equivalence we use $\tilde{A}_3 \approx C_4$, which easily follows from (11) with $\alpha = \frac{r}{1-r}$. Hence, we have $\tilde{A}_3 + B_2^* \lesssim C_4 + C_5$.

Conversely, integration by parts gives

$$
\begin{aligned}
C_5 &= \bigg( \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} \bigg( \int_t^b w(s) \Big( \int_s^b u \Big)^{\frac{r}{q}} ds \bigg)^{\frac{r}{1-r}} w(t) \Big( \int_t^b u \Big)^{\frac{r}{q}} V_p(a,t)^{\frac{r}{1-r}} \, dt \bigg)^{\frac{1-r}{r}} \\
&\lesssim \bigg( \sum_{k=-\infty}^{M-1} V_p(a,x_k)^{\frac{r}{1-r}} \bigg[ \bigg( \int_{x_k}^b w(t) \Big( \int_t^b u \Big)^{\frac{r}{q}} dt \bigg)^{\frac{1}{1-r}} \\
&\quad - \bigg( \int_{x_{k+1}}^b w(t) \Big( \int_t^b u \Big)^{\frac{r}{q}} dt \bigg)^{\frac{1}{1-r}} \bigg] \bigg)^{\frac{1-r}{r}} \\
&\quad + \bigg( \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} \bigg( \int_t^b w(s) \Big( \int_s^b u \Big)^{\frac{r}{q}} ds \bigg)^{\frac{1}{1-r}} d\Big[ V_p(a,t)^{\frac{r}{1-r}} \Big] \bigg)^{\frac{1-r}{r}} \\
&\approx B_2^* + \bigg( \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} \bigg( \int_t^{x_{k+1}} w(s) \Big( \int_s^b u \Big)^{\frac{r}{q}} ds \bigg)^{\frac{1}{1-r}} d\Big[ V_p(a,t)^{\frac{r}{1-r}} \Big] \bigg)^{\frac{1-r}{r}} \\
&\qquad + \bigg( \sum_{k=-\infty}^{M-2} \bigg( \int_{x_{k+1}}^b w(t) \Big( \int_t^b u \Big)^{\frac{r}{q}} dt \bigg)^{\frac{1}{1-r}} \int_{x_k}^{x_{k+1}} d\Big[ V_p(a,t)^{\frac{r}{1-r}} \Big] \bigg)^{\frac{1-r}{r}} \\
&\lesssim B_2^* + \bigg( \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} \bigg( \int_t^{x_{k+1}} w(s) \Big( \int_s^b u \Big)^{\frac{r}{q}} ds \bigg)^{\frac{1}{1-r}} d\Big[ V_p(a,t)^{\frac{r}{1-r}} \Big] \bigg)^{\frac{1-r}{r}}.
\end{aligned}
$$

By integrating by parts again, we obtain that

$$
\begin{aligned}
C_5 &\lesssim B_2^* + \bigg( \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} \bigg( \int_t^{x_{k+1}} w(s) \Big( \int_s^b u \Big)^{\frac{r}{q}} ds \bigg)^{\frac{r}{1-r}} \\
&\qquad \times w(t) \Big( \int_t^b u \Big)^{\frac{r}{q}} V_p(a,t)^{\frac{r}{1-r}} \, dt \bigg)^{\frac{1-r}{r}} \\
&\le B_2^* + \bigg( \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} \Big( \int_t^{x_{k+1}} w \Big)^{\frac{r}{1-r}} w(t) \Big( \int_t^b u \Big)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}} \, dt \bigg)^{\frac{1-r}{r}} \\
&\le B_2^* + \bigg( \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} \Big( \int_t^{x_{k+1}} w \Big)^{\frac{r}{1-r}} \\
&\quad \times w(t) \bigg( \operatorname*{ess\,sup}_{s \in (t,b)} \Big( \int_s^b u \Big)^{\frac{r}{q(1-r)}} V_p(a,s)^{\frac{r}{1-r}} \bigg) dt \bigg)^{\frac{1-r}{r}} \\
&\lesssim B_2^* + \tilde{A}_3. \qquad (28)
\end{aligned}
$$

Consequently, we arrive at $\tilde{A}_3 + B_2^* \lesssim C_4 + C_5 \lesssim \tilde{A}_3 + B_2^*$.

(iv) Let $p \le 1, r < 1, q < 1$. We know from Theorem 8(iv) that $C \approx A_4^* + B_2^*$. First we will prove that $A_4^* + B_2^* \approx \tilde{A}_4 + B_2^*$, where

$$\tilde{A}_4 := \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^{b} \left( \int_t^b u \right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}}\, dt \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}}.$$

It is clear that $A_4^* \le \tilde{A}_4$. On the other hand, since $\max\{A_4^*, B_2^*\} < \infty$ implies that $\max\{A_2^*, B_1^*\} < \infty$, by using the same argument we applied in case (ii), (23) holds. Then, (8) combined with (23) and (24) yields that

$$\begin{aligned}
\tilde{A}_4 &= \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} \left( \int_t^b u \right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}}\, dt \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}} \\
&\approx \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_t^b u \right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}}\, dt \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}} \\
&\lesssim \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_t^b u \right)^{\frac{1}{1-q}} d\left[ V_p(a,t)^{\frac{q}{1-q}} \right] \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}} \\
&\quad + \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t \in (x_k, x_{k+1})} \left( \int_t^b u \right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&\lesssim \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_t^{x_{k+1}} u \right)^{\frac{1}{1-q}} d\left[ V_p(a,t)^{\frac{q}{1-q}} \right] \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}} \\
&\quad + \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t \in (x_k, x_{k+1})} \left( \int_t^{x_{k+1}} u \right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&\quad + \left( \sum_{k=-\infty}^{M-2} 2^{\frac{k}{1-r}} \left( \int_{x_{k+1}}^b u \right)^{\frac{r}{q(1-r)}} V_p(a,x_{k+1})^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}}.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
\tilde{A}_4 &\lesssim \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_t^{x_{k+1}} u \right)^{\frac{q}{1-q}} u(t) V_p(a,t)^{\frac{q}{1-q}}\, dt \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}} \\
&\quad + \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t \in (x_k, x_{k+1})} \left( \int_t^{x_{k+1}} u \right)^{\frac{r}{q(1-r)}} V_p(a,t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&\quad + \left( \sum_{k=-\infty}^{M-2} 2^{\frac{k}{1-r}} \left( \int_{x_{k+1}}^b u \right)^{\frac{r}{q(1-r)}} V_p(a,x_{k+1})^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
&=: \tilde{A}_{4,1} + \tilde{A}_{4,2} + \tilde{A}_{4,3}.
\end{aligned}$$

It is easy to see that $\tilde{A}_{4,3} \lesssim B_2^*$. On the other hand, observe that (26) yields $\tilde{A}_{4,2} \lesssim \tilde{A}_{4,1}$. Moreover, using (18), we have

$$\begin{aligned}\tilde{A}_{4,1} &\lesssim A_4^* + \left( \sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left( \int_{x_k}^{x_{k+1}} u \right)^{\frac{r}{q(1-r)}} V_p(a, x_k)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ &\lesssim A_4^* + B_2^*.\end{aligned}$$

Thus, we arrive at $\tilde{A}_4 \lesssim A_4^* + B_2^*$.

We proceed by proving $\tilde{A}_4 + B_2^* \approx C_5 + C_4$. It is clear by using (11) with $\alpha = \frac{r}{1-r}$ that $\tilde{A}_4 \approx C_6$. On the other hand, using (26), we conclude that $C_4 \lesssim C_6$ and $\tilde{A}_3 \lesssim \tilde{A}_4$. Thus, taking (27) into consideration, we have $\tilde{A}_3 + B_2^* \lesssim \tilde{A}_4 + B_2^* \lesssim C_5 + C_4 \lesssim C_5 + C_6$. It remains to prove that $C_5 \lesssim \tilde{A}_4 + B_2^*$. We have already proved in (28) that $C_5 \lesssim \tilde{A}_3 + B_2^*$. Combining these estimates we arrive at $\tilde{A}_4 + B_2^* \lesssim C_5 + C_6 \lesssim \tilde{A}_4 + B_2^*$, and the proof is complete. □

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## Authors and Affiliations

**Amiran Gogatishvili[1] · Zdeněk Mihula[2,3] · Luboš Pick[3] · Hana Turčinová[3] · Tuğçe Ünver[1,4]**

Tuğçe Ünver
unver@math.cas.cz ; tugceunver@kku.edu.tr

Amiran Gogatishvili
gogatish@math.cas.cz

Zdeněk Mihula
mihulzde@fel.cvut.cz ; mihulaz@karlin.mff.cuni.cz

Luboš Pick
pick@karlin.mff.cuni.cz

Hana Turčinová
turcinova@karlin.mff.cuni.cz

[1] Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic

[2] Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Praha 6, Czech Republic

[3] Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

[4] Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

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Full Length Article

# Different degrees of non-compactness for optimal Sobolev embeddings ☆

Jan Lang [a,b], Zdeněk Mihula [c,*]

[a] Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210-1174, United States of America
[b] Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8, Czech Republic
[c] Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Technická 2, 166 27 Praha 6, Czech Republic

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The structure of non-compactness of optimal Sobolev embeddings of $m$-th order into the class of Lebesgue spaces and into that of all rearrangement-invariant function spaces is quantitatively studied. Sharp two-sided estimates of Bernstein numbers of such embeddings are obtained. It is shown that, whereas the optimal Sobolev embedding within the class of Lebesgue spaces is finitely strictly singular, the optimal Sobolev embedding in the class of all rearrangement-invariant function spaces is not even strictly singular.



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* Corresponding author.
*E-mail addresses:* lang.162@osu.edu (J. Lang), mihulzde@fel.cvut.cz (Z. Mihula).

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## 1. Introduction

Sobolev spaces and their embeddings into Lebesgue or Lorentz spaces (on an open set $\Omega \subseteq \mathbb{R}^d$) keep a prominent position in the theory of partial differential equations, and any information about structure of such embeddings is far-reaching.

There is a vast amount of literature devoted to study of conditions under which Sobolev embeddings are compact. Quality of compactness is often studied by the speed of decay of different $s$-numbers, which is connected to spectral theory of corresponding differential operators and provides estimates of the growth of their eigenvalues (see [11]). However, much less literature is devoted to study of the structure of non-compact Sobolev embeddings, which is related to the shape of essential spectrum (see [9]).

There are three common ways under which Sobolev embeddings can become non-compact:

(i) when the underlying domain is unbounded (see [1], cf. [12]);
(ii) when the boundary $\partial\Omega$ of $\Omega$ is too irregular (see [16,17,24,25]);
(iii) when the target space is optimal or "almost optimal" (see [15,20] and references therein).

In this paper we will focus on the third case. We will obtain new information about the structure of non-compactness of two optimal Sobolev embeddings—namely

$$I\colon V_0^{m,p}(\Omega) \to L^{p^*}(\Omega) \tag{1.1}$$

and

$$I\colon V_0^{m,p}(\Omega) \to L^{p^*,p}(\Omega), \tag{1.2}$$

where $1 \le m < d$, $p \in [1, d/m)$ and $p^* = dp/(d - mp)$. Here $\Omega$ is a bounded open set, and the subscript 0 means that the (ir)regularity of $\Omega$ is immaterial (see Section 2 for precise definitions). The target spaces in both embeddings are in a sense optimal. The Lebesgue space $L^{p^*}(\Omega)$ is well known to be the optimal target space in (1.1) among all Lebesgue spaces—that is, $L^{p^*}(\Omega)$ is the smallest Lebesgue space $L^q(\Omega)$ such that $I\colon V_0^{m,p}(\Omega) \to L^q(\Omega)$ is valid. However, it is also well known ([26]) that (1.1) can be improved to (1.2) if one allows not only Lebesgue spaces but also Lorentz spaces, which form a richer class of function spaces. Since $L^{p^*,p}(\Omega) \subsetneq L^{p^*}(\Omega)$, the latter is indeed an improvement. Furthermore, the Lorentz space $L^{p^*,p}(\Omega)$ is actually the optimal target space in (1.2) among all rearrangement-invariant function spaces (see [14])—that is, if $Y(\Omega)$ is a rearrangement-invariant function space (e.g., a Lebesgue space, a Lorentz space, or an Orlicz space, to name a few customary examples) such that $I\colon V_0^{m,p}(\Omega) \to Y(\Omega)$ is valid, then $L^{p^*,p}(\Omega) \subseteq Y(\Omega)$.

Not only are both embeddings (1.1) and (1.2) non-compact, but they are also in a sense "maximally non-compact" as their measures of non-compactness (in the sense of [9, Definition 2.7]) are equal to their norms. This was proved in [5,13]. Moreover, even when $L^{p^*}(\Omega)$ is enlarged to the weak Lebesgue space $L^{p^*,\infty}(\Omega)$, which satisfies $L^{p^*,p}(\Omega) \subsetneq L^{p^*}(\Omega) \subsetneq L^{p^*,\infty}(\Omega)$, the resulting Sobolev embedding is still maximally non-compact. This was proved in [19]. These results may suggest that the "quality" and the structure of these non-compact embeddings should be the same.

However, there are other possible points of view on the quality of non-compactness. One of them is the question of whether a non-compact Sobolev embedding is strictly singular or even finitely strictly singular. Strictly singular operators and finitely strictly singular ones are important classes of operators as spectral properties of such operators are very close to those of compact ones. In this regard, it follows from [6] that the Sobolev embedding $I\colon V_0^{1,1}(\Omega) \to L^{d/(d-1)}(\Omega)$, which is a particular case of (1.1) with $m = p = 1$, is finitely strictly singular. Furthermore, it was also shown there that the almost optimal critical Sobolev embedding $I\colon V_0^{d,1}((0,1)^d) \to L^\infty((0,1)^d)$ is finitely strictly singular, too. Finally, the same was proved in [18] for the optimal first-order Sobolev embedding into the space of continuous functions on a cube. These results suggest a hypothesis that non-compact Sobolev embeddings could be finitely strictly singular or at least strictly singular.

In this paper we will show that this hypothesis is correct for the "almost optimal" Sobolev embedding (1.1), but it is wrong for the "really optimal" Sobolev embedding (1.2). In other words, (1.2) is an example of a Sobolev embedding whose target space is optimal among all rearrangement-invariant function spaces that is not a singular map (i.e., there exists an infinite dimensional subspace on which the embedding is invertible), but if the target space is slightly enlarged to an "almost optimal" one (i.e., the target space is optimal only in the smaller class of Lebesgue spaces), then the resulting Sobolev embedding (1.1) is finitely strictly singular (i.e., its Bernstein numbers are decaying to zero). In the case of (1.1), we prove a two-sided estimate of the Bernstein numbers corresponding to the embedding—the estimate is sharp up to multiplicative constants. In the case of (1.2), we show that all its Bernstein numbers coincide with the norm of the embedding.

The paper is structured as follows. In the next section, we recall definitions and notation used in this paper, as well as some background results. In Section 3, we start with a couple of auxiliary results, which may be of independent interest, then we focus on the "almost optimal" embedding (Theorem 3.3), and finally on the "really optimal" one (Theorem 3.4).

## 2. Preliminaries

Here we establish the notation used in this paper, and recall some basic definitions and auxiliary results.

Any rule $s\colon T \to \{s_n(T)\}_{n=1}^\infty$ that assigns each bounded linear operator $T$ from a Banach space $X$ to a Banach space $Y$ (we shall write $T \in B(X,Y)$) a sequence $\{s_n(T)\}_{n=1}^\infty$ of nonnegative numbers having, for every $n \in \mathbb{N}$, the following properties:

(S1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$;
(S2) $s_n(S+T) \leq s_n(S) + \|T\|$ for every $S \in B(X,Y)$;
(S3) $s_n(BTA) \leq \|B\| s_n(T) \|A\|$ for every $A \in B(W,X)$ and $B \in B(Y,Z)$, where $W, Z$ are Banach spaces;
(S4) $s_n(I\colon E \to E) = 1$ for every Banach space $E$ with $\dim E \geq n$;
(S5) $s_n(T) = 0$ if $\operatorname{rank} T < n$;

is called a *strict s-number*. Notable examples of strict $s$-numbers are *the approximation numbers* $a_n$, *the Bernstein numbers* $b_n$, *the Gelfand numbers* $c_n$, *the Kolmogorov numbers* $d_n$, *the isomorphism numbers* $i_n$, or *the Mityagin numbers* $m_n$. For their definitions and the difference between strict $s$-numbers and 'non-strict' $s$-numbers, we refer the reader to [10, Chapter 5] and references therein. We say that a (strict) $s$-number is *injective* if the values of $s_n(T)$ do not depend on the codomain of $T$. More precisely, $s_n(J_N^Y \circ T) = s_n(T)$ for every closed subspace $N \subseteq Y$ and every $T \in B(X,N)$, where $J_N^Y\colon N \to Y$ is the canonical embedding operator.

In this paper, we will only need the definition of the Bernstein numbers. The $n$-th Bernstein number $b_n(T)$ of $T \in B(X,Y)$ is defined as

$$b_n(T) = \sup_{X_n \subseteq X} \inf_{\substack{x \in X_n \\ \|x\|_X = 1}} \|Tx\|_Y,$$

where the supremum extends over all $n$-dimensional subspaces of $X$. The Bernstein numbers are the smallest injective strict $s$-numbers ([28, Theorem 4.6]), that is,

$$b_n(T) \leq s_n(T) \tag{2.1}$$

for every injective strict $s$-number $s$, for every $T \in B(X,Y)$, and for every $n \in \mathbb{N}$.

An operator $T \in B(X,Y)$ is said to be *strictly singular* if there is no infinite dimensional closed subspace $Z$ of $X$ such that the restriction $T|_Z$ of $T$ to $Z$ is an isomorphism of $Z$ onto $T(Z)$. Equivalently, for each infinite dimensional (closed) subspace $Z$ of $X$,

$$\inf \{\|Tx\|_Y : \|x\|_X = 1, x \in Z\} = 0.$$

An operator $T \in B(X,Y)$ is said to be *finitely strictly singular* if it has the property that given any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $E$ is a subspace of $X$ with $\dim E \geq N(\varepsilon)$, then there exists $x \in E$, $\|x\|_X = 1$, such that $\|Tx\|_Y \leq \varepsilon$. This can be expressed in terms of the Bernstein numbers of $T$. The operator $T$ is finitely strictly singular if and only if

$$\lim_{n\to\infty} b_n(T) = 0.$$

The relations between these two notions and that of compactness of $T$ are illustrated by the following diagram:

$$T \text{ is compact } \implies T \text{ is finitely strictly singular } \implies T \text{ is strictly singular;}$$

moreover, each reverse implication is false in general. For further details and general background information concerning these matters we refer the interested reader to [2], [21] and [29].

Throughout the rest of this section, $X$ denotes a Banach space. The operator norm of the projection $Q\colon L^2([0,1],X) \to L^2([0,1],X)$ defined as

$$Qf(t) = \sum_{j=1}^{\infty} \left( \int_0^1 f(s) r_j(s)\,\mathrm{d}s \right) r_j(t),\ \ f \in L^2([0,1],X), \tag{2.2}$$

is called the *$K$-convexity constant* of $X$. Here $\{r_j\}_{j=1}^{\infty}$ are the Rademacher functions. The $K$-convexity constant of $X$ is denoted by $K(X)$. If $\dim X = n$, then (e.g., see [3, Theorem 6.2.4])

$$K(X) \le c \log(1 + d(X, \ell_2^n)). \tag{2.3}$$

Here $c$ is an absolute constant and $d(X, \ell_2^n)$ is the Banach–Mazur distance, that is,

$$d(X, \ell_2^n) = \inf\{\|T\| \|T^{-1}\| \colon T \text{ is a linear isomorphism of } X \text{ onto } \ell_2^n\}.$$

We say that $X$ is of *cotype* 2 if there is a constant $\gamma$ such that

$$\left( \sum_{j=1}^{m} \|x_j\|_X^2 \right)^{1/2} \le \gamma \int_0^1 \Big\| \sum_{j=1}^{m} x_j r_j(t) \Big\|_X \,\mathrm{d}t \tag{2.4}$$

for every $\{x_j\}_{j=1}^m \subseteq X$, $m \in \mathbb{N}$. We denote the least such a $\gamma$ by $C_2(X)$.

Let $Y$ be a Banach space such that $X \subseteq Y$. We say that the inclusion is 2-*absolutely summable* if there is a constant $\gamma$ such that

$$\left( \sum_{j=1}^{m} \|x_j\|_Y^2 \right)^{1/2} \le \gamma \sup \left\{ \left( \sum_{j=1}^{m} |x^*(x_j)|^2 \right)^{1/2} : \|x^*\|_{X^*} \le 1 \right\} \tag{2.5}$$

for every $\{x_j\}_{j=1}^m \subseteq X$, $m \in \mathbb{N}$. We denote the least such a $\gamma$ by $\pi_2(X \hookrightarrow Y)$.

Let $A, B$ be subsets of $X$. We denote the minimum number of points $x_1, \dots, x_m \in X$ such that

$$A \subseteq \bigcup_{j=1}^{m} (x_j + B) \tag{2.6}$$

by $E(A,B)$. In general, it may happen that $E(A,B)=\infty$, but in our case it will always be a finite number. $\bar{E}(A,B)$ denotes the minimum number of points $x_1,\dots,x_m \in A$ such that (2.6) holds.

Let $(R,\mu)$ be a nonatomic measure space and $p\in[1,\infty)$. As usual, $L^p(R,\mu)$ denotes the *Lebesgue space* endowed with the norm

$$\|f\|_{L^p(R,\mu)} = \left(\int_R |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}},\ f\in L^p(R,\mu).$$

Let $q\in[1,p]$. The *Lorentz space* $L^{p,q}(R,\mu)$ is the Banach space of all $\mu$-measurable functions $f$ in $R$ for which the functional

$$\|f\|_{L^{p,q}(R,\mu)} = \left(\int_0^\infty t^{\frac{q}{p}-1} f^*(t)^q \,\mathrm{d}t\right)^{\frac{1}{q}}$$

is finite—the norm on $L^{p,q}(R,\mu)$ is given by the functional. The function $f^*\colon(0,\infty)\to[0,\infty]$ is the (right-continuous) *nonincreasing rearrangement* of $f$, that is,

$$f^*(t) = \inf\{\lambda>0\colon \mu(\{x\in R\colon |f(x)|>\lambda\})\le t\},\ t\in(0,\infty).$$

Note that $f^*(t)=0$ for every $t\in[\mu(R),\infty)$. Furthermore, we have (see [4, Chapter 2, Proposition 1.8])

$$\|\cdot\|_{L^{p,p}(R,\mu)} = \|\cdot\|_{L^p(R,\mu)}.$$

When $R\subseteq\mathbb{R}^d$ and $\mu$ is the $d$-dimensional Lebesgue measure, we write $L^p(R)$ and $L^{p,q}(R)$ instead of $L^p(R,\mu)$ and $L^{p,q}(R,\mu)$, respectively, and $|R|$ instead of $\mu(R)$ for short. We refer the interested reader to [27, Chapter 8] for more information on Lorentz spaces. Assume that $(R,\mu)$ is probabilistic. We denote by $L^{\psi_2}(R,\mu)$ the *Orlicz space* generated by the Young function

$$\psi_2(t)=\exp(t^2)-1,\ t\in[0,\infty).$$

The norm on $L^{\psi_2}(R,\mu)$ is given by

$$\|f\|_{L^{\psi_2}(R,\mu)} = \inf\left\{\lambda>0\colon \int_R \psi_2\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}\mu(x)\le 1\right\}.$$

We have (e.g., see [3, Lemma 3.5.5])

$$c\sup_{p\in[1,\infty)}\frac{\|f\|_{L^p(R,\mu)}}{\sqrt{p}} \le \|f\|_{L^{\psi_2}(R,\mu)} \le \tilde{c}\sup_{p\in[1,\infty)}\frac{\|f\|_{L^p(R,\mu)}}{\sqrt{p}} \tag{2.7}$$

for every $f \in L^{\psi_2}(R,\mu)$. Here $c$ and $\tilde{c}$ are absolute constants. In particular, $L^{\psi_2}(R,\mu)$ is continuously embedded in $L^p(R,\mu)$ for every $p \in [1,\infty)$.

Throughout the entire paper, we assume that $d \in \mathbb{N}$, $d \geq 2$. Let $G \subseteq \mathbb{R}^d$ be a nonempty bounded open set. For $m \in \mathbb{N}$ and $p \in [1,\infty)$, $V^{m,p}(G)$ denotes the vector space of all $m$-times weakly differentiable functions in $G$ whose $m$-th order weak derivatives belong to $L^p(G)$. By $V_0^{m,p}(G)$ we denote the Banach space of all functions from $V^{m,p}(G)$ whose continuation by 0 outside $G$ is $m$-times weakly differentiable in $\mathbb{R}^d$ equipped with the norm $\|u\|_{V_0^{m,p}(G)} = \||\nabla^m u|_{\ell_p}\|_{L^p(G)}$. By $\nabla^m$ we denote the vector of all $m$-th order weak derivatives. When $G$ is regular enough (for example, Lipschitz), $V_0^{m,p}(G)$ coincides with the usual Sobolev space $W_0^{m,p}(G)$, up to equivalent norms.

## 3. Different degrees of noncompactness

An important property of both optimal Sobolev embeddings (1.1) and (1.2), which we will exploit in both cases, is that their norms are homothetic invariant.

**Proposition 3.1.** *Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty bounded open set, $m \in \mathbb{N}$, $1 \leq m < d$, and $p \in [1, d/m)$. Let $p^* = dp/(d-mp)$ and $q \in [p, p^*]$. Denote by $I$ the identity operator $I\colon V_0^{m,p}(\Omega) \to L^{p^*,q}(\Omega)$. For every $0 < \lambda < \|I\|$ and every $\varepsilon > 0$, there exist a system of functions $\{u_j\}_{j=1}^\infty \subseteq V_0^{m,p}(\Omega)$ and a system of open balls $\{B_{r_j}(x_j)\}_{j=1}^\infty \subseteq \Omega$ with the following properties.*

- *(i) The balls $\{B_{r_j}(x_j)\}_{j=1}^\infty$ are pairwise disjoint.*
- *(ii) $\|u_j\|_{V_0^{m,p}(\Omega)} = 1$ and $\|u_j\|_{L^{p^*,q}(\Omega)} = \lambda$ for every $j \in \mathbb{N}$.*
- *(iii) $\operatorname{supp} u_j \subseteq B_{r_j}(x_j)$ for every $j \in \mathbb{N}$.*
- *(iv) For every sequence $\{\alpha_j\}_{j=1}^\infty \subseteq \mathbb{R}$, we have*

$$\left\| \sum_{j=1}^\infty \alpha_j u_j \right\|_{L^{p^*,q}(\Omega)} \geq \frac{\lambda}{(1+\varepsilon)^{\frac{1}{q}}} \left( \sum_{j=1}^\infty |\alpha_j|^q \right)^{\frac{1}{q}}. \tag{3.1}$$

**Proof.** It is known that

$$\|I\| = \|I_G\colon V_0^{m,p}(G) \to L^{p^*,q}(G)\| \qquad \text{for every open set } \emptyset \neq G \subseteq \Omega. \tag{3.2}$$

Indeed, arguing as in the proof of [19, Proposition 3.1], we observe that the proof of (3.2) amounts to showing that, if $u \in V_0^{m,p}(B_r(0))$ and $0 < s < r$, then

$$\frac{\|u(\kappa\,\cdot)\|_{L^{p^*,q}(B_s(0))}}{\|u(\kappa\,\cdot)\|_{V_0^{m,p}(B_s(0))}} = \frac{\|u\|_{L^{p^*,q}(B_r(0))}}{\|u\|_{V_0^{m,p}(B_r(0))}}, \tag{3.3}$$

where $\kappa = r/s$. It is a matter of simple straightforward computations to show that

$$\|u(\kappa\,\cdot)\|_{L^{p^*,q}(B_s(0))} = \kappa^{m-\frac{d}{p}}\|u\|_{L^{p^*,q}(B_r(0))}$$

and

$$\|\nabla(u(\kappa\,\cdot))\|_{L^p(B_s(0))} = \kappa^{m-\frac{d}{p}}\|\nabla u\|_{L^p(B_r(0))},$$

whence (3.3) immediately follows.

We now start with construction of the desired systems. We will use induction. First, using (3.2), we find a ball $B_{r_1}(x_1) \subseteq \overline{B}_{r_1}(x_1) \subseteq \Omega$ and a function $u_1 \in V_0^{m,p}(\Omega)$ such that $\operatorname{supp} u_1 \subseteq B_{r_1}(x_1)$, $\|u_1\|_{L^{p^*,q}(\Omega)} = \lambda$ and $\|u_1\|_{V_0^{m,p}(\Omega)} = 1$. Set $\delta_0 = |B_{r_1}(x_1)|$. By the monotone convergence theorem, there is $0 < \delta_1 < \delta_0$ such that

$$(1+\varepsilon)\int\limits_{\delta_1}^{\delta_0} \left(t^{\frac{1}{p^*}-\frac{1}{q}}u_1^*(t)\right)^q \,\mathrm{d}t \geq \int\limits_{0}^{\delta_0} \left(t^{\frac{1}{p^*}-\frac{1}{q}}u_1^*(t)\right)^q \,\mathrm{d}t = \|u_1\|_{L^{p^*,q}(\Omega)}^q.$$

Next, assume that we have already found functions $u_j \in V_0^{m,p}(\Omega)$, pairwise disjoint balls $B_{r_j}(x_j) \subseteq \overline{B}_{r_j}(x_j) \subseteq \Omega$, and $0 < \delta_k < \cdots < \delta_1 < \delta_0$, $j = 1, \dots, k$, where $k \in \mathbb{N}$, such that $\|u_j\|_{V_0^{m,p}(\Omega)} = 1$ and $\|u_j\|_{L^{p^*,q}(\Omega)} = \lambda$, $\operatorname{supp} u_j \subseteq B_{r_j}(x_j)$, and

$$(1+\varepsilon)\int\limits_{\delta_j}^{\delta_{j-1}} \left(t^{\frac{1}{p^*}-\frac{1}{q}}u_j^*(t)\right)^q \,\mathrm{d}t \geq \|u_j\|_{L^{p^*,q}(\Omega)}^q. \tag{3.4}$$

Take any ball $B_{r_{k+1}}(x_{k+1})$ such that $B_{r_{k+1}}(x_{k+1}) \subseteq \overline{B}_{r_{k+1}}(x_{k+1}) \subseteq \Omega \setminus \bigcup_{j=1}^k \overline{B}_{r_j}(x_j)$ and $|B_{r_{k+1}}(x_{k+1})| < \delta_k$. Thanks to (3.2), we find a function $u_{k+1} \in V_0^{m,p}(\Omega)$ such that $\operatorname{supp} u_{k+1} \subseteq B_{r_{k+1}}(x_{k+1})$, $\|u_{k+1}\|_{L^{p^*,q}(\Omega)} = \lambda$ and $\|u_{k+1}\|_{V_0^{m,p}(\Omega)} = 1$. By the monotone convergence theorem again, there is $0 < \delta_{k+1} < \delta_k$ such that

$$(1+\varepsilon)\int\limits_{\delta_{k+1}}^{\delta_k} \left(t^{\frac{1}{p^*}-\frac{1}{q}}u_{k+1}^*(t)\right)^q \,\mathrm{d}t \geq \int\limits_{0}^{\delta_k} \left(t^{\frac{1}{p^*}-\frac{1}{q}}u_{k+1}^*(t)\right)^q \,\mathrm{d}t = \|u_{k+1}\|_{L^{p^*,q}(\Omega)}^q.$$

This finishes the inductive step.

Clearly, the constructed systems $\{u_j\}_{j=1}^\infty \subseteq V_0^{m,p}(\Omega)$ and $\{B_{r_j}(x_j)\}_{j=1}^\infty \subseteq \Omega$ have the properties (i)–(iii), and (3.4) is valid for every $j \in \mathbb{N}$. It remains to verify that (iv) is also valid. Let $\{\alpha_j\}_{j=1}^\infty \subseteq \mathbb{R}$. Since the functions $\{\alpha_j u_j\}_{j=1}^\infty$ have pairwise disjoint supports, we have

$$\left|\left\{x \in \Omega\colon \left|\sum_{j=1}^\infty \alpha_j u_j(x)\right| > \gamma\right\}\right| = \sum_{j=1}^\infty |\{x \in \Omega\colon\ |\alpha_j u_j(x)| > \gamma\}|$$

for every $\gamma > 0$. It follows that

$$\left(\sum_{j=1}^{\infty}\alpha_j u_j\right)^* \geq \sum_{j=1}^{\infty}|\alpha_j|u_j^*\chi_{(\delta_j,\delta_{j-1})}. \tag{3.5}$$

Indeed, suppose that there is $t \in (0, |\Omega|)$ such that

$$\left(\sum_{j=1}^{\infty}\alpha_j u_j\right)^*(t) < \sum_{j=1}^{\infty}|\alpha_j|u_j^*(t)\chi_{(\delta_j,\delta_{j-1})}(t).$$

Plainly, there is a unique index $k$ such that $t \in (\delta_k, \delta_{k-1})$. By the definition of the nonincreasing rearrangement, there is $\gamma > 0$ such that

$$\left|\left\{x \in \Omega\colon \left|\sum_{j=1}^{\infty}\alpha_j u_j(x)\right| > \gamma\right\}\right| \leq t \qquad \text{and} \qquad \gamma < |\alpha_k|u_k^*(t).$$

Consequently, using the definition again, we have

$$|\{x \in \Omega\colon\ |\alpha_k u_k(x)| > \gamma\}| > t,$$

however. Thus we have reached a contradiction, and so (3.5) is proved.

Finally, using (3.4) and (3.5), we observe that

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty}\alpha_j u_j\right\|_{L^{p^*,q}(\Omega)}^q &= \int_0^{\infty}\left(t^{\frac{1}{p^*}-\frac{1}{q}}\left(\sum_{j=1}^{\infty}\alpha_j u_j\right)^*(t)\right)^q \mathrm{d}t \\
&\geq \int_0^{\infty}\left(t^{\frac{1}{p^*}-\frac{1}{q}}\sum_{j=1}^{\infty}|\alpha_j|u_j^*(t)\chi_{(\delta_j,\delta_{j-1})}(t)\right)^q \mathrm{d}t \\
&\geq \sum_{j=1}^{\infty}|\alpha_j|^q\int_{\delta_j}^{\delta_{j-1}}\left(t^{\frac{1}{p^*}-\frac{1}{q}}u_j^*(t)\right)^q \mathrm{d}t \\
&\geq \frac{1}{1+\varepsilon}\sum_{j=1}^{\infty}|\alpha_j|^q\|u_j\|_{L^{p^*,q}(\Omega)}^q \\
&= \frac{\lambda^q}{1+\varepsilon}\sum_{j=1}^{\infty}|\alpha_j|^q. \quad \square
\end{aligned}$$

We start with the Lebesgue case. The following lemma of independent interest is a key ingredient for the proof of the fact that the embedding (1.1) is finitely strictly singular. Its proof is inspired by that of [6, Lemma 2.9].

**Lemma 3.2.** *Let $(R, \mu)$ be a probability measure space and $p \in [1, \infty)$. Let $X_n$ be a $n$-dimensional subspace of $L^p(R, \mu)$. There is a positive $\mu$-measurable function $g$ on $R$ and a linear isometry $L\colon L^p(R, \mu) \to L^p(R, \nu)$ defined as $Lf = g^{-1/p} f$, where $\mathrm{d}\nu = g \, \mathrm{d}\mu$, with the following properties. The measure $\nu$ is probabilistic, and in every subspace $Y \subseteq X_n$ with $\dim Y \geq n/2$, there exists a function $h \in Y$ such that*

$$\|h\|_{L^p(R,\mu)} = 1$$

*and*

$$\sup_{q \in [1,\infty)} \frac{\|Lh\|_{L^q(R,\nu)}}{\sqrt{q}} \leq C.$$

*Here $C$ is an absolute constant depending only on $\min\{p, 2\}$.*

**Proof.** Let $X_n$ be a $n$-dimensional subspace of $L^p(R, \mu)$. Thanks to [22] (cf. [31, Theorem 2.1]), there exists a positive $\mu$-measurable function $g$ on $R$ such that $\|g\|_{L^1(R,\mu)} = 1$ and the following is true: Upon setting $\mathrm{d}\nu = g \, \mathrm{d}\mu$ and defining $Lf = g^{-1/p} f$, $f \in L^p(R, \mu)$, the subspace $\widetilde{X}_n = LX_n$ of $L^p(R, \nu)$ has a basis $\{\psi_1, \dots, \psi_n\}$ that is orthonormal in $L^2(R, \nu)$ and satisfies

$$\sum_{j=1}^{n} |\psi_j|^2 \equiv n \quad \mu\text{-a.e. on } R. \tag{3.6}$$

Note that, since $\widetilde{X}_n$ has a basis consisting of functions from $L^2(R, \nu)$, we have $\widetilde{X}_n \subseteq L^2(R, \nu)$ even for $p \in [1, 2)$.

Let $Y$ be a subspace of $X_n$ with $\dim Y \geq n/2$. Set

$$B_p(Z) = \{f \in Z \colon \|f\|_{L^p(R,\nu)} \leq 1\}$$

and

$$B_{exp}(Z) = \{f \in Z \colon \|f\|_{L^{\psi_2}(R,\nu)} \leq 1\},$$

in which $Z$ is $\widetilde{X}_n$ or $\widetilde{Y} = LY$. By [7, Lemma 9.2], we have

$$\log E(B_2(\widetilde{X}_n), tB_{exp}(\widetilde{X}_n)) \leq c_1 \frac{1}{t^2} n \quad \text{for every } t \geq 1; \tag{3.7}$$

here $c_1$ is an absolute constant, which is independent of $n$ and $t$. Since $\dim \widetilde{Y} \geq n/2$, we have

$$\log E(B_p(\widetilde{Y}), \frac{1}{4} B_p(\widetilde{Y})) \geq n \log 2 \tag{3.8}$$

by a standard volumetric argument (e.g., see [8, (1.1.10)]).

We start with the case $p \in [2, \infty)$, which is simpler. Since $B_p(\widetilde{Y}) \subseteq B_2(\widetilde{Y}) \subseteq B_2(\widetilde{X}_n)$, it follows from (3.7) that

$$\log E(B_p(\widetilde{Y}), tB_{exp}(\widetilde{X}_n)) \leq c_1 \frac{1}{t^2} n \quad \text{for every } t \geq 1. \tag{3.9}$$

Moreover, since $E(B_p(\widetilde{Y}), 2tB_{exp}(\widetilde{Y})) \leq E(B_p(\widetilde{Y}), tB_{exp}(\widetilde{X}_n))$ (e.g., see [3, Fact 4.1.9]), we actually have

$$\log E(B_p(\widetilde{Y}), 2tB_{exp}(\widetilde{Y})) \leq c_1 \frac{1}{t^2} n \quad \text{for every } t \geq 1.$$

Therefore, we can find $t_0 \geq 1$, not depending on $n$, so large that

$$\log E(B_p(\widetilde{Y}), 2t_0 B_{exp}(\widetilde{Y})) \leq \frac{n \log 2}{2}.$$

It follows that $2t_0 B_{exp}(\widetilde{Y}) \nsubseteq \frac{1}{4} B_p(\widetilde{Y})$. Indeed, if $2t_0 B_{exp}(\widetilde{Y}) \subseteq \frac{1}{4} B_p(\widetilde{Y})$, then

$$\log E(B_p(\widetilde{Y}), \frac{1}{4} B_p(\widetilde{Y})) \leq \log E(B_p(\widetilde{Y}), 2t_0 B_{exp}(\widetilde{Y})) \leq \frac{n \log 2}{2},$$

which would contradict (3.8). Hence there is a function $h_0 \in \widetilde{Y}$ such that

$$\|h_0\|_{L^{\psi_2}(R,\nu)} \leq 2t_0 \qquad \text{and} \qquad \|h_0\|_{L^p(R,\nu)} > \frac{1}{4}.$$

Then $h = L^{-1} h_0 / \|h_0\|_{L^p(R,\nu)}$ is the desired function thanks to (2.7).

We now turn our attention to the case $p \in [1, 2)$. Assume for the moment that we know that

$$\log E(B_p(\widetilde{X}_n), tB_2(\widetilde{X}_n)) \leq c_2 \frac{\log^2(1+t)}{t^2} n \quad \text{for every } t \geq 1. \tag{3.10}$$

Here $c_2$ is a constant depending only on $p$. Clearly

$$\begin{aligned} E(B_p(\widetilde{Y}), tB_{exp}(\widetilde{X}_n)) &\leq E(B_p(\widetilde{X}_n), tB_{exp}(\widetilde{X}_n)) \\ &\leq E(B_p(\widetilde{X}_n), sB_2(\widetilde{X}_n)) \cdot E(sB_2(\widetilde{X}_n), tB_{exp}(\widetilde{X}_n)) \\ &= E(B_p(\widetilde{X}_n), sB_2(\widetilde{X}_n)) \cdot E(B_2(\widetilde{X}_n), \frac{t}{s} B_{exp}(\widetilde{X}_n)). \end{aligned}$$

Consequently, using (3.7) and (3.10), we obtain

$$\log E(B_p(\widetilde{Y}), tB_{exp}(\widetilde{X}_n)) \leq c_3 \left( \frac{\log^2(1+s)}{s^2} + \frac{s^2}{t^2} \right) n$$

for every $1 \le s \le t$. Here $c_3$ is a constant depending only on $p$. Plugging $s = \sqrt{t}$ into this inequality, we arrive at

$$\log E(B_p(\widetilde{Y}), tB_{exp}(\widetilde{X}_n)) \le c_3 \frac{1 + \log^2(1 + \sqrt{t})}{t} n \quad \text{for every } t \ge 1.$$

Since $\lim_{t\to\infty} \frac{1+\log^2(1+\sqrt{t})}{t} = 0$, we can now proceed in the same way as in the case $p \in [2, \infty)$, using this inequality instead of (3.9). Therefore, the proof will be complete once we prove (3.10)—to that end, we adapt the argument of [7, Proposition 9.6]. The proof of (3.10) is divided into several steps.

First, observe that

$$\begin{aligned} &E(B_p(\widetilde{X}_n), tB_2(\widetilde{X}_n)) \\ &\quad \le E(B_p(\widetilde{X}_n), 2B_p(\widetilde{X}_n) \cap 2tB_2(\widetilde{X}_n)) \cdot E(B_p(\widetilde{X}_n) \cap tB_2(\widetilde{X}_n), \frac{t}{2} B_2(\widetilde{X}_n)). \end{aligned} \tag{3.11}$$

Since $(\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)})$ is a Hilbert space, $2tB_2(\widetilde{X}_n)$ is a multiple of its unit ball and $B_p(\widetilde{X}_n)$ is a (nonempty) closed convex subset of $(\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)})$, we have $E(B_p(\widetilde{X}_n), 2tB_2(\widetilde{X}_n)) = \bar{E}(B_p(\widetilde{X}_n), 2tB_2(\widetilde{X}_n))$ (e.g., see [3, Fact 4.1.4]). Consequently, if $B_p(\widetilde{X}_n) \subseteq \bigcup_{k=1}^m (u_k + 2tB_2(\widetilde{X}_n))$, where $u_k \in B_p(\widetilde{X}_n)$, then $B_p(\widetilde{X}_n) \subseteq \bigcup_{k=1}^m (u_k + 2B_p(\widetilde{X}_n) \cap 2tB_2(\widetilde{X}_n))$. Hence

$$E(B_p(\widetilde{X}_n), 2B_p(\widetilde{X}_n) \cap 2tB_2(\widetilde{X}_n)) \le E(B_p(\widetilde{X}_n), 2tB_2(\widetilde{X}_n)). \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\begin{aligned} &\log E(B_p(\widetilde{X}_n), tB_2(\widetilde{X}_n)) \\ &\quad \le \log E(B_p(\widetilde{X}_n), 2tB_2(\widetilde{X}_n)) \\ &\qquad + \log E(B_p(\widetilde{X}_n) \cap tB_2(\widetilde{X}_n), \frac{t}{2} B_2(\widetilde{X}_n)). \end{aligned} \tag{3.13}$$

Now, thanks to (3.6) and the fact that $p < 2$, we have

$$\begin{aligned} \|u\|^2_{L^2(R,\nu)} &= \int_R \Big|\sum_{j=1}^n \alpha_j \psi_j\Big|^2 \, d\nu = \int_R \Big|\sum_{j=1}^n \alpha_j \psi_j\Big|^{2-p} \Big|\sum_{j=1}^n \alpha_j \psi_j\Big|^p \, d\nu \\ &\le \Big(\sum_{j=1}^n |\alpha_j|^2\Big)^{\frac{2-p}{2}} \int_R \Big(\sum_{j=1}^n |\psi_j|^2\Big)^{\frac{2-p}{2}} \Big|\sum_{j=1}^n \alpha_j \psi_j\Big|^p \, d\nu \\ &= \|u\|^{2-p}_{L^2(R,\nu)} n^{\frac{2-p}{2}} \|u\|^p_{L^p(R,\nu)} \end{aligned}$$

for every $u = \sum_{j=1}^n \alpha_j \psi_j \in \widetilde{X}_n$, whence it follows that

$$\|u\|_{L^2(R,\nu)} \le n^{\frac{2-p}{2p}} \|u\|_{L^p(R,\nu)} \quad \text{for every } u \in \widetilde{X}_n.$$

Clearly, this implies that $B_p(\widetilde{X}_n) \subseteq r B_2(\widetilde{X}_n)$ for every $r \ge n^{\frac{2-p}{2p}}$; hence

$$\log E(B_p(\widetilde{X}_n), 2^k t B_2(\widetilde{X}_n)) = 0$$

for every $k \in \mathbb{N}$ such that $2^k t \ge n^{\frac{2-p}{2p}}$. Therefore, iterating (3.13) with $t$ replaced by $2^k t$, $k \in \mathbb{N}$, we arrive at

$$\log E(B_p(\widetilde{X}_n), tB_2(\widetilde{X}_n)) \le \sum_{k=0}^{\infty} \log E(B_p(\widetilde{X}_n) \cap 2^k t B_2(\widetilde{X}_n), 2^{k-1} t B_2(\widetilde{X}_n)). \tag{3.14}$$

Second, we claim that

$$\log E(B_p(\widetilde{X}_n) \cap s B_2(\widetilde{X}_n), \frac{s}{2} B_2(\widetilde{X}_n)) \le c_4 \frac{\log^2(1+s)}{s^2} n \tag{3.15}$$

for every $s \ge 1$. Here $c_4$ is a constant depending only on $p$. Fix $s \ge 1$. Let $Z$ denote $\widetilde{X}_n$ endowed with the norm

$$\|u\|_Z = \max\left\{ \|u\|_{L^p(R,\nu)}, \frac{1}{s}\|u\|_{L^2(R,\nu)} \right\}.$$

Note that $B_p(\widetilde{X}_n) \cap s B_2(\widetilde{X}_n)$ is the unit ball of $Z$. Owing to [3, (9.1.7) together with Lemma 9.1.3] combined with [7, Lemma 4.4], we have

$$\begin{aligned} &\log E(B_p(\widetilde{X}_n) \cap s B_2(\widetilde{X}_n), \frac{s}{2} B_2(\widetilde{X}_n)) \\ &\quad \le c_5 K(Z)^2 C_2(Z)^2 \pi_2^2(Z \hookrightarrow (\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)})) \frac{1}{s^2}. \end{aligned} \tag{3.16}$$

Here $c_5$ is an absolute constant, and the quantities $K(Z)$, $C_2(Z)$, $\pi_2(Z \hookrightarrow (\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)}))$ are defined below (2.2), (2.4), (2.5), respectively.

As for $K(Z)$, we have

$$K(Z) \le c_6 \log(1 + d(Z, \ell_2^n)) \tag{3.17}$$

by (2.3). Here $c_6$ is an absolute constant.

We claim that $d(Z, \ell_2^n) \le s$. To this end, consider the linear isomorphism $T\colon Z \to \ell_2^n$ defined as

$$Tf = \{\alpha_j\}_{j=1}^n, \; f = \sum_{j=1}^n \alpha_j \psi_j \in Z.$$

Clearly, $T$ is onto $\ell_2^n$, and we have

$$\|Tf\|_{\ell_2^n} = \|f\|_{L^2(R,\nu)} \le s\|f\|_Z \tag{3.18}$$

for every $f \in Z$. On the other hand, using (3.6) and the fact that $p < 2$, we obtain

$$\begin{aligned}
\|T^{-1}(\{\alpha_j\}_{j=1}^n)\|_{L^p(R,\nu)}^p &= \int_R \left(\sum_{j=1}^n \alpha_j \psi_j(t)\right)^p \, \mathrm{d}\nu(t) \\
&= \int_R \left(\sum_{j=1}^n \alpha_j \psi_j(t)\right)^{p-2} \left(\sum_{j=1}^n \alpha_j \psi_j(t)\right)^2 \, \mathrm{d}\nu(t) \\
&\le \int_R \left(\|\{\alpha_j\}_{j=1}^n\|_{\ell_2^n} \sqrt{n}\right)^{p-2} \left(\sum_{j=1}^n \alpha_j \psi_j(t)\right)^2 \, \mathrm{d}\nu(t) \\
&\le \|\{\alpha_j\}_{j=1}^n\|_{\ell_2^n}^{p-2} \Big\| \sum_{j=1}^n \alpha_j \psi_j \Big\|_{L^2(R,\nu)}^2 \\
&= \|\{\alpha_j\}_{j=1}^n\|_{\ell_2^n}^p
\end{aligned}$$

for every $\{\alpha_j\}_{j=1}^n$. Furthermore, we plainly have

$$\frac{1}{s}\|T^{-1}(\{\alpha_j\}_{j=1}^n)\|_{L^2(R,\nu)} \le \|T^{-1}(\{\alpha_j\}_{j=1}^n)\|_{L^2(R,\nu)} = \|\{\alpha_j\}_{j=1}^n\|_{\ell_2^n}.$$

Therefore $\|T^{-1}\| \le 1$. By combining this with (3.18), it follows that

$$d(Z, \ell_2^n) \le s.$$

Plugging this into (3.17), we obtain

$$K(Z) \le c_6 \log(1+s). \tag{3.19}$$

As for $C_2(Z)$, we claim that

$$C_2(Z) \le c_7, \tag{3.20}$$

where $c_7$ is a constant depending only on $p$. To this end, recall that

$$\max\{C_2(L^p(R,\nu)), C_2(L^2(R,\nu))\} < \infty,$$

and that $C_2(L^p(R,\nu))$ depends only $p$ (e.g., see [2, Theorem 6.2.14]). Now, since

$$\left(\sum_{j=1}^m \|f_j\|_Z^2\right)^{1/2} \le \left(\sum_{j=1}^m \|f_j\|_{L^p(R,\nu)}^2\right)^{1/2} + \frac{1}{s}\left(\sum_{j=1}^m \|f_j\|_{L^2(R,\nu)}^2\right)^{1/2}$$

$$\leq 2\max\left\{C_2(L^p(R,\nu)), C_2(L^2(R,\nu))\right\} \int_0^1 \Big\| \sum_{j=1}^m f_j r_j(t) \Big\|_Z \, \mathrm{d}t$$

for every $\{f_j\}_{j=1}^m \subseteq Z$, $m \in \mathbb{N}$, thanks to (2.4), the claim immediately follows.

As for $\pi_2(Z \hookrightarrow (\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)}))$, note that $\|f\|_{L^1(R,\nu)} \leq \|f\|_{L^p(R,\nu)} \leq \|f\|_Z$ for every $f \in Z$, and so the unit ball of $(\widetilde{X}_n, \|\cdot\|_{L^1(R,\nu)})^*$ is contained in the unit ball of $Z^*$. It follows that

$$\pi_2(Z \hookrightarrow (\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)})) \leq \pi_2((\widetilde{X}_n, \|\cdot\|_{L^1(R,\nu)}) \hookrightarrow (\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)})).$$

By [7, Proof of Lemma 4.5], we have

$$\pi_2((\widetilde{X}_n, \|\cdot\|_{L^1(R,\nu)}) \hookrightarrow (\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)})) \leq c_8\sqrt{n}.$$

Here $c_8$ is an absolute constant. Hence

$$\pi_2(Z \hookrightarrow (\widetilde{X}_n, \|\cdot\|_{L^2(R,\nu)})) \leq c_8\sqrt{n}. \tag{3.21}$$

The desired estimate (3.15) now follows by combining (3.19), (3.20) and (3.21) with (3.16).

Finally, now that we have (3.15) at our disposal, the rest is simple. Combining (3.15) with (3.14), we obtain

$$\begin{aligned}
\log E(B_p(\widetilde{X}_n), tB_2(\widetilde{X}_n)) &\leq c_4 \sum_{k=0}^{\infty} \frac{\log^2(1+2^k t)}{4^k t^2} n \\
&\leq 2c_4 \left( \sum_{k=0}^{\infty} \frac{k^2 \log^2 2 + \log^2(1+t)}{4^k} \right) \frac{1}{t^2} n \\
&\leq 2c_4 \left( \sum_{k=0}^{\infty} \frac{k^2+1}{4^k} \right) \frac{\log^2(1+t)}{t^2} n.
\end{aligned}$$

This finishes the proof of (3.10). □

We are now in a position to prove the main result concerning the embedding (1.1).

**Theorem 3.3.** *Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty bounded open set, $m \in \mathbb{N}$, $1 \leq m < d$, and $p \in [1, d/m)$. Denote by $I$ the identity operator $I\colon V_0^{m,p}(\Omega) \to L^{p^*}(\Omega)$, where $p^* = dp/(d-mp)$. There exists $n_0 \in \mathbb{N}$, depending only on $d$ and $m$, such that*

$$C_1 n^{-\frac{m}{d}} \leq b_n(I) \leq C_2 n^{-\frac{m}{d}} \qquad \text{for every } n \geq n_0. \tag{3.22}$$

*Here $C_1$ and $C_2$ are constants depending only on $d$, $m$ and $p$.*

*In particular, $I$ is finitely strictly singular.*

**Proof.** First, we prove the upper bound on $b_n(I)$. Set $l = d^m$. We may without loss of generality assume that $|\Omega| = 1/l$; otherwise we replace $\mathrm{d}x$ with $\mathrm{d}x/(l|\Omega|)$. We start with a few definitions. By $G\colon V_0^{m,p}(\Omega) \to \bigoplus_{j=1}^l L^p(\Omega)$ we denote the linear isometric operator defined as

$$Gu = \nabla^m u,\ \ u \in V_0^{m,p}(\Omega).$$

Here $\bigoplus_{j=1}^l$ stands for the $\ell_p$-direct sum, and the way in which the vector $\nabla^m u$ is ordered is completely immaterial—we fix arbitrary order. Furthermore, let $R = \bigoplus_{j=1}^l \Omega^{(j)}$ consist of $l$ disjoint copies of $\Omega$, each endowed with the Lebesgue measure. We denote the corresponding probabilistic measure space by $(R, \mu)$. Finally, $S\colon \bigoplus_{j=1}^l L^p(\Omega) \to L^p(R, \mu)$ denotes the linear isometry defined as

$$S(f_1, \dots, f_l) = \sum_{j=1}^{l} f_j \chi_{\Omega^{(j)}}, \quad (f_1, \dots, f_l) \in \bigoplus_{j=1}^{l} L^p(\Omega).$$

Let $c_1$ be the Besicovitch constant in $\mathbb{R}^d$. Recall that $c_1$ depends only on $d$. Set $c_2 = \binom{d+m-1}{m-1}$. Note that $c_2$ is the dimension of the vector space of polynomials in $\mathbb{R}^d$ of degree at most $m-1$, which we will denote by $\mathcal{P}_{m-1}(\mathbb{R}^d)$. Assume that $n \geq 2c_1c_2$. Let $X_n$ be a $n$-dimensional subspace of $V_0^{m,p}(\Omega)$, and $\widetilde{X}_n \subseteq L^p(R,\mu)$ its image under the linear isometric operator $S \circ G$. Clearly, $\dim \widetilde{X}_n = n$. Let $L$, $g$ and $\nu$ be those from Lemma 3.2 applied to $\widetilde{X}_n$. Since $\Omega$ is bounded and $\|g\|_{L^1(R,\mu)} = 1$, for each $x \in \Omega$ we can find $r_x \in (0, \operatorname{diam} \Omega]$ such that

$$\int\limits_{\bigoplus_{j=1}^l B_{r_x}^{(j)}(x)} g \,\mathrm{d}\mu = \frac{2c_1c_2}{n}. \tag{3.23}$$

Here $B_{r_x}^{(j)}(x)$ are disjoint copies of $B_{r_x}(x)$ in $\Omega^{(j)}$. Using the Besicovitch covering lemma, we find a countable subcollection $\{B_{r_k}(x_k)\}_{k=1}^M$ such that

$$\Omega \subseteq \bigcup_{k=1}^{M} \bar{B}_{r_k}(x_k) \tag{3.24}$$

and

$$\sum_{k=1}^{M} \chi_{B_{r_k}(x_k)} \leq c_1. \tag{3.25}$$

We claim that

$$M \leq \frac{n}{2c_2}. \tag{3.26}$$

Indeed, we have

$$\begin{aligned} M\frac{2c_1c_2}{n} &= \sum_{k=1}^{M} \int\limits_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)} g \,\mathrm{d}\mu \\ &\leq \Big\| \sum_{k=1}^{M} \chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)} \Big\|_{L^\infty(R,\mu)} \|g\|_{L^1(R,\mu)} \\ &\leq c_1. \end{aligned}$$

Recall that, for every $u \in V^{m,p}(B_{r_k}(x_k))$, $k = 1, \dots, M$, there is a polynomial $P_{u,k} \in \mathcal{P}_{m-1}(\mathbb{R}^d)$, depending on $u$ and $B_{r_k}(x_k)$, such that

$$\|u - P_{u,k}\|_{L^{p^*}(B_{r_k}(x_k))} \leq c_3 \|\nabla^m u\|_{L^p(B_{r_k}(x_k))}; \tag{3.27}$$

moreover, the dependence of $P_{u,k}$ on $u$ is linear. Here $c_3$ depends only on $d$, $m$ and $p$. This follows easily by iterating the classical Sobolev–Poincaré inequality on balls (e.g., see [23, Corollary 1.64]).

We claim that there is a subspace $Y$ of $X_n$ with $\dim Y \geq n/2$ such that

$$\|u\|_{L^{p^*}(B_{r_k}(x_k))} \leq c_3 \|\nabla^m u\|_{L^p(B_{r_k}(x_k))} \quad \text{for every } u \in Y \text{ and } k. \tag{3.28}$$

Indeed, set $Y_0 = X_n$, and let $Y_1$ be the kernel of the linear operator $Y_0 \ni u \mapsto P_{u,1} \in \mathcal{P}_{m-1}(\mathbb{R}^d)$. By the rank-nullity theorem, we have

$$\dim Y_1 \geq n - \dim \mathcal{P}_{m-1}(\mathbb{R}^d) = n - c_2.$$

Now, let $Y_2$ be the kernel of the linear operator $Y_1 \ni u \mapsto P_{u,2}$. It follows that $\dim Y_2 \geq n - 2c_2$. Proceeding in the obvious way, we find a subspace $Y = Y_M$ of $X_n$ with $\dim Y \geq n - Mc_2$ such that $P_{u,k} \equiv 0$ for every $u \in Y$ and $k = 1, \dots, M$. The claim now immediately follows from (3.26) and (3.27).

Let $\widetilde{Y}$ be the image of $Y$ under the linear isometric operator $S \circ G$. Thanks to Lemma 3.2, there is $\tilde{u} \in \widetilde{Y} \subseteq \widetilde{X}_n$ such that

$$\|u\|_{V_0^{m,p}(\Omega)} = \|\tilde{u}\|_{L^p(R,\mu)} = 1 \tag{3.29}$$

and

$$\sup_{q \in [1,\infty)} \frac{\|L\tilde{u}\|_{L^q(R,\nu)}}{\sqrt{q}} \leq c_4, \tag{3.30}$$

where $u = (SG)^{-1}\tilde{u} \in V_0^{m,p}(\Omega)$ and $\mathrm{d}\nu = g\,\mathrm{d}\mu$. Here $c_4$ is a constant depending only on $p$. By (3.24) and (3.28), we have

$$\begin{aligned}
\|u\|_{L^{p^*}(\Omega)}^{p^*} &\le \sum_{k=1}^{M} \|u\|_{L^{p^*}(B_{r_k}(x_k))}^{p^*} \le c_3^{p^*} \sum_{k=1}^{M} \|\nabla^m u\|_{L^{p}(B_{r_k}(x_k))}^{p^*} \\
&= c_3^{p^*} \sum_{k=1}^{M} \|\tilde{u}\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)}\|_{L^p(R,\mu)}^{p^*} \\
&= c_3^{p^*} \sum_{k=1}^{M} \|(L\tilde{u})\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)} g^{1/p}\|_{L^p(R,\mu)}^{p^*} \\
&= c_3^{p^*} \sum_{k=1}^{M} \|(L\tilde{u})\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)}\|_{L^p(R,\nu)}^{p^*}. \qquad (3.31)
\end{aligned}$$

Furthermore, by the Hölder inequality combined with the identity $1/p^* = 1/p - m/d$, (3.23), (3.24) combined with (3.25), and (3.30), we have

$$\begin{aligned}
&\sum_{k=1}^{M} \|(L\tilde{u})\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)}\|_{L^p(R,\nu)}^{p^*} \\
&\le \sum_{k=1}^{M} \|(L\tilde{u})\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)}\|_{L^{p^*}(R,\nu)}^{p^*} \|\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)}\|_{L^{\frac{d}{m}}(R,\nu)}^{p^*} \\
&= \sum_{k=1}^{M} \|(L\tilde{u})\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)}\|_{L^{p^*}(R,\nu)}^{p^*} \|\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)}\|_{L^{1}(R,\nu)}^{\frac{mp}{d-mp}} \\
&= \left(\frac{2c_1c_2}{n}\right)^{\frac{mp}{d-mp}} \sum_{k=1}^{M} \|(L\tilde{u})\chi_{\bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k)}\|_{L^{p^*}(R,\nu)}^{p^*} \\
&\le c_1 \left(\frac{2c_1c_2}{n}\right)^{\frac{mp}{d-mp}} \|L\tilde{u}\|_{L^{p^*}(R,\nu)}^{p^*} \\
&\le c_1 \left(\frac{2c_1c_2}{n}\right)^{\frac{mp}{d-mp}} c_4^{p^*}(p^*)^{p^*/2}. \qquad (3.32)
\end{aligned}$$

Combining (3.31) and (3.32), we obtain

$$\|u\|_{L^{p^*}(\Omega)}^{p^*} \le C_2^{p^*} n^{-\frac{mp}{d-mp}}. \qquad (3.33)$$

Here $C_2^{p^*} = c_1(2c_1c_2)^{\frac{mp}{d-mp}}(c_3c_4\sqrt{p^*})^{p^*}$ depends only on $d$, $m$ and $p$. The desired upper bound in (3.22) now follows immediately from (3.29) and (3.33).

Finally, we turn our attention to the lower bound in (3.22), whose proof is simpler. To that end, recall that we have (e.g., see [30, Remark 7])

$$b_n(\ell_p \to \ell_{p^*}) = n^{\frac{p-p^*}{pp^*}} = n^{-\frac{m}{d}} \quad \text{for every } n \in \mathbb{N}. \qquad (3.34)$$

Here we used the fact that $1 \le p < p^*$. Let $0 < \lambda < \|I\|$ and $\varepsilon > 0$. By (3.34), there is a subspace $E_n$ of $\ell_p$ with $\dim E_n = n$ such that

$$\inf_{\substack{\{\alpha_j\}_{j=1}^\infty \in E_n \\ \|\{\alpha_j\}_{j=1}^\infty\|_{\ell_p}=1}} \|\{\alpha_j\}_{j=1}^\infty\|_{\ell_{p^*}} \ge n^{-\frac{m}{d}} - \varepsilon. \tag{3.35}$$

Furthermore, let $\{u_j\}_{j=1}^\infty$ and $\{B_j\}_{j=1}^\infty$ be systems whose existence is guaranteed by Proposition 3.1 with $q = p^*$. Note that the linear operator $T\colon \ell_p \to V_0^{m,p}(\Omega)$ defined as

$$T(\{\alpha_j\}_{j=1}^\infty) = \sum_{j=1}^\infty \alpha_j u_j,\ \{\alpha_j\}_{j=1}^\infty \in \ell_p,$$

is well defined and isometric. Indeed, since the functions $u_j$ have mutually disjoint supports and $\|u_j\|_{V_0^{m,p}(\Omega)} = 1$, we have

$$\Big\|\sum_{j=1}^\infty \alpha_j u_j\Big\|_{V_0^{m,p}(\Omega)}^p = \Big\|\sum_{j=1}^\infty \alpha_j \nabla u_j\Big\|_{L^p(\Omega)}^p = \sum_{j=1}^\infty |\alpha_j|^p \|\nabla u_j\|_{L^p(\Omega)}^p = \sum_{j=1}^\infty |\alpha_j|^p.$$

In particular, $T$ is injective. Furthermore, we also have

$$\Big\|\sum_{j=1}^\infty \alpha_j u_j\Big\|_{L^{p^*}(\Omega)} = \lambda \|\{\alpha_j\}_{j=1}^\infty\|_{\ell_{p^*}} \quad \text{for every } \{\alpha_j\}_{j=1}^\infty \in \ell_{p^*} \tag{3.36}$$

since $\|u_j\|_{L^{p^*}(\Omega)} = \lambda$ for every $j \in \mathbb{N}$. Set $X_n = TE_n$. We have $\dim X_n = \dim E_n = n$. Combining (3.35) and (3.36) with the fact that $T$ is isometric, we arrive at

$$\begin{aligned} b_n(I) &\ge \inf_{\substack{u \in X_n \\ \|u\|_{V_0^{m,p}(\Omega)}=1}} \|u\|_{L^{p^*}(\Omega)} = \inf_{\substack{\{\alpha_j\}_{j=1}^\infty \in E_n \\ \|\{\alpha_j\}_{j=1}^\infty\|_{\ell_p}=1}} \Big\|\sum_{j=1}^\infty \alpha_j u_j\Big\|_{L^{p^*}(\Omega)} \\ &= \lambda \inf_{\substack{\{\alpha_j\}_{j=1}^\infty \in E_n \\ \|\{\alpha_j\}_{j=1}^\infty\|_{\ell_p}=1}} \|\{\alpha_j\}_{j=1}^\infty\|_{\ell_{p^*}} \ge \lambda(n^{-\frac{m}{d}} - \varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0^+$ and $\lambda \to \|I\|^-$, we obtain

$$b_n(I) \ge \|I\| n^{-\frac{m}{d}}.$$

Note that this is actually the desired lower bound in (3.22) because we can take $C_1 = \|I\|$. Indeed, the norm of the embedding $V_0^{m,p}(\Omega) \to L^{p^*}(\Omega)$ depends only on $d$, $m$ and $p$ but not on $\Omega$. This follows from the simple observation that

$$\begin{aligned} \|I\colon V_0^{m,p}(B) \to L^{p^*}(B)\| &\le \|I\colon V_0^{m,p}(\Omega) \to L^{p^*}(\Omega)\| \\ &\le \|I\colon V_0^{m,p}(\tilde{B}) \to L^{p^*}(\tilde{B})\|, \end{aligned}$$

where $B$ and $\tilde{B}$ are (any) open balls in $\mathbb{R}^d$ such that $B \subseteq \Omega \subseteq \tilde{B}$, and from the fact that $\|I\colon V_0^{m,p}(B) \to L^{p^*}(B)\|$ is constant for every open ball $B \subseteq \mathbb{R}^d$ and depends only on $d, m$ and $p$—to that end, recall (3.2). $\square$

We conclude with the Lorentz case. The following theorem tells us that the "really optimal" Sobolev embedding (1.2) is not strictly singular (let alone finitely strictly singular); moreover, all its Bernstein numbers coincide with its norm.

**Theorem 3.4.** *Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty bounded open set, $m \in \mathbb{N}$, $1 \le m < d$, and $p \in [1, d/m)$. Denote by $I$ the identity operator $I\colon V_0^{m,p}(\Omega) \to L^{p^*,p}(\Omega)$, where $p^* = dp/(d-mp)$. We have*

$$b_n(I) = \|I\| \qquad \text{for every } n \in \mathbb{N}, \tag{3.37}$$

*where $\|I\|$ denotes the operator norm.*

*Furthermore, $I$ is not strictly singular.*

**Proof.** Thanks to the property (S1) of (strict) $s$-numbers, it is sufficient to show that

$$b_n(I) \ge \|I\| \qquad \text{for every } n \in \mathbb{N}.$$

Let $\varepsilon > 0$ and $0 < \lambda < \|I\|$, and $\{u_j\}_{j=1}^\infty \subseteq V_0^{m,p}(\Omega)$ be a system of functions from Proposition 3.1 with $q = p$. Let $X_n$ be the subspace of $V_0^{m,p}(\Omega)$ spanned by the functions $\{u_j\}_{j=1}^n$. Since the functions $u_j$ have mutually disjoint supports, it follows that $\dim X_n = n$, and that we have, for every $u = \sum_{j=1}^n \alpha_j u_j \in X_n$,

$$\|u\|^p_{V_0^{m,p}(\Omega)} = \sum_{j=1}^n |\alpha_j|^p.$$

Furthermore, thanks to (3.1),

$$\|u\|^p_{L^{p^*,p}(\Omega)} = \Big\| \sum_{j=1}^n \alpha_j u_j \Big\|^p_{L^p(\Omega)} \ge \frac{\lambda^p}{1+\varepsilon} \sum_{j=1}^n |\alpha_j|^p.$$

Hence

$$b_n(I) \ge \inf_{u \in X_n \setminus \{0\}} \frac{\|u\|_{L^{p^*,p}(\Omega)}}{\|u\|_{V_0^{m,p}(\Omega)}} \ge \frac{\lambda}{(1+\varepsilon)^{\frac{1}{p}}}.$$

Since this holds for every $\varepsilon > 0$ and $0 < \lambda < \|I\|$, it follows that $b_n(I) \ge \|I\|$.

Finally, to show that $I$ is not strictly singular, it is sufficient to take any $\varepsilon > 0$ and $0 < \lambda < \|I\|$ and consider the infinite dimensional subspace of $V_0^{m,p}(\Omega)$ spanned by the functions $u_1, u_2, \dots$ Arguing as above, we immediately see that $I$ is bounded from below on this infinite dimensional subspace. Therefore, $I$ is not strictly singular. $\square$

**Remark 3.5.** In light of (2.1), (3.37) actually tells us that, in the case of the "really optimal" Sobolev embedding (1.2), we have

$$s_n(I) = \|I\|$$

for every $n \in \mathbb{N}$ and every injective strict $s$-number $s$.

## Data availability

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Full length article

# Measure of noncompactness of Sobolev embeddings on strip-like domains ☆

David E. Edmunds[a], Jan Lang[b], Zdeněk Mihula[c,d,*]

[a] Department of Mathematics, University of Sussex, Falmer, Brighton, BN1 9QH, UK
[b] Department of Mathematics, The Ohio State University, Columbus, OH, 43210-1174, USA
[c] Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
[d] Department of Mathematics, Faculty of Electric Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague 6, Czech Republic

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**Abstract**

We compute the precise value of the measure of noncompactness of Sobolev embeddings $W_0^{1,p}(D) \hookrightarrow L^p(D)$, $p \in (1, \infty)$, on strip-like domains $D$ of the form $\mathbb{R}^k \times \prod_{j=1}^{n-k}(r_j, q_j)$. We show that such embeddings are always maximally noncompact, that is, their measure of noncompactness coincides with their norms. Furthermore, we show that not only the measure of noncompactness but also all strict $s$-numbers of the embeddings in question coincide with their norms. We also prove that the maximal noncompactness of Sobolev embeddings on strip-like domains remains valid even when Sobolev-type spaces built upon general rearrangement-invariant spaces are considered. As a by-product we obtain the explicit form for the first eigenfunction of the pseudo-$p$-Laplacian on an $n$-dimensional rectangle.


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* Corresponding author at: Department of Mathematics, Faculty of Electric Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague 6, Czech Republic.

*E-mail addresses:* davideedmunds@aol.com (D.E. Edmunds), lang@math.osu.edu (J. Lang), mihulaz@karlin.mff.cuni.cz, mihulzde@fel.cvut.cz (Z. Mihula).

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## 1. Introduction

Let $T$ be a bounded linear map between Banach spaces $X$ and $Y$. For each $k \in \mathbb{N}$ the $k$th entropy number $e_k(T)$ of $T$ is defined by

$$e_k(T) = \inf\left\{\varepsilon > 0 \colon T\,(B_X) \text{ can be covered by } 2^{k-1} \text{ balls in } Y \text{ with radius } \varepsilon\right\},$$

where $B_X$ is the closed unit ball in $X$. Since $T$ is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$, this limit is called the measure of noncompactness of $T$; we denote it by $\beta(T)$. Plainly $0 \le \beta(T) \le \|T\|$; if $\beta(T) = \|T\|$ we say that $T$ is *maximally noncompact.* The definition of entropy numbers has its roots in the notion of the metric entropy of a set, introduced by Kolmogorov in the 1930s and which, in its different variants, has proved useful in numerous branches of mathematics and theoretical computer science. Sharp upper and lower estimates of $e_k(T)$ are known in many cases when $T$ is compact: information of this type is useful in connection with the estimation of eigenvalues and is especially complete when $T$ is an embedding of one function space into another (see, for example, [8]).

It is a different story when $T$ is not compact. If $T$ is an embedding map between function spaces on an open subset $\Omega$ of $\mathbb{R}^n$, possible reasons for noncompactness include
(i) $\Omega$ is unbounded;
(ii) some bad behavior of the boundary $\partial\Omega$ if $\Omega$ is bounded, or the norm on the domain or target function space is too weak or too strong.

An example of (ii) was provided by Hencl [9], who considered the case in which $k \in \mathbb{N}$, $p \in [1,\infty)$, $kp < n$, $1/q = 1/p - k/n$ and, in standard notation, $id\colon W_0^{k,p}(\Omega) \to L^q(\Omega)$ is the natural embedding. He showed that $id$ is maximally noncompact, so that $e_k(id) = \|id\|$ for all $k \in \mathbb{N}$. Further work in this direction, involving Sobolev spaces based on Lorentz spaces and maximally noncompact embeddings, is contained in [4] and [11].

Much less seems to be known in cases of type (i), even in quite basic situations (unless $\Omega = \mathbb{R}^n$, in which case a lot is known, see e.g. [13,18,19] and references therein). For example, suppose that $n = 2$, $\Omega = \mathbb{R}\times(0,\pi)$ and $I\colon W_0^{1,2}(\Omega) \to L^2(\Omega)$ is the natural embedding. Then it is known that $I$ is not compact, so that $\beta(I) > 0$, but although this example could hardly be simpler, the exact value of $\beta(I)$ appears to be unknown up to this point. It was shown in [9, Remark 3.8] that the embedding $I$ is maximally noncompact, but the precise value of its measure of noncompactness remained unknown. Here we settle this question by establishing a more general result in which $n \ge 2$, $k \in \{1,\dots,n-1\}$, $p \in (1,\infty)$, $-\infty < q_j < r_j < \infty$ for each $j \in \{1,\dots,n-k\}$, and

$$D = \mathbb{R}^k \times \prod_{j=1}^{n-k}(q_j, r_j);$$

the norm on $W_0^{1,p}(D)$ is defined by

$$\left(\|u\|_{L^p(D)}^p + \big\|\,|\nabla u|_{\ell_p}\big\|_{L^p(D)}^p\right)^{1/p}.$$

We show that the natural embedding $I_p : W_0^{1,p}(D) \to L^p(D)$ is maximally noncompact and

$$\beta\left(I_p\right) = \left\|I_p\right\| = \left(1 + (p-1)\left(\frac{2\pi}{p\sin(\pi/p)}\right)^p \sum_{j=1}^{n-k}\left(r_j - q_j\right)^{-p}\right)^{-1/p}.$$

For the particularly elementary illustration involving $I$ that was mentioned immediately above this gives the attractive formula

$$\beta(I) = \|I\| = 1/\sqrt{2}.$$

It is also shown that the strict $s$-numbers of $I_p$ (that is, the approximation, isomorphism, Gelfand, Bernstein, Kolmogorov and Mityagin numbers) exhibit the same behavior as the entropy numbers: the $k$th such strict $s$-number coincides with $\left\|I_p\right\|$ for all $k \in \mathbb{N}$. The proof of these assertions relies on properties of the pseudo-$p$-Laplacian and the $p$-trigonometric functions. Furthermore, we show that the embedding of the Sobolev-type space $W_0^1 X(D)$ built upon a general rearrangement-invariant space $X(D)$ to a rearrangement-invariant space $Y(D)$ is always maximally noncompact provided that the space $Y(D)$ has absolutely continuous norm. Precise definitions are contained in the following section.

## 2. Background material

In this section, we fix the notation used throughout this paper and collect the fundamental theoretical background needed later in Section 3. Let $\Omega \subseteq \mathbb{R}^n$ be an open set (throughout this paper, we assume that $n \geq 2$).

We denote the set of all continuous functions that are compactly supported in $\Omega$ by $\mathcal{C}_c(\Omega)$. The set of all smooth (i.e., infinitely differentiable) functions that are compactly supported in $\Omega$ is denoted by $\mathcal{C}_0^\infty(\Omega)$.

For $p \in [1, \infty)$, $W^{1,p}(\Omega)$ stands for *the classical first-order Sobolev space* on $\Omega$ endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \|\,|\nabla u|_{\ell_p}\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, \quad u \in W^{1,p}(\Omega),$$

where $|\nabla u|_{\ell_p}$ is the $\ell_p$-norm of the (weak) gradient of $u$, that is,

$$|\nabla u|_{\ell_p} = \left(\sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}\right|^p\right)^{\frac{1}{p}}.$$

We denote the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ by $W_0^{1,p}(\Omega)$.

We shall also work with Sobolev-type spaces built upon function spaces more general than the Lebesgue spaces. We say that a functional $\varrho\colon \mathfrak{M}^+(\Omega) \to [0, \infty]$, where $\mathfrak{M}^+(\Omega)$ is the set of all nonnegative measurable functions on $\Omega$, is a rearrangement-invariant Banach function norm if, for all $f, g, f_j \in \mathfrak{M}^+(\Omega)$, $j \in \mathbb{N}$, for all $\alpha \in [0, \infty)$, and for all measurable $E \subseteq \Omega$, the following properties hold:

- $\varrho(f) = 0$ if and only if $f = 0$ a.e., $\varrho(\alpha f) = \alpha\varrho(f)$, $\varrho(f + g) \leq \varrho(f) + \varrho(g)$;
- if $0 \leq g \leq f$ a.e., then $\varrho(g) \leq \varrho(f)$;
- if $0 \leq f_j \nearrow f$ a.e., then $\varrho(f_j) \nearrow \varrho(f)$;
- if $|E| < \infty$, then $\varrho(\chi_E) < \infty$;
- if $|E| < \infty$, then there is a constant $C_E$, which may depend only on $E$ and $\varrho$, such that $\int_E f(x)\,\mathrm{d}x \leq C_E\varrho(f)$;
- $\varrho(f) = \varrho(g)$ whenever $f$ and $g$ are equimeasurable, that is, $|\{x \in \Omega\colon f(x) > \lambda\}| = |\{x \in \Omega\colon g(x) > \lambda\}|$ for every $\lambda > 0$.

If $\varrho$ is a rearrangement-invariant Banach function norm, then the set $X(\Omega) = \{f \in \mathfrak{M}(\Omega) \colon \varrho(|f|) < \infty\}$, where $\mathfrak{M}(\Omega)$ is the set of all measurable functions on $\Omega$, endowed with the norm $\|\cdot\|_{X(\Omega)}$ defined as

$$\|f\|_{X(\Omega)} = \varrho(|f|), \ \ f \in X(\Omega),$$

is called a rearrangement-invariant Banach function space. A rearrangement-invariant Banach function space (we shall write just 'a *rearrangement-invariant space*') is a Banach space. Textbook examples of rearrangement-invariant spaces are the Lebesgue spaces $L^p$ ($p \in [1, \infty]$), the (two-parametric) Lorentz spaces $L^{p,q}$ (for appropriate values of the parameters, see, e.g., [3, pp. 218–220]) or the Orlicz spaces. We say that a function $f \in X(\Omega)$ *has absolutely continuous norm* in $X(\Omega)$ if $\lim_{k\to\infty} \|f\chi_{E_k}\|_{X(\Omega)} = 0$ for every sequence $\{E_k\}_{k=1}^{\infty}$ of measurable sets $E_k \subseteq \Omega$ such that $\lim_{k\to\infty} \chi_{E_k}(x) = 0$ for a.e. $x \in \Omega$. We say that $X(\Omega)$ has absolutely continuous norm if every $f \in X(\Omega)$ has absolutely continuous norm in $X(\Omega)$. For example, the Lebesgue space $L^p(\Omega)$ has absolutely continuous norm if and only if $p < \infty$.

Comprehensive accounts of the theory of rearrangement-invariant spaces can be found, e.g., in [3] or [15].

We denote *the (first-order) Sobolev-type space built upon a rearrangement-invariant space* $X(\Omega)$, that is, the set of all weakly differentiable functions from $X(\Omega)$ whose gradients also belong to $X(\Omega)$, by $W^1X(\Omega)$. We equip $W^1X(\Omega)$ with the norm

$$\|u\|_{W^1X(\Omega)} = \|u\|_{X(\Omega)} + \| \, |\nabla u|_{\ell_1} \|_{X(\Omega)}, \ \ u \in W^1X(\Omega),$$

which turns $W^1X(\Omega)$ into a Banach space. Note that we have $W^{1,p}(\Omega) = W^1L^p(\Omega)$ in the set-theoretical sense, but their norms are merely equivalent (unless $p = 1$). The closure of $\mathcal{C}_0^{\infty}(\Omega)$ in $W^1X(\Omega)$ is denoted by $W_0^1X(\Omega)$.

Any rule $s \colon T \to \{s_m(T)\}_{m=1}^{\infty}$ that assigns to each bounded linear operator $T$ from a Banach space $X$ to a Banach space $Y$ (we shall write $T \in B(X, Y)$) a sequence $\{s_m(T)\}_{m=1}^{\infty}$ of nonnegative numbers having, for every $m \in \mathbb{N}$, the following properties:

(S1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$;
(S2) $s_m(S + T) \leq s_m(S) + \|T\|$ for every $S \in B(X, Y)$;
(S3) $s_m(BTA) \leq \|B\| s_m(T) \|A\|$ for every $A \in B(W, X)$ and $B \in B(Y, Z)$, where $W, Z$ are Banach spaces;
(S4) $s_m(id \colon E \to E) = 1$ for every Banach space $E$ with $\dim E \geq m$;
(S5) $s_m(T) = 0$ if $\operatorname{rank} T < m$;

is called a *strict $s$-number*. Notable examples of strict $s$-numbers are *the approximation numbers $a_m$, the isomorphism numbers $i_m$, the Gelfand numbers $c_m$, the Bernstein numbers $b_m$, the Kolmogorov numbers $d_m$* or *the Mityagin numbers $m_m$*. For their definitions and the difference between strict $s$-numbers and 'standard' $s$-numbers, we refer the reader to [7, Chapter 5]. In this paper, we will only need the definition of the isomorphism numbers. The $m$th isomorphism number $i_m(T)$ of $T \in B(X, Y)$ is defined as

$$i_m(T) = \sup\{\|A\|^{-1}\|B\|^{-1}\} \tag{2.1}$$

where the supremum is taken over all Banach spaces $G$ with $\dim(G) \geq m$ and all bounded linear operators $A \colon Y \to G$, $B \colon G \to X$ such that $ATB$ is the identity on $G$. The isomorphism numbers are the smallest strict $s$-numbers [16, Theorem 3.4], that is,

$$s_m(T) \geq i_m(T) \tag{2.2}$$

for every strict $s$-number $s$, for every $T \in B(X, Y)$, and for every $m \in \mathbb{N}$.

If $T \in B(X, Y)$, then its *mth entropy number* $e_m(T)$, $m \in \mathbb{N}$, already defined in the introductory section, satisfies

$$e_m(T) = \inf\left\{\varepsilon > 0\colon \text{ there are } y_1, \dots, y_{2^{m-1}} \in Y \text{ such that } T(B_X) \subseteq \bigcup_{j=1}^{2^{m-1}} \left(y_j + \varepsilon B_Y\right)\right\},$$

where $B_X$ and $B_Y$ are the closed unit balls of $X$ and $Y$, respectively. Note that entropy numbers are not (strict) $s$-numbers (e.g., property (S4) is violated [6, Chapter 2, Proposition 1.3]) even though they possess similar properties. Through entropy numbers, we define *the measure of noncompactness* $\beta(T)$ of $T \in B(X, Y)$ as

$$\beta(T) = \lim_{m\to\infty} e_m(T).$$

Note that the limit always exists because the sequence $\{e_m(T)\}_{m=1}^{\infty}$ is nonincreasing. Furthermore, we have $0 \le \beta(T) \le \|T\|$, and the operator $T$ is compact if and only if $\beta(T) = 0$. We say that the operator $T$ is *maximally noncompact* if $\beta(T) = \|I\|$. Thanks to the monotonicity of $\{e_m(T)\}_{m=1}^{\infty}$, the operator $T$ is maximally noncompact if and only if $e_m(T) = \|T\|$ for every $m \in \mathbb{N}$.

A great deal of information on (strict) $s$-numbers and entropy numbers can be found in the pioneering work of Pietsch [16,17] as well as, e.g., in books [5,6] and references therein.

Lastly, let us briefly recall *the generalized $p$-trigonometric functions*. For $p \in (1, \infty)$, $\sin_p$ is defined on $[0, \frac{\pi_p}{2}]$ as the inverse function to the increasing function

$$[0, 1] \ni t \mapsto \int_0^t \left(1 - s^p\right)^{-\frac{1}{p}} \,\mathrm{d}s,$$

where

$$\pi_p = 2\int_0^1 \left(1 - s^p\right)^{-\frac{1}{p}} \,\mathrm{d}s.$$

We extend $\sin_p$ to $[-\pi_p, \pi_p]$ by defining $\sin_p(t) = \sin_p(\pi_p - t)$, $t \in \left[\frac{\pi_p}{2}, \pi_p\right]$, and $\sin_p(t) = -\sin_p(-t)$, $t \in [-\pi_p, 0]$. Finally, we extend $\sin_p$ to the whole real line in such a way that the resulting function is $2\pi_p$-periodic. The function $\sin_p$ is continuously differentiable on $\mathbb{R}$, and its derivative is denoted by $\cos_p$. Note that $\sin_2 = \sin$, $\cos_2 = \cos$ and $\pi_2 = \pi$. The interested reader can find more information on properties of the generalized $p$-trigonometric functions as well as their connection with the theory of the $p$-Laplacian, e.g., in [7,12,14].

## 3. Noncompactness

Although the following proposition concerning the density of smooth compactly supported functions in rearrangement-invariant spaces having absolutely continuous norms on open sets $\Omega \subseteq \mathbb{R}^n$ is folklore, the only reference that we could find is [10, Lemma 2.10], which deals with the particular case $\Omega = \mathbb{R}$. For the reader's convenience, we sketch a proof of the assertion in full generality.

**Proposition 3.1.** *Let $\Omega \subseteq \mathbb{R}^n$ be a (nonempty) open set and let $X(\Omega)$ be a rearrangement-invariant space on $\Omega$. If $X(\Omega)$ has absolutely continuous norm, then smooth compactly supported functions on $\Omega$ are dense in $X(\Omega)$.*

**Proof.** Let $u \in X(\Omega) \setminus \{0\}$. Since $X(\Omega)$ has absolutely continuous norm, bounded functions supported in sets of finite measure are dense in $X(\Omega)$ (e.g. [3, Chapter 1, Proposition 3.10 and Theorem 3.13]). Therefore, we may assume, without loss of generality, that $u$ is bounded on $\Omega$. Let $\varepsilon > 0$ be given. Set $\Omega_k = \Omega \cap \{x \in \mathbb{R}^n : |x| < k\}$ for $k \in \mathbb{N}$. Clearly, $\chi_{\Omega\setminus\Omega_k} \to 0$ as $k \to \infty$. Hence, since $u$ has absolutely continuous norm in $X(\Omega)$, there is $k \in \mathbb{N}$ such that

$$\|u\chi_{\Omega\setminus\Omega_k}\|_{X(\Omega)} < \frac{\varepsilon}{3}. \tag{3.1}$$

Moreover, we may assume that $\Omega_k \neq \emptyset$. Furthermore, since $X(\Omega)$ has absolutely continuous norm, there is $\delta > 0$ such that

$$\|\chi_E\|_{X(\Omega)} < \frac{\varepsilon}{6\|u\|_{L^\infty(\Omega)}} \tag{3.2}$$

for every measurable $E \subseteq \Omega$ such that $|E| < \delta$.

Thanks to Luzin's theorem (recall that $\Omega_k$ is bounded and locally compact and that $u$ is a bounded measurable function), there is a compact set $F \subseteq \Omega_k$ and a continuous, compactly supported function $f \in \mathcal{C}_c(\Omega_k)$ such that

$$u = f \quad \text{on } F, \tag{3.3}$$

$$\sup_{x\in\Omega_k} |f(x)| \le \sup_{x\in\Omega_k} |u(x)|, \tag{3.4}$$

$$|\Omega_k \setminus F| < \delta. \tag{3.5}$$

Note that (3.5) together with (3.2) implies that

$$\|\chi_{\Omega_k\setminus F}\|_{X(\Omega)} < \frac{\varepsilon}{6\|u\|_{L^\infty(\Omega)}}. \tag{3.6}$$

Furthermore, since $f$ is continuous and compactly supported in the open set $\Omega_k$, we can employ a standard mollification argument to find a smooth compactly supported function $g \in \mathcal{C}_0^\infty(\Omega_k)$ such that

$$\sup_{x\in\Omega_k} |f(x) - g(x)| < \frac{\varepsilon}{3\|\chi_{\Omega_k}\|_{X(\Omega)}} \tag{3.7}$$

(note that $0 < \|\chi_{\Omega_k}\|_{X(\Omega)} < \infty$ because $\Omega_k$ has finite positive measure).

Finally, combining (3.1), (3.3), (3.4), (3.6) and (3.7), we arrive at

$$\begin{aligned}\|u - g\|_{X(\Omega)} &\le \|u\chi_{\Omega\setminus\Omega_k}\|_{X(\Omega)} + \|u\chi_{\Omega_k} - f\|_{X(\Omega)} + \|f - g\|_{X(\Omega)} \\ &< \frac{\varepsilon}{3} + \frac{2\|u\|_{L^\infty(\Omega_k)}}{6\|u\|_{L^\infty(\Omega)}}\varepsilon + \frac{\varepsilon}{3} \le \varepsilon. \quad \square\end{aligned}$$

**Remark 3.2.** Since the rearrangement invariance of $X(\Omega)$ was not used at all, Proposition 3.1 is actually valid even when $X(\Omega)$ is just a Banach function space, a function space defined through a functional $\varrho \colon \mathfrak{M}^+(\Omega) \to [0, \infty]$ that has all properties of a rearrangement-invariant Banach function norm but the last one.

The following theorem shows that the Sobolev embedding $W_0^1 X(D) \hookrightarrow Y(D)$ on a strip-like domain $D$ is always maximally noncompact whatever the rearrangement-invariant spaces $X(D)$ and $Y(D)$ are provided that the target space $Y(D)$ has absolutely continuous norm.

**Theorem 3.3.** *Let $k \in \{1, \dots, n-1\}$ and $-\infty < q_j < r_j < \infty$, $j = 1, \dots, n-k$. Set $D = \mathbb{R}^k \times \prod_{j=1}^{n-k}(q_j, r_j) \subseteq \mathbb{R}^n$. Let $X(D)$, $Y(D)$ be rearrangement-invariant spaces on $D$.*

*Assume that $W_0^1X(D) \hookrightarrow Y(D)$. If $Y(D)$ has absolutely continuous norm, then*

$$e_m(W_0^1X(D) \hookrightarrow Y(D)) = \|W_0^1X(D) \hookrightarrow Y(D)\| \quad \text{for every } m \in \mathbb{N},$$

*that is, the embedding $W_0^1X(D) \hookrightarrow Y(D)$ is maximally noncompact.*

**Proof.** Throughout this proof, we denote the identity operator governing the embedding $W_0^1X(D) \hookrightarrow Y(D)$ by $I$. Suppose that there is $m \in \mathbb{N}$ such that $e_m(I) < \|I\|$. Let $r, \tilde{r} > 0$ be such that $e_m(I) < r < \tilde{r} < \|I\|$. As $r > e_m(I)$, there are functions $g_j \in Y(D)$, $j = 1, \dots, 2^{m-1}$, such that

$$\forall u \in W_0^1X(D), \|u\|_{W^1X(D)} \le 1, \exists j \in \{1, \dots, 2^{m-1}\}\colon \|u - g_j\|_{Y(D)} \le r. \tag{3.8}$$

Furthermore, since (smooth) compactly supported functions are dense in $Y(D)$ by Proposition 3.1, there are functions $\tilde{g}_j \in \mathcal{C}_0^\infty(D)$ such that

$$\|g_j - \tilde{g}_j\|_{Y(D)} < \tilde{r} - r \quad \text{for every } j \in \{1, \dots, 2^{m-1}\}. \tag{3.9}$$

For every $l > 0$, set $D_l = (-l, l)^k \times \prod_{j=1}^{n-k}(q_j, r_j)$. Since

$$\|I\| = \lim_{l\to\infty} \sup_{\substack{u\in\mathcal{C}_0^\infty(D_l)\\ u\neq 0}} \frac{\|u\|_{Y(D)}}{\|u\|_{W^1X(D)}}$$

and the functions $\tilde{g}_j$ are compactly supported in $D$, there is $l > 0$ such that

$$\sup_{\substack{u\in\mathcal{C}_0^\infty(D_l)\\ u\neq 0}} \frac{\|u\|_{Y(D)}}{\|u\|_{W^1X(D)}} > \tilde{r} \tag{3.10}$$

and

$$\bigcup_{j=1}^{2^{m-1}} \operatorname{spt} \tilde{g}_j \subseteq D_l, \tag{3.11}$$

where $\operatorname{spt} \tilde{g}_j$ denotes the support of $\tilde{g}_j$, that is, the closure of the set $\{x \in D\colon \tilde{g}_j(x) \neq 0\}$.

Combining (3.10) with the translation invariance of the $Y(D)$ and $W^1X(D)$ norms in the first $k$-directions, which follows immediately from the rearrangement invariance of the $X(D)$ and $Y(D)$ norms, we see that there is a function $u \in \mathcal{C}_0^\infty(\tilde{D}_l)$ where $\tilde{D}_l = (l, 3l)^k \times \prod_{j=1}^{n-k}(q_j, r_j)$ such that $\|u\|_{W^1X(D)} = 1$ and

$$\|u\|_{Y(D)} > \tilde{r}. \tag{3.12}$$

Furthermore, since $\tilde{g}_j$ are compactly supported in $D_l$ owing to (3.11) and $D_l \cap \tilde{D}_l = \emptyset$, we have $\tilde{g}_j \equiv 0$ on $\tilde{D}_l$ for every $j = 1, \dots, 2^{m-1}$; consequently

$$\left\| \left(u - \tilde{g}_j\right) \chi_{\tilde{D}_l} \right\|_{Y(D)} = \|u \chi_{\tilde{D}_l}\|_{Y(D)} = \|u\|_{Y(D)} > \tilde{r} \quad \text{for every } j = 1, \dots, 2^{m-1}, \tag{3.13}$$

in which we also used (3.12) and the fact that $u$ is (compactly) supported in $\tilde{D}_l$.

Finally, using (3.9) and (3.13), we see that

$$\begin{aligned}\|u - g_j\|_{Y(D)} &\ge \|u - \tilde{g}_j\|_{Y(D)} - \|\tilde{g}_j - g_j\|_{Y(D)} \\ &\ge \left\| \left(u - \tilde{g}_j\right) \chi_{\tilde{D}_l} \right\|_{Y(D)} - \|\tilde{g}_j - g_j\|_{Y(D)} \\ &> \tilde{r} - (\tilde{r} - r) = r\end{aligned}$$

for every $j = 1, \dots, 2^{m-1}$, which contradicts (3.8) (recall that $\|u\|_{W^1X(D)} = 1$). Hence $e_m(I) \ge \|I\|$, and so $e_m(I) = \|I\|$. $\square$

**Remark 3.4.** Note that our choice of the norm on $W^1X(D)$ is immaterial in Theorem 3.3 and the theorem remains valid even when $W^1X(D)$ is endowed with any equivalent norm. In particular, the assertion of the theorem is also true for the standard Sobolev embeddings $W_0^{1,p}(D) \hookrightarrow L^q(D)$ with either $p \in [1, n)$ and $q \in \left[p, \frac{np}{n-p}\right]$ or $p \in [n, \infty)$ and $q \in [p, \infty)$ (cf. [1, Theorem 4.12]).

In the case when both domain and target space are the Lebesgue space $L^p$, not only do we know that the corresponding Sobolev embedding is maximally noncompact, but we also know the exact value of the norm of the embedding.

**Proposition 3.5.** *Let $p \in (1, \infty)$. Let $k \in \{0, 1, \dots, n-1\}$ and $-\infty < q_j < r_j < \infty$, $j = 1, \dots, n-k$. Set $D = \mathbb{R}^k \times \prod_{j=1}^{n-k}(q_j, r_j) \subseteq \mathbb{R}^n$. The norm of the embedding $W_0^{1,p}(D) \hookrightarrow L^p(D)$ satisfies*

$$\|W_0^{1,p}(D) \hookrightarrow L^p(D)\| = \left(1 + \pi_p^p(p-1)\sum_{j=1}^{n-k}\frac{1}{(r_j - q_j)^p}\right)^{-\frac{1}{p}}. \tag{3.14}$$

**Proof.** Since

$$\begin{aligned}\|W_0^{1,p}(D) \hookrightarrow L^p(D)\|^p &= \sup_{\substack{u\in W_0^{1,p}(D)\\ u\neq 0}} \frac{\|u\|^p_{L^p(D)}}{\|u\|^p_{L^p(D)} + \|\,|\nabla u|_{\ell_p}\|^p_{L^p(D)}} \\ &= \sup_{\substack{u\in W_0^{1,p}(D)\\ u\neq 0}} \frac{1}{1 + \frac{\|\,|\nabla u|_{\ell_p}\|^p_{L^p(D)}}{\|u\|^p_{L^p(D)}}},\end{aligned}$$

we clearly have that

$$\|W_0^{1,p}(D) \hookrightarrow L^p(D)\| = \left(1 + \inf_{\substack{u\in W_0^{1,p}(D)\\ u\neq 0}} \frac{\|\,|\nabla u|_{\ell_p}\|^p_{L^p(D)}}{\|u\|^p_{L^p(D)}}\right)^{-\frac{1}{p}}. \tag{3.15}$$

Let $\lambda$ denote the infimum in (3.15). We shall show that

$$\lambda = \pi_p^p(p-1)\sum_{j=1}^{n-k}\frac{1}{(r_j - q_j)^p}. \tag{3.16}$$

Assume that $k > 0$. For each $l > 0$, we set $D_l = (-l, l)^k \times \prod_{j=1}^{n-k}(q_j, r_j)$ and define the function $u_l$ as

$$u_l(x, y) = \left(\prod_{j=1}^{k}\sin_p\left(\frac{\pi_p x_j}{l}\right)\right)\left(\prod_{j=1}^{n-k}\sin_p\left(\frac{\pi_p(y_j - q_j)}{r_j - q_j}\right)\right)$$

for every $(x, y) = (x_1, \dots, x_k, y_1, \dots, y_{n-k}) \in D_l$, and extend it outside the rectangle $D_l$ by zero. Since $u_l \in W_0^{1,p}(D_l)$, we have that $u_l \in W_0^{1,p}(D)$. It follows from basic properties of the $p$-trigonometric functions ([12], also [7, Chapter 2, (2.22), (2.23)]) and Fubini's theorem that

$$\|u_l\|^p_{L^p(D)} = \left(\frac{2l}{p}\right)^k\prod_{j=1}^{n-k}\frac{r_j - q_j}{p} \quad \text{and}$$

$$\| \, |\nabla u_l|_{\ell_p} \|_{L^p(D)}^p = \frac{\pi_p^p}{p' p^{n-k-1}} \left(\frac{2l}{p}\right)^k \left(\prod_{j=1}^{n-k} (r_j - q_j)\right) \left(\frac{k}{l^p} + \sum_{j=1}^{n-k} \frac{1}{(r_j - q_j)^p}\right),$$

whence

$$\lambda \leq \pi_p^p (p-1) \left(\frac{k}{l^p} + \sum_{j=1}^{n-k} \frac{1}{(r_j - q_j)^p}\right).$$

Hence, since $l > 0$ was arbitrary, we obtain that

$$\lambda \leq \pi_p^p (p-1) \sum_{j=1}^{n-k} \frac{1}{(r_j - q_j)^p}. \tag{3.17}$$

Next, it is well known ([14, page 28], also [7, Theorem 3.3]) that

$$\inf_{\substack{v \in W_0^{1,p}((a,b)) \\ v \neq 0}} \frac{\|v'\|_{L^p((a,b))}^p}{\|v\|_{L^p((a,b))}^p} = \pi_p^p \frac{p-1}{(b-a)^p} \quad \text{for every } -\infty < a < b < \infty. \tag{3.18}$$

Since smooth compactly supported functions are dense in $W_0^{1,p}(D)$, we have that

$$\lambda = \inf_{\substack{u \in \mathcal{C}_0^\infty(D) \\ u \neq 0}} \frac{\| \, |\nabla u|_{\ell_p} \|_{L^p(D)}^p}{\|u\|_{L^p(D)}^p}.$$

Let $u \in \mathcal{C}_0^\infty(D)$. Since the function $(q_j, r_j) \ni t \mapsto u(x, y_1, \dots, y_{j-1}, t, y_{j+1}, \dots, y_{n-k})$ is in $\mathcal{C}_0^\infty((q_j, r_j))$ for each $j \in \{1, \dots, n-k\}$ and every fixed $x \in \mathbb{R}^k$, $y_i \in (q_i, r_i)$, $i \in \{1, \dots, n-k\} \setminus \{j\}$, it follows from (3.18) that

$$\begin{aligned} &\int_{q_j}^{r_j} \left| \frac{\partial u}{\partial t}(x, y_1, \dots, y_{j-1}, t, y_{j+1}, \dots, y_{n-k}) \right|^p \, \mathrm{d}t \\ &\quad \geq \pi_p^p \frac{p-1}{(r_j - q_j)^p} \int_{q_j}^{r_j} \left| u(x, y_1, \dots, y_{j-1}, t, y_{j+1}, \dots, y_{n-k}) \right|^p \, \mathrm{d}t. \end{aligned}$$

Hence, thanks to Fubini's theorem, $\|\frac{\partial u}{\partial y_j}\|_{L^p(D)}^p \geq \pi_p^p \frac{p-1}{(r_j - q_j)^p} \|u\|_{L^p(D)}^p$ for every $j \in \{1, \dots, n-k\}$. Therefore,

$$\begin{aligned} \lambda &\geq \inf_{\substack{u \in \mathcal{C}_0^\infty(D) \\ u \neq 0}} \frac{\left\| \left(\sum_{j=1}^{n-k} \left|\frac{\partial u}{\partial y_j}\right|^p\right)^{\frac{1}{p}} \right\|_{L^p(D)}^p}{\|u\|_{L^p(D)}^p} = \inf_{\substack{u \in \mathcal{C}_0^\infty(D) \\ u \neq 0}} \frac{\sum_{j=1}^{n-k} \|\frac{\partial u}{\partial y_j}\|_{L^p(D)}^p}{\|u\|_{L^p(D)}^p} \\ &\geq \pi_p^p (p-1) \sum_{j=1}^{n-k} \frac{1}{(r_j - q_j)^p}, \end{aligned}$$

which combined with (3.17) implies (3.16).

Finally, Eq. (3.14) follows from Eqs. (3.15) and (3.16).

The case where $k = 0$ is actually simpler and can be proved along the same lines, and so we omit its proof. $\square$

**Remark 3.6.** By using the identity $\pi_p = \frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)}$ (e.g. [7, (2.7)]), the norm of the embedding $W_0^{1,p}(D) \hookrightarrow L^p(D)$ can be expressed by means of the standard trigonometric functions as

$$\|W_0^{1,p}(D) \hookrightarrow L^p(D)\| = \left(1 + (p-1)\left(\frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)}\right)^p \sum_{j=1}^{n-k} \frac{1}{(r_j - q_j)^p}\right)^{-\frac{1}{p}}.$$

**Corollary 3.7.** *Let $p \in (1,\infty)$ and $R = \prod_{j=1}^n (q_j, r_j)$ where $-\infty < q_j < r_j < \infty$. The norm of the embedding $W_0^{1,p}(R) \hookrightarrow L^p(R)$ is equal to*

$$\left(1 + (p-1)\left(\frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)}\right)^p \sum_{j=1}^{n} \frac{1}{(r_j - q_j)^p}\right)^{-\frac{1}{p}}$$

*and is attained by the function $u$ defined as*

$$u(x) = \prod_{j=1}^n \sin_p\left(\frac{\pi_p(x_j - q_j)}{r_j - q_j}\right), \quad x \in R. \tag{3.19}$$

*Moreover, $u$ is the unique positive maximizer up to a positive multiplicative constant.*

**Proof.** The norm of the embedding is given by (3.14) with $k = 0$. Since a function $u \in W_0^{1,p}(R)$ maximizes $\sup_{\substack{u\in W_0^{1,p}(R)\\ u\neq 0}} \frac{\|u\|_{L^p(R)}^p}{\|u\|_{L^p(R)}^p + \|\,|\nabla u|_{\ell_p}\|_{L^p(R)}^p}$ if and only if it minimizes $\inf_{\substack{u\in W_0^{1,p}(R)\\ u\neq 0}} \frac{\|\,|\nabla u|_{\ell_p}\|_{L^p(R)}^p}{\|u\|_{L^p(R)}^p}$, the fact that the function defined by (3.19) is a positive maximizer follows immediately from the proof of Proposition 3.5. The uniqueness (up to a positive multiplicative constant) of the positive minimizer of the Rayleigh quotient

$$\inf_{\substack{u\in W_0^{1,p}(D)\\ u\neq 0}} \frac{\|\,|\nabla u|_{\ell_p}\|_{L^p(R)}^p}{\|u\|_{L^p(R)}^p}$$

was proved in [2, Lemma 2.1]. □

**Remark 3.8.** It can routinely be shown that the extreme function for the Rayleigh quotient $\inf_{\substack{u\in W_0^{1,p}(D)\\ u\neq 0}} \frac{\|\,|\nabla u|_{\ell_p}\|_{L^p(R)}^p}{\|u\|_{L^p(R)}^p}$ is the first eigenvalue of the pseudo-$p$-Laplacian operator with Dirichlet boundary conditions, i.e.:

$$\tilde{\Delta}_p u = \tilde{\lambda}_p |u|^{p-2} u, \quad \text{with } u = 0 \text{ on } \partial R, \tag{3.20}$$

where

$$\tilde{\Delta}_p u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right).$$

Then it follows from Corollary 3.7 that the first eigenfunction for the Dirichlet problem (3.20) for the pseudo-$p$-Laplacian operator on the domain $R$ is the function defined by (3.19). The corresponding first eigenvalue for the Dirichlet problem (3.20) is equal to

$$\pi_p^p (p-1) \sum_{j=1}^n \frac{1}{(r_j - q_j)^p} = \left(\frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)}\right)^p (p-1) \sum_{j=1}^n \frac{1}{(r_j - q_j)^p}$$

(see (3.16) with $k = 0$). In the case where $R$ is a cube, this was already observed in [2, Example 2.4].

This coincides well with the classical result for $p = 2$. However, whether all eigenfunctions for (3.20) on the domain $R$ are of the form

$$\prod_{j=1}^{n} \sin_p \left( \frac{\pi_p k_j (x_j - q_j)}{r_j - q_j} \right), \quad (x_1, \dots, x_n) \in R, \text{ for some } k_j \in \mathbb{N},$$

remains an open question if $p \neq 2$.

Not only is the embedding $W_0^{1,p}(D) \hookrightarrow L^p(D)$ maximally noncompact, but also all its strict $s$-numbers coincide with the norm of the embedding.

**Theorem 3.9.** *Let $p \in (1, \infty)$. Let $k \in \{1, \dots, n-1\}$ and $-\infty < q_j < r_j < \infty$, $j = 1, \dots, n-k$. Set $D = \mathbb{R}^k \times \prod_{j=1}^{n-k} (q_j, r_j) \subseteq \mathbb{R}^n$. We have that*

$$\begin{aligned} a_m(I) &= b_m(I) = c_m(I) = d_m(I) = i_m(I) = m_m(I) \\ &= e_m(I) = \|I\| = \left( 1 + \pi_p^p (p-1) \sum_{j=1}^{n-k} \frac{1}{(r_j - q_j)^p} \right)^{-\frac{1}{p}} \end{aligned} \tag{3.21}$$

*for every $m \in \mathbb{N}$, where $I$ stands for the identity operator governing the embedding $W_0^{1,p}(D) \hookrightarrow L^p(D)$.*

*In particular,*

$$s_m(I) = \|I\| = \left( 1 + \pi_p^p (p-1) \sum_{j=1}^{n-k} \frac{1}{(r_j - q_j)^p} \right)^{-\frac{1}{p}}$$

*for each strict $s$-number $s$ and every $m \in \mathbb{N}$, and the embedding $W_0^{1,p}(D) \hookrightarrow L^p(D)$ is maximally noncompact.*

**Proof.** The last two equalities in (3.21) were already proved in Theorem 3.3 and Proposition 3.5, respectively. As for the other equalities, it suffices to show that $i_m(I) \geq \|I\|$ for all $m \in \mathbb{N}$, where $i_m(I)$ is the $m$th isomorphism number of $I$ defined by (2.1), thanks to the fact that the isomorphism numbers are the smallest strict $s$-numbers (see (2.2)) and property (S1).

Let $\varepsilon > 0$ be given. Since smooth compactly supported functions are dense in $W_0^{1,p}(D)$, there are $l > 0$ and a function $u \in \mathcal{C}_0^\infty(D_l)$ such that

$$\|u\|_{L^p(D)} > \|I\| - \varepsilon \quad \text{and} \quad \|u\|_{W^{1,p}(D)} = 1 \tag{3.22}$$

where $D_l = (-l, l)^k \times \prod_{j=1}^{n-k} (q_j, r_j)$. For $i = 1, 2, \dots, m$, we define

$$D_l^i = ((2i-3)l, (2i-1)l)^k \times \prod_{j=1}^{n-k} (q_j, r_j)$$

and

$$u_i(x, y) = u(x_1 - 2(i-1)l, \dots, x_k - 2(i-1)l, y), \ (x, y) \in D_l^i \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}.$$

Clearly, for every $i = 1, 2, \dots, m$,

$$\text{the rectangles } D_l^i \text{ are mutually disjoint and } u_i \in \mathcal{C}_0^\infty(D_l^i), \tag{3.23}$$

$$\| |\nabla u_i|_{\ell_p} \|_{L^p(D)} = \| |\nabla u|_{\ell_p} \|_{L^p(D)} \text{ and } \|u_i\|_{L^p(D)} = \|u\|_{L^p(D)}. \tag{3.24}$$

Now, we define $B\colon \ell_p(\mathbb{R}^m) \to W_0^{1,p}(D)$ by

$$B(\{\alpha_j\}_{j=1}^m) = \sum_{j=1}^m \alpha_j u_j.$$

The operator $B$ is a well-defined linear operator and $\|B\| = 1$. Indeed, $\sum_{j=1}^m \alpha_j u_j \in W_0^{1,p}(D)$ and

$$\|B(\{\alpha_j\}_{j=1}^m)\|_{W^{1,p}(D)}^p = \left\| \sum_{j=1}^m \alpha_j u_j \right\|_{W^{1,p}(D)}^p = \sum_{i=1}^m |\alpha_j|^p \|u_j\|_{W^{1,p}(D)}^p = \sum_{j=1}^m |\alpha_j|^p \tag{3.25}$$

owing to (3.22), (3.23) and (3.24).

Next, note that the functions $u_i$, $i = 1, \dots, m$, are linearly independent in $L^p(D)$ because they have mutually disjoint supports. Hence, for each $i \in \{1, \dots, m\}$, the linear functionals $\tilde{\gamma}_i\colon \operatorname{span}\{u_1, \dots, u_m\} \to \mathbb{R}$ defined as

$$\tilde{\gamma}_i \left( \sum_{j=1}^m \beta_j u_j \right) = \beta_i$$

are well defined. Moreover,

$$\left| \tilde{\gamma}_i \left( \sum_{j=1}^m \beta_j u_j \right) \right| = |\beta_i| = \frac{|\beta_i| \|u_i\|_{L^p(D)}}{\|u\|_{L^p(D)}} \leq \frac{1}{\|u\|_{L^p(D)}} \left\| \sum_{j=1}^m \beta_j u_j \right\|_{L^p(D)}$$

thanks to (3.23) and (3.24). Therefore, by virtue of the Hahn–Banach theorem, there are functionals $\gamma_i\colon L^p(D) \to \mathbb{R}$, $i = 1, \dots, m$, such that $\gamma_i = \tilde{\gamma}_i$ on $\operatorname{span}\{u_1, \dots, u_m\}$ and

$$\|\gamma_i\| \leq \frac{1}{\|u\|_{L^p(D)}}. \tag{3.26}$$

We define $A\colon L^p(D) \to \ell_p(\mathbb{R}^m)$ as

$$Av = \left( \gamma_1 \left( v \chi_{D_l^1} \right), \dots, \gamma_m \left( v \chi_{D_l^m} \right) \right).$$

The operator $A$ is clearly linear, for the functionals $\gamma_i$ are linear. Furthermore

$$\|A\| \leq \frac{1}{\|u\|_{L^p(D)}}. \tag{3.27}$$

Indeed,

$$\|Av\|_{\ell_p(\mathbb{R}^m)}^p = \sum_{i=1}^m \left| \gamma_i \left( v \chi_{D_l^i} \right) \right|^p \leq \frac{1}{\|u\|_{L^p(D)}^p} \sum_{i=1}^m \|v \chi_{D_l^i}\|_{L^p(D)}^p \leq \frac{1}{\|u\|_{L^p(D)}^p} \|v\|_{L^p(D)}^p$$

in view of (3.26) and (3.23).

Finally, upon observing that $AIB$ is the identity on $\ell_p(\mathbb{R}^m)$ because $u_i \chi_{D_l^i} = u_i$ for every $i \in \{1, \dots, m\}$ thanks to (3.23), we see that

$$i_m(I) \geq \|A\|^{-1} \|B\|^{-1} \geq \|u\|_{L^p(D)} > \|I\| - \varepsilon$$

owing to (3.25), (3.27) and (3.22), whence $i_m(I) \geq \|I\|$ since $\varepsilon > 0$ may be chosen arbitrarily small. $\square$

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