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Faculty of Electrical Engineering
Department of Cybernetics

Algebraic and State-Space Properties of Quantum Logics

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Declaration

I declare that the presented work was developed independently and that I have listed all sources of information used within it in accordance with the methodical instructions for observing the ethical principles in the preparation of university theses.

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Abstract

This text contributes to the theory of quantum logics (the algebraic theory of the orthomodular posets).

Chapter 1 introduces basic notions of quantum theories and their link with quantum logics.

Chapter 2 rigorously recalls the basic notions of the up-to-date quantum logics.

Chapter 3 studies the algebraic generation in the lattice QLs. As a main result, we see how the fact of being locally finite allows for an extension of states. Then a Cartesian product of locally finite lattice QLs is investigated in view of the permanence property.

In Chapter 4, we investigate the QLs that are endowed with a symmetric difference. Main results obtained are finding examples of an irregular compatibility, proving the extension of $\mathbb{Z}_2$-valued states and presenting a construction of an example with a small state space and a big degree of non-compatibility.

In Chapter 5, we endow the Abbott algebras with a symmetric difference (a kind of a $\mathbb{XOR}$ operation). We find that the Abbott $\mathbb{XOR}$ algebras are categorically equivalent to the class of lattice QLs with a symmetric difference. Another result is a description of compatibility. Also, Boolean algebras are characterized among the $\mathbb{XOR}$ Abbott algebras and an appropriate definition of a state is formulated and applied.

Chapter 6 asks whether each set-representable quantum logic can be made point-distinguishing. We answer this question in the positive by considering an appropriate equivalence relation and, alternatively, by relating the problem to the Stone representation technique.

In Chapter 7 we summarize the results and comment on the matters studied.

Keywords: quantum logic, set-representable quantum logic, state on quantum logic, compatibility relation, lattice quantum logic, Boolean algebra, generation in the lattice quantum logic, quantum logic with a symmetric difference, lattice quantum logic with a symmetric difference, $\triangle$-state, $\mathbb{Z}_2$-state, Abbott algebra, Abbott $\mathbb{XOR}$ algebra, categorical equivalence, point-distinguishing quantum logic, generalized Stone representation

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Abstrakt

Tento text přispívá k teorii kvantových logik (algebraické teorii ortomodulárních uspořádaných množin).

Kapitola 1 představuje základní pojmy kvantových teorií a jejich vztah s kvantovými logikami.

Kapitola 2 formálně zavádí základní pojmy teorie kvantových logik.

Kapitola 3 se zabývá algebraickým generováním ve svazových kvantových logikách. Jedním z výsledků je zjištění, že vlastnost lokální konečnosti umožňuje rozšiřování stavů. Poté je zkoumán kartézský součin lokálně konečných svazových kvantových logik.

V kapitole 4 zkoumáme kvantové logiky dovolující zavedení pojmu symetrické diference. Hlavními dosaženými výsledky jsou nalezení příkladu neregulární kompatibility v těchto strukturách, důkaz rozšiřování $\mathbb{Z}_2$-stavů a konstrukce příkladu s malým stavovým prostorem a vysokým stupněm nekompatibility.

V kapitole 5 opatříme Abbottovy algebry symetrickou diferencí (lze ji chápat jako operaci $\mathbb{XOR}$). Ukazujeme, že Abbottovy algebry s operací $\mathbb{XOR}$ jsou kategoriálně ekvivalentní se svazovými kvantovými logikami se symetrickou diferencí. Dalším výsledkem je popis kompatibility a zavedení pojmu $\triangle$-stav. Dále je podána zajímavá charakterizace Booleovy algebry v řeči symetrické diference.

V kapitole 6 se ptáme, zda lze každou množinově reprezentovatelnou kvantovou logiku učinit isomorfní s množinově reprezentovatelnou kvantovou logikou, která dovoluje oddělování bodů. Na tuto otázku odpovídáme kladně pomocí zavedení vhodné ekvivalence a spojení s technikou Stoneovy reprezentace.

V kapitole 7 shrhujeme hlavní výsledky práce.

Klíčová slova: kvantová logika, množinově reprezentovatelná kvantová logika, stav na kvantové logice, relace kompatibility, svazová kvantová logika, Booleova algebra, generování ve svazové kvantové logice, kvantová logika se symetrickou diferencí, $\triangle$-stav, $\mathbb{Z}_2$-stav, Abbottova algebra, Abbottova $\mathbb{XOR}$ algebra, ekvivalence kategorií, bodově rozlišitelné kvantové logiky, zobecněná Stoneova reprezentace

Překlad názvu: Algebraické a stavové vlastnosti kvantových logik

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BACHELOR‘S THESIS ASSIGNMENT

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Guidelines:

(1) The research aim is a review of the student's results already accepted for publication (see (1) and (2) in the recommended literature enclosed) and the extension of the results and a further analysis. The newly expected contribution finds itself in the endowing of Abbott algebras with a symmetric difference and the equivalence with the quantum logics. Another envisaged investigation area will be the intrinsic properties of set-representable quantum logics and their state-space properties, in particular point-distinguishing properties and the character of states.
(2) To reach this aim, a requirement is the study of topical research articles that are relevant to the problems adressed and an analysis of fundamental accomplishments published in the adequate monographs.
(3) A creative component of the research envisaged should be reflected by the expected new contributions and/or publications of the student.
(4) The quality of the already obtained results should be acknowledged by the impact factor of publications. The quality of the newly added results should be judged by theoretical analysis and formal proofs.

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Chapter 1

Introduction

The aim of this thesis is to contribute to the quantum logic theory. This theory can be viewed as a part of mathematical physics, or, seen from the structural angle, as a part of the theory of ordered structures.

Let us indicate how quantum logics came into existence and how they are seen to be associated to quantum experiments. The way of this association developed a long time and it is complicated as quantum physics itself, so we have to simplify the matters in places. The subject is so complex that even the monographs ([25, 39, 17, 36, 12, 11]) devoted to this subject do not cover it completely. Hence let us only sketch the basic ideas and milestones that led to the up-to-date notions and the directions of research.

A standard presentation begins with the name of J. von Neumann. The quantum mechanics in the sense of J. von Neumann would become a type of probability theory. However, in contrast to the Kolmogorovian classical probability, the laws of classical probability cannot be obeyed. This circumstance is usually attributed to the so-called Heisenberg uncertainty principle that states, roughly, that the measurement of the position and the momentum of a particle cannot be performed accurately [13]. J. von Neumann suggested to consider the projection operators as the representatives of the propositional logic of a quantum experiment (this attitude is defended by many physicists even today). If a proposition is verified by an experiment, one evaluates it by 1, if it is false, one evaluates it by 0. This became the initial idea behind the abstract "quantum logics" and "states".

After several debates and critical essays of quantum physicists, the original idea by J. von Neumann became less trustful. There occured a need for a more natural and more verifiable axiomatics. Paradoxically, J. von Neumann himself became an advocate of a generalized approach - in the famous paper with G. Birkhoff [3] adopted the "logico-algebraic" approach. This was a preliminary step to the notion of quantum logic studied in this text.

Let us step aside for a moment and recall basic notions of Boolean algebras and lattices. If we consider the classical propositional logic, we speak of a structure closed under the formation of the negation, $^{\perp}$, the disjunction $\vee$ and the conjunction $\wedge$. The operations are supposed to satisfy some natural conditions. We then obtain a Boolean algebra (the Lindenbaum-Tarski algebra). A Boolean algebra obeys the distributivity law, $a \vee (b \wedge c) = (a \wedge b) \vee (a \wedge c)$.

If we go back to the concept of G. Birkhoff and J. von Neumann, they in fact associate to any quantum experiment an algebraic generalization L of a Boolean algebra - an ordered structure with an orthocomplement operation - as a kind of a "logic" (a Boolean algebra is not appropriate because of the Heisenberg uncertainty relation). The elements of L can be viewed as representations of verifiable propositions on the experiment. The ordering of L defines an implication, the orthocomplement operation defines a negation. If the experiment is classical (Newtonian), then L is a Boolean algebra. If the experiment is quantum, L is assumed to be modular. So L is generally non-Boolean but it is easily seen that L consists of many Boolean subalgebras that cover L. These Boolean subalgebras are associated to the arrangements of the experiment and their elements can be verified. The notions then became subject to the final revision. Since $L(\mathbb{H})$ - the projectors in a Hilbert space $\mathbb{H}$ - is modular only when $\dim(\mathbb{H})$ is finite, the appropriate conceptual "umbrella" became orthomodular partially ordered sets. A partially ordered set is said to be orthomodular if $a \leq b \implies b = a \vee (b \wedge a^{\perp})$. By the discoveries of today, the orthomodular law has much more abstract and general roots than the projection logics [18].

As it usually happens, the theory of quantum logics started to live its own life within the realm of ordered structures. The results obtained may then in turn have a bearing on theoretical physics.

According to the state of the art, quantum logics are supposed to be orthomodular posets and states are supposed to be probability measures on them. They should relatively well model some problems of quantum systems. A purpose of this thesis is bringing a solution of some natural questions that occur therein.

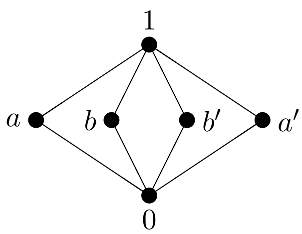
Chapter 2

Basic notions

2.0.1 Definition (quantum logic)

Let $P = (S, \leq, ^{\perp}, 0, 1)$, where $\leq$ is a partial ordering on S with a least and a greatest elements $0, 1$, and where $^{\perp} : S \to S$ is an orthocomplementation mapping ($a \leq b \implies b^{\perp} \geq a^{\perp}$, $(a^{\perp})^{\perp} = a$, $a \vee a^{\perp} = 1$ and $a \wedge a^{\perp} = 0$). If P satisfies the orthomodular law, $a \leq b \implies b = a \vee (b \wedge a^{\perp})$, then P is said to be a **quantum logic (QL)**.

In the algebraic language QLs are called orthomodular posets. The typical example of a QL is a **horizontal sum** of Boolean algebras (a disjoint union of Boolean algebras with 0 and 1 identified in all Boolean algebras). Another fundamental example is a so-called **Greechie diagram**. Both constructions are demonstrated by the figures below.



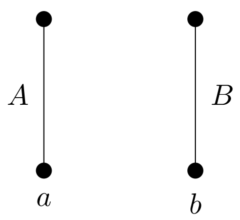


Figure 2.1: Hasse (left) and Greechie (right) diagrams of a horizontal sum of two 4-element Boolean algebras, usually denoted by MO_2 [31]

2.0.2 Definition (lattice quantum logic)

If a QL, P, is a lattice with respect to $\leq$, then P is said to be a **lattice QL**.

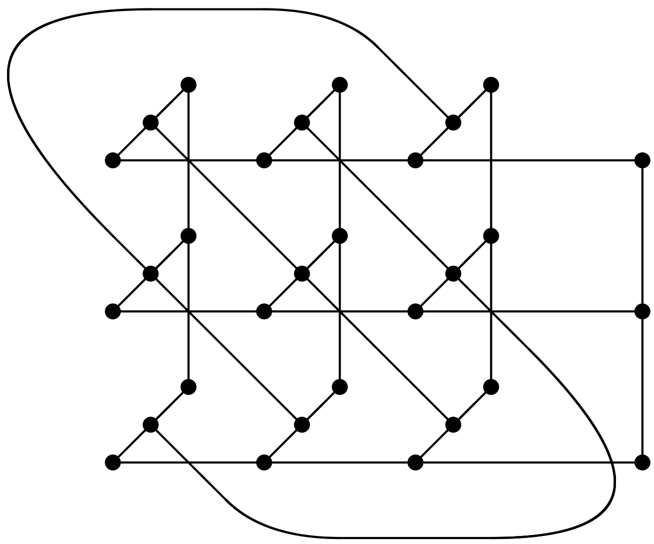


Figure 2.2: Greechie diagram of R. Mayet's example of a QL without states [31]

In the algebraic language the lattice QLs are called orthomodular lattices. Typical example is a **Boolean algebra** or the horizontal sum of Boolean algebras, and the **lattice of projectors** in a Hilbert space. We shall deal with many others.

2.0.3 Definition (set-representable QL and set-representable lattice QL)

Let S be a set and let $\mathbf{\Delta}$ be a collection of subsets of S ($\mathbf{\Delta} \subseteq \exp S$). Suppose that $\mathbf{\Delta}$ is subject to the following requirements:

1. $S \in \mathbf{\Delta}$,
2. if $A \in \mathbf{\Delta}$, then $A^\perp \in \mathbf{\Delta}$ ($A^\perp = S \setminus A$),
3. if $A, B \in \mathbf{\Delta}$ and $A \cap B = \emptyset$, then $A \cup B \in \mathbf{\Delta}$.

Then $(S, \mathbf{\Delta})$ is said to be a **set-representable QL (SR-QL)**.

If $\mathbf{\Delta}$ is a lattice with respect to the inclusion ordering, the couple $(S, \mathbf{\Delta})$ is said to be a **set-representable lattice QL (lattice SR-QL)**.

A typical example of SR-QLs are the **Gudder logics** (for instance, if the number of elements of a set is divisible by k, one takes for $\mathbf{\Delta}$ the subsets whose number of elements is a multiple of k).

A typical example of a lattice SR-QL is a Boolean algebra (the Stone theorem) or the Gudder QL on the set of 4 elements and its Cartesian products.

2.0.4 Definition (quantum logics closed under the formation of a symmetric difference)

Let $P = (S, \leq, ^\perp, 0, 1, \triangle)$, where $P = (S, \leq, ^\perp, 0, 1)$ is a QL and $\triangle : S^2 \to S$ is a binary operation. Then P is said to be a **quantum logic with a symmetric difference (QLS)** if P satisfies the following conditions ($x, y, z \in S$):

1. $x\triangle(y\triangle z) = (x\triangle y)\triangle z$,
2. $x\triangle 1 = x^\perp, 1\triangle x = x^\perp$,
3. $x \leq z, y \leq z \implies x\triangle y \leq z$.

If a QLS is a lattice with respect to $\leq$ then it is called a **lattice QLS**.

A typical example of both of these structures is a Boolean algebra. An easy example of a QLS is the even-number-subsets of a set with an even number of elements. The latter example on the 4-point set is a lattice QLS and so is also any Cartesian product of this example with a Boolean algebra. The class of lattice QLSs forms a variety of algebras that, in a sense, lies in between Boolean algebras and lattice QLs.

Regarding the interpretation of QLSs in quantum axiomatics, it may be debatable as other matters in quantum theories. It seems to assume that (at least some) quantum experiments may be associated with QLSs.

A prominent position of QLSs take the set-representable ones. We will define them in the next definition.

2.0.5 Definition (set-representable QLS)

Let $(S, \mathbf{\Delta})$ be a SR-QL. If $\mathbf{\Delta}$ is closed under the formation of the set-theoretic difference, i.e. if $A, B \in \mathbf{\Delta}$ then $A\triangle B = (A \setminus B) \cup (B \setminus A) \in \mathbf{\Delta}$, then $(S, \mathbf{\Delta})$ is said to be a **set-representable QLS (SR-QLS)**.

Obviously, a SR-QLS is a QLS with $\leq$ being the inclusion relation and $^\perp$ being the set complement operation.

Let us take up another important notion in QL. Again, it is supposed to model well what we could call a state in a quantum experiment.

2.0.6 Definition (state)

Let $P = (S, \leq, ^\perp, 0, 1)$ be a quantum logic.Let $s : P \to [0, 1]$ be a mapping that satisfies the following conditions:

1. $s(1) = 1$ (Completeness),

2. $s(a \vee b) = s(a) + s(b)$, provided $a \leq b^{\perp}$ (Exclusivity).

Then s is called a **state** on P.

2.0.7 Definition (pure state)

A state s is said to be a **pure state** if it cannot be expressed as a convex combination of other states. Formally, s is pure if the state equality $s = \alpha t + (1 - \alpha)u$ implies that $\alpha = 0$ or $\alpha = 1$.

Let us remark the importance of pure states - by the classical Krein-Milman theorem each state can be "reached" by the topological closure of the convex hull of pure states.

2.0.8 Definition (two-valued state, alias hidden variable)

The state $s : P \to \{0, 1\}$ is said to be a **two-valued** state. Following a concept of the philosophy of physics, a two-valued state is sometimes called a **hidden variable**, see [17, 42].

Let us only note that the absence of hidden variables in the projector logics is a matter of long lasting dispute of physicists.

2.0.9 Definition ($\triangle$-state)

Let P be a QLS, $P = (S, \leq, ^{\perp}, 0, 1, \triangle)$. Let $s : P \to [0, 1]$ be a state such that for any $a, b \in P$ we require $s(a \triangle b) \leq s(a) + s(b)$.
Then s is called a **$\triangle$-state** on P.

2.0.10 Definition ($\mathbb{Z}_2$-state)

Let P be a QLS, $P = (S, \leq, ^{\perp}, 0, 1, \triangle)$. A mapping $s : P \to \mathbb{Z}_2$ (where $\mathbb{Z}_2$ is the two-element group with the group operation $\oplus$) is said to be a **$\mathbb{Z}_2$-state** on P, if the following conditions are satisfied $(a, b \in P)$:

1. $s(1) = 1$,
2. $s(a \vee b) = s(a) \oplus s(b)$, $a, b \in P$.

2.0.11 Definition (state space)

Let us denote by $\mathcal{S}(P)$ the set of all states on a QL, P. Obviously, $\mathcal{S}(P)$ can be naturally understood with its affine and topological (compact) structure. Thus, $\mathcal{S}(P)$ can be called the **state space** of P.

2.0.12 Definition (space of two-valued states)

Let us denote by $\mathcal{S}_2(P)$ the set of all two-valued states on a QL, P. Obviously, $\mathcal{S}_2(P)$ can be naturally understood with its topological (compact) structure.

2.0.13 Definition ($\triangle$-state space)

Let us denote by $\mathcal{S}_{\triangle}(P)$ the set of all $\triangle$-states on a QLS, P. Obviously, $\mathcal{S}_{\triangle}(P)$ can also be naturally understood with its affine and topological (compact) structure. Thus, $\mathcal{S}_{\triangle}(P)$ can be called the **$\triangle$-state space** of P.

Chapter 3

Locally finite lattice quantum logics

This chapter contributes to some questions on the generation of lattice quantum logics by their subsets. Let us present a few results apparently not dealt with in the relevant literature.

This chapter is composed upon the published paper in an IF journal co-authored by the author of this text [8].

3.1 Basic notions of this chapter

It is the lattice QLs that are the subject of this chapter.

We will address the question on the generation of a lattice QL by its subset. There are still several open questions in this area (for instance, there is no characterization of the free lattice QL over 3 generators). We would like to contribute to some open questions. Let us say that a lattice QL, P, is **locally finite** if each finite subset of P generates in P a finite lattice subQL. So, in the generation we take into account both the QL structure and the lattice structure.

Let us denote the class of locally finite lattice QLs by $\mathcal{LF}$.

3.2 Results

A first result gives us basic information on $\mathcal{LF}$.

3.2.1 Theorem (locally finite lattice QL)

1. Each finite lattice QL and each Boolean algebra belongs to $\mathcal{LF}$,
2. $\mathcal{LF}$ is closed under finite products,
3. $\mathcal{LF}$ is closed under epimorphic images,
4. There are infinite non-Boolean set-representable lattice QLs that belong to $\mathcal{LF}$,
5. There are lattice QLs of $\mathcal{LF}$ with preassigned centers.

Proof. Statements 1 and 2 are obvious, a finitely generated Boolean algebra possesses finitely many atoms.

As for 3, recall (see [25]) that if $L_\alpha, \alpha \in I$, is a collection of lattice QLs, then by the horizontal sum $\mathrm{Hor}(L_\alpha, \alpha \in I)$ one means the lattice QL obtained by identifying zeros and ones of the disjoint union of all $L_\alpha, \alpha \in I$. Obviously, if $L_\alpha \in \mathcal{LF}$ for any $\alpha \in I$, then $\mathrm{Hor}(L_\alpha, \alpha \in I) \in \mathcal{LF}$, too.

Further, verifying 4, let $K \in \mathcal{LF}$ and $e : K \to L$ be an epimorphism in lattice QLs. Let $\{a_i, i \leq n\} \subseteq L$. Take $b_i \in K$ with $e(b_i) = a_i$. Then $\{b_i, i \leq n\}$ generates a finite lattice subQL of K, some $\tilde{K}$, and so $e(\tilde{K})$ is a finite lattice subQL of L. Since $e(\tilde{K})$ contains $\{a_i, i \leq n\}$, property 4 follows.

Finally, approaching property 5, recall that the centre $C(K)$ means the set of all "absolutely compatible" elements of K (see [25]). Let us first take the horizontal sum $\mathrm{Hor}(L, \mathrm{MO}_2)$, where $\mathrm{MO}_2 = \{0, 1, a, a^\perp, b, b^\perp\}$. Then the centre $C(\mathrm{Hor}(L, \mathrm{MO}_2))$ is trivial, $C(\mathrm{Hor}(L, \mathrm{MO}_2)) = \{0, 1\}$, and, obviously, L can be embedded in $\mathrm{Hor}(L, \mathrm{MO}_2)$. To complete the argument, let us form the so-called Boolean sum of $\mathrm{Hor}(L, \mathrm{MO}_2)$ and B (see e.g. [35]). Recall the construction of a Boolean sum, the rest is then easily seen. Let us view B as a collection $\mathcal{B}$ of subsets of a set, $B = (D, \mathcal{B})$ (the Stone Theorem). Let $\mathcal{P}$ be the collection of all partitions of B (by a partition of B we mean a family $Q = \{A_i \mid A_i \in B, i \leq n\}$ of disjoint non-empty sets A_i with $\bigcup\limits_{i \leq n} A_i = D$). If L is a lattice QL and $B = (D, \mathcal{B})$ is a Boolean algebra, a Boolean sum of L and B is the set of all functions $f : D \to L$ with the property that there is a partition $\{A_i \mid i \leq n\}$ of B such that f is constant on every $A_i, i \leq n$ (the "B step functions" on D with values in L). Obviously, a Boolean sum of $\mathrm{Hor}(L, \mathrm{MO}_2)$ and B satisfies property 5. $\square$

The next result points out the interesting state property of the lattice QLs of $\mathcal{LF}$. Let us "en passant" note that the state space of a QL may be very complicated or even somewhat bizarre (see e.g. [15, 32] and [38]).

3.2.2 Theorem (state extension)

Suppose that $P \in \mathcal{LF}$. If $\mathcal{S}(K) \neq \emptyset$ for any finite lattice subQL K of P, then $\mathcal{S}(P) \neq \emptyset$.

Proof. By a standard application of Tychonoff's theorem on the product of compact topological spaces, the set $[0,1]^P$ of all functions $f : P \to [0,1]$ is compact in the topology of pointwise convergence. Let us consider, for each finite lattice subQL K of P, the set of all functions $g : P \to [0,1]$ that are states on K when restricted to K. Denote this set of functions by $\mathcal{T}(K)$. Obviously, each $\mathcal{T}(K)$ is compact, too. Since the family of all finite lattice subQL of P is directed when ordered by inclusion, we see that $\{\mathcal{T}(K),\ K$ finite lattice subQL of $P\}$ is a centered family of closed subsets of $[0,1]^P$. By our assumption, each $\mathcal{T}(K)$ is non-void. If $\mathcal{K}$ denotes the set of all finite lattice subQL of P, then the intersection $\bigcap\limits_{K \in \mathcal{K}} \mathcal{T}(K)$ is non-void (the compactness of $[0,1]^P$ applies). Let $s \in \bigcap_{K \in \mathcal{K}} \mathcal{T}(K)$. Then $s \in \mathcal{S}(P)$ and this completes the proof. $\square$

When restricted to the two-valued states in the interpretation of [17], if $P \in \mathcal{LF}$, then P does not have a hidden variable provided there is a locally finite lattice subQL of P that does not have a hidden variable either. Of course, in general this does not hold (consider the projection logic $L(\mathbb{R}^3)$), each finite lattice subQL K of $L(\mathbb{R}^3)$ is set-representable and therefore $\mathcal{S}_2(K) \neq \emptyset$ but $\mathcal{S}_2(L(\mathbb{R}^3)) = \emptyset$, see [26]).

We saw in Theorem 3.2.1 2 that $\mathcal{LF}$ is closed under finite products. It is conjecturable that this is not true for infinite products. This is indeed the case even in the class of set-representable lattice QLs. Let us present the required construction that seems to be a novelty in lattice QLs and might be valuable in its own right.

3.2.3 Constructions with set-representable lattice QLs (Cartesian product, Kalmbach embedding, etc.)

Let us ask if $\mathcal{LF}$ contains all set-representable lattice QLs. Of course, the free lattice QL over three generators is not set-representable. In [20] the author shows that this lattice QL contains the free lattice QL over countably many generators and if the latter lattice QL was set-representable, then so would be all "small" lattice QLs. This is obviously not the case (see [15]).

However, we can answer the above formulated question in the negative in other ways contributing thus to our theme and adding to Theorem 3.2.1 2. The first way utilizes the so-called Kalmbach embedding of lattices. Recall that by [25] each lattice L can be lattice-theoretically embedded into a lattice QL. Let us denote it by $K(L)$. It can be shown (see [21]) that $K(L)$ is always set-representable. Hence if L is not locally finite as a lattice, and such a lattice is easy to find, then $K(L)$ is a set-representable lattice QL that is not locally finite.

Another example related to Theorem 3.2.1 could be obtained by showing that the class $\mathcal{LF}$ is not closed under the formation of the product of countably many (finite set-representable) lattice QLs. Indeed, it suffices to find finite set-representable lattice QLs L_n, $n \in \mathbb{N}$, such that L_n is generated by three elements a_1^n, a_2^n and a_3^n and, moreover, the cardinality of L_n is finite and greater than or equal to n (one can for instance use the Kalmbach embedding again or consult [16]). Let us take the Cartesian product $\prod_{n \in \mathbb{N}} L_n$. Then the triple $b_i \in \prod_{n \in \mathbb{N}} L_n$, $i \leq 3$, defined by $b_1 = (a_1^n)$, $b_2 = (a_2^n)$, $b_3 = (a_3^n)$, $n \in \mathbb{N}$, generates an infinite subset of $\prod_{n \in \mathbb{N}} L_n$ (the n-th coordinate "supplies" at least n elements). Since the product of set-representable lattice QLs is again set-representable (see e.g. [30]), we see that the product $\prod_{n \in \mathbb{N}} L_n$ is the example we looked for.

Chapter 4

Quantum logics enriched with a symmetric difference.

In this chapter we address some questions on QLSs and SR-QLSs.

We first disprove a naturally looking conjecture on the important relation of compatibility in SR-QLSs (the notion of compatibility captures, in theoretical form, the common measurability question of quantum events [17, 36]). Then we contribute to the $\triangle$-state extension problem and the $\mathbb{Z}_2$-state extension problem, where the latter problem is answered in the positive. We then construct an example motivated by the independence question of the compatibility and the state space in quantum theories - we construct a SR-QLS with a small $\triangle$-state space and a big degree of non-compatibility.

This chapter is composed upon the published paper in an IF journal coauthored by the author of this text [7] and the conference contribution of the author of this text [6].

4.1 Basic notions of this chapter

Let us assume that $(S, \mathbf{\Delta})$ is a SR-QLS. Recall that a subset M of $(S, \mathbf{\Delta})$ is said to be **compatible** if there is a Boolean subalgebra B of $\mathbf{\Delta}$ such that $M \subseteq B$.

Let us say that $(S, \mathbf{\Delta})$ is **compatibility regular (compreg)** if the following implication holds true. If $A = \{A_1, A_2, \ldots, A_n\} \subseteq \mathbf{\Delta}$, and if any subset of A with strictly less than n elements is compatible, then so is A. It is known that $(S, \mathbf{\Delta})$ is compreg provided $\mathbf{\Delta}$ is a lattice (see e.g. [36]). But there are several compreg non-lattice QLs, too. For instance such are several finite set-representable quantum logics (see e.g. [19]) and logics of splitting subspaces of prehilbert spaces ([37]). Since QLSs are richer than general QLs, it is conceivable that QLSs are compreg.

4.2 Results

4.2.1 Theorem (irregular compatibility)

Let $S = \{1, 2, \ldots, 2^n - 1, 2^n\}$, $n \geq 3$. Then there is such a QLS $(S, \mathbf{\Delta})$ that is not compreg.

Proof. Let $\mathbf{\Delta}$ be the collection of all subsets of S that have an even number of elements. Then $(S, \mathbf{\Delta})$ is a QLS. Let $f\colon \{0,1\}^n \to S$ be an isomorphism and let $\tilde{A}_i$, $i \leq n$, be the subsets of $\{0,1\}^n$ of all elements whose i-th coordinate is 1. Let $A_i = f(\tilde{A}_i)$, $i \leq n$. The collection $\{A_i \mid i \leq n\} = A$ is not compatible in $(S, \mathbf{\Delta})$. Indeed, $\bigcap_{i \leq n} A_i$ is a singleton, $\{p\}$, and $\{p\} \notin \mathbf{\Delta}$. However, each subset C, $C \subseteq A$, with less than n elements is compatible in $\mathbf{\Delta}$. This is easy to see since the atoms of the Boolean algebra of subsets of S generated by C consist of the intersections of the elements of C or their complements. Because the number of sets of C is strictly less than n, all these atoms belong to $\mathbf{\Delta}$. $\square$

4.2.2 Theorem (extension of $\triangle$-states and $\mathbb{Z}_2$-states)

There is a SR-QLS, $P = (S, \mathbf{\Delta}')$ and a two-valued mapping $s : \mathbf{\Delta}' \to \{0,1\}$ such that s is both a $\triangle$-state and a $\mathbb{Z}_2$-state and, moreover, if $\mathbf{\Delta} = \exp S$, then s cannot be extended over $\mathbf{\Delta}$ as a $\triangle$-state but **can** be extended over $\mathbf{\Delta}$ as a $\mathbb{Z}_2$-state. In fact, for each $(S, \mathbf{\Delta}')$ and each $\triangle$-state s on this $(S, \mathbf{\Delta}')$ the $\triangle$-state s can always be extended over a bigger SR-QLS.

Proof. Let $S = \{1, 2, \ldots, 9, 10\}$. Let

$$\begin{aligned} A &= \{2,3,4,5\}, \\ B &= \{4,6,8,9\}, \\ C &= \{1,2,4,8\}, \\ D &= \{4,5,6,7\}. \end{aligned}$$

Let $(S, \mathbf{\Delta}')$ be generated by A, B, C and D. Then there is a two-valued $\triangle$-state s on $\mathbf{\Delta}'$ that is also a $\mathbb{Z}_2$-state and, moreover, s cannot be extended as a two-valued $\triangle$-state over $\exp S$ while s can be extended over $\exp S$ as a $\mathbb{Z}_2$-state.

Indeed, set

$$\begin{aligned} s(S) &= 1, \\ s(A) &= 0, \\ s(B) &= 1, \\ s(C) &= 1, \\ s(D) &= 1, \\ s(A \triangle B) &= 1, \\ s(A \triangle C) &= 1, \\ s(A \triangle D) &= 1, \\ s(B \triangle C) &= 0, \\ s(B \triangle D) &= 0, \\ s(C \triangle D) &= 0, \end{aligned}$$

$$s(A\triangle B\triangle C) = 0,$$
$$s(A\triangle B\triangle D) = 0,$$
$$s(A\triangle C\triangle D) = 0,$$
$$s(B\triangle C\triangle D) = 1,$$
$$s(A\triangle B\triangle C\triangle D) = 1.$$

The values of s on the complements are determined by the additivity of s. It can be easily checked that $(S, \mathbf{\Delta})$ is correctly defined (it is a lattice and it has 32 elements) and that s is a state as well as a $\mathbb{Z}_2$-state. Let us first show that s cannot be extended over $\exp S$. Let us argue by contradiction.

If t is such an extension, t must be given by a partition of unity $f\colon S \to [0,1]$ such that for each X, $X \subseteq S$ we have $t(X) = \sum_{x\in X} f(x)$. This means that $t(B\triangle C) = t(\{1,2,6,9\}) = 0$ and, also, $t(\{1,2,5,6,8,9\}) = 1$. It follows that $t(\{5,9\}) = f(5) + f(9) = 1$. But $f(5) = t(\{5\}) = t(\{2,3,4,5\}) = 0$ and therefore $f(9) = t(\{9\}) = 1$. But $t(\{1,2,6,9\}) = 0$. This is a contradiction. Let us take up the case of $\mathbb{Z}_2$-state. Since s is also a $\triangle$-state, we know that it must be given by a partition of unity, g. We set

$$g(\{4\}) = g(\{5\}) = g(\{6\}) = g(\{9\}) = 1$$

and

$$g(\{1\}) = g(\{2\}) = g(\{3\}) = g(\{7\}) = g(\{8\}) = g(\{10\}) = 0,$$

we see that g extends s as a $\mathbb{Z}_2$-state on $\exp S$.

Let us consider the second part of Theorem 4.2.2. Suppose that $(S, \mathbf{\Delta}')$ is a subSR-QLS of some $(S, \mathbf{\Delta})$. Observe that $(S, \mathbf{\Delta})$ (and $(S, \mathbf{\Delta}')$ as well) can be viewed as a linear space (and a linear subspace) over the field $\mathbb{Z}_2$ (this has been observed in [33] in connection with the game Nim). Indeed, if we view the sets of $\mathbf{\Delta}$ as their characteristic functions with values is $\mathbb{Z}_2$, then the sets of $\mathbf{\Delta}$ can be identified with a linear space—if χ_1, χ_2 are characteristic functions of A_1, A_2 then $\chi_1 \oplus \chi_2$ is a characteristic function of $A_1 \triangle A_2$. Moreover, let us observe in this interpretation that a $\mathbb{Z}_2$-state on $(S, \mathbf{\Delta}')$ is a linear form. This form can be extended over $(S, \mathbf{\Delta})$ by a standard method of linear algebra (the infinite case argues by the application of Zorn's lemma). The extension then becomes the $\mathbb{Z}_2$-state extension over $\mathbf{\Delta}$. $\square$

The first result is found purely "out of curiosity" and out of the effort to strengthen the intuition on the SR-QLSs. Also, there is a link with the "hidden variable" hypothesis in the interpretation of [17]. The second result on the extension of the $\mathbb{Z}_2$-states was supposed to prepare the stage for (generalized) propositional logics in the class of QLSs. Another fact is that a SR-QLS can be viewed as a kind of a code and therefore there is a relation with the coding theory. This is still under investigation.

The next result concerns QLSs that are not set-representable. Such QLSs do exist as we show in the following observation. As a little surprise, such a QLS is easy to construct in comparison with an analogous construction in QLs (this phenomenon would deserve more attention as indicated in the presentation of the author of this thesis [6]).

4.2.3 Observation (non-set-representable QLS)

Take the horizontal sum of all 4-element Boolean subalgebras of $\exp\{1,2,3\}$. Then it is easily seen that this horizontal sum can be made a QLS and this QLS is not set-representable [27].

The final result of this chapter has an explicit bearing on quantum theories. It shows that though QLSs are "almost-Boolean", they allow for a construction of a QLS that has an arbitrarily high degree of non-compatibility and a "small" $\triangle$-state space. This shows that state spaces and compatibility relations are independent in contrast to some other models of quantum theories (for instance, in contrast with the projector approach in von Neumann algebras).

4.2.4 Theorem (big QLS with a small $\triangle$-state space)

Let K be a simplex in $\mathbb{R}^n, n \in \mathbb{N}$. Let k be a cardinal number. Then there is a QLS, P, such that P contains at least k non-compatible pairs and the $\triangle$-state space $\mathcal{S}_\triangle(P)$ of P is affinely homeomorphic to K.

Proof. There is a QLS, Q, such that $\mathcal{S}_\triangle(Q) = \emptyset$ (see [43] and [41]). Then the Cartesian product Q^k obviously has at least k non-compatible pairs (The class of QLSs forms a quasivariety [28] and therefore they are closed under the formation of Cartesian products). Moreover, $\mathcal{S}_\triangle(Q^k) = \emptyset$. Indeed, Q can be naturally embedded in Q^k and if there is no $\triangle$-state on Q, there is obviously no $\triangle$-state on Q^k. Further, $Q^k \times \{0,1\}$, where $\{0,1\}$ is viewed as a QLS, possesses exactly one $\triangle$-state—it suffices to set $s(a,0) = 0$ and $s(a,1) = 1$, $a \in Q^k$. Write $V = Q^k \times \{0,1\}$ and consider $P = V^{n+1}$, where $n+1$ is the number of the extreme points of K. Then P has precisely $n+1$ pure $\triangle$-states and therefore $\mathcal{S}_\triangle(P) = K$. $\square$

This result (based on an advanced combinatorics of QLSs) is founded on the construction of Q with $\mathcal{S}_\triangle(Q) = \emptyset$ (see [33, 41] and [43]). The final result in this direction that is hoped for is a construction of QLS with a given state space and a given degree of non-compatibility but it seems to be a matter of a long research.

Chapter 5

The categorical equivalence of the XOR Abbott algebras with the quantum logics enriched with a symmetric difference.

The content of this chapter is formulated by the title above. It extends the investigation of quantum logics that have a symmetric difference (compare with **Chapter 4**). The chapter is written upon the accepted publication of the author of this thesis [5] in an IF journal.

Though there were a few clerical mistakes in the paper, these overlookings did not affect the validity of the results.

5.1 Basic notions of this chapter.

Considering the inference rules in generalized logics, J.C. Abbott (see [1, 2]) introduced an algebra that was found appropriate for an application in mathematical logic. His definition reads as follows.

5.1.1 Definition (Abbott algebra)

Let $(A, \cdot)$ be an algebra with a binary operation, $\cdot$. Then $(A, \cdot)$ is said to be an **Abbott algebra** if for any $a, b, c \in A$, the following identities hold:

1. $(ab)a = a$,
2. $(ab)b = (ba)a$,
3. $a\big((ba)c\big) = ac$.

Prior to proving a main result, let us recall the identities valid in the calculus of Abbott algebras ([1] and [2]).

5.1.2 Proposition (Abbott's lemma)

Let $(A, \ \cdot)$ be an Abbott algebra. Then the following statements hold true $(a, b, c \in A)$:

1. $aa = bb$; let us denote the element aa by the symbol 1,

2. $1a = a$,
3. $a1 = 1$,
4. $ab = ba \implies a = b$,
5. $a(ba) = 1$,
6. $ab = 1 \implies a(bc) = ac$,
7. $ab = 1 \implies (ba)(ac) = 1$.

A main result of J.C. Abbott was to show that these algebras allow for a partial ordering with maximal element 1 in which the upper segments are orthomodular. With the intention to make the Abbott algebras "nearly Boolean" (meaning to add a natural symmetric difference), we apply the following definition.

5.1.3 Definition (Abbott $\mathbb{XOR}$ algebra)

Let $(A, 0, \cdot, \triangle)$ be an algebra, where 0 is a nullary operation, $\triangle$ is a binary operation and $\cdot$ is a binary operation. We say that $(A, 0, \cdot, \triangle)$ is an **Abbott $\mathbb{XOR}$ algebra** if for any $a, b, c \in A$ we have the following identities:

1. $(ab)a = a$,
2. $(ab)b = (ba)a$,
3. $a((ba)c) = ac$,
4. $0a = bb$,
5. $(a\triangle b)\triangle c = a\triangle(b\triangle c)$,
6. $a\triangle bb = a0$,
7. $bb\triangle a = a0$,
8. $(a\triangle b)((ab)b) = aa$.

5.2 Results

We are ready to formulate the main result of this chapter.

5.2.1 Theorem (equivalence of lattice QLSs and Abbott $\mathbb{XOR}$ algebras)

The category of lattice QLSs is categorically equivalent to the category of the Abbott $\mathbb{XOR}$ algebras.

Proof. Let us denote by $\mathcal{D}$ the category of lattice QLSs and by $\mathcal{A}$ the category of the Abbott $\mathbb{XOR}$ algebras. Let $A \in \mathcal{A}$ and let us see how we can view A as an object of $\mathcal{D}$. Let us first endow A with a partial ordering. Let us introduce the partial ordering in A by requiring $a \leq b$ if $ab = 1$. Then $0 \leq a \leq 1$ for all $a \in A$ because $0a = 1$ and $a1 = 1$. Let us show that $\leq$ is a partial ordering with a least (resp. greatest) element 0 (resp. 1). Indeed, $a \leq a$ since $aa = 1$, and if $a \leq b$ and $b \leq a$, then $ab = 1 = ba$ and therefore $a = b$. Further, if $a \leq b$ and $b \leq c$ then $ab = 1$ and $bc = 1$. It follows from Proposition 5.1.2 7 that $(ba)(ac) = 1$. Then $bc \leq ac$ but $bc = 1$ and therefore $1 \leq ac$. So $ac = 1$ and therefore $a \leq c$.

Let us see that A is a lattice with respect to $\leq$. We claim that $a \vee b = (ab)b$. To see that, we have $a\big((ab)b\big) = a\big((ba)a\big) = 1$ and therefore $a \leq (ab)b$ which means that $a \leq a \vee b$. Analogously, $b \leq a \vee b$. Moreover, if $a \leq c$ and $b \leq c$ then $ac = 1$ and $bc = 1$. Considering $ac = 1$ (and correcting [1]), we infer that $(cb)(ab) = 1$. This implies that $\big((ab)b\big)\big((cb)b\big) = 1$ and therefore $(ab)b \leq (cb)b$. So $a \vee b = (ab)b \leq (cb)b = (bc)c = 1c = c$ and hence $a \vee b \leq c$. This shows that A is a lattice.

With the intention to restructure A to make it an orthocomplemented lattice, let us set $a^{\perp} = a0$. We are to verify that $(a^{\perp})^{\perp} = a, a \leq b \implies b \leq a$ and that both equalities $a \vee a^{\perp} = 1, a \wedge a^{\perp} = 0$ are valid. Obviously, $(a^{\perp})^{\perp} = (a0)0 = a \vee 0 = a$. Further, if $a \leq b$ then $b^{\perp} = b0 \leq a0 = a^{\perp}$. Let us also see that $a \vee a^{\perp} = 1$ and $a \wedge a^{\perp} = 0$. We have $a \vee a^{\perp} = a^{\perp} \vee a = (a0)a)a = aa = 1$. As regards the condition on the infimum, one uses the de Morgan law to obtain

$$a \wedge a^{\perp} = a^{\perp} \wedge a = (a \vee a^{\perp})^{\perp} = (a \vee a^{\perp})0 = \Big((a(a0))(a0)\Big)0 = (a \vee a^{\perp})0 = 10 = 0.$$

It remains to verify the orthomodular law. Suppose that $a \leq b$. So we have $b0 \leq a0$ and we see by Definition 5.1.1 3 that

$$b = b \vee 0 = (b0)0 = (b0)\big((a0)0\big) = (b0)(a \vee 0) = (b0)a.$$

Since $a(ba) = 1$ by Proposition 5.1.2 5, then $a \leq ba$ and therefore we have $ba = \big((ba)0\big)a$. In order to verify the orthomodular law, we are to prove that $b = a \vee (a^{\perp} \wedge b)$. Let us consider the right-hand side of this equality. We obtain

$$\begin{aligned} a \vee (a^{\perp} \wedge b) &= a \vee ((a0) \wedge b) = a \vee \Big((a \vee (b0))0\Big) = a \vee \Big(((b0) \vee a)0\Big) \\ &= a \vee \Big(((b0)a)a\Big)0 = a \vee ((ba)0) = \Big(((ba)0)a\Big)a = (ba)a = b \vee a = b. \end{aligned}$$

Finally, let us check the conditions of the operation $\triangle$. The operation $\triangle$ is associative by definition. Further, $a\triangle(bb) = a\triangle 1 = (a0) = a^{\perp}$ and $(bb)\triangle a = (1\triangle a) = a0 = a^{\perp}$. To end up the verification, we obtain $aa = 1 = (a\triangle b)\big((ab)b\big)$. This means that $a\triangle b \leq (ab)b$. But $(ab)b = a \vee b$ Therefore we see that $a\triangle b \leq a \vee b$.

In the considerations above, we have defined an assignment $F : \mathcal{A} \to \ \mathcal{D}$ as a potential functor on the objects of $\mathcal{A}$ (the assignment F preserves the underlying set). Let us see that this assignment is functorial. Suppose that $f : A \to \ B$

is a morphism in $\mathcal{A}$. So $f(ab) = f(a)f(b)$, $f(0) = 0$ and $f(a \triangle b) = f(a) \triangle f(b)$. We have to check that f is a morphism in $\mathcal{D}$. For that, suppose that $a \vee b = c$ in $\mathcal{A}$. So it means that $c = (ab)b$. Thus $f(c) = (f(a)f(b))f(b)$. This implies that $f(c) = f(a) \vee f(b)$. Further, we have to check that $f(a^\perp) = f(a)^\perp$. But $a^\perp = (a0)$ and therefore $f(a^\perp) = f(a0) = f(a)f(0) = f(a)0$, and hence $f(a)^\perp = f(a^\perp)$. Thus we have checked that F is indeed a functor from $\mathcal{A}$ to $\mathcal{D}$.

We shall now construct a functor, G, $G : \mathcal{D} \to \mathcal{A}$. Let us take $D \in \mathcal{D}$. Then $G(D)$ remains with the same underlying set. We define the object $G(D)$ as follows: If $a \in G(D)$ and $b \in G(D)$, then $ab = (a \vee b)^\perp \vee b$, and $a \triangle b$ is copied from D. Let us check that $G(D)$ sends a morphism of $\mathcal{D}$ into a morphism of $\mathcal{A}$. We first have to check that the axioms of $G(D)$ make it an Abbott $\mathbb{XOR}$ algebra.

1. $(ab)a = a$; we have

$$\Big(((a \vee b)^\perp \vee b) \vee a\Big)^\perp \vee a = ((a \vee b) \wedge b^\perp \wedge a^\perp) \vee a$$
$$= ((a \vee b) \wedge (a \vee b)^\perp) \vee a = 0 \vee a = a.$$

2. $(ab)b = (ba)a$; we have $(ab)b = ((a \vee b)^\perp \vee b)^\perp \vee b = ((a \vee b) \wedge b^\perp) \vee b$. Since the triple $b, b^\perp, a \vee b$ is compatible in D, we can use distributivity (see e.g. [25]). Hence, the latter formula gives us $(a \vee b \vee b) \wedge (b^\perp \vee b) = a \vee b$. Analogously, $((b \vee a)^\perp \vee a)^\perp \vee a = b \vee a$ and so the equality is valid.

3. $a\big((ba)c\big) = ac$. Prior to verifying this condition, let us make a preliminary observation. Consider elements x and $y^\perp \wedge x$. Then $x \geq y^\perp \wedge x$. So the orthomodular law gives us

$$x = (y^\perp \wedge x) \vee ((y^\perp \wedge x)^\perp \wedge x) = (y^\perp \wedge x) \vee ((y \vee x^\perp) \wedge x).$$

Let us verify the axiom proper. We have

$$a\big((ba)c\big) = a\Big(((b \vee a)^\perp \vee a)c\Big) = a\Big(((b^\perp \wedge a^\perp) \vee a)c\Big).$$

For the sake of transparency, let us write $y = b^\perp \wedge a^\perp$.

Hence we have $a\big((ba)c\big) = a((y \vee a)c) = a\Big(((y \vee a) \vee c)^\perp \vee c\Big)$

$$= \Big(a \vee \Big((y \vee a)^\perp \wedge c^\perp\Big) \vee c\Big)^\perp \vee \Big(\Big((y \vee a)^\perp \wedge c^\perp\Big) \vee c\Big)$$
$$= \Big(a^\perp \wedge \Big(((y \vee a)^\perp \wedge c^\perp) \vee c\Big)^\perp\Big) \vee \Big(((y \vee a)^\perp \wedge c^\perp) \vee c\Big)$$
$$= \Big((a^\perp \wedge c^\perp) \wedge ((y \vee a)^\perp \wedge c^\perp)^\perp\Big) \vee ((y^\perp \wedge a^\perp \wedge c^\perp) \vee c)$$
$$= \Big((a^\perp \wedge c^\perp) \wedge (y \vee a \vee c) \vee (y^\perp \wedge a^\perp \wedge c^\perp)\Big) \vee c.$$

So we have

$$a((ba)c) = \Big((a^\perp \wedge c^\perp) \wedge \big(((b^\perp \wedge a^\perp) \vee a) \vee c)\big)\Big) \vee \Big(((b^\perp \wedge a^\perp)^\perp \wedge a^\perp) \wedge c^\perp\Big) \vee c.$$

Let us set

$$u = (a^\perp \wedge c^\perp) \wedge ((b^\perp \wedge a^\perp) \vee a \vee c)) \vee ((b^\perp \wedge a^\perp)^\perp \wedge a^\perp \wedge c^\perp).$$

Writing $x = a^\perp \wedge c^\perp$ and $y = b^\perp \wedge a^\perp$, let us use the orthomodular law formula derived at the beginning of this proof.

We obtain

$$a^\perp \wedge c^\perp = x = (y^\perp \vee x) \vee ((y^\perp \wedge x)^\perp \wedge x) = u.$$

As a result, we have $a((ba)c) = (a^\perp \wedge c^\perp) \vee c = ac$, which we wanted to prove.

4. $0a = bb$; we have

$$0a = (0 \vee a)^\perp \vee a = (1 \wedge a^\perp) \vee a = a^\perp \vee a = 1 = (1 \wedge b^\perp) \vee b = bb.$$

5. $(a \triangle b) \triangle c = a \triangle (b \triangle c)$, the operation $\triangle$ is associative in A as well as in the corresponding orthomodular lattice.

6. $a \triangle bb = a0$; we have $a0 = (a \vee 0)^\perp \vee 0 = a^\perp = a \triangle 1 = a \triangle bb$.

7. $bb \triangle a = a0$; we have $a0 = 1 \triangle a = bb \triangle a$.

8. $(a \triangle b)((ab)b) = aa$; we have $(ab)b = a \vee b$ and therefore $(a \triangle b) \leq a \vee b$. And, also, $(a \triangle b)(a \vee b) = 1 = aa$ and we have derived the required equality.

Finally, we have to show that G is a functor. Indeed, suppose that $f : L_1 \to L_2$ is a morphism in $\mathcal{D}$. We have to check that it is a morphism in $\mathcal{A}$. It means that we have to verify that $f(ab) = f(a)f(b)$. So we have to see that $f((a \vee b)^\perp \vee b) = \big(f(a) \vee f(b)\big)^\perp \vee f(b)$. But this is obvious since f preserves the suprema and the operation $^\perp$. □

This result links the structure of an Abbott algebra (an object of the mathematical logic) to a lattice QLS (an object of mathematical physics).

The proof technique uses the Abbott basic lemma ([1] corrected at a place). The explicit contribution of the author is finding the axioms to obtain the equivalence, the verification of the equivalence of the respective morphisms and the proof technique of the way "from QLSs to Abbott $\mathbb{XOR}$ algebras". In the latter reasoning, one uses the operation $a \cdot b = (a \vee b)^\perp \vee b$ that is bringing about a specific calculus in orthomodular lattices. Further, the notion of compatibility is introduced in Abbott algebras that implies the following characterization of Boolean subalgebras in terms of the operation $\triangle$.

5.2.2 Theorem (compatibility in the Abbott $\mathbb{XOR}$ algebras)

Let A be an Abbott $\mathbb{XOR}$ algebra and let $a, b \in A$. Then a, b are compatible in A if either of the following two conditions is satisfied:

1. $a = \Bigg(\bigg(\Big(\big(((a0)(b0))(b0)\big)0\Big)\Big(\big(((a0)b)b\big)0\Big)\bigg)\bigg(\Big(((a0)b)b\Big)0\bigg)$,

2. $a\triangle b = \Bigg(\bigg(\Big(\big(((a0)b)b\big)0\Big)\Big(\big((a(b0))(b0)\big)0\Big)\bigg)\bigg(\Big((a(b0))(b0)\Big)0\bigg)$.

A corollary: A is a **Boolean algebra** if and only if either of the above equalities is valid for any $a, b \in A$.

Proof. If we rewrite the equality 1 in the corresponding lattice QLS, we obtain

$$a = (a^\perp \vee b^\perp)^\perp \vee (a^\perp \vee b)^\perp = (a \wedge b) \vee (a \wedge b^\perp).$$

Analogously, we can derive that $a\triangle b = (a\wedge b^\perp)\vee(b\wedge a^\perp)$. Either of the above formulas for a and $a\triangle b$ guarantees that a, b are compatible (see e.g. [27]). $\square$

In the further considerations we introduce the notion of a $\triangle$-state in an Abbott algebra. This makes Abbott algebras closer to the realm of quantum theories. Using then the properties of $\triangle$-states, we obtain some other structural properties of Abbott algebras (the set representation of Abbott algebras, a characterization of free Abbott algebras over 2 generators, etc.).

5.2.3 Definition ($\triangle$-state in the Abbott XOR algebra)

Let A be an Abbott $\mathbb{XOR}$ algebra. Let $s : A \to [0, 1]$ be a mapping that satisfies the following conditions ($a, b, c \in A$):

1. $s(aa) = 1$,

2. if $a(b0) = bb$, then $s((ab)b) = s(a) + s(b)$,

3. $s(a\triangle b) \leq s(a) + s(b)$.

Then s is said to be a **$\triangle$-state in** A.

5.2.4 Theorem (equivalence of $\triangle$-states)

Let $\mathcal{A}$ be equivalent to $\mathcal{D}$ in the sense of Theorem 5.2.1. If $A \in \mathcal{A}$ and s is a $\triangle$-state of A then s can be viewed as a QL $\triangle$-state of $F(A)$, and vice versa.

Proof. It is easy to see that the $\triangle$-state space of A is isomorphic (via the isomorphism of Theorem 5.2.1) with the $\triangle$-state space of $D = F(A)$. $\square$

Let us summarize main results of this chapter. The category $\mathcal{A}$ of the Abbott $\mathbb{XOR}$ algebras is equivalent to the category $\mathcal{D}$ of the QLSs, and the respective $\triangle$-state spaces are isomorphic. So the knowledge we have acquired on $\mathcal{D}$ and on its $\triangle$-state space can be translated into the corresponding category $\mathcal{A}$. For instance, since we know the characterization of the set-representable objects of $\mathcal{D}$, and these are precisely those that have an "abundance" of two-valued $\triangle$-states (see [27]), we easily derive the set-representability characterization of the Abbott $\mathbb{XOR}$ algebras. In a similar vein, we can find Abbott $\mathbb{XOR}$ algebras without any $\triangle$-state or with a precisely one $\triangle$-state (see [15] and [43]). Also, we find that the free Abbott $\mathbb{XOR}$ algebra over 2 generators contains precisely 128 elements and the free algebra over 3 generators is infinite (see [29]).

Chapter 6

Point-distinguishing quantum logics

In this chapter we shall exclusively deal with SR-QLs.

We ask whether each SR-QL allows for an isomorphic copy with the following property (the point-distinguishing representation): If $x, y \in S$ and $x \neq y$, then there is a set A, $A \in \mathbf{\Delta}$ such that $x \in A$ and $y \in S \setminus A$. We answer the question in the affirmative. This may simplify technical tasks in dealing with algebraic and state conditions of SR-QLs. We then see how the two-valued states may serve to enable us the construction of point-distinguishing representations. We consider the generalized form of the Stone representation technique borrowed from the theory of Boolean algebras. We obtain several point-distinguishing representations in this way, the extreme one guarantees all two-valued states being Dirac states.

This study is upon a paper submitted for publication coauthored by the author of this text [9].

6.1 Basic notions of this chapter

We intend to show that each SR-QL is isomorphic to the point-distinguishing SR-QL. We will employ the following definition.

6.1.1 Definition (point-distinguishing QL)

Let $(S, \mathbf{\Delta})$ be an SR-QL. Let us call $(S, \mathbf{\Delta})$ **point-distinguishing** provided for each couple of distinct elements $x, y \in S$ there is a set $A, A \in \mathbf{\Delta}$ such that $x \in A$ and $y \in S \setminus A = A^{\perp}$.

6.2 Results

6.2.1 Theorem (natural equivalence relation)

Let $(S, \mathbf{\Delta})$ be a SR-QL. Then the relation $\mathcal{R}$ on S defined in Definition 6.1.1 is an equivalence relation. Moreover, if $\{R_\alpha \mid \alpha \in I\}$ is the decomposition of S by the classes of the equivalence $\mathcal{R}$, then the following implication holds

true:
If $A \in \mathbf{\Delta}$ and R_α is a class of $\mathcal{R}$, then either $R_\alpha \subseteq A$ or $R_\alpha \subseteq A^\perp$.

Proof. The relation $\mathcal{R}$ is obviously reflexive and symmetric. As regards the transitivity, let us suppose that $x\mathcal{R}y$ and $y\mathcal{R}z$ $(x, y, z \in S)$. Then $x\mathcal{R}z$. Indeed, if there is a set $A \in \mathbf{\Delta}$ such that $x \in A$ and $z \in A^\perp$, then either $y \in A^\perp$ in which case $x(\text{non } \mathcal{R})y$, or $y \in A$ in which case $y(\text{non } \mathcal{R})z$. Further, if $A \in \mathbf{\Delta}$, then R_α cannot intersect both A and $A^\perp$ since R_α would not be a class of $\mathcal{R}$. □

In order to formulate our results, let us recall that a both injective and surjective SR-QL-morphism is said to be **an SR-QL-isomorphism** if f^{-1} is an SR-QL-morphism, too.

The following definition simplifies the formulation of our results:

6.2.2 Definition (natural point-distinguishing representation)

Let $(S, \mathbf{\Delta})$ be a SR-QL. Let $\mathcal{R}$ be the equivalence introduced in Theorem 6.2.1 and let $\{R_\alpha \mid \alpha \in I\}$ be the decomposition of P given by the classes of $\mathcal{R}$. Consider the following couple $(\tilde{S},\ \tilde{\mathbf{\Delta}})$, where $\tilde{S} = \{R_\alpha \mid \alpha \in I\}$ and $\tilde{\mathbf{\Delta}}$ is the following collection of subsets of $\tilde{S}$: $\tilde{A} \in \tilde{\mathbf{\Delta}}$ if there is a set $A \in \mathbf{\Delta}$ such that $\tilde{A} = \{R_\alpha \mid R_\alpha \subseteq A\}$. Let us call $(\tilde{S},\ \tilde{\mathbf{\Delta}})$ **the natural point-distinguishing representation** of $(S, \mathbf{\Delta})$.

The summary of that we have completed so far can be formulated in the following result.

6.2.3 Theorem (point distinguishing representation)

Let $(S, \mathbf{\Delta})$ be a SR-QL and let $(\tilde{S}, \tilde{\mathbf{\Delta}})$ be the natural point-distinguishing representation of $(S, \mathbf{\Delta})$.

1. If $f : \mathbf{\Delta} \to\ \tilde{\mathbf{\Delta}}$ assigns to any $A \in \mathbf{\Delta}$ the set $f(A) = \{R_\alpha \mid R_\alpha \subseteq A\} \in \tilde{\mathbf{\Delta}}$, then f is a SR-QL-isomorphism,
2. If $\mathbf{\Delta}$ is a lattice SR-QL, then $f : \mathbf{\Delta} \to\ \tilde{\mathbf{\Delta}}$ defined above is a lattice SR-QL-isomorphism,
3. If $(S, \mathbf{\Delta})$ is closed under the formation of symmetric difference, then so is $(\tilde{S}, \tilde{\mathbf{\Delta}})$ and both f and f^{-1} preserve the respective symmetric differences,
4. If $(S, \mathbf{\Delta})$ is a Boolean algebra then so is $(\tilde{S}, \tilde{\mathbf{\Delta}})$ and f is a Boolean isomorphism.

Proof. It easily follows from Theorem 6.2.1.

□

It could be observed that the natural point-distingishing representation of $(S, \mathbf{\Delta})$ can be considered "to live on a subset of S". Indeed, we can choose a point in each R_α (Axiom of Choice) and "copy" $\tilde{\mathbf{\Delta}}$ on the set of the points chosen. It is worthwhile realizing that the state-space of $(S, \mathbf{\Delta})$ allows us to construct several point-distinguishing representations. We use the appropriate modification of the Stone technique.

Prior to that let us recall that if $(S, \mathbf{\Delta})$ is a SR-QL, then we denote by $\mathcal{S}_2(S, \mathbf{\Delta})$ the set of all two-valued states on $(S, \mathbf{\Delta})$.

6.2.4 Definition (Dirac state)

A two-valued state s on $(S, \mathbf{\Delta})$ is said to be a **Dirac state** if there is a point $p \in S$ such that $s(A) = 1$ exactly when $p \in A$.

The following result generalizes the Boolean Stone representation theorem. We style it for our purpose.

6.2.5 Theorem (constructing point-distinguishing representation by the Stone technique)

Let $(S, \mathbf{\Delta})$ be a SR-QL. Let $\mathcal{P} \subseteq \mathcal{S}_2(S, \mathbf{\Delta})$ and let $\mathcal{P}$ have the following property:

If $A, B \in \mathbf{\Delta}$ and $A \nsubseteq B$, then there is $s \in \mathcal{P}$ with $s(A) = 1$ and $s(B) = 0$.
Let $\mathcal{U}$ be the collection of the subsets U of $\mathcal{P}$ determined as follows:

$U \in \mathcal{U}$ exactly when there is a set $V \in \mathbf{\Delta}$ such that $U = \{s \in \mathcal{P} \mid s(V) = 1\}$.
Then $(\mathcal{P}, \mathcal{U}) \in$ SR-QL and $(\mathcal{P}, \mathcal{U})$ is point-distinguishing. Moreover, $(\mathcal{P}, \mathcal{U})$ is SR-QL-isomorphic to $(S, \mathbf{\Delta})$. If $\mathcal{P}$ consists of all Dirac states, then $(\mathcal{P}, \mathcal{U})$ is SR-QL-isomorphic to the natural point-distinguishing representation $(\tilde{S}, \tilde{\mathbf{\Delta}})$ of $(S, \mathbf{\Delta})$. If $\mathcal{P} = \mathcal{S}_2(S, \mathbf{\Delta})$ then each two-valued state on $(\mathcal{P}, \mathcal{U})$ is a Dirac state.

Proof. If we define a mapping $e : \mathbf{\Delta} \to \mathcal{U}$ by setting
$e(D) = \{s \in \mathcal{P} \mid s(D) = 1\}, D \in \mathbf{\Delta}$, we can easily prove verbatim the Stone representation theorem that e is a SR-QL isomorphism (see also [40] for related considerations). Moreover, if $s_1 \neq s_2$ then there is a set A, $A \in \mathbf{\Delta}$ such that $s_1(A) = 1$ and $s_2(A) = 0$. If we take $U_1 = \{s \in \mathcal{P} \mid s(A) = 1\}$, then $U_1 \in \mathcal{U}$ and $s_1 \in U_1$ whereas $s_2 \notin U_1$. Further, suppose that $\mathcal{P} = \{s \in \mathcal{S}_2(S, \mathbf{\Delta}) \mid s$ is a Dirac state$\}$. Then if $s_1 \in \mathcal{P}$ and $s_2 \in \mathcal{P}$ with s_1 and s_2 given by $p_1 \in S$ and $p_2 \in S$, we see that $s_1 = s_2$ precisely when $p_1 \mathcal{R} p_2$ in the equivalence $\mathcal{R}$ defined in Definition 6.2.2. Hence in this case the SR-QL $(\mathcal{P}, \mathcal{U})$ is SR-QL-isomorphic to the natural point-distinguishing representation $(\tilde{S}, \tilde{\mathbf{\Delta}})$ of $(S, \mathbf{\Delta})$. Finally, suppose that $\mathcal{P} = \mathcal{S}_2(S, \mathbf{\Delta})$. Let t be a two-valued state on $(\mathcal{P}, \mathcal{U})$. Applying the SR-QL-isomorphism $e : \mathbf{\Delta} \to \mathcal{U}$, let us consider the two-valued state, s, on $(S, \mathbf{\Delta})$ such that $s = te$. So $s \in \mathcal{S}_2(S, \mathbf{\Delta})$ and we conclude that t is a Dirac state given by s. Indeed, suppose that $t(U) = 1$ for some $U \in \mathcal{U}$.

Write $U = \{t \mid t \in \mathcal{S}_2(S, \mathbf{\Delta})$ such that $t(S) = 1$ for some $T \in \mathbf{\Delta}\}$. Then $te(T) = 1$ and we conclude that t is a Dirac state given by s. The proof is complete. □

Let us shortly comment on the results obtained. First, for the property of having all two-valued states Dirac states one has to pay the price of having to lift up the cardinality of the underlying set. In principal, the cardinality of $\mathcal{S}_2(S, \mathbf{\Delta})$ might be $2^{2^{\mathrm{card} S}}$. In the case of S being finite, the set $\mathcal{S}_2(S, \mathbf{\Delta})$ is finite, too, and this can be used in the analysis of states of finite SR-QLs (see e.g. [4]).

Another question concerns the algebraic structure of those elements of SR-QL on which all two-valued states are Dirac states. This class is closed under finite products. On the other hand, this class is not closed under substructures. Indeed, if we e.g. take for $(S, \mathbf{\Delta})$ the SR-QL 6_{even} of all subsets of $\{1, 2, 3, 4, 5, 6\}$ with an even cardinality, then each two-valued state on 6_{even} is a Dirac state (see [10]). The SR-QL 4_{even} defined analogously on $\{1, 2, 3, 4\}$ can be viewed as a substructure of 6_{even} (the atoms $\{1, 2\}, \{3, 4\}$ and $\{1, 3\}$ could be mapped to $\{1, 2\}$, $\{3, 4, 5, 6\}$ and $\{1, 3\}$) but 4_{even} has a two-valued state that is not a Dirac state (it suffices to set $s(\{1, 2\}) = s(\{1, 3\}) = s(\{1, 4\}) = 0$).

In the last remark, let us observe that even for the elements of SR-QL that are closed under symmetric difference there can be established a link of Theorem 6.2.3 3 with a generalized Stone representation. This can be done in the analogy to the link of Theorem 6.2.3 1 to Theorem 6.2.5. It suffices to consider the two-valued $\triangle$-states instead of the mere two-valued states (see also [10]; the state is said to be a $\triangle$-state if $s(A \triangle B) \leq s(A) + s(B)$, $A, B \in \mathbf{\Delta}$). We thus among others obtain the class of the elements of SR-QL that are closed under symmetric difference isomorphic with the class of the point-distinguishing elements of SR-QL on which all two-valued $\triangle$-states are Dirac states.

Chapter 7

Conclusion

The thesis addresses some open questions of the quantum logic theory (the theory of orthomodular partially ordered sets). Obviously, the questions are motivated by quantum theory. It is supposed that quantum logics are associated with quantum experiments and, in turn, the theoretical results obtained may help in understanding the quantum experiments better.

The main results of the thesis are highlights of four research papers of the author - one paper as the only author and three with a co-author. Here is the review of the results contained in the thesis.

The first type of results concerns the generation in the lattice QLs by their subsets (in particular, the so-called local finiteness is studied - the property that finite subsets generate finite substructures). The consideration is motivated by the classical result of projective geometry - 3 vectors in the lattice quantum logic $L(\mathbb{R}^3)$ of projections in $\mathbb{R}^3$ may generate an infinite substructure [22]. The situation in the lattice QL is analysed and it is shown how infinite set-representable lattice QLs can be generated by 3 elements.

Chapters 3 and 4 are devoted to the QLs that allow for a natural kind of a $\mathbb{XOR}$ operation. The main results therein are as follows:
The absence of regular compatibility, the results on the extensions of states and $\mathbb{Z}_2$-states and the construction of a rather instructive example, from the angle of quantum theories, of a QL with a small state space and a big degree of non-compatibility. Chapter 4 studies the Abbott algebras enriched with a $\mathbb{XOR}$ operation. Several results on the intrinsic properties of these algebras are proved, for instance a characterization of Boolean subalgebras is found. The principal result obtained is that there is an equivalence (including the morphisms) of the Abbott $\mathbb{XOR}$ algebras with the lattice QLSs. This situates the Abbott algebras close to quantum theories.

In Chapter 5 one shows that set-representable QLs can be made isomorphic with the ones that distinguish points and make all two-valued states Dirac states.

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