Bachelor Thesis

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F3

Faculty of Electrical Engineering
Department of Cybernetics

The Interplay of Triangular Norms and Their Generators

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Specialisation: Artificial Intelligence and Computer Science
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BACHELOR‘S THESIS ASSIGNMENT

I. Personal and study details

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II. Bachelor’s thesis details

Bachelor’s thesis title in English:

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Vztah trojúhelníkových norem a jejich generátorů

Guidelines:

Associative binary operations can be often understood as addition or multiplication on a scale modified by a unary function (generator). This is the case of triangular norms, which interpret conjunctions in fuzzy logic [2]. Although the basic correspondence is known for long, we lack an intuitive relation between properties of triangular norms and their generators. Probably the only step towards this is described in [1], approximation of data by a triangular norm is in [3], [4]. The aim is an extension of our knowledge by illustrative rules of the interplay of triangular norms and their generators.

1. Summarize contemporary results about the relations between of triangular norms and their generators. Try to extend them or collect arguments why these attempts fail.
2. Demonstrate the effect of local properties and changes of generators on the shape of the corresponding triangular norms.
3. A generator is not uniquely determined by a triangular norm and can be chosen in infinitely many ways. Investigate which properties of generators are and which are not invariant to this choice.

Bibliography / sources:

[1] Navara, M., Petrík, M.: Generators of fuzzy logical operations. In: H.T. Nguyen, V. Kreinovich (eds.), Algebraic Techniques andnTheir Use in Describing and Processing Uncertainty, Studies in Computational Intelligence 878, Springer, 89–112, 2020. DOI 10.1007/978-3-030-38565-1_8

[2] Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms.Trends in Logic, vol. 8. Kluwer Academic Publishers, Dordrecht, Netherlands, 2000.

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[4] Popovič, A.: Approximating by Fuzzy Conjunctions. Bachelor Thesis, Czech Technical University in Prague, 2020.

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prof. Ing. Mirko Navara, DrSc. Supervisor’s signature	prof. Ing. Tomáš Svoboda, Ph.D. Head of department’s signature	prof. Mgr. Petr Páta, Ph.D. Dean’s signature

III. Assignment receipt

The student acknowledges that the bachelor's thesis is an individual work. The student must produce her thesis without the assistance of others, with the exception of provided consultations. Within the bachelor's thesis, the author must state the names of consultants and include a list of references.

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Date of assignment receipt

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I would like to thank prof. Mirko Navara for his patience, time, and invaluable insights. My family for supporting me to the best of their abilities throughout my studies. My friends for creating an excellent environment at the university and motivating me to learn more and more each day. Jiří Kropáč, for being there for me when I needed it.

Declaration

I declare that the presented work was developed independently and that I have listed all sources of information used within it in accordance with the methodical instructions for observing the ethical principles in the preparation of university theses.

Prague, 24th May 2023

Abstract

You might have already heard of the sorites paradox. We know that a million grains of sand is a heap of sand. We also know that if we take away one grain from a heap, it is still a heap.
However, if we apply mathematical induction to those two statements enough times, there will only remain one grain of sand and logically, we should say that this grain is also a heap.

Unlike classical logic, where the only truth values are 0 and 1, fuzzy logic's truth values are in interval $[0, 1]$, which makes them infinitely many. This allows statements to be partially true and resolves the sorites paradox. However, it brings up a question: if both a and b are partially true, how much true will be their conjunction? That is why we need fuzzy conjunctions, in other words, triangular norms or t-norms.

Many research papers were written on the topic of triangular norms, which can be constructed through generators. Nevertheless, it is still not clear how a little change in one affects the other.

We study derivatives of both to clear this up. We introduce the notion of a balanced generator that, if defined, is unique to a given strict t-norm. Then, we try to widen a discovery that links a t-norm to its multiplicative generator to other t-norms.

Afterward, we inspect the interplay of the derivatives of fuzzy conjunctions and their generators at the edges of the domain. Moreover, we provide some visual examples of this interplay and of some other interesting t-norms.

In the end, we look at the interrelationship between the diagonal of a strict t-norm and its multiplicative generator.

Keywords: fuzzy conjunction, triangular norm, strict fuzzy conjunction, additive generator, multiplicative generator, balanced generator, diagonal

Supervisor: prof. Ing. Mirko Navara, DrSc.

Abstrakt

Možná už jste někdy slyšeli o paradoxu hromady. Víme, že milion zrnek písku je hromada písku. Také víme, že když z hromady odeberme jedno zrnko, pořád to je hromada. Když ale na tyto dva výroky dostatečněkrát použijeme matematickou indukci, zbude nám už jen jedno zrnko písku a z pohledu logiky bychom měli říct, že toto zrnko je také hromada.

Na rozdíl od klasické logiky, ve které jsou pouze pravdivostní hodnoty 0 a 1, pravdivostní hodnoty fuzzy logiky jsou v intervalu $[0, 1]$, takže jich je nekonečně mnoho. Díky tomu mohou být výroky částečně pravdivé a paradox hromady je tím vyřešený. Nicméně se nabízí otázka: pokud jsou a i b částečně pravdivé, jak moc pravdivá bude jejich konjunkce? Proto potřebujeme fuzzy konjunkce, jinými slovy trojúhelníkové normy či t-normy.

Na téma t-norem, které mohou být vytvořeny skrze generátory, už bylo napsáno hodně článků. Pořád však není jasně známo, jak malá změna generátoru ovlivní t-normu a naopak.

Abychom to osvětlili, zabýváme se derivacemi generátorů i t-norem. Představujeme pojem vyvážený generátor, který, pokud je definovaný, je pro danou striktní t-normu jedinečný. Poté se snažíme rozšířit objev, který spojuje t-normu s jejím multiplikativním generátorem pro další t-normy.

Následně zkoumáme vztah derivací fuzzy konjunkcí a jejich generátorů na okrajích definičního oboru. Navíc zajišťujeme vizuální ukázky tohoto vztahu a i některých jiných zajímavých t-norem.

Nakonec se podíváme na vztah mezi diagonálou striktní t-normy a jejím multiplikativním generátorem.

Klíčová slova: fuzzy konjunkce, trojúhelníková norma, striktní fuzzy konjunkce, aditivní generátor, multiplikativní generátor, vyvážený generátor, diagonála

Překlad názvu: Vztah trojúhelníkových norem a jejich generátorů

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Chapter 1

Introduction

What drew me to this topic is undoubtedly fuzzy logic. In computer science, we often classify data. When we want the computer to help us to solve a task, we need to lay out the situation at hand first. Usually, we pretend that the world is just black and white. Is this e-mail spam or not, is this stuffed animal an animal or not? Sometimes we do not even have the answer ourselves. We think about which classification could cause more damage and then decide that the e-mail probably is not spam because we fear the repercussions of not delivering an important e-mail. Fuzzy logic allows us to be precise and say that the statement "This teddy bear is an animal" is partially true. Fuzzy control is a practical use of fuzzy logic in computers. It allows us to introduce nuanced truth values in computer programs.

The goal of this thesis is to study the relationship between fuzzy conjunction, which is a function of two inputs, and its generator. Although a generator has only one input, it fully determines the fuzzy conjunction. If you still hesitate about how important this topic is, let me emphasize that Hájek proved that a triangular norm is enough to determine all the other fuzzy operations in an instance of fuzzy logic [1].

There have been many attempts to approximate a fuzzy conjunction from data. A fuzzy conjunction can be approximated using the spline function method to model its corresponding additive generator [2]. Another work focused on the least squares method [3]. In this paper, we will focus on the derivatives of a t-norm and its generator and their interplay. We hope it might encourage someone to study Hermite interpolation in relation to fuzzy conjunctions.

Let us summarize the structure of this thesis. First of all, we will clear up the basic terms and definitions. Then we will introduce the notion of balanced generators with many examples. Afterward, we will take a closer look at a formula that allows us to find a multiplicative generator for some known triangular norms [4]. We will attempt to expand on this conclusion. In the following two chapters, we will explore how the derivative at 1 or at 0 of a multiplicative generator, the corresponding additive generator, and the triangular norm relate to each other. We will also comment on the effect of local changes in the derivative of the multiplicative generator on the generated t-norm. Then we will explore the shape of a t-norm with a derivative equal to 0 or ∞ in its domain. Finally, we will highlight some interesting facts relating to the diagonal of t-norms in the last chapter.

Chapter 2

Definitions and basic notions

In this chapter, we offer an overview of generally known definitions related to the subject of this thesis. We not only define the terms but also explain a little about how they relate to each other and to fuzzy logic. It should be enough for a reader, who is unfamiliar with this field, to gain the understanding needed to understand the next parts of the thesis.

2.1 Fuzzy logic

In mathematics, there is a field called multi-valued logic. It is very similar to propositional logic with the exception of having more than two truth values. It is comprised of finite-valued logics, such as the Łukasiewicz Ł3 or Kleene logic, and infinite-valued logics with examples of fuzzy logic and probability logic.

Fuzzy logic has only been founded in the 20th century, so it is to an extent an emerging field of study. There are infinitely many truth values and each fuzzy set has a membership function χ, which tells us to what extent a point belongs to the fuzzy set.

Let us say that a young couple is looking at buying a flat and their deciding factors are price and size. They usually need to input discrete boundaries into the website and they either choose to be very lenient and get a multitude of offers that do not interest them, or they miss out on a great deal because they cut it off [5]. With fuzzy logic, it is easy for computers to qualify how intriguing an option is to you. You just need to create function χ_1 that specifies how satisfied you are with which price and χ_2 specifies satisfaction with size. Then you choose a fuzzy conjunction (or, more likely, the vendor chooses one that he knows works well for his clients) and you get a great option of compromise between the price and the size right away!

2.2 Triangular norms

Fuzzy conjunctions determine to which degree is a and b true in fuzzy logic. It is obvious that in fuzzy logic, we cannot define fuzzy conjunction by a table of truth values, which would usually be the case. Instead, we defined them by a few axioms and we allow the user to choose his or her own particular fuzzy conjunction.

Triangular norms, otherwise known as t-norms, have originated in statistical metric spaces. They were first mentioned by Karl Menger in 1942 [6]. Over the years they found their place in multiple other areas of mathematics. With the rise of machine learning, using triangular norms and by extent fuzzy logic in deep neural networks is also considered [7].

Definition 2.2.1. *Triangular norms* are binary functions $T\colon [0,1]^2 \to [0,1]$ that for arguments a and b satisfy four axioms - commutativity, associativity, monotonicity, and boundary condition $T(a,1) = a$.

Properties derived directly from those axioms include the fact that $T(0,x) = T(x,0) = 0$ and $T(1,x) = x$ [8].

There have been other ways than Definition 2.2.1 to define continuous fuzzy conjunctions [9]. In this thesis, we use the terms triangular norm and fuzzy conjunction synonymously.

Definition 2.2.2. Let us call a continuous fuzzy conjunction T *Archimedean* if $\forall x \in]0,1[\colon T(x,x) < x$.

Definition 2.2.3. An Archimedean fuzzy conjunction T is either *strict* if

$$\forall a \in]0,1]\, \forall b,c \in [0,1] : b < c \implies T(a,b) < T(a,c)$$

or *nilpotent* otherwise [5].

In this thesis, unless stated otherwise, we will deal with strict t-norms.

2.3 Generators

There are many ways to construct a t-norm. One of those is by a generator. We take a previously known function that fits the definition of a generator and use it in a specific binary function. There are two types of generators - multiplicative generators and additive generators.

Definition 2.3.1. A *multiplicative generator* is an increasing bijection $\theta : [0,1] \to [0,1]$ [8].

Notice that this implies that $\theta(0) = 0$ and $\theta(1) = 1$. Then a strict t-norm is constructed using an instance of a multiplicative generator as

$$T(a,b) = \theta^{-1}\left(\theta\left(a\right) \cdot \theta\left(b\right)\right). \tag{2.1}$$

Definition 2.3.2. An *additive generator* is a strictly decreasing bijection $t : [0,1] \to [0,\infty]$ [5].

Notice that this implies $t(1) = 0$. Then a strict t-norm is constructed using an additive generator t as

$$T\left(a,b\right) = t^{-1}\left(t\left(a\right) + t\left(b\right)\right). \tag{2.2}$$

More generally, a fuzzy conjunction T_1 is constructed using another already known fuzzy conjunction T_2 as $T_1(a,b) = \theta^{-1}\left(T_2\left(\theta\left(a\right), \theta\left(b\right)\right)\right)$. However, the world of fuzzy conjunctions is split into classes of elements. If you take the Łukasiewicz conjunction $T_Ł(a,b) = \max(0,\, a+b-1)$, you can only create another nilpotent conjunction. If you take product conjunction $T_P(a,b) = a \cdot b$, you can only create other strict conjunction, etc. [5]. Since T_P is the best-known of strict t-norms and we will usually compare other t-norms to it, we displayed it in Figure 2.1.

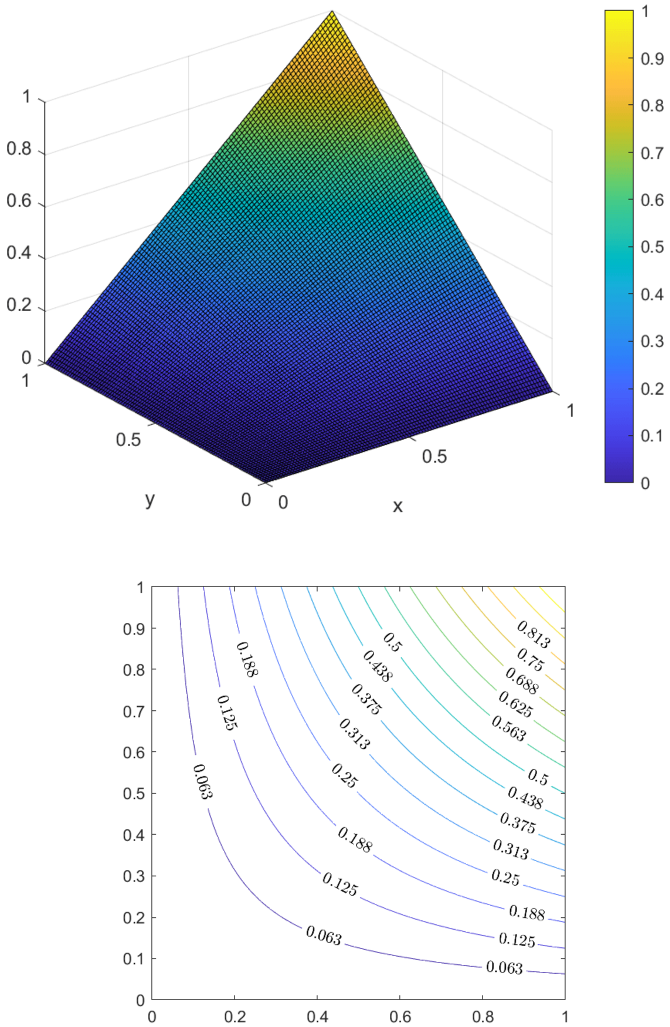


Figure 2.1: Product t-norm T_P

Multiplicative generators are determined up to a positive finite exponent. In other words, two multiplicative generators θ_1, θ_2 generate the same T if and only if there exists an $r \in]0, \infty[$ such that [4]

$$\theta_1(x) = \theta_2^r(x)\,. \tag{2.3}$$

Additive generators are determined up to a positive finite multiple. In other words, t-norms T_1 and T_2 are equal if and only if they are generated by continuous additive generators t_1, t_2 of the form [10]

$$t_1(x) = c \cdot t_2(x) \text{ for some c} \in]0, \infty[\,. \tag{2.4}$$

We can easily switch between one type of generator and the other. If we already know the multiplicative generator θ, we define the corresponding additive generator

$$t(x) = -\ln \theta(x). \tag{2.5}$$

On the other hand, if we know t, we can say

$$\theta(x) = \mathrm{e}^{-t(x)}\,. \tag{2.6}$$

Every strict t-norm has a multiplicative and an additive generator that can generate it. However, not all t-norms can be constructed through a generator. For example the standard conjunction $T_S(a,b) = \min(a,b)$ and the drastic conjunction $T_D(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$
do not have any generator [5].

Chapter 3

Balanced generators

For continuous Archimedean triangular norm T, there exists a notion of normed additive generator $[0,1] \to [0,1]$. It is uniquely determined by $t(0) = 1$ and we can find it from other additive generators of T via formula $t(x) := \frac{t(x)}{t(0)}$ [11]. This is only possible for nilpotent t-norms because only those have $t(0)$ in the interval $]0, \infty[$. In this chapter, we introduce a notion of balanced generators for strict t-norms T. If they exist for a T, then they are also uniquely determined.

3.1 Characterization

Definition 3.1.1. Let us call a multiplicative generator θ_* of a strict t-norm T *balanced* if it has a nonzero finite right derivative at 0.

According to the definition of a derivative, this means

$$\theta_*'(0) = \lim_{x \to 0+} \frac{\theta_*(x)}{x} \text{ is in the interval }]0, \infty[\,,$$

or, equivalently,

$$\lim_{x \to 0+} \frac{x \cdot \theta_*'(0)}{\theta_*(x)} = 1\,. \tag{3.1}$$

We reserve the notation θ_* for balanced generators. Not all strict t-norms have balanced generators. We know that a multiplicative generator is strictly increasing, so it is impossible for its derivative to be negative. However, the derivative need not exist, it can be 0 or ∞ for all generators. Specific examples will be discussed in section 3.4.

3.2 Methods of finding balanced generators

Proposition 1. *Suppose that a t-norm T has a balanced multiplicative generator θ_*. We can find it if we know either a t-norm T, or any generator of T.*

The first method is as follows. If $\theta'(0) \in]0, \infty[$, then we can find θ_* as [4]

$$\theta(b) = \lim_{a \to 0+} \frac{T(a,b)}{a} = \theta_*(b) \text{ for all } b \in [0,1]. \tag{3.2}$$

What was not evident at the time is that this method gives us not just any generator but a balanced one. It is true because (3.2) works if and only if $\theta'(0) \in]0, \infty[$, which is also the definition of balanced generator.

Let us present the second method of finding a balanced generator from any known generator. We know from Definition 2.3 that any other multiplicative generator θ of T is of the form

$$\theta(x) = \theta_*^r(x) \text{ for some } r \in]0, \infty[\,. \tag{3.3}$$

We assume that θ and r are as in (3.3) and that x is in an interval $]0, \epsilon[$ in which the derivative θ' is defined (for some small ϵ, if such an ϵ exists). Further, we assume that the derivative θ' is continuous at 0. Then we can write the derivative of expression (3.3) as follows

$$\theta'(x) = r\,\theta_*^{r-1}(x)\,\theta_*'(x)\,,$$

and from there we can express the derivative θ_*' as

$$\theta_*'(x) = \frac{\theta'(x)}{r\,\theta_*^{r-1}(x)} = \frac{\theta'(x)\,\theta_*(x)}{r\,\theta_*^{r}(x)} = \frac{\theta'(x)\,\theta_*(x)}{r\,\theta(x)}\,.$$

We know that this expression has a finite limit at 0,

$$\theta_*'(0) = \lim_{x\to 0+} \theta_*'(x) = \lim_{x\to 0+} \frac{\theta'(x)\,\theta_*(x)}{r\,\theta(x)}\,.$$

The rightmost limit can be expanded by a unit limit (3.1):

$$\begin{aligned}\theta_*'(0) &= \lim_{x\to 0+} \frac{\theta'(x)\,\theta_*(x)}{r\,\theta(x)} \cdot \underbrace{\frac{x\,\theta_*'(0)}{\theta_*(x)}}_{\to 1} \\ &= \lim_{x\to 0+} \frac{x\,\theta'(x)\,\theta_*'(0)}{r\,\theta(x)}\,.\end{aligned}$$

We conclude that

$$r = \lim_{x\to 0+} \frac{x\,\theta'(x)}{\theta(x)} \tag{3.4}$$

and the balanced generator of T (if it exists) can be found from $\theta(x)$ as

$$\theta_*(x) = \theta^{1/r}(x) \text{ for all } x \in [0, 1]. \tag{3.5}$$

As a consequence, every t-norm has at most one balanced multiplicative generator.

Remark 1. We do not need to check all the conditions. It is possible to start by calculating r. If it leads to $r = 0$ or $r = \infty$, the generated t-norm does not have a balanced generator. Otherwise, we can construct the balanced generator as previously stated and if the generated t-norm is the same, we have found the balanced generator.

3.3 T-norms with balanced generators

This section will show some examples of t-norms with balanced generators. In the first example, we will go through the process very slowly to demonstrate how to find the balanced generator from its non-balanced counterparts.

Example 1. Let us take a continuous multiplicative generator $\theta(x) = \dfrac{\cos\left(\frac{\pi x}{8}\right) - 1}{\cos\left(\frac{\pi}{8}\right) - 1}$ with derivative $\theta'(x) = -\dfrac{\pi \sin\left(\frac{\pi x}{8}\right)}{8\left(\cos\left(\frac{\pi}{8}\right) - 1\right)}$. When we take $\theta'(0)$ it is equal to 0, so this generator is not a balanced one according to Definition 3.1.1.

We will now use equation (3.4) to find the parameter r that will help us to find the balanced generator, [1]

$$
\begin{aligned}
r &= \lim_{x\to 0+} \frac{x\,\theta'(x)}{\theta(x)} = \lim_{x\to 0+} -\frac{x\,\pi \sin\left(\frac{\pi x}{8}\right)}{8\left(\cos\left(\frac{\pi x}{8}\right) - 1\right)} \\
&\overset{\text{LH}}{=} \lim_{x\to 0+} -\frac{\pi \sin\left(\dfrac{\pi x}{8}\right) + \dfrac{\pi^2 x \cos\left(\frac{\pi x}{8}\right)}{8}}{-\pi \sin\left(\dfrac{\pi x}{8}\right)} = \lim_{x\to 0+} 1 + \frac{\pi^2 x \cos\left(\frac{\pi x}{8}\right)}{8\,\pi \sin\left(\dfrac{\pi x}{8}\right)} \\
&\overset{\text{LH}}{=} 1 + \lim_{x\to 0+} \frac{\pi^2 \cos\left(\dfrac{\pi x}{8}\right) - \dfrac{\pi^3 x \sin\left(\frac{\pi x}{8}\right)}{8}}{\pi^2 \cos\left(\dfrac{\pi x}{8}\right)} = 2.
\end{aligned}
$$

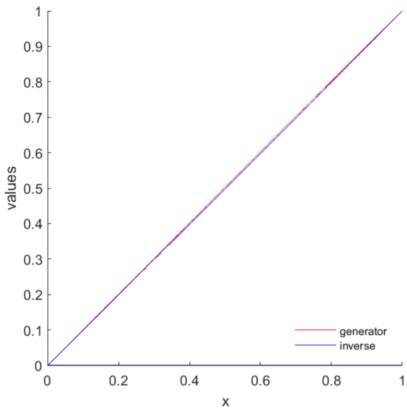


Figure 3.1: The balanced generator $\sqrt{\dfrac{\cos\left(\frac{\pi x}{8}\right) - 1}{\cos\left(\frac{\pi}{8}\right) - 1}}$ from Example 1

According to (3.5), the balanced generator is $\theta_*(x) = \sqrt{\dfrac{\cos\left(\frac{\pi x}{8}\right) - 1}{\cos\left(\frac{\pi}{8}\right) - 1}}$ and we can see it in Figure 3.1. Its inverse function is $\theta_*^{-1}(x) = \dfrac{8 \arccos\left(x^2\ \left(\cos\left(\frac{\pi}{8}\right) - 1\right) + 1\right)}{\pi}$

[1] We will use the symbol $\overset{\text{LH}}{=}$ to indicate the use of L'Hôpital's Rule.

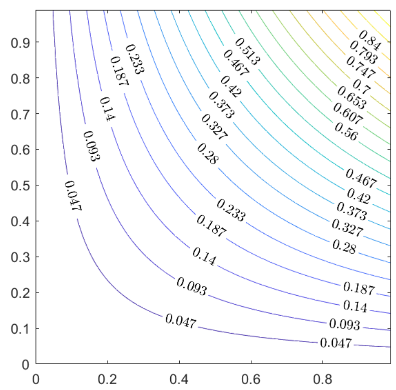


Figure 3.2: The t-norm (3.6) from Example 1

and the formula of the t-norm that they generate is

$$T(a,b) = \frac{8\arccos\left(\frac{\left(\cos\left(\frac{\pi a}{8}\right)-1\right)\cdot\left(\cos\left(\frac{\pi b}{8}\right)-1\right)}{\cos\left(\frac{\pi}{8}\right)-1}\left(\cos\left(\frac{\pi}{8}\right)-1\right)+1\right)}{\pi}$$
$$= \frac{8\arccos\left(\left(\cos\left(\frac{\pi a}{8}\right)-1\right)\left(\cos\left(\frac{\pi b}{8}\right)-1\right)+1\right)}{\pi}, \quad (3.6)$$

as shown in Figure 3.2.

Example 2. Now take a continuous multiplicative generator $\theta(x) = 2^x - 1$ shown in Figure 3.3. This generator is balanced according to Definition 3.1.1 because $\theta_*'(0) = \ln 2 \cdot 2^0 = \ln 2 \in \ ]0, \infty[$.

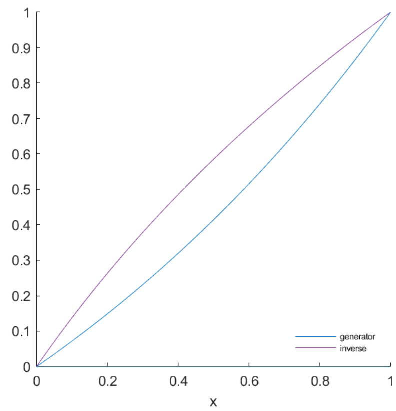


Figure 3.3: The balanced generator $2^x - 1$

The triangular norm (3.7) corresponding to the balanced generator $\theta_*(x) = 2^x - 1$ is shown in Figure 3.4

$$T(a,b) = \frac{\ln\left((2^a-1)\cdot\left(2^b-1\right)+1\right)}{\ln 2}. \quad (3.7)$$

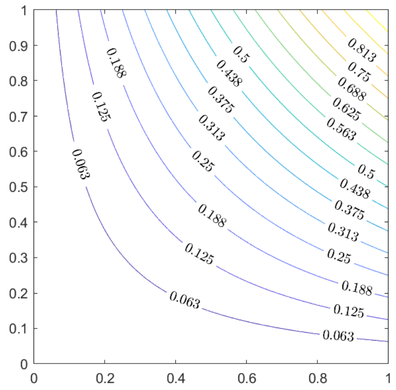


Figure 3.4: T-norm with balanced generator $2^x - 1$

Example 3. Other examples of t-norms with balanced generators are Frank t-norms. For parameter $\lambda \in (0, \infty) \setminus \{1\}$ they have multiplicative generators

$$\theta_\lambda(x) = \frac{\lambda^x - 1}{\lambda - 1} \tag{3.8}$$

with derivatives $\theta_\lambda(x)' = \dfrac{\lambda^x \ln \lambda}{\lambda - 1}$.

This time, we will show that $\theta_\lambda(x)$ are balanced by calculating $r = 1$ using equation (3.4),

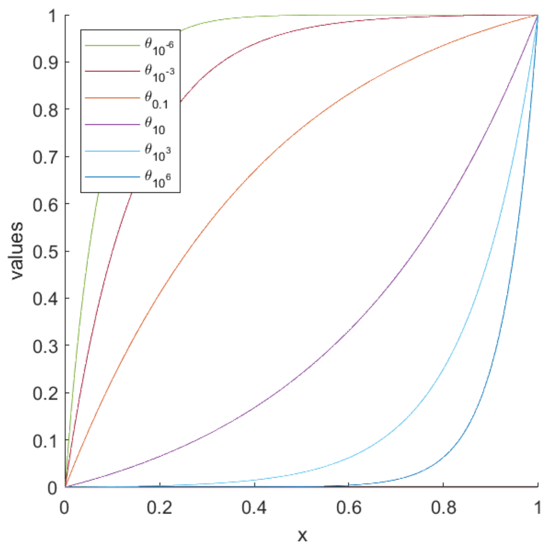


Figure 3.5: Frank multiplicative generators θ_λ

$$r = \lim_{x\to 0+} \frac{x\,\theta_\lambda'(x)}{\theta_\lambda(x)} = \lim_{x\to 0+} \frac{x\,\dfrac{\lambda^x \cdot \ln\lambda}{\lambda-1}}{\dfrac{\lambda^x-1}{\lambda-1}} = \ln\lambda \lim_{x\to 0+} \frac{x\,\lambda^x}{\lambda^x-1}$$
$$\stackrel{\text{LH}}{=} \ln\lambda \lim_{x\to 0+} \frac{\lambda^x \cdot (\ln\lambda \cdot x + 1)}{\lambda^x \ln\lambda} = 1.$$

We see that for parameters specified above, the multiplicative generators of Frank t-norms $\theta_\lambda(x)$ are balanced.

Example 4. We have also Einstein product $E(a,b) = \frac{x\,y}{2-(x+y-x\,y)}$ with multiplicative generator $\theta_*(x) = \dfrac{x}{2-x}$ [12] and $\theta_*'(x) = \dfrac{2}{(x-2)^2}$. Again, we calculate parameter r,

$$r = \lim_{x\to 0+} \frac{x\,\theta_*'(x)}{\theta_*(x)} = \lim_{x\to 0+} \frac{-2}{(x-2)} = 1$$

and conclude that this generator is also balanced.

Example 5. Our last example is $\theta_*(x) = \dfrac{\sin\left(\dfrac{\pi x}{4}\right)}{\cos\left(\frac{\pi x}{4}\right)}$. In this case, we have $\theta_*'(x) = \dfrac{\pi \sin^2\left(\frac{\pi x}{4}\right)}{4\cos^2\left(\frac{\pi x}{4}\right)} + \dfrac{\pi}{4}$, so $\theta_*'(0) = \frac{\pi}{4} \in \left]0,\infty\right[$. Therefore θ_* is indeed balanced.

3.4 T-norms without balanced generators

In some cases, after calculating the parameter r of a multiplicative generator θ we find out that $r = 0$ or $r = \infty$. If we use the formula for finding a balanced generator (3.5), we get a function that does not satisfy the axioms of a multiplicative generator. We say that t-norms created by these generators do not have any balanced generator.

3.4.1 Generator with right derivative at zero equal to 0

Example 6. A t-norm generated by $\theta(x) = -\frac{1}{\ln x - 1}$ does not have a balanced generator and (3.4) will always get $r = 0$.

We take a multiplicative generator $\theta(x) = -\frac{1}{\ln x - 1}$ with derivative $\theta'(x) = \frac{1}{x\cdot(\ln x - 1)^2}$ and we calculate a parameter r that should help us to find the balanced generator,

$$r = \lim_{x\to 0+} \frac{x\,\theta'(x)}{\theta(x)} = \lim_{x\to 0+} \frac{-1}{\ln x - 1} = 0.$$

We will use the inverse function $\theta^{-1}(x) = \begin{cases} \mathrm{e}^{\frac{x-1}{x}} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$

$$T(a,b) = \mathrm{e}^{\frac{\frac{1}{(\ln a-1)\,(\ln b-1)}-1}{\frac{1}{(\ln a-1)\,(\ln b-1)}}} = \mathrm{e}^{\frac{((\ln a-1)\,(\ln b-1))\,((\ln a-1)\,(\ln b-1)-1)}{(\ln a-1)\,(\ln b-1)}}$$
$$= \mathrm{e}^{((\ln a-1)\,(\ln b-1)-1)} = \frac{\mathrm{e}^{(\ln a)(\ln b)}}{ab} \quad \text{for } a, b \neq 0, \text{ otherwise } 0.$$

You can see this t-norm in Figure 3.6 and its generator in Figure 3.7.

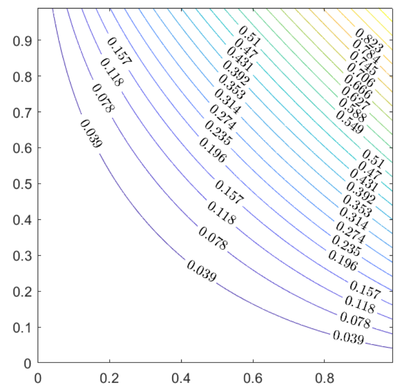


Figure 3.6: T-norm generated by $-\frac{1}{\ln x-1}$

3.4.2 Generator with right derivative at zero equal to infinity

Hamacher product $T(a,b) = \frac{a\,b}{a+b-a\,b}$ does not have a balanced generator and (3.4) will always get $r = \infty$.

Example 7. We will take a multiplicative generator of the Hamacher product, which you can see in Figure 3.7, $\theta(x) = \mathrm{e}^{\frac{x-1}{x}}$ and its derivative $\theta'(x) = \frac{\mathrm{e}^{\frac{x-1}{x}}}{x^2}$. We try to calculate the parameter r to show that it will be equal to ∞,

$$r = \lim_{x\to 0+} \frac{x\,\theta'(0)}{\theta(x)} = \lim_{x\to 0+} \frac{x\,\frac{\mathrm{e}^{\frac{x-1}{x}}}{x^2}}{\mathrm{e}^{\frac{x-1}{x}}} = \lim_{x\to 0+} \frac{x}{x^2} = \lim_{x\to 0+} \frac{1}{x} = \infty.$$

Now we can just take the inverse function $\theta^{-1}(x) = -\frac{1}{\ln x-1}$ and Hamacher product in Figures 3.9 with the formula

$$T(a,b) = -\frac{1}{\ln(\mathrm{e}^{\frac{a-1}{a}} \cdot \mathrm{e}^{\frac{b-1}{b}}) - 1} = -\frac{1}{\frac{a-1}{a} + \frac{b-1}{b} - 1}$$
$$= -\frac{1}{\frac{a\,b-a-b}{a\,b}} = \frac{a\,b}{a+b-a\,b} \quad \text{for } a, b \neq 0, \text{ otherwise } 0.$$

The following generators have all been taken from [12], where they are featured in the form of additive generators. We have turned them into

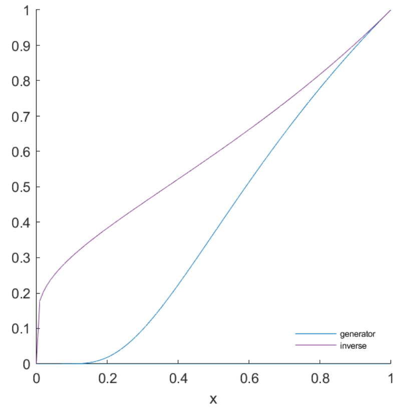


Figure 3.7: Multiplicative generator of Hamacher product $\mathrm{e}^{\frac{x-1}{x}}$ and its inverse $\frac{1}{\ln x - 1}$, the generator of Example 6

multiplicative generators via (2.6). We will calculate their derivative and show that when we try to calculate parameter r using (3.4) that normally tells us which exponent we should use to find the balanced generator, we will get $r = \infty$. Therefore, triangular norms generated by those generators do not have any balanced generator.

Example 8. Dombi's t-norms,

$$D_\lambda(a,b) = \frac{1}{1 + \left(\left(\frac{1}{a} - 1\right)^\lambda + \left(\frac{1}{b} - 1\right)^\lambda\right)^{\frac{1}{\lambda}}},$$

with parameter $\lambda > 0$, generator $\theta(x) = e^{-\left(\frac{1-x}{x}\right)^\lambda}$ and $\theta'(x) = -\dfrac{\lambda \cdot \left(\frac{1-x}{x}\right)^\lambda \mathrm{e}^{-\left(\frac{1-x}{x}\right)^\lambda}}{(x-1)\,x}$,

$$r = \lim_{x \to 0+} \frac{x\,\theta'(x)}{\theta(x)} = \lim_{x \to 0+} -\frac{\lambda \cdot \left(\frac{1-x}{x}\right)^\lambda}{x - 1} = \infty.$$

Example 9. Schweizer's second family of t-norms parameter $\lambda > 0$,

$$S^2_\lambda(a,b) = \frac{1}{\left(\frac{1}{a^\lambda} + \frac{1}{b^\lambda} - 1\right)^{\frac{1}{\lambda}}},$$

$\theta(x) = \mathrm{e}^{1 - \frac{1}{x^\lambda}}$ and $\theta'(x) = \lambda x^{-\lambda-1}\mathrm{e}^{1-\frac{1}{x^\lambda}}$,

$$r = \lim_{x \to 0+} \frac{x\,\theta'(x)}{\theta(x)} = \lim_{x \to 0+} x\,\lambda x^{-\lambda-1} = \lim_{x \to 0+} \frac{\lambda}{x^\lambda} = \infty.$$

Example 10. Mizumoto's first t-norm,

$$M_1(a,b) = \frac{2\,\mathrm{arccot}\left(\cot\left(\frac{\pi a}{2}\right) + \cot\left(\frac{\pi b}{2}\right)\right)}{\pi}, \tag{3.9}$$

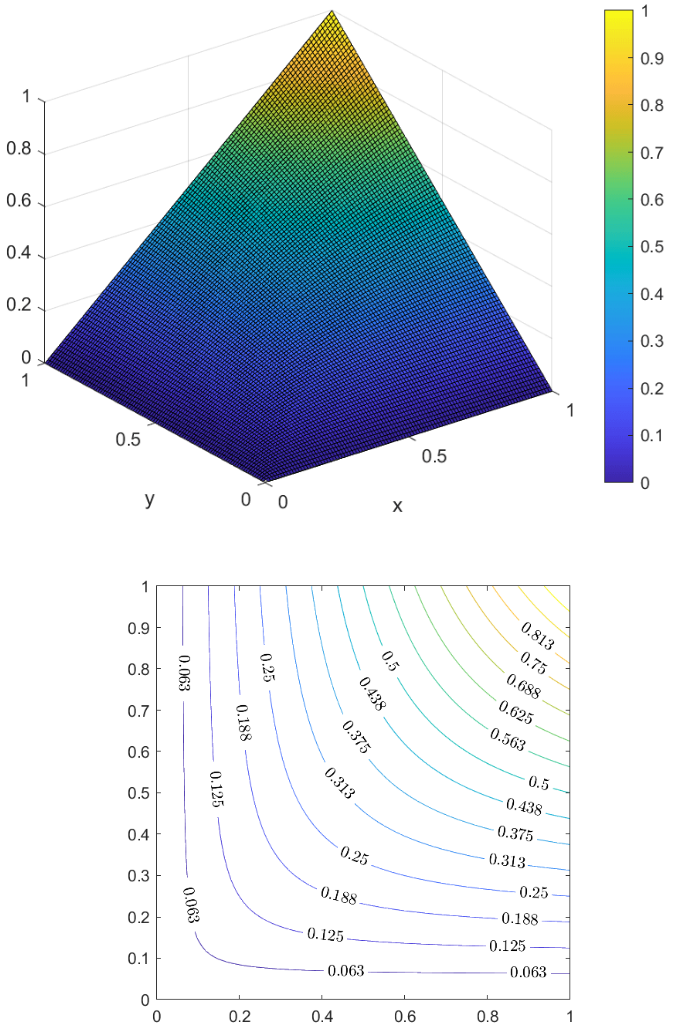


Figure 3.9: Hamacher product

has $\theta(x) = \mathrm{e}^{-\cot\left(\frac{\pi x}{2}\right)}$ and $\theta'(x) = \dfrac{\pi \mathrm{e}^{-\cot\left(\frac{\pi x}{2}\right)} \csc^2\left(\frac{\pi x}{2}\right)}{2}$,

$$r = \lim_{x \to 0+} \frac{x\,\theta'(x)}{\theta(x)} = \lim_{x \to 0+} x \cdot \frac{\pi \csc^2\left(\frac{\pi x}{2}\right)}{2} = \infty.$$

Example 11. Mizumoto's eighth family of t-norms,

$$M_\lambda^8(a, b) = \frac{1}{\log_\lambda\left(\lambda^{\frac{1}{a}} + \lambda^{\frac{1}{b}} - \lambda\right)}, \tag{3.10}$$

with parameter $\lambda > 1$, $\theta(x) = \mathrm{e}^{\lambda - \lambda^{\frac{1}{x}}}$ and $\theta'(x) = \dfrac{\lambda^{\frac{1}{x}} \ln \lambda \cdot \mathrm{e}^{\lambda - \lambda^{\frac{1}{x}}}}{x^2}$,

$$r = \lim_{x \to 0+} \frac{x\,\theta'(x)}{\theta(x)} = \lim_{x \to 0+} \frac{\lambda^{\frac{1}{x}} \ln \lambda}{x} = \infty \text{ for } \ln \lambda > 0.$$

Example 12. Mizumoto's ninth family of t-norms,

$$M^9_\lambda(a,b) = \frac{\lambda}{\ln\left(\mathrm{e}^{\frac{\lambda}{b}} + \mathrm{e}^{\frac{\lambda}{a}} - \mathrm{e}^{\lambda}\right)}, \tag{3.11}$$

has parameter $\lambda > 0$, $\theta(x) = \mathrm{e}^{\mathrm{e}^{\lambda} - \mathrm{e}^{\frac{\lambda}{x}}}$ with derivative $\theta'(x) = \dfrac{\lambda \mathrm{e}^{-\mathrm{e}^{\frac{\lambda}{x}} + \frac{\lambda}{x} + \mathrm{e}^{\lambda}}}{x^2}$,

$$r = \lim_{x\to 0+} \frac{x\,\theta'(x)}{\theta(x)} = \lim_{x\to 0+} \frac{\lambda \mathrm{e}^{\frac{\lambda}{x}}}{x} = \infty \text{ for } \lambda > 0.$$

Example 13. Mizumoto's tenth family of t-norms,

$$M^{10}_\lambda(a,b) = \frac{1}{\ln^{\frac{1}{\lambda}}\left(\mathrm{e}^{\frac{1}{a^\lambda}} + \mathrm{e}^{\frac{1}{b^\lambda}} - \mathrm{e}\right)}, \tag{3.12}$$

has parameter $\lambda > 0$, $\theta(x) = \mathrm{e}^{\mathrm{e} - \mathrm{e}^{\frac{1}{\lambda x}}}$. In this case, we have $\theta'(x) = \dfrac{\mathrm{e}^{-\mathrm{e}^{\frac{1}{\lambda x}} + \frac{1}{\lambda x} + \mathrm{e}}}{\lambda x^2}$,

$$r = \lim_{x\to 0+} \frac{x\,\theta'(x)}{\theta(x)} = \lim_{x\to 0+} \frac{\mathrm{e}^{\frac{1}{\lambda x}}}{\lambda\, x} = \infty \text{ for } \lambda > 0.$$

Chapter 4

Derivatives of t-norm

We were looking for a specific t-norm T. It would have a first-order partial derivative at zero with respect to the first argument going to zero or to infinity. It should also have a second-order partial derivative in the interval $]0, \infty[$. From the commutativity of t-norms, we assume that those derivatives would have the same properties with respect to the second argument. From (2.1), we see that those derivatives will be in fact symmetrical. Therefore, we may only consider derivatives with respect to one argument without loss of generality.

Once we find such a t-norm, we hope to be able to widen the discovery of the possibility of finding a generator θ of a known strict t-norm T (3.2), given that the generator's second derivative approaching zero from the right exists and is in $]0, \infty[$. Ideally, we would find a formula similar to (3.2) using second derivatives of T for t-norms with no balanced generator.

4.1 First derivative

$$\frac{\partial T(a,b)}{\partial a} = \frac{1}{\theta'\left(\theta^{-1}\left(\theta(a)\,\theta(b)\right)\right)} \cdot \theta'(a) \cdot \theta(b) \tag{4.1}$$

We can only use this form of the derivative if we have the closed-form expression of the inverse function of the generator. Otherwise, it could be very hard to find the value of $\theta'\left(\theta^{-1}\left(\theta(a)\,\theta(b)\right)\right)$. To guarantee (4.1) equals zero, one of the multipliers needs to equal zero. The last one will change depending on where we are in the interval and will only equal zero for $b = 0$.

Another way to ensure that $\frac{\partial T(a,b)}{\partial a} = 0$ is if $\theta'\left(\theta^{-1}\left(\theta(a)\,\theta(b)\right)\right) = \pm\infty$ for a, b approaching 0. The final possibility of insuring nullity is $\theta'(a) = 0$.

To guarantee (4.1) approaches infinity, one of the multipliers needs to approach infinity. The $\theta(b)$'s range is $[0, 1]$, so that is not a candidate. The second multiplier, $\theta'(a)$ cannot approach infinity throughout the whole domain because the multiplicative generator θ is defined as an increasing bijection. The only way to ensure that $\frac{\partial T(a,b)}{\partial a} = \infty$ is if $\theta'\left(\theta^{-1}\left(\theta(a)\,\theta(b)\right)\right) = 0$ for a and b approaching 0.

4.2 Second derivative

$$\frac{\partial^2 T(a,b)}{\partial^2 a} = \frac{\theta(b)}{\theta'\left(\theta^{-1}\left(\theta(a)\,\theta(b)\right)\right)} \cdot \left(\theta''(a) - \frac{\theta'(a)^2\,\theta(b)}{\theta'\left(\theta^{-1}(\theta(a)\,\theta(b))\right)}\right) \tag{4.2}$$

We know from our assumption that $\frac{\partial T(a,b)}{\partial a} = 0$, so we can simplify the last expression and we get

$$\frac{\partial^2 T(a,b)}{\partial^2 a} = \frac{\theta(b)}{\theta'(\theta^{-1}(\theta(a)\,\theta(b)))} \cdot \theta''(a) \tag{4.3}$$

Straight away, we notice two things in (4.3). The first is that the denominator remains the same as in the last section, so we cannot use it to ensure $\frac{\partial T(a,b)}{\partial a} \neq \infty$ and we will not be able to find a t-norm with first-order partial derivative at zero with respect to the first argument that would be equal to infinity. Furthermore, we cannot use it to ensure the nullity of (4.1) either because the second-order partial derivative would then also equal zero. The multiplier $\theta(b)$ will still equal 0 for $b = 0$ because T is strict. This still leaves the possibility of $\theta'(x) = 0$ ensuring the nullity. However, we know from Definition 2.3.1 that a multiplicative generator is an increasing bijection $[0,1] \to [0,1]$. This means that $\theta'(x) = 0$ cannot ensure the nullity of (4.1) throughout the whole interval.

The next thing we noticed is that the only difference between (4.3) and (4.1) is that $\theta'(x)$ becomes $\theta''(x)$. We could say that we also need $\theta''(x) \neq 0$. If we continue and differentiate this expression, we will get a similar result to (4.2). Then after simplification due to the first and second derivatives being equal to zero, we will probably get the same expression as in (4.1) and (4.3) with $\theta^{(n)}$ for an nth derivative.

We conclude that the desired triangular norm does not exist. Moreover, higher-order derivatives will probably not be helpful in finding a formula that specifies θ for a given strict T with no balanced generator (3.2).

Chapter 5

Derivatives at 1

5.1 Interrelationship between additive and multiplicative generator's derivatives at 1

Let T be a strict t-norm, t its additive generator, and θ its multiplicative generator. Let us take a derivative of θ at 1. We can rewrite it using t as

$$\lim_{x\to 1-} \theta'(x) = \lim_{x\to 1-} \left(\mathrm{e}^{-t(x)}\right)' = \lim_{x\to 1-} -\mathrm{e}^{-t(x)}\, t'(x) = \lim_{x\to 1-} -t'(x) \tag{5.1}$$

Notice that $\lim_{x\to 1-} \mathrm{e}^{-t(x)} = 1$ because $t(1) = 0$ from Definition 2.3.2. The other direction is similar, we used $\theta(1) = 1$ from Definition 2.3.1.

$$\lim_{x\to 1-} t(x)' = \lim_{x\to 1-} (-\ln\theta(x))' = \lim_{x\to 1-} -\theta'(x)/\theta(x) = \lim_{x\to 1-} -\theta'(x) \tag{5.2}$$

We clearly see that the following two are equivalent.

1. $0 < \theta'(1) < \infty$
2. $0 < -t'(1) < \infty$

Remark 2. Since t is strictly decreasing, the negation of its derivative, $-t'(1)$, will never be less than 0. Therefore $\theta'(1)$ will never be less than 0.

Now let us explore whether this bears an influence on the total differential of the generated triangular norm T.

We took the formula of the first derivative of T from (4.1).

$$\begin{aligned}\left.\frac{\partial T(a,b)}{\partial a}\right|_{a,b=1} &= \left.\frac{\theta'(a)\cdot\theta(b)}{\theta'\left(\theta^{-1}\left(\theta(a)\,\theta(b)\right)\right)}\right|_{a,b=1} = \left.\frac{\theta'(a)\cdot\theta(b)}{\theta'\left(\theta^{-1}(1)\right)}\right|_{a,b=1}\\ &= \left.\frac{\theta'(a)\cdot\theta(b)}{\theta'\left(1\right)}\right|_{a,b=1} = 1 \text{ for } 0 < \theta'(1) < \infty\,, \text{ otherwise undefined.}\end{aligned} \tag{5.3}$$

Since the function T is commutative,

$$\left.\frac{\partial T(a,b)}{\partial b}\right|_{a,b=1} = 1 \text{ for } 0 < \theta'(1) < \infty\,, \text{ otherwise undefined.}$$

The total differential d of T with $\triangle$ signifying a small difference is

$$d(\triangle a, \triangle b) = \frac{\partial T(a,b)}{\partial a}(a_0, b_0) \cdot \triangle a + \frac{\partial T(a,b)}{\partial b}(a_0, b_0) \cdot \triangle b. \tag{5.4}$$

At $(a_0, b_0) = (1, 1)$, (5.4) is equal to $\triangle a + \triangle b$ if and only if $0 < \theta'(1) < \infty$. Otherwise, the total differential does not exist there.

Now let us take the Taylor polynomial f of degree 1 at $(a_0, b_0) = (1, 1)$. It will also be defined if and only if $0 < \theta'(1) < \infty$ and its value is

$$f(a,b) = f(a_0,b_0) + \frac{\partial T(a,b)}{\partial a}(a_0, b_0) \cdot (a - a_0) + \frac{\partial T(a,b)}{\partial b}(a_0, b_0) \cdot (b - b_0)$$
$$= 1 + (a-1) + (b-1) = a + b - 1 \text{ for } 0 < \theta'(1) < \infty \text{, otherwise undefined.} \tag{5.5}$$

We summarize as follows.

Theorem 1. *The following three statements are equivalent. Notice also that this holds for either all the generators or none [13].*

1. $0 < \theta'(1) < \infty$
2. $0 < -t'(1) < \infty$
3. T *is differentiable at* $(1, 1)$. *It has there a total differential* $(a, b) \mapsto a + b$ *and a Taylor polynomial of degree 1:* $(a, b) \mapsto a + b - 1$

Remark 3. The famous nilpotent Łukasiewicz t-norm is defined as $T(a, b) = \max(a + b - 1, 0)$. At $(1, 1)$, this locally approaches all triangular norms that have $0 < \theta'(1) < \infty$, which includes some strict t-norms, e.g., all strict Frank t-norms.

This is not easy to see in Figure 5.2 from [8], where they use additive counterparts to the multiplicative generators with formula (3.8). That is why we created our own depiction with additive generators that generate the same Frank t-norms and all have derivatives at one equal to minus one. Using (2.5) we got the additive counterparts of (3.8). Finally, we divided the generators by the absolute value of their derivatives at one. We could do this because in Example 3 we showed that point 1 from Theorem 1 holds, therefore point 2 also holds and the derivative at one is between zero and minus infinity. See equation (2.4) for reassurance that our generators generate the same triangular norms.

In our Figure 5.1, we see that when we get closer to one, the generators of Frank t-norms start to resemble the generator of Łukasiewicz t-norm, t_∞.

At the same time, any continuous t-norm can be approximated by a strict t-norm with arbitrary precision [14]. For example, the minimum t-norm is the limit case of Frank t-norms for parameter $\lambda \to 0$, although it is not differentiable at $(1, 1)$ and does not have any generator.

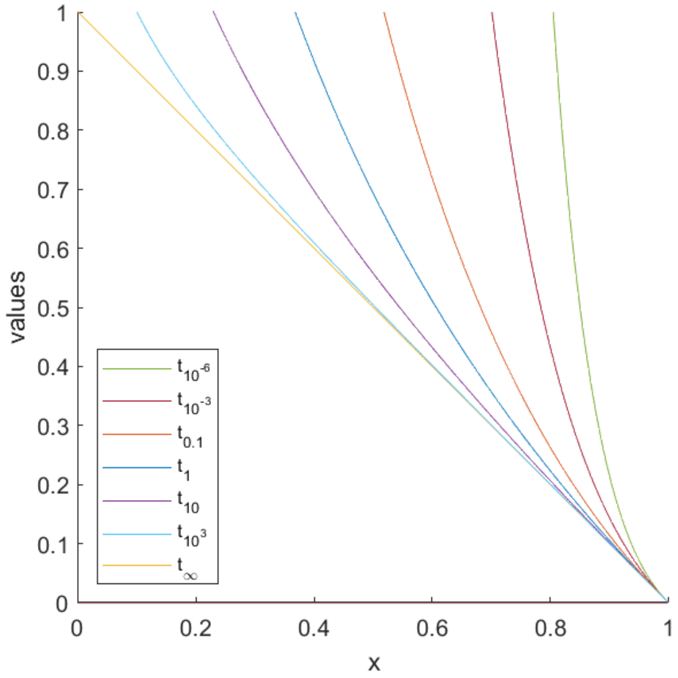


Figure 5.1: Additive generators of Frank t-norms with the first derivative at one equal to minus one

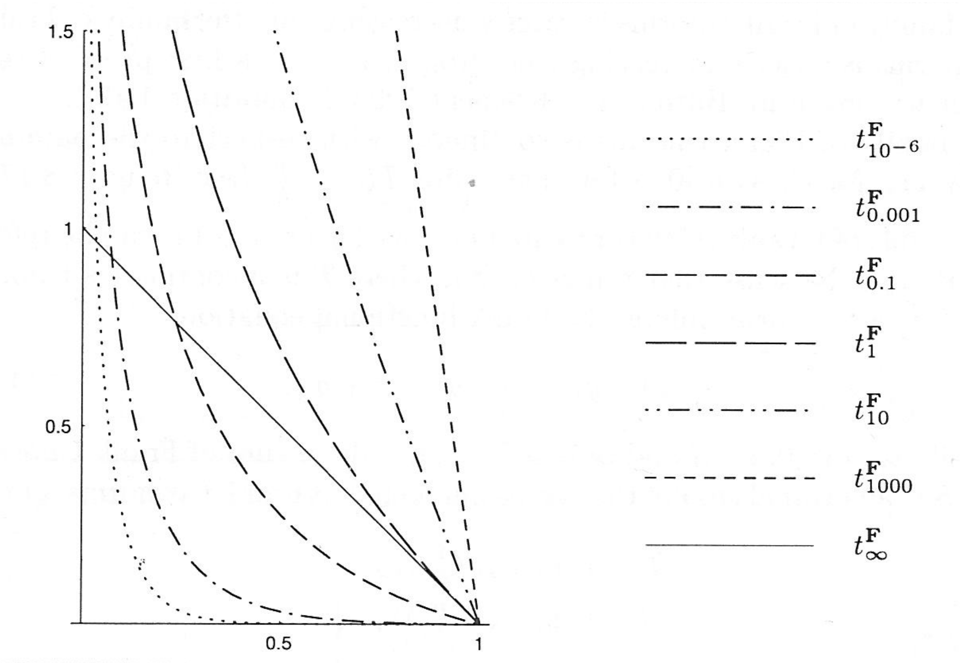


Figure 5.2: Additive generators of some Frank t-norms

5.2 Illustrating examples

In this section, we will be looking for multiplicative generators that we could easily compare to each other. We will make generators $\theta_{n,1}$ that only differ in the derivative near the right edge of their domain, meaning around $x = 1$. Other than that they will be modeled by the product t-norm, which is generated by $\theta_P(x) = x$. Then we will plot their corresponding t-norms T and talk about how the difference in the derivative at 1 of a generator affects the resulting T.

5.2.1 Choosing illustrating multiplicative generators

We will want this set of generators $\theta_{n,1}$ to be identical to the generator of product t-norm θ_P on a significant portion of the domain, they will only differ in interval $]0.9, 1[$. We want to model this part of the domain by a function that has a value at 0.9 equal to 0.9 and a value at one equal to one. This is because we want the multiplicative generator to be continuous so that the additive generator would be continuous and most importantly the T would also be continuous [10]. We decided we would use polynomial functions to get different derivatives at 1.

$$\theta_{n,1}(x) = \begin{cases} x & x \in [0, 0.9[\\ 0.9 + (x - 0.9)^n \cdot 10^{n-1} & x \in [0.9, 1], n > 0. \end{cases} \tag{5.6}$$

To speak about the generators, we will split them into two groups. The first group of four generators $\theta_{n,1}$ is for an exponent $n > 1$. Concretely, we will show generators with $n \in \{2, 3, 4, 5\}$. In other words, the generators with all values bigger than or equal to θ_P.

The second group of four generators has an exponent $1 > n > 0$. We will plot only generators with values $n \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ with values that are all less than or equal to θ_P. The argument of the root function will be positive on our interval, so the function is defined there.

The generators from the first group are inverse to those from the second. For example, $\theta_{2,1}$ is the inverse of $\theta_{\frac{1}{2},1}$. The two inverse counterparts are plotted in the same color in Figure 5.3 so that you can visually link them to one another. Notice that they are all balanced.

5.2.2 Resulting triangular norms

When we change the generator's values in the interval $]0.9, 1[$ the resulting t-norm T is impacted in two intersecting strips. One strip has values x in $]0.9, 1[$, y in $]0, 1[$ and the other x in $]0, 1[$, y in $]0.0, 1[$. In the area outside these two strips, $[0, 0.9] \times [0, 0.9]$, the newly generated t-norms will have identical values as T_P. This all is evident from (2.1) and shown in Figure 5.4.

Sadly, we cannot fully trust 3D-generated images in this thesis. They are better at instantly giving a general idea of the situation. Nonetheless, we discovered a flaw that we just could not fix in the context of this thesis.

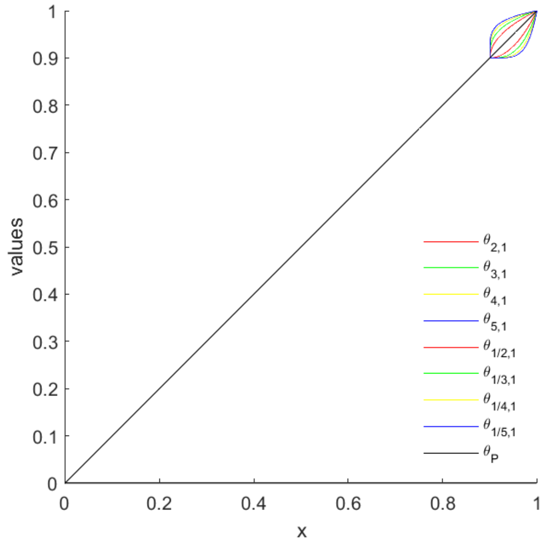
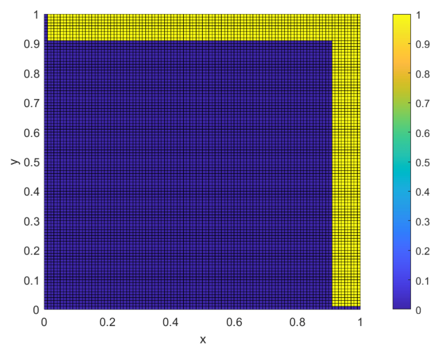
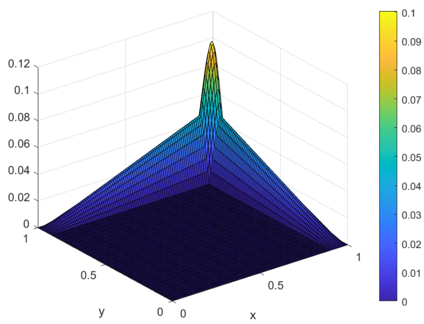


Figure 5.3: Generators edited near one

Although the boundary condition $T(a, 1) = a$ is necessary for t-norm to be a t-norm as we know from Definition 2.2.1, in Figures 5.7 it is evidently violated around $(1, 1)$. We could get rid of this flaw by using fewer points to generate the image but then we would also lose a lot of other information. By extent, in Figure 5.4, if either of the variables is equal to 1, the difference between a generated triangular norm and the product norm has to be zero because all those values are common to all the existing t-norms. Thankfully, this time we can give the Figures the benefit of the doubt. For 5.4a the one value may have gotten lost in the sea of all the others and for 5.4b we might simply be oblivious to the drop because of the chosen angle.



(a) : The area that has changed



(b) : The concrete difference for the t-norm generated by $\theta_{8,1}, 3D$

Figure 5.4: Change in t-norm generated by $\theta_{n,1}$ in relation to T_P

We can clearly see the twists along the edges of these two strips. For the first four generators shown in Figures 5.5 and 5.7a, we see that the values in the area outside these two strips seem to get lower than the values inside.

In other words, in the same way that values of generators with $n > 1$ are always greater than or equal to θ_P, the values of t-norms generated by those generators are also always greater than or equal to those of T_P.

There is even a little semicircle in the intersection of the two dips, facing away from the tip of the t-norm. These changes start smoother and they get more pronounced as we get to the last generator of this group $\theta_{5,1}$ because the change in generator gets more pronounced with each generator, too.

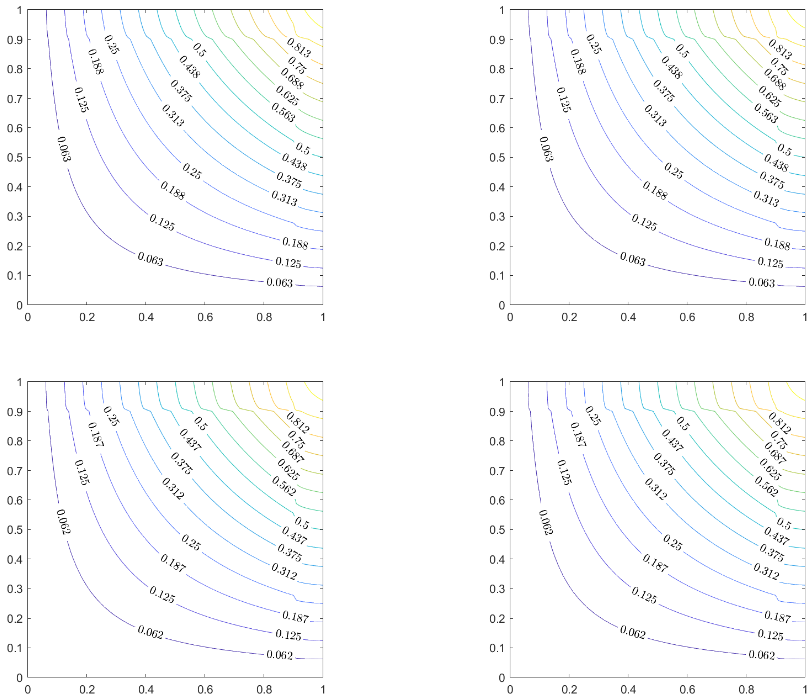


Figure 5.5: T-norms with generator changing around one, $\theta_{2,1}, \theta_{3,1}, \theta_{4,1}, \theta_{5,1}$,

On the other hand, for the second four generated t-norms, shown in Figures 5.6 and 5.7b, we see that the values outside the two strips seem to get higher than those inside. We can say that the values of t-norms generated by n that is in between 0 and 1 are always less than or equal to those of T_P. Just as it was for the generators.

Once more, there is a little semicircle in the intersection of the two dips. This time it is leaning towards the tip of the t-norm. The changes start out smooth and they get more prominent as we get to the last generator $\theta_{\frac{1}{5},1}$. To make the changes even more visible, we plotted the two t-norms with the most distinct generator in 3D in Figure 5.7.

It seems that the more pronounced a change of a generator is, the more pronounced the change in T. However, this is only valid for changes in the form of polynomials or root functions if we only change a part of the interval at a time. If we changed the whole interval as in (2.3), the T would stay the same.

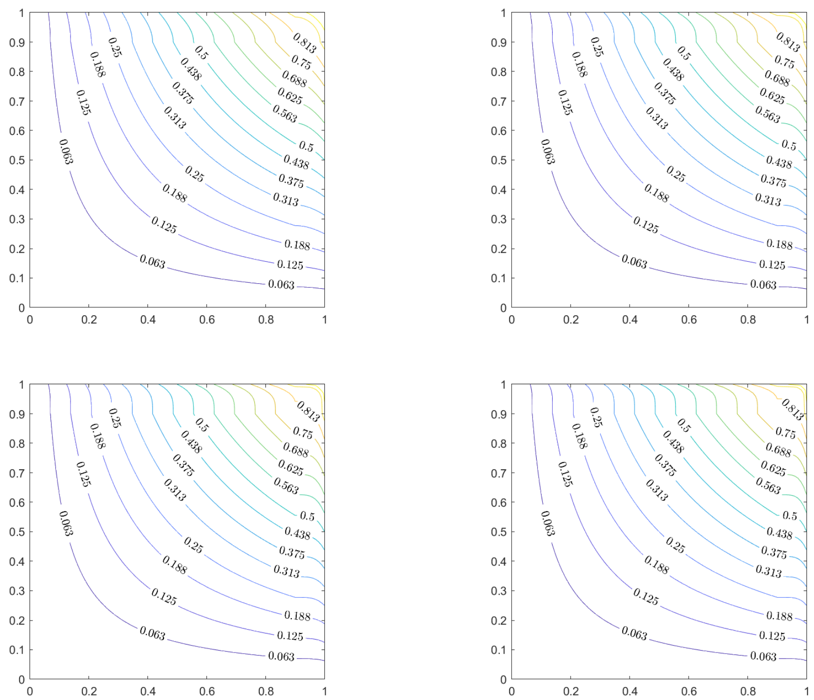
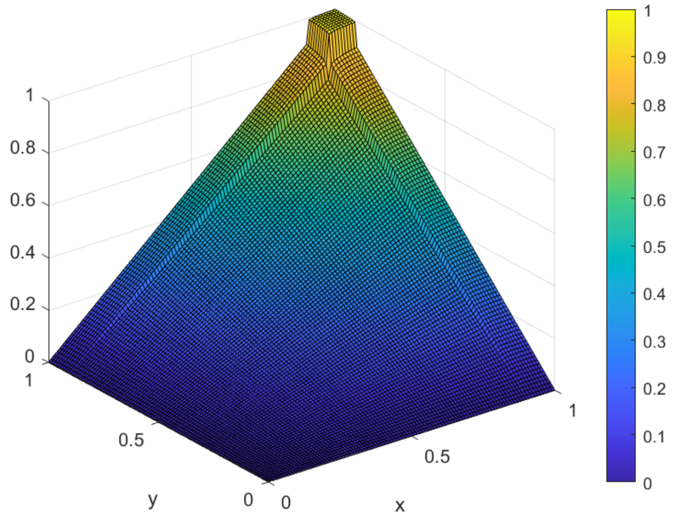
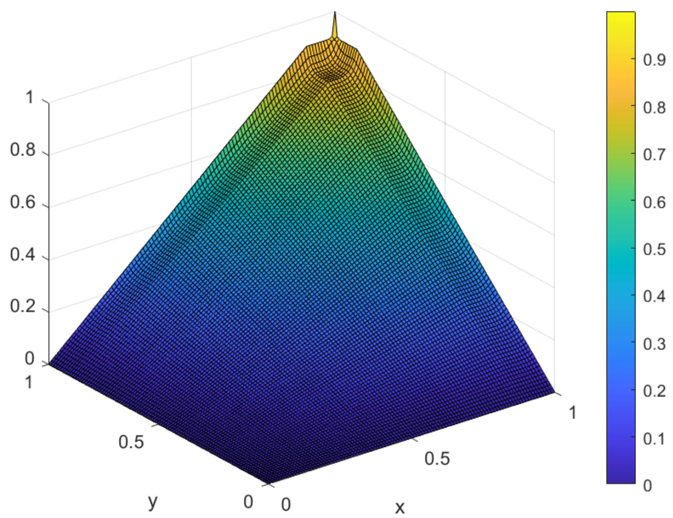


Figure 5.6: T-norms with generator changing around one, $\theta_{\frac{1}{2},1}, \theta_{\frac{1}{3},1}, \theta_{\frac{1}{4},1}, \theta_{\frac{1}{5},1}$



(a) : T-norm generated by $\theta_{5,1}, 3D$



(b) : T-norm generated by $\theta_{\frac{1}{5},1}, 3D$

Figure 5.7: 3D t-norms with generators altered around 1

Chapter 6

Derivatives at 0

6.1 Underlying theory

It is not as useful to study derivatives of additive generators as it was in section 5.1. We could look at its convergence to $-\log x$, but that is too abstract. The derivative of a multiplicative generator at 0 is more interesting since it enables us to define a balanced generator [13].

Let us calculate the total differential at $(a_0, b_0) = (0, 0)$ for strict t-norms T with $0 < \theta'(0) < \infty$, in other words, for t-norms with balanced generators. Let us start by identifying partial derivatives. We again used the formula of the first derivative of T from (4.1).

$$\left.\frac{\partial T(a,b)}{\partial a}\right|_{a,b=0} = \left.\frac{\theta'(a)\cdot\theta(b)}{\theta'\left(\theta^{-1}\left(\theta(a)\,\theta(b)\right)\right)}\right|_{a,b=0} = \left.\frac{\theta'(a)\cdot\theta(b)}{\theta'\left(\theta^{-1}(0)\right)}\right|_{a,b=0}$$
$$= \left.\frac{\theta'(a)\cdot\theta(b)}{\theta'(0)}\right|_{a,b=0} = 0 \text{ for } 0 < \theta'(0) < \infty\,,\text{ otherwise undefined.} \tag{6.1}$$

At $(a_0, b_0) = (0, 0)$, the total differential of T (5.4) is equal to 0 if and only if $0 < \theta'(0) < \infty$. Otherwise, T is not differentiable and the total differential does not exist. We can say even more:

Theorem 2. *Let T be a strict t-norm with a balanced generator θ_* such that $\theta_*'(0) = c \in \,]0, \infty[$. Then [13]*

$$\lim_{(a,b)\to(0,0)} \frac{T(a,b)}{c\,a\,b} = 1\,.$$

Proof. If the T has a balanced generator with $\theta_*'(0) = c \in \,]0, \infty[$, then we can approximate θ_* by a linear function θ_a as $\theta_a(x) = c \cdot x$. The inverse of θ_a is $\theta_a^{-1}(x) = \frac{x}{c}$. Therefore,

$$\lim_{(a,b)\to(0,0)} T(a,b) = \lim_{(a,b)\to(0,0)} \theta_a^{-1}\left(\theta_a(a)\cdot\theta_a(b)\right) \tag{6.2}$$
$$= \lim_{(a,b)\to(0,0)} \frac{c\,a\cdot c\,b}{c} = \lim_{(a,b)\to(0,0)} c\,a\,b$$

When we divide both sides of (6.2) by $c\,a\,b$, we get the equation from Theorem 2. □

Remark 4. We noticed in the last chapter that strict t-norms with $0 < \theta'(1) < \infty$ approach Łukasiewicz t-norm at $(1,1)$, see Remark 3. This time we see that t-norms with $0 < \theta'(0) < \infty$ approach a multiple of the product t-norm at $(0,0)$.

Finally, we managed to link the generators to each other in an interesting way,

$$\lim_{x\to 0+} t'(x)\,\theta(x) = \lim_{x\to 0+} \left(-\ln\theta(x)\right)'\,\theta(x) = \lim_{x\to 0+} -\frac{\theta'(x)\,\theta(x)}{\theta(x)} = \lim_{x\to 0+} -\theta'(x). \tag{6.3}$$

If we take a balanced generator, $\lim_{x\to 0+} -\theta'_*(x) \in]-\infty, 0[$. This is the right-hand side of equation (6.3), therefore the left-hand side $\lim_{x\to 0+} t'(x)\,\theta(x)$ will also be in interval $]-\infty, 0[$. Therefore we can multiply both sides of the equation by -1 and get (6.4).

$$\lim_{x\to 0+} -t'(x)\,\theta_*(x) = \lim_{x\to 0+} \theta'_*(x) \tag{6.4}$$

6.2 Illustrating examples

As in the last chapter, we are going to investigate the effect of changing derivative on the resulting t-norm T. This time, we will be changing the derivative of generators $\theta_{n,0}$ around $x = 0$. The rest of the domain of the multiplicative generators will be determined by the generator of product t-norm θ_P. To get them, we will need to create polynomials, which will only differ in our desired interval $]0, 0.1[$. They also need to have $\theta_{n,0}(0) = 0$ and $\theta_{n,0}(0.1) = 0.1$ to ensure continuity of $\theta_{n,0}$ and of the t-norm T generated by $\theta_{n,0}$ [10]. There will also be two groups of generators split by the values of the exponents.

The first one will create four generators that are less then or equal to θ_P on the whole domain, the second one will be greater then or equal to θ_P on the whole domain.

6.2.1 Making multiplicative generators

We will multiply exponents of x in such way that instead of being $[0,1] \to [0,1]$ they will be defined on $[0, 0.1] \to [0, 0.1]$. This will guarantee that $\theta_{n,0}(0) = 0$ and $\theta_{n,0}(0.1) = 0.1$. We do not even have to move this part of the generator as we did in section 5.2.1.

$$\theta_{n,0}(x) = \begin{cases} x^n \cdot 10^{n-1} & x \in [0, 0.1[\\ x & x \in [0.1, 1]. \end{cases} \tag{6.5}$$

We will use equation (6.5) for $n \in \{2,3,4,5\}$ to generate the first groups of generators. The inside of the root function is defined for $x \in [0, 0.1[$, so $\theta_{n,0}$ is a defined generator for $n > 0$. The second group of generators is generated by $n \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$. We could again say more generally that the first group has an exponent $1 < n$ and the second one has an exponent $0 < n < 1$.

The first four generators are inverses of the second four, as you can see in Figure 6.1. The inverse functions are plotted in the same color. This time, none of the generators we created is balanced. Nevertheless, they should all have some balanced counterparts.

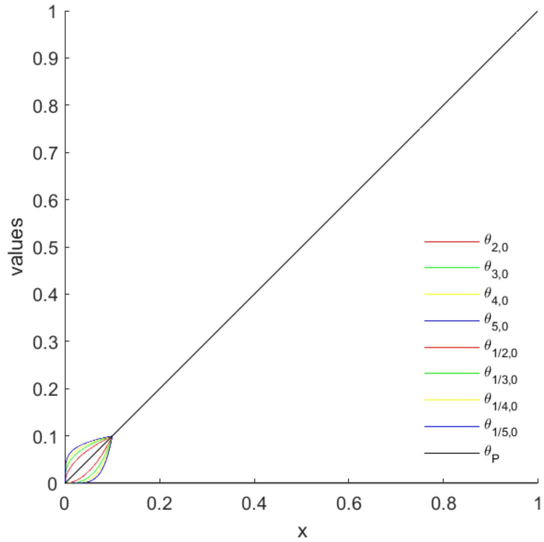


Figure 6.1: Generators edited near zero

6.2.2 Generated triangular norms

In addition to changes in the strips similar to what we noticed in subsection 5.2.2, we will also be dealing with a change resulting from the inversion in (2.1) as we can see in Figure 6.2. The first strip has values $x \in]0, 0.1[, y \in]0, 1[$ and the second strip has values $x \in]0, 1[, y \in]0, 0.1[$. However, the biggest change seems to be a result of inversion from (2.1). Another interesting thing is that whilst in the last chapter the difference in values of $\theta_{\frac{1}{5},0}$ was in Figure 5.4b up to 0.1, here the difference is only a little over 0.08 as shown in 6.2.

We will plot more contours than we did in Figure 2.1, so that we have a better understanding of what is going on in the areas of the strips.

When looking at the first four generated t-norms in Figures 6.3 and 6.5a, we notice that values in the affected area are all bigger than in the product t-norm. In the contours, we see that as the n gets bigger, the contours from affected areas approach axes $x = 0$ and $y = 0$. When it comes to the t-norm generated by $\theta_{5,0}$, there are two contours that both seemingly merge with the axes. The first has the value 0.03 and the second one has the value 0.059.

The second thing we notice is that there are some wave-like twists in the third contour with a value of 0.089 in all the Figures. It is not particularly prominent in the first Figure and resembles the shift that was present in Figures 5.5. In the next ones we can see little twists as if the contour was drawn to the axes with a magnet and the force was the greatest at points

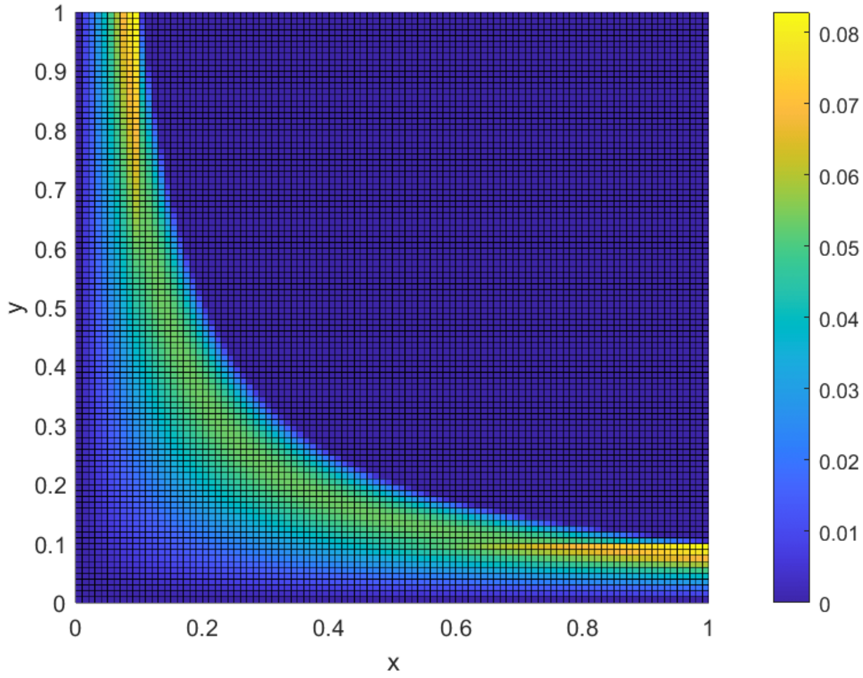


Figure 6.2: Change in t-norm generated by $\theta_{0,8}$ in relation to T_P

$(0, 1)$ and $(1, 0)$.

As for the second four functions in Figures 6.4 and 6.5b, it is as if someone switched the poles of the magnet. All contours from the affected area move away from the axes, so the values stay lower for a longer time. This corresponds with the t-norms generated by $\theta_{n,1}$ with $0 < n < 1$, which also stay below or equal to the values of product t-norm. And again, the change is more pronounced in the area of the corners $(0, 1)$ and $(1, 0)$.

We plotted in 3D two t-norms generated by generators that are the most unlike the product t-norm in Figure 6.5 to get a good visual interpretation of what is going on. We did not want to plot all the t-norms in 3D and compare them like this because the differences are subtler and would be harder to spot. Furthermore, the problem of 3D-generated images not respecting the definition of t-norm reappears in both subfigures of Figure 6.5. Neither of the t-norms respects the boundary condition in the images.

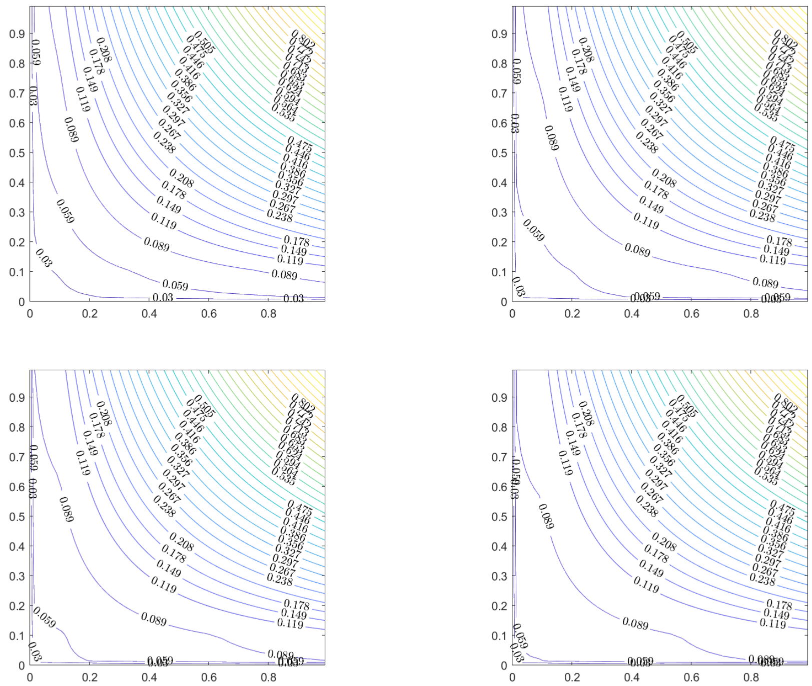


Figure 6.3: T-norms with generator changing around zero; $\theta_{2,\,0}, \theta_{3,\,0}, \theta_{4,\,0}, \theta_{5,\,0}$

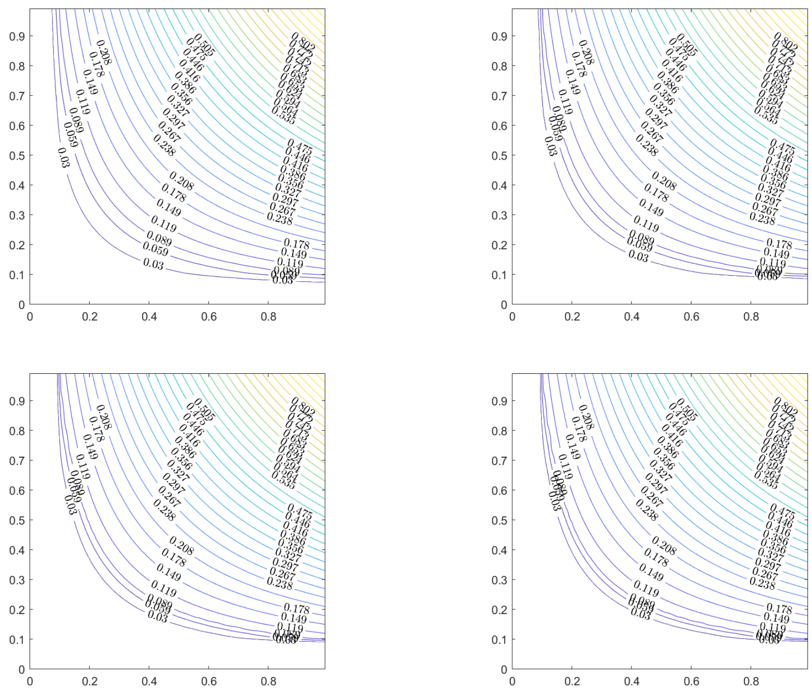
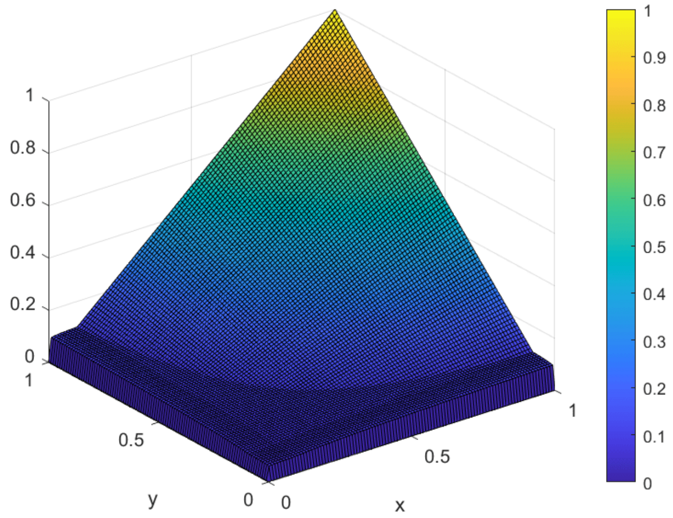
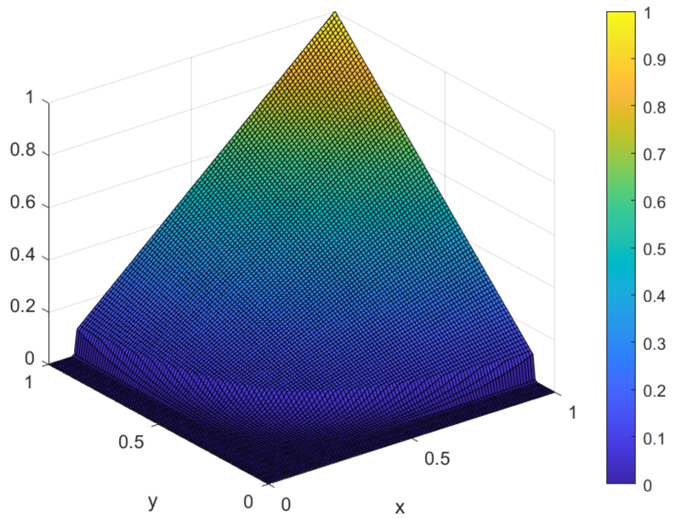


Figure 6.4: T-norms with generator changing around zero; $\theta_{\frac{1}{2},\,0}, \theta_{\frac{1}{3},\,0}, \theta_{\frac{1}{4},\,0}, \theta_{\frac{1}{5},\,0}$



(a) : T-norm generated by $\theta_{5,0}, 3D$



(b) : T-norm generated by $\theta_{\frac{1}{5},0}, 3D$

Figure 6.5: 3D t-norms with generators altered around 0

Chapter 7

Derivatives zero or infinity

In this chapter, we will be exploring the effect of a derivative equal to 0 or ∞ inside the domain of a generator on the corresponding t-norm. We will plot and compare the generators and t-norms as we did in the last two chapters. Nonetheless, this time we will not focus on only changing a generator locally.

7.1 Illustrations

In this section, we will take functions with interesting derivatives in the middle of the interval. There will be only 4 generators. Again, the first two will be the inversions of the second two and vice-versa.

7.1.1 Generators with derivative at 0.5 equal to 0 or infinity

To build the first group of generators, we will use monomials with an exponent that is odd and bigger than 1. Odd because they have both positive and negative values, with a nice curve in the middle. We will move the curve up to the middle of the interval and then adjust the function so that it has $\theta(0) = 0$ and $\theta(1) = 1$.

$$\theta_n(x) = (0.5^n + (x - 0.5)^n) \cdot 2^{n-1}. \tag{7.1}$$

We will only plot two functions to represent the first group. Those functions will be generated by (7.1) with parameter $n \in \{3, 5\}$. Their derivatives at $x = 0.5$ equal 0.

For the second group of functions, we will reuse the formula (7.1) to make generators that are inverse to the first group's generators. This time, we will use an exponent that is $0 < n < 1$ and that can always be written as $n = \frac{1}{m}$ for a parameter m that is odd. Concretely, we will use parameters $n \in \{\frac{1}{3}, \frac{1}{5}\}$ Their derivative at $x = 0.5$ equals ∞.

We need to use the odd roots because there will be negative numbers inside the root function. If we wanted to use even roots, we would need to rewrite the function and use the absolute value and signum function as follows

$$\theta_n(x) = (0.5^n + \operatorname{sign}(x - 0.5) \cdot (|x - 0.5|)^n) \cdot 2^{n-1}. \tag{7.2}$$

We can see them all in Figure 7.1.

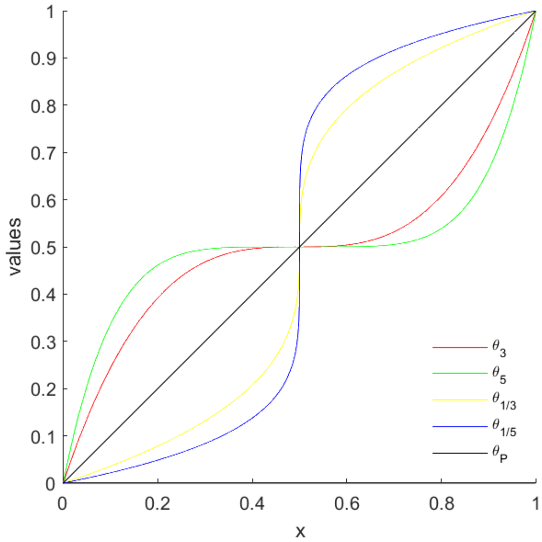


Figure 7.1: Generators with derivative at 0.5 equal to 0 or ∞

7.1.2 Corresponding t-norms

This group of t-norms is the most different from all those plotted previously in this thesis. Their generators have only three points in common with the standard product's generator: $(0,0),(0.5,0.5)$ and $(1,1)$. In the resulting t-norms, we see that the contours near the middle of the t-norm curl into a semicircle facing away from the middle.

For the group with a derivative at 0.5 equal to zero, we see in Figure 7.2 that most of the contours are crowded near the edges, where either $x = 1$ or $y = 1$.

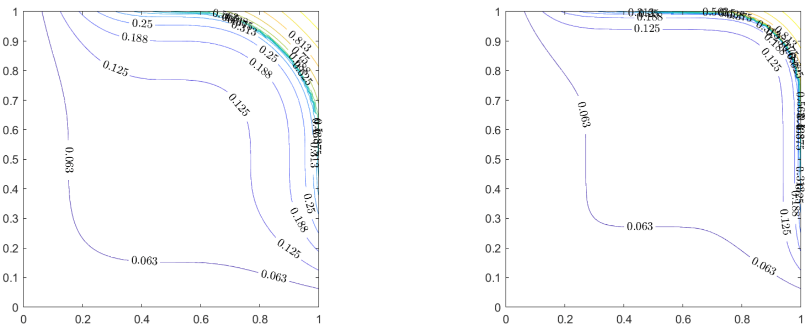


Figure 7.2: T-norms with generators with derivative at 0.5 equal to zero θ_3, θ_5

For the other group with a derivative at 0.5 equal to infinity, we have a similar situation, as we can see in Figure 7.3. Most of the contours are crowded near the edges where $x = 0$ or $y = 0$. However, we have more contours nearing point $(1,1)$ than we had contours nearing point $(0,0)$ in the last case.

Another interesting thing regarding the second group is that it sometimes seems as if the contours did not start at the right points. As we recall from the definition of t-norm, $\forall a \in [0,1] : T(a,1) = a$. Some of the contours get away from those original points so quickly that the graphical tool does not even draw the connecting line.

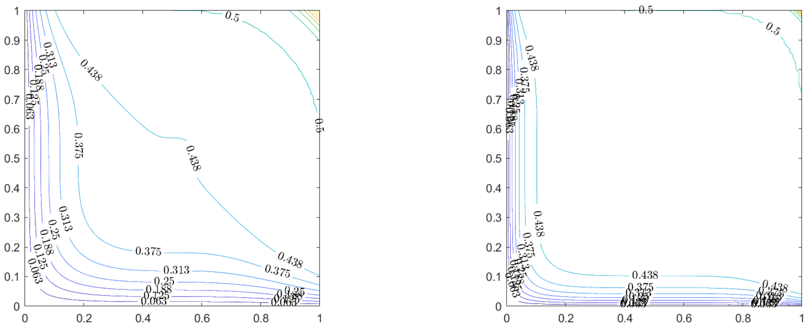


Figure 7.3: T-norms with generators with derivative at 0.5 equal to infinity $\theta_{\frac{1}{3}}, \theta_{\frac{1}{5}}$

Chapter 8

Diagonals of t-norms

This chapter will be about diagonals of t-norms T with regard to the multiplicative generator θ.

Definition 8.0.1. A *diagonal* Δ of a t-norm T, also known as the second power of T, is the unary function $\Delta(x) = T(x, x)$.

A t-norm is not uniquely defined by its diagonal [11]. We can construct a multiplicative generator θ of a t-norm with diagonal Δ as follows: [13]

1. Choose a point of the graph $(x, \theta(x)) \in]0, 1[^2$.
2. The diagonal determines θ on a countable infinite set $M \subset]0, 1[$.
3. The set M is not dense. Therefore, choose two subsequent elements $a, b \in M$ such that $a < b$ and $]a, b[\cap M = \emptyset$. The restriction $\theta|_{]a,b[}$ can be any function such that $\theta|_{[a,b]}$ is strictly increasing and continuous.
4. The remaining values of θ are uniquely determined by Δ and the preceding steps.

Proposition 2. *The range of possible values of the multiplicative generator is limited by the difference $x - \Delta(x) > 0$.*

Proof. We know from the definition of Archimedean t-norm T that $x > T(x, x)$ for all $x \in]0, 1[$. Let us rewrite that as $x > \Delta(x)$, from which we get $x - \Delta(x) > 0$ by subtracting $\Delta(x)$ from both sides. $\square$

Notice that this is the same as $x - \theta^{-1}(\theta(x)\,\theta(x)) > 0$ or as $x > \theta^{-1}(\theta(x)\,\theta(x))$.

The closer the diagonal is to the identity, the closer must be all t-norms with this diagonal. This restriction may be helpful if the diagonal is close to the identity [13]. For example, Frank t-norms' diagonals Δ_λ^F get closer and closer to the function x as the parameter λ gets smaller, as we can see in Figure 8.1. In fact, the diagonal that is the closest to the function x has parameter $\lambda = 10^{-6}$, which is the smallest one displayed there. Then according to Proposition 2, if we construct multiplicative generators of Frank t-norms knowing only their diagonal, the smaller λ we would try to do it for, the smaller approximated error we would get.

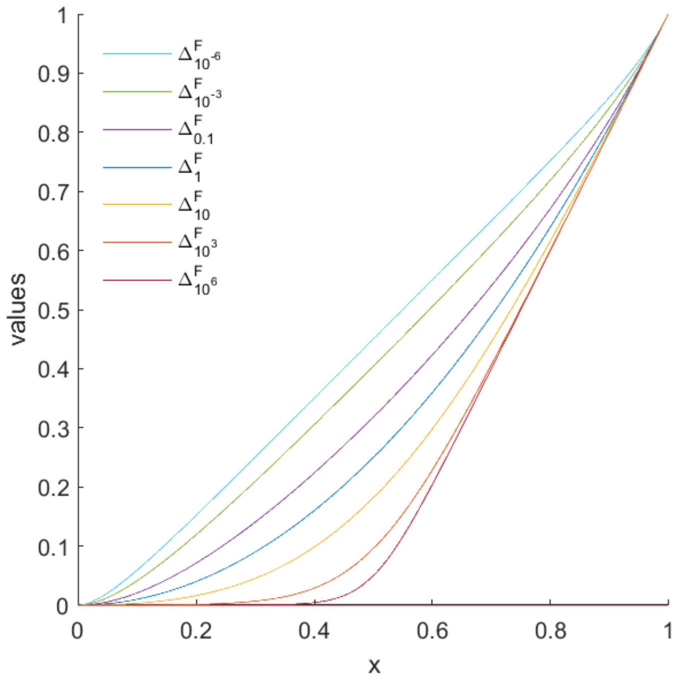


Figure 8.1: Diagonals of some strict Frank t-norms

Chapter 9

Conclusions

Let us start this chapter with a summary of how we met the goal presented in the assignment of this thesis. Throughout this thesis, we studied the relationship between fuzzy conjunction and its generator. As for the first point of our goal, we summarized contemporary results about the relationship in chapters 2, 4, and 8. In chapter 4, we showed why our efforts to extend (3.5) by using higher-level derivatives failed. In chapter 8, we managed to describe a relationship between the multiplicative generator of a strict t-norm and its diagonal.

As for the second point, we demonstrated the effect of local properties and changes of generators on the corresponding t-norms. In chapters 5 and 6, we only changed the generators in certain intervals. In chapter 7, we focused on the derivative of a generator at 0.5. In all these chapters, we first described how we chose the generators and some of their qualities. Then we moved on and discussed the t-norms they generated and how they relate to each other.

The third and last point of the goal was treated mainly in chapters 3 and in the theoretical part of chapters 5 and 6. In chapter 3, we proved that a balanced generator is unique. Only t-norms with no balanced generator have generators with derivatives that are either always equal to 0 or always equal to ∞. In chapter 6, we found that a differential of a t-norm at $(0, 0)$ only exists for those that have a balanced generator. In chapter 5, we showed how the differential of t-norm at $(1, 1)$ and the derivatives of its generators at 1 are intertwined.

We encountered an interesting issue when plotting a triangular norm using its generator. There is no graphic library for fuzzy operations that we know of, so we had to make our own tools. It turned out that plotting t-norms was harder than imagined and we had to take the generated images with a grain of salt. To remind the reader of the spotted flaw, the values of lines $x = 1$ and $y = 1$ should be equal to y and x respectively. Nevertheless, Figures 5.7a, 5.7b, 6.5a and 6.5b do not meet this requirement, even though the t-norms that are displayed there do.

We did not have enough time to correct this problem in this thesis. Nevertheless, we do think it would be useful if someone really looked into the causes and possible solutions to this problem.

Afterward, we took a closer look at formula (3.2). We tried to extend it to other t-norms. However, we found that taking higher derivatives of t-norms with the first derivative at 0 equal to 0 or ∞ is not the way. We still believe this result is worth trying to extend because it might enable us to extend the notion of the balanced generator to other strict t-norms. Moreover, we hope that new ways of using balanced generators will be discovered. For example, it would be interesting to take more balanced generators and compare them and their corresponding t-norms to each other as we did in section 5.2.

Finally, we also noticed that if a generator has a derivative of 0 or ∞, it usually has a very unusual corresponding t-norm. It would be interesting to get a deeper understanding of this effect.

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