Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Computer Science and Engineering

Geometric Algebra

Bachelor thesis

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Field of study: Software Engineering and Technology
Supervisor: Ing. Matěj Dostál, Ph.D.

Prague, 2023

ZADÁNÍ BAKALÁŘSKÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

Příjmení: **Delov** Jméno: **Teodor** Osobní číslo: **492214**

Fakulta/ústav: **Fakulta elektrotechnická**

Zadávající katedra/ústav: **Katedra počítačů**

Studijní program: **Softwarové inženýrství a technologie**

II. ÚDAJE K BAKALÁŘSKÉ PRÁCI

Název bakalářské práce:

Geometrická algebra

Název bakalářské práce anglicky:

Geometric algebra

Pokyny pro vypracování:

The student will read basic literature related to geometric algebra. The student will introduce geometric algebra informally with geometric examples from low-dimensional spaces R^n in the text of the bachelor thesis. He will then show a formal construction of geometric algebra over R^n (with the standard scalar product), and finally show a simple application of geometric algebra of his choice.

Seznam doporučené literatury:

L. Dorst, D. Fontijne, S. Mann, Geometric Algebra for Computer Science. Morgan Kaufmann, 2nd printing, 2009
E. Chisolm, Geometric Algebra, available online: arXiv:1205.5935
A. Macdonald, Linear and Geometric Algebra. CreateSpace Independent Publishing Platform, 2011
A. Macdonald, An elementary construction of the geometric algebra, 2016, available online:
http://www.faculty.luther.edu/~macdonal/GAConstruct.pdf

Jméno a pracoviště vedoucí(ho) bakalářské práce:

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Jméno a pracoviště druhé(ho) vedoucí(ho) nebo konzultanta(ky) bakalářské práce:

Datum zadání bakalářské práce: **15.09.2022** Termín odevzdání bakalářské práce: **26.05.2023**

Platnost zadání bakalářské práce: **19.02.2024**

Ing. Matěj Dostál, Ph.D. podpis vedoucí(ho) práce	podpis vedoucí(ho) ústavu/katedry	prof. Mgr. Petr Páta, Ph.D. podpis děkana(ky)

III. PŘEVZETÍ ZADÁNÍ

Student bere na vědomí, že je povinen vypracovat bakalářskou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací.
Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v bakalářské práci.

Declaration

I hereby declare that I completed the presented thesis independently and that all used sources are quoted in accordance with the Methodological Instructions that cover the ethical principles for writing an academic thesis.

In Prague, 2023

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Teodor Delov

Acknowledgements

I would like to express my gratitude to the supervisor of this thesis, Ing. Matěj Dostál, Ph.D. for the interesting assignment and for all the help he provided, especially for keeping me on the right course during this thesis. Finally, gratitude has to be expressed to my mother Margarita and father Goran, for inspiring and motivating me.

Abstract

The thesis focuses on geometric algebra. We start by an informal introduction to the necessary concepts of geometric algebra such as: oriented lengths, reflections, rotations. We also show the fundamentality of geometric algebra, the geometric product. We then show a simple application of the use of geometric algebra and explore the Cramer's rule as a method of determining solutions. Finally we show the construction of geometric algebra.

Keywords: oriented lengths, geometric product, reflections, rotations, Cramer's rule.

Abstrakt

Práce se zaměřuje na geometrickou algebru. Začneme neformálním úvodem do pojmů geometrické algebry, jako jsou: orientované délky, odrazy, rotace. Ukazujeme také podstatu geometrické algebry, geometrického součinu. Poté ukážeme jednoduchou aplikaci použití geometrické algebry a prozkoumáme Cramerovo pravidlo jako metodu určování řešení. Nakonec ukážeme konstrukci geometrické algebry.

Klíčová slova: orientované délky, geometrický součin, odrazy a rotace, Cramerovo pravidlo.

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Chapter 1

Introduction

This is a bachelor's thesis project which can be interpreted as an introduction to geometric algebra. It introduces the basic terms, concepts, properties and their use in geometric algebra, the way the the geometric algebra is constructed and some of it's applications. Geometric algebra or also called Clifford Algebra provides a generalized theory that surrounds many mathematical topics such as vectors, complex numbers, matrix algebra etc. It is also called Clifford algebra, because William Kingdon Clifford united the inner and outer product into a single geometric product, which is the fundamental identity of geometric algebra. In this thesis we prefer the term geometric algebra because that was also Clifford's choice. Geometric algebra is mainly about vector multiplication and the geometric product and how we interpret it. It is a tool which helps to express geometrical relationships through algebraic equations. The use of geometric algebra in the past years has proven to be helpful in solving geometrical problems in subfields of computer science where these problems occur: computer graphics, robotics and computer vision.We usually talk about the concept of a *vector* represented as a one-dimensional segment of a line with direction, orientation and magnitude. In this project, for example we are also going to get familiar with the term *bivector*, which is like a two-dimensional vector. A *trivector* which is like a three-dimensional vector. This is a very short insight in how geometric algebra works, it takes the concept of a vector as a one-dimensional segment and extends this concept to multiple dimensions.

Chapter 2

Informal Introduction

2.1 Basic terms and introduction to GA

In this chapter, general theory and geometry terms will be introduced in order to build an apparatus for achieving the goals of this work.

2.1.1 Scalars

Definition 2.1.1 (*Scalars*)**.** [1] A *scalar* is physical quantity that is completely described by it's magnitude. In this thesis we will represent scalars as a, b, c

Example. [*Scalars*] Examples of scalars are volume, density, speed, etc.

2.1.2 Vectors

Definition 2.1.2 (*Vector*)**.** [1] A *vector* is a quantity that has both magnitude and direction.

. In this thesis we will represent vectors as $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

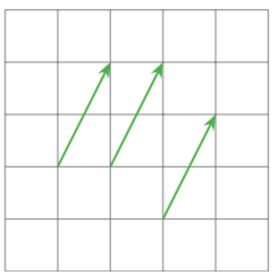

Figure 2.1: Example of vectors
[1]

By representing the vectors as *oriented lengths*, we can use that later to help us introduce the operations *vector addition* and *scalar multiplication*, which will also help us in defining the term *vector space*. We define the concept of a vector space below:

2.1.3 Vector Space

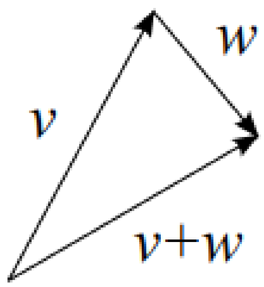

Figure 2.2: Vector addition [2]

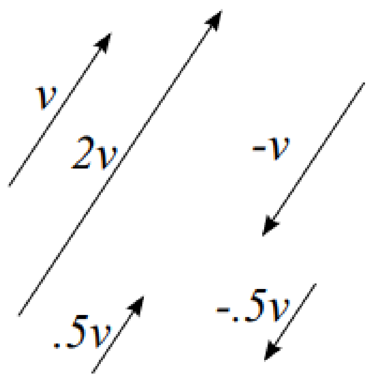

Figure 2.3: Scalar multiplication [2]

Definition 2.1.3 (*Vector Space*)**.** A *vector space* consists of a set V (it's objects are called vectors) and a field $\mathbb{F}$ (it's object are called scalars), where vector addition and scalar multiplication are defined [2]. For it to be a *vector space* the vector axioms mentioned below must be satisfied. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b scalars. Here for further reference we list the vector and algebra axioms:

V1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,

V2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$,

V3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$,

V4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$,

V5. $1\mathbf{u} = \mathbf{u}$,

V6. $(ab)\mathbf{u} = a(b\mathbf{u})$,

V7. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$,

V8. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

A1. $\mathbf{u}(\mathbf{v}\mathbf{w}) = (\mathbf{u}\mathbf{v})\mathbf{w}$,

A2. $\mathbf{u}(\mathbf{v} + \mathbf{w}) = \mathbf{u}\mathbf{v} + \mathbf{u}\mathbf{w}$
$(\mathbf{v} + \mathbf{w})\mathbf{u} = \mathbf{v}\mathbf{u} + \mathbf{w}\mathbf{u}$,

A3. $(a\mathbf{u})\mathbf{v} = \mathbf{u}(a\mathbf{v}) = a(\mathbf{u}\mathbf{v})$,

A4. $1\mathbf{u} = \mathbf{u}1 = \mathbf{u}$,

We define *vector addition* and *scalar multiplication* as the following :

- Let vectors $v, w \in V$. *Vector addition* is an operation which produces a third vector, $v + w \in V$.

- Let vector $v \in V$, and scalar $c \in \mathbb{F}$. *Scalar multiplication* is an operation which produces a third vector, $cv \in V$.

Let **V** be the set of two $n \times 1$ matrices, consisting of real numbers and let the field of scalars be $\mathbb{R}^n$. In $\mathbb{R}^n$ we can define the vector addition and scalar multiplication as the following:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \qquad c\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}$$

The verification of the vector axioms is straightforward but we will still show a few examples to show that axioms are satisfied.

$$\begin{aligned}
[V2] \left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}\right) + \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} &= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \\
&= \begin{pmatrix} (x_1 + y_1) + z_1 \\ (x_2 + y_2) + z_2 \\ \vdots \\ (x_n + y_n) + z_n \end{pmatrix} \\
&= \begin{pmatrix} x_1 + (y_1 + z_1) \\ x_2 + (y_2 + z_2) \\ \vdots \\ x_n + (y_n + z_n) \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \left(\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}\right)
\end{aligned}$$

$$[V6](ab)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}(ab)x_1\\(ab)x_2\\\vdots\\(ab)x_n\end{pmatrix}=\begin{pmatrix}a(bx_1)\\a(bx_2)\\\vdots\\a(bx_n)\end{pmatrix}=a\left(b\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\right)$$

$$[V8](a+b)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}(a+b)x_1\\(a+b)x_2\\\vdots\\(a+b)x_n\end{pmatrix}=\begin{pmatrix}ax_1+bx_1\\ax_2+bx_2\\\vdots\\ax_n+bx_n\end{pmatrix}=a\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}+b\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$$

Oriented lengths are not the only objects that can be represented as vectors. The term *vectors* can also be used to represent for example *functions* or *infinite sequences.* We can not visualize them as arrows with a certain length, but they also create a vector space. This might make it harder to comprehend the concept of a vector but it greatly increases the applications of the theory. So geometric algebra introduces new vector spaces that are of significance to geometry.

2.1.4 Oriented lengths

[3] As stated before for now we will refer to the common vectors as oriented lengths. The oriented length has a beginning point and an end point with an arrow at the end. These oriented lengths have properties. We formally define the properties below:

Definition 2.1.4.

Attitude - Two oriented lengths have the same attitude if they are aligned or are parallel to each other.

Orientation - The orientation is the direction in which the arrow of the length is pointing. Two oriented lengths have the same orientation if they have the same attitude and if the direction in which their arrows point is the same.

norm - The term *norm* is used to represent the length or size of the oriented lengths. The norm will always have non-negative value. It does not take into consideration the orientation. We denote the norm of $\mathbf{v}$ as $\|\mathbf{v}\|$

Example. For example, let $\mathbf{u}$ be an oriented length, then $-2\mathbf{u}$, will have the same attitude but opposite orientation and twice the norm.

Oriented lengths are used to construct oriented areas. We define the term *oriented area below*. Later in this thesis, oriented lengths will also help us define the term *inner product* and *outer product*.

2.1.5 Oriented Areas

Definition 2.1.5 (*Oriented Area*). [3] An oriented area is a plane segment that has weight and orientation. It's orientation and weight must be well-defined.

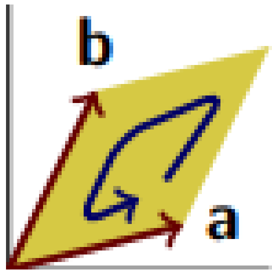

Figure 2.4: Example of oriented area

We will use the uppercase bold letter $\mathbf{B}$ to denote oriented areas. In GA the term *bivector* is synonymous with the term *oriented area*. Similarly as the oriented lengths, we also define formally the properties of the oriented area:

Definition 2.1.6.

Attitude - Two oriented areas $\mathbf{B_1}$ and $\mathbf{B_2}$ have the same attitude if they are on the same or on parallel planes.

Orientation - The term orientation is used to describe the rotational direction of the oriented area. It is used to compare areas with the same attitude.

norm - The term *norm* is used to represent the measure of the area of $\mathbf{B}$. The norm will always have non-negative value. It does not take into consideration the orientation. It is a measure for areas with the same attitude. we denote the norm of $\mathbf{B}$ as $\|\mathbf{B}\|$

Example. Oriented areas do not have any shape, on 2.1.5, the shown shapes are the same element. We say that two oriented areas $\mathbf{B_1}$ and $\mathbf{B_2}$ are equivalent if they have the same attitude, norm and orientation.

Figure 2.5: Representation of two equivalent oriented areas. [3]

2.1.6 Oriented Volumes

After talking about oriented lengths and oriented areas, we now mention and define the term *oriented volumes*.

Definition 2.1.7 (*Oriented Volume*). [3] An *oriented volume* represents a three-dimensional space spanned by its factors, with a weight (volume) and orientation.

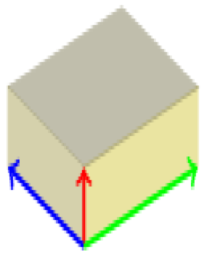

Figure 2.6: Example of an oriented volume.

In the thesis we will use the uppercase bold letter $\mathbf{T}$ to denote oriented volumes. In GA the term *trivector* is synonymous with the term *oriented volume*. Oriented volumes have the following geometric properties:

Definition 2.1.8.

Attitude - In 3-D space, every volume has only one attitude.

Orientation - The term orientation is used to describe the direction of the volume in space, it is define by the oriented lengths that are perpendicular to the volume.

Norm - It is used to describe the measure of the volume of the oriented volume .

2.1.7 The Geometric Algebra $\mathbb{G}^n$

The Geometric Algebra $\mathbb{G}^n$ is a 2^n vector space containing $\mathbb{R}^n$ [4]. We explain how is a 2^n vector space in more detail in section 4.3.1. We can reserve the term *vector* for vectors in $\mathbb{R}^n$, so that In $\mathbb{G}^n$ we can use the term *multivectors*.

$\mathbb{G}^3$

Definition 2.1.9 (*Objects in* $\mathbb{G}^3$)**.** The objects in $\mathbb{G}^3$ have the form of

$$M = s + \mathbf{v} + \mathbf{B} + \mathbf{T}, \tag{2.1}$$

where s is a scalar, $\mathbf{v}$ a *vector*, $\mathbf{B}$ a *bivector*, $\mathbf{T}$ a *trivector*. M is called a *multivector*.

2.2 Inner, Outter, Geometric Product

2.2.1 Inner Product

Definition 2.2.1 (*Inner Product*)**.** [3] For given vectors $\mathbf{u}$ and $\mathbf{v}$, the *inner product* is a *scalar*:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos\theta, 0 \leq \theta \leq \pi, \tag{2.2}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ when one is moved parallel to itself so that the initial points of $\mathbf{u}$ and $\mathbf{v}$ coincide.

Theorem 2.2.1 (*Inner Product Properties*)**.** The three following properties for the inner product are true. We provide this theorem without proof:

- $\mathbf{u} \parallel \mathbf{v} \rightarrow \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$ [1],
- $\mathbf{u} \perp \mathbf{v} \rightarrow \mathbf{u} \cdot \mathbf{v} = 0$ [2],
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

2.2.2 Outer Product

Two oriented lengths can be used to create the outer product. We formally define the term *outer product* below:

[1] $\mathbf{u} \parallel \mathbf{v}$ means that $\mathbf{u}$ and $\mathbf{v}$ are parallel
[2] $\mathbf{u} \perp \mathbf{v}$ means that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular

Definition 2.2.2 (*Outer Product*). [3] The outer product combines oriented lengths (vectors), to create higher dimensional elements such as oriented areas and oriented volumes. It is denoted by the symbol $\wedge$ called *wedge*. Vectors **u** and **v**, create the outer product $u \wedge v$. It is an operation that has the following properties. Let a be scalar and u, v, and w be vectors, then:

antisymmetry $u \wedge v = -v \wedge u$

distributive $u \wedge (v + w) = u \wedge v + u \wedge w$,
$(v + w) \wedge u = v \wedge u + w \wedge u$

homogeneity $(au) \wedge v = u \wedge (av) = a(u \wedge v)$

parallel $u \wedge v = 0$

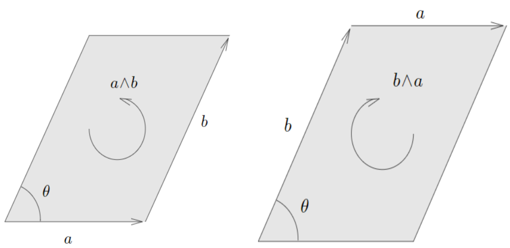

Figure 2.7: The outer product. By changing the order of the vectors, we reverse the orientation and introduce a minus sign in the product [5].

Example. Let's say we want to find the area of a parallelogram spanned [3] by the vectors $\mathbf{u} = e_1 + 2e_2$ and $\mathbf{v} = -e_1 - e_2$, relative to the area of $e_1 \wedge e_2$. First we would start by determining the outer product $\mathbf{u} \wedge \mathbf{v}$ and using it's properties ($u_i \wedge u_i = 0$ and $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$).

$$
\begin{aligned}
\mathbf{u} \wedge \mathbf{v} & \\
&= (e_1 + 2e_2) \wedge (-e_1 - e_2) \\
&= -e_1 \wedge e_1 - e_1 \wedge e_2 - 2e_2 \wedge e_1 - 2e_2 \wedge e_2 \\
&= -2e_1 \wedge e_2 - 2e_2 \wedge e_1
\end{aligned}
$$

[3] The term *span* is used informally here. For more details see book [3]

Then the area relative to $e_1 \wedge e_2$ would be:

$$\begin{aligned}\frac{\|\mathbf{u} \wedge \mathbf{v}\|}{\|e_1 \wedge e_2\|} \\ &= \frac{\| -2e_1 \wedge e_2 - 2e_2 \wedge e_1\|}{\|e_1 \wedge e_2\|} \\ &= \frac{\|0\|}{\|e_1 \wedge e_2\|} \\ &= 0.\end{aligned}$$

The area of a parallelogram relative to $e_1 \wedge e_2$ is 0.

2.2.3 Geometric Product

We talked about the inner and outer product. Now we will talk about the *geometric product*. The *geometric product* is the fundamental identity of geometric algebra, it is what makes geometric algebra powerful as a mathematical tool.

Definition 2.2.3 (*Geometric Product*)**.** [3] The *geometric product* is defined as a sum of the inner product and outer product.

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}. \tag{2.3}$$

Definition 2.2.4. [*Geometric Product Properties*]

The two following properties for the geometric product are true:

- $\mathbf{u} \parallel \mathbf{v} \rightarrow \mathbf{uv} = \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$,
- $\mathbf{u} \perp \mathbf{v} \rightarrow \mathbf{uv} = \mathbf{u} \wedge \mathbf{v}$,
- $(a\mathbf{u})\mathbf{v} = \mathbf{u}(a\mathbf{v}) = a(\mathbf{uv})$
- $\mathbf{u}(\mathbf{v} + \mathbf{w}) = \mathbf{uv} + \mathbf{uw}$,
  $(\mathbf{v} + \mathbf{w})\mathbf{u} = \mathbf{vu} + \mathbf{wu}$
- $(\mathbf{uv})\mathbf{w} = \mathbf{u}(\mathbf{vw})$

By reversal of the order of the elements in the *geometric product*, we will get:

$$\mathbf{vu} = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \wedge \mathbf{u} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \wedge \mathbf{v}. \tag{2.4}$$

Let $\mathbf{u}$ be a nonzero vector. Then $\mathbf{uu} = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \wedge \mathbf{u}$. From the definition of the properties of the outer product, we know that $\mathbf{u} \wedge \mathbf{u} = 0$. After further simplification, we get: $\mathbf{u} \cdot \mathbf{u} + 0 = \mathbf{u}^2$. After this we can write:

$$1 = \frac{\mathbf{uu}}{\mathbf{uu}} = \frac{1}{\mathbf{u^2}}\mathbf{uu} = \mathbf{u}\frac{1}{\mathbf{u^2}}\mathbf{u}$$

After further simplification we define the multiplicative inverse of a vector as:

Definition 2.2.5 (*multiplicative inverse of a vector*)**.**

$$\mathbf{u^{-1}} = \frac{1}{u} = \frac{u}{\mathbf{u}^2}. \tag{2.5}$$

2.3 Complex Numbers

This part was inspired from the book [4], and more details can be found there.

2.3.1 Complex numbers

GA complex numbers represent a multivector which combines a scalar and a vector component. The scalar represents the real part while the vector represents the imaginary part. Let a, b be scalars and $\mathbf{i}$ the unit pseudoscalar mentioned in subsection 2.3.2, then we write the complex number as:

$$a + b\mathbf{i}$$

2.3.2 Pseudoscalars

The *pseudoscalar* or also called *unit pseudoscalar* is a geometric unit that has a unique orientation in the vector space. It contains the word scalar because it behaves similarly but it is more complex. It plays a big role in GA because it helps us later to define rotations and reflections. This is because it captures the orientation of the vector space.

Definition 2.3.1 (*Pseudoscalars*)**.** We denote the pseudoscalar as $\mathbf{I}$. $\mathbf{I} = e_1 e_2 \cdots e_n$ It represents the highest dimensional subspace in $\mathbb{G}^n$. In 2-D we denote it as $\mathbf{i}$.

An important property of the unit pseudoscalar is it's square. The square of unit pseudoscalar is:

$$\mathbf{I}^2 = e_{123}e_{123} = (e_1e_2e_3)(e_1e_2e_3) = -1. \tag{2.6}$$

2.4 Rotation, Reflection

2.4.1 Reflection

We are starting to see the power of geometric algebra when we consider reflections and rotations. The geometric product of vectors **u** and **v** is uv. We start by multiplying the right side by the vector v^{-1}, which is the inverse of v. After we finish, we should be left with parts that are parallel and perpendicular to **v**. This is achieved by:

$$\begin{aligned} u = (uv)v^{-1} \\ &= (u\dot{v} + u \wedge v)v^{-1} \\ &= (u\dot{v})v^{-1} + (u \wedge v)v^{-1} \end{aligned} \tag{2.7}$$

from here we see:

$$u_{\|_v} = (u\dot{v})v^{-1},\, u_{\perp_v} = (u \wedge v)v^{-1}. \tag{2.8}$$

. The equation for $u_{\|_v}$ is the projection of u on to **v**, while $u_{\perp_v}$ is the perpendicular component (sometimes called rejection). By switching the sign of $(u \wedge v)v^{-1}$, we get the vector:

$$u' = u_{\|} - u_{\perp} = (u\dot{v})v^{-1} - (u \wedge v)v^{-1}. \tag{2.9}$$

The vector u' is the reflection. This is the same as if we would multiply the left side of the geometric product with v^{-1}. In the end we would be left with an equation that is true in general for any reflection.

In short the reflection is a transformation that mirrors a geometric object across a plane.

2.4.2 Rotation

Before we define *rotations*, we need to define what is an angle in GA:

Definition 2.4.1 (The angle $\mathbf{I}\theta$)**.** [4] The angle represents a bivector $\mathbf{I}\theta$. The unit pseudoscalar $\mathbf{I}$ describes the plane of rotation, while θ describes the amount of the rotation.

Definition 2.4.2 (*Exponential* $e^{\mathbf{I}\theta}$)**.**

$$e^{\mathbf{I}\theta} = \cos\theta + \mathbf{I}\sin\theta. \tag{2.10}$$

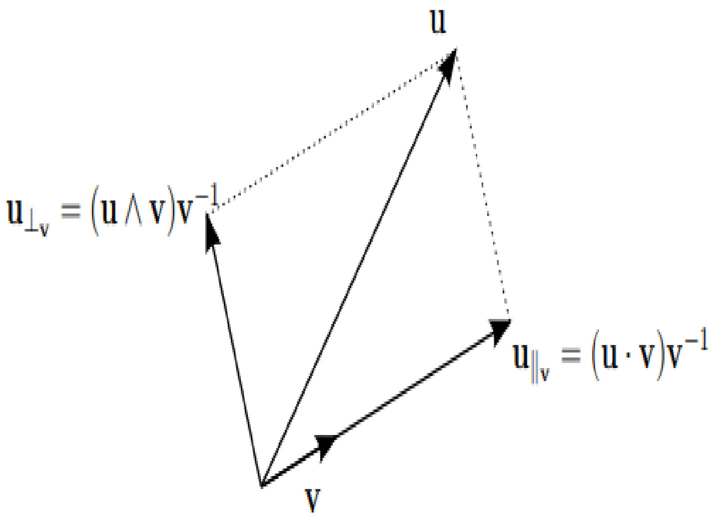

Figure 2.8: Example of reflection. [2]

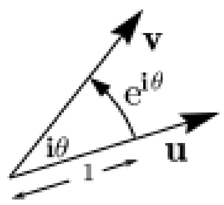

Figure 2.9: Rotation by angle $\mathbf{I}\theta$. [4]

***Rotation in the plane* I**

We define the rotation in the plane $\mathbf{I}$ by angle θ. Let's imagine that the rotation moves the vector $\mathbf{u}$ to the vector $\mathbf{v}$. Then u and v are in the plane of rotation, and then we have:

$$\begin{aligned}
\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} \mathbf{u} \wedge \mathbf{v} \\
&= \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta) + \|\mathbf{u}\|\|\mathbf{v}\|\mathbf{I} \sin(\theta) \\
&= \|\mathbf{u}\|\|\mathbf{v}\|(\cos(\theta) + \mathbf{I} \sin(\theta)) \\
&= \|\mathbf{u}\|\|\mathbf{v}\| e^{\mathbf{I}\theta}
\end{aligned} \tag{2.11}$$

We can further manipulate 2.11 by multiplying it with $\mathbf{u}$ and in the end we should be left with: $\mathbf{v} = \mathbf{u}e^{\mathbf{I}\theta}$. This shows the rotation of $\mathbf{u}$ to $\mathbf{v}$.

Chapter 3

Linear Algebra

We will now show some simple applications of geometric algebra that are explained in more detail in the survey article [4] by Macdonald.

3.1 Linear Independence

The outer product obtained from the geometric product can be used to characterize linear independence of vectors:

Theorem 3.1.1. The vectors $\mathbf{u_1}, \mathbf{u_2} \cdots \mathbf{u_n}$ are linearly independent if and only if

$$\mathbf{u_1} \wedge \mathbf{u_2} \wedge ... \wedge \mathbf{u_n} \neq 0.$$

Geometrically, this means that the vectors span a (possibly degenerate) parallelepiped. The oriented n-volume of such parallelepiped is nonzero precisely when the vectors defining the parallelepiped are linearly independent.

3.2 Determinant

In geometrical context the *determinant* represents a scalar value which describes the factor by which it multiplies n-volumes.

We can calculate the determinant by multiplying the result from the outer product of the vectors by the inverse of the unit pseudoscalar $\mathbf{I}$. We provide the following theorem without proof [4]. Let $\mathbf{u_1}, \mathbf{u_2} \cdots \mathbf{u_n}$ be a set of vectors in $\mathbb{R}^n$ and $\{e_1, e_2, \cdots e_n\}$ an orthonormal basis for $\mathbb{R}^n$ Then:

Theorem 3.2.1.

$$u_1 \wedge u_2 \wedge ... \wedge u_n = \det \begin{bmatrix} e_1u_1 & e_1u_2 & \dots & e_1u_n \\ \vdots & \ddots & & \\ e_nu_1 & e_nu_2 & \dots & e_nu_n \end{bmatrix} \mathbf{I}, \tag{3.1}$$

The determinant also gives us information on invertibility and orientation of a multivector. Let's imagine a multivector in 3-D space. The absolute value of the determinant tells us whether the multivector 'shrinks' or 'expands'. If the determinant is negative, that means that the orientation of the multivector is negative. If it is positive, the orientation is positive. If it is nonzero, it tells us that the multivector has an inverse.

3.3 Cramer's rule

Cramer's rule is used to simplify solving systems of linear equations. Let A be a matrix with n rows and columns. It is said that Cramer's rule is usually practical for solving simple systems, for example a system with three equations and with three unknowns and where the integers are small. For more complex systems a Gauss's Method based approach is preferred because it is faster. If $\det(A) \neq 0$ then the system: $A\vec{x} = \vec{b}$ has a unique solution $\chi_i = \det(B_i)/\det(A)$. B_i is formed by substituting the i column with the solution vector $\vec{b}$.

Example. Here we have an example of a system with three equations and three unknown variables x, y, z. We are going to solve for z.

$$\begin{cases} 2x + y + z = 1 \\ 3x + z = 4 \\ x - y - z = 2 \end{cases}$$

By simple calculation we can see that $\det(A) = 3$. After this we replace the third column with the solution vector $\vec{b}$ to calculate the determinant of the new matrix and we get $\det(B_3) = 3$. We have $\det(A)/\det(B_3) = 1$. $z = 1$.

In elementary linear algebra we would view the expression for x as three equations for u_1, u_2 and u_3, in components:

$$a_1u_1 + b_1u_2 + c_1u_3 = x_1$$
$$a_2u_1 + b_2u_2 + c_2u_3 = x_2$$
$$a_3u_1 + b_3u_2 + c_3u_3 = x_3$$

Geometrically the ratio of bivectors and trivectors is actually the ratio of their weights.

The weights can be expressed using the determinant.

$$x_1 = \frac{\det(\begin{Vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{Vmatrix})}{\det(\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix})} \quad x_2 = \frac{\det(\begin{Vmatrix} a_1 & x_1 & c_1 \\ a_2 & x_2 & c_2 \\ a_3 & x_3 & c_3 \end{Vmatrix})}{\det(\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix})} \quad x_3 = \frac{\det(\begin{Vmatrix} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ a_3 & b_3 & x_3 \end{Vmatrix})}{\det(\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix})}$$

Example. Problem: To solve for c_3 in $\mathbb{R}^4, v = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4$ we can multiply the equation by $u_1 \wedge u_2$ on the left and u_4 on the right. And use $u_i \wedge u_i = 0$:

$$u_1 \wedge u_2 \wedge v \wedge u_4 =$$
$$c_1(u_1 \wedge u_2 \wedge u_1 \wedge u_4) + c_2(u_1 \wedge u_2 \wedge u_2 \wedge u_4) + c_3(u_1 \wedge u_2 \wedge u_3 \wedge u_4) + c_4(u_1 \wedge u_2 \wedge u_4 \wedge u_4) =$$

$$u_1 \wedge u_2 \wedge v \wedge u_4 = c_3(u_1 \wedge u_2 \wedge u_3 \wedge u_4)$$

$$c_3 = \frac{u_1 \wedge u_2 \wedge v \wedge u_4}{u_1 \wedge u_2 \wedge u_3 \wedge u_4}$$

If the u_i are linearly independent, the outer product on the right is invertible, and we have a formula for c_2.

Chapter 4

Construction of GA

In this chapter we will show an elementary construction of the geometric algebra on R^n.

To be able to show the construction, we will need to introduce some mathematical prerequisites. In section 4.1 we will introduce equivalence relations and their properties. In section 4.2 we will define and explain formal linear combinations and the construction of a linear space of all formal linear combinations.

In the whole chapter we will be dealing with vector spaces over the real numbers – the field of scalars will always be $\mathbb{R}$.

4.1 Equivalence relations

Before talking about the construction GA(n) 4.5 basic knowledge of discrete mathematics is needed. The construction by Macdonald's approach uses equivalence classes. That is why in this part we define what equivalence relations and equivalence classes are. This will help us later in defining the construction of the vector space GA(n). The definitions found in this section, can also be found in the book [6] with maybe different notations.

Definition 4.1.1 (*Binary Relation*)**.** Let us have sets A and B. The binary relation R from set A to set B is a set of ordered pairs (a, b), where $a \in A$ and $b \in B$. R is a subset of the Cartesian product $A \times B$, $R \subset A \times B$. In the case $A = B$ we speak of a binary relation on the set A. If the elements a, b are in the relation R, we write $(a, b) \in R$ or aRb .

Example. Let A be a set of all the people dining in a restaurant, and B a set of all the items in the menu. It is not possible to write the relation between the elements of A and B with a functional dependency, because it is unlikely that every client will have at most only one item from the menu. And that one item from the menu will be ordered at most from one client. To describe the relation we need a more general term. All the information are gathered in the set:

$$R = \{(a, b) \in A \times B \mid \text{client a has item b}\} \tag{4.1}$$

The elements in R represent ordered pairs and they are the basis of the term *relation.*

Definition 4.1.2 (*Reflexivity*)**.** Let R be a relation on the set A. The relation R is reflexive if for every $a \in A$ holds that aRa.

Reflexivity described with words means that every element in A is in a relation with itself.

Example. One of the easiest examples of a reflexive relation is "=" on the set of real numbers, because every real number is always equal to itself.

Definition 4.1.3 (*Symmetry*)**.** Let R be a relation on the set A. The relation R is symmetric if for every $a, b \in A$ holds the implication $aRb \implies bRa$.

Described with words this means that element a is in a relation with b, and b is in a relation with a.

Example. Also one of the easiest examples of a symmetric relation is "=" on the set of real numbers, because if $a = b \implies b = a$.

Definition 4.1.4 (*Transitivity*)**.** Let R be a relation on the set A. The relation R is transitive if for every $a, b, c \in A$ holds aRc, whenever aRb and bRc.

This means that if the first element is in a relation to the second and the second is in a relation with the third, then the first must also be in a relation with the third element.

Example. An example of a transitive relation is "=" on the set of real numbers, because if $a = b$ and $b = c$, then $a = c$.

Definition 4.1.5 (*Equivalence relation*)**.** A relation R on a set A is called an equivalence relation if the relation is reflexive, symmetric and transitive. In the case of equivalence relation we use $a \sim b$ instead of aRb.

Example. From the previous examples we can see that the relation "is equal to" fulfils all of the requirements and it is an equivalence relation.

Definition 4.1.6 (*Equivalence class*)**.** Let A be a set with an equivalence relation $\sim$. The equivalence class determined by the element $a \in A$ is the set:

$$[\,a\,] := \{b \in A \mid b \sim a\}. \tag{4.2}$$

Here we refer to the equation (4.2).

Example. While calculating mod 5, the equivalence class determined by element 1 is equal to:

$$[1] = \{5k + 1 \mid k \in Z\}$$

Similarly $[0] = \{5k \mid k \in Z\}$, $[2] = \{5k + 2 \mid k \in Z\}$, $[3] = \{5k + 3 \mid k \in Z\}$, $[4] = \{5k + 4 \mid k \in Z\}$. We can see that we have five different equivalence classes. By adding them together we get the set of integers $\mathbb{Z}$.

4.2 Formal linear combinations

In linear algebra, a formal linear combination is typically used to describe a sum of vectors, where each vector is multiplied by a scalar coefficient. Generally, formal linear combinations are expressions that are generated by elements of a given set by multiplying those elements by scalar coefficients and summing them together.

The vector space of formal linear combinations is the set of all possible formal linear combinations that can be formed using a given set of (formal) vectors. It is equipped with the operations of multiplication by scalars and addition so that the axioms of a vector space are satisfied. It is a fundamental concept in linear algebra and provides a way to describe and analyze vector spaces.

Definition 4.2.1. Let A be a set. The set of formal linear combinations over A, denoted by V, is defined as follows:

$$V = \{c_1a_1 + c_2a_2 + ... + c_na_n \mid n \in \mathbb{N}, c_1, c_2, \ldots, c_n \in \mathbb{R}, a_1, a_2, \ldots, a_n \in A\}$$

In this definition, $c_1, c_2, \ldots, c_n$ are scalar coefficients, $a_1, a_2, \ldots, a_n$ are elements of the set A, and n is a non-negative integer.

Formal linear combinations can be added and multiplied by scalars.

Definition 4.2.2. For any two formal linear combinations $u = c_1a_1 + c_2a_2 + \cdots + c_nd_n$ and $v = d_1a_1 + d_2a_2 + \cdots + d_na_n$ in V, their sum $u + v$ is defined to be another formal linear combination in V, given by

$$u + v = (c_1 + d_1)a_1 + (c_2 + d_2)a_2 + \cdots + (c_n + d_n)a_n.$$

For any scalar α and any formal linear combination $u = c_1a_1 + c_2a_2 + \cdots + c_nd_n$ in V, their product αu is defined to be a formal linear combination in V, given by

$$\alpha u = \alpha(c_1a_1 + c_2a_2 + \cdots + c_na_n) = (\alpha c_1)a_1 + (\alpha c_2)a_2 + \cdots + (\alpha c_n)a_n.$$

These definitions ensure that the vector space of formal linear combinations over the set A satisfies the axioms of a vector space.

4.3 Introduction to the construction

In the rest of this chapter we show a formal construction of the Geometric Algebra $GA(n)$ over $\mathbb{R}^n$ with the standard inner product. It is inspired by Macdonald [7]. In most literature the description of geometric algebra uses advanced concepts, such as tensor products or usually skips the proofs of existence of the algebra. Macdonald gives an elementary and direct approach of the geometric algebra over $\mathbb{R}^n$. With his approach, only elementary knowledge of $\mathbb{R}^n$ and discrete mathematics is needed.

It is our task to add to the structure of the vector space $\mathbb{R}^n$ a way to multiply vectors – to add an *algebra structure* to $\mathbb{R}^n$. This multiplication is the geometric product from Chapter 2. Moreover, we need to enforce the defining properties of the geometric product: these are the two fundamental identities that distinguish $GA(n)$: for any pair u, v of orthonormal vectors in $\mathbb{R}^n$,

$$ee = 1, \tag{4.3}$$

$$ef = -fe. \tag{4.4}$$

Of course, we see that the result of the geometric product does not need to be a vector. The construction of the geometric algebra $GA(n)$ therefore adds structure to the vector space $\mathbb{R}^n$ that we need to describe precisely in the construction. Just stating the axioms from subsection 2.1.3 is practical, but that offers no guarantees that a mathematical structure GA(n) really exists, because the axioms might be inconsistent.

4.3.1 The canonical basis of $GA(3)$

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathbb{R}^3$. The basis of the geometric algebra $GA(3)$ consists of the vectors described in the table below.

The index of the products of e's increases. All of the products are in the basis. By rearranging the order, we only change the sign (4.4). This means that the original product and the new product are linearly dependant. In section 4.5 we stated that $\mathbb{G}^n$ has a dimension of 2^n.

$$
\begin{array}{ccc}
 & 1 & \\
e_1 & e_2 & e_3 \\
e_1e_2 & e_1e_3 & e_2e_3 \\
 & e_1e_2e_3 &
\end{array}
$$

Table 4.1: Canonical basis for $\mathbb{G}^3$

Example. Let us try and form a 4-vector in $\mathbb{G}^3$, e.g., $e_1e_2e_3e_2$, according to the product rules in (4.3) and (4.4), this is equal to $-e_1e_3$. There are no 4-vectors in $\mathbb{G}^3$. A member of the basis contains an e or it does not. Thus $\mathbb{G}^3$ has dimension $2^3 = 8$. More generally, the geometric algebra $GA(n) = \mathbb{G}^n$ will have a dimension of 2^n.

4.4 Multiplication of basis vectors

Given a vector space $\mathbb{R}^n$, we are supposed to 'freely add' an operation of a product that satisfies the identities (4.3) and (4.4).

Given an orthonormal basis $\{e_1, e_2, ..., e_n\}$ for $\mathbb{R}^n$, we therefore need to have a way of multiplying at least the basis vectors. Taking any sequence $(e_{i_1}, e_{i_2}, \ldots, e_{i_r})$ of the orthonormal basis vectors, we can therefore formally multiplying them by considering the sequence itself as a product:

$$\mathbf{E} = e_{i_1}e_{i_2}\cdots e_{i_r}. \tag{4.5}$$

In the given sequence any e can occur more than once and the order for the e's is important.

The idea for the construction of $GA(n)$ is to take the formal products of the basis vectors as the basis for a new vector space and set $GA(n)$ to be this space. However, in this way the space $GA(n)$ would have infinite dimension and the required identities (4.3) and (4.4) would not hold. We therefore have to identify some of the sequences (formal products) of basis vectors as equal. This is done by defining an equivalence relation on the set of all formal products of basis vectors.

Given a sequence E as in (4.5) we define two ways of obtaining a new sequence called E' from E. A new sequence can be obtained by a finite number of steps of two types:

1. by exchanging pairs of adjacent and unequal e's in E, and
2. by inserting or deleting pairs of adjacent and equal e's.

Let us denote the transformation of E into E' by $E' = T(E)$. Depending of the number of exchanges we made to create the new sequence we can have $T(E) = E'$ be odd or even.

Example. If $E = e_2e_1e_1e_3$, we show some of the many possible ways to obtain a new sequence E' from E. By removing the pair e_1e_1 from E we would obtain $E' = e_2e_3$. By adding a pair e_2e_2 we can obtain $E' = e_2e_1e_2e_2e_1e_3$. The pairs e_2e_1 and e_1e_3 can be exchanged to give a sequence $E' = e_1e_2e_3e_1$.

We can now start identifying some pairs formal products of basis vectors:

1. the identity (4.3) requires that inserting or deleting pairs of adjacent equal vectors does not change the product,

2. the identity (4.4) requires that exchanging pairs of adjacent and unequal vectors should change the sign of the product.

It should therefore be sufficient to identify those sequences E and E' for which there is an even transformation T such that $T(E) = E'$, and if $T(E) = E'$ for an odd transformation T, the sequence E should be identified with $-E'$ in the resulting vector space.

However, for this approach to work, we need to show that whenever there are two transformations T and T' such that $T(E) = T'(E) = E'$, these two transformations are of the same parity (they are both either even or odd). Otherwise our construction would not be well defined. This is solved by the following lemma.

Lemma.

Any two transformations T, T' with $T(E) = T(E') = E'$ are both either even or odd.

Proof. Let $G(E)$ be the number of times where the index of a vector e in the sequence E is greater than the index of the vector e that is to its right. Observe now two facts:

1. Given a sequence F and obtaining F' by adding or removing a pair of adjacent and equal vectors v does not change the quantity G: $G(F) = G(F')$.

2. Given a sequence F and obtaining F' by exchanging a pair of adjacent and unequal vectors, $G(F') = G(F) \pm 1$.

The transformations T and T' are given by a finite number of steps of the above two types. When applied to E, both transformations give E'. This means that $G(T(E)) = G(E') = G(T'(E))$. However, the quantity G can be used to compute the parity of a transformation: if the numbers $G(E)$ and $G(T(E))$ are the same, the transformation T is even, otherwise it is odd. Since $G(T(E)) = G(T'(E))$, both T and T' are either even or odd, which concludes the proof. ∎

4.5 Construction of the vector space GA(n)

Let us now define the vector space $GA(n)$. Starting with $\mathbb{R}^n$ (with the standard dot product), and taking its orthonormal basis $(e_1, e_2, \ldots, e_n)$, we first construct the set

$$A = \{(e_{i_1}, \ldots, e_{i_k}) \mid k \in \mathbb{N}, i_1, \ldots, i_k \in \{1, \ldots, n\}\}$$

of all formal products (sequences) of the basis vectors. Write any sequence $E = (e_{i_1}, \ldots, e_{i_k})$ as a formal product $e_{i_1} \ldots e_{i_k}$. Form an equivalence relation $\sim$ on A by requiring $E \sim E'$ whenever there is an even transformation T for which $T(E) = E'$.

Remark. The relation $\sim$ is an equivalence relation. We quickly show this.

1. The relation $\sim$ is reflexive, since for every E the identity transformation I, for which $I(E) = E$, has an even parity (we perform 0 swaps of vectors in the formal product). Therefore $E \sim E$.

2. The relation $\sim$ is symmetric. If $E \sim E'$, so for some pair of sequences E and E' there is an even transformation T such that $T(E) = E'$, we need to show that $E' \sim E$. That is, we need to find an even transformation T' such that $T'(E') = E$. Such a transformation exists: take the steps performed in the transformation T and perform them 'backwards'.

3. The relation $\sim$ is transitive. If $E \sim E'$ and $E' \sim E''$ for some sequences E, E', E'', this means that there are two even transformations T and T' such that $T(E) = E'$ and $T'(E') = E''$. We need to show that $E \sim E''$. This means that we need to find an even transformation T'' for which $T(E) = E''$ holds. Of course one such transformation is $T' \circ T$, the composition of the steps of T followed by T'.

We are now almost in a position to define the vector space $GA(n)$. The idea is to take the equivalence classes on A formed by $\sim$ and form the vector space of formal linear combinations over these classes. The equivalence classes would then form the basis of the space. Intuitively, the equivalence class formalizes the fact that e.g. the vectors $e_1e_2e_3$ and $e_3e_1e_2$ are to be identified in $GA(n)$. However, this idea solves only a half of the problem. In the vector space $GA(n)$ we also need to identify e.g. the vector e_1e_2 with the vector $-e_2e_1$ (due to equation (4.4)). We describe how this is done below.

Definition 4.5.1. Let $B = A/\sim$. Let U be the set of formal linear combinations over the set B equipped with the operation of addition of formal linear combinations and with the operation of multiplication by scalar as defined in section 4.2. We define $GA(n)$ to be the vector space formed from U by identification of the following pairs of vectors: given any sequence E and an odd transformation T, $[T(E)] = -[E]$.

We will show that this construction of $GA(n)$ satisfies the vector space axioms V1-V8 (2.1.3).

Let $u, v, w \in GA(n)$. We have $u = x_1U_1 + x_2U_2 + \ldots + x_nU_n$, $v = y_1U_1 + y_2U_2 + \ldots + y_nU_n$, $w = z_1U_1 + z_2U_2 + \ldots + z_nU_n$, where the scalars are real numbers and the symbols U_i refer to the equivalence classes of the sequences E generated by the equivalence relation $\sim$.

- The axiom [V1]:

$$\begin{aligned}
u + v &= (x_1 + y_1)U_1 + \ldots + (x_n + y_n)U_n \\
&= (y_1 + x_1)U_1 + \ldots + (y_n + x_n)U_n \\
&= (y_1U_1 + \ldots + y_nU_n) + (x_1U_1 + \ldots + x_nU_n) \\
&= v + u.
\end{aligned}$$

- The axiom [V2]:

$$\begin{aligned}
(u + v) + w &= ((x_1 + y_1)U_1 + \cdots + (x_n + y_n)U_n) + (z_1U1 + \cdots + z_nU_n) \\
&= ((x_1 + y_1) + z_1)U_1 + \cdots + ((x_n + y_n) + z_n)U_n \\
&= (x_1U_1 + \cdots + x_nU_n) + ((y_1 + z_1)U_1 + \cdots + (y_n + z_n)U_n) \\
&= u + ((y_1U_1 + \cdots + y_nU_n) + (z_1U_1 + \cdots + z_nU_n)) \\
&= u + (v + w).
\end{aligned}$$

- The axiom [V3]:

$$\begin{aligned}
u + 0 &= (x_1U_1 + \cdots + x_nU_n) + (0U_1 + \cdots + 0U_n) \\
&= (x_1 + 0)U_1 + \cdots + (x_n + 0)U_n \\
&= x_1U_1 + \cdots + x_nU_n \\
&= u
\end{aligned}$$

- The axiom [V4]:

$$\begin{aligned}
u + (-u) &= (x_1U_1 + \cdots + x_nU_n) + ((-x_1)U_1 + \cdots + (-x_n)U_n) \\
&= (x_1 + (-x_1))U_1 + \cdots + (x_n + (-x_n))U_n \\
&= 0U_1 + \cdots + 0U_n \\
&= 0.
\end{aligned}$$

- The axiom [V5]:

$$\begin{aligned}
1u &= 1(x_1U_1 + \cdots + x_nU_n) \\
&= 1(x_1U_1) + \cdots + (x_nU_n) \\
&= (1x_1)U_1 + \cdots + (1x_nU_n) \\
&= x_1U_n + \cdots + x_nU_n \\
&= u.
\end{aligned}$$

- The axiom [V6]:

$$\begin{aligned}
a(bu) &= a(b(x_1U_1 + \cdots + x_nU_n)) \\
&= a((bx_1)U_1 + \cdots + (bx_n)U_n) \\
&= a(bx_1)U_1 + \cdots + a(bx_n)U_n \\
&= ((ab)x_1)U_1 + \cdots + ((ab)x_n)U_n \\
&= (ab)(x_1U_1) + \cdots + (ab)(x_nU_n) \\
&= ab(x_1U_1 + \cdots + x_nU_n) \\
&= abu
\end{aligned}$$

- The axiom [V7]:

$$\begin{aligned}
a(u+v) &= a((x_1U_1 + \cdots + x_nU_n) + (y_1U_1 + \cdots + y_nU_n)) \\
&= a((x_1 + y_1)U_1 + \cdots + (x_n + y_n)U_n) \\
&= (a(x_1 + y_1))U_1 + \cdots + (a(x_n + y_n))U_n \\
&= (ax_1 + ay_1)U_1 + \cdots + (ax_n + ay_n)U_n \\
&= (a(x_1U_1) + \cdots + a(x_nU_n)) + (a(y_1U_1) + \cdots + a(y_nU_n)) \\
&= a(x_1U_1 + \cdots + x_nU_n) + a(y_1U_1 + \cdots + y_nU_n) \\
&= au + av
\end{aligned}$$

- The axiom [V8]:

$$
\begin{aligned}
&(a+b)u \\
&\quad = (a+b)(x_1U_1 + \cdots + x_nU_n) \\
&\quad = ((a+b)x_1)U_1 + \cdots + ((a+b)x_n)U_n \\
&\quad = (ax_1 + bx_1)U_1 + \cdots + (ax_n + bx_n)U_n \\
&\quad = ((ax_1)U_1 + \cdots + (ax_n)U_n) + ((bx_1)U_1 + \cdots + (bx_n)U_n) \\
&\quad = au + bu
\end{aligned}
$$

Observe that the identification of E and E' whenever $E' = T(E)$ for an odd transformation T does not pose any problems and the fact that $GA(n)$ is well defined as a vector space follow from the fact that the set of all formal linear combinations over a given set naturally forms a vector space.

4.6 Construction of the GA

Having defined the vector space $GA(n)$, we now show how to equip it with a product (the geometric product) so that we get the geometric algebra $GA(n)$.

The idea is to first define the product naturally on the basis vectors of $GA(n)$, and then extend the definition by linearity. Given two sequences E and F, consisting of basis vectors, where $E = e_{i_1}e_{i_2}\cdots e_{i_r}$, and $F = e_{j_1}e_{j_2}\cdots e_{j_s}$, Macdonald defines the geometric product as a concatenation of the two sequences.

$$EF = (e_{i_1}e_{i_2}\cdots e_{i_r})(e_{j_1}e_{j_2}\cdots e_{j_s}) = e_{i_1}e_{i_2}\cdots e_{i_r}e_{j_1}e_{j_2}\cdots e_{j_s} \tag{4.6}$$

This product is well defined on the equivalence classes $[E]$ of sequences: whenever $E \sim E'$ and $F \sim F'$, then $EF \sim E\prime F\prime$. This follows immediately from the definition of the equivalence relation $\sim$. Also observe that because the product is defined by concatenation of sequences, we obtain that the product is associative (on the basis vectors) for free. We can thus write the product of two sequences without parentheses.

The extension of the product on all elements of $GA(n)$ is done by linearity as follows: given two vectors $u = \sum_i a_iE_i$ and $v = \sum_j b_jF_j$, their product is defined as

$$(\sum_i a_iE_i)(\sum_j b_jF_j) = \sum_{i,j} a_ib_jE_iF_j \tag{4.7}$$

The equation can be informally explained: every vector in $GA(n)$ is a formal linear combination of the basis vectors – equivalence classes of sequences. Since we know how to multiply the basis vectors, write down the formal linear combinations that define the vectors u and v and define the product by formal expansion of the formal linear combinations. Forming the product in this way is well defined since it is well defined on the basis vectors.

The axioms A1-A4 are all satisfied. To verify that they hold in $GA(n)$ is straightforward, but we show the proof for some of them below.

Given $u = a_1U_1 + a_2U_2 + \ldots + a_iU_i$, $v = b_1V_1 + b_2V_2 + \ldots + b_nV_n$, $w = c_1W_1 + c_2W_2 + \ldots + c_eW_e$, where the scalars are real numbers, the axiom [A1] holds by the following computation:

$$\begin{aligned}
u(vw) &= (\sum_i a_iU_i)((\sum_j b_jV_j)(\sum_e c_eW_e)) \\
&= (\sum_i a_iU_i)(\sum_{j,e} b_jc_e(V_jW_e)) \\
&= \sum_{i,j,e} a_i(b_jc_e)U_i(V_jW_e) \\
&= \sum_{i,j,e}(a_ib_j)c_e(U_iV_j)W_e \\
&= (\sum_{i,j}(a_ib_jU_iV_j))(\sum_e(c_eW_e)) \\
&= (uv)w.
\end{aligned}$$

The axiom [A3] holds by the following computation:

$$\begin{aligned}
&(\alpha u)v \\
&\qquad = \alpha(\sum_i a_iU_i)(\sum_j b_jV_j) \\
&\qquad = (\sum_i(\alpha a_i)U_i)(\sum_j b_jV_j) \\
&\qquad = \sum_{i,j}(\alpha a_i)b_jU_iV_j \\
&\qquad = \sum_{i,j}\alpha(a_ib_j)U_iV_j \\
&\qquad = \alpha\sum_{i,j}(a_ib_j)U_iV_j \\
&\qquad = \alpha((\sum_i a_iU_i)(\sum_j b_jV_j)) \\
&\qquad = \alpha(uv).
\end{aligned}$$

Moreover, observe that the equations (4.3) and (4.4) hold in $GA(n)$ by definition: this is precisely why we defined the equivalence relation $\sim$ the way we did. This concludes the construction of $GA(n)$.

Chapter 5

Conclusion

The goal of this thesis is aimed to explain the fundamental mathematical principles used in geometric algebra. The thesis consists of three main chapters, where we primarily focused on describing the foundations of geometric algebra, the construction of geometric algebra and it's applications, so that we can have a better understanding of this mathematical framework. In chapter 2 we started by establishing the basic terms and definitions necessary to understand the following chapters. We defined scalars, vectors and the vector space. We then explored the inner, outer and geometric product. We saw how and why they represent an important feature of geometric algebra. The inner product grasps the dot product of vectors, while the outer product extends it to capture the concept of oriented lengths, oriented areas and oriented volumes. Then we introduced the fundamentality of geometric algebra, the geometric product. The geometric product is a sum of the inner and outer product. We then show the integration of complex numbers in geometric algebra.After this by leveraging the complex number we introduced the transformations called reflection and rotation, and how to perform these operations using simple algebraic operations. In chapter 3 we continue to dive deeper into the concepts of geometric algebra and show a simple application based on the foundation we laid in chapter 2. We explore how can linear independence be proved in the concept of geometric algebra. Then investigate the role of the determinants and emphasize their role in studying oriented areas and volumes. This leads us to the Cramer's rule which represents an elegant method for finding solutions of systems of linear equations. In the chapter 4, we focused on the construction of geometric algebra, showing it's underlying structures and building blocks and also showing it's consistency. In the end readers of the thesis should have the basic knowledge to be able to continue exploring more advanced concepts and topics of the geometric algebra and it's geometrically intuitive approach to problem-solving. This geometrically intuitive approach increases the clarity and insight researchers, engineers and mathematicians can use to solve problems related to computer graphics, robotics, and computer vision.

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