Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Department of Physics

Generalized uncertainty relation and its use in cosmology

Zobecněný princip neurčitosti a jeho aplikace v kosmologii

Bachelor's Degree Project

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KATEDRA FYZIKY

ZADÁNÍ BAKALÁŘSKÉ PRÁCE

Akademický rok: 2022/2023

Student: Jan Masák

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Specializace: Matematická fyzika

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Název práce: (anglicky) Generalized uncertainty relation and its use in cosmology

Jazyk práce: Angličtina

Pokyny pro vypracování:

1) Nastudujte si doporučenou literaturu o zobecněných relacích neurčitosti (ZRN) a specielně věnujte pozornost kosmologickým aplikacím.

2) Analyzujte odvození zobecněného Schwarzschildova a Kerrova řešení pro ZRN černé díry.

3) Odvoďte vypařovací formuli (hmotnost-teplotní formuli) pro třídu kvadratických ZRN černých děr a diskutejte další související problémy.

Doporučená literatura:

[1] A. Tawfik and A. Diab, Generalized uncertainty principle: Approaches and applications International Journal of Modern Physics D 23, 1430025 (2014)

[2] F. Skara and L. Perivolaropoulos, Primordial power spectra of cosmological fluctuations with generalized uncertainty principle and maximum length quantum mechanics, Physical Review D 100, 123527 (2019)

[3] Petr Jizba, Hagen Kleinert, Fabio Scardigli, Uncertainty relation on a world crystal and its applications to micro black holes, Physical Review D 81, 084030 (2010)

[4] Giovanni Amelino-Camelia, Laurent Freidel, Jerzy Kowalski-Glikman, and Lee Smolin, Principle of relative locality, Physical Review D 84, 084010 (2011)

Jméno a pracoviště vedoucího bakalářské práce:

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Doba platnosti zadání je dva roky od data zadání.

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prohlašuji, že jsem bakalářskou práci s názvem:

Zobecněný princip neurčitosti a jeho aplikace v kosmologii

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V Praze dne 2.8.2023

Jan Masák
....
podpis

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Název práce:

Zobecněný princip neurčitosti a jeho aplikace v kosmologii

Autor: Jan Masák

Studijní program: Matematické inženýrství

Specializace: Matematická fyzika

Druh práce: Bakalářská práce

Vedoucí práce: doc. Ing. Mgr. Petr Jizba, Ph.D., Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská ČVUT v Praze

Abstrakt: Zobecněné relace neurčitosti jsou zobecněním standardních Heisenbergových relací z kvantové mechaniky. Snaží se zachycovat některé aspekty kvantové gravitace, jako je například existence minimální délky. Protože jsou relace neurčitosti jedním ze základních pilířů kvantové mechaniky, jejich deformace ovlivňuje mnoho fyzikálních jevů. V této práci se zabýváme deformací geodetického pohybu skrze kvantování. Odvozujeme rovnice modifikované geodetického pohybu v obecném časoprostoru, a pak je aplikujeme v pseudo-Newtonovské limitě na Kerrovo a Schwarzschildovo řešení. Nakonec prezentujeme odvození zobecněné teplotní formule pro vypařování černých děr skrze Hawkingovo záření.

Klíčová slova: černé díry, Hawkingovo záření, kosmologie, kvantová gravitace, zobecněné relace neurčitosti

Title:

Generalized uncertainty relation and its use in cosmology

Author: Jan Masák

Abstract: The generalised uncertainty relations are generalisation of the standard Heisenberg relations from quantum mechanics. They attempt to capture some aspects of quantum gravity, such as the existence of a minimum length. Since uncertainty relations are one of the fundamental pillars of quantum mechanics, their deformation affects many physical phenomena. In this paper, we study the deformation of geodesic motion through quantisation. We derive equations of modified geodesic motion in general spacetime and then apply them in a pseudo-Newtonian limit to Kerr and Schwarzschild solutions. Finally, we present the derivation of a generalised temperature formula for black hole evaporation via Hawking radiation.

Key words: black holes, cosmology, generalized uncertainty relation, Hawking radiation, quantum gravity

Contents

Introduction

In 1927 the German physicist Werner Heisenberg formulated his famous uncertainty principle. Today, it is considered one of the cornerstone principles of quantum physics. It restricts the amount of information we can gain about two complementary observables that are measured in the same quantum state. For instance, it is not possible to know the momentum and position of a particle in a given quantum state with arbitrary precision. In principle, either of these can be determined arbitrarily precisely, but then the uncertainty in the other has to increase accordingly. This is not due to the inaccuracy of our measurement devices, but is a fundamental constraint that is inherent to a quantum state in which a system is prepared. According to Heisenberg, uncertainty relations do not represent just a limitation on our knowledge of the observables involved, but they pose a restriction on their very existence. This reflected his belief that the only things that can be considered real are those that can be measured [1]. If one adopts this philosophy, the uncertainty relation tells us something profound about the structure of reality.

Heisenberg developed the uncertainty principle by examining various thought experiments, such as Heisenberg's microscope and Young's double-slit experiment. A more precise mathematical formulation is derived in Section 1.1 using the postulate of quantum mechanics (QM) that observables are represented by self-adjoint operators that live on a Hilbert space together with the Cauchy-Schwarz inequality. In this way, the Heisenberg uncertainty principle is deeply rooted in the very foundations of quantum theory.

The generalised uncertainty principle (GUP) or generalised uncertainty relation is a modification of the uncertainty principle that has garnered significant interest in physics. There are many reasons for such a deformation which are subject to Chapter 1. For instance, GUP may reflect the existence of a minimal length scale (such as the Planck scale) in the universe [2]. Minimal length is an important concept in physics, which is also discussed in detail, as it carries profound implications not only for quantum physics but also for the structure of spacetime itself. In this context, for example, a lattice-like model of spacetime based on the existence of a minimal length has been proposed and is discussed in Section 1.6. Numerous more or less heuristic thought experiments suggest the existence of minimal length, or GUP directly. However, even sophisticated candidates for the theory of quantum gravity, such as string theory, predict GUP. GUP has also been linked to various approaches that modify important theories in physics. For example, doubly special relativity (DSR) deforms special relativity (SR), while curved momentum space alters the Hamiltonian dynamics, and non-commutative geometry introduces modifications to quantum physics. Moreover, these concepts are interconnected in interesting ways and have far-reaching implications.

Although the uncertainty principle is an aspect of the quantum realm, GUP would likely affect both the atomic and macroscopic worlds. The existence of GUP would imply the existence of some larger theory, of which it is a part. From the form of the deformed relation we could then gain some insight into this theory. Furthermore, the deformation of uncertainty relations is probably related to the deformation of other physical laws. The main contribution of this work is the deformation of geodesic motion based on GUP and its application to Schwarzschild and Kerr solutions. Chapter 2 gives some motivations for the form of our deformation based on the quantisation procedure and models from curved momentum space and non-commutative geometry. In Chapter 3 we apply this deformation to a general geodesic motion and in Chapter 4 we study the form that this deformation can take in Schwarzschild and Kerr spacetime in a pseudo-Newtonian limit. We shall see that GUP might violate some foundational principles like locality or standard Lorentz invariance.

In Chapter 5 we examine black hole thermodynamics, a fascinating area of physics, which has the power to connect quantum physics, black holes, and thermodynamics. Through this connection, we study the effects of GUP on black hole evaporation. We will see that GUP can resolve the infinite temperature at the end of a black hole's life and some forms even predict massive black hole remnants.

Chapter 1

Motivations for Generalised Uncertainty Principle

In this chapter, we first delve deeper into the meaning of the classical Heisenberg uncertainty principle in Section 1.1. After that, different types, motivations and consequences of GUP are illustrated from the theoretical point of view.

An essential concept in the study of GUP is the minimal length scale. This notion is generally described in Section 1.3 but continues to play an essential role throughout the entirety of the chapter. Similarly, other topics, such as non-commutative geometry and curved momentum space, are mutually deeply interconnected, and it was sometimes hard for me to separate sections apart.

1.1 Heisenberg uncertainty principle

The Introduction has already briefly described the uncertainty principle, but let us expand on that in this chapter. Specifically, we make sense here of the uncertainty principle in the mathematical formalism of QM.

In quantum physics, observables (measurable quantities such as position, angular momentum, etc.) are represented by self-adjoint operators on a Hilbert space $\mathscr{H}$ and states of a system by state vectors $|\psi\rangle \in \mathscr{H}$. Throughout this work, bra-ket notation is used. Using this notation, a linear form or "bra" $\langle\psi|$ is a covector to a vector or "ket" $|\psi\rangle$. $\langle\phi|\psi\rangle$ is then an inner product of a state vectors $|\phi\rangle$ and $|\psi\rangle$. The expectation value of an operator a in a state $|\psi\rangle$ is defined as $\langle a\rangle = \langle\psi| a |\psi\rangle$ and its standard deviation as in statistics $\Delta a = \sqrt{\langle(a - \langle a\rangle)^2\rangle}$.

We are now ready to prove the uncertainty principle for two observables, a and b using the method in [3]. Let us define an operator $\overline{a} = a - \langle a\rangle$, so that $\Delta a = \sqrt{\langle\overline{a}^2\rangle}$. Similarly, for the operator b. Cauchy-Schwarz inequality states that

$$(\forall |\psi\rangle, |\phi\rangle \in \mathscr{H}) \left(\sqrt{\langle\psi|\psi\rangle} \cdot \sqrt{\langle\phi|\phi\rangle} \geq |\langle\psi|\phi\rangle|\right) . \tag{1.1}$$

If we define the two vectors to be

$$\begin{aligned} |\psi\rangle &= \overline{a}\,|\xi\rangle \ , \\ |\phi\rangle &= \overline{b}\,|\xi\rangle \ , \end{aligned} \tag{1.2}$$

for arbitrary vector $|\xi\rangle \in \mathscr{H}$. The Cauchy-Schwarz inequality becomes

$$\Delta a \Delta b \geq |\langle\overline{a}\,\overline{b}\rangle| . \tag{1.3}$$

As for any pair of operators, we have

$$\overline{a}\,\overline{b} = \frac{1}{2}[\overline{a},\overline{b}] + \frac{1}{2}\{\overline{a},\overline{b}\}\,, \tag{1.4}$$

where $[\overline{a},\overline{b}] = \overline{a}\,\overline{b} - \overline{b}\,\overline{a}$ is the commutator and $\{\overline{a},\overline{b}\} = \overline{a}\,\overline{b} + \overline{b}\,\overline{a}$ the anticommutator of $\overline{a}$ and $\overline{b}$. Now we can use the fact that $[\overline{a},\overline{b}] = [a,b]$ is anti-Hermitian $[a,b] = -[a,b]^\dagger$ and $\{\overline{a},\overline{b}\}$ Hermitian $\{\overline{a},\overline{b}\} = \{\overline{a},\overline{b}\}^\dagger$, therefore $\langle [a,b]\rangle$ is purely imaginary and $\langle\{\overline{a},\overline{b}\}\rangle$ purely real. From this and (1.4)

$$|\,\langle \overline{a}\,\overline{b}\rangle\,| = \sqrt{\frac{1}{4}|\,\langle [a,b]\rangle\,|^2 + \frac{1}{4}|\,\langle\{\overline{a},\overline{b}\}\rangle\,|^2}\,. \tag{1.5}$$

If we now substitute this into (1.3) and use a lower estimate for (1.5), we get

$$\Delta a \Delta b \geq \sqrt{\frac{1}{4}|\,\langle [a,b]\rangle\,|^2 + \frac{1}{4}|\,\langle\{\overline{a},\overline{b}\}\rangle\,|^2} \geq \frac{1}{2}|\,\langle [a,b]\rangle\,|\,. \tag{1.6}$$

To conclude, we have the famous inequality

$$\Delta a \Delta b \geq \frac{1}{2}|\,\langle [a,b]\rangle\,|\,. \tag{1.7}$$

This is sometimes called the 'general uncertainty principle,' but let us not confuse it with GUP. However, (1.7) is very important in the following discussion of GUP as it enables us to connect a specific uncertainty relation with its commutator.

1.2 Planck scale

The Planck scale is essential to us mainly because the effects that take place on this scale motivate many of the GUP approaches discussed in later sections. This short section introduces it and provides an overview of some of its properties.

Let us start with the definition based on the Planck units. For example, an object is considered at the Planck length scale when it has a length on the order of the Planck length. The same principle applies to the dimensions of time, mass, and temperature. The Planck units are then defined as units that comprise solely of the universal constants in Table 1.1a, and when the universal constants are expressed in these units, they take the value of 1. How this is achieved can be seen in Table 1.1.

Max Planck first introduced this system of units at the end of his 1900 paper [4]. He viewed it as a universal system of units that everyone would agree on because it is based on the fundamental properties of nature and not on specific substances (for example, the kilogramme was at this time defined through the International Prototype of the Kilogramme). However, it would be strange to use these units, as the Planck scale is so far from our experience. The Planck scale is so extreme that our understanding of physics is thought to break down on this scale, and a complete theory of quantum gravity is needed. This makes sense as these units combine $\hbar$, a fundamental constant in quantum physics, and G from general relativity (GR), but it is better seen from the following argument from [5].

We get quantum field theory (QFT) when considering both QM and SR. Quantum mechanical effects cannot be neglected when we are on the scale of an object's de Broglie wavelength $\lambda = \frac{h}{p}$ and special relativity has to be considered at high velocities where the dispersion relation $E^2 = m^2c^4 + p^2c^2$ is needed. When we substitute the momentum from the de Broglie wavelength to the SR dispersion relation, we get the Compton wavelength $\lambda_c = \frac{h}{mc}$. This should be the scale where QFT is important. Another argument for λ_c being the scale for QFT is that the energy needed to determine the position of a particle of mass m to the precision of its λ_c is sufficient to create another particle of that mass. The spontaneous creation of particles from energy is not predicted by QM, so we must use QFT. We can then consider how massive a particle must be so that its Schwarzschild radius $R_s = \frac{2Gm}{c^2}$ is the same as its Compton wavelength and get the mass $\sqrt{\pi}m_p$. As a result, we can conclude that the Planck scale is the mass scale where we have to account for both QFT and GR.

Another way to arrive at the Planck scale is discussed in [6]. Let us consider two particles with mass m and elementary charge e. These can be, for example, highly energetic proton and electron that have large effective masses. Now we set the electric and gravitational force between them to be equal

$$\frac{Gm^2}{r^2} = k_c \frac{e^2}{r^2}, \tag{1.8}$$

where k_c is the Coulomb constant. The fine structure constant is defined as $\alpha = k_c e^2/\hbar c \simeq 1/137$. We can express m in terms of α and m_p using (1.8) to get

$$M = \sqrt{\alpha} m_p \simeq \frac{m_p}{12}. \tag{1.9}$$

So, we see that the mass is of the order of the Planck mass.

Scattering calculations in quantum electrodynamics (QED) make use of integrals that involve arbitrarily high energies, and most of them diverge. However, it is clear from the previous calculations that we cannot reliably use QFT at energies larger than the Planck energy $E_p = m_p c^2$. This gives us hope that these results will be finite in the theory of quantum gravity. Also, a grand unified theory suggests that electromagnetic, weak and strong forces are unified at energies just three orders of magnitude below the Planck scale (10^{16} GeV). The previous result then hints that all the forces, including gravity, are comparable and maybe unified at the Planck scale.

1.3 Minimal length

Since the dawn of physics, people have wondered what the structure of reality is. It is one of the main goals (if not the main one) of physics, after all. In ancient Greece, the philosopher Democritus hypothesised that all matter comprises the indivisible building blocks he called 'atoms' ('a' stands for 'not' and 'tomos' cut). Since then, many scientists have pursued this idea and devoted their lives to the search for atoms. However, nowadays, it seems that fundamentals are maybe not discrete particles but continuous fields.

As time passed, physicists became also increasingly interested in the structure of space and time itself. They began asking questions such as, 'Are there any "atoms" of spacetime?'. In answering this question, the concept of minimal length is essential. Minimal length can have ontological or epistemological meaning. The ontological meaning would be to imagine spacetime being discretised as some sort of a world lattice [7]. Epistemologically, on the other hand, we could interpret the minimal length as the minimal distance we can measure, which does not necessarily imply that the spacetime itself is discrete. In both cases, there is a problem with the notion of an absolute minimum length because of Lorentz contraction in relativity, which is discussed in Section 1.7, but let us leave it for now.

	Approximate value (in SI units)
Gravitational constant (G)	$6.7 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Speed of light (c)	$3 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Reduced Planck constant ($\hbar$)	$6.6 \times 10^{-33}\,\mathrm{m^2\,kg\,s^{-1}}$
Boltzmann constant (k_B)	$1.4 \times 10^{-23}\,\mathrm{m^2\,kg\,s^{-2}\,K^{-1}}$

(a) The universal constants

	Expression	Approximate value (in SI units)
Planck length (ℓ_p)	$\sqrt{\frac{\hbar G}{c^3}}$	$1.6 \times 10^{-35}\,\mathrm{m}$
Planck time (t_p)	$\sqrt{\frac{\hbar G}{c^5}}$	$5.4 \times 10^{-44}\,\mathrm{s}$
Planck mass (m_p)	$\sqrt{\frac{\hbar c}{G}}$	$2.2 \times 10^{-8}\,\mathrm{kg}$
Planck temperature (T_p)	$\sqrt{\frac{\hbar c^5}{G k_B^2}}$	$1.4 \times 10^{32}\,\mathrm{K}$

(b) Planck units

Table 1.1: Table (b) shows how Planck units are formed from the universal constants in table (a). Both tables also show approximate values of the constants.

Dealing with uncertainty relations, we are here more interested in the minimal measurable length, but also some ontological approaches are mentioned in Section 1.6 and Subsection 1.9.1. Interestingly, even today, physicists imagine spacetime lattice, for example, in quantum chromodynamics computations to eliminate ultraviolet divergences without the need for renormalisation [8]. This is, of course, just a mathematical trick, and the physically relevant quantities are extracted only in the continuum limit. Still, it would be nice for it to turn out to be real.

Heisenberg himself greatly supported the existence of a minimal length for a similar reason. He believed that it could save the non-renormalizability of Fermi's theory of β-decay. He proposed that the minimal length should not be much smaller than the classical electron radius (of the order $10^{-13}\,\mathrm{m}$) [9]. Today many scientists believe there to be a minimum length (in the ontological or epistemological sense) but nowadays is the most common candidate the Planck length discussed in Section 1.2.

The minimal measurable length has also been motivated by some candidates for quantum theory of gravity, such as string theory or loop quantum gravity, many thought experiments, and other theories. These can be found throughout this chapter. In this section, we present one of the arguments for minimum length and the general form of GUP that is connected this concept.

1.3.1 Minimal length from the gravitational collapse of a test particle

Let us first present a device-independent argument for a minimum measurable length following [10] also discussed in [9]. Here, we do not derive any form of GUP, but it serves as a motivation for the concept, which is crucial in many GUPs.

We consider a free non-relativistic test particle with mass M. Therefore, the time evolution of the position operator in the Heisenberg picture is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{i}{\hbar}[H, x] = \frac{p}{M}\,. \tag{1.10}$$

Here the momentum operator p is constant in time, therefore

$$x(t) = x(0) + p\frac{t}{M}\,. \tag{1.11}$$

This gives us the commutator

$$[x(0), x(t)] = i\hbar\frac{t}{M}\,, \tag{1.12}$$

and substituting into (1.7) we have

$$\Delta x(0)\Delta x(t) \geq \frac{\hbar t}{2M}\,. \tag{1.13}$$

Since we need two position measurements to determine the distance, the uncertainty in the distance becomes

$$\Delta d = \max\{\Delta x(0), \Delta x(t)\} \geq \sqrt{\frac{\hbar t}{2M}}\,. \tag{1.14}$$

We aim to resolve the distance below ℓ_p, which we could theoretically achieve by choosing a large enough mass. However, as the mass becomes significant, we must also consider gravitational effects. The hoop conjecture states that if an object is compressed so that it can be enclosed into a sphere with its Schwarzschild radius, it necessarily collapses into a black hole. This conjecture remains unproven, but there is much evidence for its validity (for example, from computer simulations) [11].

To preserve causality, nothing outside the region with radius ct can affect the measurement. Simultaneously to prevent the formation of a black hole (in which case all the information would be lost to us), the inequality $ct \geq \frac{2GM}{c^2}$ must hold. If we substitute this into (1.14) we finally arrive to

$$\Delta d \geq \ell_p\,. \tag{1.15}$$

Therefore, there is no experiment we could perform to measure a distance smaller than the Planck length using the free non-relativistic test particle, considering that the Hoop conjecture holds.

1.3.2 Form of the relation

We now consider the easiest general form that the GUP relation can take to introduce a minimal length [12]. If we restrict ourselves to one dimension, we can use

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2) = \frac{\hbar}{2}\left(1 + \left(\frac{\beta_0 \ell_p}{\hbar}\Delta p\right)^2\right)\,, \tag{1.16}$$

where $\beta \in \mathbb{R}$ and $\beta_0 \in \mathbb{C}$ are a deformation parameters. Notice that β_0 is dimensionless. It is easy to see that the minimum uncertainty in position introduced by this inequality is $\Delta x_0 = \hbar\sqrt{\beta} = \beta_0 \ell_p$. Different GUP derivations arrive at different values for β_0, but generally, it is of the order of one. For example, in Subsection 1.4.2 we arrive at $\beta_0 = \sqrt{2}$, in Section 1.5 we have $\beta_0 = \frac{\sqrt{2K\alpha'}}{\ell_p}$, and if we set $\ell = \ell_p$ in Section 1.6 we get $\beta_0 = \frac{i}{\sqrt{2}}$.

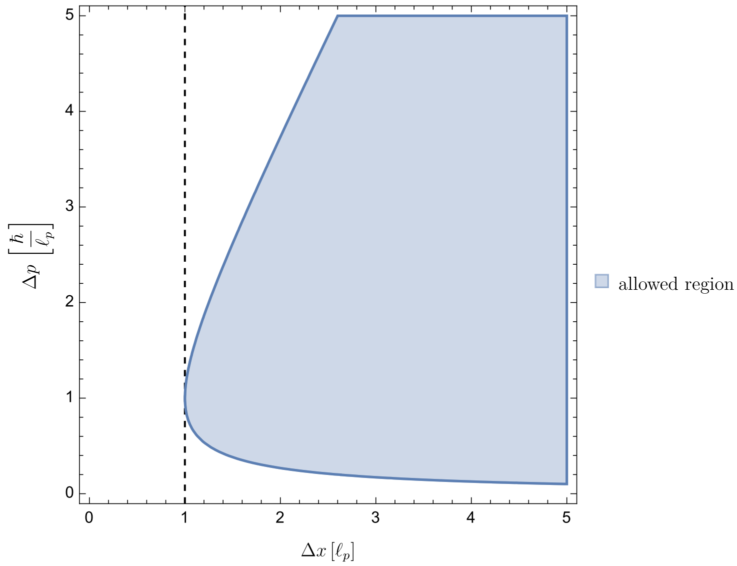

Figure 1.1: Plot of the inequality (1.16) for the case $\beta_0 = 1$. As you can see, Δp can take on any value, whereas Δx is always bigger than 1. From this, we have minimal length uncertainty in position ℓ_p.

1.4 Thought experiments

1.4.1 Heisenberg microscope

We start considering the thought experiment that Heisenberg presented when first introducing the uncertainty principle. Or, to be precise, we discuss here a later refined version of the experiment given in his Chicago lectures of 1930 [13]. The original argument is based only on Compton scattering without a microscope [1]. Hopefully, this will be interesting to the reader for historical reasons and for later arguments using some of the ideas presented here.

Imagine an apparatus schematically represented in Figure 1.2. The microscope is here, just in the form of one lens. We want to use it to determine the position of any other particle that would interact with light. For simplicity, let us consider an electron. The electron moves at such a distance from the microscope that the cone of photons scattered from it through the lens has an angular opening ε. When the wavelength of the light is λ, then the uncertainty in the x-direction Δx is due to the diffraction limit

$$\Delta x = \frac{\lambda}{\sin\varepsilon}\,. \tag{1.17}$$

To determine the position of an electron, at least one photon must be scattered from the electron. This creates uncertainty in the momentum because the electron receives from the photon Compton recoil, and we do not know the direction of the photon within the angle ε. It can be shown that due to this, the uncertainty in the momentum in the x-direction is about

$$\Delta p \simeq \frac{h}{\lambda}\sin\varepsilon\,. \tag{1.18}$$

If we now combine (1.17) and (1.18) we get

$$\Delta x\Delta p \simeq h\,. \tag{1.19}$$

This is not precisely the fundamental limit where $\hbar/2$ appears instead of h, but it is close to it.

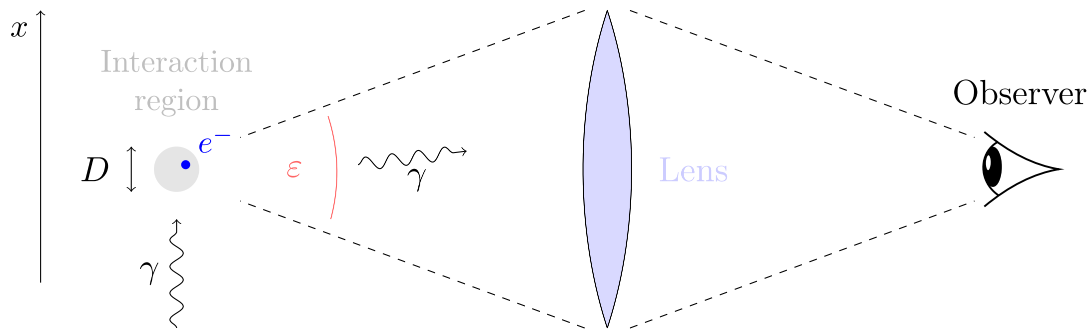

Figure 1.2: Schematic depiction of the Heisenberg microscope thought experiment. In this experiment, photon γ is sent towards an electron e^-. The main part of the gravitational interaction between the photon and the electron takes place in a region of diameter D, which is discussed in Subsection 1.4.2. The photon is then scattered off the electron and is observed by an observer on the right side through a lens with an angular opening ε.

1.4.2 Heisenberg microscope with Newtonian gravity

Here we present a very crude heuristic argument for GUP of the form (1.16) following [9], which can also be found in [14]. Although not very rigorous, this derivation contains essential ideas used in more detailed calculations and gives the same answer. These more rigorous derivations using not only Newtonian gravity but also GR can also be found in [14].

Let us imagine the same situation as in Subsection 1.4.1, but now we also consider gravity. We assume that the interaction between the photon and the electron occurs in an interaction region with diameter D as in Figure 1.2. Therefore, the distance between photon and electron during the interaction is at maximum D. Let us also treat the photon as a classical particle with mass p/c, which we get from the energy of the photon $E = pc$ divided by c^2. The acceleration a that the electron experiences due to photons gravity is then

$$a \geq \frac{Gp}{cD^2}\,. \tag{1.20}$$

The interaction time should be about D/c. We consider the uncertainty in the electron's position due to the gravitational acceleration to be approximately

$$\Delta x_G \gtrsim \frac{Gp}{cD^2}\left(\frac{D}{c}\right)^2 = \frac{Gp}{c^3}\,. \tag{1.21}$$

Let us assume that the uncertainty in the electron's momentum is on the order of the photon's momentum. From this

$$\Delta x_G \gtrsim \frac{G}{c^3}\Delta p = \frac{\ell_p^2}{\hbar}\Delta p\,. \tag{1.22}$$

If we add to Δx_G position uncertainty from the original uncertainty principle $\frac{\hbar}{2\Delta p}$ and multiply the resulting inequality by Δp we have

$$\Delta x \Delta p \gtrsim \frac{\hbar}{2}\left(1 + \left(\frac{\sqrt{2}\ell_p}{\hbar}\Delta p\right)^2\right)\,, \tag{1.23}$$

but this is approximately GUP of the form (1.16) where $\beta_0 = \sqrt{2}$. Realising this, we see that this predicts minimal uncertainty in the position

$$\Delta x \gtrsim \sqrt{2}\ell_p \geq \ell_p\,. \tag{1.24}$$

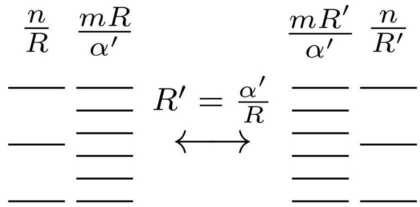

Figure 1.3: The energy spectrum of string of size R (left) is dual to spectrum of string with $R' = \alpha'/R$ (right). These states are equivalent in string theory

1.5 String theory

String theory is one of the leading candidates for the quantum theory of gravity. It does not predict minimal length in the ontological sense, but it predicts minimal measurable length and GUP similar to the form (1.16). These predictions can be based on high-energy string scattering [15, 16]. The corresponding expression for the GUP is

$$\Delta x \Delta p \geq \hbar + K\frac{\alpha'}{\hbar}(\Delta p)^2\,, \tag{1.25}$$

where α' is called 'string tension' (units are m^2, not Newtons as for classical tension) and is a fundamental string theory parameter. K is a suitable constant so that the minimal uncertainty in position is of order $\sqrt{\alpha'}$.

Because the rigorous derivation is long and complicated, we will consider at least an intuitive motivating argument for a minimal length in string theory from [17]. Let us imagine that the space in which our strings live has one periodic dimension with a period $2\pi R$, or we could say that the space is wrapped in a circle of radius R. The string energy spectrum is then composed of two parts. One is due to wrapping around the periodic dimension. If the string wraps around the periodic dimension n-times, the corresponding energy is

$$E_n = \frac{n}{R}\,. \tag{1.26}$$

The second part of the spectrum comes from vibration. The string can vibrate only in certain energy modes, and energy of m^{th} mode is

$$\tilde{E}_m = \frac{mR}{\alpha'}\,. \tag{1.27}$$

If we exchange R with α'/R, the two parts of the spectrum exchange places, but the overall energy levels are the same as seen in Figure 1.3. In string theory, these two states are therefore taken to be equivalent. Similar arguments can be made for any circle in spacetime where R determines the string size; therefore, if we try to compress string below the size where $R \leq \sqrt{\alpha'}$ it becomes equivalent to string with $R' \geq \sqrt{\alpha'}$.

To see how minimal length arises in this situation, consider an experiment similar to the Heisenberg microscope (see Subsection 1.2), where we determine the position of a test particle by scattering some other particles of it. In string theory, particles are replaced by strings. However, according to the previous discussion, as we increase the energy of the strings to increase the resolution, they expand, leaving us just with large propagating strings. This effectively gives us a minimal measurable length of order $\sqrt{\alpha'}$.

1.6 World crystal

One of the ontological approaches to minimal length is world crystal studied in [7], which is source of this section. As we have already mentioned, physicists have begun to realise that there is probably something wrong with our understanding of spacetime as a continuum. This is suggested, for example, by infinities arising in QFT and our inability to quantise gravity [17]. One way to discretise spacetime is to imagine it as a lattice (or world crystal). We have also discussed in Section 1.3 that this is used in lattice quantum chromodynamics as a computational tool, but here it is taken to represent reality.

For simplicity, let us use work just in one dimension. We define the world crystal as a sequence $M = \{n\ell\}_{n\in\mathbb{Z}}$, where ℓ is the minimal length. We take an integral over M to be

$$\sum_{x\in M} f(x)\,\ell\,. \tag{1.28}$$

Using this, we can consider modified quantum mechanics on the lattice. Hilbert space $\mathscr{H}_\ell$ is now also space of all complex square integrable functions on M, but now with the new integration (1.28) and the scalar product naturally defined as

$$\langle\psi|\phi\rangle_\ell := \sum_{x\in M} \psi^*(x)\phi(x)\,\ell\,. \tag{1.29}$$

We choose the lattice position and momentum operators to be

$$\begin{aligned}
(X_\ell\,\psi)(y) &:= x\psi(x)\,,\\
(P_\ell\,\psi)(x) &:= -i\hbar\frac{\psi(x+\ell)-\psi(x-\ell)}{2\ell}\,,
\end{aligned} \tag{1.30}$$

where $x \in M$. These are self-adjoint operators on $\mathscr{H}_\ell$. Moreover, we see that $X_\ell \xrightarrow{\ell\to 0} X$ and $P_\ell \xrightarrow{\ell\to 0} P$, where X and P are the standard position and momentum operators $(X\psi)(x) = x\psi(x)$, $(P\psi)(x) = -i\hbar\frac{d\psi}{dx}(x)$.

The commutator of X_ℓ and P_ℓ can be shown to be

$$([X_\ell, P_\ell]\psi)(x) = i\hbar\frac{\psi(x+\ell)+\psi(x-\ell)}{2} := i\hbar(I_\ell\,\psi)(x)\,, \tag{1.31}$$

where I_ℓ is a lattice-version of the identity operator given by the average over two adjacent sites. Using 1.7 we can find the uncertainty relation corresponding to the commutator (1.31) to be

$$\Delta X_\ell\,\Delta P_\ell \geq \frac{1}{2}|\,\langle[X_\ell, P_\ell]\rangle\,| = \frac{\hbar}{2}|\,\langle I_\ell\rangle\,|\,. \tag{1.32}$$

In [7] it is shown that the identity operator can be rewritten in terms of the standard momentum operator as

$$I_\ell = i\hbar\cos\left(\frac{\ell P}{\hbar}\right). \tag{1.33}$$

Moreover, based on the assumption of long wavelengths $\langle P\rangle << \frac{\hbar}{\ell}$ and the Taylor expansion of cosine, they were able to obtain

$$\langle I_\ell\rangle \simeq 1 - \frac{\ell^2 p^2}{2\hbar}\,, \tag{1.34}$$

where $p^2 := \langle P^2 \rangle$. We can use well known equality from statistics, which for mirror symmetric states ($\langle P \rangle = 0$) becomes

$$\langle P^2 \rangle = (\Delta P)^2 + \langle P \rangle^2 = (\Delta P)^2 \,, \tag{1.35}$$

and write

$$\Delta X_\ell \Delta P_\ell \gtrsim \frac{\hbar}{2} \left| 1 - \frac{\ell^2}{2\hbar^2} (\Delta P)^2 \right| \,. \tag{1.36}$$

Because ℓ is the minimum length and, hence, small, we can leave out the absolute value and approximate $\Delta P \simeq \Delta P_\ell$ to get

$$\Delta X_\ell \Delta P_\ell \gtrsim \frac{\hbar}{2} \left(1 - \frac{\ell^2}{2\hbar^2} (\Delta P_\ell)^2 \right) \,. \tag{1.37}$$

This is a GUP of the form 1.16, but notice that $\beta < 0$ and therefore the minimum uncertainty in position predicted by this GUP is imaginary $\Delta x_0 = \hbar\sqrt{\beta} = i\frac{\ell}{\sqrt{2}}$. However, we cannot reliably use 1.37 to make such conclusions, because we assumed long wavelengths.

1.7 Doubly special relativity

In the first years of the twentieth century, an Italian physicist Amelino-Camelia came up with the idea of DSR in papers [18, 19]. In this section, we offer a brief introduction to this theory drawing from [20].

The basic idea of the theory is that, apart from the invariant velocity scale defined by the speed of light, there is also an observer independent length scale ℓ, hence 'doubly' special relativity. This solves the problem discussed in Section 1.3, that theories with minimal observable length would have to have otherwise some preferred class of observers, which is seldom mentioned.

The introduction of DSR is similar to that of special relativity. When people discovered that the Maxwell equations contain an observer-independent velocity c, some thought that there has to be some preferred inertial frame to which it is connected. However, it turned out that there is no preferred frame and, instead, the rules of transformation between inertial frames are deformed in a way characterised by c. Notice that there are three types of observer-invariant quantities in relativity. That is, quantities defining some invariant physical laws, like the Planck constant $\hbar$ and the gravitational constant G; quantities connected to some specific frame of reference, such as the rest mass or the rest charge density; and ones defining the transformation between frames. It is important to understand that the length ℓ in DSR is of the third kind and, therefore, it is not a version of special relativity, but a different theory.

The formulation of the theory can be made more precise with these three 'principles' of DSR from [20]:

- **Principle of relativity:** *The laws of physics take the same form in all inertial frames.*
- **Invariance of ℓ:** *The laws of physics, and in particular the laws of transformation between inertial observers, involve an observer-independent length scale ℓ, which can be measured by each inertial observer.*
- **Invariance of c:** *The laws of physics, and in particular the laws of transformation between inertial observers, involve an observer-independent velocity scale c, which can be measured by each inertial observer as the speed of light in a limit $\frac{\lambda}{\ell} \to \infty$, where λ is wavelength of the light.*

Notice that in the second principle there is not specified the procedure for measuring ℓ, because it is yet to be determined.

One of the, in principle, measurable predictions of DSR, which could be used to measure ℓ, is a deformed relativistic energy-momentum dispersion relation [18]. Let us stress that not all specific formulations of DSR predict these deformed relations, but many do. The situation is again similar to the advent of special relativity. In Galilean relativity, the dispersion relation is $E = \frac{p^2}{2m}$, and in special relativity it is replaced by $E^2 = (cp)^2 + (mc^2)^2$. The form of a deformed dispersion relation in DSR could be, for example, $E^2 = (cp)^2 - f(E, p, \ell)$, where the leading ℓ dependence of f is $f(E, p, \ell) \simeq \ell cp^2 E$.

We have so far not used many mathematical formulae, because there is no agreed mathematical formalism for the theory. Promising candidates include the Hopf algebras (specifically, for example, κ-Poincaré Hopf algebra with κ-Minkowski non-commutative spacetime) and the spacetime non-commutative algebras discussed in Section 1.9. Also interesting is the connection between DSR and curved momentum spacetime, which is related to non-commutative geometry. Momentum spaces of constant curvature can, for instance, serve as a model of DSR [21, 22].

1.8 Curved momentum space

The arena of Hamiltonian mechanics is phase space which consists of position and momentum space. As a symplectic manifold, phase space only has an intrinsic notion of curvature once we introduce additional structures like a metric. This is done in general relativity for position space, but what would happen if we implemented it at momentum space? As we know from general relativity, position space curvature introduces much new and exciting physics, but arguably curved momentum space brings an even more surprising reality.

The idea of a curved momentum space was first proposed by Born in 1938 [23]. He noticed the equal role of position and momentum in quantum physics and considered that their spaces could have independent geometries. This idea was seen as a potential route towards quantum gravity, offering a way to renormalise QED. Born was able to obtain finite self-energies in QED and, in some sense, a discrete structure of space by assuming that momentum space has the geometry of a hypersphere.

Another who introduced curved momentum space, again in quantum physics, was Snyder [24]. As discussed in Subsection 1.9.1, he introduces a non-commutative quantum algebra on phase space. In his paper, Snyder does not explicitly talk about curved momentum space, but implicitly uses it to construct his algebra. As we will see, curved momentum space can often serve as a motivation for non-commutative algebra. This motivation is essential to us because we use the Snyder algebra for the deformation of the geodesic motion in Chapter 3. In addition, other algebras in deformed special relativity can be based on a curved momentum space; this is illustrated in [22].

The implementation of curved momentum space in classical physics can be found, for example, in the paper titled 'The principle of relative locality' [21]. In this paper, the authors propose that we take a phase space as not a mere tool for describing physics in spacetime but as more fundamental than spacetime. They argue that what we observe is not position space, but momentum space. For example, we see by receiving information about the energies and directions of photons of light. In this reality, a curved momentum space would amount to different observers constructing different spacetimes. This would mean that distant observers do not have to agree if interacting particles are at the same point in spacetime. This means that locality is relative, hence the name principle of relative locality. The exact formulation of the principle is

Physics takes place in phase space, and there is no invariant global projection that gives a description of processes in spacetime. From their measurements, local observers can construct descriptions of particles moving and interacting in a spacetime, but different observers construct different spacetimes, which are observer-dependent slices of phase space.

In the mathematical formalism of the theory, coordinates are viewed as elements of cotangent spaces to the momentum space. Cotangent spaces of a curved space are generally not the same, which is the main reason why different observers construct different spacetimes.

Although it is a classical theory, through some heuristic arguments, it is suggested that the Principle of relative locality implies some sort of GUP of the form

$$\Delta x \geq \frac{\hbar}{2p} + |x||\Gamma|p\,. \tag{1.38}$$

x is the position of a particle with momentum p and the term $\frac{\hbar}{2p}$ comes from classical Heisenberg's uncertainty principle. Γ represents a connection on the momentum space, which is determined by the deformation of the combination rule for momenta. It is again similar to the minimal length GUP (1.16). Perhaps the biggest difference is $|x|$, which has the effect of the uncertainty from the principle of relative locality increasing linearly with the particle's distance from the coordinates' origin.

1.9 Non-commutative geometry

Non-commutative geometry is a concept of great importance in mathematics and mathematical physics. It was devised mainly as a mathematical theory, but motivated by physics. Many ideas of the theory arise from a mathematician and theoretical physicist, Alain Connes, and are covered in the so-called 'Red Book' [25], which is written from the point of view of a mathematician with the goal of application to the Standard Model of particle physics.

The general notion behind non-commutative geometry is the correspondence of geometric space and algebras of functions on that space [26]. This relationship was already established in algebraic geometry for commutative algebras ($fg = gf$), but non-commutative geometry also extends it to non-commutative algebras ($fg \neq gf$). The study of non-commutative geometry is beneficial for two main reasons. First, it allows us to study non-commutative spaces using algebra which can be more natural. An example of this is the phase space in QM. The other advantage is extending standard tools corresponding to commutative algebras to non-commutative spaces. [25]

In quantum physics, non-commutative geometry manifests mainly as nonzero commutators of space(time) coordinates ($[x^\mu, x^\nu] \neq 0$). This was first conceived as early as 1946 by Snyder [24] with coordinates commutator

$$[x^\mu, x^\nu] = \frac{ia^2}{\hbar}\left(x^\mu p^\nu - x^\nu p^\mu\right)\,, \tag{1.39}$$

and this model will be the subject of Subsection 1.9.1.

One of the theories motivating non-commutative geometry is also string theory [27]. There is used the commutation relation

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,, \tag{1.40}$$

where $\theta^{\mu\nu}$ is a real-valued anti-symmetric tensor of dimension length squared. This relation also found applications in QFT where it might, for example, improve the renormalizability at short distances [28]. The commutator can be substituted into (1.7) to obtain the corresponding generalised uncertainty principle

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}|\,. \tag{1.41}$$

We can interpret (1.41) as giving us a minimal observable area in the $\mu\nu$-plane [9]. This indicates that the theory might not have a well-defined concept of a point in space, and in [29], it is shown that it is indeed so. There are also other possibly problematic aspects of (1.40). One is that applied to quantum field theories; it leads to non-locality at area scales of the order of $\theta^{\mu\nu}$. On the other hand, this can be welcomed as a theory of quantum gravity is suspected to be non-local. For example, string theory is not local in any sense, which we now understand [28]. Another problem is that the Lorentz invariance is broken if $\theta^{\mu\nu}$ is to be a constant matrix. However, this can again lead to interesting physics, as we can take (1.40) as an invariant for DSR [30].

Between GUP and non-commutative geometry, the motivation goes both ways. Non-commutative geometry can motivate GUP, as well as GUP can motivate non-commutative geometry. For example, the higher dimensional case of general minimal length GUP from [12] motivates a commutation relation of the form

$$[x_i, x_j] = 2i\hbar\beta(p_i x_j - p_j x_i)\,, \tag{1.42}$$

where β is the deformation parameter of the minimal length GUP.

From the previous discussion, it might seem that non-commutative geometry motivates only a generalised uncertainty principle between position coordinates. That is because we considered only non-commutative geometries of physical spaces. However, practically any quantum mechanical commutator algebra gives rise to non-commutative geometry if we consider phase space. The commutation relation of spatial coordinates is often part of larger commutator algebra, where we also deform a commutation relation of coordinates and momenta, as with the Snyder model.

There is also a connection between non-commutativity in space(time) coordinates and curved momentum space discussed in Section 1.8. In some cases this can be viewed as another way of describing the same physical space. However, although all curved momentum spaces give rise to non-commutative geometry [21], not all non-commutative geometries come from or can be described by a curved momentum space [31].

1.9.1 Snyder model

In a 1946 paper [24], Snyder introduced his famous model. It gave rise to the first model of non-commutative geometry, though at the time, non-commutative geometry as a theory did not exist. It is important to us mainly because it is a well-known commutator algebra for a relativistic theory. This is useful because, in Chapter 3, we are interested in modifying the geodesic motion of a relativistic particle.

However, the goal of Snyder was different. He wanted a quantum Lorentz invariant spacetime with a minimal length to help remove infinities from QFT. He claims to have achieved this in the following way. Let us consider all the spacetime coordinates to be self-adjoint operators and, from now on, call them spacetime operators. The spectra of the spacetime operators give us possible values of the spacetime coordinates. If we want our theory to be Lorentz invariant, it should hold that the spectra of spacetime operators do not change when we move to another inertial frame. The requirement of minimal length is then equivalent to the spectra of spacetime operators being discrete. According to Snyder, it can be shown that if we want discrete spectra of spacetime operators, we must allow non-commutativity between them. Otherwise, Lorentz invariance implies continuous spectra.

Snyder then embarks on finding operators that meet these conditions. The exact form of the operators is not of immediate interest to us, but we are interested in the spectra and the commutator algebra. Each of the space operators found by Snyder has a spectrum $\{m\ell\}_{m\in\mathbb{Z}}$, where ℓ is the minimal length, and the time operator has a continuous spectrum extending from $-\infty$ to $+\infty$. Commutator algebra is the following[1]

$$[x^\mu, p_\nu] = i\hbar\left(\delta^\mu_\nu - \left(\frac{\ell}{\hbar}\right)^2 p^\mu p_\nu\right), \qquad [x^\mu, x^\nu] = -i\hbar\left(\frac{\ell}{\hbar}\right)^2 (x^\mu p^\nu - p^\mu x^\nu), \qquad [p_\mu, p_\nu] = 0\,, \tag{1.43}$$

where $x^\mu p^\nu - p^\mu x^\nu$ are generators of the Lorentz group.

It is perhaps worth mentioning that Snyder's quantised spacetime is not invariant under the whole standard Poincaré group. It is invariant under the Lorentz group, that is, spatial rotations and boosts, but we cannot allow continuous translation as they would imply continuous spectra. However, this can be solved by deforming the translations to better match the discrete nature of spacetime (as Snyder did).

At the beginning of this section, we already pointed out that sometimes a space of non-commutative geometry can be characterised by a curved momentum space. This is true for the Snyder model. We want this space to be Poincaré invariant, which means 4 translations, 3 spatial rotations, and 3 boosts, which restricts our space to be maximally symmetric [32]. There are just three such Lorentzian manifolds, flat Minkowski space, de Sitter space with positive curvature and anti-de Sitter space with negative curvature. The momentum space of the Snyder model can have either negative or positive curvature, but to recover commutator algebra (1.43), we must have de Sitter space. This 4D de Sitter momentum space can be described as a subspace of a flat 5D space with signature (1, -1, -1, -1, -1) and the Minkowski coordinates ξ^μ where $\mu \in \{0, 1, 2, 3, 4\}$, which is defined by the constraint $\xi^\mu \xi_\mu = -\hbar^2/\ell^2$ [31]. 4D de Sitter spaces have a topology of $\mathbb{R}\times S^3$ and can be therefore 'visualised' as a 3D sphere with the dimension of time. Interested readers can find more information in [33, 34].

Although the Snyder model is a model of special relativity with a minimum length, it is important to note that it is not a case of DSR. This is because Lorenz transformations remain unchanged as Snyder intended, and this goes against the second principle of DSR from Section 1.7. However, it is true that translations are modified in a way that contains ℓ, but this is probably not enough to consider it a DSR theory. Also, Amelino-Camelia suggests re-examining Snyder's proposal with modern tools of symmetry analysis because he may have used some incorrect assumptions [20, 36]. Perhaps after this analysis, it would be discovered that the Snyder model leads to a DSR scenario after all.

It is also interesting to note that in [12], they managed to arrive at an algebra very similar to that of Snyder. They considered a natural generalisation of the algebra corresponding to the minimal length GUP (1.16) to n-dimensions, preserving the rotational symmetry, and obtained

$$[x_i, p_j] = i\hbar\delta_{ij}(1 + \beta p^k p_k)\,, \qquad [x_i, x_j] = 2i\hbar\beta(p_i x_j - p_j x_i)\,, \qquad [p_i, p_j] = 0\,, \tag{1.44}$$

where $i, j, k \in \hat{n}$. One of these commutators was already mentioned before in this section.

[1] Throughout this work we employ the metric signature $(+, -, -, -)$ and the Einstein summation convention.

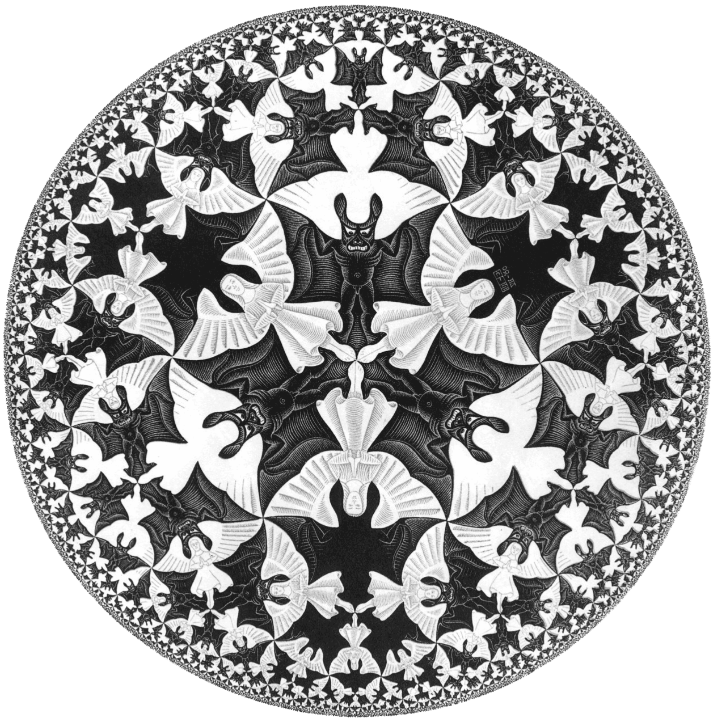

Figure 1.4: Circle Limit IV by M. C. Escher can be seen as representing a conformal projection of a spatial slice (according to signature we take one coordinate to represent time) of 3D anti-de Sitter space [35]. Image downloaded from favpng.com in July 2023.

Chapter 2

Motivations for our Deformation

In this chapter, we explore the motivation behind the form of deformation of geodesic motion based on GUP, which is used in Chapter 3. Although geodesic motion is a classical concept and GUP comes from quantum physics, we will connect the two through the quantisation of constrained system.

2.1 Generalised Hamiltonian dynamics

To better understand the ideas of Section 2.2, we introduce the generalised (or constrained) Hamiltonian dynamics. This formalism was first introduced by Dirac in his 1950 article [37] mainly for relativistic quantisation. However, the approach found its use, among others, also in electromagnetism and other theories with gauge freedom [38]. In the first three subsections, we introduce the formalism of the theory following [39] with the help of [37], in subsection 2.1.4 we discuss the problems with quantisation of constraint systems and in the last subsection 2.1.5 we apply it to a free relativistic particle.

2.1.1 Generalised Hamilton's equation

Let us consider a manifold M of dimension n (all indices in this section run from 1 to n) to be a configuration space of our system and TM the tangent bundle of this space. For a Lagrange function, $L : TQ \to \mathbb{R}$ the action of a trajectory through configuration space $q(t)$ reads

$$S[q(t)] = \int_{t_A}^{t_B} L(q(t), \dot{q}(t))\, dt\,, \qquad \dot{q}(t) = \frac{dq}{dt}(t)\,. \tag{2.1}$$

Realised is the trajectory where the variation of the action takes a stationary value with respect to the variations with fixed ends $\delta q(t_A) = 0 = \delta q(t_B)$. In local coordinates $q = (q^1, \ldots, q^n)$, these trajectories can be found also through Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(q, \dot{q})\right) - \frac{\partial L}{\partial q^i}(q, \dot{q}) = 0\,. \tag{2.2}$$

These are the equations of motion of Lagrangian mechanics.

If we want to move to the Hamiltonian formalism, we can use the Legendre map $\mathcal{L} : TM \to T^*M$ (T^*M denotes the cotangent bungle to M) defined as

$$\mathcal{L}(q, \dot{q}) = (q, p)\,, \qquad p_i = \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q})\,. \tag{2.3}$$

When $\det(W_{ij}) \neq 0$ where $W_{ij} = \frac{\partial p_j}{\partial \dot{q}^i} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$, then L is called *regular (otherwise singular)* and $\mathscr{L}$ is a local diffeomorphism. Therefore we can express $\dot{q} = \dot{q}(q,p)$ using $\mathscr{L}^{-1}$, and obtain

$$H(q,p) = p_i \dot{q}^i(q,p) - L(q,\dot{q}(q,p)) . \tag{2.4}$$

Generalised Hamiltonian dynamics gives us the tools to deal also with singular Lagrange functions. Even if L is singular, we can still write

$$H(q,\dot{q},p) = p_i \dot{q}^i - L(q,\dot{q}) , \tag{2.5}$$

but this time $H : TM \oplus T^*M \to \mathbb{R}$. We assume $\mathscr{L}$ is not a local diffeomorphism, but that there exist r smooth functions $\varphi_\alpha : T^*M \to \mathbb{R}$ such that $\mathscr{L}(TM) = \{(q,p) \in T^*M | (\forall \alpha \in \hat{r})(\varphi_\alpha(q,p) = 0)\}$ ($\dim \mathscr{L}(TM) = 2n - r$) and $d\varphi_\alpha$ are linearly independent on $\mathscr{L}(TM)$. Equations $\varphi_\alpha = 0$ are called *primary constraints*.

The partial derivatives of H are no longer Hamilton's equations, but

$$\frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i} , \qquad \frac{\partial H}{\partial p_i} = \dot{q}^i , \qquad \frac{\partial H}{\partial \dot{q}^i} = p_i - \frac{\partial L}{\partial \dot{q}^i} . \tag{2.6}$$

Let us introduce a new notation. We denote the graph of the Legendre map $\Gamma_{\mathscr{L}} = \{(q,\dot{q},p) \in TM \oplus T^*M | (q,p) = \mathscr{L}(q,\dot{q})\}$, and all equations that hold only on this graph are called *weak equations*. In weak equations, we use '$\approx$' instead of '$=$'. Using this notation

$$\frac{\partial H}{\partial \dot{q}^i} \approx 0 , \tag{2.7}$$

in other words H is independent of $\dot{q}$ on $\Gamma_{\mathscr{L}}$. Therefore, we suppose that there exists a smooth function H_C defined on an open superset of $\mathscr{L}(TM)$, such that $H(q,\dot{q},p) \approx H_C(q,p)$[1]. To have the functions defined on the same space, we set $\tilde{H} = H_C \circ \pi_{T^*M}$ and $\tilde{\varphi}_\alpha = \varphi_\alpha \circ \pi_{T^*M}$, where $\pi_{T^*M} : TM \oplus T^*M \to T^*M$ is the canonical submersion.

Let us now use the theorem for constrained extremes from differential geometry [40]. For that, we need that $\Gamma_{\mathscr{L}}$ is a submanifold of $TM \oplus T^*M$ and

$$(\forall (q,\dot{q},p) \in \Gamma_{\mathscr{L}}) \left(\bigcap_{\alpha=1}^{r} \ker d\tilde{\varphi}_\alpha \subset \ker d(H - \tilde{H}) \right) . \tag{2.8}$$

The theorem then gives us

$$dH(q,\dot{q},p) \approx d\tilde{H}(q,\dot{q},p) + \sum_{\alpha=1}^{r} v_\alpha(q,\dot{q},p) d\tilde{\varphi}_\alpha(q,\dot{q},p) . \tag{2.9}$$

We used the theorem at every point of $\Gamma_{\mathscr{L}}$ separately, therefore, the Lagrange multipliers depend on that point.

If we combine (2.6), (2.7), (2.9), and primary constraints, we have

$$\begin{aligned} \frac{\partial \tilde{H}}{\partial q^i} &\approx -\frac{\partial L}{\partial q^i} - \sum_{\alpha=1}^{r} v_\alpha \frac{\partial \tilde{\varphi}_\alpha}{\partial q^i} , \\ \frac{\partial \tilde{H}}{\partial p_i} &\approx \dot{q}^i - \sum_{\alpha=1}^{r} v_\alpha \frac{\partial \tilde{\varphi}_\alpha}{\partial p_i} . \end{aligned} \tag{2.10}$$

[1]For H_C to exist, we may have to restrict the domain of $\mathscr{L}$, but we will not go into all the details.

From this we see that v_α can depend only on the velocities $\dot{q}$. It is because (2.10) gives us $2n$ variables q and $\dot{q}$ in terms of $2n + r$ variables q, p and v_α, which we assume to be related by r primary constraints. Therefore, q, p and v_α cannot be related in other ways, because that would mean that q and $\dot{q}$ are not independent.

Now we substitute from (2.2) to obtain the generalised Hamilton's equations

$$\begin{aligned} \frac{dq^i}{dt} &\approx \frac{\partial H_C}{\partial p_i}(q,p) + \sum_{\alpha=1}^{r} v_\alpha(\dot{q}) \frac{\partial \varphi_\alpha}{\partial p_i}(q,p) = \frac{\partial H_T}{\partial p_i}(q,\dot{q},p)\,, \\ \frac{dp_i}{dt} &\approx -\frac{\partial H_C}{\partial q^i}(q,p) + \sum_{\alpha=1}^{r} v_\alpha(\dot{q}) \frac{\partial \varphi_\alpha}{\partial q^i}(q,p) = \frac{\partial H_T}{\partial q^i}(q,\dot{q},p)\,, \end{aligned} \tag{2.11}$$

where we have defined the total Hamiltonian to be $H_T := H_C + \sum_{\alpha=1}^{r} v_\alpha \varphi_\alpha$. These equations are fulfilled for all solutions of the Euler-Lagrange equations for the right choice of functions $v_\alpha(\dot{q})$. The form of these functions is constrained by the consistency conditions discussed in the following subsection.

2.1.2 Consistency conditions

We define the Poisson bracket of functions f and g on T^*M as

$$\{f, g\} := \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\,. \tag{2.12}$$

Using it and the generalised Hamilton's equations (2.11), we can write the time evolution of any smooth phase space function f, which does not explicitly depend on time as

$$\frac{df}{dt} \approx \{f, H_T\}\,. \tag{2.13}$$

All constraints must be satisfied at all times, therefore $\varphi(q(t), p(t)) = 0$. This is fulfilled if initial conditions are in $\mathcal{L}(TM)$, and (assuming that constraints do not explicitly depend on time)

$$(\forall \beta \in \hat{r}) \left(\frac{d\varphi_\beta}{dt} \approx \{\varphi_\beta, H_C\} + \sum_{\alpha=1}^{r} v_\alpha \{\varphi_\beta, \varphi_\alpha\} = 0 \right)\,. \tag{2.14}$$

These equations are called the *compatibility conditions.*

There are several possibilities for the form of these equations. In general, there might be some impossible equations (for example $1 = 0$), but these do not arise in physical systems. On the other hand, some equalities can be weakly satisfied $\left(\frac{d\varphi_\beta}{dt} \approx 0\right)$. However, they can also restrict the Lagrange multipliers v_α and introduce new constraints $\phi_\beta = 0$ called the *secondary constraints.* These conditions further restrict our constraint space, but do not enter the generalised Hamilton equations. Now we have to repeat this also for the secondary constraints and solve the equations $\frac{d\phi_\beta}{dt} \approx 0$ (the weak equality is taken to be on the new constraint space). These might produce additional secondary constraints, and we repeat this process until all the conditions are met.

This gives us all the restrictions on v_α. They still do not have to be determined uniquely, which would correspond to gauge degrees of freedom, and therefore these v_α could be chosen freely. We also have a new set of s constraints ϕ_β. We assume that all $d\varphi_\alpha$ and $d\phi_\beta$ are linearly independent and that they define a final constraint space Ω ($\dim \Omega = 2n - r - s$), which is a submanifold of T^*M. If additional conditions were to be added, they should be treated as secondary constraints, and if other consistency conditions would arise, they should be dealt with in the same way.

Now we have all that is needed for the time evolution of the system. In the next two sections we discuss the Dirac bracket and its use in quantisation.

2.1.3 Dirac bracket

Let use introduce the important concept of Dirac bracket. First, we introduce some terminology. We call a function *first-class* if its Poisson bracket with all the constraints φ_α and ϕ_β, from the previous section, vanishes; otherwise, it is called *first-class*.

Now we take the function φ_α and make a linear transformation of the form

$$\vartheta_\alpha = \sum_{\beta=1}^{r} L_{\alpha\beta}\varphi_\beta \,, \tag{2.15}$$

where $L_{\alpha\beta}$ are functions of q and p and $\det L \not\approx 0$. Also, the transformation should be such that we are left with the maximum number of first-class constraints $\vartheta_\alpha^{(1)}$. We then do similar transformation to the secondary constraints ϕ_β

$$\theta_\beta = \sum_{\alpha=1}^{s} \tilde{L}_{\beta\alpha}\phi_\beta + \sum_{\alpha=1}^{r} \hat{L}_{\beta\alpha}\varphi_\alpha \,, \tag{2.16}$$

where $\tilde{L}$ and $\hat{L}$ have the same properties as L. Again, we should obtain the maximum number of first-class constraints $\theta_\beta^{(1)}$.

Suppose that we have l second-class constraints $\vartheta_\alpha^{(2)}$ and m constraints $\theta_\beta^{(2)}$. If we denote $\Theta_i := (\theta_1^{(2)}, \ldots, \theta_l^{(2)}, \vartheta_1^{(2)}, \ldots, \vartheta_m^{(2)})$, we can construct a matrix $M_{ij} := \{\Theta_i, \Theta_j\}$ using the Poisson bracket. It can be proved that $\det M \not\approx 0$. Notice that because the Poisson bracket is antisymmetric (hence the matrix M), this means that $l + m$ must be an even number. We can now define the Dirac bracket of function f and g as

$$\{f, g\}_D = \{f, g\} - \sum_{i=1}^{l+m}\sum_{j=1}^{l+m} \{f, \Theta_i\} M_{ij}^{-1} \{\Theta_j, g\} \,, \tag{2.17}$$

where M^{-1} is the inverse matrix to M.

Nice property of this new Poisson bracket is that for any function f

$$\{f, \Theta_k\}_D = \{f, \Theta_k\} - \sum_{i=1}^{l+m}\sum_{j=1}^{l+m} \{f, \Theta_i\} M_{ij}^{-1} \{\Theta_j, \Theta_k\} = \{f, \Theta_k\} - \sum_{i=1}^{l+m} \{f, \Theta_i\}\delta_{ik} = 0 \,. \tag{2.18}$$

We know that $\Theta_i \approx 0$, but using this property, we can set $\Theta_i = 0$. These equations can be used to simplify the Hamiltonian and they will be important in the next Subsection 2.1.4.

From the consistency conditions, we know that $\{\Theta_i, H_T\} \approx 0$, because Θ_i are linear combinations of constraints. Thanks to this, we can also write the time evolution of function f (therefore also the generalised Hamilton's equations) using the Dirac bracket

$$\frac{df}{dt} \approx \{f, H_T\} \approx \{f, H_T\}_D \,, \tag{2.19}$$

which reduces the effective number of degrees of freedom.

We have seen that the Dirac bracket has many useful properties and that it can serve as a generalisation of the Poisson bracket. Its importance will become even clearer in the next section.

2.1.4 Quantisation of constrained systems

Through this subsection we are drawing from [38]. In the standard procedure of canonical quantisation, we replace functions on a phase space by self-adjoint linear operators on a Hilbert space. Poisson bracket has the important properties of

- **bilinearity:** $(\forall a, b \in \mathbb{C})(\forall f, g, h \in C^\infty(T^*M))(\{af + bg, h\} = a\{f, h\} + \{g, h\})$
- **anticommutativity:** $(\forall f, g \in C^\infty(T^*M))(\{f, g\} = -\{g, f\})$
- **Jacobi identity:** $(\forall f, g, h \in C^\infty(T^*M))(\{f, \{g, h\}\} + \{h, \{f, g\}\} = 0)$

which give $C^\infty(T^*M)$ with the Poisson bracket the status of Lie algebra. Because the same is true for the commutator on the space of observables, in canonical quantisation we promote the Poisson bracket to the commutator as

$$\{a, b\} \to \frac{1}{i\hbar}[a, b]\,, \tag{2.20}$$

where a and b are observables. This gives us the standard equation of the time evolution in the Heisenberg picture for an observable A that is not explicitly dependent on time

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]\,. \tag{2.21}$$

However, there are some problems with this procedure. To a classical observable, is not always associated a unique operator and also there is problem with the ordering of operators.

For constrained systems, quantisation is even more problematic. One way to do this is to fix all the Lagrange multipliers (in classical physics) and then solve the Schrödinger equation with the total Hamiltonian H_T as a quantum operator. We promote the Poisson bracket to the commutator as in canonical quantisation. However, we have to restrict the Hilbert space by the constraints to obtain the physical Hilbert space $\mathscr{H}_p$. We say that $|\phi\rangle\,, |\psi\rangle \in \mathscr{H}_p$ if they satisfy relations

$$\langle\phi| C_i |\psi\rangle = 0\,, \tag{2.22}$$

where C_i are operators corresponding to all the constraints. There is a problem with assigning quantum operators to the constraints, and in this way we also have to know the full spectrum of C_i to separate the physical states.

We can try to replace (2.22) by

$$C_i |\psi\rangle = 0\,, \tag{2.23}$$

but this has problems, if want to quantise the Poisson bracket. This is because for consistency we require

$$[C_i, C_j] |\psi\rangle = 0 \tag{2.24}$$

but this does not have to generally hold. For the first-class constraints, we have $\{\Theta_i^{(1)}, \Theta_j^{(1)}\} = \sum_k f_{ijk} C_k$ (here C_k denotes all the classical constraints), but the coefficients do not have to remain to the left upon quantisation. For the second-class constraints, it is not even true that $\{\Theta_i, \Theta_j\} \approx 0$.

This problem can be solved using the Dirac bracket. In [37] Dirac proves that the Dirac bracket has the properties of bilinearity, anticommutativity, and Jacobi identity, as the Poisson bracket. This makes it a good classical analogue for the commutator. From Subsection 2.1.3 we also know that when using the Dirac bracket we take the second-class to vanish. Also, it can be shown that all first-class constraints are generators of gauge transformations [41], therefore, if we impose gauge-fixing conditions, all the constraints become second-class. This gives us the right to replace (2.22) with $C_i = 0$ when quantising the Dirac bracket.

2.1.5 Free relativistic particle

In this subsection we apply the principles from this section to free relativistic particle in flat spacetime, drawing from [38]. This is important to us because it will serve as a basis for discussion about our deformation of geodesic motion. The standard action for a massive relativistic particle reads

$$S = -mc\int_{\tau_A}^{\tau_B} \sqrt{\dot{x}^\mu \dot{x}_\mu}\, d\tau\,, \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau}\,. \tag{2.25}$$

Interested readers can see the beginning of Section 3.1 for some motivation of (2.25). Because we consider only time-like trajectories $\dot{x}^\mu \dot{x}_\mu > 0$. Lagrange function corresponding to the action S is $L = -mc\sqrt{\dot{x}^\mu \dot{x}_\mu}$.

Notice that L is singular, because

$$W_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} \tag{2.26}$$

has rank three (we are in four dimensions), where the associated null-eigenvector is $\dot{x}^\mu$. Therefore, we are looking for one constraint. The canonical momenta are

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -mc\frac{\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}\,, \tag{2.27}$$

from which we have the constraint $p_\mu - \frac{\partial L}{\partial \dot{x}^\mu} = 0$. We can multiply this by $\dot{x}^\mu$ to obtain

$$0 = \dot{x}^\mu p_\mu - \dot{x}^\mu \frac{\partial L}{\partial \dot{x}_\mu} = \dot{x}^\mu p_\mu + mc\sqrt{\dot{x}^\mu \dot{x}_\mu} = \dot{x}^\mu p_\mu - L = H_C\,. \tag{2.28}$$

From this we see that $H_C = 0$, but if we continue after the third equal sign and use (2.27), we also get

$$0 = \dot{x}^\mu p_\mu + mc\sqrt{\dot{x}^\mu \dot{x}_\mu} = -\frac{1}{mc}\sqrt{\dot{x}^\nu \dot{x}_\nu} p^\mu p_\mu + mc\sqrt{\dot{x}^\nu \dot{x}_\nu} = -\frac{1}{mc}\sqrt{\dot{x}^\nu \dot{x}_\nu}(p^\mu p_\mu - m^2c^2)\,. \tag{2.29}$$

Now we see that we can write the constraint also as

$$0 = \varphi = p^\mu p_\mu - m^2c^2\,. \tag{2.30}$$

The total Hamiltonian becomes $H_T = v(p^\mu p_\mu - m^2c^2)$.

Because we have only one constraint and $H_C = 0$, there are no restrictions on the choice of v. This corresponds to one gauge degree of freedom, which is the parametrisation invariance. The fact that the action is independent of the change of world-line parameter $\sigma = \sigma(\tau)$, when $\sigma(\tau)$ is a strictly increasing function, can be seen from

$$\int_{\tau_A}^{\tau_B} \sqrt{\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}}\, d\tau = \int_{\tau_A}^{\tau_B} \sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}}\frac{d\sigma}{d\tau}\, d\tau = \int_{\sigma_A}^{\sigma_B} \sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}}\, d\sigma\,. \tag{2.31}$$

This can be easily generalised to strictly monotonic functions. From the discussion in the last Subsection 2.1.4 we also know that because φ is a first-class constraint, it generates gauge transformations and therefore reparametrisation.

We can fix the gauge by introducing a new constraint ϕ. In Subsection 2.1.2 we have seen that when introducing a new constraint, one has to check the consistency conditions. If we consider that ϕ can depend explicitly on τ, the consistency conditions become

$$0 = \frac{d\phi}{d\tau} \approx \frac{\partial \phi}{\partial \tau} + \{\phi, H_T\} \approx \frac{\partial \phi}{\partial \tau} + 2vp^\mu \frac{\partial \phi}{\partial x^\mu} \,. \tag{2.32}$$

From this we have that if we want to fix the gauge and determine v, then $\partial_\mu \phi \not\approx 0$, and if v should not vanish (then we would have no time evolution), it must hold that $\partial_\tau \phi \not\approx 0$. We choose two gauge constraints, $\phi_1 = x^0 - \tau$ and $\phi_2 = \frac{\ell x^\mu p_\mu}{\hbar} - \tau$, which are similar to the ones chosen in [22]. Let us just note that $x^\mu p_\mu$ is the generator of dilation in quantum physics as discussed in Subsection 3.2.2.

Because $\{\varphi, \phi_{1,2}\} \not\approx 0$ these constraints are both second-class. This is exactly what we expect when the gauge is fixed. Therefore, we can calculate the Dirac brackets. Because we have just two constraints, the formula for Dirac bracket simplifies to

$$\{f, g\}_{D1,2} = \{f, g\} - \{f, \phi_{1,2}\} \frac{1}{\{\varphi, \phi_{1,2}\}} \{\varphi, g\} + \{f, \phi_{1,2}\} \frac{1}{\{\varphi, \phi_{1,2}\}} \{\phi_{1,2}, g\} \,. \tag{2.33}$$

If we substitute the constraints, we get

$$\{x^\mu, x^\nu\}_{D1} = 0 \qquad \{x^\mu, p_\nu\}_{D1} = \delta^\mu_\nu - \frac{p^\mu}{p^0} \delta^0_\nu \qquad \{p_\mu, p_\nu\}_{D1} = 0 \tag{2.34}$$

for the constraint ϕ_1, and

$$\{x^\mu, x^\nu\}_{D2} = -\left(\frac{\ell}{\hbar}\right)^2 (x^\mu p^\nu - x^\nu p^\mu) \qquad \{x^\mu, p_\nu\}_{D2} = \delta^\mu_\nu - \left(\frac{\ell}{\hbar}\right)^2 p^\mu p_\nu \qquad \{p_\mu, p_\nu\}_{D2} = 0 \tag{2.35}$$

for ϕ_2.

As discussed in Subsection 2.1.4 one way to quantise this system is to promote the Dirac brackets to the commutator. Notice that the quantum analogue of (2.35) is exactly the quantum Snyder algebra (1.43). This interesting 'coincidence' is subject to Subsection 2.2.1. Other way to quantise the system would be to have

$$(p^\mu p_\mu - m^2 c^2) \left|\psi\right\rangle = 0 \tag{2.36}$$

in the place of $\varphi = 0$. We discussed that there are some problems with this approach, but they do not manifest themselves if we do not fix the gauge and have just one first-class constraint. It can be seen that this works quite well in this case, because (2.36) is just the Klein-Gordon equation in momentum representation.

2.2 Motivations from Snyder model

Our deformed Poisson algebra of a free relativistic particle reads for four-position x^μ and corresponding canonical momentum p_μ

$$\{x^\mu, x^\nu\} = -\frac{1}{p^2} (x^\mu p^\nu - x^\nu p^\mu) \,, \qquad \{x^\mu, p_\nu\} = \delta^\mu_\nu - \frac{1}{p^2} p^\mu p_\nu \,, \qquad \{p_\mu, p_\nu\} = 0 \,, \tag{2.37}$$

where $p := m_p c$. The reader may recognise this from Subsection 1.9.1. Indeed it is just the Snyder algebra (1.43) with the choice $\ell = {\ell_p}^2$ and the canonical quantisation of Poisson bracket (2.20) (here of course in reverse). In this section, we show how the use of this algebra is motivated based on some considerations surrounding the Snyder model. These also include a model of deformed special relativity with a curved momentum space, and the discussion from Subsection 2.1.5.

2.2.1 Possible gauge dependence of Snyder model

We have seen in Subsection 2.1.5, that we can reproduce the Dirac bracket algebra corresponding to the Snyder commutator algebra considering a standard relativistic particle. This is probably not just a coincidence, and in this subsection we will consider what the reasons and implications might be.

One reason for this might be that in both cases we are dealing with a special-relativistic algebra. The symmetries of special relativity significantly restrict the possibilities for such an algebra and therefore it was not difficult to find the parameterisation for relativistic particle which reproduces the Snyder algebra.

Another, more radical explanation that comes to mind is that in reality, the Snyder algebra should be seen, for some reason, to correspond directly to the Dirac brackets (2.35). It could also correspond to deformation of a Dirac bracket algebra in different parameterisation, but we would not know which one. The first case would mean that the Snyder algebra depends on the parameterisation (gauge) of the relativistic particle world-line because the form of Dirac brackets is gauge dependent. Therefore, the Snyder algebra would not introduce anything new. However, this seems to us to go against Snyder's intentions of introducing a fundamental model for position and momentum operators in quantum field theory [24].

From our point of view, Snyder's commutator algebra should for this reason correspond upon quantisation of a relativistic particle to the Poisson bracket algebra. The Poisson brackets have the same form in terms of positions and canonical momenta in all parametrisations; however, in different parameterisations, they have a different meaning. Therefore, we have to choose some parameterisation corresponding to the Snyder model, and the most natural candidate seems to us to be the proper time, which is also used for the deformed geodesic motion in Chapter 3.

2.2.2 Curved momentum space of deformed special relativity

In this subsection, we present another argument for the deformed Poisson algebra (2.37), which can also be found in [22]. First, we consider a relativistic particle in a 5D Minkowski space. Then we introduce a constraint that restricts the 10D phase space of the particle to an 8D phase space with a de Sitter momentum space. At the end we impose a gauge-fixing condition, and because we start in the 10D phase space with Minkowski coordinates, this amounts to some choice of coordinates on the 8D phase space. The Poisson algebra of the final coordinates is then our desired algebra. This approach reproduces also some other algebras of deformed special relativity and is motivated in various ways by the authors in [22].

Drawing heavily from Subsection 2.1.5, we write the action of the 5D particle as[3]

$$S_{5d} = \int_{\tau_A}^{\tau_B} (\dot{X}_A P^A - \lambda H_{5d} - \mu H_{4d}), d\tau, , \tag{2.38}$$

[2] This choice slightly reduces the generality of our discussion, but it is not difficult at any point to move on to the case of a general ℓ.

[3] The 5D variables in this subsection are indexed and represented by capital letters, and they are also subject to the Einstein summation convention.

where the dot represents the derivative with respect to the parameter τ, and X, P are the Minkowski coordinates on the flat 5D space with the standard Poisson bracket algebra. H_{5d} and H_{4d} are Hamiltonian constraints. The first constraint reads $H_{5d} = P^A P_A + \rho^2$, and it restricts the initial 5D momentum space to a 4D de Sitter space. The second constraint is $H_{4d} = P_4 - Mc$, where M is for now just some constant which role we will see later. This condition introduces the mass-shell condition and serves as a Hamiltonian for the reduced phase space.

As in Subsection 2.1.5 we introduce an additional gauge-fixing constraint

$$\phi = X^A P_A - \tau = 0 \,. \tag{2.39}$$

This makes H_{5d} a second-class constraint and together with ϕ they reduce the phase space from 10D to 8D. We can choose coordinates on the new phase space as functions that commute with H_{5d} and ϕ

$$\{x^\mu, H_{5d}\} = \{x^\mu, \phi\} = \{p_\mu, H_{5d}\} = \{p_\mu, \phi\} = 0 \,. \tag{2.40}$$

It follows that the Poisson bracket of these coordinates with any phase-space function $f(X, P)$ is equal to their Dirac bracket. This is an advantage in quantisation, since we can directly quantise the Poisson bracket when using these variables. This is one of the reasons why the Poisson algebra of these variables should be our deformed algebra. Because we have 10 functions H_{5d}, ϕ, x^μ, and p_μ, we can also invert this system to obtain X^A and P_A in terms of these variables. Furthermore, since H_{5d} and ϕ commute with x^μ, p_μ, we can fix them and express the 5D variables just in terms of x^μ, p_μ, ρ and τ. From this we find that any function $g(X, P)$ commuting with the constraints can be rewritten just in terms of x^μ, p_μ, hence these variables form a closed algebra under the Poisson bracket.

Based on the previous discussion, we choose the new coordinates to be

$$\begin{aligned} p_\mu &= \rho \frac{P_\mu}{P_4} \,. \\ x^\mu &= \frac{1}{\rho}(X^\mu P^4 - X^4 P^\mu) \,. \end{aligned} \tag{2.41}$$

Now, using (2.41) and the standard Poisson algebra of the 5D coordinates, we compute the algebra of x^μ and p_μ to be exactly 2.37.

We have achieved our main goal, but let us continue to obtain the equations of motion. First, we express H_{4d} in the new coordinates. From $H_{5d} = 0$ we have $P_4 = \sqrt{\rho^2 - P_\mu P^\mu}$, which can be rewritten as

$$P_4 = \frac{\rho}{\sqrt{1 - \frac{p^\mu p_\mu}{\rho^2}}} \,. \tag{2.42}$$

We have therefore that the condition $H_{4d} = 0$ is equivalent to

$$H = \frac{1}{2m}(p^\mu p_\mu - m^2 c^2) = 0 \,, \tag{2.43}$$

where

$$m = m_p \sqrt{1 - \frac{m_p^2}{M^2}} \tag{2.44}$$

is the rest mass of the particle. (2.44) also defines the constant $M > 0$ in terms of m and m_p. Finally, the equations of motion are

$$\dot{x}^\mu \approx \tilde{\lambda}\{x^\mu, H\} + \mu\{x^\mu, H_{5d}\} = \tilde{\lambda}\{x^\mu, H\} = \tilde{\lambda}\frac{1}{m}\left(1 - \frac{p^\mu p_\mu}{p^2}\right) , \tag{2.45}$$

$$\dot{p}_\mu \approx 0 \,. \tag{2.46}$$

If we impose the constraint $H = 0$, we can rewrite (2.45) as

$$\dot{x}^\mu \approx \tilde{\lambda}\frac{1}{m}\left(1 - \frac{m^2}{m_p^2}\right) . \tag{2.47}$$

This means that a Planck mass particle would be frozen in spacetime as discussed also in Subsection 3.2.2.

Chapter 3

Deformation of the Geodesic Motion

3.1 Geodesic equation in Hamiltonian mechanics

In this section, we embark on finding the equation of geodesic motion using Hamilton's formalism. Geodesic between two points A and B can be defined as the path of the shortest distance measured by a metric. Let us consider a particle with rest mass $m \neq 0$ on a time-like path parameterised by proper time τ. The spacetime length ℓ of a trajectory $x^\mu(\tau)$[1] between A and B can be calculated as

$$\ell = \int_{\tau_A}^{\tau_B} \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\tau\,, \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau}\,, \tag{3.1}$$

where $g_{\mu\nu}$ is a metric tensor, $x^\mu(\tau_A) = A$ and $x^\mu(\tau_B) = B$.

If we want to find the shortest trajectory, we can take ℓ as our action. In string theory, $S_{NG} = -mc\ell$ is known as the Nambu-Goto action but we are for simplicity instead using the so-called Polyakov action [42]

$$S_P = \frac{1}{2}\int_{\tau_A}^{\tau_B} (h_{\tau\tau}^{-1} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2c^2)\,\sqrt{h_{\tau\tau}} d\tau\,, \tag{3.2}$$

where $dl = \sqrt{h_{\tau\tau}}d\tau$ is a one-dimensional volume element of the world-line invariant under reparametrisation. This way we got rid of the nasty square root, and it can be shown that this action reproduces the Nambu-Goto action for massive particles. If we choose our parameter to be the proper time we have $h_{\tau\tau} = \frac{1}{m^2}$ and (3.2) reduces to

$$S = \frac{1}{2}m\int_{\tau_A}^{\tau_B} (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - c^2)\, d\tau\,. \tag{3.3}$$

In the theory of general relativity, the four-velocity is normalised to c^2 ($g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2$), which is an additional constraint that we impose only at the end.

From (3.3), we see that the Lagrange function corresponding to action S is

$$L = \frac{1}{2}m(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - c^2)\,, \tag{3.4}$$

[1]We do not differentiate between a vector and its components (for example, x^μ can mean vector x as well as the μ-component of this vector), and we continue to not distinguish between a point and its coordinates.

with the canonical momenta

$$p_\sigma = \frac{\partial L}{\partial \dot{x}^\sigma} = \frac{1}{2} m\, 2 g_{\mu\nu} \delta^\mu_\sigma \dot{x}^\nu = m g_{\sigma\nu} \dot{x}^\nu \,. \tag{3.5}$$

Now we can follow the standard procedure for transitioning into Hamiltonian formalism to obtain Hamiltonian

$$H = \frac{1}{2m} (g^{\mu\nu} p_\mu p_\nu - m^2 c^2) \,. \tag{3.6}$$

We calculate Hamilton's equations to be

$$\begin{aligned} \dot{x}^\sigma &= \frac{1}{m} g^{\sigma\nu} p_\nu \,, \\ \dot{p}_\sigma &= -\frac{1}{2m} \partial_\sigma g^{\mu\nu} p_\mu p_\nu \,. \end{aligned} \tag{3.7}$$

Of course, as we mentioned earlier, solutions to these equations have to be normalised so $g^{\mu\nu} p_\mu p_\nu = m^2 c^2$ to obtain actual free particle trajectories.

Let us see if we can recover the standard form of the geodesic equation from (3.7). First, we do the following auxiliary calculation.

$$\begin{aligned} \partial_\mu g^{\sigma\nu} = \partial_\mu (g^{\sigma\alpha} g^{\nu\beta} g_{\alpha\beta}) = g^{\nu\beta} g_{\alpha\beta} \partial_\mu g^{\sigma\alpha} + g^{\sigma\alpha} g_{\alpha\beta} \partial_\mu g^{\nu\beta} + g^{\sigma\alpha} g^{\nu\beta} \partial_\mu g_{\alpha\beta} = \\ = g^{\sigma\alpha} g^{\nu\beta} \partial_\mu g_{\alpha\beta} + 2 \partial_\mu g^{\sigma\nu} \end{aligned}$$

From this, we have

$$\partial_\mu g^{\sigma\nu} = -g^{\sigma\alpha} g^{\nu\beta} \partial_\mu g_{\alpha\beta} \,. \tag{3.8}$$

Now we take the time derivative of the first equation from (3.7) to obtain

$$\ddot{x}^\sigma = \frac{1}{m} \partial_\mu g^{\sigma\nu} \dot{x}^\mu p_\nu + \frac{1}{m} g^{\sigma\nu} \dot{p}_\nu \,. \tag{3.9}$$

If (3.8) and (3.5) are substituted into (3.9), everything is rearranged, and indexes are renamed, one gets

$$\ddot{x}^\sigma = -g^{\sigma\rho} \partial_\mu g_{\rho\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} g^{\sigma\rho} \partial_\rho g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \,.$$

It holds that $g^{\sigma\rho} \partial_\mu g_{\rho\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\rho\nu} \dot{x}^\mu \dot{x}^\nu + \partial_\nu g_{\rho\mu} \dot{x}^\nu \dot{x}^\mu)$ because the expression is symmetric in μ and ν, so that we can write

$$\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \,, \tag{3.10}$$

which is the standard geodesic equation in affine parameterisation.

3.2 Application of deformation to general geodesic motion

In Section 2, we discussed the motivation for Snyder model Poisson algebra deformation of the form 2.37. Let us now apply this algebra to the geodesic motion. In Section 3.1, we derived that the Hamiltonian should be 3.6. Hamilton's equations in terms of Poisson brackets are

$$\begin{aligned} \dot{x}^\mu &= \{x^\mu, H\} \,, \\ \dot{p}_\mu &= \{p_\mu, H\} \,. \end{aligned} \tag{3.11}$$

We can expand these for our Hamiltonian into

$$\begin{aligned}\dot{x}^\sigma &= \frac{1}{2m}(\{x^\sigma, g^{\mu\nu}\}p_\mu p_\nu + 2\{x^\sigma, p_\mu\}g^{\mu\nu}p_\nu)\,,\\ \dot{p}_\sigma &= \frac{1}{2m}(\{p_\sigma, g^{\mu\nu}\}p_\mu p_\nu + 2\{p_\sigma, p_\mu\}g^{\mu\nu}p_\nu)\,.\end{aligned} \tag{3.12}$$

To fully make use of our deformed Poisson algebra, we need the following lemma.

Lemma. *If $\{g,\cdot\}$ is a continuous linear function on $C^\omega(\mathbb{R}^4)$*[2] *that carries the Leibniz rule, i.e., $\{g, fh\} = \{g, f\}h + f\{g, h\}$ and $f \in C^\omega(\mathbb{R}^4)$, then $\{g, f(\boldsymbol{x})\} = \partial_\nu f(\boldsymbol{x})\{g, x^\nu\}$.*

Proof. First, let us introduce the notation $D_1^n = \partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}$. Importantly, if any of the α_i indices appear again, the Einstein summation convention is used. Using this, we can express the multivariable Taylor expansion as

$$f(\boldsymbol{x}) = f(\boldsymbol{0}) + \sum_{n=1}^{\infty}\frac{1}{n!}D_1^n f(\boldsymbol{0})\prod_{i=1}^{n} x^{\alpha_i}\,. \tag{3.13}$$

If we substitute this into the Poisson bracket we get

$$\{g, f(\boldsymbol{x})\} = \sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{1}{n!}D_1^n f(\boldsymbol{0})\{g, x^{\alpha_j}\}\prod_{i\neq j} x^{\alpha_i} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\partial_\nu D_2^n f(\boldsymbol{0})\prod_{i=2}^{n} x^{\alpha_i}\{g, x^\nu\}\,. \tag{3.14}$$

In the first equality, we took advantage of the continuity and of the fact that the Poisson bracket with a constant is 0 because of the Leibniz rule and linearity. In the second equality, we renamed the dummy indices and used the equality of mixed partials. Similarly we can arrive to

$$\partial_\nu f(\boldsymbol{x}) = \sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{1}{n!}D_1^n f(\boldsymbol{0})\delta_\nu^{\alpha_j}\prod_{i\neq j} x^{\alpha_i} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\partial_\nu D_2^n f(\boldsymbol{0})\prod_{i=2}^{n} x^{\alpha_i}\,. \tag{3.15}$$

Combining (3.14) and (3.15) we have the desired equality. □

Supposing that all the assumptions in this lemma are met, we can rewrite the Poisson brackets from (3.12) involving the metric as

$$\begin{aligned}\{x^\sigma, g^{\mu\nu}\} &= \partial_\rho g^{\mu\nu}\{x^\sigma, x^\rho\} = -\frac{1}{p^2}\partial_\rho g^{\mu\nu}\left(x^\sigma p^\rho - x^\rho p^\sigma\right)\,,\\ \{p_\sigma, g^{\mu\nu}\} &= \partial_\rho g^{\mu\nu}\{p_\sigma, x^\rho\} = \frac{1}{p^2}\partial_\rho g^{\mu\nu}p^\rho p_\sigma - \partial_\sigma g^{\mu\nu}\,.\end{aligned} \tag{3.16}$$

If we insist on standard normalisation of four-momenta (these are no longer the standard four-momenta, rather canonical momenta, but we will still call them the same), it should be added to equations (3.17) as in Section 3.1. This, of course, does not have to be the case. Normalisation of four-momenta defines the relativistic dispersion relation, which is for example in DSR modified [18].

[2] $C^\omega(D)$ denotes the set of all real analytic functions on D.

Now we substitute (2.37) and (3.16) into (3.12) and use the standard normalisation, which we assume to hold to get

$$\boxed{\begin{aligned}\dot{x}^\sigma &= \frac{1}{m}\left(1-\frac{m^2}{m_p^2}\right)p^\sigma - \frac{1}{2p^2 m}\partial_\rho g^{\mu\nu}\left(x^\sigma p^\rho - x^\rho p^\sigma\right)p_\mu p_\nu\,,\\ \dot{p}_\sigma &= \frac{1}{2p^2 m}\partial_\rho g^{\mu\nu}p^\rho p_\sigma p_\mu p_\nu - \frac{1}{2m}\partial_\sigma g^{\mu\nu}p_\mu p_\nu\,.\end{aligned}} \tag{3.17}$$

This is the final form of deformed Hamilton's geodesic equations, whose properties we study in the following subsections.

3.2.1 Violation of relativistic principles

There are two postulates that the theory of general relativity holds sacred. These are the equivalence principle and the principle of covariance. Here, we examine to what extent our deformation violates these principles.

Let us first consider the principle of equivalence. There are three main variants of this principle. Formulations of these principles may differ, but we are using these definitions

- **Weak Equivalence Principle (WEP):** *Freely falling test particles, characterised only by their mass, follow locally laws of special relativity in an unaccelerated reference frame, with their inertial mass equal to their gravitational mass.*
- **Einstein Equivalence Principle (EEP):** *The outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.*
- **Strong Equivalence Principle (SEP):** *The outcome of any local experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.*

EEP principle adds to WEP independence of all other physical laws like electromagnetism, that is apart from gravitational self-interaction, which is contained only in SEP. In our deformation of geodesic motion, we consider just point-like test particles whose only intrinsic property is mass; therefore, we are concerned only with WEP.

We know that in a local inertial frame, all first derivatives of the metric vanish and the equations (3.17) reduce to

$$\begin{aligned}\dot{x}^\sigma &= \frac{1}{m}\left(1-\frac{m^2}{m_p^2}\right)p^\sigma\,,\\ \dot{p}_\sigma &= 0\,.\end{aligned} \tag{3.18}$$

Compared to a standard relativistic free particle, there is an extra factor of $1-\frac{m^2}{m_p^2}$. From this we see that the role of gravitational mass plays $m_d = m\left(1-\frac{m^2}{m_p^2}\right)^{-1}$. The violation of WEP now depends only on the chosen form of the inertial mass. If we consider the inertial mass to be only m, then the WEP is indeed violated. On the other hand, if special relativity is also modified so that the inertial mass is defined as m_d, then WEP is saved. Also, if we take elementary particles as our test particle, the deformed inertial mass is very close to the standard one, because the mass of any elementary particle is much smaller than the Planck mass. For example, the heaviest elementary particle, the top quark, has a mass approximately 10^{-17} times smaller than the Planck mass. This implies that the deformed inertial mass would be reduced by just 10^{-32}%, which is far beyond our precision of measurements.

Now we move to the principle of covariance. The main idea of the principle is the following: *All physical laws expressed as equations should be possible to reformulate as equations of the form 'tensor=tensor'.* Unfortunately equations (3.17) are not in the form 'tensor=tensor,' as the reader can verify through a tedious calculation. It can be easily suspected that this is indeed the case because they contain derivatives of the metric in a manner that is not manifestly tensorial. One might hope that this can be resolved after all, because the same is true for equations (3.7), and combined they stand for the standard geodesic equation, which is, of course, tensorial. Maybe it is the same for equations (3.17). Let us see why this is not the case. If we employ a similar strategy as in Section 3.1 and take a time derivative of the first equation, apart from four-momenta derivatives, we also get four-momenta. The problem is that we cannot globally obtain an explicit expression for four-momenta from these equations. A similar problem with the four-position would arise if we tried to take the time derivative of the second equation and express everything in four-momenta. Also, when the time derivative is taken, the second derivative of the metric appears. These do not have to vanish if we go to a local inertial frame. This would conflict with (3.18) and the WEP.

To conclude, in our deformed geodesic motion, the violation of WEP depends on the form of the inertial mass that we are considering. If we chose to deform just the gravitational mass, the effects due to the violation of WEP would still be immeasurable. On the other hand, the principle of covariance is probably irreparably lost. This forces us to choose a coordinate system in which the deformed Hamilton's equations of geodesic motion take the form (3.17). We choose this to be the Minkowski coordinates. This is because our motivations for the deformation come from special relativity and these are also the coordinates used in a weak field limit in the Chapter 4.

3.2.2 Motion of Planck mass test particle

Let us examine the case of a particle with the Planck mass m_p. We also discuss the form of $\frac{d}{d\tau}(x^\mu p_\mu)$, which, for our deformation, is particularly simple.

An important term in equations (3.17) is $1-\frac{m^2}{m_p^2}$. We already commented on it in the previous Subsection 3.2.1. In a local inertial frame or in a flat spacetime, the equations take the form (3.18) and hence for a Planck mass particle where $1-\frac{m^2}{m_p^2}=0$ the motion stops. This can also be viewed as the deformed inertial mass m_d (defined in Subsection 3.2.1) being infinite. Notice that the particle is frozen not only in space $x^i = const.$ but also in time $t = const.$. It is also interesting that due to the normalisation of four-momenta $p_\mu \neq 0$, even though the particle does not move and $\dot{x}^\mu = 0$. This is possible because four-momentum and four-velocity are not connected in the standard way.

Now we look at the term $\frac{d}{d\tau}(x^\mu p_\mu) = \dot{x}^\mu p_\mu + x^\mu \dot{p}^\mu$. We use equations (3.17) and multiply the first equation by p_μ, second by x^μ and add them together. If we also use our chosen normalisation of four-momenta $p^\mu p_\mu = m^2c^2$, we obtain

$$\frac{d}{d\tau}(x^\mu p_\mu) = \left(1-\frac{m^2}{m_p^2}\right)\left(mc^2 - \frac{1}{2m}x^\sigma \partial_\sigma g^{\mu\nu} p_\mu p_\nu\right) = \left(1-\frac{m^2}{m_p^2}\right)\frac{d}{d\tau}(\tilde{x}^\mu \tilde{p}_\mu)\,, \tag{3.19}$$

where $\tilde{x}^\mu$ and $\tilde{p}_\mu$ are just four-position and four-momentum from the original Hamilton's equations for geodesic motion (3.7). The simplicity of (3.19) might be related to the fact that the algebra of Dirac brackets of a free relativistic particle, for the choice of parameterisation

$\tau = \frac{\ell x^\mu p_\mu}{\hbar}$, is exactly equal to the deformed Poisson algebra (2.37), if $\ell = \ell_p$. But we do not know exactly why there might be any connection. One can also see that for a Planck mass particle, the right-hand side of (3.19) vanishes. This means that $x^\mu p_\mu$ is conserved along the deformed geodesic motion.

Unfortunately, we do not know how to interpret this term in classical physics. However, in quantum physics, it is a generator of dilation. Let us illustrate this in one dimension in the x-representation on a dilation by e^α.

$$\begin{aligned} e^{-\frac{i}{\hbar}\alpha xp} f(x) = e^{-\alpha x \frac{d}{dx}} f(x) = \left\{ y = \ln(x),\ \frac{d}{dy} = \frac{1}{\frac{dy}{dx}}\frac{d}{dx} = x\frac{d}{dx} \right\} = \\ = e^{-\alpha\frac{d}{dy}} f(e^y) = \{g(y) := f(e^y)\} = \sum_{n=0}^{\infty} \frac{g^{(n)}(y)}{n!}(-\alpha)^n = g(y-\alpha) = f(e^{-\alpha}x) \end{aligned} \tag{3.20}$$

This procedure holds, of course, just for positive x, so the logarithm is defined. Still, for negative x, we would use the substitution $y = \ln(-x)$ instead, and for 0, it holds trivially.

Chapter 4

Deformed Schwarzschild and Kerr Solutions

In Chapter 3 we obtained and discussed deformed Hamilton's equations of geodesic motion in general spacetime. In this chapter, we focus on their application in Schwarzshild and Kerr solutions.

4.1 Schwarzschild spacetime

Following [43], we introduce the first nontrivial exact solution to Einstein field equations. It was found by Karl Schwarzschild just about a month after the release of the theory of general relativity. During World War I, Schwarzschild announced the discovery to Einstein in a letter, concluding with: 'As you see, the war is kindly disposed toward me, allowing me, despite fierce gunfire at a decidedly terrestrial distance, to take this walk into this your land of ideas.' The Schwarzschild solution describes the spacetime around an uncharged spherically symmetric source, such as a non-rotating black hole. In an approximation, it can also be applied to systems like our solar system. Later in this section, we also mention the Reissner-Nordström metric for charged spherically symmetric sources.

We do not present here the detailed derivation of the Schwarzschild metric but let us just briefly summarise the main ideas behind it. We are looking for a spherically symmetric vacuum solution to the Einstein equations. First, we make use of the symmetry. It can be shown that every spherically symmetric metric can be expressed as

$$ds^2 = g_{tt}(t,r)dt^2 + g_{rr}(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{4.1}$$

where θ and ϕ are the angles of spherical coordinates, r is the so called 'area radius' and t is of course time coordinate. The name area radius comes from the fact that the proper area of a sphere centred on the word-line $r = 0$, which has an invariant meaning, is given by $4\pi r^2$.

The famous Einstein field equations read

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)\,, \tag{4.2}$$

where $R_{\mu\nu}$ is the Ricci tensor encoding curvature, $T_{\mu\nu}$ is the energy-momentum tensor representing sources, T its trace and Λ is the cosmological constant. We are interested in a vacuum solution without the cosmological constant, therefore we are left with

$$R_{\mu\nu} = 0\,. \tag{4.3}$$

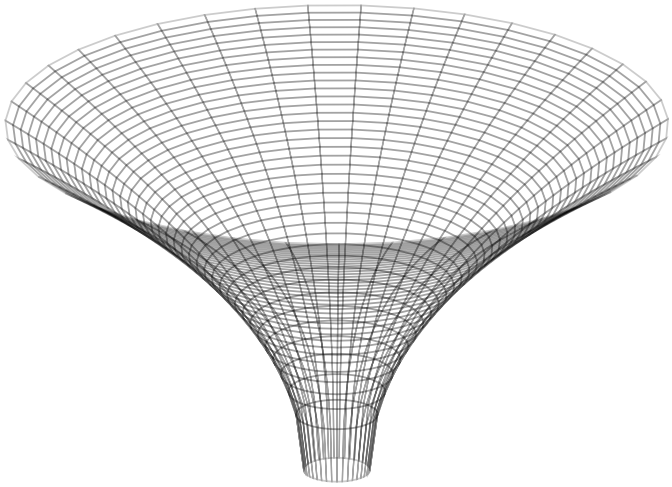

Figure 4.1: Flamm's paraboloid, a surface that has its metric equal to that of Schwarzschild with ($\theta = \frac{\pi}{2}$,t=const.). It can be used to visualise the spatial curvature of the Schwarzschild solution. [44]

By substituting (4.1) into (4.3) a series of differential equations can be obtained and solving these we get the following form.

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{4.4}$$

Notice that we need not assume stationarity because it follows from the Einstein equations. The integration constant r_s can be determined from the Newtonian limit because it must hold that $g_{tt} = 1 + 2\frac{\Phi}{c^2} = 1 + \frac{2GM}{rc^2}$ where M is the mass of the object producing the gravitational field with the Newtonian gravitational potential Φ. From this, we see that the constant is $r_s = \frac{2GM}{c^2}$. r_s is called the Schwarzschild radius and it is the radius of the event horizon of a Schwarzschild black hole with mass M.

To conclude (4.4) is the Schwarzschild metric in its full glory. From our treatment, we can see how the following theorem holds.

Theorem (Birkhoff)**.** *In general, relativity, if we take the cosmological constant to be 0, every vacuum spherically symmetric solution to (4.2) can be described by (4.4).*

Notice also that in the absence of source $M = 0$ the metric reduces to that of flat Minkowski space in standard spherical coordinates

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \,. \tag{4.5}$$

The same happens in the limit $\frac{M}{r} \to 0$, which can be understood as asymptotic flatness.

Let us just mention the generalisation of Schwarzschild solution for charged sources. This is a solution to the Einstein field equations together with the Maxwell equation on a curved spacetime. It is no longer a vacuum solution because the source produces an electromagnetic field which has to be included in the energy-momentum tensor. It can be shown that the final metric has the same form as Schwarzschild's if we make the substitution

$$r_s r \to r_s r - r_Q^2 \,, \qquad r_Q = \frac{kQ^2G}{c^4} \,, \tag{4.6}$$

where Q is the charge of the source, and k is the Coulomb constant. This metric is called the Reissner–Nordström metric and it can describe also a charged black hole. However, real black holes can not accumulate a large amount of charge, because the charge would quickly attract opposite changes and the black hole would get neutralised.

Now, let us discuss the application of this metric to our deformed geodesic motion. Of course, we could just substitute the metric into (3.17). However, as the reader can verify for himself, this yields a not-very-interesting mess that is probably not analytically solvable. Unfortunately, we lost the ability to solve the system using constants of geodesic motion as is normally done for the Schwarzschild solution, because we are no longer dealing with the standard geodesic equation. We know of one constant of motion that we imposed as a constraint, that is, the normalisation of four-momenta $p_\mu p^\mu = m^2c^2$ and in the case of the Planck mass particle, also $x^\mu p_\mu$ is conserved, as has been shown in Subsection 3.2.2. However, because of the complicated symplectic structure, it would be very hard to find any additional constants of motion in Hamiltonian formalism. Therefore, to be able to further analyse the system, we resort to a pseudo-Newtonian limit.

4.1.1 Introduction of the weak field limit

We proceed to formulate a weak field limit for a metric derived from [45]. Let us have two spacetime manifolds. First, a background spacetime M_b. This is the spacetime that we are perturbing. For us, it is a space with the Minkowski metric $\eta_{\mu\nu}$ and the Minkowski coordinates, but it could be anything. For example, if we would like to know how gravitational waves propagate in an expanding universe, we would probably take a spacetime with the Friedmann–Lemaître–Robertson–Walker metric as our background spacetime. The second spacetime M_p with a metric $\tilde{g}_{\mu\nu}$ is a physical one, which we can imagine as a sum of the background and our perturbation. We also introduce a diffeomorphism $\phi : M_b \to M_p$. If M_p is in some sense a sum of M_b and the perturbation, we would say that we obtain the perturbation by subtracting M_b from M_p. This is made rigorous by

$$h_{\mu\nu} := (\phi^*\tilde{g})_{\mu\nu} - \eta_{\mu\nu}\,, \tag{4.7}$$

where $g_{\mu\nu} := (\phi^*\tilde{g})_{\mu\nu}$ is the pullback of $\tilde{g}_{\mu\nu}$ and $h_{\mu\nu}$ is the perturbation to the metric. In the weak field limit, there exists some diffeomorphism such that the components of the perturbation are small with its derivatives (the field also changes slowly) $|h_{\mu\nu}|, |\partial_\sigma h_{\mu\nu}| << 1$.

If the metric $\tilde{g}_{\mu\nu}$ is required to be a solution of the Einstein equations, then $g_{\mu\nu}$ solves the pulled-back equations. We have $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ from (4.7) and define the inverse (in reality, we define $h^{\mu\nu}$) as $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ because from now on we are working to first order in $h_{\mu\nu}$ and $\partial_\sigma h_{\mu\nu}$. If we substitute this into the Einstein equations and neglect all higher-order terms, we obtain the linearised field equations, where the linearised Einstein tensor is equal to

$$G_{\mu\nu} = \frac{1}{2}(\partial_{\sigma\nu}h^{\sigma}{}_{\mu} + \partial_{\sigma\mu}h^{\sigma}{}_{\nu} - \partial_{\mu\nu}h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_{\rho\lambda}h^{\rho\lambda} + \eta_{\mu\nu}\Box h)\,, \qquad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu\,. \tag{4.8}$$

Let us now have a vector field $\xi^\mu(x)$ on M_b. The flow generated by this vector field can be seen as one parameter family of diffeomorphisms $\psi_\varepsilon : M_b \to M_b$. If $h_{\mu\nu}$ was small for some diffeomorphism ϕ, it is also small for $\phi\circ\psi_\varepsilon$ if we take ε to be sufficiently small. This freedom in the chosen diffeomorphism is essentially a gauge invariance of $h_{\mu\nu}$. It can be shown that the change in diffeomorphism $\phi \to \phi \circ \psi_\varepsilon$ results in the change in perturbation $h_{\mu\nu} \to h^{(\varepsilon)}_{\mu\nu} = h_{\mu\nu} + \varepsilon \pounds_\xi \eta_{\mu\nu}$, where $\pounds_\xi$ is a Lie derivative in the direction of ξ^μ. The physical properties of a theory should not depend on a gauge, and indeed, it is so. It can be shown that the linearised Riemann tensor is gauge invariant and, therefore, the curvature does not change. We are now free to choose our gauge and this is going to be in the Lorenz gauge

$$\partial^\mu \bar{h}_{\mu\nu} = 0\,, \qquad \bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\,. \tag{4.9}$$

In this gauge, the linearised field equations simplify to

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}\,. \tag{4.10}$$

We assume that the rest energy density $T_{tt} = \rho$ is the dominant term of $T_{\mu\nu}$. In the real world, it is mostly the case and, certainly, it is true for objects where the Schwarzschild solution is normally used like stars, planets, or non-rotating black holes. From this assumption and (4.10), it follows that $|\bar{h}_{tt}| >> |\bar{h}_{ij}|$ and

$$h = -\bar{h} \simeq -\bar{h}_{tt}\,. \tag{4.11}$$

The standard Newtonian limit tells us that $h_{tt} = -\frac{r_s}{r}$. If we combine it with (4.11) and the definition of $\bar{h}$, we have

$$\bar{h}_{tt} \simeq 2h_{tt} = -2\frac{r_s}{r}\,. \tag{4.12}$$

From (4.12), the definition of $\bar{h}_{\mu\nu}$ and the negligibility of other components of $\bar{h}_{\mu\nu}$ follow also

$$\begin{aligned} h_{it} &= \bar{h}_{it} - \frac{1}{2}\eta_{it}\bar{h} \simeq 0\,, \\ h_{ij} &= \bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h} \simeq \frac{r_s}{r}\delta_{ij}\,. \end{aligned} \tag{4.13}$$

Now we can conclude that the metric $g_{\mu\nu}$ of the weak field limit takes in the Minkowski coordinates the form

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 + \frac{r_s}{r}\right)(dx^2 + dy^2 + dz^2)\,, \qquad r = \sqrt{x^2 + y^2 + z^2}\,. \tag{4.14}$$

4.1.2 Weak field limit of the Schwarzschild metric

In this subsection, we demonstrate how we can also get (4.14) by taking some approximations to the Schwarzschild metric. This way the metric (4.14) will be well motivated; we would have confidence that the approximations of the Schwarzshild metric were reasonable and that the coordinates can be seen as the Minkowski coordinates.

Let us first rewrite (4.4) in the so-called isotropic coordinates using the substitution

$$r = \tilde{r}\left(1 + \frac{r_s}{4\tilde{r}}\right)^2\,. \tag{4.15}$$

We have

$$\begin{aligned} 1 - \frac{r_s}{r} &= \left(1 - \frac{r_s}{4\tilde{r}}\right)^2 \left(1 + \frac{r_s}{4\tilde{r}}\right)^{-2}\,, \\ \frac{dr}{d\tilde{r}} &= \left(1 - \frac{r_s}{4\tilde{r}}\right)\left(1 + \frac{r_s}{4\tilde{r}}\right)\,, \end{aligned} \tag{4.16}$$

and from this

$$ds^2 = \frac{\left(1 - \frac{r_s}{4\tilde{r}}\right)^2}{\left(1 + \frac{r_s}{4\tilde{r}}\right)^2} c^2 dt^2 - \left(1 - \frac{r_s}{4\tilde{r}}\right)^4 (d\tilde{r}^2 + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2\theta\, d\phi^2)\,. \tag{4.17}$$

$d\tilde{r}^2 + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2\theta \, d\phi^2$ is just the metric of 3D flat space in spherical coordinates and can be rewritten in Cartesian coordinates as $dx^2 + dy^2 + dz^2$ if we set $\tilde{r} = \sqrt{x^2 + y^2 + z^2}$. When we are far from the source where the gravitational field is weak we can consider $\frac{r_s}{\tilde{r}}$ to be small and use the Taylor expansion to first order to make the following approximations.

$$\begin{aligned}
&\left(1 - \frac{r_s}{4\tilde{r}}\right)^2 \left(1 + \frac{r_s}{4\tilde{r}}\right)^{-2} \simeq \left(1 - \frac{r_s}{2\tilde{r}}\right)\left(1 - \frac{r_s}{2\tilde{r}}\right) \simeq 1 - \frac{r_s}{\tilde{r}} \\
&\left(1 - \frac{r_s}{4\tilde{r}}\right)^4 \simeq 1 - \frac{r_s}{\tilde{r}}
\end{aligned} \tag{4.18}$$

If we substitute all this into (4.17) and rename $\tilde{r} \to r$, we arrive exactly at (4.14).

4.1.3 Deformed geodesic motion in a pseudo-Newtonian limit

Here, we finally apply (4.14) to our deformed geodesic equations (3.17). However, first, we introduce Hamilton's equations of motion with Newtonian gravity to have something to compare it with. The corresponding Lagrange function is[1].

$$L = \frac{1}{2} m \dot{\boldsymbol{x}}^2 + \frac{GMm}{r}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \qquad \dot{\boldsymbol{x}} = \frac{dx^i}{dt}. \tag{4.19}$$

Canonical momenta are just $p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i$ so the Hamiltonian is

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{GMm}{r}, \tag{4.20}$$

and Hamilton's equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = \frac{p^i}{m}, \tag{4.21}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = \frac{GMm}{r^3} x_i \,. \tag{4.22}$$

Here, we should just note that the dot in this paragraph meant derivative with respect to coordinate time and, from now on, again means derivative with respect to a proper time.

For the following discussion, we will need one more assumption of the Newtonian limit in addition to the weak field, that is, slow motion. Specifically, $|\frac{dx^i}{dt}| << c \implies |\dot{x}^i| << |c\dot{t}|$ is required. If we add the standard normalisation of four-velocity and the weak field limit, it follows that $\dot{t} \simeq 1$. It is because

$$\begin{aligned}
g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu &= \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\mu + h_{\mu\nu}\dot{x}^\mu \dot{x}^\mu = c^2 \,, \\
c^2\dot{t}^2 - \delta_{ij}\dot{x}^i \dot{x}^j &+ h_{ij}\dot{x}^i \dot{x}^j + h_{tt}c^2\dot{t}^2 = c^2 \,,
\end{aligned}$$

and the dominant term is $c^2\dot{t}^2$ due to h_{tt} also being small (we have also used that $h_{\mu\nu}$ is diagonal).

Now, with the promise that we will soon discuss (3.17), we also check that (3.7) behave correctly in the Newtonian limit. After substituting (4.14) into the first equation of (3.7) we obtain

$$c\dot{t} = \frac{p^t}{m}, \tag{4.23}$$

$$\frac{dx^i}{dt} \simeq \frac{dx^i}{dt}\dot{t} = \frac{dx^i}{d\tau} = \frac{p^i}{m}. \tag{4.24}$$

[1]The bold letters in this chapter always stand for three-vectors.

Note that (4.24) becomes exactly (4.21) after the approximation. We can make this approximation because we have the normalisation of four-momenta, which implies in this case due to (3.5) also the normalisation of four-velocity. The second equation of (3.7) becomes

$$\dot{p}_t = 0\,, \tag{4.25}$$

and

$$\dot{p}_i = -\frac{1}{2m}\partial_i g^{\mu\nu} p_\mu p_\nu = \frac{1}{2m} g^{\mu\alpha} g^{\nu\beta} \partial_i h_{\alpha\beta} p_\mu p_\nu \simeq \frac{1}{2m}\partial_i h_{tt} (p^t)^2 = \frac{GM}{c^2 r^3 m} x_i (p^t)^2\,. \tag{4.26}$$

where in (4.26) we have used $|p^i| << |p^t|$ and neglected the spatial terms. This can be further simplified using (4.23) to

$$\begin{aligned} \frac{dp_i}{d\tau} &= \frac{GMm}{r^3} x_i \dot{t}^2\,, \\ \frac{dp_i}{dt} &= \frac{GMm}{r^3} x_i \dot{t} \simeq \frac{GMm}{r^3} x_i \end{aligned} \tag{4.27}$$

and again, this is the same as (4.22). Therefore, equations (3.7) have the correct Newtonian limit.

Finally, we move to the deformed geodesic motion. First, let us alter the assumptions a little bit. The weak field limit stays the same. However, we will now assume that $|p^i| << |p^t|$ rather than $|\frac{dx^i}{dt}| << c$. The latter no longer implies the former because our canonical momenta differ from the standard four-momenta. Let us now substitute (4.14) into (3.17) and again neglect all the terms of second or higher order in p_i to get

$$\begin{aligned} \dot{x}^\sigma &\simeq \frac{p^\sigma}{m_d} + \frac{r_s}{2mr^3}(\boldsymbol{x}\cdot\boldsymbol{p}\,x^\sigma - r^2 p^\sigma)\left(\frac{p^t}{p}\right)^2\,, \\ \dot{p}_t &\simeq \frac{r_s}{2mr^3}\boldsymbol{x}\cdot\boldsymbol{p}\left(\frac{p^t}{p}\right)^2 p_t\,, \\ \dot{p}_i &\simeq \frac{r_s}{2mr^3}\boldsymbol{x}\cdot\boldsymbol{p}\left(\frac{p^t}{p}\right)^2 p_i - \frac{r_s}{2mr^3}(p^t)^2 x_i\,. \end{aligned} \tag{4.28}$$

Some of the steps in the derivation were similar to those of (4.26). We have also used the definition of the deformed rest mass m_d from Subsection 3.2.1. Finally the equations (4.28) can be combined and simplified into

$$\boxed{\begin{aligned} \dot{x}^\sigma &= \frac{p^\sigma}{m_d} + (\dot{\ln p_t})x^\sigma - \frac{r_s}{2mr}\left(\frac{p^t}{p}\right)^2 p^\sigma\,, \\ (\dot{\ln p_t}) &= \frac{r_s}{2mr^3}\boldsymbol{x}\cdot\boldsymbol{p}\left(\frac{p^t}{p}\right)^2\,, \\ \dot{p}_i &= (\dot{\ln p_t})p_i + \frac{r_s}{2mr^3}(p^t)^2 x_i\,. \end{aligned}} \tag{4.29}$$

4.2 Kerr spacetime

Drawing from [45] and [43], we begin by shortly introducing the Kerr metric. This is another exact solution to the Einstein equations describing an uncharged rotating black hole. We are also going to briefly mention the most general Kerr–Newman solution. Following that, we explore the application of our proposed deformation within this spacetime.

So far we have seen the Schwarzschild solution which can describe a perfectly spherically symmetric black hole. However, there are probably no such black holes in nature. All astrophysical black holes have some rotation which they get from the in-falling material and these are described exactly by the Kerr metric. Einstein released his field equation in the year 1915 and as mentioned in Section 4.1 it did not take long to discover the Schwarzschild solution. Also, a generalisation of the Schwarzschild metric known as the Reissner–Nordström metric for charged black holes followed shortly after that. However, it was not until 1963 that the Kerr solution was presented. The reason for this is that both Schwarzschild and Reissner–Nordström spacetimes are spherically symmetric, whereas Kerr's is just axially symmetric, which introduced a lot of complexity. Because of the diabolical difficulty of the derivation of the Kerr metric, we are not going to attempt to present it here and will be content with studying its properties.

The Kerr metric in the Boyer-Lindquist coordinates reads

$$\begin{aligned} ds^2 = \left(1 - \frac{r_s r}{\Sigma}\right) c^2 dt^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2 \\ + \frac{r_s r a}{\Sigma}\sin^2\theta c\left(dt d\theta + d\theta dt\right), \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} \Delta(r) &= r^2 - r_s r + a^2\,, \\ \Sigma(r,\theta) &= r^2 + a^2\cos^2\theta\,, \end{aligned} \tag{4.31}$$

and M and a are constants specifying the source. If we again make the substitution (4.6) we get the most general asymptotically flat, stationary solution of the Einstein equations together with the Maxwell equations on a curved background. This is called the Kerr-Newman metric and it can describe a rotating charged black hole, which is the most general case of a black hole as is discussed in Subsection 5.1.1. However, we will only go on to discuss Kerr black holes; any generalisation to Kerr-Newman black holes is not difficult.

The Kerr metric is a stationary axially symmetric vacuum solution, and therefore it possesses two Killing vector fields. These have in Boyer-Lindquist coordinates the simple form ∂_t (stationarity) and ∂_ϕ (axial symmetry), as is simply seen from the fact that the metric components are independent of these coordinates. Unlike the Schwarzschild metric, Kerr's is not static. This is because it contains non-zero off-diagonal terms $g_{t\phi}$ and $g_{\phi t}$.

Let us now discover the meaning of constants M and a. If we set $a = 0$ we get exactly the Schwarzschild metric (4.4) and therefore we see that M represents the mass. Because the only physical difference between Schwarzschild and Kerr black holes should be rotation, we deduce that a must have something to do with it. Specifically, it can be found that

$$a = \frac{J}{Mc}\,, \tag{4.32}$$

where J is the Komar angular momentum, which is a relativistic analogue of standard angular momentum.

It is also interesting to see that as one makes the limit $M \to 0$ while keeping a fixed Kerr metric reduces to that of flat space in ellipsoidal coordinates

$$ds^2 = c^2 dt^2 - \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2} dr^2 - (r^2 + a^2\cos^2\theta) d\theta^2 - (r^2 + a^2)\sin^2\theta d\phi^2\,. \tag{4.33}$$

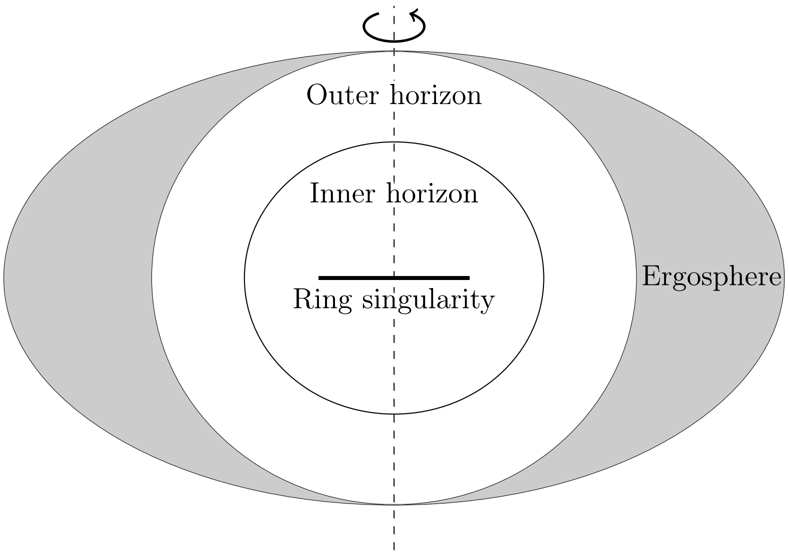

Figure 4.2: Schematic depiction of Kerr black hole as viewed from a side. In the centre is the ring-like singularity enclosed by inner and outer event horizons. The shape of the ergosphere surrounding the black hole can be for slow rotating black holes approximated by a spheroid (as is depicted). The dotted line represents the axis of rotation and symmetry.

These are defined as

$$\begin{aligned} x &= \sqrt{r^2+a^2}\cos\phi\sin\theta\,, \\ y &= \sqrt{r^2+a^2}\sin\phi\sin\theta\,, \\ z &= r\cos\theta\,. \end{aligned} \tag{4.34}$$

Now we turn to coordinate singularities. These come from $\Delta = 0$ or $\Sigma = 0$. Solutions of $\Delta(r) = 0$ are just coordinate singularities and represent event horizons, surfaces past which it becomes impossible to return to a certain region of spacetime. Depending on the value of a, $\Delta(r) = 0$ has a different number of solutions.

- **Standard case:** $a < \frac{r_s}{2}$ ... 2 solutions $r_\pm = \frac{r_s}{2} \pm \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2}$
- **Extremal black hole:** $a = \frac{r_s}{2}$ ... 1 solution $r_+ = r_- = \frac{r_s}{2}$
- **Naked singularity:** $a > \frac{r_s}{2}$ ... 0 solutions

Here r_+ is called the outer and r_- the inner event horizon. The real world black holes discovered so far are well in the category of a standard case. No extremal black holes or naked singularities have been observed. The horizon structure of the standard case is schematically depicted in Figure 4.2.

The case when $\Sigma(r,\theta) = 0$ represents on the other hand real physical singularity as can be seen from the Kretschmann scalar

$$K = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} = \frac{12\,r_s}{\Sigma^6}(r^2 - a^2\cos^2\theta)(\Sigma^2 - 16r^2a^2\cos^2\theta)\,. \tag{4.35}$$

We will not delve deeper into this, but it can be shown that the singularity has the topology of a circle. This ring is represented in Figure 4.2.

As in Section 4.1, we are not going to substitute the metric (4.30) directly into our deformed geodesic equations (3.17) for the same reasons, which are even more vivid for this metric. Instead, we are going to again resort to a pseudo-Newtonian limit using principles similar to that of Section 4.1.

4.2.1 Deformed geodesic motion in a pseudo-Newtonian limit

The goal of this subsection is to derive an approximate version of (3.17) with the Kerr metric using the methods of the Newtonian limit. First, we make some approximations to (4.30) inspired by the weak field limit of Schwarzschild from Subsection 4.1.2 and then we use an assumption of slow motion again similar to that of Subsection 4.1.3.

Let us assume that we are far from the source so that $r >> a$ and also $r >> r_s$. Using the first assumption we can make the following approximation to first order in $\frac{a}{r}$

$$\begin{aligned} &\Delta \simeq r^2 - r_s r\,, \\ &\Sigma \simeq r^2\,, \\ &g_{\phi\phi} \simeq \left(r^2 + a^2 + \frac{r_s a^2}{r}\sin^2\theta\right)\sin^2\theta \simeq r^2\sin^2\theta\,, \end{aligned} \tag{4.36}$$

and substitute them into (4.30) [43]. We obtain

$$ds_K^2 = ds_S^2 + \frac{r_s a}{r}\sin^2\theta c\,(dt d\theta + d\theta dt)\,, \tag{4.37}$$

where ds_K^2 is the approximated Kerr metric and ds_S^2 is the standard Schwarzschild metric (4.4).

Now we want to transfer to the isotropic coordinates as in Subsection 4.1.2. Problem is that we have seen at the beginning of this section that Boyer-Lindquist coordinates are not spherical, rather ellipsoidal. But because we are working to first order in $\frac{a}{r}$ we can take the coordinates to be spherical and write

$$ds_K^2 = ds_S^2 + \frac{r_s a}{r^3}\left(1 - \frac{r_s}{4r}\right)^{-2} c\,(x(dtdy + dydt) - y(dtdx + dxdt))\,, \qquad r = \sqrt{x^2 + y^2 + z^2}\,. \tag{4.38}$$

If we make use of the second assumption $r >> r_s$, we can make the following approximation.

$$\frac{r_s a}{r^3}\left(1 - \frac{r_s}{4r}\right)^{-2} \simeq \frac{r_s a}{r^3}\left(1 + \frac{r_s}{2r}\right) \simeq \frac{r_s a}{r^3}\,. \tag{4.39}$$

We also approximate the Schwarzschild metric by (4.14) as in Subsection 4.1.2 and substitute into (4.38) to get

$$ds^2 = \left(1 - \frac{r_s}{r}\right)c^2 dt^2 - \left(1 + \frac{r_s}{r}\right)(dx^2 + dy^2 + dz^2) + \frac{r_s a}{r^3} c\,(x(dtdy + dydt) - y(dtdx + dxdt))\,. \tag{4.40}$$

This is the final metric that we use in (3.17). Note that it is in the form of the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

However, let us first introduce the assumption of slow motion. It is the same as in Subsection 4.1.3, that is $|p^i| << |p^t|$. Due to this and because we have our metric (4.40) in the form of the weak field limit, we have

$$\begin{aligned}
\partial_i g^{\mu\nu} p_\mu p_\nu = -\partial_i h_{\mu\nu} p^\mu p^\nu &\simeq -\partial_i h_{tt}(p^t)^2 - 2\partial_i h_{xt} p^x p^t - 2\partial_i h_{yt} p^y p^t \,, \\
-\partial_\rho g^{\mu\nu}(x^\sigma p^\rho - x^\rho p^\sigma) p_\mu p_\nu &\simeq (\partial_i h_{tt}(p^t)^2 + 2\partial_i h_{xt} p^x p^t + 2\partial_i h_{yt} p^y p^t)(p^i x^\sigma - x^i p^\sigma) \simeq \\
&\simeq \partial_i h_{tt}(p^i x^\sigma - x^i p^\sigma)(p^t)^2 - 2(\partial_i h_{xt} p^x p^t + \partial_i h_{yt} p^y p^t) x^i p^\sigma \,, \\
\partial_\rho g^{\mu\nu} p^\rho p_\sigma p_\mu p_\nu &\simeq -(\partial_i h_{tt}(p^t)^2 + 2\partial_i h_{xt} p^x p^t + 2\partial_i h_{yt} p^y p^t) p^i p_\sigma \simeq -\partial_i h_{tt}(p^t)^2 p^i p_\sigma \,.
\end{aligned} \tag{4.41}$$

Using the form of the metric (4.40), we prepare also

$$\begin{aligned}
\partial_i h_{xt} &= 3y \frac{r_s a}{r^5} x_i + \frac{r_s a}{r^3} \delta_i^y \,, \\
\partial_i h_{yt} &= -3x \frac{r_s a}{r^5} x_i + \frac{r_s a}{r^3} \delta_i^x \,.
\end{aligned} \tag{4.42}$$

Now we finally substitute everything into (3.17) and rearrange to get

$$\begin{aligned}
\dot{t} &\simeq \frac{p^t}{m_d} + \frac{r_s}{m}\left[\frac{1}{2r^3}(\boldsymbol{x}\cdot\boldsymbol{p}\, t - r^2 p^t) + \frac{a}{r^3}(2xp^y - 4yp^x)\right]\left(\frac{p^t}{p}\right)^2 \,, \\
\dot{x}^i &\simeq \frac{p^i}{m_d} + \frac{r_s}{2mr^3}(\boldsymbol{x}\cdot\boldsymbol{p}\, x^i - r^2 p^i)\left(\frac{p^t}{p}\right)^2 \,, \\
\dot{p}_t &\simeq \frac{r_s}{2mr^3}\boldsymbol{x}\cdot\boldsymbol{p}\left(\frac{p^t}{p}\right)^2 p_t \,, \\
\dot{p}_i &\simeq \frac{r_s}{2mr^3}\boldsymbol{x}\cdot\boldsymbol{p}\left(\frac{p^t}{p}\right)^2 p_i + \frac{r_s}{m}\left[\frac{p^t}{2r^3} 2x_i + \frac{x_i}{r^3}(3yp^x - 3xp^y) + \frac{a}{r^3}(\delta_i^y p^x + \delta_i^x p^y)\right] p^t \,.
\end{aligned} \tag{4.43}$$

which can be further simplified into what is the final form.

$$\boxed{\begin{aligned}
&\dot{t} \simeq \frac{p^t}{m_d} + (\dot{\ln p_t})t + \frac{r_s}{m}\left[\frac{a}{r^3}(2xp^y - 4yp^x) - \frac{p^t}{2r}\right]\left(\frac{p^t}{p}\right)^2 \\
&\dot{x}^i \simeq \frac{p^i}{m_d} + (\dot{\ln p_t})x^i - \frac{r_s}{2mr}\left(\frac{p^t}{p}\right)^2 p^i \\
&(\dot{\ln p_t}) \simeq \frac{r_s}{2mr^3}\boldsymbol{x}\cdot\boldsymbol{p}\left(\frac{p^t}{p}\right)^2 \\
&\dot{p}_i \simeq (\dot{\ln p_t})p_i + \frac{r_s}{m}\left[\frac{p^t}{2r^3} 2x_i + \frac{x_i}{r^3}(3yp^x - 3xp^y) + \frac{a}{r^3}(\delta_i^y p^x + \delta_i^x p^y)\right] p^t
\end{aligned}} \tag{4.44}$$

Chapter 5

Modified Hawking Radiation

In this chapter, we uncover a fascinating bridge between the physics of black holes, thermodynamics, and quantum fields.

5.1 Introduction to black hole thermodynamics

Throughout this section, we are drawing from [43], if not mentioned otherwise. During the 1970s physicists noticed some interesting similarities between black hole physics and thermodynamics. The early study of this analogy is summarised in the four laws of black hole dynamics. These read

1. *The surface gravity κ is on a station horizon of a black hole everywhere the same.*
2. *For perturbations of a stationary black hole it holds that*

$$\delta(Mc^2) = \frac{\kappa c^2}{8\pi G}\delta A + \omega\delta J + \varphi\delta Q\,, \tag{5.1}$$

   where M is the mass, J is the angular momentum, Q is the charge, A is the proper area of the horizon, ω is the angular speed with respect to infinity and φ is the electric potential of the black hole.
3. *The proper area of a black hole horizon can never decrease .*
4. *The surface gravity of a black hole can not be reduced to 0 in a finite number of steps.*

Notice that if we exchange the surface gravity with temperature, the area with entropy and stationarity with thermal equilibrium, we get basically the laws of thermodynamics.

In the beginning, this correspondence was seen as just a coincidence, and many thought that there is no real connection between black holes and thermodynamics. For example, people did not understand in what sense a black hole can have a temperature. However, the physicist Jacob Bekenstein insisted that this connection goes deeper. He had many reasons for his belief, some of which are discussed in the following subsection.

5.1.1 Wheeler's demon

First, let us mention the black hole no-hair theorem. Already in Section 4.2 we commented on the generality of the Kerr-Newman solution. This can be made rigorous by the following theorem.

Theorem (no-hair)**.** *Every isolated stationary black hole in an asymptotically flat space-time, which contains no singularities and no closed time-like curves elsewhere than possibly under the horizon, is necessarily of the Kerr-Newman type.*

We have also seen in Section 4.2 that the Kerr-Newman metric is fully characterised by the mass M, the charge Q and the angular momentum J of a black hole. This is very surprising because it means that nearly all the information about the object that falls into a black hole is lost to the outside world, and hence black holes have no 'hair'. As we will see, this can be exploited by Wheeler's demon.

Jacob Bekenstein was a student of Archibald Wheeler, and in 1971 Wheeler came to Bekenstein with an interesting problem. Suppose that a nefarious creature wants to commit a crime against the second law of thermodynamics. Bekenstein called it Wheeler's demon [46]. If the creature takes a box full of radiation with some entropy and throws it into a black hole, the entropy of the outside universe is decreased. Because of the no-hair theorem, we cannot know how much entropy is in the black hole, and therefore we do not know if the second law was violated. After this, strictly speaking, we could no longer call the second law of thermodynamics a 'law'.

Let us go beyond Wheeler's original argument by following [47]. Suppose that the demon continues his insidious plans and wants to construct a perpetual motion machine of the second kind. He has inside his box thermal radiation of temperature T and connects it to a rope attached to an engine doing work. As he lowers the box towards a Schwarzschild black hole, he converts the potential energy of the radiation into work. At the event horizon, all the energy mc^2 of the radiation is extracted. He dumps the radiation into the black hole and repeats the process. From this, it seems that all of the heat in the radiation was converted into useful work, thus violating the send law of thermodynamics.

However, we did not take into account the fact that no part of the box can cross the event horizon. Inside the box, there is thermal radiation, which has an average wavelength of about $\frac{\hbar c}{k_B T}$ and, therefore, the box must have a side length $a \simeq \frac{\hbar c}{kT}$. If one end of the box is at the event horizon, the centre of gravity will be $\frac{d}{2}$ above the event horizon and the potential energy will be $U = -mc^2 + m\kappa\frac{d}{2}$ where κ is the surface gravity. The exact relativistic value for surface gravity agrees with the Newtonian and that is $\kappa \simeq \frac{GM}{r_s^2} = \frac{c^4}{4GM}$.

The work done by the engine must therefore be reduced by $mg\frac{d}{2}$, hence the second law is saved at last. We have the efficiency

$$\eta = \frac{W}{mc^2} = 1 - \frac{\kappa d}{2c^2} \simeq 1 - \frac{\kappa\hbar}{2ckT}\,. \tag{5.2}$$

We can compare this with the definition of the absolute temperature scale from the efficiency of the Carnot heat engine given by

$$\eta = 1 - \frac{T_C}{T_H}\,, \tag{5.3}$$

where T_H is the temperature of a heat source and T_C is the temperature of a heat sink. If we choose $T_C = T_{BH}$ and $T_H = T$, we see that the black hole has the temperature

$$T_{BH} \simeq \frac{\kappa\hbar}{2ck_B} \simeq \frac{\hbar c^3}{8GMk_B}\,. \tag{5.4}$$

This is not the exact result that we derive in the next section, but as we will see, it is very close. Notice also that we foiled the demon's second plan but not the first originally proposed by Wheeler. It will not be until Subsection 5.2.2 that this paradox is resolved.

5.2 Hawking radiation

Wheeler's demon and other thought experiments motivated Bekenstein to think that the entropy of a black hole is very real. But it was not until Hawking discovered that black holes should also emit thermal radiation that the physics community was convinced. This is the subject of this section, where we follow [45], if not stated otherwise.

In Section 5.1.1 we derived an approximate form of the temperature of the black hole (5.4). We see that this temperature is a concept from quantum gravity because $T_{BH} \xrightarrow{\hbar\to 0} 0$ and $T_{BH} \xrightarrow{\kappa\to 0} 0$. Therefore, to arrive at the exact value of this temperature, we would need a quantum theory of gravity, which is not yet fully developed. However, Hawking argued that we can obtain a very good approximation using the quantum field theory on curved spacetime and calculated the corresponding temperature formula [48]. We are not going to present here Hawking's approach, but let us show at least a simplified picture of the basic ideas of how Hawking radiation follows from quantum field theory.

In an attempt to understand the Hawking effect, physicist William Unruh discovered that a similar effect also occurs in flat Minkowski space for an accelerated observer. Let us see why. Very roughly speaking, the quantum vacuum can be thought of as a sea of negative and positive frequency modes of quantum fields that cancel out. However, for an accelerated observer, an event horizon appears as seen in Figure 5.1, which introduces a certain cut-off in the modes of the quantum fields. This effectively creates a thermal spectrum of particles called Unruh radiation. For an unaccelerated observer, there is of course no Unruh radiation, therefore, different observers disagree on the particle content of the quantum vacuum. This might seem contradictory, but this paradox has a resolution, because the unaccelerated observer will also see particles hitting the accelerated observer, but for a very different reason, which we will not delve into.

Now we apply this to a Schwarzschild black hole. There are reasons to expect that, from the point of view of a freely falling observer near the event horizon, the vacuum looks like the Minkowski vacuum for an unaccelerated observer - empty. Also, we assume that for static observers close to the event horizon, spacetime can be approximated as flat on their spacetime scales. But these observers can be seen as accelerated because they resist the pull of gravity, therefore, they see the Unruh radiation. Observers in asymptotic infinity can no longer approximate the spacetime as flat, but they will see the radiation seen by the observers near the event horizon, but redshifted so the temperature appears to be

$$T = \frac{\kappa\hbar}{2\pi c k_B} = \frac{\hbar c^3}{8\pi G M k_B}\,. \tag{5.5}$$

This is exactly the famous Hawking temperature. Notice that compared to (5.4) there is a factor $\frac{1}{\pi}$.

5.2.1 Hawking radiation from the uncertainty relations

At the beginning of this section, we discussed how is Hawking temperature derived from the quantum field theory. However, we did not show any details. In contrast, in this subsection we derive the Hawking effect heuristically but in detail, drawing from [49, 50]. This method is also important if we want to later obtain a deformed Hawking temperature from GUP.

Imagine that in a quantum vacuum, virtual particle-antiparticle pairs are constantly being created, and almost instantaneously they annihilate. As the name suggests, these virtual particles cannot be observed directly. However, if such a pair is created on the event horizon of a

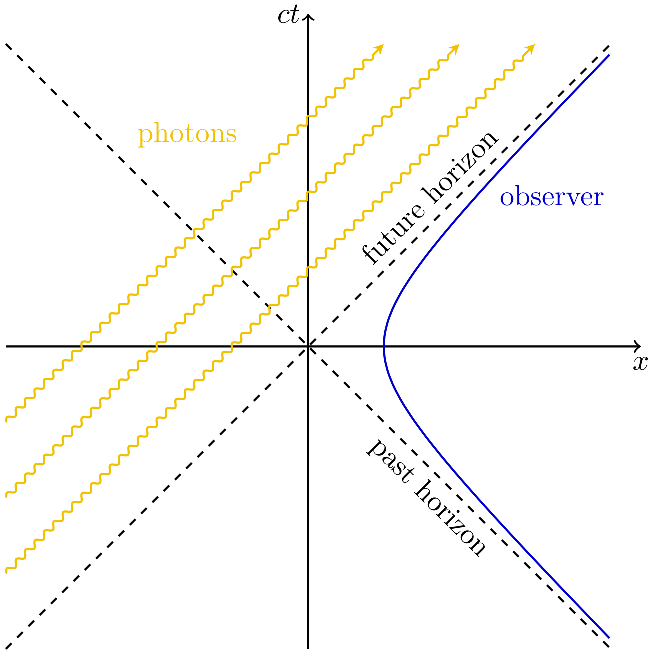

Figure 5.1: Spacetime diagram for an accelerated observer with past and future event horizons. The photons represent some light-like trajectories. In this diagram, the acceleration is uniform and acts for an infinitely long time, but it does not have to be so for the Unruh radiation to appear.

Schwarzschild black hole and one particle falls into the black hole, they can no longer annihilate and the other particle becomes real. If the created particle had energy E, the other must have had negative energy $-E$ in order to conserve energy. Therefore, the black hole loses energy. This analogy with virtual particles was even used by Hawking, but he also warned against taking it too seriously because of the elusive nature of virtual particles.

Because the particles created in this process can appear anywhere around the horizon, their uncertainty in position is approximately the circumference of the black hole.

$$\Delta x \simeq 2\pi r_s \tag{5.6}$$

From this we can estimate the uncertainty in momentum using the Heisenberg uncertainty principle to obtain

$$\Delta p \simeq \frac{\hbar}{2\Delta x} \simeq \frac{\hbar}{4\pi r_s}\,. \tag{5.7}$$

We make the assumption that the radiation is thermal black-body radiation in the form of photons. It follows from the equipartition theorem, that the average energy of photons is

$$\Delta E = k_B T = c\Delta p\,, \tag{5.8}$$

where we also used that the energy of the photon is $E = pc$. If we substitute (5.7) into (5.8), we finally get

$$T \simeq \frac{\hbar}{4\pi r_s k_B} = \frac{\hbar c^3}{8\pi G M k_B}\,, \tag{5.9}$$

which is exactly (5.5).

We have already mentioned that through the emission of Hawking radiation the black hole loses energy and must therefore shrink because we are dealing with a Schwarzschild black hole. But that is a violation of the second law of black hole dynamics. Indeed, we have to replace the second law, and this is the subject of the next subsection.

5.2.2 Generalised second law of thermodynamics

We have seen that black holes do have temperature in a very real sense throughout this section and in Subsection 5.1.1. This suggests that they should also have real thermodynamical entropy. In this section, we derive a formula for the entropy of a black hole and introduce the generalised second law of thermodynamics following [46, 50]. This will solve the paradoxes of Subsections 5.1.1 and 5.2.1.

If we compare (5.1) with the first law of thermodynamics and use (5.5), we have

$$\begin{aligned} T dS &= \frac{\kappa c^2}{8\pi G} dA = T \frac{c^3 k_B}{4G\hbar} dA\,, \\ \frac{dS}{dA} &= \frac{c^3 k_B}{4G\hbar}\,. \end{aligned} \tag{5.10}$$

This determines the entropy up to a constant, which we set to zero in order to $S \xrightarrow{A\to 0} 0$ and therefore

$$S = \frac{k_B c^3 A}{4G\hbar}\,. \tag{5.11}$$

This is the Bekenstein-Hawking entropy. It quantifies the entropy of the most general Kerr-Newman black hole. Notice that (5.11) contains four fundamental constants and thus connects the fields of general relativity, quantum physics, and thermodynamics.

In his 1974 paper, Bekenstein proposed that we should treat black hole entropy as thermodynamical entropy and add it to the second law of thermodynamics to obtain the generalised law of thermodynamics [51]. This law states that the entropy of an isolated system never decreases if the Bekenstein-Hawking entropy of black holes is taken into account. Bekenstein showed that this resolves the paradox from Subsection 5.1.1 because due to the in-falling object, the entropy of the black hole increases, so that the generalised second law holds. It can also be shown that the loss of black hole entropy due to Hawking radiation is compensated by the corresponding increase of entropy in the surrounding environment.

To this day black hole thermodynamics continues to play an important role in physics. It serves as an inspiration in the quest for the quantum theory of gravity. For example, it led to development of the holographic principle.

5.3 Hawking radiation with GUP

In Subsection 5.2.1, we have seen how to derive the Hawking temperature formula using the uncertainty principle. We will build on this subection and derive how the black hole temperature changes in the case of a minimal length GUP of the form (1.16). In this section, we are drawing from [7], where a more rigorous derivation for the world crystal GUP can be found. They also present a derivation based on the Landauer principle.

The situation is the same as in Subsection 5.2.1 until we used (5.6). Instead of this, we now have

$$\Delta p \simeq \frac{\hbar}{2\Delta x}(1+\beta(\Delta p)^2) \simeq \frac{\hbar}{4\pi r_s}(1+\beta(\Delta p)^2)\,. \tag{5.12}$$

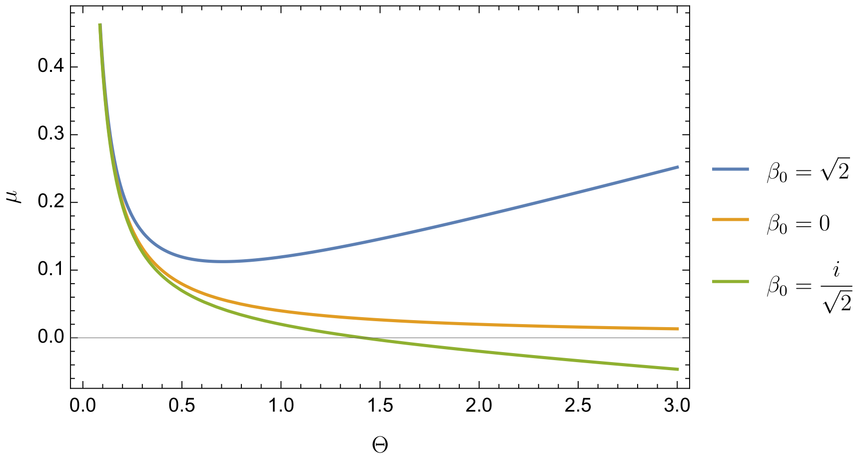

Figure 5.2: Plot of the function (5.14) for the cases of $\beta_0 = \sqrt{2}$, corresponding to Heisenberg microscope in Newtonian gravity or string theory for specific choice of K and α'; $\beta_0 = 0$, representing the standard Hawking's formula; and for $\beta_0 = \frac{i}{\sqrt{2}}$, which comes from the world crystal. This plot is nearly identical to that of [7].

As before, we substitute (5.8) to obtain

$$T \simeq \frac{\hbar}{4\pi r_s}\left(\frac{c}{k_B} + \beta\frac{k_B T^2}{c}\right) = \frac{c^2\hbar}{8\pi GM}\left(\frac{c}{k_B} + \beta\frac{k_B T^2}{c}\right) . \tag{5.13}$$

Let us introduce a dimensionless temperature as $\Theta := \frac{T}{T_p}$ and a dimensionless mass as $\mu := \frac{M}{m_p}$. Using these quantities, we can rewrite (5.13) as

$$\mu(\Theta) = \frac{1}{8\pi\Theta} + \frac{\beta_0^2}{8\pi}\Theta \tag{5.14}$$

Plot of this function is shown in Figure 5.2 for different values of β_0 presented in Subsection 1.3.2.

The standard Hawking's formula for black hole evaporation predicts infinite temperature at the end of a black hole's life. This troublesome aspect is solved by both the deformation coming from string theory (or Heisenberg microscope in Newtonian gravity) and that from the world crystal. However, whereas the theory of world crystal predicts that the black hole fully disappears, the temperature formula of string theory predicts the existence of massive black hole remnants. The presence of black hole remnants can be seen from the fact that 5.14 for $\mu(\Theta)$ reaches a minimum at

$$\Theta_{min} = \frac{1}{\beta_0}, \qquad \mu(\Theta_{min}) = \frac{\beta_0}{4\pi}, \tag{5.15}$$

for $\Theta > 0$. Notice that from this one can also predict that there are no remnants when $\beta_0 = 0$ and $\beta_0 = \frac{i}{\sqrt{2}}$. Black hole remnants are a controversial topic in physics, while some would welcome their existence (for example, to explain the origin of dark matter) some argue it would cause more problems than it would solve.

Finally, let us just note that the temperature formulae are different for different types of black holes. In this paper, we have assumed Schwarzschild black holes all along. A derivation of the temperature formulas for Reissner-Nordström, (anti-)de Sitter black holes using uncertainty relations can be found in [52]. The deformation of these temperature formulas with GUP might be the subject of further studies.

Conclusion

In Chapter 1 of this thesis, we introduced important concepts surrounding the study of GUP and presented some motivations for different forms of GUP. We also tried to highlight the interconnectedness of different theories and concepts. In particular, the deeply related theories of DSR, curved momentum space, and non-commutative geometry were important for our work. The basis of our deformation of geodesic motion then became the non-commutative algebra of the Snyder model, coming from non-commutative geometry.

A discussion of what a deformation based on Snyder's model should look like was then part of Chapter 2. We introduced the theory of generalised Hamiltonian dynamics, which then played a key role in our analysis. Finally, we concluded that the Snyder algebra could correspond to the deformed algebra of Poisson brackets. We presented several motivations for this decision, one of which was a model of deformed special relativity with curved momentum space.

Finally, we applied our modified Poisson algebra to geodesic motion in Chapter 3. We obtained Hamilton's equations of deformed geodesic motion in general spacetime and studied some of its aspects. In Chapter 4 we considered our deformed geodesic motion in the spacetimes of Schwarzschild and Kerr. We have decided that the full version of our equations would be too complicated and resorted to a pseudo-Newtonian limit. In this limit, we obtained the form of our equations for both Schwarzschild and Kerr solutions. Our theory can be experimentally tested in principle on the orbits of massive bodies in our solar system, which can be approximated by the Schwarzschild solution or around rotating black holes described by the Kerr solution. However, this is no longer the subject of this thesis.

We concluded with a discussion of the thermal emission from black holes in Chapter 5. We first introduced the topic of black hole thermodynamics and then demonstrated how to derive the temperature formula using uncertainty relations. We then discussed the modification of this formula in the case of GUP.

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