The Spectrum of Triangle-free Graphs

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Abstract

Denote by $q_n(G)$ the smallest eigenvalue of the signless Laplacian matrix of an n-vertex graph G. Brandt conjectured in 1997 that for regular triangle-free graphs $q_n(G) \leq \frac{4n}{25}$. We prove a stronger result: If G is a triangle-free graph then $q_n(G) \leq \frac{15n}{94} < \frac{4n}{25}$. Brandt's conjecture is a subproblem of two famous conjectures of Erdős:

(1) Sparse-Half-Conjecture: Every n-vertex triangle-free graph has a subset of vertices of size $\lceil \frac{n}{2} \rceil$ spanning at most $n^2/50$ edges.

(2) Every n-vertex triangle-free graph can be made bipartite by removing at most $n^2/25$ edges.

In our proof we use linear algebraic methods to upper bound $q_n(G)$ by the ratio between the number of induced paths with 3 and 4 vertices. We give an upper bound on this ratio via the method of flag algebras.

1 Introduction

We prove a result on eigenvalues of triangle-free graphs which is motivated by the following two famous conjectures of Erdős.

Conjecture 1.1 (Erdős' *Sparse Half Conjecture* [9, 10])**.** *Every triangle-free graph on n vertices has a subset of vertices of size $\lceil \frac{n}{2} \rceil$ vertices spanning at most $n^2/50$ edges.*

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Erdős offered a \$250 reward for proving this conjecture. There has been progress on this conjecture in various directions [4, 12, 14, 15, 17]. Most recently, Razborov [17] proved that every triangle-free graph on n vertices has an induced subgraph on $n/2$ vertices with at most $(27/1024)n^2$ edges.

For a graph G, denote by $D_2(G)$ the minimum number of edges which have to be removed to make G bipartite.

Conjecture 1.2 (Erdős [9])**.** *Let G be a triangle-free graph on n vertices. Then $D_2(G) \leq n^2/25$.*

There also has been work on this conjecture [1, 3, 11, 13, 18], most recently, Balogh, Clemen and Lidický [3] proved $D_2(G) \leq n^2/23.5$.

Brandt [5] found a surprising connection between these two conjectures and the eigenvalues of triangle-free graphs. Denote by $\lambda_n(G) \leq \ldots \leq \lambda_1(G)$ the eigenvalues of the adjacency matrix of an n-vertex graph G. Brandt [5] proved that

$$D_2(G) \geq \frac{\lambda_1(G) + \lambda_n(G)}{4} \cdot n \tag{1}$$

for regular graphs and conjectured the following.

Conjecture 1.3 (Brandt [5])**.** *Let G be a triangle-free regular n-vertex graph. Then*

$$\lambda_1(G) + \lambda_n(G) \leq \frac{4}{25} \cdot n.$$

Towards this conjecture, Brandt [5] proved a bound $\lambda_1(G)+\lambda_n(G) \leq (3-2\sqrt{2})n \approx 0.1715n$ for regular triangle-free graphs, which was very recently shown to hold also in the non-regular setting by Csikvári [7]. Brandt also noted that $\lambda_1(G_{HS})+\lambda_n(G_{HS}) = 0.14n$ for the so-called Higman-Sims graph G_{HS}, which is the unique strongly regular graph with parameters $(n, d, t, k) = (100, 22, 0, 6)$. Recall that an (n, d, t, k)-*strongly regular graph* is an n-vertex d-regular graph, where the number of common neighbors of every pair of adjacent vertices is t and the number of common neighbors of a non-adjacent pair of vertices is k.

The value $4/25$ is motivated by the fact that if either of Conjectures 1.1 or 1.2 were true, it would imply Conjecture 1.3. As observed by Brandt [5], Conjecture 1.1 implies Conjecture 1.3 by applying the following version of the Expander Mixing Lemma for a set $S \subset V(G)$ of size $n/2$ with $e(S) \leq n^2/50$.

Lemma 1.4 (Bussemaker-Cvetković-Seidel [6], Alon-Chung [2])**.** *Let G be an n-vertex d-regular graph. Then, for every $S \subseteq V(G)$, we have*

$$e(S) \geq |S| \cdot \frac{|S|d + (n - |S|)\lambda_n(G)}{2n}.$$

Given a graph G, denote by $Q = A + D$ the *signless Laplacian matrix* of G, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G. Let $q_n(G) \leq \ldots \leq q_1(G)$ be the eigenvalues of Q. By considering the signless Laplacian matrix, De Lima, Nikiforov and Olivera [8] extended (1) beyond regular graphs as follows.

Theorem 1.5 (De Lima, Nikiforov and Olivera [8])**.** *For every n-vertex graph G we have*

$$D_2(G) \geq \frac{q_n(G)}{4} \cdot n.$$

By Theorem 1.5, if Conjecture 1.2 holds then $q_n(G) \le \frac{4n}{25}$ for every triangle-free n-vertex graph G. Motivated by this observation De Lima, Nikiforov and Olivera [8] proposed investigating upper bounds on $q_n(G)$, and proved $q_n(G) \le \frac{2n}{9}$ for n-vertex triangle-free graphs G. Our main result is an improvement of this bound, which solves Conjecture 1.3.

Theorem 1.6. *If G is a triangle-free n-vertex graph, then*

$$q_n(G) \le \frac{15}{94} \cdot n < 0.1596n.$$

Note that, if G is d-regular, then $\lambda_1(G) = d$ and $q_n(G) = \lambda_n(G) + d = \lambda_n(G) + \lambda_1(G)$. Thus Theorem 1.6 implies that $\lambda_1(G) + \lambda_n(G) < 0.1596n < \frac{4n}{25}$ for every regular triangle-free n-vertex graph G, confirming Conjecture 1.3 in strong form.

It remains open to determine a sharp upper bound for $q_n(G)/n$ for triangle-free n-vertex graph G. While we only prove Theorem 1.6 with the constant $\frac{15}{94} \approx 0.1596$, a larger flag algebra computation yields $q_n(G) < 0.15467n$. Also, one can additionally assume that G is regular and use flag algebras to show a slightly stronger bound $q_n(G) = \lambda_1(G) + \lambda_n(G) < 0.15442n$. As we believe neither of these two bounds are sharp (see Section 3), we omit presenting their proofs.

2 Proof of Theorem 1.6

Our proof is based on bounding the ratio between the number of induced paths with 3 and 4 vertices in triangle-free graphs. On one hand, we upper bound $q_n(G)$ in terms of this ratio in Lemma 2.1 and Corollary 2.2. On the other hand, Lemma 2.3, which is proved using flag algebras, gives a sufficiently good bound on the ratio.

For an edge $e = xy$ of a graph G, let m_{xy} be the number of edges $uv \in E(G)$ such that $ux, vy \in E(G)$. For a vertex $x \in V(G)$, let w_x to be the number of walks of length two starting in x, i.e. w_x is the number of edges $uv \in E(G)$ such that $xu \in E(G)$.

Lemma 2.1. *If G is an n-vertex triangle-free graph and $xy \in E(G)$, then*

$$(\deg(x) + \deg(y)) \cdot q_n(G) \le w_x + w_y - 2m_{xy}\,. \tag{2}$$

Proof. Define a vector $z = (z_v)_{v \in V(G)} \in \mathbb{R}^{V(G)}$ by

$$z_v = \begin{cases} +1, & \text{if } xv \in E(G), \\ -1, & \text{if } yv \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The vector z is well-defined since G is triangle-free. Also note that $\|z\|^2 = \deg(x) + \deg(y)$. Let Q be the signless Laplacian matrix of G. We have

$$\begin{aligned} z^T Q z &= \sum_{u,v \in V(G)} Q_{uv} z_u z_v = \sum_{u \in V(G)} (z_u)^2 \deg(u) + 2 \cdot \sum_{uv \in E(G)} z_u z_v \\ &= w_x + w_y + 2 \cdot \sum_{uv \in E(G)} z_u z_v = w_x + w_y - 2m_{xy}, \end{aligned}$$

where in the last equality we used that G is triangle-free. Since Q is symmetric, $q_n(G)$ is upper bounded by the Rayleigh-Ritz quotient of z, i.e.

$$q_n(G) \leq \frac{z^T Q z}{\|z\|^2} = \frac{w_x + w_y - 2m_{xy}}{\deg(x) + \deg(y)},$$

as desired. ∎

A map $\varphi : V(H) \to V(G)$ is *a strong homomorphism* from a graph H to a graph G if for every pair of vertices $u, v \in V(H)$ we have $uv \in E(H)$ if and only if $\varphi(u)\varphi(v) \in E(G)$. Let $\hom_s(H, G)$ denote the number of strong homomorphisms from H to G. Let P_k denote the k-vertex path. Summing the bound from Lemma 2.1 over all the edges of G yields the following.

Corollary 2.2. *If G is an n-vertex triangle-free graph, then*

$$\hom_s(P_3, G) \cdot q_n(G) \leq \hom_s(P_4, G). \tag{3}$$

Proof. First, note that

$$\sum_{xy \in E(G)} (\deg(x) + \deg(y)) = \sum_{x \in V(G)} \deg^2(x) = \hom_s(P_3, G), \tag{4}$$

where in the last equality we used that G is triangle-free. Meanwhile, $\sum_{xy \in E(G)} (w_x + w_y)$ is equal to the number of walks of length three in G, i.e. the number of maps $\phi : \{1, 2, 3, 4\} \to V(G)$ such that $\{\phi(1)\phi(2), \phi(2)\phi(3), \phi(3)\phi(4)\} \subset E(G)$. Similarly, the expression $2\sum_{xy \in E(G)} m_{xy}$ is equal to the number of maps $\phi : \{1, 2, 3, 4\} \to V(G)$ such that $\{\phi(1)\phi(2), \phi(2)\phi(3), \phi(3)\phi(4), \phi(4)\phi(1)\} \subset E(G)$. It follows that $\sum_{xy \in E(G)} (w_x + w_y - 2m_{xy})$ counts the maps $\psi : \{1, 2, 3, 4\} \to V(G)$ such that $\{\psi(1)\psi(2), \psi(2)\psi(3), \psi(3)\psi(4)\} \subset E(G)$ and $\psi(4)\psi(1) \notin E(G)$, i.e.,

$$\sum_{xy \in E(G)} (w_x + w_y - 2m_{xy}) = \hom_s(P_4, G). \tag{5}$$

Summing (2) over all $xy \in E(G)$ and using (4) and (5), we obtain (3). ∎

Theorem 1.6 is an immediate consequence of the above corollary and the following lemma which is proved using standard, albeit computer-assisted flag-algebra calculation.

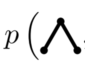
Lemma 2.3. *If G is an n-vertex triangle-free graph, then*

$$\hom_s(P_4, G) \leq \frac{15n}{94} \cdot \hom_s(P_3, G). \tag{6}$$

Proof. Suppose the lemma is false, and let G be an n-vertex triangle-free graph such that

$$\hom_s(P_4, G) = \frac{15n}{94} \cdot \hom_s(P_3, G) + \varepsilon n^4, \tag{7}$$

for some $\varepsilon > 0$. Let $G^{(b)}$ be the b-blowup of G, obtained by replacing every vertex of G by b pairwise non-adjacent vertices. Then $\hom_s(P_k, G^{(b)}) = \hom_s(P_k, G) \cdot b^k$ for $k = 3, 4$. In particular, for every $b \in \mathbb{N}$, the graph $G^{(b)}$ satisfies the analogue of (7) as well.

Let us now reformulate (7) in the flag algebra language [16]. Given a graph H, let $p\Big($$, H\Big)$ be the probability that a 3-vertex subset of $V(H)$ chosen uniformly at random induces exactly

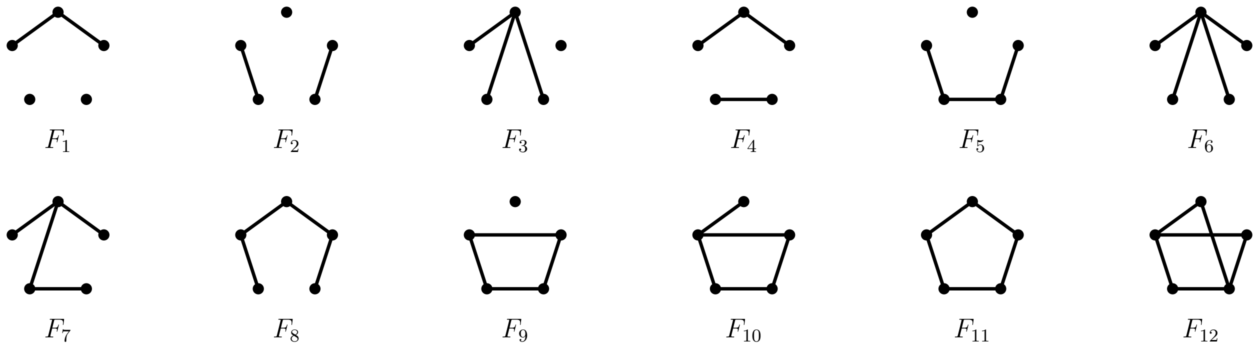


Figure 1: The set $\mathcal{F}$ of 5-vertex triangle-free graphs with at least 2 edges.

two edges. Analogously, let $p\left(\sqcup, H\right)$ be the probability that a randomly chosen 4-vertex subset induces a path of length 3.

For every fixed ℓ-vertex graph F and a k-vertex graph H, only $O(k^{\ell-1})$ maps $V(F) \to V(H)$ are non-injective. Therefore, $\hom_s(F,H) = |\operatorname{Aut}(F)| \cdot p(F,H) \cdot \binom{k}{\ell} + O(k^{\ell-1})$, so in particular every k-vertex triangle-free graph H satisfies

$$\hom_s(P_4, H) = \frac{k^4}{12} \cdot p\left(\sqcup, H\right) + O\left(k^3\right) \quad \text{and} \quad \hom_s(P_3, H) = \frac{k^3}{3} \cdot p\left(\wedge, H\right) + O\left(k^2\right).$$

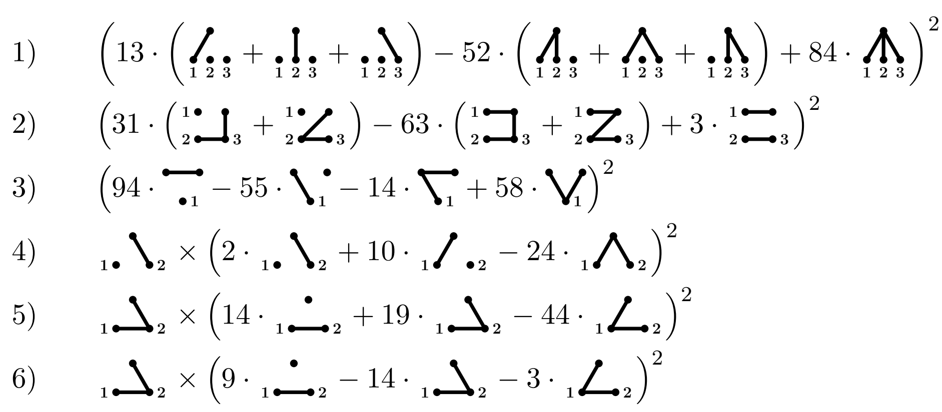
Recall that $G^{(b)}$ satisfies (7). Multiplying it by $564/(bn)^4$ and rearranging yields that

$$\lim_{b\to\infty} 30 \cdot p\left(\wedge, G^{(b)}\right) - 47 \cdot p\left(\sqcup, G^{(b)}\right) = -564\varepsilon^4.$$

In order to derive a contradiction, we present a flag algebra computation proving that an inequality

$$30 \cdot \wedge - 47 \cdot \sqcup \geq 0 \tag{8}$$

asymptotically holds in the theory of triangle-free graphs. To see that, consider the following 6 flag-algebra expressions, which are all non-negative:



Let $\mathcal{F}$ be the set of all the 5-vertex triangle-free graphs with at least 2 edges. A case analysis yields $|\mathcal{F}| = 12$; see Figure 1. Now observe that averaging over all choices of the labelled vertices in each of the 6 expressions yields a linear combination of subgraph densities, where every term has

5 vertices and at least 2 edges. Thus a flag algebra argument yields that the average of the i-th expression is equal to the i-th coordinate of $M \cdot (v_{\mathcal{F}})^T$, where $v_{\mathcal{F}} = (F_1, \ldots, F_{12})$ and

$$M = \frac{1}{30} \times \begin{pmatrix} 507 & 2028 & 0 & -4056 & -3549 & 0 & 1248 & 8112 & 16224 & -13104 & 0 & 21168 \\ 0 & 0 & 2883 & 381 & 961 & 0 & -3906 & -4098 & 3844 & 63 & 19845 & 0 \\ 12100 & -23688 & -19140 & -23620 & 12172 & 20184 & -37248 & 17486 & 47664 & 2956 & 86730 & -7392 \\ 0 & 0 & 6 & 140 & 0 & 0 & -48 & -100 & 0 & 358 & -1200 & 0 \\ 196 & 0 & 798 & 196 & -420 & 2166 & 762 & -1036 & -2464 & -702 & -3080 & 792 \\ 81 & 0 & -378 & 81 & 54 & 1176 & -165 & 27 & -108 & -87 & -135 & 279 \end{pmatrix}.$$

On the other hand, another flag algebra argument yields that the left-hand side of (8) is equal to

$$3 \cdot F_1 + 9 \cdot F_3 + 3 \cdot F_4 - \frac{17}{5} \cdot F_5 + 18 \cdot F_6 - \frac{34}{5} \cdot F_7 - \frac{49}{5} \cdot F_8 + 12 \cdot F_9 - \frac{4}{5} \cdot F_{10} - 32 \cdot F_{11} + 27 \cdot F_{12}.$$

A tedious yet straightforward calculation reveals the following coordinate-wise inequality

$$\left(\frac{1}{33}, \frac{12}{209}, \frac{3}{1147}, \frac{231}{163}, \frac{17}{84}, \frac{12}{293}\right) \cdot M < \left(3, 0, 9, 3, -\frac{17}{5}, 18, -\frac{34}{5}, -\frac{49}{5}, 12, -\frac{4}{5}, -32, 27\right),$$

which in turn shows that (8) asymptotically holds in the theory of triangle-free graphs. □

The flag algebra calculations used in the proof of Lemma 2.3 can be independently verified by a SAGE script, which is available as an ancillary file of the arXiv version of this manuscript.

3 Concluding remarks

As we have already mentioned in the introduction, a significantly larger flag algebra computation than the one used in our proof yields that $q_n(G) < 0.15467n$ for every triangle-free n-vertex graph. Similarly, assuming that G is regular allows us to show $\lambda_1(G) + \lambda_n(G) < 0.15442n$. On the other hand, our method will be able to get neither of the coefficients below $42/275 = 0.15\overline{27}$.

Indeed, consider the Higman-Sims graph G_{HS}. It is edge-transitive so $m_{xy} = 21 \cdot 6 + 22$ for every $xy \in E(G_{HS})$, and $w_x = 22^2$ for every $x \in V(G)$, where m_{xy} and w_x are defined as before Lemma 2.1. Therefore,

$$\frac{w_x + w_y - 2m_{xy}}{(\deg(x) + \deg(y)) \cdot |V(G_{HS})|} = \frac{2(22^2 - 21 \cdot 6 - 22)}{2 \cdot 22 \cdot 100} = \frac{42}{275},$$

for every $xy \in E(G_{HS})$, and so Lemma 2.1 only yields $q_n(G_{HS}) \leq \frac{42}{275} \cdot |V(G_{HS})|$. However, we have $q_n(G_{HS}) = \lambda_1(G_{HS}) + \lambda_n(G_{HS}) = 0.14 \cdot |V(G_{HS})|$. It might be that $q_n(G) \leq 0.14n$ holds for every triangle-free graph G on n vertices.

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