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QUALITATIVE SIGN STABILITY OF LINEAR TIME INVARIANT DESCRIPTOR SYSTEMS

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Abstract. This article discusses assessing the instability of a continuous linear homogeneous time-invariant descriptor system. Some necessary conditions and sufficient conditions are derived to establish the stability of a matrix pair by the fundamentals of qualitative ecological principles. The proposed conditions are derived using only the qualitative (sign) information of the matrix pair elements. Based on these conditions, the instability of a matrix pair can easily be determined, without any magnitude information of the matrix pair elements and without numerical eigenvalues calculations. With the proposed theory, Magnitude Dependent Stable, Magnitude Dependent Unstable, and Qualitative Sign Stable matrix pairs can be distinguished. The consequences of the proposed conditions and some illustrative examples are discussed.

Keywords: Descriptor systems, stability of a matrix pair, qualitative sign instability, interactions and interconnections, characteristic polynomial.

1. Introduction

The concept of stability of a matrix and a matrix pair is very fundamental to the control theory, and it is an important property to be analysed for all practical control systems. A continuous homogeneous linear time-invariant descriptor system, i.e. differential algebraic equations (DAEs) can be written as:

$$E\dot{x}(t) = Ax(t)\,, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $E, A \in \mathbb{R}^{n\times n}$ are the constant matrices [1]. When $E = I$ (identity matrix), system (1) is well known as a state space system. System (1) is called regular if $\det(\lambda E - A)$ is not identically zero as a polynomial of λ [2, 3]. A regular system (1) is said to be stable if and only if the matrix pair (E, A) is a stable matrix pair, i.e. all of its eigenvalues have negative real parts. In order to find the eigenvalues of the matrix pair (E, A), we have to determine the roots of the characteristic equation $\det(\lambda E - A) = 0$. It is remarkable that when matrix E is singular, the number of eigenvalues of the matrix pair (E, A) is less than n. This numerical eigenvalue calculation of a matrix pair of a higher order is a computationally intensive effort. To overcome this drawback, economists have introduced the concept of 'qualitative stability' and ecologists have derived some necessary and sufficient conditions for the stability of a matrix using only the sign information of matrix elements. Nonetheless, in the literature, this problem is addressed only for state space systems, i.e. when $E = I$, where eigenvalues of only matrix A are checked. This paper extends these results for checking the eigenvalues of matrix pair (E, A). In this paper, the word 'quantitative' is used for both magnitude and sign information, and the word 'qualitative' strictly for the sign information with no magnitude information of matrix elements. Matrix pairs which are stable, independent of their magnitudes with only sign information are denoted as Qualitative Sign Stable (QLSS) matrix pairs and Qualitative Sign Unstable are denoted as 'QLSU' matrix pairs. Matrix pairs, whose stability/instability depend upon the magnitude information of the matrix pair elements, are denoted as Magnitude Dependent Stable/Unstable (MDS/U) matrix pairs. With the knowledge of qualitative sign structure, we can now discuss the stability of a matrix pair.

The analysis of stability of matrices has evoked various research directions. The way non-engineers, such as ecologists and economists, have tackled this problem without even having any magnitude information is fascinating. In [4], the stability problem of a matrix is studied in a purely qualitative environment assuming that quantitative information is unavailable. Article [5] provides some sufficient conditions for the qualitative stability of an ecosystem by simply concluding the mutual qualitative effects on member species via signed digraphs, whereas necessary conditions for the qualitative stability are presented in [6]. In [7], linear systems are studied based on the qualitative theory. Some conditions concerning the structural qualitative stability of a system are proposed in [8]. A graph-theoretic analysis based on sign patterns of a real square matrix is used to conclude the stability of a linear system in [9]. In [10], it has been shown that in a complex ecological system, when species interact as predator-prey, the system can still be stable. In [11, 12], ecological-sign stability pricnciples of a matrix are transformed into mathematical principles to encounter stability problems in engineering control systems. The qualitative analysis of control systems is explained in [13]. The stability of the continuous-time linear state space system is explained in [14] and the stability of discrete-time system is explained in [15]. The series of papers [16], [17], and [18] were attempts to find the conditions for stability/instability of real matrices using qualitative reasoning. In [16], few conditions for qualitative sign instability of a matrix are derived in terms of the nature of interactions and interconnections, taken from ecological principles. The stability analysis of a matrix using these conditions requires only the qualitative (sign) information of the matrix elements (no need for any quantitative information). In [17], an alternative sufficient condition is proposed by combining the concepts of both quantitative (magnitude and sign) as well as the qualitative (only sign) information of the matrix elements. This condition possesses a convexity promotion property with respect to stability. A new necessary and sufficient condition is proposed in [18], for the stability of any real matrix that does not need the information of the characteristic polynomial, and it is based on matrix entries' sign information only. Article [19] studies asymptotic stability criteria for time-delayed systems. Remarkable works have been done in [20, 21] on matrices with stable sign patterns. However, all the existing research is focused on the stability of a matrix confining its utility to only linear normal state space systems, and improving on these papers, this paper generalises some of these conditions for qualitative stability/instability of a matrix pair that have a relatively broader scope in the analysis of linear square descriptor systems in control engineering. To the best of our knowledge, this is the first work discussing the qualitative sign stability for a matrix pair.

For a proper perspective, let us consider the following matrix pairs $(E_i, A_i), i = 1, 2$:

$$E_1 = \begin{bmatrix} -3 & -2 & 0.1 & 0.4 \\ -1 & 0.9 & -0.3 & -8 \\ -0.7 & 5 & 3 & 1 \\ 0.1 & 2 & 1.4 & 5 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 2 & -1 & -3 & 0.5 \\ 0.3 & -1 & 0.4 & -2 \\ -0.1 & 3 & -0.5 & 4 \\ -2 & -0.2 & 1 & 0.7 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1 & -0.2 & 0.8 & -3.5 & -4.1 \\ 1.9 & -2.7 & -3.3 & 4 & -0.1 \\ -1.2 & -0.4 & 2 & -0.5 & -4.8 \\ -0.3 & -1.7 & -4.3 & -3.7 & 0.2 \\ 0.6 & -2.5 & 0.9 & -5.2 & -2.3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0.2 & -3 & -0.7 & 5 \\ -0.9 & 1.3 & 2.4 & -1.2 & -3.1 \\ 4.8 & -1.1 & -2 & -0.4 & 4.3 \\ 2.7 & 1 & -1.8 & 1.7 & -0.7 \\ -5 & -3.2 & 0.3 & -2.9 & -1.5 \end{bmatrix}.$$

It is very much difficult to decide the stability/instability of the above matrix pairs with the numerical eigenvalue calculation. But with the necessary and sufficient conditions presented in this paper, we can conclude that the matrix pair (E_1, A_1) is a Qualitative Sign Unstable (QLSU) matrix pair and the matrix pair (E_2, A_2) is an MDS/U matrix pair. This is concluded just by a simple visualisation of the nature of interactions and interconnections of the matrix pair. Note that, the matrix pairs of order 2 do not have any interconnection terms and thus are trivial for our studies. Hence, we focus on matrix pairs of order 3 and higher.

In the next section, the matrix pair elemental sign structures are briefly reviewed and few basic 'Qualitative Sign Matrix Pair Indices' are developed for formulating the conditions for qualitative sign instability. In Section 3, the necessary and sufficient conditions for qualitative instability are proposed, which is the main result of this paper. Section 4 discusses the implications of these conditions and illustrates few examples for a clear visualisation of the importance of qualitative stability. In Section 5, we discuss the conclusions drawn from this paper.

2. Qualitative Sign Matrix Pair Indices

For assessing the stability, first, we have to visualise an $n \times n$ matrix pair in the following structured way. Just for a simplified view, let us illustrate the structure using a 4×4 matrix pair:

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix} \text{ and}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

The entire Matrix Pair Sign Structure is completely specified by the diagonal elements and the off-diagonal link structures (Interactions and Interconnections). This matrix pair consists of:

- Diagonal elements: e_{ii} and a_{ii},
- Interactions of the form $e_{ij}e_{ji}$ and $a_{ij}a_{ji}$,
- Interactions of the form $e_{ij}a_{ji}$,
- Interconnections of the form $b_{ij}b_{jk}\dots b_{mi}$, where $b_{ij} = e_{ij}$ or a_{ij}.

Now, we look at the signs of the entries and use the following sign convention in the rest of this paper. We use the letter 'P' for the '+' (positive) sign, the letter 'N' for the '−' (negative) sign, and '0' for the zero entry. We label the interactions of the matrix pair

Notations	
N_p	number of positive diagonal elements
N_z	number of zero diagonal elements
N_{ng}	number of negative diagonal elements
N_{pz}	number of non-negative diagonal elements

TABLE 1. Notations for number of diagonal elements.

using this sign convention. The possible off-diagonal links or interactions of a matrix pair are:

- Mutualism link: (PP) link,
- Competition link: (NN) link,
- Predation-Prey, Prey-Predation links: (PN) link and (NP) link,
- Ammensalism link: (N0) link and (0N) link,
- Commensalism link: (P0) link and (0P) link,
- Null link: (00) link.

We further categorise these links in the following way. All the Mutualism (PP) links and the Competition (NN) links are collectively labeled as 'Same Sign (SS) links'. All the Predation-Prey (PN) links and the Prey-Predation (NP) links are collectively labeled as 'Opposite Sign (OS) links'. Similarly, all the Ammensalism (N0 and 0N) links and the Commensalism (P0 and 0P) links are collectively labeled as 'Zero Sign (ZS) links'. Finally, the Null (00) links are labeled as 'Zero Zero' (ZZ) links. For a 'structural zero link', we label them as SZZ links and for Elemental Zero links, we label them as EZZ links.

- Same Sign (SS) links: PP (++) links and NN (−−) links,
- Opposite Sign (OS) links: PN (+−) links and NP (−+) links,
- Zero Sign (ZS) links: N0 links, 0N links, P0 links, and 0P links,
- Zero Zero (ZZ) links: 00 links.

Based on this sign convention, the matrix pair elemental sign structure of an $n \times n$ matrix pair are elaborated in the following ways:

2.1. DIAGONAL ELEMENTS e_{ii} AND a_{ii}

The number of diagonal elements of different signs is mentioned in Table 1. The total number of diagonal elements of the matrix pair is $2n$. The number of non-negative diagonal elements can be written as:

$$N_{pz} = N_p + N_z \,.$$

Let us assume that N_{pz} is not zero and define:

$$\eta_{pz} = \frac{N_{pz}}{2n} \,,$$
$$\eta_{ng} = \frac{N_{ng}}{2n} \,.$$

Notations	
N_{tl}	total number of links of E and A
N_{ss}	total number of SS links of E and A
N_{zs}	total number of ZS links of E and A
N_{os}	total number of OS links of E and A
N_{szz}	total number of SZZ links of E and A
N_{ezz}	total number of EZZ links of E and A
N_{lc}	total number of active links of E and A
N_{good}	number of 'Good' links of E and A
N_{bad}	number of 'Bad' links of E and A

TABLE 2. Notations for number of links of matrices E and A.

$\therefore \eta_{pz} + \eta_{ng} = 1$.

From an ecological perspective, the information about how a species affects itself is provided by the diagonal elements. The positive sign signifies that the species helps to increase its own population, zero signifies that the species has no effect on itself, and the negative sign signifies that it is self-regulatory. That is why Elemental + (positive) and 0 (zero) signs are considered as 'Bad' signs and Elemental − (negative) signs are considered as 'Good' signs in a row or column of a matrix pair [11].

2.2. INTERACTIONS OF THE FORM $e_{ij}e_{ji}$ AND $a_{ij}a_{ji}$

All the products of off-diagonal elements of the form $e_{ij}e_{ji}$ connecting only two distinct nodes (indices) of matrix E are known as Interactions of matrix E. Similarly, all the products of off-diagonal elements of the form $a_{ij}a_{ji}$ connecting only two distinct nodes (indices) of matrix A are known as Interactions of matrix A.

Table 2 includes information on the number of different links of matrices E and A. The total number of links (interactions) of this form is:

$$\begin{aligned} N_{tl} &= [1 + 2 + 3 + \ldots + (n-1)] \times 2 \\ &= n(n-1) \end{aligned}$$

and it can be expressed as:

$$N_{tl} = N_{ss} + N_{zs} + N_{os} + N_{szz} + N_{ezz} \,. \tag{2}$$

We now take out the structural zero links from any further discussion [16] and denote the total number of 'active links' as N_{lc}. Thus:

$$N_{lc} = N_{ss} + N_{zs} + N_{os} + N_{ezz} \,. \tag{3}$$

From an ecological viewpoint, it is noted that Same Sign (SS) links (i.e. PP and NN links) of this form are highly detrimental to stability whereas, Opposite Sign (OS) links (i.e. PN and NP links) of this form are conducive to stability [10, 22]. So, from the stability

Notations	
N'_{tl}	total number of links of (E, A)
N'_{ss}	total number of SS links of (E, A)
N'_{zs}	total number of ZS links of (E, A)
N'_{os}	total number of OS links of (E, A)
N'_{szz}	total number of SZZ links of (E, A)
N'_{ezz}	total number of EZZ links of (E, A)
N'_{lc}	total number of active links of (E, A)
N'_{good}	number of 'Good' links of (E, A)
N'_{bad}	number of 'Bad' links of (E, A)

Table 3. Notations for number of links of matrix pair (E, A).

point of view, SS links, ZS links, and ZZ links of this form are considered as 'Bad' links, and OS links of this form are considered as 'Good' links. Hence:

$$N_{good} = N_{os}\,, \tag{4}$$

$$N_{bad} = N_{ss} + N_{zs} + N_{ezz}\,. \tag{5}$$

Let us define:

$$\eta_{bad} = \frac{N_{bad}}{N_{lc}}$$

$$\eta_{good} = \frac{N_{good}}{N_{lc}}$$

$\therefore \eta_{bad} + \eta_{good} = 1$.

Remark 1. *The above indices are already defined for matrices. Now we are, for the first time, going to define another form of interaction and chain for matrix pairs.*

2.3. Interactions of the form $e_{ij}a_{ji}$

All the products of off-diagonal elements of the form $e_{ij}a_{ji}$ connecting one node of matrix E and the corresponding node of matrix A are known as Interactions of the matrix pair (E, A).

Table 3 lists the number of different links of matrix pair (E, A). The total number of links (interactions) of this form is:

$$N'_{tl} = n(n-1)$$

and it can be expressed as:

$$N'_{tl} = N'_{ss} + N'_{zs} + N'_{os} + N'_{szz} + N'_{ezz}\,. \tag{6}$$

We now take out the structural zero links from any further discussion and denote the total number of 'active links' as N'_{lc}. Thus:

$$N'_{lc} = N'_{ss} + N'_{zs} + N'_{os} + N'_{ezz}\,. \tag{7}$$

In ecological literature, it is realised that Same Sign (SS) links (i.e. PP and NN links) of this form are conducive to stability whereas Opposite Sign (OS) links (i.e. PN and NP links) of this form are detrimental to stability. So, SS links and ZS links of this form are considered as 'Good' links, and OS links of this form are considered as 'Bad' links. Hence:

$$N'_{good} = N'_{ss} + N'_{zs} + N'_{ezz}\,, \tag{8}$$

$$N'_{bad} = N'_{os}\,. \tag{9}$$

Let us define:

$$\eta'_{bad} = \frac{N'_{bad}}{N'_{lc}}$$

$$\eta'_{good} = \frac{N'_{good}}{N'_{lc}}$$

$\therefore \eta'_{bad} + \eta'_{good} = 1$.

Let us define ζ_{bad} as Potentially Destabilising Sign Matrix Pair Index and ζ_{good} as the Potentially Stabilising Sign Matrix Pair Index. So:

$$\zeta_{good} = \eta_{ng} + \eta_{good} + \eta'_{good}\,, \tag{10}$$

$$\zeta_{bad} = \eta_{pz} + \eta_{bad} + \eta'_{bad}\,. \tag{11}$$

Also define ζ_{net} as the Net Matrix Pair Stabilisation Index given by

$$\zeta_{net} = \zeta_{good} - \zeta_{bad}\,. \tag{12}$$

Let us define the index, known as 'Chain'. The elemental structure of the form $e_{ij}e_{ji}a_{ij}a_{ji}$ is called a 'Chain'. The chain containing at least three '+' sign is known as '+ chain' and the chain containing at least three '−' sign is known as '− chain'.

Using these 'Qualitative Sign Matrix Pair Indices', we discuss the qualitative stability of a matrix pair and derive few conditions for the qualitative sign instability.

3. Conditions for Qualitative Sign Stability and Instability

In this section, the main results are presented. Here we focus on the case of matrix pairs with diagonal elements containing only a mixture of positive and negative elements, i.e. with $N_{pz} = N_p$. Also, we assume that $N_{bad} = N_{ss}$ and $N'_{good} = N'_{ss}$. A series of necessary and sufficient conditions for stability/instability of a matrix pair are presented here.

3.1. Necessary Conditions for Qualitative Stability of Matrix Pair

For qualitative reasoning, we need to know the number and nature of the above-defined interactions and interconnections. But it is not possible to find the number and nature of all the interconnection terms of a matrix pair. So, for the qualitative sign stability, all the interconnection terms of a matrix pair need to be zero. Furthermore, any matrix pair (E, A) with $\det(A) = 0$ has always at least one zero eigenvalue, that makes the matrix pair unstable. Hence, the non-singularity of matrix A is also necessarily required for QLSS matrix pair.

Let us consider a matrix pair (E, A) with entries e_{ij} and a_{ij}, respectively. Based on the above discussion, we are enlisting the two important 'necessary' conditions for 'qualitative sign stability':

- C1: $b_{ij}b_{jk}\ldots b_{qr}b_{ri} = 0$, *where* $b_{ij} = e_{ij}$ *or* a_{ij} *for any sequences of three or more distinct indices* $i, j, k, \ldots, q, r$.
- C2: $\det(A) \neq 0$.

Here, C1 is the necessarily required condition for Qualitative Sign Stability, i.e. the condition which makes the matrix pair stable, independent of magnitudes, and C2 is the necessarily required condition for the stability of the matrix pair (E, A), independent of any qualitative or quantitative information of the matrix pair elements. More details about the concept of Qualitative Sign Stability is discussed in [11, 14].

The Net Matrix Pair Stabilisation Index ζ_{net} serves as an index to indicate the likelihood of matrix pair being stable or unstable in a Qualitative way. Negative values indicate that the matrix pair is more likely to be unstable, and positive values indicate that the matrix pair is more likely to be stable. The higher the value, the higher the probability, see [16]. The qualitative sign stability of a matrix pair depends upon the stabilising strength of the matrix pair. When the matrix pair is more potentially stabilised, then ζ_{net} is non-negative, and when it is more potentially destabilised, then ζ_{net} is negative. So, the qualitative stability is connected with the value of ζ_{net}.

With this observation, we have the following conditions:

- ζ_{net} varies in an interval given by $-3 \leq \zeta_{net} \leq 3$.
- $\zeta_{net} < 0$ specifies that the matrix pair is MDU, that means for these MDU matrix pairs, there always exist magnitudes that make this matrix pair unstable.
- $\zeta_{net} \geq 0$ specifies that the matrix pair is MDS, that means for these MDS matrix pairs, there always exist magnitudes that make this matrix pair stable.
- A given (non-QLSS) matrix pair is QLSU only if $-3 \leq \zeta_{net} < 0$ (Necessary condition for QLSU).

A matrix pair, which is not Qualitative Sign Stable (QLSS), is said to be a non-QLSS matrix pair.

3.2. A Necessary Condition for Instability of a Matrix

In a matrix pair (E, A), if we substitute E by the Identity matrix I, then the above necessary condition for QLSU will be the same as that of the matrix A. With the fundamentals of ecological principles, the matrix pair (I, A) is visualised in the following structured way. Let us illustrate the structure by using a 4×4 matrix pair.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

According to the qualitative stability concept of matrix pair:

$$\begin{aligned} \eta_{pz} &= \frac{n + N_{pz}(A)}{2n} = \frac{1}{2} + \frac{N_{pz}(A)}{2n} \\ \eta_{ng} &= \frac{N_{ng}(A)}{2n} \\ \eta_{bad} &= \frac{N_{lc}(I) + N_{bad}(A)}{N_{lc}} = \frac{1}{2} + \frac{N_{bad}(A)}{N_{lc}} \\ \eta_{good} &= \frac{N_{good}(A)}{N_{lc}} \\ \eta_{bad}^{'} &= 0 \\ \eta_{good}^{'} &= 1 \end{aligned}$$

$$\begin{aligned} \therefore \zeta_{net} &= \zeta_{good} - \zeta_{bad} \\ &= \frac{1}{2}[\frac{N_{ng}(A)}{n} + \frac{N_{good}(A)}{N_{lc}(A)} - \frac{N_{pz}(A)}{n} - \frac{N_{bad}(A)}{N_{lc}(A)}] \\ &= \frac{1}{2}[\zeta_{good}(A) - \zeta_{bad}(A)] \\ &= \frac{1}{2}\zeta_{net}(A)\,. \end{aligned}$$

Since the matrix pair (I, A) is QLSU only if $\zeta_{net} < 0$, therefore, the matrix A is QLSU only if $\zeta_{net}(A) < 0$. This is the necessary condition for a matrix to be QLSU, which is a particular case of a matrix pair. The necessary and sufficient conditions of a matrix to be QLSU are discussed extensively in [16]. In this paper, we generalise the Identity matrix I to any matrix E and propose few necessary and sufficient conditions for the qualitative sign instability for a matrix pair.

3.3. A Necessary Condition for a Matrix Pair to be QLSU

We know that ζ_{net} is a real function and it takes on discrete values. So there may be a situation that not all the diagonal elements have to be positive and not all the links have to be 'Bad' links to make the matrix pair QLSU. That means, for a matrix pair to be QLSU, the number of positive diagonal elements can be less than $2n$ and the number of Bad links in the matrix pair can be less than N_{lc}, not necessarily equal to N_{lc}. Now, we calculate the minimum number of Bad links needed to make a matrix pair unstable and denote it as N_u^*. For a matrix pair with $N_p \neq 0$, let us define N_u^* as follows:

N_u^* =the closest higher upper integer of $(\eta_{ng} + \frac{1}{2})$ times N_{lc}, including when that is itself an integer.

Let us state a theorem.

Theorem. *A matrix pair is QLSU only if the total number of bad links in it is* $\geq N_u^*$ *i.e.* $N_{bad} + N_{bad}' \geq N_u^*$.

Proof. For a QLSU matrix pair,

$$
\begin{aligned}
& \zeta_{net} < 0 \\
\implies & \eta_{ng} + \eta_{good} + \eta_{good}' < \eta_{pz} + \eta_{bad} + \eta_{bad}' \\
\implies & \eta_{ng} + 1 - \eta_{bad} + 1 - \eta_{bad}' < 1 - \eta_{ng} + \eta_{bad} + \eta_{bad}' \\
\implies & \eta_{ng} + \frac{1}{2} < \eta_{bad} + \eta_{bad}' \\
\implies & \eta_{ng} + \frac{1}{2} < \frac{N_{bad} + N_{bad}'}{N_{lc}} \\
\implies & [\eta_{ng} + \frac{1}{2}] \times N_{lc} < N_{bad} + N_{bad}' .
\end{aligned}
$$

Thus, the total number of Bad links needed to make a matrix pair QLSU is greater than or equal to N_u^*. This is a necessary condition for a matrix pair to be QLSU.

3.4. Sufficiency Guidelines For QLSU Matrix Pair

While observing the expression for the determinant of a matrix pair, we find that a diagonal element is always multiplied by the link elements surrounded by it. That means the row and column elements associated with a diagonal element play a vital role to assess the stability/instability. These observations provide us with some 'guidelines' for sufficiency for QLSU [16].

- Guideline 1: A matrix pair with $\zeta_{net} < 0$, is likely QLSU if a positive diagonal element e_{ii} or a_{ii} is surrounded by + chains.
  (The above guideline is the result of the idea that positive diagonal elements along with + chains promote instability.)
- Guideline 2: A matrix pair with $\zeta_{net} < 0$, is likely QLSU if a negative diagonal element e_{ii} or a_{ii} is surrounded by − chains.
  (The above guideline is the result of the idea that negative diagonal elements along with − chains promote instability.)

4. Illustrative examples for instability of a matrix pair

By now, we have few necessary conditions for QLSU and few guidelines for sufficiency for QLSU. Once the necessary condition $\zeta_{net} < 0$ is satisfied, we can make a QLSU matrix pair, with the appropriate placement of the 'Bad' and 'Good' links in it. Let us consider some examples.

Example 1. *Consider a* 4×4 *matrix pair with diagonal elements as shown below:*

$$
E = \begin{bmatrix} - & * & * & * \\ * & + & * & * \\ * & * & + & * \\ * & * & * & + \end{bmatrix} \text{ and } A = \begin{bmatrix} + & * & * & * \\ * & - & * & * \\ * & * & - & * \\ * & * & * & + \end{bmatrix}
$$

Suppose there is no ZZ link in the matrix pair. For the given matrix pair (E, A),

$$
\begin{aligned}
\eta_{ng} &= \frac{3}{8}, \ \eta_{pz} = \frac{5}{8} \\
N_{lc} &= 12 \\
N_u^* &= 11
\end{aligned}
$$

Thus, $N_{bad} + N_{bad}' \geq 11$.

Any matrix pair with the given diagonal element structure and satisfying the above condition have $\zeta_{net} < 0$, *and thus the matrix pair is an MDU matrix pair.*

Example 2. *Let us consider a matrix pair with the conditions given in Example 1:*

$$
E = \begin{bmatrix} - & - & + & + \\ - & + & - & - \\ - & + & + & + \\ + & + & + & + \end{bmatrix} \text{ and } A = \begin{bmatrix} + & - & - & + \\ + & - & + & - \\ - & + & - & + \\ - & - & + & + \end{bmatrix}
$$

Here

$$
\begin{aligned}
N_{bad} &= 7 , \quad N_{good} = 5 \\
N_{bad}' &= 5 , \quad N_{good}' = 7
\end{aligned}
$$

$\therefore N_{bad} + N_{bad}' = 7 + 5 = 12 > 11 = N_u^*$.

Hence, the necessary condition is satisfied, and now we check the sufficiency guidelines required for QLSU. Here, all the negative diagonal elements are surrounded by − chains. Therefore, this is a QLSU matrix pair.

It is noted that once there is at least one positive diagonal element, we can assess the Qualitative Sign Instability by computing the relative distribution of the Bad links together with the Good links.

Example 3. *Let us discuss the stability of the matrix pair given below:*

$$
E = \begin{bmatrix} - & - & - & + & - \\ - & + & + & + & + \\ + & + & + & + & - \\ - & - & + & + & + \\ + & + & + & - & - \end{bmatrix} \text{ and}
$$

$$
A = \begin{bmatrix} - & + & + & - & + \\ + & + & + & + & + \\ + & - & + & + & + \\ - & + & - & + & - \\ + & + & + & - & - \end{bmatrix}
$$

For the given matrix pair (E, A),

$$
\begin{aligned}
\eta_{pz} &= \frac{6}{10} , \quad \eta_{ng} = \frac{4}{10} \\
\eta_{bad} &= \frac{3}{5} , \quad \eta_{good} = \frac{2}{5} \\
\eta_{bad}' &= \frac{1}{2} , \quad \eta_{good}' = \frac{1}{2}
\end{aligned}
$$

$\therefore \zeta_{net} = -\frac{2}{5} < 0$.

Here all the positive diagonal elements are surrounded by + chains. Hence, as per the sufficiency guidelines for QLSU, the given matrix pair is QLSU. Thus, this particular quantitative matrix pair is unstable without any need of eigenvalue calculations.

Example 4. *Consider the sign pattern of the matrix pair given below:*

$$E = \begin{bmatrix} - & + & - \\ - & + & + \\ + & - & + \end{bmatrix} \text{ and } A = \begin{bmatrix} + & + & - \\ - & + & + \\ + & - & - \end{bmatrix}$$

This matrix pair (E, A) has

$$\eta_{pz} = \frac{4}{6}, \quad \eta_{ng} = \frac{2}{6}$$
$$\eta_{bad} = 0, \quad \eta_{good} = 1$$
$$\eta'_{bad} = 1, \quad \eta'_{good} = 0$$

making $\zeta_{net} = \frac{-1}{3} < 0$. Since, $\zeta_{net} < 0$, the necessary condition is satisfied. But the sufficient conditions are not satisfied. Hence, it is an MDU (not a QLSU) matrix pair.

It should be noted that we are not stating that the matrix pair with this Elemental Sign Structure is always unstable. We are simply stating that the elemental sign structure of the above matrix pair guarantees that there exist magnitudes which would definitely make this matrix unstable.

For example, the following matrix pair (E_1, A_1) with the above sign structure is unstable,

$$E_1 = \begin{bmatrix} -1.6132 & 2.0118 & -1.6806 \\ -0.0021 & 0.5791 & 0.2139 \\ 0.2017 & -2.1852 & 1.7419 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 0.2462 & 1.8614 & -0.7201 \\ -1.8764 & 1.4531 & 1.9056 \\ 3.4318 & -0.1567 & -1.7543 \end{bmatrix},$$

while the following matrix pair (E_2, A_2) having the same sign structure is stable.

$$E_2 = \begin{bmatrix} -1.0132 & 0.0118 & -1.0006 \\ -1.0021 & 0.4001 & 0.2110 \\ 0.0213 & -0.0411 & 0.2015 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1.8112 & 21.5624 & -1.0207 \\ -0.9016 & 1.0172 & 1.0061 \\ 9.6512 & -0.0600 & -5.0234 \end{bmatrix}.$$

Example 5. *Consider the matrix pair (E, A) with a sign structure given by:*

$$E = \begin{bmatrix} + & - & + & - & - \\ + & - & - & + & - \\ - & - & + & - & - \\ - & - & - & - & + \\ + & - & + & - & - \end{bmatrix} \text{ and}$$
$$A = \begin{bmatrix} - & + & - & - & + \\ - & + & + & - & - \\ + & - & - & - & + \\ + & + & - & + & - \\ - & - & + & - & - \end{bmatrix}$$

This matrix pair (E, A) has

$$\eta_{pz} = \frac{2}{5}, \quad \eta_{ng} = \frac{3}{5}$$
$$\eta_{bad} = \frac{2}{5}, \quad \eta_{good} = \frac{3}{5}$$
$$\eta'_{bad} = \frac{1}{5}, \quad \eta'_{good} = \frac{4}{5}$$

$\therefore \zeta_{net} = 1 > 0$.

Here, $\zeta_{net} > 0$, but the necessary condition for Qualitative Sign Stability C1 is not satisfied. Hence, it is an MDS/U matrix pair and is a non-QLSS matrix pair.

5. Conclusion

This paper addresses the issue of determining the stability/instability of a matrix pair that arises in a continuous linear homogeneous time-invariant system. The conditions for a matrix to be QLSU have already been discussed in the earlier works. In this research, we generalise the identity matrix I to any matrix E and propose few necessary conditions and sufficient conditions for qualitative sign instability of a matrix pair. The proposed conditions are very simple and are based on the number and nature of the diagonal elements and the number and nature of the off-diagonal element pairs (links). This reflects that the Elemental Sign Structure of a matrix pair is an important contributor to the stability/instability. These conditions are extremely helpful for engineers and ecologists, in solving stability-related problems.

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