Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering
Department of Physics

Graded Lie theory

Gradovaná Lieova teorie

Diploma thesis

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KATEDRA FYZIKY

ČVUT
ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

ZADÁNÍ DIPLOMOVÉ PRÁCE

Akademický rok: 2022/2023

Student: Bc. Rudolf Šmolka

Studijní program: Matematická fyzika

Název práce: (česky) Gradovaná Lieova teorie

Název práce: (anglicky) Graded Lie theory

Jazyk práce: Angličtina

Pokyny pro vypracování:

1) Nalézt vhodné definice gradovaných Lieových algeber a grup pro použití v kontextu obecných gradovaných variet.

2) Zobecnit akce Lieových grup a vyšetřit jejich vlastnosti.

3) Vymyslet definici gradovaných hlavních fibrovaných prostorů.

4) Prozkoumat možnosti zobecnění forem konexe a křivosti.

Doporučená literatura:

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[4] C Bartocci, U Bruzzo, D Hernández-Ruipérez: The geometry of supermanifolds, Vol. 71. Springer Science & Business Media, 2012

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prohlašuji, že jsem diplomovou práci s názvem:

Gradovaná Lieova teorie

vypracoval samostatně a uvedl veškeré použité informační zdroje v souladu s Metodickým pokynem o dodržování etických principů při přípravě vysokoškolských závěrečných prací.

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podpis

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Gradovaná Lieova teorie

Autor: Rudolf Šmolka

Studijní program: Matematická fyzika

Druh práce: Diplomová práce

Vedoucí práce: Ing. Jan Vysoký, Ph.D., Katedra fyziky, FJFI ČVUT v Praze

Abstrakt: V této práci se zabýváme zobecněním některých základních definic a poznatků Lieovy teorie do prostředí $\mathbb{Z}$-gradovaných variet. Uvedeme nezbytné základy teorie kategorií a teorie gradovaných variet, zavedeme ekvivalenty grup v obecných kategoriích a odvodíme o nich základní poznatky. Dále definujeme pojem gradovaná Lieova grupa a algebra, a představíme gradovanou grupu invertibilních matic. Detailně diskutujeme gradovaná levoinvariantní a fundamentální vektorová pole. Definujeme a zkoumáme gradované hlavní fibrované prostory a zkonstruujeme gradovanou hlavní fibraci repérů. Zabýváme se také vertikálními a horizontálními distribucemi v gradovaných hlavních fibrovaných prostorech a uvádíme definice forem konexe a křivosti.

Klíčová slova: gradovaná Lieova grupa, gradovaná Lieova algebra, gradovaný hlavní fibrovaný prostor

Title:

Graded Lie theory

Author: Rudolf Šmolka

Abstract: In this work, we examine some key concepts of Lie theory in the setting of $\mathbb{Z}$-graded manifolds. Necessary fundamental parts of category theory and graded manifold theory are reviewed; generalization of groups to abstract categories is discussed and some results are derived. Graded Lie groups and algebras are defined and we introduce the graded general linear group. Graded left-invariant and fundamental vector fields are examined in detail. Graded principal bundles are defined and studied and the graded frame bundle is constructed. Some treatment of vertical and horizontal distributions in graded principal bundles is presented, along with forms of connection and curvature.

Key words: graded Lie group, graded Lie algebra, graded principal bundle

Contents

Introduction

The aim of the following text is to generalize some key concepts of Lie theory — such as Lie groups and their actions, associated Lie algebras, left-invariant and fundamental vector fields, principal bundles and horizontal connections — to the setting of $\mathbb{Z}$-graded manifolds as they were introduced in [10].

Extensive work has already been done on this topic in the world of $\mathbb{Z}_2$-graded manifolds, else known as supermanifolds ([5], [1], [4], [9]). Note that in this text, the word "graded" will be reserved to mean $\mathbb{Z}$-graded. While supermanifolds and graded manifolds, as defined in [10], differ in several ways — for one, supermanifolds admit only anticommuting graded variables — they share one crucial commonality: the sections of their structure sheaf, called graded functions, are not fully determined by their values at the points of the underlying topological space. However, this is a common occurrence in another branch of mathematics: algebraic geometry. As a result, many techniques from algebraic geometry have been adopted for use in supermanifold and graded manifold theory. It is also the case that even if most techniques and result from supermanifold theory are not directly applicable to graded manifolds, they can often be suitably modified or serve as an inspiration. Indeed, much of this text was inspired by the book [2] or the article [9].

Let us lay forward the structure of this text. We begin by recalling some fundamental category theory, the basic concepts of which are a necessary prerequisite for our investigation of graded Lie theory. Notably we will state and prove the Yoneda lemma that will be used extensively in the next chapter. The main resource for these introductory passages and indeed any category theory in this work is the book [8] and we will also mostly follow the notation therein.

Next, we examine what we choose to call "group objects" which generalize the concept of a group to a very broad class of categories — those that admit finite products and contain a terminal object. Then we shift our focus to group objects in functor categories whose objects are functors and whose morphisms are natural transformations, and show that they can be reduced to a collection of "ordinary" groups and their homomorphisms. We use the Yoneda lemma to apply what we learned to a broader range of categories. The use of Yoneda lemma in the treatment of supermanifolds is abundant and appears e.g. in [2] under the name of a "functor of points" approach.

In the second chapter we give a concise review of graded manifolds as presented in [10] after which we put forth a definition of a graded Lie group as a group object in the category of graded manifolds. In $\mathbb{Z}_2$-graded setting this approach is used in [2] but appears already e.g. in [1]. We give a concrete example of a graded Lie group: the graded general linear group, which will accompany us throughout the rest of this text.

In the third chapter we briefly recall vector fields on graded manifolds [10] and examine in detail the notion of left-invariant vector fields on graded Lie groups and fundamental vector fields on graded manifolds acted upon by graded Lie groups. We also examine the correspondence of graded left-invariant vector fields and the tangent space at the unit of the graded Lie group. In the fourth and final chapter we define a graded principal $\mathcal{G}$-bundle for any graded Lie group $\mathcal{G}$ and construct a concrete example: the graded frame bundle, which is acted upon by the graded general linear group. Finally we examine, to some extent, vertical and horizontal distributions on graded principal bundles, define the form of connection and the exterior covariant derivative, and hence also the form of curvature, and illustrate them on the trivial bundle.

Chapter 1

Categorical Foundations

1.1 Prerequisites from Category Theory

Our intention here is not to give a self-sufficient introduction to category theory. An interested reader will find that and much more in the excellent book [8] or in other similar manuscript. However, we find it prudent to give a quick overview of some of the basic concepts which will be used extensively throughout this text.

The definition of a category is a very fundamental one and to be done properly it requires some preliminary set-theoretical discussion. Clearly, this is not a place for that; for our purposes a category C is a collection[1] of **objects** and **arrows** (also called **morphisms**) between these objects. To express that a is an object in C we write $a \in \mathsf{C}$. For two objects $a, b \in \mathsf{C}$ we denote as $\mathsf{C}(a, b)$ the collection of all arrows from a to b. To express that f is an arrow from a to b, we write $f \in \mathsf{C}(a, b)$ or more commonly $f : a \to b$. If $\mathsf{C}(a, b)$ is a set for any two objects $a, b \in \mathsf{C}$ we say that C is **locally small**.

For C to be a category we require, in addition, the ability to **compose arrows**: for any $f : a \to b$ and $g : b \to c$ in C there must exist an arrow $g \circ f : a \to c$ and this rule of composition must be associative. Finally, for any $a \in \mathsf{C}$ there must exist an arrow $1_a : a \to a$ such that for any objects $b, c \in \mathsf{C}$ and any arrows $f : b \to a$ and $g : a \to c$ there is $1_a \circ f = f$ and $g \circ 1_a = g$. We say that an arrow $f : a \to b$ in C is an **isomorphism** if there exists an arrow $g : b \to a$ such that $f \circ g = 1_b$ and $g \circ f = 1_a$. For any category C one obtains a so-called opposite category C^{op} by reversing all the arrows in C.

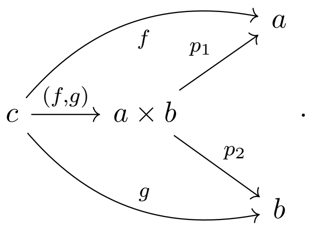
Definition 1.1 (Terminal Object)**.** Let C be a category and consider $t \in \mathsf{C}$. We say that t is a **terminal object** in C if for any $c \in C$ there exists exactly one arrow from c to t. A terminal object, if it exists, is unique up to a unique isomorphism.

As an illustration, a terminal object in the category of sets is any one-point set and the terminal object in the category of vector spaces is the zero vector space $\{0\}$. We also need the notion of a product.

Definition 1.2 (Product of Objects)**.** Let C be a category and consider two objects $a, b \in \mathsf{C}$. We say that an object $a \times b \in \mathsf{C}$ together with two arrows $p_1 : a \times b \to a$ and $p_2 : a \times b \to b$ is a **product**

[1] Notice we avoid using the word "set". A reason for that is simple: consider, for example, the category of sets. Often it is denoted as Set, its objects are sets and its arrows are ordinary maps between sets. It is a well-known fact that the collection of all sets is itself not a set.

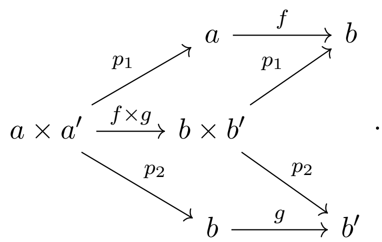
of a **and** b if it satisfies that for any object $c \in \mathsf{C}$ and any two arrows $f : c \to a$ and $g : c \to b$ there exists a unique arrow $(f, g) : c \to a \times b$ such that $p_1 \circ (f, g) = f$ and $p_2 \circ (f, g) = g$. This may be best visualized with a commutative diagram:



(1.1)

The arrows p_1 and p_2 are then called the **projections** on the first and on the second object, respectively. Note that products need not exist, but if they do, they are unique up to a unique isomorphism. Products of three or more objects are defined analogously. It can be show inductively [8] that if a product exists for any two objects, then it exists for any n objects for any $n \in \mathbb{N}$. Such categories are said to admit (contain, have) **finite products**.

As an illustration, consider that products in the category of sets are simply Cartesian products. Let us also introduce the following notation: consider four objects $a, a', b, b' \in \mathsf{C}$ such that the products $a \times a'$ and $b \times b'$ exist. Consequently, for any two arrows $f : a \to b$ and $g : a' \to b'$ there is a unique arrow, denoted as $f \times g$, fitting into the commutative diagram



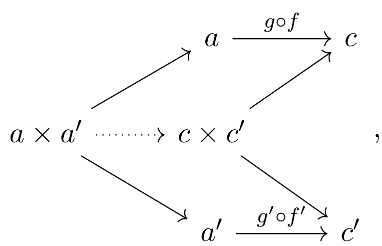
(1.2)

Of course, we have $f \times g = (f \circ p_1, g \circ p_2)$ where p_1 and p_2 are the projections from $a \times a'$.

Lemma 1.3. *Let* C *be a category with finite products and consider* $f : a \to b$*,* $g : b \to c$*,* $f' : a' \to b'$ *and* $g' : b' \to c'$ *in* C*. Then*

$$(g \circ f) \times (g' \circ f') = (g \times g') \circ (f \times f'). \tag{1.3}$$

Proof. $(g \circ f) \times (g' \circ f')$ is defined as the unique morphism fitting as the dashed arrow into the commutative diagram



(1.4)

where the unlabeled full arrows are the canonical product projections. Now consider

$$\begin{array}{ccccccc} & & a & \xrightarrow{f} & b & \xrightarrow{g} & c \\ & \nearrow & & \nearrow & & \nearrow & \\ a \times a' & \xrightarrow{f \times f'} & b \times b' & \xrightarrow{g \times g'} & c \times c' & & \\ & \searrow & & \searrow & & \searrow & \\ & & a' & \xrightarrow{f'} & b' & \xrightarrow{g'} & c' \end{array} \quad , \tag{1.5}$$

which clearly commutes. Consequently $(g \times g') \circ (f \times f')$ fits into (1.4), thus

$$(g \times g') \circ (f \times f') = (g \circ f) \times (g' \circ f'). \tag{1.6}$$

Note that in particular there is $(f \times 1) \circ (1 \times g) = f \times g = (1 \times g) \circ (f \times 1)$ for any two arrows f, g. ■

The last two fundamental concepts we need are those of functors and natural transformations. If C and D are two categories, then a **functor** F from C to D, written as $F : \mathsf{C} \to \mathsf{D}$, assigns to every object $a \in \mathsf{C}$ and object $Fa \in \mathsf{D}$ and to every arrow $f \in \mathsf{C}(a, b)$ and arrow $Ff \in \mathsf{D}(Fa, Fb)$ so that $F1 = 1$ and $F(f \circ g) = Ff \circ Fg$ for any composable arrows f and g. Given two functors $F, G : \mathsf{C} \to \mathsf{D}$ we define a **natural transformation** of F and G, written as $\eta : F \to G$, as a collection of arrows $\{\eta_a\}_{a \in C}$, where $\eta_a \in \mathsf{D}(Fa, Ga)$ which satisfies the so-called naturality condition: for any $a, b \in \mathsf{C}$ and any $f \in \mathsf{C}(a, b)$ the diagram

$$\begin{array}{ccc} Fa & \xrightarrow{\eta_a} & Ga \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ Fb & \xrightarrow{\eta_b} & Gb \end{array} \tag{1.7}$$

must commute. For any two categories C, D we denote as D^{C} the category whose objects are functors from C to D and whose arrows are their natural transformations. A category of this type is called a functor category.

1.2 Yoneda's Lemma

In this subchapter we state and prove the famous Yoneda's lemma and we include it due to the paramount importance Yoneda's lemma plays in subchapter 1.5. The contents have been largely taken from [8] with minor changes. Let Set denote the category of sets, i.e. the category whose objects are sets and whose arrows are maps between sets. Consider some locally small category C and let us define two functors $E, N : \mathsf{Set}^{\mathsf{C}} \times \mathsf{C} \to \mathsf{Set}$.

- The functor E is called the **evaluation functor** and is defined on objects as

$$E(H, x) := Hx, \tag{1.8}$$

hence the name, and for any natural transformation $\alpha : H \to K$ and $f \in \mathsf{C}(x, y)$ the arrow $E(\alpha, f)$ is defined as the arrow $Hx \to Ky$ arising from the commutative diagram

$$\begin{array}{ccc} Hx & \xrightarrow{\alpha_x} & Kx \\ \downarrow{\scriptstyle Hf} & & \downarrow{\scriptstyle Kf} \\ Hy & \xrightarrow{\alpha_y} & Ky. \end{array} \tag{1.9}$$

- The functor N is defined on objects as

$$N(H,x) = \mathrm{Nat}(\mathsf{C}(x,\cdot),H), \tag{1.10}$$

the set of all natural transformations between $\mathsf{C}(x,\cdot)$ and H. For any arrow $(\alpha : H \to K, f : x \to y)$ we have a set map $N(\alpha,f) : \mathrm{Nat}(\mathsf{C}(x,\cdot),H) \to \mathrm{Nat}(\mathsf{C}(y,\cdot),K)$ given, for every $\eta \in \mathrm{Nat}(\mathsf{C}(x,\cdot),H)$ and every $z \in \mathsf{C}$ as

$$\left(N(\alpha,f)\eta\right)_z := \alpha_z \circ \eta_z \circ f_z^*, \tag{1.11}$$

where f_z^* is the pre-composition with arrow f, or in other words, $f_z^* = \mathsf{C}(f,z)$, though it may be better to view f^* as a natural transformation $f^* : \mathsf{C}(y,\cdot) \to \mathsf{C}(x,\cdot)$. Then we have the direct formula

$$N(\alpha,f)\eta := \alpha \circ \eta \circ f^*, \tag{1.12}$$

which immediately implies naturality. That $N(\alpha,f)\eta$ is natural can also be seen by drawing explicitly the commutative diagram

$$\begin{array}{ccccccc} \mathsf{C}(y,z) & \xrightarrow{f^*} & \mathsf{C}(x,z) & \xrightarrow{\eta_z} & Hz & \xrightarrow{\alpha_z} & Kz \\ \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle Hh} & & \downarrow{\scriptstyle Kh} \\ \mathsf{C}(y,z') & \xrightarrow{f^*} & \mathsf{C}(x,z') & \xrightarrow{\eta_{z'}} & Hz' & \xrightarrow{\alpha_{z'}} & Kz' \end{array} \tag{1.13}$$

for any $h : z \to z'$ in C.

We may now state the famous lemma:

Theorem 1.4 (Yoneda's Lemma)**.** *Let C be a locally small category and E, N the functors defined above. Then the assignment $y : N \to E$ where for every $(H,x) \in \mathsf{Set}^{\mathsf{C}} \times \mathsf{C}$ the map of sets*

$$y_{(H,x)} : \mathrm{Nat}(\mathsf{C}(x,\cdot),H) \to Hx, \tag{1.14}$$

is given by

$$y_{(H,x)}(\eta) := \eta_x(1_x), \tag{1.15}$$

defines a natural isomorphism $y : N \to E$.

Proof. Let us first verify that the assignment (1.14) is a bijection for every (H,x) and then investigate naturality. Consider some element $e \in Hx$. Then every natural transformation $\eta \in \mathrm{Nat}(\mathsf{C}(x,\cdot),H)$ such that $\eta_x(1_x) = e$ must comply with the commutativity of

$$\begin{array}{ccc} \mathsf{C}(x,x) & \xrightarrow{\eta_x} & Hx \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle Hf} \\ \mathsf{C}(x,x') & \xrightarrow{\eta_{x'}} & Hx' \end{array} \tag{1.16}$$

for any $f : x \to x'$ and so for any $f \in \mathsf{C}(x,x')$ there must hold

$$\eta_{x'} f = (\eta_{x'} \circ f_*)\, 1_x = (Hf \circ \eta_x)\, 1_x = (Hf)\, e. \tag{1.17}$$

Thus, given $e \in Hx$ we may define η by (1.17). Such η is indeed natural, since for any $h : x' \to x''$ there is

$$(Hh \circ \eta_{x'})\, f = (H(h \circ f) \circ \eta_x)\, 1_x = \eta_{x''}(h \circ f) = (\eta_{x''} \circ h_*)\, f. \tag{1.18}$$

The assignment $e \mapsto \eta$ is obviously a two-sided inverse to $y_{(H,x)}$. Now for the naturality of y itself. Let us show naturality in H and in x, the result will then follow from Lemma 1.5.

Naturality in H. Consider some $\alpha : H \to H'$. We need to show that

$$\begin{array}{ccc} N(H,x) & \xrightarrow{y_{(H,x)}} & E(H,x) \\ \downarrow{\scriptstyle N(\alpha,1_x)} & & \downarrow{\scriptstyle E(\alpha,1_x)} \\ N(H',x) & \xrightarrow{y_{(H',x)}} & E(H',x) \end{array} \tag{1.19}$$

commutes. Unpacking the definitions, we find

$$\left(y_{(H',x)} \circ N(\alpha, 1_x)\right)\eta = y_{(H',x)}\left(\alpha \circ \eta \circ 1_x^*\right) = y_{(H',x)}\left(\alpha \circ \eta\right) = \left(\alpha_x \circ \eta_x\right) 1_x, \tag{1.20}$$

and the other way around:

$$\left(E(\alpha, 1_x) \circ y_{(H,x)}\right)\eta = \alpha_x(y_{(H,x)}\eta) = \left(\alpha_x \circ \eta_x\right) 1_x, \tag{1.21}$$

for every $\eta : \mathsf{C}(x, \cdot) \to H$.

Naturality in x. Consider some $f : x \to x'$. We need to show that

$$\begin{array}{ccc} N(H,x) & \xrightarrow{y_{(H,x)}} & E(H,x) \\ \downarrow{\scriptstyle N(1_H,f)} & & \downarrow{\scriptstyle E(1_H,f)} \\ N(H,x') & \xrightarrow{y_{(H,x')}} & E(H,x'), \end{array} \tag{1.22}$$

commutes. Once again unpacking the definitions gives us, for any $\eta : \mathsf{C}(x, \cdot) \to H$,

$$\left(y_{(H,x')} \circ N(1_H, f)\right)\eta = y_{(H,x')}\left(1_H \circ \eta \circ f^*\right) = \left(\eta \circ f^*\right)_{x'} 1_{x'} = \eta_{x'} f. \tag{1.23}$$

and the other way around

$$\left(E(1_H, f) \circ y_{(H,x)}\right)\eta = E(1_H, f)\left(\eta_x 1_x\right) = \left(Hf \circ \eta_x\right) 1_x = \eta_{x'} f, \tag{1.24}$$

where we used (1.17) in the last equality. ■

Lemma 1.5. *Let $\mathsf{A}, \mathsf{B}, \mathsf{C}$ be categories and $F, G : \mathsf{A} \times \mathsf{B} \to \mathsf{C}$ be two functors. Then $\eta : F \to G$ is a natural transformation if and only if for each $a \in \mathsf{A}$ the diagram*

$$\begin{array}{ccc} F(a,b) & \xrightarrow{\eta_{(a,b)}} & G(a,b) \\ \downarrow{\scriptstyle F(1_a,h)} & & \downarrow{\scriptstyle G(1_a,h)} \\ F(a,b') & \xrightarrow{\eta_{(a,b')}} & G(a,b') \end{array} \tag{1.25}$$

commutes for every $h : b \to b'$ and for each $b \in \mathsf{B}$ the diagram

$$\begin{array}{ccc} F(a,b) & \xrightarrow{\eta_{(a,b)}} & G(a,b) \\ \downarrow{\scriptstyle F(k,1_b)} & & \downarrow{\scriptstyle G(k,1_b)} \\ F(a',b) & \xrightarrow{\eta_{(a',b)}} & G(a',b) \end{array} \tag{1.26}$$

commutes for every $K : a \to a'$. In other words, η is natural if and only if it is natural in a and in b.

Proof. The only if direction is obvious, so let η be natural in a and in b. Consider some arbitrary $k : a \to a'$, $h : b \to b'$ and draw the commutative diagram

$$\begin{array}{ccc} F(a,b) & \xrightarrow{\eta_{(a,b)}} & G(a,b) \\ \downarrow{\scriptstyle F(k,1_b)} & & \downarrow{\scriptstyle G(k,1_b)} \\ F(a',b) & \xrightarrow{\eta_{(a',b)}} & G(a',b) \\ \downarrow{\scriptstyle F(1_{a'},h)} & & \downarrow{\scriptstyle F(1_{a'},h)} \\ F(a',b') & \xrightarrow{\eta_{(a',b')}} & G(a',b'). \end{array} \tag{1.27}$$

The result then follows from functorality of F and G. ■

Corollary 1.6. *The* ***Yoneda functor*** $Y : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}}$ *given for every* $f \in \mathsf{C}(y,x)$ *by*

$$Yx := \mathsf{C}(x,\cdot), \tag{1.28}$$
$$Yf := f^*, \tag{1.29}$$

is fully faithful.

Proof. Consider a natural transformation $\eta : \mathsf{C}(x,\cdot) \to \mathsf{C}(y,\cdot)$. By Yoneda's lemma, it is fully and uniquely determined by $f := \eta_x 1_x \in \mathsf{C}(y,x)$ through (1.17), that is, for any $h \in \mathsf{C}(x,z)$ we have

$$\eta_z h = C(y,h)f = h \circ f = f_* h, \tag{1.30}$$

which means that $\eta = f^*$. ■

Remark 1.7. From the proof of Corollary 1.6 we see that Yoneda's lemma (Theorem 1.4) for the particular choice $H = \mathsf{C}(y,\cdot)$ gives the set bijection as

$$f \mapsto f^*, \qquad \mathsf{C}(y,x) \to \mathrm{Nat}(\mathsf{C}(x,\cdot), \mathsf{C}(y,\cdot)). \tag{1.31}$$

Throughout this text we will need to work with functors $\mathsf{C}(\cdot,x)$ instead of $\mathsf{C}(x,\cdot)$. We feel this should pose no difficulty, but let us formalize it anyway.

Lemma 1.8. *Let* C *be a locally small category. Then for any* $x \in \mathsf{C}$*, the functors*

$$\mathsf{C}^{\mathrm{op}}(x,\cdot), \mathsf{C}(\cdot,x) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set} \tag{1.32}$$

are naturally isomorphic.

Proof. Let use use op for the required natural transformation. Obviously, for any $y \in C$ we want to define $\mathrm{op}_y : \mathsf{C}(y,x) \to \mathsf{C}^{\mathrm{op}}(x,y)$ by

$$\mathrm{op}_y f := f^{\mathrm{op}}. \tag{1.33}$$

This is clearly a bijection by definition of the opposite category, but is it natural? Consider some $h : z \to y$. We need to show that

$$\begin{array}{ccc} \mathsf{C}(y,x) & \xrightarrow{\mathrm{op}_y} & \mathsf{C}^{\mathrm{op}}(x,y) \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle (h^{\mathrm{op}})_*} \\ \mathsf{C}(z,x) & \xrightarrow{\mathrm{op}_z} & \mathsf{C}^{\mathrm{op}}(x,z), \end{array} \tag{1.34}$$

commutes. But this is simple, as for any $f \in \mathsf{C}(y,x)$ we have

$$\mathrm{op}_z h^* f = \mathrm{op}_z(fh) = (fh)^{\mathrm{op}} = h^{\mathrm{op}} f^{\mathrm{op}} = (h^{\mathrm{op}})_* f^{\mathrm{op}} = (h^{\mathrm{op}})_* \mathrm{op}_y f. \tag{1.35}$$

■

For any functor $H : \mathsf{C} \to \mathsf{Set}$ and any $x \in \mathsf{C}$ we therefore have the canonical bijection

$$\mathrm{Nat}(\mathsf{C}^{\mathrm{op}}(\cdot, x), H) \cong \mathrm{Nat}(\mathsf{C}(x, \cdot), H) \tag{1.36}$$

given by $\alpha \mapsto \alpha \circ \mathrm{op}$, which can be viewed as a natural isomorphism of functors $N', N : \mathsf{Set}^{\mathsf{C}} \times \mathsf{C} \to \mathsf{Set}$, where N is introduced in (1.10) and $N'(H,x) := N(H,x) \circ \mathrm{op}$. Therefore we have the dual versions of Yoneda's lemma and the Yoneda functor:

Corollary 1.9 (Dual Statements)**.** *For any functor $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ and $x \in \mathsf{C}$, the assignment $\eta \mapsto \eta_x 1_x$ defines a bijection*

$$\mathrm{Nat}(\mathsf{C}(\cdot, x), H) \cong Hx, \tag{1.37}$$

natural in both H and x. Consequently, the Yoneda functor $Y : \mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ given for any $f : x \to y$ by

$$\begin{aligned} Yx &:= \mathsf{C}(\cdot, x), \\ Yf &:= f_*, \end{aligned} \tag{1.38}$$

is fully faithful. In particular, for any $x, y \in \mathsf{C}$ the function $Y : \mathsf{C}(x,y) \to \mathrm{Nat}(\mathsf{C}(\cdot,x), \mathsf{C}(\cdot,y))$, $f \mapsto f_*$ is bijective.

1.3 Group Object Definitions

In this subchapter we give the definitions instrumental for the rest of this text: that of a group object, taken from [8, III.6.], and that of an action of a group object.

Definition 1.10 (Monoid Object)**.** Let C be a category with finite products and a terminal object t. Then a **monoid object** in C is a triple (c, μ, η) where $c \in \mathsf{C}$, $\mu : c \times c \to c$ and $\eta : t \to c$, such that the following diagrams commute:

$$\begin{array}{ccccc} (c \times c) \times c & \xrightarrow{\mu \times 1} & c \times c & & \\ & & & \searrow^{\mu} & \\ \downarrow{\scriptstyle \alpha} & & & & c\ , \\ & & & \nearrow_{\mu} & \\ c \times (c \times c) & \xrightarrow{1 \times \mu} & c \times c & & \end{array} \tag{1.39}$$

called the associativity diagram and

$$\begin{array}{ccccc} c & \xrightarrow{(\eta,1)} & c \times c & \xleftarrow{(1,\eta)} & c \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & c & & \end{array} \ . \tag{1.40}$$

called the unit diagram. Above, α is the canonical associativity isomorphism (also call the "associator") arising from the fact that both $(c \times c) \times c$ and $c \times (c \times c)$ are the triple products of c. Also η is considered as a morphism $\eta : c \to c$ defined as the composite arrow

$$c \longrightarrow t \xrightarrow{\eta} c, \tag{1.41}$$

where the unmarked arrow is the unique one to the terminal object. Note that here and for the rest of the text, if the context permits no confusion, we use the same symbol to denote the arrow from the terminal object and the arrow "filtering through" the terminal object. We shall call μ the **multiplication arrow** and η the **unit arrow** for obvious reasons. Note that for the special choice $\mathsf{C} = \mathsf{Set}$, the definition coincides with the classical definition of a monoid.

Definition 1.11 (Group Object)**.** Let $m \in \mathsf{C}$ be a monoid object with multiplication arrow μ and unit arrow η. We say that (m, μ, η, ι) is a **group object** in C, if $\iota : m \to m$ is arrow such that the following diagram commutes:

$$\begin{array}{ccccc} m & \xrightarrow{(\iota,1)} & m \times m & \xleftarrow{(1,\iota)} & m \\ & \searrow^{\eta} & \downarrow{\scriptstyle \mu} & \swarrow_{\eta} & \\ & & m & & \end{array} . \tag{1.42}$$

Such ι is called the **inversion arrow**.

We can define an analogue of group action for any monoid object.

Definition 1.12 (Group Object Action)**.** Let C be a category with finite products and a terminal object t, let (g, μ, η, ι) be a group object in C and consider some $c \in \mathsf{C}$. We say that an arrow $\theta : g \times c \to c$ is a **left action of the group object** g **on** c, if the following diagrams commute:

$$\begin{array}{ccc} c & \xrightarrow{(\eta,1)} & g \times c \\ & \searrow^{1} & \downarrow{\scriptstyle \theta} \\ & & c \end{array} , \tag{1.43}$$

$$\begin{array}{ccccc} (g \times g) \times c & \xrightarrow{\mu \times 1} & g \times c & & \\ & & & \searrow^{\theta} & \\ \downarrow{\scriptstyle \alpha} & & & & c \\ & & & \nearrow_{\theta} & \\ g \times (g \times c) & \xrightarrow{1 \times \theta} & g \times c & & \end{array} , \tag{1.44}$$

where α is once more the canonical associativity isomorphism. Right action would be defined similarly. Note especially that the multiplication arrow $\mu : g \times g \to g$ is automatically both a left and a right action of g on g. Along with every left action θ of g on c, we may consider the associated **shear morphism** $\Sigma : g \times c \to c \times c$ defined as the composite arrow

$$g \times c \xrightarrow{1 \times (1,1)} g \times (c \times c) \xrightarrow{\alpha} (g \times c) \times c \xrightarrow{\theta \times 1} c \times c, \tag{1.45}$$

or more concisely as $\Sigma = (\theta, p_2)$. Based on the properties of the shear morphism we say that the action θ is **free** if Σ is a monomorphism, **transitive** if Σ is an epimorphism and **regular** if Σ is an isomorphism.

Proposition 1.13. *In the category of sets the definitions of a free and transitive action agree with the usual ones.*

Proof. Consider a left group action $\theta : G \times X \to X$ for some group G and some set X. As long as we stay in the category Set we will use the notation $\theta(g, x) =: g \cdot x$ for any $g \in G$ and $x \in X$. The shear map τ_θ is now defined as $\tau_\theta(g, x) = (g \cdot x, x)$ and it is injective $\iff$ whenever $(g \cdot x, x)$ equals $(h \cdot x, x)$ we necessarily have $g = h$, which is the usual definition of a free action.

The shear map is surjective $\iff$ for every $(y, x) \in X \times X$ there exists $g \in G$ such that $g \cdot y = x$, which is the usual definition of a transitive action. ■

Definition 1.14. Let (g, μ, η, ι) be a group object in a category C with a terminal object t and let $\theta : g \to c$ be a left action of g on some $c \in \mathsf{C}$.

i.) For every $\lambda : t \to g$ let $\theta_\lambda : c \to c$ be the arrow defined by the the diagram

$$\begin{array}{ccc} c & \xrightarrow{(\lambda,1)} & g \times c \\ & \overset{\theta_\lambda}{\searrow} & \downarrow \theta \\ & & c \end{array} \quad . \tag{1.46}$$

ii.) For every $\omega : t \to c$ define the arrow $\theta^\omega : g \to c$, called **orbit arrow** of ω, by the diagram

$$\begin{array}{ccc} g & \xrightarrow{(1,\omega)} & g \times c \\ & \overset{\theta^\omega}{\searrow} & \downarrow \theta \\ & & c \end{array} \quad . \tag{1.47}$$

Remark 1.15. As discussed, the group object multiplication arrow μ is always both a left and a right action. We will denote the arrow μ_λ from (1.46) as L_λ and R_λ when μ is considered as a left and a right action, respectively.

All the facts about actions in this section are stated for a left action. However, with intuitive modifications they are also valid for a right action. For a general group object g we may also generalize the notion of conjugation in the form of a **conjugation arrow** $\varkappa : g \times g \to g$ defined as

$$\varkappa := \mu \circ (\mu, \iota \circ p_1). \tag{1.48}$$

Later on we will find that for any locally small category this arrow is a left action of g on itself.

Definition 1.16 (Equivariant Arrows)**.** Let m be a monoid object in a category C and let $\theta : m \times c \to c$ and $\theta' : m \times c' \to c'$ be two left actions of m. We say that an arrow $\varphi : c \to c'$ is **equivariant** if the following diagram commutes:

$$\begin{array}{ccc} m \times c & \xrightarrow{1 \times \varphi} & m \times c' \\ \downarrow \theta & & \downarrow \theta' \\ c & \xrightarrow{\varphi} & c' \end{array} \quad . \tag{1.49}$$

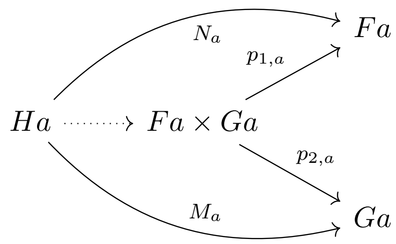
1.4 Group Objects in Functor Categories

Yoneda's lemma, or the dual thereof (Corollary 1.9) gives us a fully faithful functor $Y : \mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{op}}$ for any locally small category C. This functor is sometimes also called the Yoneda embedding and gives a natural and unique (up to an isomorphism) correspondence between objects of C and representable contravariant functors from C to Set. Following [8, III.6.], we shall examine how monoid and group objects behave under the Yoneda embedding, but first let us state some general observations for group objects F in $\mathsf{Set}^{\mathsf{B}}$ for a general category B.

Product of two functors $F, G : \mathsf{B} \to \mathsf{Set}$ is given simply by

$$(F \times G)a := Fa \times Ga, \qquad \text{and} \qquad (F \times G)h := Fh \times Gh, \tag{1.50}$$

for any arrow $h \in \mathsf{B}(a, b)$ and the projections p_1 and p_2 are defined like so: for any $a \in \mathsf{B}$, $p_{1,a}$ is the (Cartesian product) projection $p_{1,a} : Fa \times Ga \to Fa$ and for p_2 similarly. That (1.50) defines a functor and that the projections are natural transformations is clear, so let us check whether (1.50) indeed defines the product in $\mathsf{Set}^{\mathsf{B}}$. Consider $H, F, G \in \mathsf{Set}^{\mathsf{B}}$and natural transformations $N : H \to F$ and $M : H \to G$. Then for every $a \in \mathsf{B}$ there is a unique arrow (N_a, M_a) fitting into the diagram



. (1.51)

We need only show naturality, i.e. that the assignment $a \mapsto (N_a, M_a)$ defines the component maps of a natural transformation (N, M). But this follows from naturality of M and N since for every $f : a \to b$ in B we have

$$(Ff \times Gf) \circ (N_a, M_a) = (Ff \circ N_a, Gf \circ M_a) = (N_b \circ Hf, M_b \circ Hf) = (N_b, M_b) \circ Hf. \tag{1.52}$$

Consequently, $\mathsf{Set}^{\mathsf{B}}$ has all finite products. Note that the products used in the definition of $F \times G$ were products in Set, that is ordinary Cartesian products, and so it was not necessary for the category B to admit products. Let us also point out that for any two natural transformations N, M there is

$$(N \times M)_a = N_a \times M_a \qquad \text{and} \qquad (N, M)_a = (N_a, M_a), \tag{1.53}$$

for any $a \in \mathsf{B}$. Next observe that the functor $T : \mathsf{B} \to \mathsf{Set}$ which assigns to every object $b \in \mathsf{B}$ the one point set $*$ and to every arrow $h : a \to b$ the identity arrow $1 : * \to *$, is the terminal object in $\mathsf{Set}^{\mathsf{B}}$. Indeed, as $*$ is the terminal object in Set, any component map N_b of a natural transformation $N : F \to T$ is necessarily the terminal arrow $Fb \to *$. As this indeed defines a natural transformation, the result follows.

A monoid object in $\mathsf{Set}^{\mathsf{B}}$ is therefore a functor $F : \mathsf{B} \to \mathsf{Set}$ together with natural transformations $\mu : F \times F \to F$ and $\eta : T \to F$ satisfying the relevant commutative diagrams (1.39) and (1.40). A group object is a monoid object F with additional natural transformation $\iota : F \to F$ satisfying the appropriate version of (1.42).

One has another characterization of group objects in $\mathsf{Set}^{\mathsf{B}}$ stated as an exercise in [8]:

Theorem 1.17. *Let* B *be a category. Then* F *is a group object in* $\mathsf{Set}^{\mathsf{B}}$ *if and only if* Fa *is a group (a group object in* Set*) for every* $a \in \mathsf{B}$ *and* Fh *is a group homomorphism for every* $h : a \to b$*. Moreover, the correspondence is* $(F, \mu, \eta, \iota) \leftrightarrow (Fa, \mu_a, \eta_a, \iota_a)$ *for every* $a \in \mathsf{B}$*.*

Proof. Let (F, μ, η, ι) be a group object in $\mathsf{Set}^{\mathsf{B}}$. Then $(Fa, \mu_a, \eta_a, \iota_a)$ is a group. Indeed, this can be seen by applying the "evaluation at a" functor $\mathrm{ev}_a : F \mapsto Fa$ and $(N : F \to G) \mapsto (N_a : Fa \to Ga)$ to the multiplication, unit and inversion diagrams for F. That Fh is a group homomorphism follows from naturality of $\mu : F \times F \to F$. Indeed, for every $h \in \mathsf{C}(a, b)$ we have

$$\begin{array}{ccc} Fa \times Fa & \xrightarrow{\mu_a} & Fa \\ \downarrow{\scriptstyle Fh \times Fh} & & \downarrow{\scriptstyle Fh} \\ Fb \times Fb & \xrightarrow{\mu_b} & Fb \end{array} \; . \tag{1.54}$$

Conversely, let Fa be a group for every $a \in \mathsf{B}$ with some multiplication arrow μ_a, unit arrow η_a and inversion arrow ι_a. That these form components of natural transformations μ, η and ι follows from Fh being group homomorphisms (consider the required diagrams). Hence, (F, μ, η, ι) is a group object as all components of the natural transformations μ, η, ι satisfy the multiplication, unit and inversion diagrams and thus so do the natural transformations themselves. ■

There is an immediate corollary:

Corollary 1.18. *Let* B *be a category and let* (G, μ, η, ι) *be a group object in* $\mathsf{Set}^{\mathsf{B}}$*. Then*

i.) *The arrows* η *and* ι *are the unique natural transformations satisfying (1.40) and (1.42), respectively.*

ii.) *The inversion arrow* ι *is a natural isomorphism satisfying* $\iota \circ \iota = 1$*.*

iii.) *For any natural transformation* $\alpha : T \to H$ *there holds*

$$\eta = \mu(\alpha, \iota\alpha) = \mu(\iota\alpha, \alpha). \tag{1.55}$$

Proof. Ad *i)*. If $\eta' : T \to G$ is another arrow satisfying the monoid unit diagram, then for any $a \in \mathsf{B}$, both η_a and η'_a are units in the (ordinary) group Ga and hence $\eta'_a = \eta_a$. The uniqueness of ι is shown the same way.

Ad *ii)*. By Theorem 1.17, ι_a s the inversion arrow in the group Ga for any $a \in \mathsf{B}$ and thus $(\iota \circ \iota)_a = \iota_a \circ \iota_a = 1$. Since this holds for every component map of ι, we have $\iota \circ \iota = 1$.

Ad *iii)*. Again, for every $a \in \mathsf{B}$ there is $\eta_a = \mu_a(\alpha_a, \iota_a\alpha_a)$, as this is merely the restatement of the fact that $x \cdot x^{-1} = 1$ in an ordinary group Ga where $x := \alpha_a \in Ga$. By definition of compositions of natural transformations together with (1.53) we also have $\mu_a(\alpha_a, \iota_a\alpha_a) = \left(\mu(\alpha, \iota\alpha)\right)_a$, and the statement follows. ■

Let us now consider some left action θ of a group object (G, μ, η, ι) in $\mathsf{Set}^{\mathsf{B}}$ on some $H \in \mathsf{Set}^{\mathsf{B}}$. Note that naturality of θ is equivalent to

$$\begin{array}{ccc} Ga \times Ha & \xrightarrow{\theta_a} & Ha \\ \downarrow{\scriptstyle Gf \times Hf} & & \downarrow{\scriptstyle Hf} \\ Gb \times Hb & \xrightarrow{\theta_b} & Hb \end{array} \tag{1.56}$$

commuting for every $f : a \to b$. Consequently, we have the following Proposition:

Proposition 1.19. *Let* B *be a category,* $G, H \in \mathsf{Set}^{\mathsf{B}}$ *where* G *is a group object. Then* θ *is a left action of* G *on* H *if and only if* $\theta_a : Ga \times Ha \to Ha$ *is a collection of left actions such that the diagram (1.56) commutes for every* $f : a \to b$*.*

Corollary 1.20. *Let* B *be a category and* (G, μ, η, ι) *a group object in* $\mathsf{Set}^{\mathsf{B}}$*. Then*

i.) Let $\theta : G \times H \to H$ *be some left action. Then for every* $\alpha : T \to G$ *the natural transformation* $\theta_\alpha : H \to H$ *defined in (1.46) is a natural isomorphism which satisfies*

$$\theta_{\alpha,a}(x) = \theta_a(\alpha_a, x), \tag{1.57}$$

for every $a \in \mathsf{B}$ *and* $x \in Ha$*. Furthermore,*

$$\theta_\alpha^{-1} = \theta_{\iota\alpha}, \qquad \theta_\eta = 1, \qquad \text{and} \qquad \theta_\alpha\theta_\beta = \theta_{\mu(\alpha,\beta)}, \tag{1.58}$$

for any $\alpha, \beta : T \to G$*.*

ii.) The conjugation arrow, that is the natural transformation $\varkappa : G \times G \to G$ *defined as in (1.48), is a left action of* G *on itself.*

Proof. Ad *i)*. From the previous lemma we know that θ_a is an ordinary left group action of Ga on Ha, from which follows (1.57) under the usual identification of $\alpha_a : * \to Ga$ with the point $\alpha_a(*) \in G$. The relations (1.58) contain only natural transformations and so can be verified component-wise with the help of (1.57), (1.55), the above proposition and Theorem 1.17.

Ad *ii)*. From the definition of $\varkappa$ we see that for every $a \in \mathsf{B}$, $\varkappa_a$ is indeed the map

$$\varkappa_a(x, y) = xyx^{-1}, \tag{1.59}$$

for every x, y in the group Ga. In other words, $\varkappa_a$ is the ordinary conjugation map, which we know is a left action of Ga on itself. Furthermore, for every $a, b \in \mathsf{B}$ and every $f : a \to b$ the diagram (1.56) commutes due to Gf being a group homomorphism and the statement follows from Proposition 1.19. ■

Definition 1.21 (Natural Subgroup)**.** Let $F, G : \mathsf{B} \to \mathsf{Set}$ be two group objects in $\mathsf{Set}^{\mathsf{B}}$. Then F is called a **natural subgroup** of G if there exists a natural transformation $I : F \to G$ such that for every $a \in \mathsf{B}$, I_a is an injective group homomorphism.

That I_a is a group homomorphism for every $a \in \mathsf{C}$ corresponds to the commutativity of

$$\begin{array}{ccc} F \times F & \xrightarrow{\mu^F} & F \\ \Big\downarrow{\scriptstyle I\times I} & & \Big\downarrow{\scriptstyle I} \\ G \times G & \xrightarrow{\mu^G} & G \end{array} \tag{1.60}$$

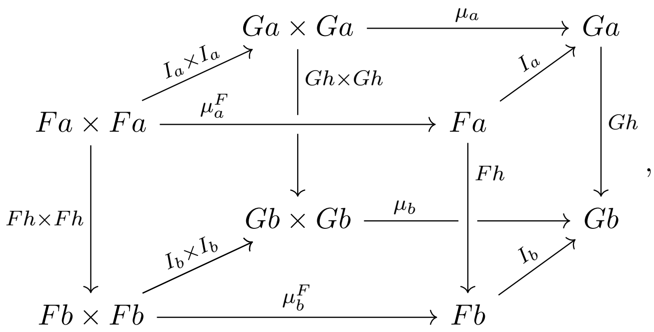
Remark 1.22. Specifically for $\mathsf{B} = \mathsf{C}^{\mathrm{op}}$ for some locally small C, $F = \mathsf{C}(\cdot, c)$ and $G = \mathsf{C}(\cdot, d)$ a natural transformation $I : F \to G$ is given uniquely as h_* for some $h : c \to d$. Injectivity of I_a then amounts to the fact that for any $f, g : a \to c$, $h \circ f = h \circ g$ implies $f = g$. In other words, $h_{*,a}$ is injective for every a if and only if h is a monic arrow in C.

Lemma 1.23. *Let* $F, G : \mathsf{B} \to \mathsf{Set}$ *be two functors such that* (G, μ, η, ι) *is a group object, and let* $I : F \to G$ *be a natural transformation such that* $I_a : Fa \to Ga$ *is injective and* $I_a(Fa)$ *is a subgroup of* G_a*. Then there is a canonical group object structure on* F *such that* F *becomes a natural subgroup of* G*.*

Proof. The injectivity of I_a together with $\operatorname{im} I_a$ being a subgroup in Ga of course means that for every $a \in \mathsf{B}$ there is an induced group structure on Fa such that

$$\begin{array}{ccc} Fa \times Fa & \xrightarrow{\mu_a^F} & Fa \\ \downarrow{\scriptstyle Ia \times Ia} & & \downarrow{\scriptstyle I_a} \\ Ga \times Ga & \xrightarrow{\mu_a} & Ga \end{array} \tag{1.61}$$

commutes, where μ_a^F denotes the induced group multiplication on Fa. Similarly, $I_a \circ \eta_a^F = \eta_a$ and $I_a \circ \iota_a = \iota_a^F \circ I_a$. We only need to show that for every $h : a \to b$ the map $Fh : Fa \to Fb$ is a group homomorphism. Consider



, (1.62)

where all the sides commute, except for the one facing us. Consequently,

$$I_b \circ Fh \circ \mu_a^F = I_b \circ \mu_b^F \circ Fh \times Fh, \tag{1.63}$$

and from injectivity of I_b it follows that $Fh \circ \mu_a^F = \mu_b^F \circ Fh \times Fh$, or in other words, that Fh is a group homomorphism. ∎

1.5 Representable Functors

In this subchapter let C be a locally small category with finite products and a terminal object. For two representable functors $F = \mathsf{C}(\cdot, c)$, $G = \mathsf{C}(\cdot, d)$ we have the canonical natural isomorphism

$$I : \mathsf{C}(\cdot, c) \times \mathsf{C}(\cdot, d) \cong \mathsf{C}(\cdot, c \times d), \tag{1.64}$$

given by $\mathsf{C}(a, c) \times \mathsf{C}(a, d) \to \mathsf{C}(a, c \times d)$, $(f, g) \mapsto (f, g)$ for every $a \in \mathsf{C}$, where the first (f, g) is an ordered pair of arrows, and the second is the arrow defined in (1.1). Note that I is natural, as for any $h : b \to a$ we have $(f, g)h = (fh, gh)$ and I_a is an isomorphism for every a due to the properties of product in C. To restate, we have

$$Yc \times Yd \cong Y(c \times d), \tag{1.65}$$

whence the projections are the images of the projection arrows in C by the Yoneda functor. We say that the Yoneda functor **preserves products**. Furthermore, under this identification there is $f_* \times g_* = (f \times g)_*$ and $(f_*, g_*) = (f, g)_*$ for all possible arrows in C.

If t is terminal in C then Yoneda embedding of t is terminal in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ as is stated in the following lemma:

Lemma 1.24. *The functor* $T := \mathsf{C}(\cdot, t) \equiv Yt$ *is the terminal object in* $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$.

Proof. Since t is terminal in C, the set $\mathsf{C}(c,t)$ is a one-point set for every $c \in \mathsf{C}$. For any functor $F \in \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ and any $c \in \mathsf{C}$ there is thus exactly one map $N_c : Fc \to Tc$ and so $N : F \to T$ defined by these maps is the unique natural transformation form F to T. Note that naturality of N follows immediately from the fact that Tc is terminal in Set for every $c \in \mathsf{C}$. ■

Now comes the first truly interesting part of this subchapter; an observation that allows us to work with the Yoneda-embedded group object instead of the original. Taken from [8, III.6. Proposition 1].

Theorem 1.25. *Let C be a locally small category with finite products and a terminal object. Then c is a group (monoid) object in C if and only if $\mathsf{C}(\cdot,c) \equiv Yc$ is a group (monoid) object in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$. Moreover, the correspondence of multiplication, unit and inversion arrows is $(\mu,\eta,\iota) \leftrightarrow (\mu_*,\eta_*,\iota_*) \equiv (Y\mu, Y\eta, Y\iota)$.*

Proof. The proof is an application of the fully faithful Yoneda functor Y on the diagrams (1.39), (1.40) and (1.42) while using the isomorphism (1.64) and the fact that $\mathsf{C}(\cdot,t)$ is the terminal object in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$. ■

This theorem has several immediate corollaries:

Corollary 1.26 (Group Object Facts)**.** *Let (g,μ,η,ι) be a group object in C. Then*

i.) The arrows η and ι are the unique arrows satisfying (1.40) and (1.42), respectively.

ii.) The inversion arrow ι is an isomorphism satisfying $\iota \circ \iota = 1$.

iii.) For any arrow $\alpha : t \to g$ there holds

$$\eta = \mu(\alpha, \iota\alpha) = \mu(\iota\alpha, \alpha). \tag{1.66}$$

Proof. The proof consists of considering the group object $\mathsf{C}(\cdot,q)$ in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ and Corollary 1.18 in the light of the above theorem and then using the fact that the Yoneda embedding is a fully faithful functor. For example, if η,η' are both unit arrows of the group object g, then η_* and η'_* are both unit arrows of the object $\mathsf{C}(\cdot,g)$ and by Corollary 1.18 there is $\eta_* = \eta'_*$. Since the Yoneda functor is fully faithful, it follows that $\eta = \eta'$.

The remaining claims are shown similarly. ■

Corollary 1.27. *An arrow θ is a left action of g on c in C if and only if $\theta_* \equiv Y\theta$ is a left action of $C(\cdot,g)$ on $\mathsf{C}(\cdot,c)$ in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$.*

Corollary 1.28 (Action Facts)**.** *Let (g,μ,η,ι) be a group object in a locally small category C. Then*

i.) Let $\theta : g \to c$ be a left action of g on some object $c \in \mathsf{C}$. Then for every $\alpha : t \to g$, the arrow $\theta_\alpha : c \to c$ defined in (1.46) is an isomorphism which satisfies

$$\theta_\alpha^{-1} = \theta_{\iota\alpha}, \qquad \theta_\eta = 1 \qquad \textit{and} \qquad \theta_\alpha\theta_\beta = \theta_{\mu(\alpha,\beta)}, \tag{1.67}$$

for any $\alpha, \beta : t \to g$.

ii.) The conjugation arrow $\varkappa : g \times g \to g$ from (1.48) is a left action of g on itself.

Proof. Ad *i)*. That $\theta_\eta = 1$ follows straight from the definition. Consider some $\alpha : t \to g$. By applying the Yoneda functor Y on the defining diagram of θ_α, we find that $(\theta_\alpha)_* = (\theta_*)_{\alpha_*}$. One need only look at Corollary 1.20 to see that

$$\left(\theta_{\mu(\alpha,\beta)}\right)_* = (\theta_*)_{\mu_*(\alpha_*,\beta_*)} = (\theta_*)_{\alpha_*} \circ (\theta_*)_{\beta_*} = (\theta_\alpha \circ \theta_\beta)_* , \tag{1.68}$$

And by faithfulness of the Yoneda functor, $\theta_{\mu(\alpha,\beta)} = \theta_\alpha \theta_\beta$. From this and (1.66) it immediately follows that θ_α is an isomorphism with $\theta_{\iota\alpha}$ as its inverse.

Ad *ii)*. Noting that the Yoneda functor preserves products, see the beginning of this subchapter, we may apply the Yoneda functor on the conjugation arrow $\varkappa$ and find that $\varkappa_*$ is the conjugation arrow for the group object $Yg \equiv \mathsf{C}(\cdot, g)$. The result now follows from Corollary 1.27. ∎

Corollary 1.29. *Let (g, μ, η, ι) be a group object in C and let $c \in \mathsf{C}$ be such that $\mathsf{C}(\cdot, c)$ is a natural subgroup of $\mathsf{C}(\cdot, g)$ by means of a natural transformation $j_* : \mathsf{C}(\cdot, c) \to \mathsf{C}(\cdot, g)$. Then there exists a unique group object structure (μ^c, η^c, ι^c) on c such that*

$$\begin{array}{ccc} c \times c & \xrightarrow{\mu^c} & c \\ \big\downarrow{\scriptstyle j \times j} & & \big\downarrow{\scriptstyle j} \\ g \times g & \xrightarrow{\mu} & g \end{array} \tag{1.69}$$

commutes. In addition, the unit and inversion arrows satisfy $\eta^c \circ j = \eta$ and $\iota^c \circ j = \iota$.

Proof. Combine Theorem 1.25 with Lemma 1.23. ∎

Chapter 2

Graded Lie Groups

2.1 Graded Manifolds Recalled

Here we face something of a conundrum. On the one hand we wish this text to be as self-contained as possible, and as such we would like to include the necessary elements from the theory of graded manifolds as presented in [10]. On the other hand, if this was to be done properly, the size of this work would grow beyond acceptable. Therefore we resort to a compromise: below we give a brief overview of the necessary concepts and invite the interested reader to consult [10] for proper introduction to $\mathbb{Z}$-graded manifolds as we use them here.

First a note about nomenclature: by the word "graded" we exclusively mean $\mathbb{Z}$-graded. Also note that when the context permits no confusion we will often omit the word "graded" entirely, since not doing so would result in its unbearable abundance.

By a **graded vector space** V we mean a sequence of vector spaces $\{V_k\}_{k\in\mathbb{Z}}$. In this text we work with *real vector spaces only.* We define the graded dimension of V as the sequence $\operatorname{gdim} V := (\dim V_k)_{k\in\mathbb{Z}}$ and the total dimension of V as $\operatorname{tdim} V := \sum_{k\in\mathbb{Z}} \dim V_k$. We say that a graded vector space is finite-dimensional if its total dimension is finite. We say that v is an element of V, written as $v \in V$, if there exists $k \in \mathbb{Z}$ such that $v \in V_k$. We say that this k is the degree of v, written as $|v| := k$. Consider two graded vector spaces V, W. We say that ϕ is a **graded linear map** of degree k from V to W if $\phi = (\phi_j)_{j\in\mathbb{Z}}$ is a sequence of linear maps $\phi_j : V_j \to W_{j+k}$. We write $\phi : V \to W$ and $|\phi| := k$. For any $v \in V$ we write simply $\phi(v)$ instead of $\phi_{|v|}(v)$. Graded vector spaces together with graded linear maps of degree zero form a category gVec. We say that V is a subspace of W if V_k is a subspace of W_k for every $k \in \mathbb{Z}$.

By a **graded algebra** of degree ℓ we mean a graded vector space $A \in \mathsf{gVec}$ equipped with a bilinear map $(\cdot,\cdot) : A \times A \to A$, by which we mean a sequence of bilinear maps $(\cdot,\cdot)_{i,j} : A_i \times A_j \to A_{i+j+\ell}$. For any $a, b \in A$ we write simply $a \cdot b$ or ab in place of $(a, b)_{|a|,|b|}$. We say that A is associative if $(ab)c = a(bc)$ for any $a, b, c \in A$. We say that an algebra A of degree zero is unital if there exists an element $1 \in A$ such that $1 \cdot a = a = a \cdot 1$ for any $a \in A$. Clearly such 1 must have degree 0. We say that A is **graded commutative** if

$$ab = (-1)^{|a||b|} ba. \tag{2.1}$$

Graded commutative, associative and unital algebras form a category gcAs where morphisms are graded linear maps ϕ of degree zero such that $\phi(ab) = \phi(a)\phi(b)$ and $\phi(1) = 1$. A subspace V of A is called an **ideal** if $av, va \in V$ for any $v \in V$ and $a \in A$. We say that $A \in \mathsf{gcAs}$ is **local** if it contains a unique maximal ideal.

Let M be a topological space. We denote the set of all open subsets of M as $\mathrm{Op}(M)$, and for any $x \in M$ we denote the set of all open neighborhoods of x as $\mathrm{Op}_x(M)$. We may make $\mathrm{Op}(M)$ into a category by saying that for any $U, V \in \mathrm{Op}(M)$ there is an arrow from V to U if $V \subseteq U$. We say that a functor $\mathscr{S} : \mathrm{Op}(M)^{\mathrm{op}} \to \mathsf{gVec}$ is a **presheaf** on M valued in gVec. In particular, for any $V \subseteq U$ open sets in M there is a morphism $\rho^U_V \in \mathsf{gVec}(\mathscr{S}(U), \mathscr{S}(V))$ called the **restriction** from U to V. For a vector $x \in \mathscr{S}(U)$ we usually write $x|_V$ instead of $\rho^U_V(x)$. We say that a presheaf $\mathscr{S}$ is a **sheaf** if for any $U \in \mathrm{Op}(M)$, any open cover $\{U_\alpha\}_{\alpha\in I}$ of U and any collection $\{x_\alpha\}_{\alpha\in I}$ of $x_\alpha \in \mathscr{S}(U_\alpha)$ of the same degree, such that $x_\alpha|_{U_\alpha\cap U_\beta} = x_\beta|_{U_\alpha\cap U_\beta}$ for every $\alpha, \beta \in I$, there exists a unique $x \in \mathscr{S}(U)$ such that $x|_{U_\alpha} = x_\alpha$ for every $\alpha \in I$. Similarly, we define presheaves and sheaves valued in gcAs.

Let $\mathscr{S}$ be a presheaf on M. Then for any $U \in \mathrm{Op}(M)$ we have a restricted presheaf $\mathscr{S}|_U$ on U defined simply as $\mathscr{S}|_U(W) := \mathscr{S}(W)$ for any $W \in \mathrm{Op}(U)$, with restrictions inherited from $\mathscr{S}$. Let N be another topological space and $\phi : M \to N$ a continuous map. Then we define the so-called **pushforward** presheaf $\phi_*\mathscr{S}$ on N by $(\phi_*\mathscr{S})(U) := \mathscr{S}(\phi^{-1}(U))$ for any $U \in \mathrm{Op}(N)$ and restrictions inherited from $\mathscr{S}$. It is not difficult to see that if $\mathscr{S}$ is a sheaf, then its restriction and pushforward are also sheaves.

Next, we need to define the so-called graded domains, which will be to graded manifolds as open subsets of $\mathbb{R}^n$ are to smooth manifolds. Let $(n_j)_{j\in\mathbb{Z}}$ be a finite sequence of non-zero integers, that is, $\sum_{j\in\mathbb{Z}} n_j =: n < \infty$. To simplify notation, we will write simply $(n_j)_{j\in\mathbb{Z}} =: (n_j)$. For any $U \in \mathrm{Op}(\mathbb{R}^{n_0})$ we define a graded commutative, associative and unital algebra $C^\infty_{(n_j)}(U) \in \mathsf{gcAs}$. This algebra is constructed rigorously in [10] using the symmetric tensor algebra over finite-dimensional graded vector spaces, but here let us introduce the elements of $\left(C^\infty_{(n_j)}(U)\right)_k$ as formal power-series of the shape

$$f := \sum_{\boldsymbol{p}\in\mathbb{N}^m_k} f_{\boldsymbol{p}}\, \xi_1^{p_1} \cdots \xi_m^{p_m}, \tag{2.2}$$

where $m := n - n_0$ and

- $(\xi_1, \ldots \xi_m)$ are the so-called graded variables (or graded coordinates), each of which is assigned a non-zero degree $|\xi_\mu| \in \mathbb{Z}$ such that $\#\{\mu : |\xi_\mu| = k\} = n_k$ for every $k \in \mathbb{Z}\setminus\{0\}$. These variables commute or anticommute according to the rule

$$\xi_\mu \xi_\nu = (-1)^{|\xi_\mu||\xi_\nu|}\, \xi_\nu \xi_\mu. \tag{2.3}$$

- The sum ranges over all multiindices $\boldsymbol{p} \equiv (p_1, \ldots, p_m) \in \mathbb{N}^m_k$ where

$$\mathbb{N}^m_k = \{\boldsymbol{q} \in (\mathbb{N}_0)^m \,:\, \sum_{\mu=1}^m q_\mu |\xi_\mu| = k \text{ and } q_\mu \in \{0,1\} \text{ whenever } |\xi_\mu| \text{ is odd }\}. \tag{2.4}$$

- For every $\boldsymbol{p} \in \mathbb{N}^m_k$, $f_{\boldsymbol{p}}$ is an ordinary smooth function on U.

We will also write $\xi^{\boldsymbol{p}} := \xi_1^{p_1} \cdots \xi_m^{p_m}$. We make $C^\infty_{(n_j)}(U)$ into a graded algebra by instituting a multiplication rule $(f, g) \mapsto f \cdot g \in C^\infty_{(n_j)}(U)_{|f|+|g|}$ by

$$\left(\sum_{\boldsymbol{r}\in\mathbb{N}^m_{|f|}} f_{\boldsymbol{r}}\, \xi^{\boldsymbol{r}}\right) \cdot \left(\sum_{\boldsymbol{q}\in\mathbb{N}^m_{|g|}} g_{\boldsymbol{q}}\, \xi^{\boldsymbol{q}}\right) := \left(\sum_{\boldsymbol{p}\in\mathbb{N}^m_{|f|+|g|}} (f \cdot g)_{\boldsymbol{p}}\, \xi^{\boldsymbol{p}}\right), \tag{2.5}$$

where for any $\boldsymbol{p} \in \mathbb{N}^m_{|f|+|g|}$ there is

$$(f \cdot g)_{\boldsymbol{p}} = \sum_{\substack{\boldsymbol{r} \in \mathbb{N}^m_{|f|} \\ \boldsymbol{r} \leq \boldsymbol{p}}} \epsilon^{\boldsymbol{r},\boldsymbol{p}} \, f_{\boldsymbol{r}} \, g_{\boldsymbol{p}-\boldsymbol{r}}, \tag{2.6}$$

where $\boldsymbol{r} \leq \boldsymbol{p}$ if and only if $r_\mu \leq p_\mu$ for every $\mu \in \{1, \ldots, m\}$ and $\epsilon^{\boldsymbol{r},\boldsymbol{p}} \in \{-1, 1\}$ is the sign obtained by rearranging $\xi^{\boldsymbol{r}} \xi^{\boldsymbol{p}-\boldsymbol{r}}$ into $\xi^{\boldsymbol{p}}$ according to the rule (2.3). Note that the sum in (2.6) is finite for every $\boldsymbol{p} \in \mathbb{N}^m_{|fg|}$ and so the multiplication is well-defined. Also note that if f, g are polynomials, i.e. $f_{\boldsymbol{r}}$ and $g_{\boldsymbol{q}}$ are non-zero only for finitely many $\boldsymbol{r} \in \mathbb{N}^m_{|f|}$ and $\boldsymbol{q} \in \mathbb{N}^m_{|g|}$, then the multiplication rule reduces to the intuitive multiplication of polynomials with graded-commutative variables.

Notice that the algebra of smooth functions on U, denoted as $C^\infty_{n_0}(U)$, is a subalgebra of $(C^\infty_{(n_j)}(U))_0$. Indeed, one need only consider the graded functions $f \in C^\infty_{(n_j)}(U)$ of degree zero such that $f_{\boldsymbol{p}} = 0$ for all $\boldsymbol{p} \in \mathbb{N}^m_0, \ \boldsymbol{p} > \boldsymbol{0}$. In [10] it is shown that $C^\infty_{(n_j)}(U)$ is in fact a graded commutative, associative and unital algebra, where the unit element is the constant function $1 \in C^\infty_{n_0}(U) \subseteq (C^\infty_{(n_j)}(U))_0$. For any $V \in \mathrm{Op}(U)$ we may define a **restriction** $\rho^U_V : C^\infty_{(n_j)}(U) \to C^\infty_{(n_j)}(V)$ by

$$\rho^U_V \left(\sum_{\boldsymbol{p} \in \mathbb{N}^m_{|f|}} f_{\boldsymbol{p}} \, \xi^{\boldsymbol{p}} \right) := \sum_{\boldsymbol{p} \in \mathbb{N}^m_{|f|}} (f_{\boldsymbol{p}})|_V \, \xi^{\boldsymbol{p}}. \tag{2.7}$$

The assignment $U \mapsto C^\infty_{(n_j)}(U)$ along with these restrictions defines a sheaf $C^\infty_{(n_j)}$ on $\mathbb{R}^{n_0}$ valued in gcAs. For any $U \in \mathrm{Op}(\mathbb{R}^{n_0})$ we define the **graded domain** $U^{(n_j)}$ as the pair $(U, C^\infty_{(n_j)}|_U)$.

Now, we may move to the definition of a graded manifold. Consider a Hausdorff, second-countable topological space M together with a sheaf $C^\infty_{\mathcal{M}}$ on M valued in gcAs. We say that $\mathcal{M} := (M, C^\infty_{\mathcal{M}})$ is a **graded manifold** if there exists a finite sequence of non-negative integers (n_j) such that there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of M such that for every α there exists a homeomorphism $\underline{\varphi}_\alpha : U_\alpha \to \hat{U}_\alpha$ for some $\hat{U}_\alpha \in \mathrm{Op}(\mathbb{R}^{n_0})$ and a sheaf isomorphism,

$$\varphi^*_\alpha : C^\infty_{(n_j)}\Big|_{\hat{U}_\alpha} \to \underline{\varphi}_{\alpha,*} \left(C^\infty_{\mathcal{M}}|_{U_\alpha} \right). \tag{2.8}$$

Recall that sheaf morphisms are merely natural transformations. The collection $\{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$ where $\varphi_\alpha := (\underline{\varphi}_\alpha, \varphi^*_\alpha)$ is called a **graded atlas** for $\mathcal{M}$, or simply just an atlas for $\mathcal{M}$. We will also refer to open sets from some atlas as coordinate patches. We call $\mathrm{gdim}\, \mathcal{M} := (n_j)$ the **graded dimension** of $\mathcal{M}$ and $\mathrm{tdim}\, \mathcal{M} := \sum_{k \in \mathbb{Z}} n_k$ the **total dimension** of $\mathcal{M}$. Note that a graded domain is clearly a special case of a graded manifold. We call elements of $C^\infty_{\mathcal{M}}(U)$, for any $U \in \mathrm{Op}(M)$, the **graded functions** on $\mathcal{M}$.

If $\mathcal{M} = (M, C^\infty_{\mathcal{M}})$ is a graded manifold, then for any $x \in M$ we have a **stalk** of $C^\infty_{\mathcal{M}}$ at x defined as

$$C^\infty_{\mathcal{M},x} = \left(\bigsqcup_{U \in \mathrm{Op}_x(M)} C^\infty_{\mathcal{M}}(U) \right) / \sim, \tag{2.9}$$

where for any $U, V \in \mathrm{Op}(M)$ and any $f \in C^\infty_{\mathcal{M}}(U)$, $g \in C^\infty_{\mathcal{M}}(V)$ there is $f \sim g$ if and only if there is some $W \in \mathrm{Op}(U \cap V)$ such that $f|_W = g|_W$. The elements of $C^\infty_{\mathcal{M},x}$ are therefore classes of equivalence $[f]_x$ represented by some $f \in C^\infty_{\mathcal{M}}(U)$. The stalks $C^\infty_{\mathcal{M},x}$ inherit the structure of a graded commutative, associative and unital algebra. In addition, every stalk of a graded manifold is a local algebra, possessing a unique maximal ideal.

Let $\mathcal{M} = (M, C^\infty_\mathcal{M})$ and $\mathcal{N} = (N, C^\infty_\mathcal{N})$ be two graded manifolds, then any morphism of graded manifolds, also called a **graded smooth map**, is a pair $\phi = (\underline{\phi}, \phi^*)$ where $\underline{\phi} : M \to N$ is a continuous map and $\phi^* : C^\infty_\mathcal{N} \to \underline{\phi}_* C^\infty_\mathcal{M}$ is a sheaf morphism which is required to satisfy the following condition: for any $x \in M$ consider the induced algebra morphism $\phi^*_x : C^\infty_{\mathcal{N},\underline{\phi}(x)} \to C^\infty_{\mathcal{M},x}$ defined on germs as $\phi^*_x([f]_{\underline{\phi}(x)}) := [\phi^*_U f]_x$, for any $f \in C^\infty_\mathcal{N}(U)$ and $U \in \mathrm{Op}_{\underline{\phi}(x)}(N)$. We require that $\phi^*_x(J_{\underline{\phi}(x)}) \subseteq J_x$, where $J_{\underline{\phi}(x)}$ is the unique maximal ideal of the stalk $C^\infty_{\mathcal{N},\underline{\phi}(x)}$ and J_x is the unique maximal ideal of the stalk $C^\infty_{\mathcal{M},x}$. Graded manifolds and graded smooth maps form the category of graded manifolds gMan^∞. A graded smooth map $\phi = (\underline{\phi}, \phi^*)$ is an isomorphism if and only if $\underline{\phi}$ is a homeomorphism and ϕ^* is a natural isomorphism. Isomorphism in gMan^∞ is also called a **graded diffeomorphism**.

Note that, if it is clear from the context, we often denote the component graded algebra morphisms $\phi^*_U : C^\infty_\mathcal{N}(U) \to C^\infty_\mathcal{M}(\underline{\phi}^{-1}(U))$ simply as ϕ^*, without the explicit mention of the open set U. We call $\underline{\phi}$ the **underlying smooth map** and ϕ^* the **pullback**. For any graded manifold $\mathcal{M}$, the underlying topological space M can be canonically assigned the structure of a smooth manifold such that if $\phi : \mathcal{M} \to \mathcal{N}$ is a graded smooth map, then $\underline{\phi} : M \to N$ is indeed a smooth map, justifying the name. Moreover, if $\{U_\alpha, \phi_\alpha\}$ is an atlas for a graded manifold $\mathcal{M}$, then $\{U_\alpha, \underline{\phi}_\alpha\}$ is an (ordinary) atlas for the smooth manifold M. Any graded function $f \in C^\infty_\mathcal{M}(U)$ can be canonically assigned a smooth function $\underline{f} \in C^\infty_\mathcal{M}(U)$ called the **body** of f. At the graded domain level, where f is of the shape (2.2), this corresponds to the assignment $f \mapsto f_\mathbf{0}$. In particular, the body of any non-zero degree graded function is necessarily the zero function (of degree $|f|$). We can also define, for any $f \in C^\infty_\mathcal{M}(U)$ and any $x \in U$ the **value of** f **at** x as $f(x) := \underline{f}(x)$.

Consider a graded manifold $\mathcal{M}$ of graded dimension (n_j) with a graded atlas $\{U_\alpha, \varphi_\alpha\}$. For any α we have the standard i-th coordinate function $x^i \in C^\infty_{n_0}(\hat{U}_\alpha)$ and the so-called graded coordinates $\xi_\mu \in C^\infty_{(n_j)}(\hat{U}_\alpha)$. We will refer to the graded functions $\varphi^* x^i$ and $\varphi^* \xi_\mu$ simply as **the coordinates on** U_α. We will often abuse notation and denote them as x^i and ξ_μ as well.

2.2 The Body Functor and the Insertion Functor

Throughout the rest of this text we will often encounter certain simple limit objects in the category of graded manifolds gMan^∞, the simplest of which is perhaps the product of two graded manifolds. It is therefore useful to have a formalized relation of these limit objects to those in the category of smooth manifolds Man^∞, so that we will immediately know e.g. that the underlying manifold of the product of graded manifolds $\mathcal{M} \times \mathcal{N}$ is the smooth manifold $M \times N$. We begin with a general category-theoretic observation about limits and adjoint functors. The uninterested reader may skip to the last paragraph where we provide a summary of this short subchapter.

Proposition 2.1. *Let* C, D *be categories and let* $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ *be functors where* F *is a left adjoint of* G. *Next, let* $L : \mathsf{J} \to \mathsf{D}$ *be a functor from some category* J *to* D, *such that there exists a limit for* L *in* D, *described by the limit object* $\varprojlim L$ *and a universal cone* λ.

Then $G(\varprojlim L)$ *and* $G\lambda$ *form a limit for* GL *in* C.

Proof. For a general adjunction $\mathsf{D}(F(\cdot), \cdot) \cong \mathsf{C}(\cdot, G(\cdot))$ let us denote by a bar the bijection $\bar{(\cdot)}$: $\mathsf{D}(Fc, d) \to \mathsf{C}(c, Gd)$ for every $c \in \mathsf{C}$ and $d \in \mathsf{D}$ and also its inverse.

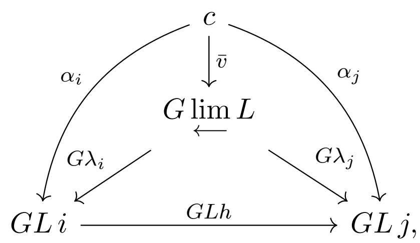
That $G\lambda$ is a cone is obvious; we need to show its universality. Consider some $c \in \mathsf{C}$ and an arbitrary cone $\alpha : c \to GL$. From the naturality of the adjunction we find that

$$\overline{\alpha}_j = \overline{GLh \circ \alpha_i} = Lh \circ \overline{\alpha_i}, \tag{2.10}$$

for every $h : i \to j$ in J. Consequently, $\overline{\alpha} := \{\overline{\alpha_i}\}_{i \in \mathsf{J}}$ is a cone from Fc to L and so there exists a unique $v : Fc \to \varprojlim L$ such that $\overline{\alpha}_i = \lambda_i \circ v$ for every $i \in \mathsf{J}$. We can apply $\overline{(\cdot)}$ to find that

$$\alpha_i = \overline{\lambda_i \circ v} = G\lambda_i \circ \overline{v}. \tag{2.11}$$

In other words, $\overline{v}$ fits into the commutative diagram



(2.12)

for every $i, j \in \mathsf{J}$. Furthermore, if $f : c \to G \varprojlim L$ is another arrow fitting into the above diagram, we find that $\lambda_j \circ \overline{f} = \overline{G\lambda_j \circ f} = \overline{\alpha_j}$ and by universality of λ we have $\overline{f} = v$ which implies $f = \overline{v}$. The arrow $\overline{v}$ is thus unique, hence $G\lambda$ is a limiting cone. ∎

Let us apply the above observation to the category of graded manifolds. We know from [10] that the assignment $\mathcal{M} \mapsto M$ and $\phi \mapsto \underline{\phi}$ for every $\phi \in \mathsf{gMan}^\infty(\mathcal{M}, \mathcal{N})$ defines a functor

$$\mathrm{B} : \mathsf{gMan}^\infty \to \mathsf{Man}^\infty \tag{2.13}$$

called the **body functor**. On the other hand, each ordinary smooth manifold can be regarded as a trivially graded smooth manifold, and every smooth map $\underline{\phi} : M \to N$ can be promoted to a graded smooth map between trivially graded M and N. We may formalize this assignment as the **insertion functor** $\mathrm{I} : \mathsf{Man}^\infty \to \mathsf{gMan}^\infty$.

Note that we usually write $\mathrm{B}\mathcal{M}$ as M and $\mathrm{I}M$ as M as well, but for the purposes of this subchapter we shall attempt to adhere to the more rigorous notation which distinguishes between ordinary smooth manifolds and trivially graded smooth manifolds.

Next, for every $\mathcal{M} \in \mathsf{gMan}^\infty$ we have the graded smooth map $i_\mathcal{M} \in \mathsf{gMan}^\infty(\mathrm{IB}\mathcal{M}, \mathcal{M})$ defined as $\underline{i_\mathcal{M}} = \mathrm{id}_M$ and $i^*_\mathcal{M} f = \underline{f}$. By [10, Proposition 3.26], we know that for every $\phi \in \mathsf{gMan}^\infty(\mathcal{M}, \mathcal{N})$ the diagram

$$\begin{array}{ccc} \mathrm{IB}\mathcal{M} & \xrightarrow{i_\mathcal{M}} & \mathcal{M} \\ {\scriptstyle \mathrm{IB}\phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ \mathrm{IB}\mathcal{N} & \xrightarrow{i_\mathcal{N}} & \mathcal{N} \end{array} \tag{2.14}$$

commutes. Note that $\mathrm{IB}\phi$ is usually denoted simply as $\underline{\phi}$. In other words, i is a natural transformation $i : \mathrm{IB} \to 1$, where 1 denotes the identity functor on gMan^∞. Furthermore, from the definition of I and B it immediately follows that $\mathrm{BI} = 1$ is the identity functor on Man^∞.

Proposition 2.2. *Using notation from above, there exists an adjunction*

$$\mathsf{gMan}^\infty(\mathrm{I}M, \mathcal{N}) \cong \mathsf{Man}^\infty(M, \mathrm{B}\mathcal{N}), \tag{2.15}$$

the counit of which is $i : \mathrm{IB} \to 1$ and the unit of which is the identity natural transformation.

Proof. As we have $\mathrm{BI} = 1$ the potential unit of the adjunction, under slightly headache-inducing notation, is $u := \mathrm{id}_{\mathrm{id}_{\mathsf{Man}^\infty}} : \mathrm{id}_{\mathsf{Man}^\infty} \to \mathrm{id}_{\mathsf{Man}^\infty} \equiv \mathrm{BI}$. To show we indeed have an adjunction, we must show that the unit and counit satisfy the so-called triangle identities, which in our case translate to the commutativity of the following two diagrams:

$$
\begin{array}{ccc}
\mathrm{I} & \xrightarrow{\mathrm{I}(u)} & \mathrm{IBI} \\
 & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle i(\mathrm{I})} \\
 & & \mathrm{I},
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{B} & \xrightarrow{u(\mathrm{B})} & \mathrm{BIB} \\
 & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mathrm{B}(i)} \\
 & & \mathrm{B},
\end{array}
\tag{2.16}
$$

But with the use of the fact that $\mathrm{BI} = 1$ it is easy to see that for any smooth manifold M both $\mathrm{I}(u)_M = \mathrm{I}u_M$ and $i(\mathrm{I})_M = i_{\mathrm{I}M}$ are identity maps on $\mathrm{I}M$. Similarly, for any graded manifold $\mathcal{M}$, both $u(\mathrm{B})_{\mathcal{M}}$ and $\mathrm{B}(i)_{\mathcal{M}}$ are identity maps on $\mathrm{B}\mathcal{M}$. Thus the commutativity of diagrams (2.16) is obvious. ■

Notably the body functor has a left adjoint, yielding the following corollary.

Corollary 2.3. *Let $L : \mathsf{J} \to \mathsf{gMan}^\infty$ be a functor from some category J, such that there exists a limit for L in gMan^∞ given as a limit graded manifold $\mathcal{L}$ together with a universal cone $\lambda : \mathcal{L} \to L$.*

Then $\mathrm{B}\mathcal{L}$ together with the cone $\mathrm{B}\lambda := \{\mathrm{B}\lambda_{\mathrm{j}}\}_{\mathrm{j}\in\mathsf{J}} : \mathrm{B}\mathcal{L} \to \mathrm{L}$ are a limit for L in Man^∞.

In particular, whenever we have a commutative diagram in the category gMan^∞ featuring products, fiber products, or other limit objects, we may apply to it the body functor and obtain the corresponding diagram[1] in the category Man^∞. Therefore we may be certain that if we make a definition using a commutative diagram, the underlying manifolds satisfy the corresponding diagram as well. For example, it is immediately apparent that the underlying manifold of a graded Lie group (Definition 2.12) is an ordinary Lie group.

2.3 Graded Lie Algebras, Graded Matrices

Definition 2.4 (Graded Lie algebra)**.** A graded Lie algebra (of degree zero) is a graded vector space $V \in \mathsf{gVec}$ equipped with a graded bilinear map $[\cdot,\cdot] : V \times V \to V$ of degree zero satisfying

1. For all $x, y \in V$,

$$[x, y] + (-1)^{|x||y|}\,[y, x] = 0 \qquad \ldots \text{graded antisymmetry.} \tag{2.17}$$

2. For all $x, y, z \in V$,

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}\,[y, [x, z]] \qquad \ldots \text{graded Jacobi identity.} \tag{2.18}$$

If A and B are two graded Lie algebras, then a graded vector space morphism $\phi : A \to B$, i.e. a graded linear map of degree zero, is called a **Lie algebra morphism** if

$$\phi\,[x, y] = [\phi\,x, \phi\,y], \tag{2.19}$$

for all $x, y \in A$.

[1] Where the objects are the underlying smooth manifolds and the arrows are the underlying smooth functions.

Remark 2.5. If A is a graded associative algebra, then A can be made into a Lie algebra by defining the Lie bracket as the **graded commutator**:

$$[x, y] := xy - (-1)^{|x||y|}\, yx. \tag{2.20}$$

Note that similarly as in the classical case, if A is graded commutative, the graded commutator is always zero.

Example 2.6. Consider some $V \in \mathsf{gVec}$. In accordance with the above remark, any subalgebra of the graded linear space of maps from V to itself, such as $\mathrm{Lin}(V)$, can be made into a Lie algebra through the introduction of the graded commutator. More generally, consider some $A \in \mathsf{gcAs}$ and some graded A-module V. Then $\mathrm{Der}(A, V)$, the graded linear space of derivations from A to V, is closed under the graded commutator and so forms a graded Lie algebra.

We begin by introducing the notion of a matrix of a graded A-linear map: consider $A \in \mathsf{gcAs}$ and V, W some freely and finitely generated A-modules[2] [10, Subsection 1.4]. In addition, suppose that A-modules have a well-defined graded rank.

Let $\varphi : V \to W$ be an A-linear map, and fix some generators $\{v_i\}_{i=1}^m$, $\{w_j\}_{j=1}^n$ of V and W, respectively. Thus, any $x \in V$ and $y \in W$ can be decomposed as

$$x = v_i x^i = (-1)^{|v_i||x^i|}\, x^i v_i \qquad \text{and} \qquad y = w_j y^j = (-1)^{|y^j||w_j|}\, y^j w_j, \tag{2.21}$$

for some unique $x^i, y^j \in A$, where $|x^i| = |x| - |v_i|$ and $|y^j| = |y| - |w_j|$ for every $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. Note that in (2.21) we a priori choose to write the coefficients of the A-linear combination *after* the generators. Similarly, for every $i \in \{1, \dots, m\}$ we have

$$\varphi(v_i) = w_j\, \varphi^j{}_i, \tag{2.22}$$

for unique $\varphi^j{}_i \in A$ of degree $|\varphi^j{}_i| = |\varphi| + |v_i| - |w_j|$. In other words, we have $\varphi^j{}_i = (\varphi(v_i))^j$. Finally, we see that for a general $x = v_i x^i \in V$ there is $\varphi(x) = \varphi(v_i)x^i = w_j \varphi^j{}_i x^i$and consequently

$$(\varphi(x))^j = \varphi^j{}_i\, x^i. \tag{2.23}$$

We may thus call $\varphi^j{}_i$ the **matrix of φ in the bases (v_i) and (w_j)** and find that the coordinates of $\varphi(x)$ in the basis w_j are simply given by acting on coordinates x^i via the matrix $\varphi^j{}_i$.

One might ask if this definition is somehow compatible with matrix multiplication. Namely, take some third freely and finitely generated A-module U with generators $\{u_k\}_{k=1}^{\ell}$ and $\psi : U \to V$ an A-linear map. Then for any $x \in U$, $x = x^k\, u_k$, we have

$$(\varphi \circ \psi\, (x))^j = \varphi^j{}_i\, (\psi(x))^i = \varphi^j{}_i\, \psi^i{}_k x^k = (\varphi \circ \psi)^j{}_k\, x^k, \tag{2.24}$$

and we immediately obtain that the matrix of a composite A-linear map is indeed obtained through the matrix multiplication. We note that, just as in classical linear algebra, once we fix a total basis for V and for W, every degree n A-linear graded map uniquely determines a degree n matrix, and vice-versa.

Let us look closer upon the matrix $\varphi^j{}_i$ for the special case $A = \mathbb{R}$. In this case, V and W are only finite-dimensional graded vector spaces and the generators v_i and w_j are their bases. As $\mathbb{R}$ is a trivially graded space, the elements $\varphi^j{}_i$ can be non-zero only when i and j satisfy

$$|w_j| - |v_i| = |\varphi|. \tag{2.25}$$

[2] For some overview of freely and finitely generated modules see also section 3.1.1.

Let us therefore relabel the bases as $\{v_i^{(k)}\}_{k\in\mathbb{Z},i=1}^{m_k} := \{v_i\}_{i=1}^{m}$ and $\{w_j^{(k)}\}_{k\in\mathbb{Z},j=1}^{n_k} := \{v_i\}_{j=1}^{n}$, where $m_k := \dim V_k$ and $n_k := \dim W_k$ and $|v_i^{(k)}| = k = |w_j^{(k)}|$. In particular, $\{v_i^{(k)}\}_{i=1}^{m_k}$ forms a basis for V_k and $\{w_j^{(k)}\}_{j=1}^{n_k}$ forms a basis for W_k. Consequently, the $n \times m$ matrix ${\varphi^j}_i$ is (possibly) non-zero only in $n_{k+|\varphi|} \times m_k$ blocks corresponding to the matrices of the constituting linear maps $\varphi_k : V_k \to W_{k+|\varphi|}$, which comes as no surprise.

Example 2.7. Let $A = \mathbb{R}$ and $V = W = \mathbb{R}^{(n_j)}$ for $n_{-1} = 1, n_0 = 2, n_1 = 1$ and $n_j = 0$ otherwise. Also the basis of $\mathbb{R}^{(n_j)}$ we shall use will be the graded standard basis, that is, $(e_1^{(-1)}, e_1^{(0)}, e_2^{(0)}, e_1^{(1)})$.

Any matrix ${\varphi^j}_i$ (matrix of any graded linear map $\varphi : \mathbb{R}^{(n_j)} \to \mathbb{R}^{(n_j)}$ in the standard basis) then takes one of the following forms, based upon its degree:

- Degree -2 matrices:
$$\begin{pmatrix} 0 & 0 & 0 & {\varphi^1}_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.26}$$
corresponding to linear maps $\mathbb{R}^{n_1} \to \mathbb{R}^{n_{-1}}$.

- Degree -1 matrices:
$$\begin{pmatrix} 0 & {\varphi^1}_2 & {\varphi^1}_3 & 0 \\ 0 & 0 & 0 & {\varphi^2}_4 \\ 0 & 0 & 0 & {\varphi^3}_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.27}$$
corresponding to pairs of linear maps $\mathbb{R}^{n_1} \to \mathbb{R}^{n_0}$ and $\mathbb{R}^{n_0} \to \mathbb{R}^{n_{-1}}$.

- Degree 0 matrices:
$$\begin{pmatrix} {\varphi^1}_1 & 0 & 0 & 0 \\ 0 & {\varphi^2}_2 & {\varphi^2}_3 & 0 \\ 0 & {\varphi^3}_2 & {\varphi^3}_3 & 0 \\ 0 & 0 & 0 & {\varphi^4}_4 \end{pmatrix}, \tag{2.28}$$
corresponding to triples of linear maps $\mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$, $\mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ and $\mathbb{R}^{n_{-1}} \to \mathbb{R}^{n_{-1}}$.

- Degree 1 matrices:
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ {\varphi^2}_1 & 0 & 0 & 0 \\ {\varphi^3}_1 & 0 & 0 & 0 \\ 0 & {\varphi^4}_2 & {\varphi^4}_3 & 0 \end{pmatrix}, \tag{2.29}$$
corresponding to pairs of linear maps $\mathbb{R}^{n_0} \to \mathbb{R}^{n_1}$ and $\mathbb{R}^{n_{-1}} \to \mathbb{R}^{n_0}$.

- Degree 2 matrices:
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ {\varphi^4}_1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.30}$$
corresponding to linear maps $\mathbb{R}^{n_{-1}} \to \mathbb{R}^{n_1}$.

Other-degree matrices are necessarily zero.

Remark 2.8. Generally $e_i^{(k)}$ will denote the i-th standard basis vector of the space $(\mathbb{R}^{(n_j)})_k \equiv \mathbb{R}^{n_k}$. For any $\mathbb{R}^{(n_j)}$, the (total) standard basis is ordered by degree and then as a classical standard basis, i.e. $e_i^{(k)} < e_j^{(\ell)}$ if $k < \ell$ or $k = \ell \wedge i < j$. The entire standard basis will then be denoted as usual by

$$(e_1, \dots, e_n) \equiv (e_1^{(a_1)}, e_2^{(a_1)}, \dots, e_{n_{a_1}}^{(a_1)}, e_1^{(a_2)}, \dots, \dots, e_{n_{a_\ell}}^{(a_\ell)}), \tag{2.31}$$

where $\{a_1, \dots, a_\ell\} = \{j \in \mathbb{Z} \,|\, n_j \neq 0\}$ and $a_i < a_{i+1}$ for all $i \in \{1, \dots, \ell - 1\}$.

Definition 2.9 (Real Graded Matrices)**.** We shall henceforth denote the graded linear space of matrices of linear maps $\mathbb{R}^{(m_j)} \to \mathbb{R}^{(n_j)}$ in the standard basis as $\mathbb{R}^{(n_j)\times(m_j)}$. We call it the **space of** $(n_j) \times (m_j)$ **matrices**. Its graded dimension $\operatorname{gdim} \mathbb{R}^{(n_j)\times(m_j)} =: (q_j)$ is given by

$$q_j := \sum_{k\in\mathbb{Z}} m_k n_{k+j}, \tag{2.32}$$

for every $j \in \mathbb{Z}$. We thus have $\mathbb{R}^{(n_j)\times(m_j)} \cong \mathbb{R}^{(q_j)}$. Note that the total dimension of $\mathbb{R}^{(n_j)\times(m_j)}$ is

$$\operatorname{tdim} \mathbb{R}^{(n_j)\times(m_j)} = \sum_j \sum_k m_k n_{k+j} = \sum_k m_k \sum_j n_{k+j} = \sum_k m_k n = mn. \tag{2.33}$$

The total basis of $\mathbb{R}^{(n_j)\times(m_j)}$ can be taken to be the matrices $\Delta^i{}_j \in \mathbb{R}^{n\times m}$, defined by

$$(\Delta^i{}_j)^k{}_\ell = \delta^i{}_\ell \delta^k{}_j, \tag{2.34}$$

i.e. $\Delta^i{}_j$ is the matrix with (i, j)-th entry equal to 1 and every other entry equal to zero. Furthermore, the degree of $\Delta^i{}_j$ can be inferred from its indices via the relation (2.25) as

$$|\Delta^i{}_j| = |e_i| - |f_j|, \tag{2.35}$$

where e_i is the i-the vector of the total standard basis of $\mathbb{R}^{(m_j)}$ (the source space) and f_j is the j-th vector of the total standard basis of $\mathbb{R}^{(n_j)}$ (the target space).

Remark 2.10. Since the sequences of integers (n_j) we work with are finite, we will sometimes write them out completely like so:

$$(n_j) = (\dots, 0, n_{-1}, n_0, n_1, 0, \dots). \tag{2.36}$$

Let us put forward an agreement that, unless otherwise specified, we will always explicitly write out an odd number of entries with the middle one being the zeroth entry n_0. Also, under this notation, all non-zero entries will be explicitly written and so all entries hidden under "…" will be zero.

Example 2.11. As we have seen for the case $(n_j) = (\dots, 0, 1, 2, 1, 0, \dots)$ in Example 2.7, the graded dimension of $\mathbb{R}^{(n_j)\times(n_j)}$ was

$$\operatorname{gdim} \mathbb{R}^{(n_j)\times(n_j)} = (\dots, 0, 1, 4, 6, 4, 1, 0, \dots). \tag{2.37}$$

2.4 Graded Lie Groups

With all the foundations laid, the definition of a graded Lie group is quite straightforward.

Definition 2.12 (Graded Lie Group)**.** We say that a group object $(\mathcal{G}, \mu, \eta, \iota) \in \mathsf{gMan}^\infty$ is a graded Lie group.

As already stated below Corollary 2.3, $(G, \underline{\mu}, \underline{\eta}, \underline{\iota})$ is an ordinary Lie group for any graded Lie group $(\mathcal{G}, \mu, \eta, \iota)$. The main body of this chapter is be devoted to several illustrative examples of graded Lie groups, the first of which is a generalization of the general linear group.

2.4.1 Graded General Linear Group

For the purposes of this section, let $(n_j)_{j\in\mathbb{Z}}$ be some fixed sequence of non-negative integers with $\sum_j n_j < \infty$. In this subchapter we give the first, and the most important, example of a graded Lie group — the graded general linear group $\mathrm{GL}((n_j)_{j\in\mathbb{Z}}, \mathbb{R})$. We begin by constructing the graded manifold $\mathbb{M}^{(n_j)}$ of $(n_j)\times(n_j)$ matrices, which we will endow with the structure of a monoid object in gMan^∞ through the virtue of a multiplication arrow and a unit arrow. We then restrict $\mathbb{M}^{(n_j)}$ to an open set of invertible (in a sense) matrices, define the inversion arrow and thus construct the graded Lie group $\mathrm{GL}((n_j), \mathbb{R})$.

Definition 2.13. We define the so-called **manifold of** $(n_j)\times(n_j)$ **matrices**, as the graded domain $\mathbb{M}^{(n_j)}$ corresponding to the linear space $\mathbb{R}^{(n_j)\times(n_j)}$ of graded $(n_j)\times(n_j)$ matrices, that is

$$\mathbb{M}^{(n_j)} := \mathrm{g}\mathbb{R}^{(n_j)\times(n_j)} \equiv (\mathbb{R}^{q_0})^{(q_{-j})}, \tag{2.38}$$

where for every $j\in\mathbb{Z}$, $q_j := \sum_{k\in\mathbb{Z}} n_k n_{k+j}$. Indeed, (q_j) is the graded dimension of the linear space of $(n_j)\times(n_j)$ matrices, see Definition 2.9. In particular, the underlying topological space of $\mathbb{M}^{(n_j)}$ is the vector space

$$\mathbb{R}^{q_0} = \mathbb{R}^{\sum_{k\in\mathbb{Z}} n_k^2} \cong \bigoplus_{k\in\mathbb{Z}} \mathbb{R}^{n_k^2}, \tag{2.39}$$

which we will think of as the direct sum of the spaces of all square $n_k \times n_k$-matrices.

Let us relabel the standard coordinates on this space to better suit our needs. In keeping with Remark 2.8, let $(e_1,\dots,e_n)$ be the standard total basis of $\mathbb{R}^{(n_j)}$. Let us introduce an embedding $\mathrm{diag} : \bigoplus_{k\in\mathbb{Z}} \mathbb{R}^{n_k^2} \to \mathbb{R}^{n^2}$ given by

$$\mathrm{diag} : (M_1,\dots,M_\ell) \mapsto \mathrm{diag}(M_1,\dots,M_\ell), \tag{2.40}$$

where $\ell = \#\{j \,|\, n_j \neq 0\}$. In other words, we simply take the ℓ matrices in $\oplus_{k\in\mathbb{Z}}\mathbb{R}^{n_k^2}$ and embed them as the blocks of a block-diagonal matrix in $\mathbb{R}^{n^2}$. Note that the image of this embedding is exactly the space $(\mathbb{R}^{(n_j)\times(n_j)})_0$, i.e. the space of degree-zero matrices, see e.g (2.28). For every $i,j\in\{1,\dots,n\}$ such that $|e_i| = |e_j|$ then define the coordinate ${x^i}_j$ on $\oplus_k\mathbb{R}^{n_k^2}$ as the (i,j)-th coordinate of the embedding, that is

$${x^i}_j(M_1,\dots,M_\ell) := {(\mathrm{diag}(M_1,\dots,M_\ell))^i}_j\,. \tag{2.41}$$

It is easy to see that this is in fact only the relabeling of the standard coordinate functions on $\oplus_k\mathbb{R}^{n_k^2}$.

To specify the graded coordinates on $\mathbb{M}^{(n_j)}$, we need to fix a total basis of $\left(\mathbb{R}_*^{(n_j)\times(n_j)}\right)^*$. For this, we choose the dual basis to matrices ${\Delta^i}_j$ from (2.34) for all $i,j\in\{1,\dots,n\}$ such that $|e_i|\neq|e_j|$. As the graded matrices ${\Delta^i}_j$ and the standard coordinates ${x^i}_j$ are indexed by complementary subsets of $\{1,\dots,n\}^2$, we will use ${x^i}_j$ for $i,j\in\{1,\dots,n\}$ to denote all our coordinates, both degree-zero and graded.

To sum up, on $\mathbb{M}^{(n_j)}$ we have n^2 coordinates $\{{x^i}_j\}_{i,j=1}^n$ with degrees

$$|{x^i}_j| = |e_j| - |e_i|, \tag{2.42}$$

where e_k is the k-th standard basis vector of $\mathbb{R}^{(n_j)}$. The degree-zero coordinates in particular are defined by (2.41). We think of the coordinates ${x^i}_j$ as true coordinates on the graded matrices.

Remark 2.14. If the context permits no confusion, we will denote the degree of the j-th standard basis vector of $\mathbb{R}^{(n_j)}$ as

$$|e_j| =: |j|. \tag{2.43}$$

To make the graded manifold $\mathbb{M}^{(n_j)}$ into a monoid in gMan^∞ we need to specify the multiplication arrow $\mu : \mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)} \to \mathbb{M}^{(n_j)}$ and the unit arrow $\eta : * \to \mathbb{M}^{(n_j)}$.

- Multiplication arrow. We know that $\mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)} \cong (\mathbb{R}^{q_0} \times \mathbb{R}^{q_0})^{(2n_j)}$ so essentially we have 2 copies of the coordinates $x^i{}_j$, denoted as $a^i{}_j$ and $b^i{}_j$ where

$$p_1^* x^i{}_j = a^i{}_j, \quad \text{and} \quad p_2^* x^i{}_j = b^i{}_j. \tag{2.44}$$

Also note that for $|x^i{}_j| = 0$ we have $a^i{}_j = x^i{}_j \circ \underline{p_1}$ and $b^i{}_j = x^i{}_j \circ \underline{p_2}$, where $\underline{p_1}$ and $\underline{p_2}$ are the classical projections from the Cartesian product $\oplus_k \mathbb{R}^{n_k^2} \times \oplus_k \mathbb{R}^{n_k^2} \to \oplus_k \mathbb{R}^{n_k^2}$.

We first define $\underline{\mu} : \oplus_k \mathbb{R}^{n_k^2} \times \oplus_k \mathbb{R}^{n_k^2} \to \oplus_k \mathbb{R}^{n_k^2}$ as the expected component-wise matrix multiplication:

$$\underline{\mu}\left((M_1, \dots, M_\ell), (N_1, \dots, N_\ell)\right) := (M_1 \cdot N_1, \dots, M_\ell \cdot N_\ell), \tag{2.45}$$

or equivalently using the diag embedding as

$$\underline{\mu}\left((M_1, \dots, M_\ell), (N_1, \dots, N_\ell)\right) = \operatorname{diag}^{-1}\left(\operatorname{diag}(M_1, \dots, M_\ell) \cdot \operatorname{diag}(N_1, \dots, N_\ell)\right). \tag{2.46}$$

Next, we need to specify the pullback μ^*. By [10, Theorem 3.13] it is enough to define pullbacks of coordinate functions. Since the graded coordinates do not generally commute, there are two intuitive ways to do this. For reasons that arise when dealing with coordinate transformations that we will encounter in the construction of the graded fiber bundle in Chapter 4, we choose to set

$$\mu^* x^i{}_j = b^k{}_j a^i{}_k. \tag{2.47}$$

We need to show that this is a valid definition. First, for every i, j, k there is (here assume no Einstein summation):

$$|b^k{}_j a^i{}_k| = |b^k{}_j| + |a^i{}_k| = |j| - |k| + |k| - |i| = |j| - |i| = |x^i{}_j|, \tag{2.48}$$

so the Einstein sum in (2.47) is justified. Next, consider the case when $|i| = |j|$, wherein

$$x^i{}_j \circ \underline{\mu}\left((M_1, \dots, M_\ell), (N_1, \dots, N_\ell)\right) = \left[\operatorname{diag}(M_1, \dots, M_\ell) \cdot \operatorname{diag}(N_1, \dots, N_\ell)\right]^i{}_j \tag{2.49}$$

$$= \sum_{k=1}^{n} \left[\operatorname{diag}(M_1, \dots, M_\ell)\right]^i{}_k \left[\operatorname{diag}(N_1, \dots, N_\ell)\right]^k{}_j \tag{2.50}$$

$$= \sum_{k:|k|=|i|} \left[\operatorname{diag}(M_1, \dots, M_\ell)\right]^i{}_k \left[\operatorname{diag}(N_1, \dots, N_\ell)\right]^k{}_j \tag{2.51}$$

$$= a^i{}_k b^k{}_j \left((M_1, \dots, M_\ell), (N_1, \dots, N_\ell)\right). \tag{2.52}$$

Consequently, for $|x^i{}_j| = 0$ we have $x^i{}_j \circ \underline{\mu} = \sum_{k:|k|=|i|} a^i{}_k b^k{}_j = \sum_{k:|k|=|i|} b^k{}_j a^i{}_k$. Note that in this case we have the decomposition

$$b^k{}_j a^i{}_k = \underbrace{\sum_{k:|k|=|i|} b^k{}_j a^i{}_k}_{\text{smooth function}} + \underbrace{\sum_{k:|k|\neq|i|} b^k{}_j a^i{}_k}_{\text{purely graded}}, \tag{2.53}$$

hence for any $|x^i{}_j| = 0$ we see that

$$\mu^* x^i{}_j = x^i{}_j \circ \underline{\mu} + \underbrace{\sum_{k:|k|\neq|i|} b^k{}_j a^i{}_k}_{\text{purely graded}}, \tag{2.54}$$

which means that μ is well defined.

- Unit arrow. This is simple, as any arrow $\eta : * \to \mathbb{M}^{(n_j)}$ is fully determined by $\underline{\eta}$, i.e. by specifying a point of the underlying topological space of $\mathbb{M}^{(n_j)}$. Furthermore, this element must be the unit element of the underlying Lie group, so

$$\underline{\eta}(*) := (1, \dots, 1) \in \bigoplus_k \mathbb{R}^{n_k^2}, \tag{2.55}$$

that is, the ℓ-tuple of identity matrices. Equivalently, $\underline{\eta}(*) := \mathrm{diag}^{-1}(1_{n\times n})$ using the diag embedding from (2.40).

Proposition 2.15 ($\mathbb{M}^{(n_j)}$ is a monoid)**.** *The graded manifold of $(n_j) \times (n_j)$ matrices $\mathbb{M}^{(n_j)}$ together with μ and η defined above is a monoid object in* gMan^∞*.*

Proof. We need to verify the commutativity of

$$\begin{array}{ccccc} (\mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)}) \times \mathbb{M}^{(n_j)} & \xrightarrow{\mu\times 1} & \mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)} & & \\ & & & \searrow^{\mu} & \\ \downarrow & & & & \mathbb{M}^{(n_j)} \\ & & & \nearrow_{\mu} & \\ \mathbb{M}^{(n_j)} \times (\mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)}) & \xrightarrow{1\times \mu} & \mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)} & & \end{array}, \tag{2.56}$$

and

$$\begin{array}{ccccc} \mathbb{M}^{(n_j)} & \xrightarrow{(\eta,1)} & \mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)} & \xleftarrow{(1,\eta)} & \mathbb{M}^{(n_j)} \\ & {}_{1}\searrow & \downarrow \mu & \swarrow_{1} & \\ & & \mathbb{M}^{(n_j)} & & \end{array}. \tag{2.57}$$

At the level of the underlying smooth maps, this is easily seen to be true from the relevant definitions (2.45) and (2.55). Let us verify it for the pullbacks. First, on the triple product $\mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)} \times \mathbb{M}^{(n_j)}$ we have 3 copies of the original coordinates $x^i{}_j$, whom we denote as $u^i{}_j, v^i{}_j$ and $w^i{}_j$. In other words,

$$u^i{}_j = p_1^* x^i{}_j, \quad v^i{}_j = p_2^* x^i{}_j, \quad w^i{}_j = p_3^* x^i{}_j. \tag{2.58}$$

Consequently, we have

$$(\mu \times 1)^* a^i{}_j = v^k{}_j u^i{}_k, \qquad (\mu \times 1)^* b^i{}_j = w^i{}_j, \tag{2.59}$$

$$(1 \times \mu)^* a^i{}_j = u^i{}_j, \qquad (1 \times \mu)^* b^i{}_j = w^k{}_j v^i{}_k. \tag{2.60}$$

Therefore,

$$(\mu \times 1)^* \mu^* x^i{}_j = (\mu \times 1)^* b^k{}_j a^i{}_k = w^k{}_j v^\ell{}_k u^i{}_\ell, \tag{2.61}$$

and

$$(1\times\mu)^*\mu^* x^i{}_j = (1\times\mu)^* b^k{}_j a^i{}_k = w^\ell{}_j v^k{}_\ell u^i{}_k, \tag{2.62}$$

which is the same. We have just shown the commutativity of the multiplication diagram (2.56).

Now for the unit diagram: let us first discuss pullback by the unit arrow. As $C^\infty_*(*) = \mathbb{R}$ contains no graded coordinates, necessarily $\eta^* x^i{}_j = 0$ whenever $|x^i{}_j| \neq 0$ and for coordinates of degree zero there is $\eta^* x^i{}_j = x^i{}_j \circ \underline{\eta}$. Since

$$x^i{}_j \circ \underline{\eta}(*) = x^i{}_j(1,\dots,1) = [\mathrm{diag}(1,\dots,1)]^i{}_j = \delta^i{}_j, \tag{2.63}$$

we find that $\eta^* x^i{}_j = \delta^i{}_j$ is the Kronecker delta, by which we mean

$$\delta^i{}_j = \begin{cases} 1 & \text{of degree } 0\text{, when } i = j, \\ 0 & \text{of degree } |j| - |i|\text{, when } i \neq j. \end{cases} \tag{2.64}$$

Consequently, we have

$$(\eta,1)^* a^i{}_j = \delta^i{}_j, \qquad (\eta,1)^* b^i{}_j = x^i{}_j, \tag{2.65}$$

$$(1,\eta)^* a^i{}_j = x^i{}_j, \qquad (1,\eta)^* b^i{}_j = \delta^i{}_j, \tag{2.66}$$

and so

$$(\eta,1)^*\mu^* x^i{}_j = (\eta,1)^* b^k{}_j a^i{}_k = x^k{}_j \delta^i{}_k = x^i{}_j, \tag{2.67}$$

while also

$$(1,\eta)^*\mu^* x^i{}_j = (1,\eta)^* b^k{}_j a^i{}_k = \delta^k{}_j x^i{}_k = x^i{}_j, \tag{2.68}$$

which shows the commutativity of the unit diagram (2.57). Recall that as η we denote both the arrow from the terminal object $\eta : * \to \mathbb{M}^{(n_j)}$ and the arrow "filtered through" the terminal object $\mathbb{M}^{(n_j)} \to * \xrightarrow{\eta} \mathbb{M}^{(n_j)}$. Indeed, either way we obtain $\eta^* x^i{}_j = \delta^i{}_j$. ■

Next we **define the graded Lie group** $\mathrm{GL}((n_j),\mathbb{R})$ by introducing the inversion arrow on the graded manifold

$$\mathrm{GL}((n_j),\mathbb{R}) := \mathbb{M}^{(n_j)}\Big|_G, \tag{2.69}$$

where G is the open set

$$G = \{(M_1,\dots,M_\ell) \mid \det(\mathrm{diag}(M_1,\dots,M_\ell)) \neq 0\}, \tag{2.70}$$

i.e. the set on which all component matrices (the blocks under the diag embedding) are invertible. We need the inversion arrow $\iota : \mathrm{GL}((n_j),\mathbb{R}) \to \mathrm{GL}((n_j),\mathbb{R})$ to preserve the commutativity of the appropriate version of diagram (1.42). For it to commute on the level of underlying smooth maps, it is both necessary and sufficient to set

$$\underline{\iota}\,(M_1,\dots,M_\ell) = (M_1^{-1},\dots,M_\ell^{-1}), \tag{2.71}$$

or equivalently as

$$\mathrm{diag}\circ\underline{\iota}\,(M_1,\dots,M_\ell) = (\mathrm{diag}(M_1,\dots,M_\ell))^{-1}, \tag{2.72}$$

using the diag embedding. Now for the pullbacks: we have

$$(\iota,1)^* a^i{}_j = \iota^* x^i{}_j, \qquad (\iota,1)^* b^i{}_j = x^i{}_j, \tag{2.73}$$

$$(1,\iota)^* a^i{}_j = x^i{}_j, \qquad (1,\iota)^* b^i{}_j = \iota^* x^i{}_j, \tag{2.74}$$

Thus, from the appropriate diagrams we see that ι is the inversion arrow if and only if

$$x^k{}_j\, \iota^*(x^i{}_k) = \delta^i{}_j, \qquad \text{and} \qquad \iota^*(x^k{}_j)x^i{}_k = \delta^i{}_j. \tag{2.75}$$

Let us put forward some motivation before we define ι. We (privately) think of $x^i{}_j$ as the coordinates on the "complete graded matrix", i.e. a matrix $n \times n$ which would be the element of $\oplus\mathbb{R}^{(n_j)\times(n_j)} \cong \mathbb{R}^{n\times n}$ and whose entries would have different degrees; the (i,j)-th entry of this matrix would have degree $|i| - |j| = -|x^i{}_j|$. Let $Q \in \mathbb{R}^{n\times n}$. Then Q decomposes as $Q_D + Q_N$ where Q_D is the block-diagonal degree-zero part of Q and Q_N is the remaining non-degree zero part. Assume the block-diagonal degree-zero part Q_D to be invertible so we can write

$$Q = Q_D\left(1 + Q_D^{-1}Q_N\right). \tag{2.76}$$

Note that

$$\left(1 + Q_D^{-1}Q_N\right)^{-1} = \sum_{n=0}^{\infty}(-1)^n(Q_D^{-1}Q_N)^n, \tag{2.77}$$

if the infinite sum on the right-hand side is well-defined (this is only a motivation, after all). In such case, we would have

$$\begin{aligned}\left(Q^{-1}\right)^i{}_j &= \left[\left(1 + Q_D^{-1}Q_N\right)^{-1} Q_D^{-1}\right]^i{}_j \\ &= \left(\sum_{n=0}^{\infty}(-1)^n(Q_D^{-1}Q_N)^i{}_{k_1}(Q_D^{-1}Q_N)^{k_1}{}_{k_2}\cdots(Q_D^{-1}Q_N)^{k_n}{}_\ell\right)(Q_D^{-1})^\ell{}_j.\end{aligned} \tag{2.78}$$

Let us now extend some notation, so that we can use Einstein summation without too much confusion. Whenever $|i| \neq |j|$, we define $x^i{}_j \circ \underline{\iota} =: 0$ of degree $|j| - |i|$ and also denote

$$\xi^i{}_j = \begin{cases} x^i{}_j, & \text{whenever } |i| \neq |j|, \\ 0, & \text{of degree } 0, \text{ whenever } |i| = |j|. \end{cases} \tag{2.79}$$

With this notation at hand, we may (still privately) think of $(Q_D^{-1})^i{}_j$ as $x^i{}_j \circ \underline{\iota}$ (the coordinates of the inverted diagonal blocks) and of $(Q_N)^i{}_j$ as $\xi^i{}_j$ (the coordinates on the "graded part" of Q). Armed with this intuition, we define

$$\Theta^i{}_j := (x^i{}_k \circ \underline{\iota})\, \xi^k{}_j, \tag{2.80}$$

$$T^i_j := \delta^i{}_j + \sum_{n=1}^{\infty}(-1)^n\, \Theta^{k_1}{}_j\Theta^{k_2}{}_{k_1}\cdots\Theta^{k_n}{}_{k_{n-1}}\Theta^i{}_{k_n}, \tag{2.81}$$

$$\iota^* x^i{}_j := T^i_k\,(x^k{}_j \circ \underline{\iota}), \tag{2.82}$$

where we are clearly motivated by (2.78). Note however, that the Θ's are in the opposite order than would follow from the motivation, which is necessary due to our definition of the multiplication arrow (2.47). We need to verify that the definition of T^i_j makes sense. First notice that $\Theta^i{}_j$ is a homogeneous polynomial in graded variables of pedigree[3] 1. Consequently for every $n \in \mathbb{N}$, $\Theta^{k_1}{}_j\Theta^{k_2}{}_{k_1}\cdots\Theta^i{}_{k_n}$ is a homogeneous pedigree n polynomial. Hence, T^i_j is a formal power series

$$T^i_j = \sum_{\boldsymbol{p}\in\mathbb{N}^n_{|j|-|i|}} (T^i_j)_{\boldsymbol{p}}\, \xi^{\boldsymbol{p}}, \tag{2.83}$$

[3]Polynomial degree, or a p-degree.

where for every $\boldsymbol{p} \in \mathbb{N}^n_{|j|-|i|}$, $\boldsymbol{p} \neq \boldsymbol{0}$ there is

$$(T^i{}_j)_{\boldsymbol{p}} = (-1)^{\omega(\boldsymbol{p})} \left(\Theta^{k_1}{}_j \Theta^{k_2}{}_{k_1} \cdots \Theta^i{}_{k_{\omega(\boldsymbol{p})}} \right)_{\boldsymbol{p}}, \tag{2.84}$$

which is clearly well-defined. For the case $\boldsymbol{p} = \boldsymbol{0}$ we simply have $(T^i{}_j)_{\boldsymbol{0}} = \delta^i{}_j$. Note that for $|x^i{}_j| = 0$, we find

$$\iota^*(x^i{}_j) = x^i{}_j \circ \underline{\iota} + \underbrace{(\sum_{n=1}^{\infty} (-1)^n \, \Theta^{k_1}{}_k \Theta^{k_2}{}_{k_1} \cdots \Theta^i{}_{k_n}) \, (x^k{}_j \circ \underline{\iota})}_{\text{purely graded}}, \tag{2.85}$$

which makes ι^* a well-defined pullback.

Proposition 2.16 (GL$((n_j), \mathbb{R})$ is a Lie group)**.** *The graded manifold* GL$((n_j), \mathbb{R})$ *together with the multiplication arrow μ, unit arrow η and inversion arrow ι defined above is a graded Lie group.*

Proof. All that remains[4] is to show the relations in (2.75). To validate the first, we write

$$\begin{aligned}
x^k{}_j \iota^*(x^i{}_k) &= x^k{}_j T^i{}_\ell (x^\ell{}_k \circ \underline{\iota}) = \sum_{k:|k|=|j|} T^i{}_\ell (x^\ell{}_k \circ \underline{\iota}) x^k{}_j + \sum_{k:|k|\neq|j|} x^k{}_j T^i{}_\ell (x^\ell{}_k \circ \underline{\iota}) \\
&= T^i{}_\ell \delta^\ell{}_j + \xi^k{}_j (x^\ell{}_k \circ \underline{\iota}) T^i{}_\ell = T^i{}_j + \Theta^\ell{}_j T^i{}_\ell \\
&= \delta^i{}_j + \left(\sum_{n=1}^{\infty} (-1)^n \, \Theta^{k_1}{}_j \Theta^{k_2}{}_{k_1} \cdots \Theta^i{}_{k_n} \right) + \Theta^i{}_j + \Theta^\ell{}_j \left(\sum_{n=1}^{\infty} (-1)^n \Theta^{k_1}{}_\ell \Theta^{k_2}{}_{k_1} \cdots \Theta^i{}_{k_n} \right) \\
&= \delta^i{}_j + \left(\sum_{n=1}^{\infty} (-1)^n \, \Theta^{k_1}{}_j \Theta^{k_1}{}_j \Theta^{k_2}{}_{k_1} \cdots \Theta^i{}_{k_n} \right) - \left(\sum_{n=1}^{\infty} (-1)^n \, \Theta^{k_1}{}_j \Theta^{k_2}{}_{k_1} \cdots \Theta^i{}_{k_n} \right) \\
&= \delta^i{}_j.
\end{aligned} \tag{2.86}$$

In order to validate the second, let us state a relation stemming from (2.76), which reads

$$x^i{}_j = \sum_{k:|k|=|i|} x^i{}_k \left(\delta^k{}_j + \Theta^k{}_j \right). \tag{2.87}$$

This holds, as for $|i| = |j|$ we have $(x^k{}_\ell \circ \underline{\iota}) \xi^\ell{}_j = 0$ for any k such that $|k| = |i|$, yielding

$$\sum_{k:|k|=|i|} x^i{}_k \delta^k{}_j = x^i{}_j \tag{2.88}$$

while for $|i| \neq |j|$ necessarily $\delta^k{}_j = 0$ for any k, $|k| = |i|$, giving us

$$\sum_{k:|k|=|i|} x^i{}_k (x^k{}_\ell \circ \underline{\iota}) \xi^\ell{}_j = \delta^i{}_\ell \xi^\ell{}_j = \xi^i{}_j = x^i{}_j. \tag{2.89}$$

Now observe that

$$\begin{aligned}
T^k{}_j \left(\delta^i{}_k + \Theta^i{}_k \right) &= T^i{}_j + \Theta^i{}_j + \left(\sum_{n=1}^{\infty} (-1)^n \Theta^{k_1}{}_j \cdots \Theta^k{}_{k_n} \right) \Theta^i{}_k \\
&= \delta^i{}_j + \sum_{n=1}^{\infty} (-1)^n \Theta^{k_1}{}_j \cdots \Theta^i{}_{k_n} + \sum_{n=1}^{\infty} (-1)^{n+1} \Theta^{k_1}{}_j \cdots \Theta^i{}_{k_n} \\
&= \delta^i{}_j,
\end{aligned} \tag{2.90}$$

[4] Round the decay of that colossal Wreck, boundless and bare...

and so

$$\iota^*(x^k{}_j)x^i{}_k = T^k{}_\ell(x^\ell{}_j \circ \underline{\iota}) \sum_{r:|r|=|i|} x^i{}_r(\delta^r{}_k + \Theta^r{}_k) = \sum_{r:|r|=|i|} (x^\ell{}_j \circ \underline{\iota})x^i{}_r\delta^r{}_\ell$$
$$= \sum_{r:|r|=|i|} (x^r{}_j \circ \underline{\iota})x^i{}_r = \delta^i{}_j \tag{2.91}$$

which together with (2.90) implies the commutativity of the defining diagram for the inversion arrow (1.42) and hence show that $\mathrm{GL}((n_j),\mathbb{R})$ is a graded Lie group. ■

As we noted already in Definition 1.12, a group object multiplication arrow is always automatically both a left and a right action. Let us now consider μ as a left action of $\mathrm{GL}((n_j),\mathbb{R})$ on itself. Arrows from the terminal object $* \in \mathsf{gMan}^\infty$ correspond uniquely to points of the underlying topological space $\times_k \mathrm{GL}(n_k,\mathbb{R})$ and are thus usually denoted by the same letter. One may therefore take some $M \in \times_k \mathrm{GL}(n_k,\mathbb{R})$ and consider the "multiplication from the left" arrow $\mu_M : \mathrm{GL}((n_j),\mathbb{R}) \to \mathrm{GL}((n_j),\mathbb{R})$ which is is the graded diffeomorphism defined in (1.46) as

$$L_M := \mu \circ (M, 1), \tag{2.92}$$

see also Remark 1.15. Note that for $|i| \neq |j|$ we have $M^*x^i{}_j = 0$ and for $|i| = |j|$ there is

$$M^*x^i{}_j = x^i{}_j(M) = (\operatorname{diag} M)^i{}_j\,. \tag{2.93}$$

Consequently one has

$$L^*_M\, x^i{}_j = (M,1)^*\mu^*\, x^i{}_j = (M,1)^*\left(b^k{}_j a^i{}_k\right) = \sum_{k:|k|=|i|} (\operatorname{diag} M)^i{}_k\, x^k{}_j. \tag{2.94}$$

Just as the classical Lie group $\mathrm{GL}(n,\mathbb{R})$ acts on the linear space $\mathbb{R}^n$ from the left by matrix multiplication $A \cdot x = Ax$, so will the graded domain $\mathrm{GL}((n_j),\mathbb{R})$ act on

$$(\mathbb{R}^{n_0})^{(n_{-j})} =: \mathrm{g}\mathbb{R}^{(n_j)}, \tag{2.95}$$

which is the graded domain associated to the graded vector space $\mathbb{R}^{(n_j)}$. Assuming our matrix intuition is still valid, we would like the left action

$$\theta : \mathrm{GL}((n_j),\mathbb{R}) \times \mathrm{g}\mathbb{R}^{(n_j)} \to \mathrm{g}\mathbb{R}^{(n_j)}, \tag{2.96}$$

see Definition 1.12, to satisfy

$$\theta^* y^i = y^j x^i{}_j, \tag{2.97}$$

where y^j are the coordinates on $\mathrm{g}\mathbb{R}^{(n_j)}$ of degree $|y^j| = -|j|$. To be more precise, y^j form the dual basis to the standard coordinates on $\mathbb{R}^{(n_j)}$. Since for any $i, j \in \{1, \ldots, n\}$ we have (no Einstein summation)

$$|y^j x^i{}_j| = -|j| + |j| - |i| = -|i| = |y^i|, \tag{2.98}$$

the relation (2.97) makes sense degree-wise. If we consider some i such that $|i| = 0$, we can decompose $\theta^* y^i$ as

$$\theta^* y^i = y^j x^i{}_j = \underbrace{\sum_{j:|j|=|i|} y^j x^i{}_j}_{\text{ordinary function}} + \underbrace{\sum_{j:|j|\neq|i|} y^j x^i{}_j}_{\text{purely graded}}. \tag{2.99}$$

Clearly we need the "ordinary function" part, or the body, of $\theta^* y^i$ to equal $y^i \circ \underline{\theta}$. Recall that the underlying smooth manifold of $\mathrm{GL}((n_j), \mathbb{R})$ is the set

$$U = \{(M_k)_{k\in\mathbb{Z}} \mid \det(M_k) \neq 0 \text{ for every } k \in \mathbb{Z} \text{ such that } n_k \neq 0\} \subset \times_{k\in\mathbb{Z}} \mathbb{R}^{n_k^2}. \tag{2.100}$$

Note that this expression for U is the same as in (2.70) though it is expressed slightly differently. We define the underlying smooth map $\underline{\theta}$ as multiplication from the left by the zeroth component of $(M_k)_{k\in\mathbb{Z}}$, i.e.

$$\underline{\theta}\left((M_k)_{k\in\mathbb{Z}}, v\right) = M_0 \cdot v, \tag{2.101}$$

for any $(M_k)_{k\in\mathbb{Z}} \in U$ and $v \in \mathbb{R}^{n_0}$. For any degree zero coordinate y^i we now have

$$y^i \circ \underline{\theta} = \sum_{j:|j|=0} y^j x^i{}_j, \tag{2.102}$$

giving, for any $|i| = 0$,

$$\theta^* y^i = y^i \circ \underline{\theta} + \sum_{j:|j|\neq|i|} y^j x^i{}_j, \tag{2.103}$$

where the second term is purely graded, which makes θ a well-defined morphism of graded domains. Of course, we need to verify that we have truly defined a left graded Lie group action.

Proposition 2.17. *The arrow θ defined above is a left Lie group action of $\mathrm{GL}((n_j), \mathbb{R})$ on the graded domain $\mathrm{g}\mathbb{R}^{(n_j)}$.*

Proof. The proof consists of validating the commutativity of the appropriate version of diagrams (1.43) and (1.44). At the level of the underlying smooth maps, this is again clear. At the level of pullbacks we have, for diagram (1.43),

$$(\eta, 1)^* \theta^* y^i = (\eta, 1)^* \left(y^j x^i{}_j\right) = y^j \delta^i{}_j = y^i. \tag{2.104}$$

For diagram (1.44) label the coordinates on $\mathrm{GL}((n_j), \mathbb{R}) \times \mathrm{GL}((n_j), \mathbb{R}) \times \mathrm{g}\mathbb{R}^{(n_j)}$ as $a^i{}_j, b^i{}_j, y^i$ with the obvious meaning. Then

$$\begin{aligned} (\mu \times 1)^* x^i{}_j &= b^k{}_j a^i{}_k, \qquad & (\mu \times 1)^* y^i &= y^i, \\ (1 \times \theta)^* x^i{}_j &= a^i{}_j, \qquad & (1 \times \theta)^* y^i &= y^k b^i{}_k, \end{aligned} \tag{2.105}$$

which gives us

$$\begin{aligned} (\mu \times 1)^* \theta^* y^i &= (\mu \times 1)^* y^j x^i{}_j = y^j b^k{}_j a^i{}_k \qquad \text{and} \\ (1 \times \theta)^* \theta^* y^i &= (1 \times \theta)^* y^k x^i{}_k = y^k b^\ell{}_k a^i{}_\ell, \end{aligned} \tag{2.106}$$

as desired. ■

Some observations:

- Whenever (n_j) has only one non-zero component, the graded Lie group $\mathrm{GL}((n_j), \mathbb{R})$ reduces to the ordinary Lie group $\mathrm{GL}(n, \mathbb{R})$. However, the action θ reduces to the ordinary left action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ *only* if the non-zero entry of the sequence (n_j) is the zeroth one, i.e. n_0. Otherwise it becomes a left action of $\mathrm{GL}(n, \mathbb{R})$, seen as trivially graded, on the domain $\mathrm{g}\mathbb{R}^{(n_j)} = \{*\}^{(n_{-j})}$.

- For a general (n_j), the underlying map $\underline{\theta}$ becomes an (ordinary) left action of $\bigtimes_{k\in\mathbb{Z}} \mathrm{GL}(n_k, \mathbb{R})$ on $\mathbb{R}^{n_0}$. However, by construction $\underline{\theta}$ only depends on the "zeroth" matrix, i.e. $\underline{\theta}((M_k)_{k\in\mathbb{Z}}, v) = \underline{\theta}(M_0, v)$ and thus "ignores" a large part of the underlying Lie group.

- In the non-graded case, one can just as easily define a *right* action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ by a matrix multiplication from the right, that is

$$v \cdot M := vM, \tag{2.107}$$

where v is now taken as a row vector. Note that in the graded setting the straightforward generalization of this would be to set $\theta^* y^i := y^i {x^i}_j$ with sum over i. But simple degree-counting argument tells us that this is not possible. One would have to define the right action of $\mathrm{GL}((n_j), \mathbb{R})$ on the graded domain $\mathrm{g}\mathbb{R}^{(n_{-j})}$ corresponding to the *dual* linear space $\left(\mathbb{R}^{(n_j)}\right)^* \cong \mathbb{R}^{(n_{-j})}$, i.e.

$$\theta : \mathrm{g}\mathbb{R}^{(n_{-j})} \times \mathrm{GL}((n_j), \mathbb{R}) \to \mathrm{g}\mathbb{R}^{(n_{-j})}. \tag{2.108}$$

Standard coordinates on the domain $\mathrm{g}\mathbb{R}^{(n_{-j})}$ are y_j with degrees $|y_j| = |j|$ corresponding to the standard basis of $\mathbb{R}^{(n_j)}$. The pullback by the right action θ would then be

$$\theta^* y_j := y_i \, {x^i}_j, \tag{2.109}$$

and the underlying smooth map would simply be multiplication of a (row) vector by the zeroth matrix from the right.

- For every point $\times_k \mathrm{GL}(n_k, \mathbb{R})$ we have the isomorphism

$$\theta_M : \mathrm{g}\mathbb{R}^{(n_j)} \to \mathrm{g}\mathbb{R}^{(n_j)} \tag{2.110}$$

defined in (1.46) by $\theta_M = \theta \circ (M, 1)$. As in (2.93), there is $M^* {x^i}_j = {x^i}_j(M) = (\mathrm{diag}\, M)^i{}_j$ and so the graded diffeomorphism θ_M is given by

$$\underline{\theta_M} v = M_0 \, v = \underline{\theta}(M, v), \tag{2.111}$$

for every $v \in \mathbb{R}^{n_0}$and

$$\theta^*_M y^i = (M, 1)^* \theta^* y^i = (M, 1)^* \left(y^j {x^i}_j\right) = \sum_{j:|j|=|i|} (\mathrm{diag} M)^i{}_j \, y^j. \tag{2.112}$$

2.4.2 Other Examples

Example 2.18 (Graded Vector Space)**.** This example shows that similarly to the non-graded case, a graded vector space — or rather the corresponding graded domain — can be considered as a graded Lie group. Let (n_j) be some finite sequence of non-zero integers and consider the graded domain $\mathrm{g}\mathbb{R}^{(n_j)}$ with coordinates $(x^1, \ldots, x^n)$ corresponding to the dual of the standard basis for $\mathbb{R}^{(n_j)}$. In particular, under the notation of Remark 2.14, we have $|x^j| = -|j|$.

The multiplication arrow μ will be "inherited" from the addition in $\mathbb{R}^{(n_j)}$, that is, the underlying smooth map is defined as

$$\underline{\mu}(x, y) := x + y, \tag{2.113}$$

for any $x, y \in \mathbb{R}^{n_0}$, and the pullback is defined as

$$\mu^* x^i = a^i + b^i, \tag{2.114}$$

where a^i, b^i denote the corresponding copies of coordinates x^i on the product $g\mathbb{R}^{(n_j)} \times g\mathbb{R}^{(n_j)} \cong g\mathbb{R}^{(2n_j)}$. For any x^i of degree zero we clearly have $\mu^* x^i = x^i \circ \underline{\mu}$ which makes μ a well-defined graded smooth map. The unit arrow will of course be the zero-vector $0 \in \mathbb{R}^{n_0}$, and the inversion arrow is given by $\underline{\iota}(x) = -x$ for any $x \in \mathbb{R}^{n_0}$ and $\iota^* x^i = -x^i$.

Example 2.19 (Two-Point Group)**.** In the classical setting finite Lie groups have no interesting smooth structure — they are merely a finite disjoint union of trivial singleton manifolds. In the graded setting, however, one can consider the graded domain $\{*\}^{(n_j)}$ for any finite sequence of integers (n_j) where $n_0 = 0$. The algebra of graded functions for such a domain consists of formal power series

$$f = \sum_{\boldsymbol{p} \in \mathbb{N}^n_{|f|}} \lambda_{\boldsymbol{p}}\, \xi^{\boldsymbol{p}}, \tag{2.115}$$

where $\lambda_{\boldsymbol{p}}$ are real numbers and $\{\xi^\nu\}_{\nu=1}^n$ are the graded coordinates on $\{*\}^{(n_j)}$. This opens up a possibility of non-trivial graded Lie groups whose underlying manifolds are finite groups.

In this example we find the most general shape of a graded Lie group $\mathcal{G}$ of graded dimension $(n_j) = (\dots, 0, 0, 0, 1, 0 \dots)$ whose underlying Lie group is $G = \mathbb{Z}_2$. We denote the points of the underlying group as $\{\bullet, \circ\}$ with the group multiplication rules

$$\bullet^2 = \bullet, \quad \bullet\circ = \circ, \quad \circ\bullet = \circ \quad \text{and} \quad \circ^2 = \bullet. \tag{2.116}$$

In the usual additive notation $\bullet$ corresponds to the unit element 0 and $\circ$ corresponds to 1. Due to the fact that $\xi^2 = 0$, the algebra of graded functions on the domain $\{*\}^{(n_j)}$ is very simple:

$$\Big(C^\infty_{(n_j)}(*)\Big)_0 = \mathbb{R}, \qquad \Big(C^\infty_{(n_j)}(*)\Big)_1 = \mathbb{R}\,\xi, \tag{2.117}$$

and $(C^\infty_{(n_j)}(*))_k = 0$ for all $k \in \mathbb{Z} \setminus \{0, 1\}$. Every global graded function f on $\mathcal{G}$ is fully determined by its restrictions to $\{\bullet\}$ and to $\{\circ\}$. On the other hand, as $\{\bullet\}$ and $\{\circ\}$ are disjoint subsets of G, every pair of graded functions $f_1 \in C^\infty_{\mathcal{G}}(\bullet)$ and $f_2 \in C^\infty_{\mathcal{G}}(\circ)$ glues together unique a global function f that restricts to f_1 on $\{\bullet\}$ and to f_2 on $\{\circ\}$.

Consider any morphism $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, such that $\underline{\mu}$ is the $\mathbb{Z}_2$ group multiplication (2.116). The pullback $\mu^* : C^\infty_{\mathcal{G}} \to \underline{\mu}_* C^\infty_{\mathcal{G}\times\mathcal{G}}$ is then fully and uniquely determined by 8 numbers $k_1, k_2, \dots, k_8 \in \mathbb{R}$ in the following way:

$$\begin{aligned} (\mu^*_\bullet \xi)|_{(\bullet,\bullet)} &= k_1\,\eta + k_2\,\theta, & (\mu^*_\bullet \xi)|_{(\circ,\circ)} &= k_3\,\eta + k_4\,\theta, \\ (\mu^*_\circ \xi)|_{(\bullet,\circ)} &= k_5\,\eta + k_6\,\theta, & (\mu^*_\circ \xi)|_{(\circ,\bullet)} &= k_7\,\eta + k_8\,\theta, \end{aligned} \tag{2.118}$$

where η, θ are the graded coordinates on the product domain $\{*\}^{(n_j)} \times \{*\}^{(n_j)} \cong \{*\}^{(2n_j)}$. In order for μ to be the product arrow, it must satisfy the following conditions.

The unit condition. The unit arrow for $\mathcal{G}$ must be the unit element of G i.e. $\bullet$, hence the requirement $\mu(1, \bullet) = 1 = \mu(\bullet, 1)$ leads to

$$\xi = (1, \bullet)^*\ (\mu^*_\bullet \xi)|_{(\bullet,\bullet)} = (1, \bullet)^*(k_1\,\eta + k_2\,\theta) = k_1\,\xi, \tag{2.119}$$

$$\xi = (1, \bullet)^*\ (\mu^*_\circ \xi)|_{(\circ,\bullet)} = (1, \bullet)^*(k_7\,\eta + k_8\,\theta) = k_7\,\xi, \tag{2.120}$$

$$\xi = (\bullet, 1)^*\ (\mu^*_\bullet \xi)|_{(\bullet,\bullet)} = (\bullet, 1)^*(k_1\,\eta + k_2\,\theta) = k_2\,\xi, \tag{2.121}$$

$$\xi = (\bullet, 1)^*\ (\mu^*_\circ \xi)|_{(\bullet,\circ)} = (\bullet, 1)^*(k_5\,\eta + k_6\,\theta) = k_6\,\xi, \tag{2.122}$$

which dictates that $k_1 = k_2 = k_6 = k_7 = 1$. Let us relabel the remaining coefficients as $k_3 =: a$, $k_4 =: b$, $k_5 =: c$ and $k_8 =: d$. The multiplication is thus forbidden to take any form other than

$$\begin{aligned} (\mu_\bullet^*\xi)|_{(\bullet,\bullet)} &= \eta + \theta, & (\mu_\bullet^*\xi)|_{(\circ,\circ)} &= a\,\eta + b\,\theta, \\ (\mu_\circ^*\xi)|_{(\bullet,\circ)} &= c\,\eta + \theta, & (\mu_\circ^*\xi)|_{(\circ,\bullet)} &= \eta + d\,\theta, \end{aligned} \tag{2.123}$$

for some $a, b, c, d \in \mathbb{R}$.

The inversion condition. As for any other group object, we must specify the inversion arrow $\iota : \mathcal{G} \to \mathcal{G}$. The underlying map is already given as the inversion in $\mathbb{Z}_2$, which is the identity map. Any graded smooth function $\iota : \mathcal{G} \to \mathcal{G}$ with the identity as its underlying smooth map is fully and uniquely determined by 2 numbers $r, s \in \mathbb{R}$, where

$$\iota_\bullet^*\,\xi = r\,\xi, \qquad \text{and} \qquad \iota_\circ^*\,\xi = s\,\xi. \tag{2.124}$$

The relation between the multiplication arrow and the inversion arrow is given in (1.42) and in this case translates to the following 4 conditions:

$$0 = (\iota, 1)^*\ (\mu_\bullet^*\xi)|_{(\bullet,\bullet)} = (\iota, 1)^*(\eta + \theta) = r\xi + \xi, \tag{2.125}$$

$$0 = (\iota, 1)^*\ (\mu_\bullet^*\xi)|_{(\circ,\circ)} = (\iota, 1)^*(a\eta + b\theta) = as\xi + b\xi \tag{2.126}$$

$$0 = (1, \iota)^*\ (\mu_\bullet^*\xi)|_{(\bullet,\bullet)} = (1, \iota)^*(\eta + \theta) = \xi + r\xi, \tag{2.127}$$

$$0 = (1, \iota)^*\ (\mu_\bullet^*\xi)|_{(\circ,\circ)} = (1, \iota)^*(a\eta + b\theta) = a\xi + bs\xi. \tag{2.128}$$

It follows that r needs to equal -1. We also know, from Corollary 1.26, that if the inversion arrow exists then it is unique. This gives us another limitation on μ, namely that $(a, b) \neq (0, 0)$, otherwise s could take any value in $\mathbb{R}$. In fact, neither a nor b can be zero: suppose, for instance, that $b = 0$. Then $a = 0$ by (2.128) which is in contradiction with the above. Hence from (2.126) and (2.128) we infer the relations

$$-\frac{a}{b} = s = -\frac{b}{a}. \tag{2.129}$$

The associativity condition. Requirement $\mu\ (1 \times \mu) = \mu\ (\mu \times 1)$ leads to 8 conditions on the parameters a, b, c, d, most of which turn out to be redundant. In fact, it is enough to state one of them. If we denote the graded coordinates on the triple product domain $\{*\}^{(n_j)} \times \{*\}^{(n_j)} \times \{*\}^{(n_j)} \cong \{*\}^{(3n_j)}$ as α, β, γ, then requiring

$$\Big((1 \times \mu)_{(\circ,\bullet)}^*\ (\mu_\circ^*\xi)|_{(\circ,\bullet)}\Big)\Big|_{(\circ,\circ,\circ)} = \Big((1 \times \mu)_{(\circ,\bullet)}^*\ (\eta + d\theta)\Big)\Big|_{(\circ,\circ,\circ)} = \alpha + d\,(a\beta + b\gamma) \tag{2.130}$$

to equal

$$\Big((\mu \times 1)_{(\bullet,\circ)}^*\ (\mu_\circ^*\xi)|_{(\bullet,\circ)}\Big)\Big|_{(\circ,\circ,\circ)} = \Big((\mu \times 1)_{(\bullet,\circ)}^*\ (c\eta + \theta)\Big)\Big|_{(\circ,\circ,\circ)} = c\,(a\alpha + b\beta) + \gamma, \tag{2.131}$$

yields three conditions: $ac = 1$, $bd = 1$and $ad = bc$, where the last condition in fact follows from the first two and (2.129). As stated, all other bounds imposed by associativity turn out to be redundant.

We conclude that the choice of a graded Lie group structure on the graded manifold $\mathcal{G}$ corresponds uniquely to the choice of a sign and a non-zero real number, i.e. an element $(\epsilon, a) \in \{-1, 1\} \times \mathbb{R} \setminus \{0\}$. The inversion arrow is then given as

$$\iota_\bullet^*\,\xi = -\xi, \qquad \text{and} \qquad \iota_\circ^*\,\xi = \epsilon\,\xi, \tag{2.132}$$

and the multiplication arrow as

$$\begin{aligned} (\mu^*_\bullet \xi)|_{(\bullet,\bullet)} &= \eta + \theta, & (\mu^*_\bullet \xi)|_{(\circ,\circ)} &= a\,\eta - \epsilon a\,\theta, \\ (\mu^*_\circ \xi)|_{(\bullet,\circ)} &= \frac{1}{a}\eta + \theta, & (\mu^*_\circ \xi)|_{(\circ,\bullet)} &= \eta - \frac{\epsilon}{a}\theta. \end{aligned} \tag{2.133}$$

Chapter 3

Associated Lie Algebra

3.1 Vector Fields on Graded Manifolds

3.1.1 Brief Overview of Graded Vector Fields

As with subchapter 2.1, this overview is not intended as a rigorous mathematical introduction but merely as a summary of the relevant concepts from [10].

Consider a graded manifold $\mathcal{M}$ and $U \in \mathrm{Op}(M)$. We say that a graded linear map $X : C^\infty_{\mathcal{M}}(U) \to C^\infty_{\mathcal{M}}(U)$ is a **graded vector field** if it satisfies the graded analogue of the Leibniz rule:

$$X(fg) = X(f)g + (-1)^{|X||f|} f X(g), \tag{3.1}$$

for any $f, g \in C^\infty_{\mathcal{M}}(U)$. We denote the graded linear space of all graded vector fields on U as $\mathscr{X}_{\mathcal{M}}(U)$. Note that graded vector fields are closed under the graded commutator

$$[X, Y] := X \circ Y - (-1)^{|X||Y|} Y \circ X. \tag{3.2}$$

For any $V \in \mathrm{Op}(U)$ one may define the restriction $X|_V \in \mathscr{X}_{\mathcal{M}}(V)$, though this is non-trivial and requires the use of partition of unity [10, Subsection 3.5]. With these restrictions, the assignment $\mathscr{X}_{\mathcal{M}} : U \mapsto \mathscr{X}_{\mathcal{M}}(U)$ becomes a sheaf on M valued in gVec.

The sheaf $\mathscr{X}_{\mathcal{M}}$ has additional structure that is of interest. One may multiply a vector field $X \in \mathscr{X}_{\mathcal{M}}(U)$ by a graded function $h \in C^\infty_{\mathcal{M}}(U)$ to obtain a vector field $h \cdot X \in \mathscr{X}_{\mathcal{M}}(U)$ of degree $|h \cdot X| = |h| + |X|$, where

$$(h \cdot X)f := h(Xf), \tag{3.3}$$

for any $f \in C^\infty_{\mathcal{M}}(U)$. In other words, $\mathscr{X}_{\mathcal{M}}(U)$ is a $C^\infty_{\mathcal{M}}(U)$-module. This module structure is compatible with restrictions, i.e. $(h \cdot X)|_V = h|_V \cdot X|_V$ and so we say that $\mathscr{X}_{\mathcal{M}}$ is a **sheaf of $C^\infty_{\mathcal{M}}$-modules**, see [10, Subsection 2.4]. Consider some sheaf $\mathscr{S}$ on M valued in gVec which is a sheaf of $C^\infty_{\mathcal{M}}$-modules. We say that the n-tuple $(s_1, \ldots, s_n)$, for some $n \in \mathbb{N}$ and $s_i \in \mathscr{S}(M)$ forms a **frame** for $\mathscr{S}$ if for every $U \in \mathrm{Op}(M)$, any $f \in \mathscr{S}(U)$ can be decomposed as

$$f = f^i \cdot s_i|_U \,, \tag{3.4}$$

for some unique $f^i \in C^\infty_{\mathcal{M}}(U)$. Note that the graded functions f^i must be of degree $|f^i| = |f| - |s_i|$. We say that $\mathscr{S}$ is freely and finitely generated if there exists a frame for $\mathscr{S}$. We say it is **locally**

freely and finitely generated if every point $x \in M$ has a neighborhood $U \in \mathrm{Op}_x(M)$ such that $\mathscr{S}|_U$ is freely and finitely generated.

The sheaf of vector fields $\mathscr{X}_\mathcal{M}$ is itself locally freely and finitely generated. Indeed, let $(n_j) := \mathrm{gdim}\,\mathcal{M}$ and $n := \dim \mathcal{M}$. Consider some graded domain $U^{(n_j)}$ where we have the **coordinate vector fields** $\{\frac{\partial}{\partial x^i}\}_{i=1}^{n_0}$ and $\{\frac{\partial}{\partial \xi_\mu}\}_{\mu=1}^{n-n_0}$ which act as

$$\frac{\partial}{\partial x^i} x^j = \delta^j{}_i, \quad \frac{\partial}{\partial x^i}\xi_\mu = 0, \quad \frac{\partial}{\partial \xi_\mu} x^i = 0, \quad \frac{\partial}{\partial \xi_\mu}\xi_\nu = \delta^\mu{}_\nu, \tag{3.5}$$

on the coordinates and "extend naturally" to all graded functions on $U^{(n_j)}$. For more details, see [10, Subsection 4.2]. Any vector field X on the graded domain $U^{(n_j)}$ is then given as

$$X = X(x^i)\frac{\partial}{\partial x^i} + X(\xi_\mu)\frac{\partial}{\partial \xi_\mu}, \tag{3.6}$$

i.e. $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n_0}}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_{n-n_0}})$ forms a frame for $\mathscr{X}_{(n_j)}|_U$, which is the sheaf of vector fields on $U^{(n_j)}$. Given that $\mathcal{M}$ is locally isomorphic to graded domains, $\mathscr{X}_\mathcal{M}$ is locally freely and finitely generated.

For any graded manifold $\mathcal{M}$ and any point $x \in M$ we also have $\mathrm{T}_x\mathcal{M} \in \mathsf{gVec}$, called the **tangent space** at x, defined as the space of all graded linear maps $v : C^\infty_{\mathcal{M},x} \to \mathbb{R}$ which satisfy another analogue of the Leibniz rule:

$$v\left([f]_x[g]_x\right) = v([f]_x)g(x) + (-1)^{|v||f|}f(x)v([g]_x), \tag{3.7}$$

for any $[f]_x, [g]_x \in C^\infty_{\mathcal{M},x}$. Recall that $f(x) \equiv \underline{f}(x)$. Here, $\mathbb{R}$ is considered as a graded linear space $(\mathbb{R})_0 = \mathbb{R}$ and $(\mathbb{R})_k = 0$ for any $k \neq 0$. The elements of $\mathrm{T}_x\mathcal{M}$ are called **tangent vectors as** x. For any vector field $X \in \mathscr{X}_\mathcal{M}(U)$ and any point $x \in U$ we have the tangent vector $X|_x$, which we call the value of X at x and which is defined as $X|_x[f]_x := [Xf]_x$ for some representative $f \in C^\infty_\mathcal{M}(U)$ of $[f]_x$. That such a representative can always be found is another consequence of the partition of unity.

A major distinction between graded vector fields and ordinary vector fields is that the values of $X \in \mathscr{X}_\mathcal{M}(U)$ at every point $x \in U$ do not determine X itself. A prototypical example is the so-called Euler vector field $E \in \mathscr{X}_\mathcal{M}(M)$ defined as $Ef := |f|f$ for any $f \in C^\infty_\mathcal{M}(M)$, which is valued as zero at every point of M.

Consider two graded manifolds $\mathcal{M}, \mathcal{N}$, a graded smooth map $\phi : \mathcal{M} \to \mathcal{N}$ and a pair of global vector fields $X \in \mathscr{X}_\mathcal{M}(M)$ and $Y \in \mathscr{X}_\mathcal{N}(N)$. We say that X and Y are **ϕ-related**, written as $X \sim_\phi Y$, if

$$\begin{array}{ccc} C^\infty_\mathcal{N}(N) & \xrightarrow{\phi^*} & C^\infty_\mathcal{M}(M) \\ \downarrow{\scriptstyle Y} & & \downarrow{\scriptstyle X} \\ C^\infty_\mathcal{N}(N) & \xrightarrow{\phi^*} & C^\infty_\mathcal{M}(M) \end{array} \tag{3.8}$$

commutes, i.e. if $X \circ \phi^* = \phi^* \circ Y$. Recall [10, Example 4.25.] that on $\mathcal{M} \times \mathcal{N}$ we have the global vector fields $X \otimes 1$ and $1 \otimes Y$ defined uniquely by the relations $X \otimes 1 \sim_{p_1} X$, $X \otimes 1 \sim_{p_2} 0$ and $1 \otimes Y \sim_{p_1} 0$, $1 \otimes Y \sim_{p_2} Y$, respectively, where p_1, p_2 are the canonical product projections.

Locally, let $\{x^i\}$ be coordinates on $U \in \mathrm{Op}(M)$ and $\{y^a\}$ be coordinates on $V \in \mathrm{Op}(N)$, with $X|_U = X^i \frac{\partial}{\partial x^i}$ and $Y|_V = Y^a \frac{\partial}{\partial y^a}$. By construction of the product, $\{p_1^*(x^i)|_{U\times V}, p_2^*(y^a)|_{U\times V}\}$ are local coordinates for $\mathcal{M} \times \mathcal{N}$ on the open set $U \times V \in \mathrm{Op}(M \times N)$. The pullbacks by projections are usually omitted and we write simply $\{x^i, y^a\}$, which yields the local decompositions

$$(X \otimes 1)|_{U\times V} = p_1^*(X^i)|_{U\times V}\,\frac{\partial}{\partial x^i}, \qquad \text{and} \qquad (1 \otimes Y)|_{U\times V} = p_2^*(Y^a)|_{U\times V}\,\frac{\partial}{\partial y^a}. \tag{3.9}$$

If ϕ is a graded diffeomorphism we write $\phi_* X := (\phi^{-1})^* \circ X \circ \phi^*$ and we call $\phi_* X$ the **pushforward** of the vector field X by the graded diffeomorphism ϕ. Note that $\phi_* X = Y$ is the same as $X \sim_\phi Y$.

3.1.2 Some Observations about Vector Fields

Let us follow up by several observations about vector fields on graded manifolds which we will find useful later in this text. The next proposition is somewhat technical, but gives a useful tool for verifying when two vector fields are ϕ-related.

Proposition 3.1. *Let $\mathcal{M}, \mathcal{N}$ be two graded manifolds, $\phi : \mathcal{M} \to \mathcal{N}$ a graded smooth map and let X be a global vector field on $\mathcal{M}$ and Y be a global vector field on $\mathcal{N}$.*

Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover by coordinate patches on $\mathcal{N}$. Then $X \sim_\phi Y$ if and only if for every $\alpha \in I$,

$$\left(X|_{\underline{\phi}^{-1}(V_\alpha)} \circ \phi^*_{V_\alpha} \right) y^a = \left(\phi^*_{V_\alpha} \circ Y|_{V_\alpha} \right) y^a, \tag{3.10}$$

for every coordinate graded function y^a on V_α.

Proof. The "only if" part follows from the extension lemma: let $X \sim_\phi Y$ and consider some $\alpha \in I$ and $W \in \mathrm{Op}(V_\alpha)$ such that $\overline{W} \subseteq V_\alpha$. Let $\hat{y}^a \in C^\infty_{\mathcal{N}}(N)$ be some extension of y^a from W, i.e. $\hat{y}^a|_W = y^a|_W$. Then

$$(\phi^*_N \circ Y)\, \hat{y}^a = (X \circ \phi^*_N)\, \hat{y}^a. \tag{3.11}$$

We may also shuffle the restrictions to find

$$\begin{aligned} ((\phi^*_N \circ Y)\hat{y}^a)|_{\underline{\phi}^{-1}(W)} &= \phi^*_W\, (Y\hat{y}^a)|_W = \phi^*_W\, Y|_W\, \hat{y}^a|_W = \phi^*_W\, Y|_W\, y^a|_W = \phi^*_W (Y|_{V_\alpha} y^a)|_W \\ &= \left((\phi^*_{V_\alpha} \circ Y|_{V_\alpha}) y^a\right)\big|_{\underline{\phi}^{-1}(W)} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} ((X \circ \phi^*_N)\hat{y}^a)|_{\underline{\phi}^{-1}(W)} &= X|_{\underline{\phi}^{-1}(W)} \left((\phi^*_N \hat{y}^a)|_{\underline{\phi}^{-1}(W)} \right) = X|_{\underline{\phi}^{-1}(W)}\, (\phi^*_W\, \hat{y}^a|_W) = X|_{\underline{\phi}^{-1}(W)}\, (\phi^*_W\, y^a|_W) \\ &= X|_{\underline{\phi}^{-1}(W)}\, (\phi^*_{V_\alpha} y^a)|_{\underline{\phi}^{-1}(W)} = \left((X|_{\underline{\phi}^{-1}(V_\alpha)} \circ \phi^*_{V_\alpha}) y^a \right)\Big|_{\underline{\phi}^{-1}(W)}. \end{aligned} \tag{3.13}$$

We see that (3.12) equals (3.13) for every $W \in \mathrm{Op}(V_\alpha)$ such that $\overline{W} \subseteq V_\alpha$. Since such sets form an open cover for V_α and hence the sets $\underline{\phi}^{-1}(W)$ form an open cover of $\underline{\phi}^{-1}(V_\alpha)$, we obtain the equality (3.10).

Conversely, assume that (3.10) holds for every α and every a and consider some $f \in C^\infty_{\mathcal{N}}(N)$. We need to show that $(X \circ \phi^*_N) f = (\phi^*_N \circ Y) f$. Since $\{\underline{\phi}^{-1}(V_\alpha)\}_{\alpha \in I}$ is an open cover for M, it is enough to show that $((X \circ \phi^*_N) f)|_{\underline{\phi}^{-1}(V_\alpha)} = ((\phi^*_N \circ Y) f)|_{\underline{\phi}^{-1}(V_\alpha)}$ for every $\alpha \in I$. Similarly as above, we have

$$((X \circ \phi^*_N) f)|_{\underline{\phi}^{-1}(V_\alpha)} = \left(X|_{\underline{\phi}^{-1}(V_\alpha)} \circ \phi^*_{V_\alpha} \right) (f|_{V_\alpha}), \tag{3.14}$$

and

$$((\phi^*_N \circ Y) f)|_{\underline{\phi}^{-1}(V_\alpha)} = \left(\phi^*_{V_\alpha} \circ Y|_{V_\alpha} \right) (f|_{V_\alpha}) \tag{3.15}$$

We will start by showing that

$$\left(\phi^*_{V_\alpha} \circ Y|_{V_\alpha} \right) h = \left(X|_{\underline{\phi}^{-1}(V_\alpha)} \circ \phi^*_{V_\alpha} \right) h, \tag{3.16}$$

for any $h \in C^\infty_{\mathcal{N}}(V_\alpha)$ which is a polynomial in variables y^a. Let us do this by induction on the pedigree of h. For pedigree zero polynomials (i.e. real numbers) this is doubtlessly true, so assume (3.16) holds for all pedigree $n-1$ polynomials, and let h be a pedigree n monomial, that is $h = py^a$ for some y^a and a pedigree $n-1$ monomial p. We see that

$$\begin{aligned}
\left(\phi^*_{V_\alpha} \circ Y|_{V_\alpha}\right)(py^a) &= \phi^*_{V_\alpha}\left((Y|_{V_\alpha} p)y^a + (-1)^{|Y||p|} p\ Y|_{V_\alpha} y^a\right) && (3.17)\\
&= \left(\phi^*_{V_\alpha} Y|_{V_\alpha} p\right)\ \phi^*_{V_\alpha} y^a + (-1)^{|Y||p|}\ \left(\phi^*_{V_\alpha} p\right)\ \left(\phi^*_{V_\alpha} Y|_{V_\alpha} y^a\right) && (3.18)\\
&= \left(X|_{\underline{\phi}^{-1}(V_\alpha)}\, \phi^*_{V_\alpha} p\right) \phi^*_{V_\alpha} y^a + (-1)^{|X||p|}\ \left(\phi^*_{V_\alpha} p\right) \left(X|_{\underline{\phi}^{-1}(V_\alpha)}\, \phi^*_{V_\alpha} y^a\right) && (3.19)\\
&= X|_{\underline{\phi}^{-1}(V_\alpha)}\left((\phi^*_{V_\alpha} p)\,(\phi^*_{V_\alpha} y^a)\right) && (3.20)\\
&= \left(X|_{\underline{\phi}^{-1}(V_\alpha)} \circ \phi^*_{V_\alpha}\right)(py^a) && (3.21)
\end{aligned}$$

Where in the third equality we used the induction hypothesis, the statement assumption (3.10), the fact that $|X| = |Y|$ which is dictated by the statement assumption and also that $|y^a| = |\phi^*_{V_\alpha} y^a|$. We have just verified the validity of (3.16) for all monomials of pedigree n and all polynomials of pedigree $n-1$, hence due to linearity it holds for all polynomials of pedigree n, as was to be shown.

The goal now is to use the graded Hadamard's Lemma [10, Lemma 3.4.]. Recall that for any $m \in \underline{\phi}^{-1}(V_\alpha)$ we have the ideal J_m of all functions vanishing at m, that is

$$J_m := \{g \in C^\infty_{\mathcal{M}}(\underline{\phi}^{-1}(V_\alpha)) \,|\, \underline{g}(m) = 0\}. \tag{3.22}$$

Also recall that if a graded function $g \in C^\infty_{\mathcal{M}}(\underline{\phi}^{-1}(V_\alpha))$ is in $(J_m)^q$ for every $q \in \mathbb{N}$ and every $m \in \underline{\phi}^{-1}(V_\alpha)$, it is necessarily zero [10, Proposition 3.5.].

Let us therefore relabel the coordinates $\{y^a\}$ on V_α as $\{y^a\} =: \{y^j, Y^\mu\}$, where $\{y^j\}$ are the degree zero coordinates and $\{Y^\mu\}$ are the purely graded coordinates, and consider some $m \in \underline{\phi}^{-1}(V_\alpha)$. By the graded Hadamard's lemma we can write $f|_{V_\alpha}$ for every $q \in \mathbb{N}$ as

$$f|_{V_\alpha} = T + R, \tag{3.23}$$

where T is a polynomial of pedigree q in variables $\{y^j - y^j(\underline{\phi}(m)), Y^\mu\}$ and $R \in (J_{\underline{\phi}(m)})^{q+1}$. Here $J_{\underline{\phi}(m)}$ denotes the ideal of graded functions in $C^\infty_{\mathcal{N}}(V_\alpha)$ vanishing at $\underline{\phi}(m)$. Evidently, T can be regarded as a pedigree q polynomial in variables y^a and by the above discussion we have

$$\left(\phi^*_{V_\alpha} \circ Y|_{V_\alpha}\right) T = \left(X|_{\underline{\phi}^{-1}(V_\alpha)} \circ \phi^*_{V_\alpha}\right) T. \tag{3.24}$$

Since $\phi^*_{V_\alpha}$ is a graded algebra morphism such that $\underline{\phi^*_{V_\alpha} g} = \underline{g} \circ \underline{\phi}$, we have

$$\phi^*_{V_\alpha}\left((J_{\underline{\phi}(m)})^q\right) \subseteq (J_m)^q, \tag{3.25}$$

for any $m \in \underline{\phi}^{-1}(V_\alpha)$ and $q \in \mathbb{N}$. On the other hand, as X and Y are vector fields, we have

$$X|_{\underline{\phi}^{-1}(V_\alpha)}\left((J_m)^{q+1}\right) \subseteq (J_m)^q \qquad \text{and} \qquad Y|_{V_\alpha}\left((J_{\underline{\phi}(m)})^{q+1}\right) \subseteq (J_{\underline{\phi}(m)})^q, \tag{3.26}$$

for any m and q. All of this together means that

$$\left(\phi^*_{V_\alpha} \circ Y|_{V_\alpha} - X|_{\underline{\phi}^{-1}(V_\alpha)} \circ \phi^*_{V_\alpha}\right) f|_{V_\alpha} \in (J_m)^q \tag{3.27}$$

for any $m \in \underline{\phi}^{-1}(V_\alpha)$ and any $q \in \mathbb{N}$ and as such must it be zero. Therefore, due to (3.14) & (3.15) there is

$$((X \circ \phi_N^*) f)|_{\underline{\phi}^{-1}(V_\alpha)} = ((\phi_N^* \circ Y) f)|_{\underline{\phi}^{-1}(V_\alpha)}, \tag{3.28}$$

and since $\{\underline{\phi}^{-1}(V_\alpha)\}_{\alpha \in I}$ forms an open cover for M and $f \in C^\infty_{\mathcal{N}}(N)$ was arbitrary, X is ϕ-related to Y. ∎

Lemma 3.2. *Let X be a global vector field on a graded manifold $\mathcal{N}$ and let $\phi : \mathcal{M}' \to \mathcal{M}$ be a graded smooth function. Then*

$$1' \otimes X \sim_{(\phi \times 1)} 1 \otimes X, \tag{3.29}$$

where $1' \otimes X$ denotes the vector field $1 \otimes X$ on the graded manifold $\mathcal{M}' \times \mathcal{N}$.

Proof. The idea is to use Proposition 3.1, hence suppose we have a coordinate patch U on $\mathcal{M}$ with coordinates $\{x^i\}_{i=1}^{\mathrm{tdim}\,\mathcal{M}}$ and a coordinate patch V on $\mathcal{N}$ with coordinates $\{y^a\}_{a=1}^{\mathrm{tdim}\,\mathcal{N}}$. This gives us a coordinate patch $U \times V$ on $\mathcal{M} \times \mathcal{N}$ with coordinates $\{(p_{1,U}^* x^i)|_{U\times V}\} \cup \{(p_{2,V}^* y^a)|_{U \times V}\}$ where p_1 and p_2 are the canonical projections. Now observe that

$$\begin{aligned}\big(1' \otimes X\big)\big|_{\underline{\phi}^{-1}(U)\times V} \circ (\phi \times 1)^*_{U\times V} \circ \varrho^{U\times N}_{U\times V} \circ p^*_{1,U} &= \big(1' \otimes X\big)\big|_{\underline{\phi}^{-1}(U)\times V} \circ \varrho^{\underline{\phi}^{-1}(U)\times N}_{\underline{\phi}^{-1}(U)\times V} \circ (\phi \times 1)^*_{U\times N} \circ p^*_{1,U} \\ &= \varrho^{\underline{\phi}^{-1}(U)\times N}_{\underline{\phi}^{-1}(U)\times V} \circ \big(1' \otimes X\big)\big|_{\underline{\phi}^{-1}(U)\times N} \circ p^*_{1,\underline{\phi}^{-1}(U)} \circ \phi^*_U = 0. \end{aligned} \tag{3.30}$$

Indeed, let us show that for every $W \in \mathrm{Op}(M')$ there is $(1' \otimes X)|_{W\times N}\, p^*_{1,W} = 0$. Let $f \in C^\infty_{\mathcal{M}'}(W)$ and let $S \in \mathrm{Op}(W)$ be such that $\overline{S} \subseteq W$. Consider an extension $\bar{f}$ of f from S to M', that is, $\bar{f}|_S = f_S$. Then, due to naturality of p_1^*, we have

$$\varrho^{M'\times N}_{S\times N}\ p^*_{1,M'}\ \bar{f} = p^*_{1,S}\ \ \bar{f}\big|_S = p^*_{1,S}\ \ f|_S = \varrho^{W\times N}_{S\times N}\ p^*_{1,W} f. \tag{3.31}$$

Consequently, for any $R \in \mathrm{Op}(N)$ such that $\overline{R} \subseteq N$ we have

$$\big(p^*_{1,M'}\ \bar{f}\big)\big|_{S\times R} = \big(p^*_{1,W}\ f\big)\big|_{S\times R}, \tag{3.32}$$

i.e. $p^*_{1,M'}\ \bar{f}$ is an extension of $p^*_{1,W}\ f$ from $S \times R$ to $M' \times N$ and thus there is

$$\big(1' \otimes X\big)\big|_{S\times R}\ \big(p^*_{1,W} f\big)\big|_{S\times R} = \big(1' \otimes X\big)\big|_{S\times R}\ \big(p^*_{1,M'} \bar{f}\big)\big|_{S\times R} = \big(1' \otimes X \circ p^*_{1,M'}\ \bar{f}\big)\big|_{S\times R} = 0. \tag{3.33}$$

As sets of the type $S \times R$ cover $W \times N$, we have shown the claim $(1' \otimes X)|_{W\times N}\, p^*_{1,W} = 0$ and hence also the relation (3.30). In much the same way we would show that $(1 \otimes X)|_{U\times V}\ \ \varrho^{U\times N}_{U\times V}\ \ p^*_{1,U} = 0$. Therefore,

$$\big(1' \otimes X\big)\big|_{\underline{\phi}^{-1}(U)\times V}\ \ (\phi \times 1)^*_{U\times V}\ \big(p^*_{1,U} x^i\big)\big|_{U\times V} = 0 = (\phi \times 1)^*_{U\times V}\ (1 \otimes X)|_{U\times V}\ \big(p^*_{1,U} x^i\big)\big|_{U\times V}. \tag{3.34}$$

Similarly one would find that

$$\begin{aligned}\big(1' \otimes X\big)\big|_{\underline{\phi}^{-1}(U)\times V}\ \ (\phi \times 1)^*_{U\times V}\ \big(p^*_{2,V} y^a\big)\big|_{U\times V} &= \big(p^*_{2,V} X^i_V\big)\big|_{U\times V} \\ &= (\phi \times 1)^*_{U\times V}\ (1 \otimes X)|_{U\times V}\ \big(p^*_{2,V} y^a\big)\big|_{U\times V}, \end{aligned} \tag{3.35}$$

and so one may indeed use Proposition 3.1 to show that $1' \otimes X \sim_{(\phi\times 1)} 1 \otimes X$. ∎

Remark 3.3. In particular, if one chooses in the above proposition as the graded manifold $\mathcal{M}'$ the terminal object $*$, one obtains that for any graded manifold $\mathcal{M}$ there is $X \sim_{(x,1)} 1 \otimes X$ for any $x \in M$.

The next lemma states that if one deals with sheaves of $C^\infty_{\mathcal{M}}$-modules, the concepts of a frame for the sheaf and a frame for the "global" module are the same.

Lemma 3.4. *Consider some graded manifold $\mathcal{M}$, let $\mathscr{S}$ be a sheaf of $C^\infty_{\mathcal{M}}$-modules and consider $(f_i)_{i=1}^n$ an n-tuple of global sections $f_i \in \mathscr{S}(M)$. Then $(f_1, \dots, f_n)$ form a frame for $\mathscr{S}(M)$ if and only if they form a frame for $\mathscr{S}$.*

Proof. The only if direction is trivial. Let $(f_1, \dots, f_n)$ form a frame for $\mathscr{S}(M)$ and consider some $U \in \mathrm{Op}(M)$ and $s \in \mathscr{S}(U)$. We may take an open cover $\{W_\alpha\}_{\alpha \in I}$ of U composed of sets that satisfy $\overline{W}_\alpha \subseteq U$. We may then construct [10, Proposition 5.5], for every α, a global section $s_\alpha \in \mathscr{S}(M)$ such that $s_\alpha|_{W_\alpha} = s|_{W_\alpha}$. By assumption we have the unique decomposition $s_\alpha = \varphi^j_\alpha f_j$ for some $\varphi^j_\alpha \in C^\infty_{\mathcal{M}}(M)$ and hence

$$s|_{W_\alpha} = \varphi^j_\alpha\big|_{W_\alpha} f^j\big|_{W_\alpha} . \tag{3.36}$$

Now let $\{\lambda_\alpha\}_{\alpha \in I}$, $\lambda_\alpha \in C^\infty_{\mathcal{M}}(U)$ be a partition of unity [10, Proposition 3.40] subordinate to $\{W_\alpha\}_{\alpha \in I}$. For any j we may form the graded functions $\varphi^j \in C^\infty_{\mathcal{M}}(U)$ as

$$\varphi^j := \sum_{\alpha \in I} \lambda_\alpha \cdot \varphi^j_\alpha\big|_{W_\alpha} . \tag{3.37}$$

If we consider some $x \in U$ and its neighborhood $U_x \in \mathrm{Op}_x(U)$ that intersects with only finitely many supports of λ_α, say those with indices $\{\alpha_1, \dots, \alpha_\ell\}$, we find that

$$\begin{aligned}(\varphi^j\, f_j|_U)\big|_{U_x} &= \varphi^j\big|_{U_x} f_j|_{U_x} = \left(\sum_{i=1}^{\ell} \lambda_{\alpha_i} \cdot \varphi^j_{\alpha_i}\big|_{W_{\alpha_i}}\right)\Bigg|_{U_x} f_j|_{U_x} = \left(\sum_{i=1}^{\ell} \lambda_{\alpha_i} \cdot \left(\varphi^j_{\alpha_i}\big|_{W_{\alpha_i}} f_j|_{W_{\alpha_i}}\right)\right)\Bigg|_{U_x} \\ &= \left(\sum_{i=1}^{\ell} \lambda_{\alpha_i} \cdot s|_{W_\alpha}\right)\Bigg|_{U_x} = s|_{U_x} \end{aligned} \tag{3.38}$$

hence $\varphi^j f_j|_U = s$. To show uniqueness, let $s = \psi^j f_j|_U$ for some other graded functions ψ^j, consider some $V \in \mathrm{Op}(U)$ such that $\overline{V} \subseteq U$ and let $\lambda \in C^\infty_{\mathcal{M}}(M)$ be a smooth bump function on V supported in U, i.e. $\lambda|_V = 1$. We may extend φ^j and ψ^j from V to M by $\tilde{\psi}^j := \lambda \cdot \psi^j$ and $\tilde{\varphi}^j := \lambda \cdot \varphi^j$, where e.g. $\lambda \cdot \psi^j$ is defined by its restriction to V as $(\lambda\, \psi^j)|_V$ and its restriction to $M \setminus \overline{V}$ as zero. We see that

$$(\tilde{\varphi}^j\, f_j)\big|_V = s|_V = \left(\tilde{\psi}^j\, f_j\right)\Big|_V , \qquad \text{and} \qquad (\tilde{\varphi}^j\, f_j)\big|_{M\setminus\overline{V}} = s|_{M\setminus\overline{V}} = \left(\tilde{\psi}^j\, f_j\right)\Big|_{M\setminus\overline{V}} , \tag{3.39}$$

thus $\tilde{\varphi}^j\, f_j = \tilde{\psi}^j\, f_j$ and hence $\tilde{\varphi}^j = \tilde{\psi}^j$ for every j. But this implies that $\varphi^j|_V = \tilde{\varphi}^j|_V = \tilde{\psi}^j|_V = \psi^j|_V$ and as we may cover U with sets like V, the uniqueness is proven. ■

3.2 Left-Invariant Vector Fields

Let us begin by stating three equivalent definitions of left-invariant vector fields in the classical setting. For the rest of this section, we will study the relation of these notions in the graded setting, where at least some of them show to be non-equivalent. We base our definition of left-invariant vector fields on the one for supermanifolds in [2], though we try to provide a more thorough discussion.

Proposition 3.5 (Equivalent Classical Definitions)**.** *Let G be an ordinary Lie group with multiplication map $\mu : G \times G \to G$ and let $X \in \mathscr{X}(G)$ be a global vector field. Then the following are equivalent:*

1. $L_{g,*}X = X$ *for every* $g \in G$.
2. $1 \otimes X \sim_\mu X$.
3. $X = (1,e)^* \circ (1 \otimes X) \circ \mu^*$.

Proof. (*1.* $\Longrightarrow$ *2.*). We assume that for all $g, h \in G$ and any $f \in C^\infty(G)$,

$$[(L_{g,*}X)f](gh) = (Xf)(gh). \tag{3.40}$$

Observe that the right-hand side can be written for our purposes as

$$(Xf)(gh) = ((Xf) \circ \mu)(g,h) = [(\mu^* \circ X)f](g,h). \tag{3.41}$$

We would like the left hand side of (3.40) to equal $(1 \otimes X) \circ \mu^*$ acting on f at the point (g,h). For this, let us move to some coordinate neighborhoods $U \in \mathrm{Op}_g(G)$ with coordinates $\{x^i\}$ and $V \in \mathrm{Op}_h(G)$ with coordinates $\{y^i\}$. This yields

$$\begin{aligned}
[((1 \otimes X) \circ \mu^*)f](g,h) &= [(1 \otimes X)(f \circ \mu)](g,h) = X^i(h) \left.\frac{\partial}{\partial y^i}\right|_{(g,h)} f \circ \mu \\
&= X^i(h) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \mu)(g, h + te_i)
\end{aligned} \tag{3.42}$$

where e_i is the i-the standard basis vector on $\mathbb{R}^n$. On the other hand we find that

$$\begin{aligned}
[(L_{g,*}X)f](gh) &= ([(L_{g,*}X)f] \circ L_g)(h) = (X(f \circ L_g))(h) = X^i(h) \left.\frac{\partial}{\partial y^i}\right|_h f \circ L_g \\
&= X^i(h) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ L_g)(h + te_i) = X^i(h) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \mu)(g, h + te_i),
\end{aligned} \tag{3.43}$$

hence for every $f \in C^\infty(G)$ and every $g, h \in G$ there holds

$$[(L_{g,*}X)f](gh) = [((1 \otimes X) \circ \mu^*)f](g,h), \tag{3.44}$$

which together with (3.41) proves the implication.

(*2.* $\Longrightarrow$ *3.*). This implication stems from the identity $(1,e)^* \circ \mu^* = 1$ and so holds for every (graded) smooth manifold.

(*3.* $\Longrightarrow$ *1.*). For any $g \in G$ and $f \in C^\infty(G)$ we have

$$\begin{aligned}
X|_g f &= (Xf)(g) = \left(\left((1,e)^* \circ 1 \otimes X \circ \mu^*\right) f\right)(g) = \left((1 \otimes X \circ \mu^*) f\right)(g,e) \\
&= X^i(e) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \mu)(g, e + te_i) = X^i(e) \cdot \left(\frac{\partial}{\partial x^i}(f \circ L_g)\right)(e) = X|_e (f \circ L_g) \\
&= ((T_e L_g)\, X|_e) f,
\end{aligned} \tag{3.45}$$

hence $L_{g,*}X = X$. ■

Any of these equivalent definitions of left-invariant vector fields in the classical setting may in principle be used to define the generalization of left-invariant vector fields in the graded setting. It turns out that the second and the third notion coincide even in the graded setting, while the first is only implied by the latter two.

Definition 3.6 (Left-Invariant Vector Fields)**.** Let $(\mathcal{G}, \mu, e, \iota)$ be a graded Lie group and X a global vector field on $\mathcal{G}$. We say that X is **left-invariant**, if

$$1 \otimes X \sim_\mu X. \tag{3.46}$$

The space of all left-invariant vector fields on $\mathcal{G}$ will be denoted as $\mathscr{X}^L_{\mathcal{G}}$. On the other hand, if X satisfies

$$L_{g,*}X = X, \tag{3.47}$$

for all $g \in G$, we say that X is **left-translation invariant**.

Similarly, X is right-invariant if $X \otimes 1 \sim_\mu X$ and the space of all right-invariant vector fields on $\mathcal{G}$ is denoted as $\mathscr{X}^R_{\mathcal{G}}$. We will explicitly investigate only left-invariant vector fields.

Proposition 3.7. *The graded commutator of two left-invariant vector fields is a left-invariant vector field. Consequently, $\mathscr{X}^L_{\mathcal{G}}$ forms a Lie subalgebra of $\mathscr{X}_{\mathcal{G}}(G)$.*

Proof. Let X, Y be two left-invariant vector fields on $\mathcal{G}$. We assume that $1 \otimes X \sim_\mu X$ and $1 \otimes Y \sim_\mu Y$, hence also $[1 \otimes X, 1 \otimes Y] \sim_\mu [X, Y]$, this follows immediately by writing out the relevant definitions. We only need to show that

$$[1 \otimes X, 1 \otimes Y] = 1 \otimes [X, Y]. \tag{3.48}$$

This is best verified locally in coordinates. Let $U, V \in \mathrm{Op}(G)$ be two coordinate patches, and let $\{x^i\}$ and $\{y^i\}$ be the coordinates on U and V, respectively. Then clearly

$$[1 \otimes X, 1 \otimes Y]|_{U\times V}\, x^i = 0 = (1 \otimes [X, Y])|_{U\times V}\, x^i \tag{3.49}$$

and

$$\begin{aligned}[1 \otimes X, 1 \otimes Y]|_{U\times V}\, y^i &= (1 \otimes X)|_{U\times V}\, \left(p^*_{2,V} Y^i\right)\big|_{U\times V} - (-1)^{|X||Y|}\, (1 \otimes Y)|_{U\times V}\, \left(p^*_{2,V} X^i\right)\big|_{U\times V} \\ &= p^*_{2,V}\left(XY^i - (-1)^{|X||Y|} YX^i\right) = p^*_{2,V}[X, Y]^i = (1 \otimes [X, Y])|_{U\times V}\, y^i, \end{aligned} \tag{3.50}$$

as was to be shown. ∎

Proposition 3.8 (Left-Invariance Implies Left-Translation Invariance)**.** *Let $(\mathcal{G}, \mu, e, \iota)$ be a graded Lie group and let $X \in \mathscr{X}_{\mathcal{G}}(G)$ be a left-invariant vector field on $\mathcal{G}$. Then*

$$L_{g,*}X = X, \tag{3.51}$$

for every $g \in G$. The converse is in general not true.

Proof. Recall that $L_g = \mu \circ (g, 1)$ for any $g \in G$. Writing out the definition of the pushforward vector field $L_{g,*}X$, we obtain

$$L_{g,*}X = (L^*_g)^{-1} \circ X \circ L^*_g = L^*_{\iota g} \circ X \circ L^*_g = (\iota g, 1)^* \circ \mu^* \circ X \circ L^*_g. \tag{3.52}$$

Using left-invariance of X and Lemma 3.2 (see also Remark 3.3) we can continue:

$$(\iota g, 1)^* \circ \mu^* \circ X \circ L_g^* = (\iota g, 1)^* \circ (1 \otimes X) \circ \mu^* \circ L_g^* = X \circ (\iota g, 1)^* \circ \mu^* \circ L_g^* = X \circ L_{\iota g}^* \circ L_g^* = X, \quad (3.53)$$

as was to be proven.

Finally, that not all left-translation invariant vector fields are left-invariant can be seen in Example 3.9. ■

Example 3.9 (The Euler Vector Field on $\mathrm{GL}((n_j), \mathbb{R})$)**.** Consider the Euler vector field E on the Lie group $\mathrm{GL}((n_j), \mathbb{R})$ defined as $Ef = |f| f$ for any graded function f. In the usual coordinates it is expressed as

$$E = |x^i{}_j| x^i{}_j \frac{\partial}{\partial x^i{}_j} \equiv \sum_{i,j=1}^{n} (|j| - |i|)\, x^i{}_j \frac{\partial}{\partial x^i{}_j}. \quad (3.54)$$

We will show that E is left-translation invariant. The relation (3.51) is an equality of two vector fields and it is therefore enough to examine how they act on coordinate functions. Take some $M \in \times_k \mathrm{GL}(n_k, \mathbb{R})$ yielding the arrow L_M, see (2.92), and observe that

$$\begin{aligned}
(L_{M,*} E) x^i{}_j &= (L_{\iota M}^* \circ E \circ L_M^*) x^i{}_j = (L_{\iota M}^* \circ E) \left(\sum_{k:|k|=|i|} \mathrm{diag}(M)^i{}_k x^k{}_j \right) \\
&= L_{\iota M}^* \left(\sum_{k:|k|=|i|} (|j| - |k|)\, \mathrm{diag}(M)^i{}_k x^k{}_j \right) = \sum_{\ell:|\ell|=|k|} \sum_{k:|k|=|i|} (|j| - |k|)\, \mathrm{diag}(M)^i{}_k \mathrm{diag}(\iota M)^k{}_\ell x^\ell{}_j \\
&= \sum_{\ell:|\ell|=|i|} (|j| - |\ell|) \delta^i{}_\ell x^\ell{}_j = (|j| - |i|)\, x^i{}_j = E\, x^i{}_j, \quad (3.55)
\end{aligned}$$

where we used the fact that $\mathrm{diag}(\iota M) = (\mathrm{diag}(M))^{-1}$ which follows from the definition of the inversion arrow on $\mathrm{GL}((n_j), \mathbb{R})$.

Let us show that E is not left-invariant: on the one hand we have

$$(\mu^* \circ E)\, x^i{}_j = (|j| - |i|) b^k{}_j a^i{}_k, \quad (3.56)$$

while on the other,

$$((1 \otimes E) \circ \mu^*)\, x^i{}_j = |b^u{}_v| b^u{}_v \frac{\partial}{\partial b^u{}_v} \left(b^k{}_j a^i{}_k \right) = |b^u{}_v|\, b^u{}_v\, \delta^k{}_u \delta^v{}_j\, a^i{}_k = (|j| - |k|) b^k{}_j a^i{}_k. \quad (3.57)$$

The difference is subtle, but profound; in fact (3.56) equals (3.57) for every $i, j \in \{1, \ldots, n\}$ if and only if all basis vectors of $\mathbb{R}^{(n_j)}$ have the same degree. But we know this leads to $\mathrm{GL}((n_j), \mathbb{R})$ being trivially graded and hence $E = 0$.

We claimed that the graded versions of the properties *2.* and *3.* in Proposition 3.5 are equivalent. The next theorem in fact states something a little stronger. However, it requires in its proof the following lemma.

Lemma 3.10. *Let $X, Y \in \mathscr{X}_{\mathcal{G}}(G)$ be two global vector fields on a graded Lie group $(\mathcal{G}, \mu, e, \iota)$ such that $X = (1, e)^* \circ 1 \otimes Y \circ \mu^*$. Then*

$$1 \otimes X = (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ (1 \times \mu)^*. \quad (3.58)$$

Proof. We know that the vector field $1 \otimes X \in \mathscr{X}_{\mathcal{G}\times\mathcal{G}}(G \times G)$ is fully determined by the relations $1 \otimes X \sim_{p_1} 0$ and $1 \otimes X \sim_{p_2} X$. The right-hand side of (3.58) is certainly a linear map of degree $|X|$. Moreover, it is a global vector field on $\mathcal{G} \times \mathcal{G}$. Indeed, the Leibniz rule can be directly verified thanks to the fact that $(1 \times \mu)^*$ is an algebra morphism, $1 \otimes (1 \otimes Y)$ is a vector field of degree $|X|$ and $(1 \times (1, e))^* \circ (1 \times \mu)^* = (1 \times ((1, e)^* \circ \mu^*)) = (1 \times 1)^* = 1$. One may therefore denote

$$Z := (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ (1 \times \mu)^* \ \in \mathscr{X}_{\mathcal{G}\times\mathcal{G}}(G \times G) \tag{3.59}$$

and verify that $Z \sim_{p_1} 0$ and $Z \sim_{p_2} X$. We find

$$Z \circ p_1^* = (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ (1 \times \mu)^* \circ p_1^* = (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ p_1^* = 0, \tag{3.60}$$

where we used that $1 \otimes (1 \otimes Y) \sim_{p_1} 0$. Note that one needs to watch closely the meaning of p_1; while it always denotes the canonical projection on the first term, we abuse the notation in that we do not specify the product manifold from which it projects. Case in point, the commutative diagram implicitly used in (3.60) is

$$\begin{array}{ccc} \mathcal{G} \times (\mathcal{G} \times \mathcal{G}) & \xrightarrow{(1\times\mu)} & \mathcal{G} \times \mathcal{G} \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p_1} \\ \mathcal{G} & \xrightarrow{\quad 1 \quad} & \mathcal{G} \end{array} . \tag{3.61}$$

This notational abuse will be common. Similarly, we find that

$$\begin{aligned} Z \circ p_2^* &= (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ (1 \times \mu)^* \circ p_2^* = (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ p_2^* \circ \mu^* \\ &= (1 \times (1, e))^* \circ p_2^* \circ 1 \otimes Y \circ \mu^* = p_2^* \circ (1, e)^* \circ 1 \otimes Y \circ \mu^* = p_2^* \circ X, \end{aligned} \tag{3.62}$$

as was to be shown. ∎

Theorem 3.11. *Let $X \in \mathscr{X}_{\mathcal{G}}(G)$ be a global vector field on a graded Lie group $(\mathcal{G}, \mu, e, \iota)$. Then X is left-invariant if and only if*

$$X = (1, e)^* \circ 1 \otimes Y \circ \mu^*, \tag{3.63}$$

for some $Y \in \mathscr{X}_{\mathcal{G}}(G)$.

Proof. The direction " $\Longrightarrow$ " follows immediately from the identity $\mu \circ (1, e) = 1$ for $Y = X$. Let us therefore focus on the direction " $\Longleftarrow$ ". It turns out that it is a consequence of associativity of μ. From Lemma 3.10 we know that we can write

$$1 \otimes X = (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ (1 \times \mu)^*. \tag{3.64}$$

Composing this with μ^* from the right yields

$$1 \otimes X \circ \mu^* = (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ (1 \times \mu)^* \circ \mu^* \tag{3.65}$$

$$= (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ \alpha^* \circ (\mu \times 1)^* \circ \mu^*, \tag{3.66}$$

where we used the associativity diagram (1.39) with the explicit mention of the canonical "associator" isomorphism

$$\alpha : \mathcal{G} \times (\mathcal{G} \times \mathcal{G}) \to (\mathcal{G} \times \mathcal{G}) \times \mathcal{G}. \tag{3.67}$$

Now we claim that

$$1 \otimes (1 \otimes Y) \circ \alpha^* = \alpha^* \circ 1' \otimes Y, \tag{3.68}$$

where $1' \otimes Y \in C^{\infty}_{(\mathcal{G}\times\mathcal{G})\times\mathcal{G}}((G\times G)\times G)$. Indeed, let us show that $(\alpha^{-1})^* \circ 1 \otimes (1 \otimes Y) \circ \alpha^* \circ p_2^* = p_2^* \circ Y$. One uses the commutative diagram

$$\begin{array}{ccc} \mathcal{G}\times(\mathcal{G}\times\mathcal{G}) & \xrightarrow{p_2} & \mathcal{G}\times\mathcal{G} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle p_2} \\ (\mathcal{G}\times\mathcal{G})\times\mathcal{G} & \xrightarrow{p_2} & \mathcal{G} \end{array}, \tag{3.69}$$

to find that

$$(\alpha^{-1})^* \circ 1 \otimes (1 \otimes Y) \circ \alpha^* \circ {p_2}^* = (\alpha^{-1})^* \circ 1 \otimes (1 \otimes Y) \circ {p_2}^* \circ p_2^* = (\alpha^{-1})^* \circ {p_2}^* \circ (1 \otimes Y) \circ p_2^*$$
$$= (\alpha^{-1})^* \circ {p_2}^* \circ p_2^* \circ Y = {p_2}^* \circ Y. \tag{3.70}$$

Similarly one would show that $(\alpha^{-1})^* \circ 1 \otimes (1 \otimes Y) \circ \alpha^* \circ p_1^* = 0$ and hence our current claim (3.68) is justified. Also note that $\alpha \circ (1 \times (1, e)) = (1', e)$, where $1'$ is the identity morphism on $\mathcal{G} \times \mathcal{G}$. With these facts in mind, we can continue in (3.66):

$$1 \otimes X \circ \mu^* = (1 \times (1, e))^* \circ 1 \otimes (1 \otimes Y) \circ \alpha^* \circ (\mu \times 1)^* \circ \mu^* \tag{3.71}$$
$$= (1 \times (1, e))^* \circ \alpha^* \circ 1' \otimes Y \circ (\mu \times 1)^* \circ \mu^* \tag{3.72}$$
$$= (1', e)^* \circ 1' \otimes Y \circ (\mu \times 1)^* \circ \mu^* \tag{3.73}$$
$$= (1', e)^* \circ (\mu \times 1)^* \circ 1 \otimes Y \circ \mu^* \tag{3.74}$$
$$= \mu^* \circ (1, e)^* \circ 1 \otimes Y \circ \mu^* \tag{3.75}$$
$$= \mu^* \circ X, \tag{3.76}$$

where between (3.73) and (3.74) we used Lemma 3.2, between (3.74) and (3.75) we used the commutative diagram

$$\begin{array}{ccc} \mathcal{G}\times\mathcal{G} & \xrightarrow{(1',e)} & (\mathcal{G}\times\mathcal{G})\times\mathcal{G} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu\times 1} \\ \mathcal{G} & \xrightarrow{(1,e)} & \mathcal{G}\times\mathcal{G} \end{array}, \tag{3.77}$$

and in the final equality we used the assumption (3.63). This concludes the proof.

Note that in particular X is left-invariant if and only if $X = (1, e)^* \circ 1 \otimes X \circ \mu^*$.

■

A very important property of left-invariant vector fields in the classical setting is that their value at any one point determines their value at every other point. In the graded case, this statement does not generally hold for left-translation invariant vector fields. Indeed, we see that the Euler vector field on $\mathrm{GL}((n_j), \mathbb{R})$ is valued as $0 \in \mathrm{T}_g\mathcal{G}$ for any $g \in G$, but it is not the zero vector field. However, we can show that left-translation invariant vector fields are uniquely determined by restrictions to any open set and that left-invariant vector fields are fully determined by their value at the unit $e \in G$.

Proposition 3.12. *Let X, Y be two left-translation invariant vector fields on a graded Lie group $\mathcal{G}$ and consider any $U \in \mathrm{Op}(G)$. Then $X|_U = Y|_U \implies X = Y$.*

Proof. Consider some $f \in C^\infty_{\mathcal{M}}(M)$. Then for any $g \in G$ we have

$$\begin{aligned}(Xf)|_{\underline{L_g}(U)} &= \left(((L^*_{\iota g})_G \circ X \circ (L^*_g)_G) f\right)\big|_{\underline{L_g}(U)} = (L^*_{\iota g})_U \left((X \circ (L^*_g)_G) f\right)\big|_U \\ &= (L^*_{\iota g})_U \circ X|_U) \left((L^*_g)_G f\right)\big|_U = (L^*_{\iota g})_U \circ Y|_U) \left((L^*_g)_G f\right)\big|_U \\ &= (Yf)|_{\underline{L_g}(U)} \,. \end{aligned} \tag{3.78}$$

The second equality follows simply from L^*_g being a sheaf morphism, and thus commuting with restrictions. Notice that

$$(L^*_{\iota g})_U : C^\infty_{\mathcal{G}}(U) \to C^\infty_{\mathcal{G}}((\underline{L_{\iota g}})^{-1}(U)) \equiv C^\infty_{\mathcal{G}}(\underline{L_g}(U)) \tag{3.79}$$

Since $\{\underline{L_g}(U)\}_{g\in G}$ form an open cover of G, the statement follows. ∎

Proposition 3.13. *Let $(\mathcal{G}, \mu, e, \iota)$ be a graded Lie group and let $X, Y \in \mathscr{X}^L_{\mathcal{G}}$ be two left-invariant vector fields on $\mathcal{G}$. Then $X|_e = Y|_e \implies X = Y$.*

Proof. Consider some $f \in C^\infty_{\mathcal{G}}(G)$ and 2 coordinate patches $U, V \in \mathrm{Op}(G)$ on $\mathcal{G}$, such that $e \in V$. Denote the degree-zero and purely graded coordinates on U as $\{x^i\}$ and $\{\xi^\mu\}$and on V as $\{y^i\}$ and $\{\theta^\mu\}$. We know that on $U \times V$ we have the coordinates $\{x^i, y^j, \xi^\mu, \theta^\nu\}$, with the precise relation given in the beginning of Subchapter 3.1. Consequently,

$$(Xf)|_U = (((1,e)^* \circ (1 \otimes X) \circ \mu^*) f)|_U = (1,e)^*_{U\times V} \, (1 \otimes X)|_{U\times V} \, (\mu^* f)|_{U\times V} \,. \tag{3.80}$$

This may not seem like much, since we don't know what $\mu^* f$ looks like for a general multiplication map μ. As with any graded function in $C^\infty_{\mathcal{G}\times\mathcal{G}}(U \times V)$ we can write $(\mu^* f)|_{U\times V}$ as a formal infinite series. In particular, we can decompose it as

$$(\mu^* f)|_{U\times V} = h_0 + h_\mu \theta^\mu + R, \tag{3.81}$$

where h_0 and h_μ are graded functions on $C^\infty_{\mathcal{G}\times\mathcal{G}}(U\times V)$ that do not contain any of the graded coordinates $\{\theta^\mu\}$ and $R \in \langle\{\theta^\mu\}\rangle^2$, where $\langle\{\theta^\mu\}\rangle \subseteq C^\infty_{\mathcal{G}\times\mathcal{G}}(U \times V)$ is the ideal generated by the set $\{\theta^\mu\}$. Recall that

$$(1,e)^*_{U\times V}\, x^i = x^i, \qquad (1,e)^*_{U\times V}\, y^i = y^i(e), \tag{3.82}$$

$$(1,e)^*_{U\times V}\, \xi^\mu = \xi^\mu, \qquad (1,e)^*_{U\times V}\, \theta^\mu = 0, \tag{3.83}$$

and that

$$(1 \otimes X)|_{U\times V} = \left(p^*_{2,V}(X^i)\right)\big|_{U\times V} \frac{\partial}{\partial y^i} + \left(p^*_{2,V}(X^\mu)\right)\big|_{U\times V} \frac{\partial}{\partial \theta^\mu}. \tag{3.84}$$

It is apparent now that $(1 \times e)^*_{U\times V}\, (1 \otimes X)|_{U\times V}\, R = 0$, which was indeed the motivation for the decomposition (3.81). Let us now continue in (3.80):

$$(Xf)|_U = (1,e)^*_{U\times V}\, (1 \otimes X)|_{U\times V}\, (h_0 + h_\mu \theta^\mu) \tag{3.85}$$

$$= (1,e)^*_{U\times V} \left(\left(p^*_{2,V}(X^i)\right)\big|_{U\times V} \frac{\partial}{\partial y^i} h_0 + h_\mu \left(p^*_{2,V}(X^\mu)\right)\big|_{U\times V} \right) \tag{3.86}$$

$$= X^i(e)\, (1,e)^*_{U\times V} \frac{\partial}{\partial y^i} h_0 + X^\mu(e)\, (1,e)^*_{U\times V} h_\mu, \tag{3.87}$$

where we used that

$$(1,e)^*_{U\times V} \left(p^*_{2,V}\, h\right)\big|_{U\times V} = h(e), \tag{3.88}$$

for any $U, V \in \mathrm{Op}(G)$ such that $e \in V$and any $h \in C^\infty_{\mathcal{G}\times\mathcal{G}}(U \times V)$. As the same procedure can be followed for Y and f and U were arbitrary, the statement follows. ∎

Before we fully investigate the correspondence between left-invariant vector fields and the tangent space at the unit, let us give an example on our model graded Lie group.

Example 3.14 (LIVFs on $\mathrm{GL}((n_j),\mathbb{R})$)**.** Note that the tangent space of $\mathcal{G} := \mathrm{GL}((n_j),\mathbb{R})$ at any point in G is isomorphic to the graded linear space $\mathbb{R}^{(n_j)\times(n_j)}$, see Definition 2.9. For any tangent vector $v \in \mathrm{T}_e\mathcal{G}$ we shall construct a left-invariant vector field X such that $X_e = v$. Inspired by (3.87), we define

$$X := v^i{}_j\,(1,e)^* \circ \frac{\partial}{\partial b^i{}_j} \circ \mu^*, \tag{3.89}$$

where $v^i{}_j$ are real numbers such that $v = v^i{}_j \frac{\partial}{\partial x^i{}_j}|_e$. Note that $v^i{}_j = 0$ whenever $|j| - |i| + |v| \neq 0$. We see that X is a graded linear map of degree $|X| = |v|$. Let us verify the Leibniz rule: for any $f, g \in C^\infty_{\mathcal{G}}(G)$ we have

$$\begin{aligned} X(fg) &= v^i{}_j\,(1,e)^* \frac{\partial}{\partial b^i{}_j}\left((\mu^* f)(\mu^* g)\right) & (3.90)\\ &= v^i{}_j\,(1,e)^* \left(\frac{\partial \mu^* f}{\partial b^i{}_j} \mu^* g + (-1)^{(|i|-|j|)|f|}\,(\mu^* f) \frac{\partial \mu^* g}{\partial b^i{}_j} \right) & (3.91)\\ &= (Xf)g + (-1)^{|X||f|} f X g, & (3.92) \end{aligned}$$

where we used that $(1,e)^*$ is a graded algebra morphism, $(1,e)^*\mu^* = 1$ and that $|v^i{}_j|$ is (possibly) non-zero only for $|i| - |j| = |X|$. The graded linear map X satisfies the Leibniz rule, hence it is a vector field; let us see what it looks like when decomposed in a coordinate frame. As for any global vector field, we have $X = X^i{}_j \frac{\partial}{\partial x^i{}_j}$ where

$$X^i{}_j = X x^i{}_j = v^k{}_r (1,e)^* \frac{\partial}{\partial b^k{}_r}(b^\ell{}_j a^i{}_\ell) = v^k{}_j (1,e)^* a^i{}_k = x^i{}_k v^k{}_j. \tag{3.93}$$

Take note that $v^i{}_j \in \mathbb{R}$ for any i, j and so it commutes with every graded function. If we consult a classical textbook such as [3] we see that $X^i{}_j$ takes formally the exact same shape as in the classical setting, which we take as a good sign regarding the validity of our definitions. Since $x^i{}_j(e) = \delta^i{}_j$ we find that indeed $X_e = v$. It remains to be shown that X is in fact left-invariant: on the one hand

$$(\mu^* \circ X)\, x^i{}_j = \mu^* X^i{}_j = b^k{}_\ell\, a^i{}_k v^\ell{}_j, \tag{3.94}$$

while on the other

$$((1 \otimes X) \circ \mu^*)\, x^i{}_j = p_2^*(X^k{}_\ell) \frac{\partial}{\partial b^k{}_\ell}(b^s{}_j a^i{}_s) = v^r{}_\ell b^k{}_r\, \delta^s{}_k\, \delta^\ell{}_j\, a^i{}_s = v^r{}_j b^s{}_r a^i{}_s. \tag{3.95}$$

As (3.94) equals (3.95), we see that $1 \otimes X \sim_\mu X$, as desired.

It may be prudent, or at least illustrative, to perform this construction on one concrete example of $\mathrm{GL}((n_j),\mathbb{R})$.

Example 3.15. Here, let[1] $(n_j) = (\dots, 0, 0, 1, 1, 0, \dots)$ and let us denote as $\mathcal{G}$ the resultant graded Lie group $\mathrm{GL}((n_j),\mathbb{R})$. We know that its graded dimension is $\operatorname{gdim}\mathcal{G} = (\dots, 0, 1, 2, 1, 0, \dots)$, the underlying Lie group is

$$G = \{(x, y) \in \mathbb{R} \oplus \mathbb{R} \mid x \neq 0 \wedge y \neq 0\} \tag{3.96}$$

[1] For the sequence notation see Remark 2.10.

with pointwise multiplication and inversion, and with the unit $e = (1,1) \in G$. We have the global coordinates ${x^i}_j$ for $i, j \in \{0,1\}$ whom we shall relabel as

$$ {x^0}_0 =: x, \quad {x^1}_1 = y, \quad {x^1}_0 =: \xi, \quad \text{and} \quad {x^0}_1 =: \eta. \tag{3.97} $$

In other words, x and y are the standard coordinates on $\mathbb{R} \oplus \mathbb{R}$, ξ is the only graded coordinate of degree -1 and η is the only graded coordinate of degree $+1$. Let us choose a total basis of the tangent space $\mathrm{T}_e\mathcal{G}$:

$$ {f_0}^1 := \frac{\partial}{\partial \eta}\bigg|_e, \quad {f_0}^0 := \frac{\partial}{\partial x}\bigg|_e, \quad {f_1}^1 := \frac{\partial}{\partial y}\bigg|_e, \quad {f_1}^0 := \frac{\partial}{\partial \xi}\bigg|_e. \tag{3.98} $$

Clearly $({f_i}^j)^k{}_\ell = \delta^j{}_\ell \delta^k{}_i$ and we may use the general formula (3.93) to find the corresponding left-invariant vector fields:

$$ ({f_0}^1)^L = {x^i}_k \delta^1{}_j \delta^k{}_0 \frac{\partial}{\partial {x^i}_j} = {x^i}_0 \frac{\partial}{\partial {x^i}_1} = x\frac{\partial}{\partial \eta} + \xi \frac{\partial}{\partial y}, \tag{3.99} $$

and similarly

$$ ({f_0}^0)^L = {x^i}_0 \frac{\partial}{\partial {x^i}_0} = x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}, \tag{3.100} $$

$$ ({f_1}^1)^L = {x^i}_1 \frac{\partial}{\partial {x^i}_1} = \eta\frac{\partial}{\partial \eta} + y\frac{\partial}{\partial y}, \tag{3.101} $$

$$ ({f_1}^0)^L = {x^i}_1 \frac{\partial}{\partial {x^i}_0} = \eta\frac{\partial}{\partial x} + y\frac{\partial}{\partial \xi}. \tag{3.102} $$

From Proposition 3.13, or indeed from the next theorem, we know that there are no other linearly independent left-invariant vector fields on this particular graded Lie group.

The next theorem generalizes the construction of left-invariant vector fields from tangent vectors at the unit to any graded manifold. This together with Proposition 3.13 yields a linear isomorphism between the graded vector space of left-invariant vector fields on $\mathcal{G}$ and the tangent space at the unit $\mathrm{T}_e\mathcal{G}$.

Theorem 3.16 (Tangent Vectors at Unit & LIVFs)**.** *Let $(\mathcal{G}, \mu, e, \iota)$ be a graded Lie group. Then for every $v \in \mathrm{T}_e\mathcal{G}$ there exists a left-invariant vector field v^L such that $(v^L)_e = v$. In fact, this assignment defines a canonical graded linear isomorphism*

$$ \mathrm{T}_e\mathcal{G} \cong \mathscr{X}^L_{\mathcal{G}}. \tag{3.103} $$

Proof. Consider some open coordinate neighborhood U of the unit $e \in G$ with coordinates $\{x^i\}$ together with some subset $V \in \mathrm{Op}_e(U)$ such that $\overline{V} \subseteq U$ and let $v \in \mathrm{T}_e\mathcal{G}$ be a tangent vector at the unit. We can write v uniquely as $v = v^i \frac{\partial}{\partial x^i}|_e$ and we can consider the vector field $v^i \frac{\partial}{\partial x^i} \in \mathscr{X}_{\mathcal{G}}(U)$. Using partition of unity, we can extend this vector field from V to G and in so doing produce a global vector field $Y \in \mathscr{X}_{\mathcal{G}}(G)$ such that

$$ Y|_V = v^i \frac{\partial}{\partial x^i}\bigg|_V. \tag{3.104} $$

It is now natural to define the vector field $X \in \mathscr{X}_{\mathcal{G}}(G)$ akin to Example 3.14 as

$$ X := (1, e)^* \circ 1 \otimes Y \circ \mu^*. \tag{3.105} $$

We immediately see that X is a linear map of degree $|v|$ and the Leibniz identity can be verified directly using the fact that pullbacks are algebra morphisms, $1 \otimes Y$ is a vector field of degree $|v|$ and $(1,e)^* \circ \mu^* = 1$. Hence X is a vector field. That X is left-invariant follows from Theorem 3.11.

We still need to verify that $X_e = v$. Note that (e,e) is an element of $\underline{\mu}^{-1}(V)$ which is an open set in the product topology, hence there exist some $U_e, V_e \in \mathrm{Op}_e(G)$ such that $U_e \times V_e \in \underline{\mu}^{-1}(V)$. Without loss of generality we may assume that $U_e, V_e \subseteq V$. Now, we have

$$X^i\big|_{U_e} = (1,e)^*_{U_e\times V_e} \circ (1 \otimes Y)|_{U_e\times V_e} \circ \left(\mu_V^* x^i\right)\big|_{U_e\times V_e} \tag{3.106}$$

Here we are going to get our hands dirty a little and study the graded smooth functions

$$h^i := \left(\mu_V^* x^i\right)\big|_{U_e\times V_e} \in C^\infty_{\mathcal{G}\times\mathcal{G}}(U_e \times V_e). \tag{3.107}$$

We know that $U_e \times V_e$ is a coordinate patch in $\mathcal{G}\times\mathcal{G}$ with coordinates "inherited" from U_e and V_e, both of which share the coordinates x^i as they are subsets of U. We shall label these coordinates as

$$a^i := \left(p^*_{1,U_e} x^i\right)\big|_{U_e\times V_e}, \quad \text{and} \quad b^i := \left(p^*_{2,V_e} x^i\right)\big|_{U_e\times V_e}. \tag{3.108}$$

In this case we will need to distinguish between degree-zero and purely graded coordinates, so let us also denote $\{x^i\}_{i=1}^n =: \{x^K\}_{K=1}^{n_0} \cup \{\xi^\mu\}_{\mu=1}^{n_*}$, $\{a^i\} =: \{a^K\} \cup \{\eta^\mu\}$ and $\{b^i\} =: \{b^K\} \cup \{\theta^\mu\}$, where the coordinates denoted by Latin letters and indexed by capital letters have degree zero and the coordinates denoted by Greek letters have a non-zero degree. Under this notation, we may write h^i as a formal power series

$$h^i \equiv \left(\mu_V^* x^i\right)\big|_{U_e\times V_e} = \sum_{(\boldsymbol{p},\boldsymbol{g})\in\mathbb{N}^{2n_*}_{|h|}} h^i_{\boldsymbol{p},\boldsymbol{q}}\, \eta^{\boldsymbol{p}}\, \theta^{\boldsymbol{q}}, \tag{3.109}$$

where $h^i_{\boldsymbol{p},\boldsymbol{q}} \in C^\infty_{G\times G}(U_e \times V_e)$ are ordinary smooth functions in coordinates a^L and b^K. With the identity $\mu \circ (e,1) = 1$ at hand, we find that

$$(e,1)^*_{U_e\times V_e} \left(\mu_V^* x^i\right)\big|_{U_e\times V_e} = \left(\varrho^V_{V_e} \circ (e,1)^*_{\underline{\mu}^{-1}(V)} \circ \mu_V^*\right) x^i = x^i\big|_{V_e}, \tag{3.110}$$

hence

$$x^K = (e,1)^*_{U_e\times V_e} \sum_{(\boldsymbol{p},\boldsymbol{g})\in\mathbb{N}^{2n_*}_{|h|}} h^K_{\boldsymbol{p},\boldsymbol{q}}\, \eta^{\boldsymbol{p}}\, \theta^{\boldsymbol{q}} = \sum_{(\boldsymbol{0},\boldsymbol{g})\in\mathbb{N}^{2n_*}_{|h|}} h^K_{\boldsymbol{0},\boldsymbol{q}}(e,\cdot)\, \xi^{\boldsymbol{q}}, \tag{3.111}$$

which implies that $h^K_{\boldsymbol{0},\boldsymbol{0}}(e,\cdot) = x^K$ and $h^K_{\boldsymbol{0},\boldsymbol{q}}(e,\cdot) = 0$ for any $\boldsymbol{q} \neq \boldsymbol{0}$. Similarly, we have

$$\xi^\mu = (e,1)^*_{U_e\times V_e} \sum_{(\boldsymbol{p},\boldsymbol{g})\in\mathbb{N}^{2n_*}_{|h|}} h^\mu_{\boldsymbol{p},\boldsymbol{q}}\, \eta^{\boldsymbol{p}}\, \theta^{\boldsymbol{q}} = \sum_{(\boldsymbol{0},\boldsymbol{g})\in\mathbb{N}^{2n_*}_{|h|}} h^\mu_{\boldsymbol{0},\boldsymbol{q}}(e,\cdot)\, \xi^{\boldsymbol{q}}, \tag{3.112}$$

from which one gleans that $h^\mu_{\boldsymbol{0},\nu}(e,\cdot) = \delta^\mu{}_\nu$ and $h^\mu_{\boldsymbol{0},\boldsymbol{q}}(e,\cdot) = 0$ for any $\boldsymbol{q} \neq \nu$ where we use the shorthand multiindex $\nu \equiv (0,\ldots,0,1,0,\ldots,0)$ with the non-zero entry at position ν. Now, we write out (3.106) in our current notation:

$$X^i\big|_{U_e} = v^K (1,e)^*_{U_e\times V_e} \frac{\partial}{\partial b^K} \sum_{(\boldsymbol{p},\boldsymbol{g})\in\mathbb{N}^{2n_*}_{|h|}} h^i_{\boldsymbol{p},\boldsymbol{q}}\, \eta^{\boldsymbol{p}}\, \theta^{\boldsymbol{q}} + v^\mu (1,e)^*_{U_e\times V_e} \frac{\partial}{\partial \theta^\mu} \sum_{(\boldsymbol{p},\boldsymbol{g})\in\mathbb{N}^{2n_*}_{|h|}} h^i_{\boldsymbol{p},\boldsymbol{q}}\, \eta^{\boldsymbol{p}}\, \theta^{\boldsymbol{q}} \tag{3.113}$$

$$= v^K \sum_{(\boldsymbol{p},\boldsymbol{0})\in\mathbb{N}^{2n_*}_{|h|}} \frac{\partial h^i_{\boldsymbol{p},\boldsymbol{0}}}{\partial b^K}(\cdot,e)\, \xi^{\boldsymbol{p}} + v^\mu \sum_{(\boldsymbol{p},\mu)\in\mathbb{N}^{2n_*}_{|h|}} (-1)^{|\theta_\mu||\xi^{\boldsymbol{p}}|}\, h^i_{\boldsymbol{p},\mu}(\cdot,e)\, \xi^{\boldsymbol{p}}, \tag{3.114}$$

which gives

$$X^i(e) = v^K \frac{\partial h^i_{\mathbf{0},\mathbf{0}}}{\partial b^K}(e,e) + v^\mu h^i_{\mathbf{0},\mu}(e,e). \tag{3.115}$$

In particular,

$$X^L(e) = v^K \frac{\partial h^L_{\mathbf{0},\mathbf{0}}}{\partial b^K}(e,e) = v^K \frac{\partial h^L_{\mathbf{0},\mathbf{0}}(e,\cdot)}{\partial x^K}(e) = v^K \frac{\partial x^L}{\partial x^K}(e) = v^K \delta^L{}_K = v^L, \tag{3.116}$$

and similarly,

$$X^\mu(e) = v^\nu h^\mu_{\mathbf{0},\nu}(e,e) = v^\nu \delta^\mu{}_\nu = v^\mu. \tag{3.117}$$

This of course means nothing else than

$$X|_e = X^i(e) \left.\frac{\partial}{\partial x^i}\right|_e = v^i \left.\frac{\partial}{\partial x^i}\right|_e = v, \tag{3.118}$$

which is what we wanted to show. By Proposition 3.13 the above construction did not depend on any choices we made, only on the choice of v, so let us label $v^L := X$. We now have a map $\mathrm{T}_e\mathcal{G} \mapsto \mathscr{X}^L_\mathcal{G}$, $v \mapsto v^L$, whose two-sided inverse is the evaluation map at the unit $X \mapsto X_e$ and since the evaluation map is linear, so is the assignment $v \mapsto v^L$. ■

Based on the above theorem, for any graded Lie group $\mathcal{G}$ we may canonically induce on its tangent space at the unit $\mathrm{T}_e\mathcal{G}$ the structure of a graded **Lie algebra** of degree zero via

$$[v,w] := [v^L, w^L]_e, \tag{3.119}$$

whence the map $v \mapsto v^L$ becomes a graded Lie algebra morphism. The Lie algebra $T_e\mathcal{G} \cong \mathscr{X}^L_\mathcal{G}$ will also be denoted as $\mathrm{Lie}\,(\mathcal{G})$ or simply as $\mathfrak{g}$.

Example 3.17 ($\mathrm{gl}((n_j),\mathbb{R})$)**.** In Example 3.14 we have explicitly constructed the assignment $v \mapsto v^L$ for the graded general linear group, so let us have a look at the induced Lie algebra structure on its tangent space at the unit. We know that any tangent vector v at the unit has the form $v = v^i{}_j \frac{\partial}{\partial x^i{}_j}|_e$, where $v^i{}_j$ can be regarded as elements of a degree $|v|$ $(n_j)\times(n_j)$ matrix, see definition 2.9, which gives a graded linear isomorphism

$$\mathrm{gl}((n_j),\mathbb{R}) := \mathrm{Lie}\,(\mathrm{GL}((n_j),\mathbb{R})) \cong \mathbb{R}^{(n_j)\times(n_j)}. \tag{3.120}$$

By definition, for any $v,w \in \mathrm{gl}((n_j),\mathbb{R})$ we have $[v,w] = [v^L,w^L]_e$. Expressing the left-invariant vector fields with the use of (3.93) yields, after some forthright calculation,

$$[v,w]^i{}_j = \left([v^L,w^L]x^i{}_j\right)(e) = v^i{}_\ell w^\ell{}_j - (-1)^{|v|\cdot|w|} w^i{}_\ell v^\ell{}_j = (v\cdot w - (-1)^{|v|\cdot|w|} w \cdot v)^i{}_j, \tag{3.121}$$

where in the last expression we regard v and w as the elements of $\mathbb{R}^{(n_j)\times(n_j)}$. In conclusion, we see that the induced Lie bracket on $\mathrm{gl}((n_j),\mathbb{R})$ is none other than the **commutator of graded matrices**.

We would like to show that akin to the non-graded case, left-invariant vector fields on any graded Lie group $\mathcal{G}$ generate the Lie algebra of all vector fields on $\mathcal{G}$. For this we require the next proposition which is due to the thesis supervisor.

Proposition 3.18. *Consider a graded manifold $\mathcal{M}$, let $\mathscr{S}$, $\mathscr{R}$ be two locally freely and finitely generated sheaves of $C^\infty_\mathcal{M}$-modules, each of constant graded rank, and let $\Phi : \mathscr{S} \to \mathscr{R}$ be their morphism. Then Φ is an isomorphism if and only if it is fiberwise bijective.*

Proof. First note that without loss of generality we may assume both $\mathscr{S}$ and $\mathscr{R}$ to be freely and finitely generated. Indeed, as both sheaves are assumed to be locally freely and finitely generated, we may cover M by sets $U \in \mathrm{Op}(M)$ such that $(s_1, \ldots, s_\ell)$ is a frame for $\mathscr{S}|_U$ and $(r_1, \ldots, r_\ell)$ is a frame for $\mathscr{R}|_U$. Then Φ is bijective if and only if $\Phi|_U : \mathscr{S}|_U \to \mathscr{R}|_U$ is bijective for any U [10, Proposition 2.6] and similarly Φ is fiberwise bijective if and only if Φ_U is fiberwise bijective for every U. Hence assume $\mathscr{S}$ and $\mathscr{R}$ to be freely and finitely generated, and so without further loss of generality assume that

$$\mathscr{S} = C^\infty_{\mathcal{M}} \otimes S, \qquad \text{and} \qquad \mathscr{R} = C^\infty_{\mathcal{M}} \otimes R, \tag{3.122}$$

for some finite-dimensional real graded vector spaces $S, R \in \mathsf{gVec}$ of the same graded dimension $(m_k)_{k\in\mathbb{Z}}$ and that $(s_1, \ldots, s_\ell)$ and $(r_1, \ldots, r_\ell)$ are total bases for S and R, respectively. Recall that for any finite-dimensional $V \in \mathsf{gVec}$, $C^\infty_{\mathcal{M}} \otimes V$ is the sheaf of $C^\infty_{\mathcal{M}}$ modules defined by the assignment $U \mapsto C^\infty_{\mathcal{M}}(U) \otimes_{\mathbb{R}} V$. The sheaf morphism Φ is therefore fully and uniquely determined by the ℓ^2 graded functions ${\varphi^\nu}_\mu$ where

$$\Phi(s_\mu) = r_\nu \, {\varphi^\nu}_\mu. \tag{3.123}$$

Let the bases $s := (s_i)$ and $r := (r_i)$ be ordered increasingly by their degree, that is, ordered so that they can be partitioned (while preserving the order) into

$$(s_1, \ldots, s_\ell) = \sqcup_{k\in\mathbb{Z}} (s_1^{(k)}, \ldots, s_{m_k}^{(k)}), \tag{3.124}$$

where for every $k \in \mathbb{Z}$, $(s_1^{(k)}, \ldots, s_{m_k}^{(k)})$ is a basis for S_k and similarly for r. For every $x \in M$ the induced linear fiber map $\Phi_x : S \to K$ acts as

$$\Phi_x(s_\mu) = r_\nu \, {\varphi^\nu}_\mu(x). \tag{3.125}$$

Now, consider some $\Psi : C^\infty_{\mathcal{M}} \otimes R \to C^\infty_{\mathcal{M}} \otimes S$, which is fully determined by graded functions ${\psi^\nu}_\mu$ where $\Psi(r_\mu) = s_\nu \, {\psi^\nu}_\mu$. The fact that Ψ is the two-sided inverse of Φ is equivalent to the following two sets of equations:

$${\varphi^\nu}_\varkappa {\psi^\varkappa}_\mu = {\delta^\nu}_\mu, \qquad \text{and} \qquad {\psi^\nu}_\varkappa {\varphi^\varkappa}_\mu = {\delta^\nu}_\mu, \tag{3.126}$$

where ${\delta^\nu}_\mu$ is the graded Kronecker delta, i.e. if $\mu = \nu$ then ${\delta^\nu}_\mu = 1$ and else ${\delta^\nu}_\mu = 0$ of degree $|s_\mu| - |s_\nu|$. Recall that with our ordering, $|s_\mu| = |r_\mu|$ for every $\mu \in \{1, \ldots, \ell\}$. We immediately see that if Φ is bijective with Ψ as its two-sided inverse, then Ψ_x is the two-sided inverse of Φ_x for every $x \in M$ and therefore Φ is fiberwise bijective. On the other hand, assume that Φ_x is a linear isomorphism for every x. This is equivalent to saying that $\underline{\varphi}(x) \in \mathbb{R}^{m,m}$, where $m := \mathrm{tdim}\, S$ and ${\underline{\varphi}(x)^\mu}_\nu = {\varphi^\mu}_\nu(x)$, is an invertible (block-diagonal) matrix for every $x \in M$. As the assignment $x \mapsto \underline{\varphi}(x)$ is a smooth map $M \to \mathrm{GL}(m, \mathbb{R})$, its "inverse matrix map"

$$\underline{\psi} : x \mapsto \big(\underline{\varphi}(x)\big)^{-1} \tag{3.127}$$

is also smooth. We must show that there exist graded smooth functions ${\psi^\mu}_\nu \in C^\infty_{\mathcal{M}}(M)$ such that ${\psi^\mu}_\varkappa {\varphi^\varkappa}_\nu = {\delta^\mu}_\nu$ and ${\varphi^\mu}_\varkappa {\psi^\varkappa}_\nu = {\delta^\mu}_\nu$. The construction of ${\psi^\mu}_\nu$ is done somewhat similarly to the construction of the inversion arrow in the graded Lie group $\mathrm{GL}((n_j), \mathbb{R})$ in Section 2.4.1. Let us keep a similar notation as there and write

$${\Theta^\mu}_\nu := {\underline{\psi}^\mu}_\varkappa {(\varphi - \underline{\varphi})^\varkappa}_\nu, \qquad \text{and} \qquad {T^\mu}_\nu := {\delta^\mu}_\nu + \sum_{n=1}^\infty (-1)^n {\Theta^\mu}_{\lambda_1} {\Theta^{\lambda_1}}_{\lambda_2} \cdots {\Theta^{\lambda_{n-1}}}_\nu, \tag{3.128}$$

and then define

$${\psi^\mu}_\nu := {T^\mu}_\varkappa {\underline{\psi}^\varkappa}_\nu. \tag{3.129}$$

Since ${\Theta^\mu}_\nu$ are by construction purely graded functions, one may use similar arguments as in Section 2.4.1 to conclude that ${T^\mu}_\nu$ is a well-defined graded function for every μ, ν. We also immediately see that the body of the graded function on the right-hand side of (3.129) equals ${\underline{\psi}^\mu}_\nu$, which justifies the notation in (3.127). Finally, we see that

$$ {\psi^\mu}_\rho {\varphi^\rho}_\nu = {T^\mu}_\varkappa {\underline{\psi}^\varkappa}_\rho \left({\underline{\varphi}^\rho}_\nu + {\varphi^\rho}_\nu - {\underline{\varphi}^\rho}_\nu \right) = {T^\mu}_\nu + {T^\mu}_\varkappa {\Theta^\varkappa}_\nu = {\delta^\mu}_\nu, \tag{3.130} $$

and also

$$ \begin{aligned} {\varphi^\mu}_\rho {\psi^\rho}_\nu &= {\delta^\mu}_\nu + {(\varphi - \underline{\varphi})^\mu}_\rho {\underline{\psi}^\rho}_\nu + {\varphi^\mu}_\rho \sum_{n=1}^\infty (-1)^n {\underline{\psi}^\rho}_{\varkappa_1} {(\varphi - \underline{\varphi})^{\varkappa_1}}_{\lambda_1} \cdots {\underline{\psi}^{\lambda_{n-1}}}_{\varkappa_n} {(\varphi - \underline{\varphi})^{\varkappa_n}}_\alpha {\underline{\psi}^\alpha}_\nu \\ &= {\delta^\mu}_\nu + {(\varphi - \underline{\varphi})^\mu}_\rho {\underline{\psi}^\rho}_\nu + {(\varphi - \underline{\varphi})^\mu}_\rho \sum_{n=1}^\infty (-1)^n {\underline{\psi}^\rho}_{\varkappa_1} {(\varphi - \underline{\varphi})^{\varkappa_1}}_{\lambda_1} \cdots {\underline{\psi}^{\lambda_{n-1}}}_{\varkappa_n} {(\varphi - \underline{\varphi})^{\varkappa_n}}_\alpha {\underline{\psi}^\alpha}_\nu \\ &\quad + \sum_{n=1}^\infty (-1)^n {\delta^\rho}_{\varkappa_1} {(\varphi - \underline{\varphi})^{\varkappa_1}}_{\lambda_1} \cdots {\underline{\psi}^{\lambda_{n-1}}}_{\varkappa_n} {(\varphi - \underline{\varphi})^{\varkappa_n}}_\alpha {\underline{\psi}^\alpha}_\nu \\ &= {\delta^\mu}_\nu + \sum_{n=1}^\infty (-1)^{n+1} {(\varphi - \underline{\varphi})^\mu}_{\varkappa_1} {\underline{\psi}^{\varkappa_1}}_{\lambda_1} \cdots {(\varphi - \underline{\varphi})^{\lambda_{n-1}}}_{\varkappa_n} {\underline{\psi}^{\varkappa_n}}_\nu \\ &\quad + \sum_{n=1}^\infty (-1)^n {(\varphi - \underline{\varphi})^\mu}_{\varkappa_1} {\underline{\psi}^{\varkappa_1}}_{\lambda_1} \cdots {(\varphi - \underline{\varphi})^{\lambda_{n-1}}}_{\varkappa_n} {\underline{\psi}^{\varkappa_n}}_\nu \\ &= {\delta^\mu}_\nu, \end{aligned} \tag{3.131} $$

as desired. ■

Corollary 3.19 (LIVFs as a Frame)**.** *Let $\mathcal{G}$ be a graded Lie group, and let $(v_1, \ldots, v_n)$ be a basis for the tangent space at the unit $\mathrm{T}_e\mathcal{G}$. Then $(v_1^L, \ldots, v_n^L)$ forms a frame for $\mathscr{X}_\mathcal{G}$.*

Proof. First, note that $(v_1^L|_g, \ldots, v_n^L|_g)$ is a basis for the tangent space $T_g\mathcal{G}$ for any $g \in G$. Indeed, any left-invariant vector field is in particular left-translation invariant, hence $v^L|_g = (\mathrm{T}_e\mathrm{L}_g)v^L|_e = (\mathrm{T}_e\mathrm{L}_g)v$ for any $v \in \mathrm{T}_e\mathcal{G}$. Since L_g is a graded diffeomorphism per Corollary 1.28, the tangent map is a linear isomorphism and $(v_1, \ldots, v_n)$ is a basis for $\mathrm{T}_e\mathcal{G}$, the conclusion follows.

Next, notice that the graded linear map $v \mapsto v^L$ can be extended to a morphism of sheaves of $C^\infty_\mathcal{G}$-modules $\Phi : C^\infty_\mathcal{G} \otimes \mathrm{T}_e\mathcal{G} \to \mathscr{X}_\mathcal{G}$ sending any element $f^\mu v_\mu$ of $C^\infty_\mathcal{G}(U) \otimes T_e\mathcal{G}$ to $f^\mu(v_\mu^L)|_U$ in $\mathscr{X}_\mathcal{G}(U)$. With the above paragraph in mind we conclude that Φ is fiberwise bijective and so an isomorphism by Proposition 3.18. Hence $(v_1^L, \ldots, v_n^L) \equiv (\Phi(v_1), \ldots, \Phi(v_n))$ is a frame for $\mathscr{X}_\mathcal{G}$. ■

3.3 Fundamental Vector Fields

As with left-invariant vector fields, let us first give an equivalent definition of fundamental vector fields on ordinary smooth manifolds which will be suitable for generalization to graded manifolds.

Proposition 3.20. *Let $\theta : M \times G \to M$ be a right action of a Lie group G on a smooth manifold M and let Y be a global vector field on M. Then Y is a fundamental vector field if and only if $1 \otimes X \sim_\theta Y$ for some left-invariant vector field X on G.*

Proof. What we suspect of course is that $1 \otimes X \sim_\theta Y$ not for some random X, but for the X such that $\#X = Y$, where $\# : \mathscr{X}_G^L \to \mathscr{X}(M)$ is the infinitesimal generator map. Let us write $\theta(p, g) := p \cdot g$ for any $p \in M$ and $g \in G$.

($\Longrightarrow$). Assume that $\#X = Y$ for some $X \in \mathscr{X}_G^L$. This is the usual definition of a fundamental vector field. In other words, for every $p \in M$ we have $Y_p = (T_e\theta^p)X_e$ where $\theta^p \equiv \theta(p, \cdot)$ is the orbit map. Observe, that for any $p \in M$ and $g \in G$ we have

$$Y_{p\cdot g} = (T_e\theta^{p\cdot g})\, X_e = (T_e(\theta^p \circ L_g))\, X_e = (T_g\theta^p \circ T_eL_g)\, X_e = (T_g\theta^p)\, X_g. \tag{3.132}$$

Hence for any $f \in C^\infty(M)$,

$$((\theta^* \circ Y)\, f)\, (p, g) = (Y f)\, (p \cdot g) = Y_{p\cdot g} f = ((T_g\theta^p)\, X_g)\, f = X_g(f \circ \theta^p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 f(p \cdot g e^{tX}) \tag{3.133}$$

equals

$$((1 \otimes X \circ \theta^*)\, f)\, (p, g) = ((1 \otimes X)(f \circ \theta))\, (p, g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (f \circ \theta)(p, g e^{tX}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 f(p \cdot g e^{tX}), \tag{3.134}$$

where we use that he^{tX} is the integral curve of X from h for any $h \in G$. The first implication is thereby shown.

($\Longleftarrow$). Conversely, assuming $1 \otimes X \sim_\theta Y$ for some $X \in \mathscr{X}_G^L$ gives the equality

$$Y_{p\cdot g} f = ((\theta^* \circ Y)\, f)\, (p, g) = ((1 \otimes X \circ \theta^*)\, f)\, (p, g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 f(p \cdot g e^{tX}), \tag{3.135}$$

for any $f \in C^\infty(M)$ and all $p \in M$, $g \in G$. For the special choice $g = e$ we thus obtain

$$Y_p f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 f(p \cdot e^{tX}) = ((T_e\theta^p)\, X_e)\, f, \tag{3.136}$$

as desired. ■

Definition 3.21 (Fundamental Vector Fields)**.** Let $\theta : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ be an action of a graded Lie group $\mathcal{G}$ on a graded manifold $\mathcal{M}$ and let $Y \in \mathscr{X}_\mathcal{M}(M)$ be a global vector field on $\mathcal{M}$. We say that Y is **fundamental** if there exists a left-invariant vector field $X \in \mathscr{X}_\mathcal{G}^L$ on $\mathcal{G}$ such that

$$1 \otimes X \sim_\theta Y. \tag{3.137}$$

We denote the graded vector space of all fundamental vector fields on $\mathcal{M}$ as $\mathscr{X}_\mathcal{M}^F$. Using the same argument as in the proof of Proposition 3.7 and the proposition itself, one finds that $\mathscr{X}_\mathcal{M}^F$ forms a **Lie subalgebra** of $\mathscr{X}_\mathcal{M}(M)$.

Proposition 3.22 (Infinitesimal Generator)**.** *Let $\theta : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ be an action of a graded Lie group $\mathcal{G}$ on a graded manifold $\mathcal{M}$. Then, for every left-invariant vector field $X \in \mathscr{X}_\mathcal{G}^L$ there exists exactly one fundamental vector field $\#X \in \mathscr{X}_\mathcal{M}^F$ such that $1 \otimes X \sim_\theta \#X$. Moreover, the assignment $\# : \mathscr{X}_\mathcal{G}^L \to \mathscr{X}_\mathcal{M}^F$ is a graded Lie algebra morphism.*

Proof. The proof shares similarities with the proof of Theorem 3.16 and thus we beg to be more succinct here. First, let us show uniqueness, that is, consider Y and $Y' \in \mathscr{X}_\mathcal{M}(M)$ such that $1 \otimes X \sim_\theta Y$ and

$1 \otimes X \sim_\theta Y'$ for some $X \in \mathscr{X}_\mathcal{G}^L$. By use of the identity $(1, e)^*\theta^*$ which comes from one of the defining diagrams for a right action[2] we obtain

$$Y = (1, e)^* \circ \theta^* \circ Y = (1, e)^* \circ (1 \otimes X) \circ \theta^* = (1, e)^* \circ \theta^* \circ Y' = Y', \tag{3.138}$$

which validates the notation $\#X$. Now we need to find $\#X$ for arbitrary $X \in \mathscr{X}_\mathcal{G}^L$ and show that this assignment is a graded Lie algebra morphism. Having a look at (3.138), the intuitive definition is

$$\#X := (1, e)^* \circ (1 \otimes X) \circ \theta^*, \tag{3.139}$$

for any $X \in \mathscr{X}_\mathcal{G}^L$. From this we immediately see that $\#X$ is a graded linear map of degree $|X|$ and that the assignment $\#$ itself is a graded linear map of degree 0. One may directly verify the Leibniz rule, by virtue of which $\#X \in \mathscr{X}_\mathcal{M}(M)$. In much the same way[3] as in the proof of Theorem 3.16 one finds that

$$\begin{aligned}
\theta^* \circ (1, e)^* \circ 1 \otimes X \circ \theta^* &= (1', e)^* \circ (\theta \times 1)^* \circ 1 \otimes X \circ \theta^* && (3.140)\\
&= (1', e)^* \circ 1' \otimes X \circ (\theta \times 1)^* \circ \theta^* && (3.141)\\
&= (1', e)^* \circ 1' \otimes X \circ \alpha^* \circ (1 \times \mu)^* \circ \theta^* && (3.142)\\
&= (1 \times (1, e))^* \circ 1 \otimes (1 \otimes X) \circ (1 \times \mu^*) \circ \theta^* && (3.143)\\
&= 1 \otimes X \circ \theta^*, && (3.144)
\end{aligned}$$

where $\alpha : (\mathcal{M} \times \mathcal{G}) \times \mathcal{G} \to \mathcal{M} \times (\mathcal{G} \times \mathcal{G})$ is the canonical associator, between (3.143) and (3.144) we used Lemma 3.10 and between (3.141) and (3.142) we used the defining "associativity diagram" for a right action, which reads

$$\begin{array}{ccccc}
(\mathcal{M} \times \mathcal{G}) \times \mathcal{G} & \xrightarrow{\theta \times 1} & \mathcal{M} \times \mathcal{G} & & \\
 & & & \searrow^{\theta} & \\
\downarrow{\scriptstyle \alpha} & & & & \mathcal{G}. \\
 & & & \nearrow_{\theta} & \\
\mathcal{M} \times (\mathcal{G} \times \mathcal{G}) & \xrightarrow{1 \times \mu} & \mathcal{M} \times \mathcal{G} & &
\end{array} \tag{3.145}$$

We see that $\#X$ is indeed fundamental; all that is left is to see whether $\#$ preserves the commutator and so is a Lie algebra morphism. For this we can take advantage of the already proven relation (3.48) whence for any two $X, Y \in \mathscr{X}_\mathcal{G}^L$ we find that

$$1 \otimes [X, Y] = [1 \otimes X, 1 \otimes Y] \sim_\theta [\#X, \#Y], \tag{3.146}$$

hence $[\#X, \#Y] = \#[X, Y]$ as was to be shown. ■

Given the canonical isomorphism $\mathrm{T}_e\mathcal{G} \cong \mathscr{X}_\mathcal{G}^L$ from Theorem 3.16, we will often consider the infinitesimal generator as a map $\# : \mathrm{T}_e\mathcal{G} \to \mathscr{X}_\mathcal{M}^F$.

[2] See (1.43) for the relevant diagram for a left action.
[3] Read (3.71) through (3.75) in the opposite direction.

Chapter 4

Graded Principal Bundles

4.1 Definitions & Properties

We begin by a straightforward graded generalization of a fiber bundle. Then, similarly as in the classical case, we will define principal bundles as a rather special case of a fiber bundle.

Definition 4.1 (Graded Fiber Bundle)**.** Let $\mathcal{B}, \mathcal{F}, \mathcal{M}$ be graded manifolds and let $\pi : \mathcal{B} \to \mathcal{M}$ be a surjective submersion. We say that $\mathcal{B}$ is a graded fiber bundle over $\mathcal{M}$ with the typical fiber $\mathcal{F}$ if there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of M such that for every $\alpha \in I$ there exists a graded diffeomorphism

$$\phi_\alpha : \mathcal{M}|_{U_\alpha} \times \mathcal{F} \to \mathcal{B}|_{\underline{\pi}^{-1}(U_\alpha)} \tag{4.1}$$

such that $\pi|_{\underline{\pi}^{-1}(U_\alpha)} \circ \phi_\alpha = p_1$ for every α. The diffeomorphisms ϕ_α are called local trivializations of $\mathcal{F}$.

Next, we need a notion of what it means that a graded Lie group acts along fibers. Consider a right action θ of a graded Lie group $\mathcal{G}$ on a graded manifold $\mathcal{P}$, together with a surjective submersion $\pi : \mathcal{P} \to \mathcal{M}$ for some graded manifold $\mathcal{M}$. We say that θ **acts along fibers** of π if

$$\begin{array}{ccc} \mathcal{P} \times \mathcal{G} & \xrightarrow{\theta} & \mathcal{P} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ \mathcal{P} & \xrightarrow{\pi} & \mathcal{M} \end{array} \tag{4.2}$$

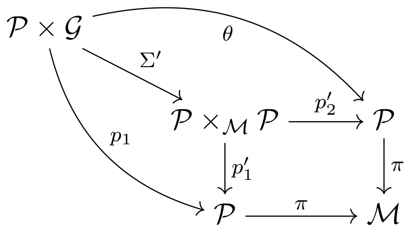
commutes. This clearly reduces to the familiar notion when all manifolds are trivially graded. Recall now [10] that since π is a submersion, there exists a **fiber product**[1] $\mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ together with arrows p'_1 and p'_2 fitting into the commutative diagram

$$\begin{array}{ccc} \mathcal{P} \times_{\mathcal{M}} \mathcal{P} & \xrightarrow{p'_2} & \mathcal{P} \\ \downarrow{\scriptstyle p'_1} & & \downarrow{\scriptstyle \pi} \\ \mathcal{P} & \xrightarrow{\pi} & \mathcal{M} \end{array} . \tag{4.3}$$

This is a fiber product in the category gMan^∞ and so is characterized by the familiar universal property: for any graded manifold $\mathcal{N}$ and any two graded smooth maps $f_1, f_2 : \mathcal{N} \to \mathcal{P}$ such that $\pi \circ f_1 = \pi \circ f_2$

[1] See e.g. [8] for the general category-theoretic definition of a fiber product.

there exists a unique smooth map $f : \mathcal{N} \to \mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ satisfying $f_1 = p'_1 \circ f$ and $f_2 = p'_2 \circ f$. Therefore, if θ acts along fibers of π then there is a unique graded smooth map $\Sigma' : \mathcal{P} \times \mathcal{G} \to \mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ such that



(4.4)

commutes. One also has the canonical arrow $(p'_1, p'_2) : \mathcal{P} \times_{\mathcal{M}} \mathcal{P} \to \mathcal{P} \times \mathcal{P}$. It has been shown [10] that this arrow makes $\mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ into a closed embedded submanifold of $\mathcal{P} \times \mathcal{P}$. Let us show that $(p'_1, p'_2) \circ \Sigma' = \Sigma$, the shear morphism (1.45) associated to θ, which is given as

$$\Sigma = (p_1, \theta). \tag{4.5}$$

Indeed, composed with the canonical product projections we find that

$$p_1 \circ (p'_1, p'_2) \circ \Sigma' = p_1 = p_1 \circ \Sigma, \tag{4.6}$$

and

$$p_2 \circ (p'_1, p'_2) \circ \Sigma' = \theta = p_2 \circ \Sigma, \tag{4.7}$$

which together means that

$$(p'_1, p'_2) \circ \Sigma' = \Sigma. \tag{4.8}$$

To summarize, whenever θ acts along the fibers of π, we may uniquely factor the shear morphism Σ through the fiber product $\mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ via Σ'. We are now ready to make the definition of a graded principal bundle.

Definition 4.2 (Graded Principal Bundle)**.** Let $\pi : \mathcal{P} \to \mathcal{M}$ be a surjective submersion of graded manifolds and $\theta : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ a right action of a graded Lie group $\mathcal{G}$. We say that $\mathcal{P}$ is a **principal $\mathcal{G}$-bundle** over $\mathcal{M}$ if

1. θ acts along fibers of π.
2. The morphism $\Sigma' : \mathcal{P} \times \mathcal{G} \to \mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ induced by the shear map Σ is a graded diffeomorphism.
3. There exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of M such that for every $\alpha \in I$ there exists a graded diffeomorphism
$$\phi_\alpha : \mathcal{M}|_{U_\alpha} \times \mathcal{G} \to \mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)}, \tag{4.9}$$
called a local trivialization of $\mathcal{P}$ over U_α, such that $\pi|_{\underline{\pi}^{-1}(U_\alpha)} \circ \phi_\alpha = p_1$ and which is equivariant with respect to the right $\mathcal{G}$-actions $1 \times \mu$ and $\theta|_{\underline{\pi}^{-1}(U_\alpha)}$ on $\mathcal{M}|_{U_\alpha} \times \mathcal{G}$ and $\mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)}$, respectively.

Let us elaborate on the last point of this definition. Equivariant arrows are introduced in Definition 1.16 and the meaning of the action $1 \times \mu$ of $\mathcal{G}$ on the graded manifold $\mathcal{M}|_{U_\alpha} \times \mathcal{G}$ is self-explanatory. Now, consider some point $(p, g) \in \underline{\theta}^{-1}(\underline{\pi}^{-1}(U_\alpha))$. We see that $p \cdot g \in \underline{\pi}^{-1}(U_\alpha)$ and as $\underline{\theta}$ acts along fibers of $\underline{\pi}$, thanks to point 1. of the definition, we know that $p \in \underline{\pi}^{-1}(U_\alpha)$. Hence $\underline{\theta}^{-1}(\underline{\pi}^{-1}(U_\alpha)) \subseteq \underline{\pi}^{-1}(U_\alpha)$ and we may restrict θ to the right action

$$\theta|_{\underline{\pi}^{-1}(U_\alpha)} : \mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)} \times \mathcal{G} \to \mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)} . \tag{4.10}$$

We of course want our definition to reduce to the usual one in the trivially graded case, but this is apparently so: the first and second point amount to the action being free and transitive along fibers and the third point is the familiar local triviality condition. Furthermore, any graded principal $\mathcal{G}$-bundle contains as its underlying smooth manifold an ordinary principal G-bundle:

Proposition 4.3. *Let $\pi : \mathcal{P} \to \mathcal{M}$ be a graded principal $\mathcal{G}$-bundle. Then $\underline{\pi} : P \to M$ is an ordinary principal G-bundle.*

Proof. If θ acts along fibers, then applying the body functor (2.13) to the defining diagram (4.2) tells us that $\underline{\theta} : P \times G \to P$ also acts along fibers. Furthermore, from Corollary 2.3 we immediately see that the underlying smooth manifold of $\mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ is $P \times_M P$, that the shear morphism of $\underline{\theta}$ is $\underline{\Sigma}$ and the induced map from $P \times G$ to $P \times_M P$ is $\underline{\Sigma'}$ and as Σ' is an isomorphism, $\underline{\Sigma'}$ is a diffeomorphism. What is more, $\underline{\phi}_\alpha$ are the relevant equivariant local trivializations.

■

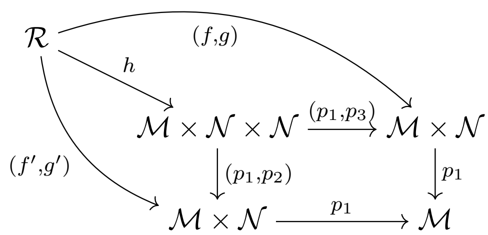
A logical first example when illustrating the definition of a principal bundle is the trivial one. Before we give it, however, we need a lemma.

Lemma 4.4. *Let $\mathcal{M}$, $\mathcal{N}$ be two graded manifolds. Then $\mathcal{M} \times \mathcal{N} \times \mathcal{N}$ together with (p_1, p_2) and (p_1, p_3) is the fiber product of $p_1 : \mathcal{M} \times \mathcal{N} \to \mathcal{M}$ and $p_1 : \mathcal{M} \times \mathcal{N} \to \mathcal{M}$.*

Proof. First we are asked to show that

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{N} \times \mathcal{N} & \xrightarrow{(p_1,p_3)} & \mathcal{M} \times \mathcal{N} \\ \big\downarrow{\scriptstyle (p_1,p_2)} & & \big\downarrow{\scriptstyle p_1} \\ \mathcal{M} \times \mathcal{N} & \xrightarrow{\quad p_1 \quad} & \mathcal{M} \end{array} \tag{4.11}$$

commutes, but this is plain to see. Next we need to prove the universality property, i.e. that for any graded manifold $\mathcal{R}$ and any two graded smooth maps $(f, g) : \mathcal{R} \to \mathcal{M} \times \mathcal{N}$ and $(f', g') : \mathcal{R} \to \mathcal{M} \times \mathcal{N}$ such that $p_1 \circ (f, g) = p_1 \circ (f', g')$ there exists a unique arrow $h : \mathcal{R} \to \mathcal{M} \times \mathcal{N} \times \mathcal{N}$ that makes the diagram

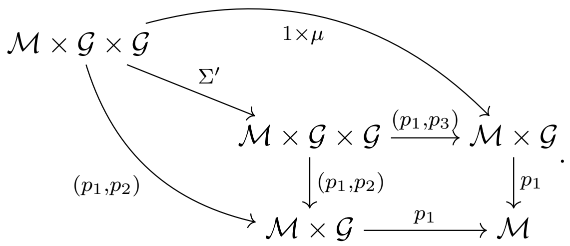


(4.12)

commute. But since $f = p_1 \circ (f, g) = p_1 \circ (f', g') = f'$, this unique arrow is obviously $h = (f, g', g)$. Hence the universality is proven and we see that indeed $\mathcal{M} \times \mathcal{N} \times \mathcal{N} = (\mathcal{M} \times \mathcal{N}) \times_{\mathcal{M}} (\mathcal{M} \times \mathcal{N})$. ■

Example 4.5 (Trivial Principal Bundle)**.** Consider some graded manifold $\mathcal{M}$ and let $\mathcal{G}$ be a graded Lie group that acts on the product manifold $\mathcal{P} := \mathcal{M} \times \mathcal{G}$ from the right by $\theta := 1 \times \mu$. we wish to show that $\mathcal{P}$ is a graded principal $\mathcal{G}$-bundle over $\mathcal{M}$. The surjective submersion π will be the projection $p_1 : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$. That θ acts along fibers is evident, as both routes along the diagram (4.2) compose to the projection on the first term in the three-fold product, that is $p_1 : \mathcal{M} \times \mathcal{G} \times \mathcal{G} \to \mathcal{M}$. The third condition in the definition of a principal bundle is satisfied trivially. One only needs to verify that the graded smooth map $\Sigma' : \mathcal{M} \times \mathcal{G} \to (\mathcal{M} \times \mathcal{G}) \times_{\mathcal{M}} (\mathcal{M} \times \mathcal{G})$ induced by the shear morphism Σ is an isomorphism.

Thanks to Lemma 4.4 we know that $(\mathcal{M} \times \mathcal{G}) \times_{\mathcal{M}} (\mathcal{M} \times \mathcal{G}) = \mathcal{M} \times \mathcal{G} \times \mathcal{G}$ with $p_1' = (p_1, p_2)$ and $p_2' = (p_1, p_3)$. The graded smooth map Σ' is then the unique morphism fitting into the commutative diagram



(4.13)

In the trivially graded case we would have $\Sigma'(m, g, h) = (m, g, gh)$ and so the inverse map would be $\Xi : (m, g, h) \mapsto (m, g, g^{-1}h)$. In the general case let us therefore define $\Xi : \mathcal{M} \times \mathcal{G} \times \mathcal{G} \to \mathcal{M} \times \mathcal{G} \times \mathcal{G}$ by

$$(p_1, p_2) \circ \Xi := (p_1, p_2), \quad \text{and} \quad p_3 \circ \Xi := \mu \circ (\iota p_2, p_3), \tag{4.14}$$

and let us show that it is indeed the two-sided inverse of Σ'. From the definitions we immediately have $p_1 \circ \Xi \circ \Sigma' = p_1$ and $p_1 \circ \Sigma' \circ \Xi = p_1$ and similarly for p_2. Working a little with the definitions and the properties of product we find that

$$\begin{aligned}
p_3 \circ \Xi \circ \Sigma' &= \mu \circ (\iota, \mu) \circ (p_2, p_3) && (4.15)\\
&= \mu \circ (1 \times \mu) \circ (\iota \times 1 \times 1) \circ ((1, 1) \times 1) \circ (p_2, p_3) && (4.16)\\
&= \mu \circ (\mu \times 1) \circ ((\iota, 1) \times 1) \circ (p_2, p_3) && (4.17)\\
&= \mu \circ (e \times 1) \circ (p_2, p_3) && (4.18)\\
&= \mu \circ (e, 1) \circ p_2 \circ (p_2, p_3) && (4.19)\\
&= p_3, && (4.20)
\end{aligned}$$

where between (4.18) and (4.19) we used that $(e \times 1) = (e, 1) \circ p_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$. Indeed, after-composition with the second projection yields the second projection and after-composition with the first projection yields the arrow $\mathcal{G} \times \mathcal{G} \to * \xrightarrow{e} G$. In much the same way, we would find that $p_3 \circ \Sigma' \circ \Xi = p_3$, hence $\Sigma' \circ \Xi = 1$ and $\Xi \circ \Sigma' = 1$ which shows Σ' to be an isomorphism and consequently $p_1 : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ to be a principal $\mathcal{G}$-bundle.

One can define local sections of a fiber bundle in the intuitive way.

Definition 4.6 (Local Sections)**.** Let $\pi : \mathcal{B} \to \mathcal{M}$ be a fiber bundle and consider some $U \in \mathrm{Op}(M)$. By a local section of $\mathcal{B}$ over U we mean a graded smooth map $\sigma : \mathcal{M}|_U \to \mathcal{B}|_{\underline{\pi}^{-1}(U)}$ satisfying $\pi|_{\underline{\pi}^{-1}(U)} \circ \sigma = 1_{\mathcal{M}|_U}$. We denote the set of all sections of $\mathcal{B}$ over U as $\Gamma_{\mathcal{B}}(U)$.

4.2 Graded Frame Bundle

In this subchapter we construct what we believe to be a reasonable generalization of the frame bundle to the graded setting. Let $(n_j)_{j\in\mathbb{Z}}$ be a finite sequence of integers and let $\mathcal{M}$ be a graded manifold of graded dimension (n_{-j}), notice the minus sign the reason for which will become apparent later. Our nascent graded frame bundle $\mathcal{P} \equiv \mathscr{F}(\mathcal{M})$ will be acted upon from the right by the graded Lie group $\mathcal{G} \equiv \mathrm{GL}((n_j),\mathbb{R})$. Let us fix an atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ for the graded manifold $\mathcal{M}$ and let us agree that whenever we discuss local coordinates on $\mathcal{M}$, say $\{x^i\}_{i=1}^n$, they will be labeled so that $|x^j| = -|j|$ where $|j|$ denotes the degree of the j-th standard basis vector of the graded vector space $\mathbb{R}^{(n_j)}$, see Remark 2.14. We will also sometimes partition these coordinates according to their degrees as $\{x^i\} =: \cup_{k\in\mathbb{Z}}\{x^j_{(k)}\}_{j=1}^{n_k}$ where $|x^j_{(k)}| = -k$ and they are labeled so that

$$(x^1_{(k_1)}, x^2_{(k_1)}, \ldots, x^{n_{k_1}}_{(k_1)}, \ldots, x^1_{(k_N)}, \ldots, x^{n_{k_N}}_{(k_N)}) = (x^1, \ldots, x^n), \tag{4.21}$$

where $\{k_1, \ldots, k_N\} := \{j \in \mathbb{Z} \,|\, n_j \neq 0\}$ is ordered increasingly.

Let us begin by constructing the **underlying smooth manifold** P which we create as a principal G-bundle over M for $G = \times_{k\in\mathbb{Z}}\mathrm{GL}(n_k,\mathbb{R})$. Mirroring the construction of an ordinary non-graded frame bundle, let us set

$$P := \bigsqcup_{m\in M}\bigsqcup_{k\in\mathbb{Z}} \mathcal{B}\,(T_m\mathcal{M})_k\,, \tag{4.22}$$

where $\mathcal{B}(T_m\mathcal{M})_k$ is the set of all the bases of the (ordinary) vector space $(T_m\mathcal{M})_k$ of degree k tangent vectors at m. Note that only finitely many of $\mathcal{B}(T_m\mathcal{M})_k$ are non-empty. According to our custom we write $n := \sum_j n_j = \mathrm{tdim}\,(\mathcal{M})$. The set P will be acted upon from the right by the group G via

$$(m, (b_k)_{k\in\mathbb{Z}}) \cdot (A_k)_{k\in\mathbb{Z}} := (m, (b_k \cdot A_k)_{k\in\mathbb{Z}}), \tag{4.23}$$

for any $m \in M$ and all $b \equiv (b_k)_{k\in\mathbb{Z}} \in \sqcup_{k\in\mathbb{Z}}\mathcal{B}(T_m\mathcal{M})_k$ where if $b_k = (v_1, \ldots, v_{n_k})$, then $b_k \cdot A_k$ denotes the basis

$$b_k \cdot A_k = (v_i\, A^i{}_1, \ldots, v_j\, A^j{}_{n_k}). \tag{4.24}$$

We define a surjective map $\underline{\pi} : P \to M$ as

$$\underline{\pi}(m, b) := m \tag{4.25}$$

Clearly the action of G, which we will denote as $\underline{\theta}$, is free and transitive along the fibers of $\underline{\pi}$. Next, for every $\alpha \in I$ we construct a bijective map

$$\underline{\phi}_\alpha : U_\alpha \times G \to P|_{\underline{\pi}^{-1}(U_\alpha)} \tag{4.26}$$

like so: let $\{x^i\}_{i=1}^n$ be the coordinates on U_α. Then for every $k \in \mathbb{Z}$ we have the coordinate basis

$$\left(\left.\frac{\partial}{\partial x^j_{(k)}}\right|_m\right)_{j=1}^{n_k} =: \partial|^{(\alpha)}_{m,k} \tag{4.27}$$

for the vector space $(T_m\mathcal{M})_k$. Let $N := \#\{k \,|\, n_k \neq 0\}$. For every $m \in U_\alpha$ there is a bijection between N-tuples of bases $b \equiv (b_k)_{k\in\mathbb{Z}} \in \sqcup_{k\in\mathbb{Z}}\mathcal{B}(T_m\mathcal{M})_k$ and N-tuples of invertible matrices $A \equiv (A_k)_{k\in\mathbb{Z}} \in G$ expressed as

$$b_k \leftrightarrow \partial|^{(\alpha)}_{m,k} \cdot A_k, \tag{4.28}$$

for every $k \in \mathbb{Z}$. Keeping this in mind, we define the bijective map $\underline{\phi}_\alpha$ as

$$\underline{\phi}_\alpha(m, (A_k)_{k\in\mathbb{Z}}) := (m, (\,\partial|^{(\alpha)}_{m,k} \cdot A_k)_{k\in\mathbb{Z}}). \tag{4.29}$$

We immediately see that $\underline{\phi}_\alpha$ is a G-equivariant map. Let us consider one more coordinate patch U_β with coordinates $\{y^i\}_{i=1}^n$ from our atlas. For every $m \in U_\alpha \cap U_\beta =: U_{\alpha\beta}$ we have

$$\underline{\phi}_\alpha^{-1}(m, (\,\partial|^{(\beta)}_{m,k})_{k\in\mathbb{Z}}) = (m, (B_k)_{k\in\mathbb{Z}}) \tag{4.30}$$

for the N-tuple $(B_k)_{k\in\mathbb{Z}}$ of matrices $B_k \in \mathrm{GL}(n_k, \mathbb{R})$ such that

$$\left(\partial|^{(\alpha)}_{m,k} \cdot B_k\right)_{k\in\mathbb{Z}} = \left(\partial|^{(\beta)}_{m,k}\right)_{k\in\mathbb{Z}} \tag{4.31}$$

Here recall how tangent vectors transform under the change of coordinates [10], i.e.

$$\left.\frac{\partial}{\partial y^i_{(k)}}\right|_m = \frac{\partial x^j_{(k)}}{\partial y^i_{(k)}}(m) \left.\frac{\partial}{\partial x^j_{(k)}}\right|_m = J_k(m)^j{}_i \left.\frac{\partial}{\partial x^j_{(k)}}\right|_m, \tag{4.32}$$

where in $J_k(m) \in \mathrm{GL}(n_k, \mathbb{R})$ we recognize the block of the Jacobi matrix corresponding to the transformation between degree $-k$ coordinates valued at m. Hence $B_k = J_k(m)$ and the transition functions $\underline{\phi}_{\alpha\beta} \equiv \underline{\phi}_\alpha^{-1} \circ \underline{\phi}_\beta : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ are therefore given as

$$\begin{aligned}\underline{\phi}_{\alpha\beta}\,(m, (A_k)_{k\in\mathbb{Z}}) &= \underline{\phi}_\alpha^{-1}\left(m, (\,\partial|^{(\beta)}_{m,k})_{k\in\mathbb{Z}} \cdot (A_k)_{k\in\mathbb{Z}}\right) = \underline{\phi}_\alpha^{-1}\left(m, (\,\partial|^{(\beta)}_{m,k})_{k\in\mathbb{Z}}\right) \cdot (A_k)_{k\in\mathbb{Z}} \\ &= (m, (J_k(m) \cdot A_k)_{k\in\mathbb{Z}})\,.\end{aligned} \tag{4.33}$$

These are apparently smooth and hence P is a well-defined G-principal bundle over M.

We may now move on to the **construction of the graded manifold** $\mathcal{P}$ itself. For this we will make use of the Gluing theorem [10, Proposition 3.33] and construct $\mathcal{P}$ by defining the transition morphisms

$$\Phi_{\alpha\beta} : \underline{\varphi}_\beta(U_{\alpha\beta})^{(n_{-j})} \times \mathcal{G} \to \underline{\varphi}_\alpha(U_{\alpha\beta})^{(n_{-j})} \times \mathcal{G}, \tag{4.34}$$

for the graded Lie group $\mathcal{G} = \mathrm{GL}((n_j), \mathbb{R})$. The underlying smooth maps are given simply by the composition of the local trivializations of P and local charts for M like so:

$$\underline{\Phi}_{\alpha\beta} = (\underline{\varphi}_\alpha \times 1) \circ \underline{\phi}_{\alpha\beta} \circ (\underline{\varphi}_\beta^{-1} \times 1), \tag{4.35}$$

or expressed directly as $\underline{\Phi}_{\alpha\beta}\left(\underline{\varphi}_\beta(m), (A_k)_{k\in\mathbb{Z}}\right) = \left(\underline{\varphi}_\alpha(m), (J_k(m) \cdot A_k)_{k\in\mathbb{Z}}\right)$. Denote the usual coordinates on the source graded domain of (4.34) as $\{y^i\}_{i=1}^n \cup \{y^j{}_k\}_{j,k=1}^n$ and on the target graded domain as $\{x^i\}_{i=1}^n \cup \{x^j{}_k\}_{j,k=1}^n$ and define the pullbacks of the transition morphisms as

$$\Phi^*_{\alpha\beta}\, x^i := \varphi^*_{\alpha\beta}\, x^i \qquad \text{and} \qquad \Phi^*_{\alpha\beta}\, x^j{}_k = y^\ell{}_k \,\frac{\partial\, \varphi^*_{\alpha\beta}\, x^j}{\partial\, y^\ell}. \tag{4.36}$$

First note that since $x^i{}_j$ and $y^i{}_j$ are the standard coordinates on the graded Lie group $\mathrm{GL}((n_j), \mathbb{R})$, their degree is $|j| - |i|$. The reason why we chose the graded dimension of $\mathcal{M}$ to be (n_{-j}) is so that the degrees in (4.36) would agree. We still don't know if $\Phi^*_{\alpha\beta}$ are well defined pullbacks; on the coordinates x^i this is clear and on the coordinates ${x^i}_j$ we find that whenever $|i| = |j|$,

$$\begin{aligned} {x^i}_j \circ \underline{\Phi}_{\alpha\beta}\ (y, (A_k)_{k\in\mathbb{Z}}) &= {x^i}_j((J_k(\underline{\varphi}_\beta^{-1}(y)) \cdot A_k)_{k\in\mathbb{Z}}) = \mathrm{diag}((J_k(\underline{\varphi}_\beta^{-1}(y)) \cdot A_k)_{k\in\mathbb{Z}})^i{}_j \\ &= \frac{\partial \varphi^*_{\alpha\beta} x^i}{\partial y^\ell}(y)\ {y^\ell}_j (A_k)_{k\in\mathbb{Z}}, \end{aligned} \tag{4.37}$$

for every $y \in \underline{\varphi}_\beta(U_{\alpha\beta})$ and $(A_k)_{k\in\mathbb{Z}} \in G$, here note that the last equality in (4.37) is ensured by the ordering (4.21). Consequently, we have ${x^i}_j \circ \underline{\Phi} = \underline{{J^i}_\ell}\, {y^\ell}_j = {y^\ell}_j\, \underline{{J^i}_\ell} = \underline{{y^\ell}_j\, {J^i}_\ell}$ where ${J^i}_\ell$ is the graded function $\frac{\partial \varphi^*_{\alpha\beta} x^i}{\partial y^\ell}$, which leads us to conclude that

$$\underline{\Phi^*_{\alpha\beta}\, {y^i}_j} = {y^i}_j \circ \underline{\Phi}_{\alpha\beta}. \tag{4.38}$$

For all $|i| = |j|$. This shows $\Phi_{\alpha\beta}$ to be a well defined morphism of graded domains. All that is left to verify, so that we may use the Gluing theorem, is the cocycle condition. Consider therefore yet another coordinate chart U_γ from our atlas and denote the coordinates on $\underline{\varphi}_\gamma(U_\gamma)^{(n_{-j})} \times \mathcal{G}$ as $\{z^i\} \cup \{{z^j}_k\}$ with the usual meaning. Then for every $m \in U_{\alpha\beta\gamma}$,

$$\underline{\Phi}_{\alpha\beta} \circ \underline{\Phi}_{\beta\gamma} \left(\underline{\varphi}_\gamma(m), A_k\right) = \underline{\Phi}_{\alpha\beta} \left(\underline{\varphi}_\beta(m), (J_k^{(\beta\gamma)}(m) \cdot A_k)_{k\in\mathbb{Z}}\right) \tag{4.39}$$

$$= \left(\underline{\varphi}_\alpha(m), (J_k^{(\alpha\beta)}(m) \cdot J_k^{(\beta\gamma)}(m) \cdot A_k)_{k\in\mathbb{Z}}\right) \tag{4.40}$$

$$= \underline{\Phi}_{\alpha\gamma} \left(\underline{\varphi}_\gamma(m), (A_k)_{k\in\mathbb{Z}}\right), \tag{4.41}$$

where

$$\left(J_k^{(\alpha\beta)}(m) \cdot J_k^{(\beta\gamma)}(m)\right)^i{}_j = \frac{\partial x^i_{(k)}}{\partial y^\ell_{(k)}}(m)\, \frac{\partial y^\ell_{(k)}}{\partial z^j_{(k)}}(m) = \frac{\partial x^i_{(k)}}{\partial z^j_{(k)}}(m) = J_k^{(\alpha\gamma)}(m), \tag{4.42}$$

so for the underlying smooth maps the condition holds. As for the pullbacks, the equality $\Phi^*_{\beta\gamma}\Phi^*_{\alpha\beta} x^i = \Phi^*_{\alpha\gamma} x^i$ follows from the cocycle condition for the transition morphisms $\varphi_{\alpha\beta}$ and

$$\Phi^*_{\beta\gamma}\Phi^*_{\alpha\beta}\, {x^i}_j = \Phi^*_{\beta\gamma} \left({y^\ell}_j\, \frac{\partial\, \varphi^*_{\alpha\beta}\, x^i}{\partial\, y^\ell}\right) = {z^k}_j\, \frac{\partial\, \varphi^*_{\beta\gamma}\, y^\ell}{\partial z^k} \varphi^*_{\beta\gamma} \left(\frac{\partial\, \varphi^*_{\alpha\beta}\, x^i}{\partial\, y^\ell}\right) = {z^k}_j\, \frac{\partial\, \varphi^*_{\alpha\gamma}\, x^i}{\partial z^k} = \Phi^*_{\alpha\gamma}\, {x^i}_j, \tag{4.43}$$

which means that the cocycle condition holds, and so there is a unique (up to an isomorphism) graded manifold structure on $\mathcal{P} \equiv \mathscr{F}(\mathcal{M})$ described by a graded atlas $\{\underline{\pi}^{-1}(U_\alpha), \Phi_\alpha\}$ for

$$\Phi_\alpha : \underline{\varphi}_\alpha(U_\alpha)^{(n_{-j})} \times \mathrm{GL}((n_j), \mathbb{R}) \to \mathscr{F}(\mathcal{M})|_{\underline{\pi}^{-1}(U_\alpha)}, \tag{4.44}$$

such that $\Phi_{\alpha\beta}$ become the transition morphisms.

We still need to make $\mathscr{F}(\mathcal{M})$ into a principal $\mathcal{G}$-bundle. We introduced the transition morphisms $\Phi_{\alpha\beta}$ and hence the local charts Φ_α as morphisms whose codomain spaces are graded domains specifically so we could use the Gluing theorem from [10]. Now however it will be more advantageous to work with local trivializations

$$\phi_\alpha : \mathcal{M}|_{U_\alpha} \times \mathrm{GL}((n_j), \mathbb{R}) \to \mathscr{F}(\mathcal{M})|_{\underline{\pi}^{-1}(U_\alpha)} \tag{4.45}$$

which we obtain by simply composing Φ_α with the local charts φ_α of $\mathcal{M}$ like so:

$$\phi_\alpha := \Phi_\alpha \circ (\varphi_\alpha \times 1). \tag{4.46}$$

From these we obtain yet another transition morphisms

$$\phi_{\alpha\beta} \equiv \phi_\alpha^{-1} \circ \phi_\beta : \mathcal{M}|_{U_{\alpha\beta}} \times \mathcal{G} \to \mathcal{M}|_{U_{\alpha\beta}} \times \mathcal{G}, \tag{4.47}$$

As any morphism between two products of graded manifolds, $\phi_{\alpha\beta}$ can be written as $\phi_{\alpha\beta} = (f_{\alpha\beta}, g_{\alpha\beta})$. We claim that

$$\phi_{\alpha\beta} = (p_1, g_{\alpha\beta}), \tag{4.48}$$

where $g_{\alpha\beta} : \mathcal{M}|_{U_{\alpha\beta}} \times \mathcal{G} \to \mathcal{G}$ is fully determined [10, Theorem 3.29] by

$$\underline{g}_{\alpha\beta} = \underline{p_2} \circ \underline{\phi}_{\alpha\beta} \qquad \text{and} \qquad g^*_{\alpha\beta}\, {x^i}_j = {y^k}_j \frac{\partial x^i}{\partial y^k}, \tag{4.49}$$

where $\{x^i\}$ are the coordinates on $\mathcal{M}|_{U_\alpha}$ induced by φ_α which reside on the codomain of $\phi_{\alpha\beta}$ and $\{y^i\}$ are the coordinates on $\mathcal{M}|_{U_\beta}$ induced by φ_β which reside on the domain of $\phi_{\alpha\beta}$. The claim (4.48) is clear on the level of the underlying morphisms due to (4.35), while on the level of pullbacks we have

$$\phi^*_{\alpha\beta} = \left((\varphi_\alpha^{-1} \times 1) \circ \Phi_\alpha^{-1} \circ \Phi_\beta \circ (\varphi_\beta \times 1)\right)^* = (\varphi_\beta \times 1)^* \circ \Phi^*_{\alpha\beta} \circ (\varphi_\alpha^{-1} \times 1)^*, \tag{4.50}$$

and the rest follows from (4.36). We see that in terms of ϕ_α and the respective transition morphisms everything is much cleaner. We may now **define the surjective submersion** $\pi : \mathscr{F}(\mathcal{M}) \to \mathcal{M}$. We already have the underlying smooth map $\underline{\pi} : P \to M$ which is defined in (4.25), we only need the pullback

$$\pi^* : C^\infty_{\mathcal{M}} \to \underline{\pi}_* C^\infty_{\mathcal{P}}. \tag{4.51}$$

For this we make use of the fact that morphisms between sheaves can be glued together [10, Proposition 2.6.]. Note that for every α we can write

$$C^\infty_{\mathcal{M}|_{U_\alpha}} = (C^\infty_{\mathcal{M}})|_{U_\alpha} \qquad \text{and} \qquad \underline{\pi}_*\left(C^\infty_{\mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)}}\right) = (\underline{\pi}_* C^\infty_{\mathcal{P}})|_{U_\alpha}\,, \tag{4.52}$$

hence we have a sheaf morphism $\pi^*_\alpha : C^\infty_{\mathcal{M}}|_{U_\alpha} \to (\underline{\pi}_* C^\infty_{\mathcal{P}})|_{U_\alpha}$ given by

$$\pi^*_\alpha = \left(p_1 \circ \phi_\alpha^{-1}\right)^*. \tag{4.53}$$

For any open set $V \subseteq U_{\alpha\beta}$ we can write

$$(\pi^*_\alpha)_V = (p_1 \circ \phi_\alpha^{-1})^* = (\phi_\alpha^{-1})^* \circ p_1^* = (\phi_\beta^{-1})^* \circ \phi^*_{\alpha\beta} \circ p_1^* = (\phi_\beta^{-1})^* \circ p_1^* = \left(\pi^*_\beta\right)_V, \tag{4.54}$$

where in the second-to-last equality we made use of (4.48). Consequently $\pi^*_\alpha|_{U_{\alpha\beta}} = \pi^*_\beta|_{U_{\alpha\beta}}$ and there exists a unique sheaf morphism $\pi^* : C^\infty_{\mathcal{M}} \to \underline{\pi}_* C^\infty_{\mathcal{P}}$ such that $\pi^*|_{U_\alpha} = \pi^*_\alpha$. We set $\pi := (\underline{\pi}, \pi^*)$. The underlying map $\underline{\pi}$ is surjective by construction, and locally we have $\pi|_{\underline{\pi}^{-1}(U_\alpha)} = p_1 \circ \phi_\alpha^{-1}$, which is a composition of a graded diffeomorphism ϕ_α^{-1} and a submersion p_1, therefore π itself is a surjective submersion. Also note that manifestly $\pi|_{\underline{\pi}^{-1}(U_\alpha)} \circ \phi_\alpha = p_1$.

The only piece of the puzzle left is to define **the action** θ **of** $\mathrm{GL}((n_j), \mathbb{R})$ **on the entire graded manifold** $\mathscr{F}(M)$. Aiming to use the same approach as for the definition of π, we notice that for every α we may define the graded smooth map

$$\theta_\alpha : \mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)} \times \mathcal{G} \to \mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)} \tag{4.55}$$

as the composite arrow

$$\mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)} \times \mathcal{G} \xrightarrow{\phi_\alpha^{-1}\times 1} \mathcal{M}|_{U_\alpha} \times \mathcal{G} \times \mathcal{G} \xrightarrow{1\times\mu} \mathcal{M}|_{U_\alpha} \times \mathcal{G} \xrightarrow{\phi_\alpha} \mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)} . \tag{4.56}$$

One can use the associativity and unit diagrams for μ to see that θ_α is a right $\mathcal{G}$-action on $\mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)}$ and from the fact that $\underline{\pi} : P \to M$ is an ordinary principal G bundle we find that $\underline{\theta}_\alpha = \underline{\theta}|_{\underline{\pi}^{-1}(U_\alpha)}$. Since $\underline{\theta}$ is transitive along fibers, for every α we may consider θ_α^* as a sheaf morphism

$$\theta_\alpha^* : C^\infty_{\mathcal{P}}|_{\underline{\pi}^{-1}(U_\alpha)} \to \underline{\theta}_* C^\infty_{\mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)}\times G} = \big(\underline{\theta}_* C^\infty_{\mathcal{P}\times\mathcal{G}}\big)\big|_{\underline{\pi}^{-1}(U_\alpha)} . \tag{4.57}$$

As before, we need to verify that $\theta_\alpha^*|_{\underline{\pi}^{-1}(U_{\alpha\beta})} = \theta_\beta^*|_{\underline{\pi}^{-1}(U_{\alpha\beta})}$. When restricted to $\underline{\pi}^{-1}(U_{\alpha\beta})$, but without explicitly writing the restrictions, we can compose θ_β with the isomorphisms $(\phi_\alpha^{-1})^*$ and $(\phi_\alpha \times 1)^*$ to form a pullback

$$(\phi_\alpha \times 1)^* \circ \theta_\beta^* \circ (\phi_\alpha^{-1})^* = (\phi_{\beta\alpha} \times 1)^* \circ (1 \times \mu)^* \circ \phi_{\alpha\beta}^*. \tag{4.58}$$

Which, when applied on the global cordinates $\{x^i\}$, $\{x^i{}_j\}$ on $\mathcal{M}|_{U_{\alpha\beta}} \times \mathcal{G}$, gives

$$(\phi_{\beta\alpha} \times 1)^* \circ (1 \times \mu)^* \circ \phi_{\alpha\beta}^* \; x^i = x^i = (1 \times \mu)^* x^i, \tag{4.59}$$

and

$$\begin{aligned}(\phi_{\beta\alpha} \times 1)^* \circ (1 \times \mu)^* \circ \phi_{\alpha\beta}^* \; x^i{}_j &= (\phi_{\beta\alpha} \times 1)^* \circ (1 \times \mu)^* \left(x^k{}_j \frac{\partial x^i}{\partial y^k} \right) = (\phi_{\beta\alpha} \times 1)^* \left(c^\ell{}_j b^k{}_\ell \frac{\partial x^i}{\partial y^k} \right) \\ &= c^\ell{}_j a^s{}_\ell \frac{\partial y^k}{\partial x^s} \frac{\partial x^i}{\partial y^k} = b^\ell{}_j a^s{}_\ell \delta^i{}_s = b^\ell{}_j a^i{}_\ell = (1 \times \mu)^* x^i{}_j, \end{aligned} \tag{4.60}$$

where $a^i{}_j, b^i{}_j, c^i{}_j$ and $x^i{}_j$ are merely convenient labels for the standard coordinates on $\mathcal{G}$. As a result,

$$(\phi_\alpha \times 1)^* \circ \theta_\beta^* \circ (\phi_\alpha^{-1})^* = (1 \times \mu)^* = (\phi_\alpha \times 1)^* \circ \theta_\alpha^* \circ (\phi_\alpha^{-1})^*, \tag{4.61}$$

hence $\theta_\beta^*|_{\underline{\pi}^{-1}(U_{\alpha\beta})} = \theta_\alpha^*|_{\underline{\pi}^{-1}(U_{\alpha\beta})}$ as desired, and there exists a graded smooth map $\theta : \mathcal{P} \times \mathcal{G} \to \mathcal{G}$ such that $\theta_V^* = \theta_{\alpha,V}^*$ for every $V \subseteq \underline{\pi}^{-1}(U_\alpha)$. Note that here in (4.61) lies the reason why we chose to define the multiplication μ on $\mathrm{GL}((n_j), \mathbb{R})$ the way we did in (2.47).

Is this glued-together morphism θ a right action of $\mathcal{G}$ on $\mathcal{P}$? Let us verify the commutativity of the "associativity" diagram (1.44). On the level of the underlying smooth maps this is clear, as $\underline{\theta}$ is a right action of G on P. Thanks to the fact that $\underline{\theta}$ acts along fibers, we have $\underline{\theta}^{-1}(\underline{\pi}^{-1}(U_\alpha)) = \underline{\pi}^{-1}(U_\alpha) \times \mathcal{G}$ for every α and so we can use the fact that θ_α is an action to write

$$\begin{aligned}((\theta \times 1)^* \circ \theta^*)|_{\underline{\pi}^{-1}(U_\alpha)} &= (\theta \times 1)^*|_{\underline{\pi}^{-1}(U_\alpha)\times G} \circ \theta^*|_{\underline{\pi}^{-1}(U_\alpha)} = (\theta_\alpha \times 1)^* \circ \theta_\alpha^* \\ &= (1 \times \mu)^* \circ \theta_\alpha^* = ((1 \times \mu)^* \circ \theta^*)|_{\underline{\pi}^{-1}(U_\alpha)} , \end{aligned} \tag{4.62}$$

which means that (1.44) commutes on the level of pullbacks as well. Commutativity of the "unit" diagram (1.43) can be shown similarly. Clearly, this action θ was constructed so that the local trivializations ϕ_α would be equivariant with respect to $\theta|_{\underline{\pi}^{-1}(U_\alpha)}$ and $1 \times \mu$.

Before we conclude that $\pi : \mathscr{F}(\mathcal{M}) \to \mathcal{G}$ is a principal $\mathrm{GL}((n_j), \mathbb{R})$ bundle according to Definition 4.2, there are two more things left to verify: that θ acts along fibers and that the shear map Σ' is an isomorphism. Again, both of these properties will be verified locally. For any α, one has

$$\begin{aligned}(\pi \circ \theta)|_{\underline{\pi}^{-1}(U_\alpha)\times G} &= \pi|_{\underline{\pi}^{-1}(U_\alpha)} \circ \theta|_{\underline{\pi}^{-1}(U_\alpha)\times G} = p_1 \circ \phi_\alpha^{-1} \circ \theta_\alpha = p_1 \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ (1 \times \mu) \circ (\phi_\alpha^{-1} \times 1) \\ &= p_1 \circ (1 \times \mu) \circ (\phi_\alpha^{-1} \times 1) = p_1 \circ \phi_\alpha^{-1} \circ p_1 = \pi|_{\underline{\pi}^{-1}(U_\alpha)} \circ p_1 = (\pi \circ p_1)|_{\underline{\pi}^{-1}(U_\alpha)\times G} , \end{aligned} \tag{4.63}$$

which means that θ acts along fibers. The last remaining point requires slightly more attention if it is to be treated properly. Let us denote

$$\underline{\pi}^{-1}(U_\alpha) =: V_\alpha, \qquad \text{and} \qquad \underline{\Sigma}'(\underline{\pi}^{-1}(U_\alpha) \times G) =: W_\alpha. \tag{4.64}$$

We need the following lemma, which states that the fiber product behaves as expected when restricted to sets W_α.

Lemma 4.7. *Using the notation of this subchapter, the diagram*

$$\begin{array}{ccc} (\mathcal{P} \times_{\mathcal{M}} \mathcal{P})|_{W_\alpha} & \xrightarrow{p_2'|_{W_\alpha}} & \mathcal{P}|_{V_\alpha} \\ \big\downarrow{\scriptstyle p_1'|_{W_\alpha}} & & \big\downarrow{\scriptstyle \pi|_{V_\alpha}} \\ \mathcal{P}|_{V_\alpha} & \xrightarrow{\pi|_{V_\alpha}} & \mathcal{M}|_{U_\alpha} \end{array} \tag{4.65}$$

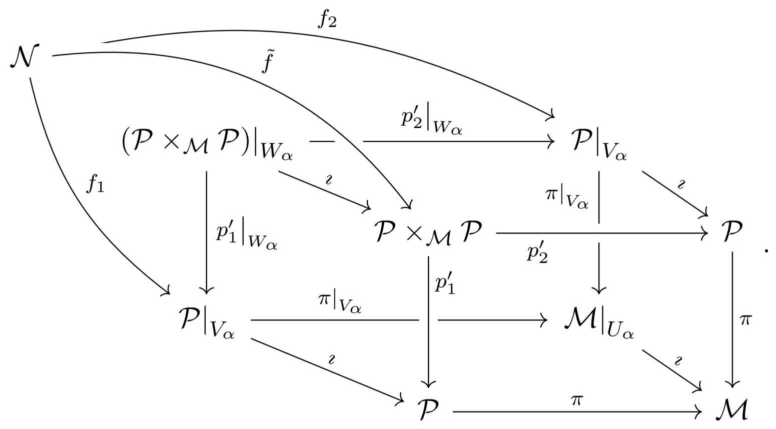
is a pullback square in gMan^∞ *for every* α. *In particular, there is a canonical isomorphism*

$$(\mathcal{P} \times_{\mathcal{M}} \mathcal{P})|_{\underline{\Sigma}'(\underline{\pi}^{-1}(U_\alpha)\times G)} \cong \mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)} \times_{\mathcal{M}|_{U_\alpha}} \mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)}\,. \tag{4.66}$$

Proof. Let us recall that $P \times_M P = \{(p,q) \in P \times P \,|\, \underline{\pi}(p) = \underline{\pi}(q)\}$ and that the fiber product projections $\underline{p}_1'$, $\underline{p}_2'$ are merely classical projections on the first and second component, respectively. Also the induced shear $\underline{\Sigma}'$ is given simply as $\underline{\Sigma}'(p,g) = (p, p\cdot g)$ for any $p \in P$ and $g \in G$. Hence we have

$$(\underline{p}_1')^{-1}(V_\alpha) = (\underline{p}_2')^{-1}(V_\alpha) = W_\alpha = \{(p,q) \in V_\alpha \times V_\alpha \,|\, \underline{\pi}(p) = \underline{\pi}(q)\}, \tag{4.67}$$

so the restrictions of the morphisms in (4.65) make sense and also in the trivially graded case the lemma clearly holds, i.e. $(P \times_M P)|_{W_\alpha} \cong P|_{V_\alpha} \times_{M|_{U_\alpha}} P|_{V_\alpha}$. The proof consists of showing the universal fiber product property, so let $\mathcal{N}$ be a graded manifold together with two graded smooth maps $f_1, f_2 : \mathcal{N} \to \mathcal{P}|_{V_\alpha}$ such that $\pi|_{V_\alpha} \circ f_1 = \pi|_{V_\alpha} \circ f_2$. One can consider the inclusion morphisms $\imath : \mathcal{M}|_{U_\alpha} \to \mathcal{M}$ and $\imath : \mathcal{P}|_{V_\alpha} \to \mathcal{P}$ where $\underline{\imath}$ is the usual set inclusion and the pullbacks are the sheaf restrictions. By composing them with the arrows f_1 and f_2 we obtain $\imath \circ f_1, \imath \circ f_2 : \mathcal{N} \to \mathcal{P}$ satisfying $\pi \circ \imath \circ f_1 = \pi \circ \imath \circ f_2$ and from the universal property of $\mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ there exists a unique arrow $\tilde{f} : \mathcal{N} \to \mathcal{P} \times_{\mathcal{M}} \mathcal{P}$ fitting into the commutative diagram



(4.68)

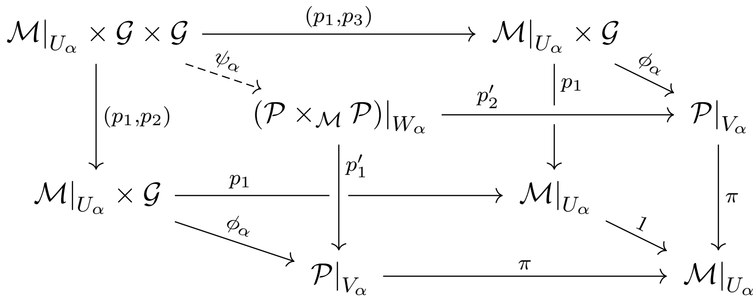
If we apply the body functor to this diagram while keeping in mind $(P \times_M P)|_{W_\alpha} \cong P|_{V_\alpha} \times_{M|_{U_\alpha}} P|_{V_\alpha}$, we find that necessarily $\underline{\tilde{f}}(N) \subseteq W_\alpha$. Consequently, there is a graded smooth map $f : \mathcal{N} \to (\mathcal{P} \times_{\mathcal{M}} \mathcal{P})|_{W_\alpha}$

such that $\imath \circ f = \tilde{f}$. Let h be another morphism such that $p'_1|_{W_\alpha} \circ h = f_1$ and $p'_2|_{W_\alpha} \circ h = f_2$, then $\imath \circ h$ fits in (4.65) in the place of $\tilde{f}$ and by uniqueness of $\tilde{f}$ we find that $\imath \circ f = \imath \circ h$. As the inclusion morphism is a monomorphism in the category gMan^∞, it follows that $h = f$ which proves uniqueness. The same argument shows that the introduction of f into (4.65) preserves the commutativity of the diagram, hence the lemma is proven. ■

With the above lemma in hand, one can consider the unique graded smooth map

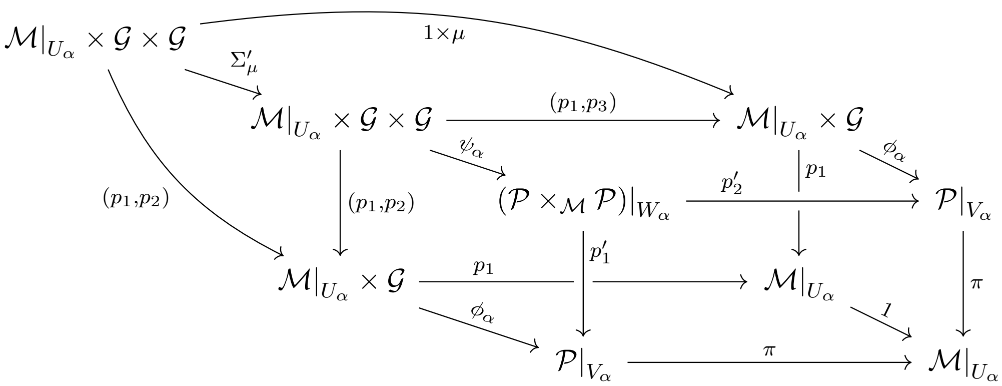
$$\psi_\alpha : \mathcal{M}|_{U_\alpha} \times \mathcal{G} \times \mathcal{G} \to (\mathcal{P} \times_\mathcal{M} \mathcal{P})|_{W_\alpha} \tag{4.69}$$

fitting into the commutative diagram



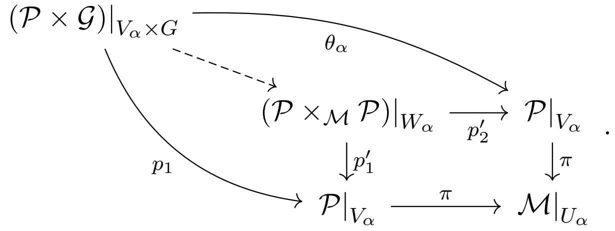
, (4.70)

where we omitted the explicit writing of the restrictions of the morphisms. We see that ψ_α is an isomorphism, since one may invert all full diagonally drawn arrows in (4.70) to obtain the definition for ψ_α^{-1}. Let us now recall Example 4.5, the trivial principal bundle, and denote as $\Sigma'_\mu : \mathcal{M}|_{U\alpha} \times \mathcal{G} \times \mathcal{G} \to \mathcal{M}|_{U\alpha} \times \mathcal{G} \times \mathcal{G}$ the induced shear morphism for the trivial $\mathcal{G}$-bundle $\mathcal{M}|_{U_\alpha} \times \mathcal{G}$. We know Σ'_μ to be an isomorphism and we may combine the defining diagram (4.13) of Σ'_μ with the diagram (4.70) to form



. (4.71)

Recall now the definition (4.56) of the "local" action $\theta_\alpha = \theta|_{V_\alpha \times G}$ and note that $(\mathcal{P} \times \mathcal{G})|_{V_\alpha \times G} = \mathcal{P}|_{V_\alpha} \times \mathcal{G}$. From the diagram (4.71) it then follows that both the composite morphism $\psi_\alpha \circ \Sigma'_\mu \circ (\phi_\alpha^{-1} \times 1)$ and the restriction of the shear morphism $\Sigma'|_{\underline{\pi}^{-1}(U_\alpha)\times G}$ fit as the dashed arrow in the commutative diagram



(4.72)

By universality of the fiber product therefore $\Sigma'|_{\underline{\pi}^{-1}(U_\alpha)\times G} = \psi_\alpha \circ \Sigma'_\mu \circ (\phi_\alpha^{-1} \times 1)$ and as the latter is a composition of graded diffeomorphisms, the former is a graded diffeomorphism as well. Consequently Σ' itself is a graded diffeomorphism, as can be seen e.g. from [10, Proposition 4.31]. We have verified that **the graded frame bundle $\pi : \mathscr{F}(\mathcal{M}) \to \mathcal{M}$ is a principal $\mathrm{GL}((n_j), \mathbb{R})$-bundle.**

Similarly as in the non-graded setting, local sections of the frame bundle $\mathscr{F}(\mathcal{M})$ correspond to local frames for vector fields on $\mathcal{M}$.

Proposition 4.8 (Hence the Name)**.** *Let $U \in \mathrm{Op}(M)$. Then there is a canonical one-to-one correspondence between local sections of $\mathscr{F}(M)$ over U and global frames for the $C^\infty_{\mathcal{M}}(U)$-module $\mathscr{X}_{\mathcal{M}}(U)$.*

Proof. Consider some $\sigma \in \Gamma_{\mathcal{P}}(U)$. If U is a subset of one of the coordinate patches U_α, then we may produce a frame $\{f_i\}_{i=1}^n$ for $\mathscr{X}_{\mathcal{M}}|_U$ like so:

$$f_i := \left(\sigma^* a^k{}_i\right) \frac{\partial}{\partial x^k}, \tag{4.73}$$

where $\{x^i\}$ and $\{a^i{}_j\}$ are the coordinates on $\mathcal{P}|_{\underline{\pi}^{-1}(U)}$ induced by the chart Ψ_α as in (4.44). Note that $|f_i| = |i|$. To show that the sections f_i form a frame, we may use the inversion arrow ι of the Lie group $\mathrm{GL}((n_j), \mathbb{R})$ to find

$$\left(\sigma^* \iota^* a^i{}_j\right) f_i = \left(\sigma^* \iota^* a^i{}_j\right) \left(\sigma^* a^k{}_i\right) \frac{\partial}{\partial x^k} = \sigma^* \left(\iota^*(a^i{}_j) a^k{}_i\right) \frac{\partial}{\partial x^k} = \delta^k{}_j \frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^j}, \tag{4.74}$$

where we used (2.75). If U is not a subset of any U_α, we may construct a collection of local frames $\{\{f_i^{(\alpha)}\}_{i=1}^n\}_{\alpha\in I}$ where $\{f_i^{(\alpha)}\}_{i=1}^n$ is a frame for $\mathscr{X}_{\mathcal{M}}(U \cap U_\alpha)$ defined as in (4.73). Denote the familiar coordinates on $\mathcal{P}|_{\underline{\pi}^{-1}(U_\alpha)}$ as $\{x^i\} \cup \{a^i{}_j\}$ and on $\mathcal{P}|_{\underline{\pi}^{-1}(U_\beta)}$ as $\{y^i\} \cup \{b^i{}_j\}$. Using the transition maps of $\mathscr{F}(\mathcal{M})$ we find that

$$\begin{aligned} f_i^{(\alpha)}\Big|_{U\cap U_{\alpha\beta}} &= \sigma^*(a^k{}_i) \frac{\partial}{\partial x^k} = \sigma^* \left(b^s{}_i \frac{\partial x^k}{\partial y^s}\right) \frac{\partial y^\ell}{\partial x^k} \frac{\partial}{\partial y^\ell} = \sigma^*(b^s{}_i) \frac{\partial x^k}{\partial y^s} \frac{\partial y^\ell}{\partial x^k} \frac{\partial}{\partial y^\ell} = \sigma^*(b^\ell{}_i) \frac{\partial}{\partial y^\ell} \\ &= f_i^{(\beta)}\Big|_{U\cap U_{\alpha\beta}}, \end{aligned} \tag{4.75}$$

where in the third equality we used that $\sigma^* \pi^* = 1$. Therefore $\{f_i^{(\alpha)}\}_{\alpha\in I}$ glue together a vector field $f_i \in \mathscr{X}_{\mathcal{M}}(U)$ for every i. Moreover, $\{f_i\}_{i=1}^n$ form a frame for $\mathscr{X}_{\mathcal{M}}(U)$ since their restrictions $\{f_i|_{U_\alpha\cap U}\}_{i=1}^n$ form a frame for $\mathscr{X}_{\mathcal{M}}(U \cap U_\alpha)$ for every α.

On the other hand, let $\{f_i\}_{i=1}^n$ be a frame for $\mathscr{X}_{\mathcal{M}}(U)$ ordered so that $|f_i| = |i|$. This ordering ensures that for every $m \in U$, the n-tuple $(f_{i,m})_{i=1}^n$ of linearly-independent tangent vectors $f_{i,m} \equiv (f_i)_m$ in $\mathrm{T}_m\mathcal{M}$ can be partitioned into

$$(f_{i,m})_{i=1}^n = \sqcup_{k\in\mathbb{Z}} (f_{i,m}^{(k)})_{i=1}^{n_k}, \tag{4.76}$$

where $(f^{(k)}_{i,m})^{n_k}_{i=1}$ is a basis for $(\mathrm{T}_m\mathcal{M})_k$. In this way, $(f_{i,m})^n_{i=1}$ can be regarded as an element of P. We may therefore define a smooth function $\underline{\sigma} : U \to \underline{\pi}^{-1}(U)$ by

$$\underline{\sigma}(m) := (f_{i,m})^n_{i=1}. \tag{4.77}$$

Furthermore, for every α there are unique graded functions $\gamma^i{}_{j,\alpha} \in C^\infty_{\mathcal{M}}(U \cap U_\alpha)$ such that $f_i|_{U\cap U_\alpha} = \gamma^k{}_{i,\alpha}\frac{\partial}{\partial x^k}$, where $\{x^j\}$ are the coordinates on U_α. Consequently we may define a local pullback $\sigma^*_\alpha : C^\infty_{\mathcal{M}}|_{U\cap U_\alpha} \to (\underline{\sigma}_* C^\infty_{\mathcal{P}})|_{\underline{\pi}^{-1}(U\cap U_\alpha)}$ by specifying the pullbacks of coordinate functions

$$\sigma^*_\alpha x^i = x^i \qquad \text{and} \qquad \sigma^*_\alpha\, a^i{}_j := \gamma^i{}_{j,\alpha}. \tag{4.78}$$

Note that the pullbacks of x^i are dictated to take this form by the requirement $\pi\sigma = 1$. If we denote the local coordinates the same way as above, we find that

$$\gamma^i{}_{j,\alpha}\frac{\partial}{\partial x^i} = f_j|_{U\cap U_{\alpha\beta}} = \gamma^\ell{}_{j,\beta}\frac{\partial}{\partial y^\ell} = \gamma^\ell{}_{i,\beta}\frac{\partial x^i}{\partial y^\ell}\frac{\partial}{\partial x^i} \qquad \Longrightarrow \qquad \gamma^i{}_{j,\alpha} = \gamma^\ell{}_{j,\beta}\frac{\partial x^i}{\partial y^\ell} \tag{4.79}$$

and so

$$\sigma^*_\alpha|_{U\cap U_{\alpha\beta}}(a^i{}_j) = \gamma^i{}_{j,\alpha} = \gamma^\ell{}_{j,\beta}\frac{\partial x^i}{\partial y^\ell} = \sigma^*_\beta(b^\ell{}_j\frac{\partial x^i}{\partial y^\ell}) = \sigma^*_\beta|_{U\cap U_{\alpha\beta}}(a^i{}_j), \tag{4.80}$$

and we may glue σ^* together from $\{\sigma^*_\alpha\}_{\alpha\in I}$. Clearly $\sigma = (\underline{\sigma}, \sigma^*) \in \Gamma_{\mathcal{P}}(U)$ and the described correspondence $\sigma \leftrightarrow \{f_i\}^n_{i=1}$ is bijective. ■

Example 4.9 (Fundamental Vector Fields)**.** In Example 3.14 we learned the general shape of left-invariant vector fields on $\mathcal{G} \equiv \mathrm{GL}((n_j),\mathbb{R})$ and in Proposition 3.22 we defined the infinitesimal generator for any graded Lie group action. Let us now examine the fundamental vector fields $\mathscr{X}^F_{\mathcal{P}}$ on the newly minted graded frame bundle $\mathcal{P} \equiv \mathscr{F}(\mathcal{M})$. Consider some $v \in \mathrm{T}_e\mathcal{G}$. In the standard coordinates it is written as $v^i{}_j\frac{\partial}{\partial x^i{}_j}|_e$ and the corresponding left-invariant vector field $X \in \mathscr{X}^L_{\mathcal{G}}$ is given by its component functions

$$X^i{}_j = x^i{}_k v^k{}_j. \tag{4.81}$$

From the definition of $\#X \in \mathscr{X}^F_{\mathcal{P}}$, see (3.139), we may discern its local form: let U_α be one of the coordinate patches for $\mathcal{M}$ whom we use throughout this subchapter. That is, $\underline{\pi}^{-1}(U_\alpha)$ is a coordinate patch for $\mathscr{F}(\mathcal{M})$ with coordinates $\{x^i\}^n_{i=1} \cup \{x^i{}_j\}^n_{i,j=1}$ inherited from $\mathcal{M}|_{U_\alpha}$ and $\mathcal{G}$, respectively. We find that

$$(\#X)|_{\underline{\pi}^{-1}(U_\alpha)}\, x^i = (1,e)^*(1\otimes X)\theta^* x^i = (1,e)^*(1\otimes X)x^i = 0, \tag{4.82}$$

and

$$\begin{aligned}(\#X)|_{\underline{\pi}^{-1}(U_\alpha)}\, x^i{}_j &= (1,e)^*(1\otimes X)\theta^* x^i{}_j = (1,e)^*\left(b^\ell{}_k v^k{}_u\frac{\partial}{\partial b^\ell{}_u} b^s{}_j a^i{}_s\right) = (1,e)^*\left(b^s{}_k v^k{}_j a^i{}_s\right)\\ &= \delta^s{}_k v^k{}_j x^i{}_s = v^k{}_j x^i{}_k,\end{aligned}$$

hence the local shape of $\#X$ is

$$(\#X)|_{\underline{\pi}^{-1}(U_\alpha)} = v^k{}_j x^i{}_k \frac{\partial}{\partial x^i{}_j}. \tag{4.83}$$

4.3 Principal Connection

Differential forms on any graded manifold $\mathcal{M}$ have been described in [11]. Let $V \in \mathsf{gVec}$ be a graded, finite-dimensional real vector space. For any $p \in \mathbb{N}_0$ and any $U \in \mathrm{Op}(M)$ we define the space of **p-forms valued in** V on U as

$$\Omega^p_{\mathcal{M}}(U, V) := V \otimes_{\mathbb{R}} \Omega^p_{\mathcal{M}}(U). \tag{4.84}$$

It follows that if $\{v_\mu\}_{\mu=1}^{\mathrm{tdim}\, V}$ is a total basis of V, then any $\omega \in \Omega^p_{\mathcal{M}}(U, V)$ can be uniquely written as $\omega = v_\mu \otimes \omega^\mu$ for some $\omega^\mu \in \Omega^p_{\mathcal{M}}(U)$ of degree $|\omega^\mu| = |\omega| - |v_\mu|$. We will omit the tensor product sign and simply write $\omega = v_\mu \omega^\mu$. Any $\omega \in \Omega^p(U, V)$ acts on the p-tuple of vector fields $X_1, \ldots, X_p \in \mathscr{X}_{\mathcal{M}}(U)$ as

$$(v_\mu\, \omega^\mu)(X_1, \ldots, X_p) = v_\mu\, \omega^\mu(X_1, \ldots, X_p). \tag{4.85}$$

From the properties of graded differential forms we find that

$$\omega(f X_1, \ldots, X_p) = (-1)^{|f||\omega|} f\, \omega(X_1, \ldots, X_p), \qquad \text{and} \tag{4.86}$$

$$\omega(X_1, \ldots, X_j, X_{j+1}, \ldots, X_p) = (-1)^{|X_j||X_{j+1}|}\, \omega(X_1, \ldots, X_{j+1}, X_j, \ldots, X_p), \tag{4.87}$$

for any $f \in C^\infty_{\mathcal{M}}(U)$. We may also expand some of the usual operations on differential forms to $\Omega^p_{\mathcal{M}}(U, V)$. For their definition on $\Omega^p_{\mathcal{M}}(U)$ refer to [11].

- The **exterior derivative** $d : \Omega^p_{\mathcal{M}}(U, V) \to \Omega^{p+1}_{\mathcal{M}}(U, V)$ is a graded linear map of degree $|d| = 0$ defined by $d\omega := (-1)^{|v_\mu|}\, v_\mu (d\omega^\mu)$.
- For any vector field $X \in \mathscr{X}_{\mathcal{M}}(U)$, the **interior product** $i_X : \Omega^p_{\mathcal{M}}(U, V) \to \Omega^{p-1}_{\mathcal{M}}(U, V)$ is a graded linear map of degree $|i_X| = |X|$ defined as $i_X \omega := (-1)^{(|X|-1)|v_\mu|}\, v_\mu (i_X \omega^\mu)$.
- For any vector field $X \in \mathscr{X}_{\mathcal{M}}(U)$ the **Lie derivative** $\mathcal{L}_X : \Omega^p_{\mathcal{M}}(U, V) \to \Omega^p_{\mathcal{M}}(U, V)$ is a graded linear map of degree $|\mathcal{L}_X| = |X|$ defined as $\mathcal{L}_X \omega = (-1)^{|X||v_\mu|}\, v_\mu (\mathcal{L}_X \omega^\mu)$.
- If, in addition, V is equipped with a graded bilinear map $V \times V \to V$, $(v, w) \mapsto v \cdot w$ of degree zero, making it a graded algebra, we can extend the **exterior product** of differential forms to a graded bilinear map $\dot{\wedge} : \Omega^p_{\mathcal{M}}(U, V) \times \Omega^q_{\mathcal{M}}(U, V) \to \Omega^{p+q}_{\mathcal{M}}(U, V)$ of degree $|\,\dot{\wedge}\,| = 0$ by

  $$\omega \,\dot{\wedge}\, \tau := (-1)^{(p+|\omega^\mu|)|v_\nu|}\, v_\mu \cdot v_\nu\; \omega^\mu \wedge \tau^\nu, \tag{4.88}$$

  for any $\omega \in \Omega^p_{\mathcal{M}}(U, V)$ and $\tau \in \Omega^q_{\mathcal{M}}(U, V)$. Notice that every summand of the Einstein sum over μ and ν carries with it a sign determined by the degree of τ^ν and v_μ. This is to ensure that if V is a graded commutative algebra, then

  $$\omega \,\dot{\wedge}\, \tau = (-1)^{(p+|\omega|)(q+|\tau|)}\, \tau \,\dot{\wedge}\, \omega, \tag{4.89}$$

  which is the known commutativity relation for the exterior product of "ordinary" graded differential forms in [11]. In particular, if V is a **Lie algebra**, see Definition 2.4, we denote the resulting product as $\omega \,\dot{\wedge}\, \tau =: [\omega \wedge \tau]$ and there holds

  $$[\omega \wedge \tau] = -(-1)^{(p+|\omega|)(q+|\tau|)}\, [\tau \wedge \omega]. \tag{4.90}$$

Example 4.10 (Maurer-Cartan Form)**.** For any graded Lie group $\mathcal{G}$ and its associated Lie algebra $\mathfrak{g}$, we may construct a canonical 1-form $\omega_{\mathrm{MC}} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ by its action on the frame $(v_1^L, \ldots, v_n^L)$ where $(v_1, \ldots, v_n)$ is some basis of $\mathfrak{g}$, see Corollary 3.19, by

$$\omega_{\mathrm{MC}}(v_\mu^L) := v_\mu, \tag{4.91}$$

for any $\mu \in \{1, \ldots, n\}$. We call this 1-form the graded left Maurer-Cartan form. One may use explicit formulas for the evaluation of the exterior product and exterior derivative of graded differential forms, together with our definitions, to find that the graded Maurer-Cartan form satisfies the graded version of the Maurer-Cartan equation

$$d\,\omega_{\mathrm{MC}} + \frac{1}{2}\left[\omega_{\mathrm{MC}} \wedge \omega_{\mathrm{MC}}\right] = 0. \tag{4.92}$$

Consider now any graded manifold $\mathcal{M}$ and recall [10] that, similarly to the classical case, any locally freely and finitely generated sheaf of $C^\infty_{\mathcal{M}}$-modules of constant rank corresponds uniquely (up to an isomorphism) to a graded vector bundle over $\mathcal{M}$.

We may define graded distributions the same way as in [9]:

Definition 4.11 (Graded Distribution)**.** Let $\mathcal{M}$ be a graded manifold and let $\mathscr{D}$ be a subsheaf of $\mathscr{X}_{\mathcal{M}}$ such that $\mathscr{D}(U)$ is a $C^\infty_{\mathcal{M}}(U)$-submodule of $\mathscr{X}_{\mathcal{M}}(U)$ for any $U \in \mathrm{Op}(M)$ of constant graded rank. We say that $\mathscr{D}$ is a **graded distribution on** $\mathcal{M}$ if $\mathscr{D} = 0$ or it satisfies the following condition: there exists an integer $\ell \in \{1, \ldots, n\}$ and a neighborhood $U \in \mathrm{Op}_m(M)$ for any $m \in M$ such that there is a frame $(f_1, \ldots, f_n)$ for $\mathscr{X}_{\mathcal{M}}(U)$ where $(f_1, \ldots, f_\ell)$ is a frame for $\mathscr{D}(U)$.

Note: with the identification of a vector bundle with its sheaf of sections, this definition is equivalent to $\mathscr{D}$ being a vector subbundle of $\mathscr{X}_{\mathcal{M}}$. See [10, Section 5].

Definition 4.12 (Vertical Vector Fields)**.** Let $\pi : \mathcal{B} \to \mathcal{M}$ be a graded fiber bundle and let $V \in \mathrm{Op}(B)$. We say that $X \in \mathscr{X}_{\mathcal{B}}(V)$ is vertical if $X \circ \rho_V^{\pi^{-1}(\underline{\pi}(V))} \circ \pi^*_{\underline{\pi}(V)} = 0$. We denote the set of vertical vector fields over V as $\mathrm{Ver}_{\mathcal{B}}(V)$.

Lemma 4.13. *Let $\pi : B \to M$ be a classical fiber bundle with a typical fiber F and let V be an open set in B. Then for every $p \in V$ there exists $W \in \mathrm{Op}_p(V)$ such that both $\overline{W} \subseteq V$ and $\overline{\pi(W)} \subseteq \pi(V)$.*

Proof. Let $U \in \mathrm{Op}(M)$ be a trivializing open set with a local trivialization $\phi : U \times F \to \pi^{-1}(U)$ such that $p \in \pi^{-1}(U)$. The set $(U \times F) \cap \phi^{-1}(V)$ is an open neighborhood of $\phi^{-1}(p)$, and by the definition of the product topology there exist open sets $R' \subseteq M$ and $S' \subseteq F$ such that

$$\phi^{-1}(p) \in R' \times S' \subseteq (U \times F) \cap \phi^{-1}(V). \tag{4.93}$$

As M and F are smooth manifolds, there exist open subsets $R \subseteq R'$ and $S \subseteq S'$ such that $\overline{R} \subseteq R'$, $\overline{S} \subseteq S'$ and $\phi^{-1}(p) \in R \times S$. Let us argue that the set $W := \phi(R \times S)$ has the desired properties. Apparently W is an open set such that $p \in W$. Since ϕ is a diffeomorphism,

$$\overline{W} \subseteq \overline{\phi(R \times S)} \subseteq \phi(\overline{R} \times \overline{S}) \subseteq \phi(R' \times S') \subseteq V \tag{4.94}$$

and also

$$\overline{\pi(W)} = \overline{\pi\left(\phi(R \times S)\right)} = \overline{p_1(R \times S)} = \overline{R} \subseteq R' = p_1(R' \times S') = \pi\left(\phi(R' \times S')\right) \subseteq \pi(V). \tag{4.95}$$

■

Proposition 4.14 (Vertical Distribution)**.** *The assignment $U \mapsto \mathrm{Ver}_{\mathcal{B}}(U)$ defines a graded distribution on any graded fiber bundle $\pi : \mathcal{B} \to \mathcal{M}$. Moreover, if $\mathcal{F}$ is the typical fiber of $\mathcal{B}$ with graded dimension (f_j) then the graded rank of $\mathrm{Ver}_{\mathcal{B}}$ is (f_{-j}).*

Proof. From the definition it is apparent that $\mathrm{Ver}_{\mathcal{B}}(U)$ is a $C^\infty_{\mathcal{B}}(U)$-submodule of $\mathscr{X}_{\mathcal{B}}(U)$ for any $U \in \mathrm{Op}(B)$. Let us show that $\mathrm{Ver}_{\mathcal{B}}$ is a subsheaf of $\mathscr{X}_{\mathcal{B}}$, namely that a restriction of a vertical vector field is vertical and that a vector field glued from vertical vector fields is vertical. Consider some $X \in \mathrm{Ver}_{\mathcal{B}}(U)$, $V \in \mathrm{Op}(U)$, $f \in C^\infty_{\mathcal{M}}(\underline{\pi}(V))$ and a point $p \in V$. By Lemma 4.13 there is an open neighborhood W of p such that $\overline{W} \subseteq V$ and $\overline{\underline{\pi}(W)} \subseteq \underline{\pi}(V)$. Consequently there is an extension $\tilde{f} \in C^\infty_{\mathcal{M}}(\underline{\pi}(U))$ of f from $\underline{\pi}(W)$, i.e. $\tilde{f}|_{\underline{\pi}(W)} = f|_{\underline{\pi}(W)}$. But then

$$\left(\left(\pi^*_{\underline{\pi}(V)} f\right)\Big|_V\right)\Big|_W = \left(\pi^*_{\underline{\pi}(W)}\left(f|_{\underline{\pi}(W)}\right)\right)\Big|_W = \left(\pi^*_{\underline{\pi}(W)}\, \tilde{f}\Big|_{\underline{\pi}(W)}\right)\Big|_W = \left(\left(\pi^*_{\underline{\pi}(U)}\tilde{f}\right)\Big|_U\right)\Big|_W, \tag{4.96}$$

and thus, by the definition of restrictions of vector fields,

$$\left(X|_V \left(\pi^*_{\underline{\pi}(V)} f\right)\Big|_V\right)\Big|_W = \left(X\left(\pi^*_{\underline{\pi}(U)}\tilde{f}\right)\Big|_U\right)\Big|_W = 0. \tag{4.97}$$

As W was a neighborhood of an arbitrary point and f was an arbitrary graded function, we conclude that $X|_V \in \mathrm{Ver}_{\mathcal{B}}(V)$. Similarly, consider some $U \in \mathrm{Op}(B)$ with its open cover $\{U_\alpha\}$ and let $X \in \mathscr{X}_{\mathcal{B}}(U)$ be such that $X|_{U_\alpha} \in \mathrm{Ver}_{\mathcal{B}}(U_\alpha)$ for every α. Then for any $f \in C^\infty_{\mathcal{M}}(\underline{\pi}(U))$ we find

$$\left(X\left(\pi^*_{\underline{\pi}(U)} f\right)\Big|_U\right)\Big|_{U_\alpha} = X|_{U_\alpha}\left(\pi^*_{\underline{\pi}(U_\alpha)}\, f|_{\underline{\pi}(U_\alpha)}\right)\Big|_{U_\alpha} = 0, \tag{4.98}$$

for any α, hence $X \in \mathrm{Ver}_{\mathcal{B}}(U)$. We see that $\mathrm{Ver}_{\mathcal{B}}$ is a sheaf of $C^\infty_{\mathcal{B}}$-submodules of $\mathscr{X}_{\mathcal{B}}$.

Next we will show that every point $p \in B$ has a neighborhood $W \in \mathrm{Op}_p(B)$ such that $\mathrm{Ver}_{\mathcal{B}}(W)$ is freely generated by some f vector fields from a frame for $\mathscr{X}_{\mathcal{B}}(W)$, where $f = \mathrm{tdim}\,\mathcal{F}$ and $\mathcal{F}$ is the typical fiber of $\mathcal{B}$. Let us fix some $p \in B$ and let $U \in \mathrm{Op}(M)$ be a trivializing open subset containing $\underline{\pi}(p)$, which is also a coordinate patch on $\mathcal{M}$ with coordinates $\{x^i\}_{i=1}^m$, where $m := \mathrm{tdim}\,\mathcal{M}$. We have the local trivialization $\phi : \mathcal{M}|_U \times \mathcal{F} \to \mathcal{B}|_{\underline{\pi}^{-1}(U)}$ satisfying $\pi \circ \phi = p_1$. Also consider some coordinate patch V on $\mathcal{F}$ with coordinates $\{y^\mu\}_{\mu=1}^f$ such that $\underline{p}_2(\underline{\phi}^{-1}(p)) \in V$. Hence $W := \underline{\phi}(U \times V) \in \mathrm{Op}_p(B)$, and we may restrict ϕ to a graded diffeomorphism $\phi|_{U\times V} : \mathcal{M}|_U \times \mathcal{F}|_V \to \mathcal{B}|_W$, whom we shall nevertheless still denote ϕ. We find that $X \in \mathscr{X}_{\mathcal{B}}(W)$ is vertical if and only if $\hat{X} := \phi^* \circ X \circ (\phi^{-1})^* \in \mathscr{X}_{\mathcal{M}\times\mathcal{F}}(U \times V)$ satisfies

$$\hat{X} \circ \rho^{U\times F}_{U\times V} \circ p^*_{1,U} = 0. \tag{4.99}$$

As $\hat{X} = \hat{X}^i \frac{\partial}{\partial x^i} + \hat{X}^\mu \frac{\partial}{\partial y^\mu}$, we see that (4.99) holds if and only if $\hat{X}^i = 0$ for every $i \in \{1, \dots, m\}$. Since $\partial_i := (\phi^{-1})^* \circ \frac{\partial}{\partial x^i} \circ \phi^*$ and $\partial_\mu := (\phi^{-1})^* \circ \frac{\partial}{\partial y^\mu} \circ \phi^*$ form a frame for $\mathscr{X}_{\mathcal{B}}(W)$, we conclude that X is vertical if and only if $X = X^\mu \partial_\mu$ for some (freely chosen) $X^\mu \in C^\infty_{\mathcal{B}}(W)$, and so $\{\partial_\mu\}_{\mu=1}^f$ is a frame for $\mathrm{Ver}_{\mathcal{B}}(W)$. Thus we have shown that $\mathrm{Ver}_{\mathcal{B}}$ is a graded distribution. ∎

Proposition 4.15 (Frame for $\mathrm{Ver}_{\mathcal{P}}$). *Let $\pi : \mathcal{P} \to \mathcal{M}$ be a principal $\mathcal{G}$-bundle and let $(t_1, \dots, t_\ell)$ be a total basis of $\mathrm{T}_e\mathcal{G}$. Then $(\#t_1, \dots, \#t_\ell)$ forms a frame for the vertical distribution $\mathrm{Ver}_{\mathcal{P}}$.*

Proof. First we need to verify that $\#t_\mu \in \mathscr{X}_{\mathcal{P}}(P)$ is vertical for any μ. By definition of the infinitesimal generator $\#$, see Proposition 3.22, we have $1 \otimes t^L_\mu \sim_\theta \#t_\mu$ and so we may write

$$\theta^* \circ \#t_\mu \circ \pi^* = (1 \otimes t^L_\mu) \circ \theta^* \circ \pi^* = (1 \otimes t^L_\mu) \circ p_1^* \circ \pi^* = 0, \tag{4.100}$$

where in the second equality we used that θ is assumed to act along the fibers of π, see diagram (4.2), and in the last equality we used that $1 \otimes X \sim_{p_1} 0$ for any vector field X. Since we have $(1,e)^* \circ \theta^* = 1$, it follows that $\#t_\mu \circ \pi^* = 0$ ergo $\#t_\mu$ is vertical.

Consider some local trivialization $\phi : \mathcal{M}|_U \times \mathcal{G} \to \mathcal{P}|_{\underline{\pi}^{-1}(U)}$ and let us examine the pushforward vector field $(\phi^{-1})_*(\# t_\mu|_{\underline{\pi}^{-1}(U)})$. Note that

$$(\# t_\mu)|_{\underline{\pi}^{-1}(U)} = (1,e)^*_{\underline{\pi}^{-1}(U)\times G} \circ \left(1 \otimes t_\mu^L\right)\big|_{\underline{\pi}^{-1}(U)\times G} \circ \theta^*_{\underline{\pi}^{-1}(U)}, \tag{4.101}$$

which can be gleaned using similar arguments as those used in the proof of Theorem 3.16. Now denote $V := \underline{\pi}^{-1}(U)$ and observe that

$$\begin{aligned}
(\phi^{-1})_* \left(\# t_\mu|_V\right) &= \phi_V^* \circ (\# t_\mu)|_V \circ (\phi^{-1})^*_{U\times G} && (4.102)\\
&= \phi_V^* \circ (1,e)^*_{V\times G} \circ \left(1 \otimes t_\mu^L\right)\big|_{V\times G} \circ \theta_V^* \circ (\phi^{-1})^*_{U\times G} && (4.103)\\
&= \phi_V^* \circ (1,e)^*_{V\times G} \circ \left(1 \otimes t_\mu^L\right)\big|_{V\times G} \circ (\phi^{-1}\times 1)^*_{U\times G\times G} \circ (1\times\mu)^*_{U\times G} && (4.104)\\
&= \phi_V^* \circ (1,e)^*_{V\times G} \circ (\phi^{-1}\times 1)^*_{U\times G\times G} \circ \left(1' \otimes t_\mu^L\right)\big|_{U\times G\times G} \circ (1\times\mu)^*_{U\times G} && (4.105)\\
&= (1',e)^*_{U\times G\times G} \circ \left(1' \otimes t_\mu^L\right)\big|_{U\times G\times G} \circ (1\times\mu)^*_{U\times G} && (4.106)\\
&= \left(1 \otimes t_\mu^L\right)\big|_{U\times G}, && (4.107)
\end{aligned}$$

where after (4.103) we used the equivariance of ϕ, after (4.104) we used Lemma 3.2, after (4.105) we used Lemma 1.3 and after (4.106) we used Lemma 3.10. We may now consider the infinitesimal generator as a morphism of sheaves of $C^\infty_{\mathcal{P}}$-modules $\# : C^\infty_{\mathcal{P}} \otimes \mathfrak{g} \to \mathrm{Ver}_{\mathcal{P}}$, wherein

$$\#(f^\mu t_\mu) := f^\mu \left(\# t_\mu\right)|_V, \tag{4.108}$$

for any $f^\mu t_\mu \in C^\infty_{\mathcal{P}}(V) \otimes \mathfrak{g}$. Since $\{t_\mu^L|_g\}_{\mu=1}^\ell$ are linearly independent vectors for each $g \in G$, so too are $\{(1 \otimes t_\mu^L)|_{(m,g)}\}_{\mu=1}^\ell$ linearly independent for each $m \in M$ and $g \in G$. Indeed, one must merely recall their shape in local coordinates. As a result, $\{\# t_\mu|_p\}_{\mu=1}^\ell$ are also linearly independent at every $p \in P$ because

$$\# t_\mu|_p = \left(\phi_* \left(1 \otimes t_\mu^L\right)\right)|_p = \left(\mathrm{T}_{\underline{\phi}^{-1}(p)}\phi\right)\left(1 \otimes t_\mu^L\right)|_{\underline{\phi}^{-1}(p)}. \tag{4.109}$$

This means that $\#$ is fiber-wise injective, and as $C^\infty_{\mathcal{P}} \otimes \mathfrak{g}$ and $\mathrm{Ver}_{\mathcal{P}}$ have the same graded rank, the result follows from Proposition 3.18. ■

Definition 4.16 (Horizontal Distribution). i.) Let $\pi : \mathcal{B} \to \mathcal{M}$ be a fiber bundle. We say that a distribution $\mathscr{D}$ on $\mathcal{B}$ is horizontal if it satisfies $\mathscr{X}_{\mathcal{B}} = \mathrm{Ver}_{\mathcal{B}} \oplus \mathscr{D}$. Every horizontal distribution is also called an **Ehresmann connection**.

ii.) Let $\pi : \mathcal{P} \to \mathcal{M}$ be a principal $\mathcal{G}$-bundle with the right action of $\mathcal{G}$ on $\mathcal{P}$ denoted as θ. If $U \in \mathrm{Op}(M)$, then a vector field $X \in \mathscr{X}_{\mathcal{P}}(\underline{\pi}^{-1}(U))$ is called **θ-invariant** if $X \otimes 1 \sim_\theta X$. We say that a horizontal distribution $\mathscr{D}$ on $\mathcal{P}$ is a **principal connection** if there exists an open cover $\{U_\alpha\}_{\alpha\in I}$ of M such that for every α there is a frame for $\mathscr{D}(\underline{\pi}^{-1}(U_\alpha))$ consisting of θ-invariant vector fields. Just as in the classical case we may, for any principal connection $\mathscr{D}$, define a **form of connection** $A \in \Omega^1_{\mathcal{P}}(P, \mathfrak{g})$ by

$$A|_{\mathscr{D}} = 0 \qquad \text{and} \qquad A(\# t_\mu) = t_\mu, \tag{4.110}$$

where $(t_1, \ldots, t_\ell)$ is some basis of the Lie algebra $\mathfrak{g}$.

Note that in writing $X \otimes 1 \sim_\theta X$ in the above definition we consider X as the global vector fields on the graded manifold $\mathcal{P}|_{\underline{\pi}^{-1}(U)}$, and the right action θ restricted to $\theta : \mathcal{P}|_{\underline{\pi}^{-1}(U)} \times \mathcal{G} \to \mathcal{P}|_{\underline{\pi}^{-1}(U)}$. As a side note we may point out that the Maurer-Cartan form ω_{MC} from Example 4.10 is a form of connection on the trivial principal $\mathcal{G}$-bundle $\mathcal{P} = \{*\} \times \mathcal{G}$, where $\theta = \mu$ taken as a right action with the principal connection $\mathscr{D} = 0$.

Proposition 4.17. *The definition of a principal connection in Definition 4.16 reduces to the standard one in the trivially graded case.*

Proof. It can be seen, using similar arguments as in the proof of Proposition 3.5, that in the trivially graded case $X \otimes 1 \sim_\theta X$ if and only if for any $g \in G$ there is $\theta_{g,*}X = X$ hence our definition of θ-invariant vector fields reduces to the usual one.

Consider now an ordinary principal G-bundle $\pi : P \to M$ with right the action of G on P denoted as θ. By the standard definition of a principal connection $\mathscr{D}$ we mean that $\mathscr{D}$ is a smooth distribution on P such that $\mathscr{X}_M = \mathrm{Ver}_P \oplus \mathscr{D}$ and $\theta_g \mathscr{D}_p = \mathscr{D}_{p\cdot g}$ for every $g \in G$ and $p \in P$. If $\mathscr{D}$ is a principal connection by the standard definition, then the horizontal lifts $(f_1^h, \dots, f_n^h)$ of any any local frame $(f_1, \dots, f_n)$ for $\mathscr{X}_M(U)$ form a θ-invariant frame for $\mathscr{D}(\underline{\pi}^{-1}(U))$.

On the other hand, let $\mathscr{D}$ be a distribution on P which satisfies $\mathscr{X}_P = \mathrm{Ver}_P \oplus \mathscr{D}$, and let $U \in \mathrm{Op}(M)$ be an open set such that there exists a frame $(X_1, \dots, X_n)$ for $\mathscr{D}(\underline{\pi}^{-1}(U))$ made up of θ-invariant vector fields. Then for every $p \in P$, $g \in G$ and $v = v^i X_i|_p \in \mathrm{T}_p P$ we have

$$(\mathrm{T}_p\theta_g)\, v = v^i\, (\mathrm{T}_p\theta_g)\, X|_p = v^i (\theta_{g,*}X_i)|_{p\cdot g} = v^i X_i|_{p\cdot g} \in \mathscr{D}_{p\cdot g}, \tag{4.111}$$

as desired. ■

Remark 4.18 (Horizontal and Vertical Projectors)**.** It follows immediately from the definition that for any fiber bundle $\mathcal{B}$ with a horizontal distribution $\mathscr{D}$ and for any $V \in \mathrm{Op}(B)$ we have a **horizontal projector** i.e. a graded $C^\infty_{\mathcal{B}}(V)$-linear map of degree zero

$$\mathrm{Hor}_V : \mathscr{X}_{\mathcal{B}}(V) \to \mathscr{D}(V) \tag{4.112}$$

which assigns to any vector field its horizontal part — the part lying in $\mathscr{D}$. It is easy to see that the collection $\{\mathrm{Hor}_V\}_{V\in\mathrm{Op}(B)}$ behaves naturally with respect to restrictions and so comprise a morphism of sheaves of $C^\infty_{\mathcal{B}}$-modules $\mathrm{Hor} : \mathscr{X}_{\mathcal{B}} \to \mathscr{D}$. In much the same way we have the **vertical projector** which we shall denote as $\mathrm{Ver} : \mathscr{X}_{\mathcal{B}} \to \mathrm{Ver}_{\mathcal{B}}$.

Proposition 4.19 (Horizontal Lift)**.** *Let $\pi : \mathcal{B} \to \mathcal{M}$ be a fiber bundle bundle with a horizontal distribution $\mathscr{D}$ and consider some $U \in \mathrm{Op}(M)$. Then for any $X \in \mathscr{X}_{\mathcal{M}}(U)$ there exists a unique horizontal vector field $X^h \in \mathscr{D}(\underline{\pi}^{-1}(U))$ such that $X^h \sim_\pi X$. We call X^h the **horizontal lift** of X.*

Furthermore, if $\mathcal{B}$ is a principal $\mathcal{G}$-bundle with the right action θ and $\mathscr{D}$ is a principal connection, then every horizontal lift of a vector field is θ-invariant.

Proof. Consider some trivializing open cover $\{U_\alpha\}_{\alpha\in I}$ of M with local trivializations $\phi_\alpha : \mathcal{M}|_{U_\alpha} \times \mathcal{G} \to \mathcal{B}|_{\underline{\pi}^{-1}(U_\alpha)}$. Let us first assume that $U \subseteq U_\alpha$ for some $\alpha \in I$. In this case we have $X \otimes 1 \sim_{p_1} X$ and therefore

$$(\phi_{\alpha,*}\,(X \otimes 1)) \circ \pi^* = (\phi_\alpha^{-1})^* \circ X \otimes 1 \circ \phi_\alpha^* \circ \pi^* = (\phi_\alpha^{-1})^* \circ X \otimes 1 \circ p_1^* = (\phi_\alpha^{-1})^* \circ p_1^* \circ X = \pi^* \circ X, \tag{4.113}$$

i.e. $\phi_{\alpha,*}(X \otimes 1) \sim_\pi X$. Denote $Y := \phi_{\alpha,*}(X \otimes 1)$. As with all vector fields on $\mathcal{B}$ we can decompose it into its vertical and horizontal parts $Y = \mathrm{Ver}Y + \mathrm{Hor}\,Y$, then

$$\pi^* \circ X = Y \circ \pi^* = \mathrm{Ver}Y \circ \pi^* + \mathrm{Hor}\,Y \circ \pi^* = \mathrm{Hor}\,Y \circ \pi^*, \tag{4.114}$$

which means that also $\mathrm{Hor}\,Y \sim_\pi X$. Hence let us choose

$$X^h := \mathrm{Hor}\;\phi_{\alpha,*}(X \otimes 1) \in \mathscr{D}(\underline{\pi}^{-1}(U)). \tag{4.115}$$

Now assume that $U \in \mathrm{Op}(M)$ was arbitrary. Then by the above construction we have, for every α, the vector field $X^h_\alpha := \mathrm{Hor}\ \phi_{\alpha,*}(X|_{U\cap U_\alpha} \otimes 1)$ which is π-related to $X|_{U\cap U_\alpha}$. Let us show that X^h_α agree on overlaps: denote $V_{\alpha\beta} := U \cap U_{\alpha\beta}$. Similarly as in (4.114), we see that

$$\big(\phi_{\alpha,*}(X|_{V_{\alpha\beta}} \otimes 1) - \phi_{\beta,*}(X|_{V_{\alpha\beta}} \otimes 1)\big) \circ \pi^* = \pi^* \circ (X|_{V_{\alpha\beta}} - X|_{V_{\alpha\beta}}) = 0, \tag{4.116}$$

which means that

$$\phi_{\alpha,*}(X|_{V_{\alpha\beta}} \otimes 1) - \phi_{\beta,*}(X|_{V_{\alpha\beta}} \otimes 1) \in \mathrm{Ver}_{\mathcal{B}}(\underline{\pi}^{-1}(V_{\alpha\beta})) \tag{4.117}$$

and therefore $X^h_\alpha|_{\underline{\pi}^{-1}(V_{\alpha\beta})} = X^h_\beta|_{\underline{\pi}^{-1}(V_{\alpha\beta})}$. Consequently $\{X^h_\alpha\}_{\alpha\in I}$ glue together a horizontal vector field $X^h \in \mathscr{D}(\underline{\pi}^{-1}(U))$ such that $X^h \sim_\pi X$. Indeed, for any $f \in C^\infty_{\mathcal{M}}(U)$ there is

$$\Big((X^h \circ \pi^*)f\Big)\Big|_{U\cap U_\alpha} = \Big(X^h|_{\underline{\pi}^{-1}(U\cap U_\alpha)} \circ \pi^*_{U\cap U_\alpha}\Big)\, f|_{U\cap U_\alpha} = \Big(X^h_\alpha \circ \pi^*_{U\cap U_\alpha}\Big)\, f|_{U\cap U_\alpha} \tag{4.118}$$

$$= \big(\pi^*_{U\cap U_\alpha} \circ X|_{U\cap U_\alpha}\big)\, f|_{U\cap U_\alpha} = ((\pi^* \circ X)f)|_{U\cap U_\alpha}, \tag{4.119}$$

for any α, hence $X^h \circ \pi^* = \pi^* \circ X$.

As for uniqueness of X^h, let $Y \in \mathscr{D}(\underline{\pi}^{-1}(U))$ be another vector field such that $Y \sim_\pi X$. Then $(X^h - Y) \circ \pi^* = \pi^* \circ (X - X) = 0$, which means that $X^h - Y \in \mathrm{Ver}_{\mathcal{B}}(\underline{\pi}^{-1}(U))$. But since $X^h - Y$ is a horizontal vector field, necessarily $X^h - Y = 0$.

Next, assume that $\pi : \mathcal{B} \to \mathcal{M}$ is a principal $\mathcal{G}$-bundle. We need to show that X^h is θ-invariant. Without loss of generality we may assume that for every α there is a frame for $\mathscr{D}(\underline{\pi}^{-1}(U_\alpha))$ consisting of θ-invariant vector fields. Since X^h is θ-invariant if and only if $X^h|_{\underline{\pi}^{-1}(U\cap U_\alpha)}$ is θ-invariant for every α, we may further assume that $U \subseteq U_\alpha$ for some α and also that U is a coordinate patch on $\mathcal{M}$. Denote the θ-invariant frame for $\mathscr{D}(\underline{\pi}^{-1}(U))$ as $(D_1, \ldots, D_n)$. We know that $X^h = f^j D_j$ for some $f^j \in C^\infty_{\mathcal{B}}(\underline{\pi}^{-1}(U))$ and so we can write

$$\theta^* \circ X^h - \Big(X^h \otimes 1\Big) \circ \theta^* = \theta^*\left(f^j\right) \cdot (\theta^* \circ D_j) - p_1^*\left(f^j\right) \cdot ((D_j \otimes 1) \circ \theta^*)$$

$$= \left(\theta^*\left(f^j\right) - p_1^*\left(f^j\right)\right) \cdot (\theta^* \circ D_j). \tag{4.120}$$

On the other hand, from the fact that $X^h \sim_\pi X$ we find

$$\theta^* \circ X^h \circ \pi^* = \theta^* \circ \pi^* \circ X = p_1^* \circ \pi^* \circ X = p_1^*\, X^h \circ \pi^* = X^h \otimes 1 \circ p_1^* \circ \pi^*$$

$$= \Big(X^h \otimes 1\Big) \circ \theta^* \circ \pi^*, \tag{4.121}$$

which implies

$$\Big(\Big(X^h \otimes 1\Big) \circ \theta^* - \theta^* \circ X^h\Big) \circ \pi^* = \left(\theta^*\left(f^j\right) - p_1^*\left(f^j\right)\right) \cdot (\theta^* \circ D_j \circ \pi^*) = 0. \tag{4.122}$$

We will show that for any $g^j \in C^\infty_{\mathcal{B}\times\mathcal{G}}(\underline{\pi}^{-1}(U) \times G)$ there is

$$g^j \cdot (\theta^* \circ D_j \circ \pi^*) = 0 \qquad \Longrightarrow \qquad \forall j,\ g^j = 0. \tag{4.123}$$

Since U is assumed to be a trivializing open set with with coordinates $\{x^i\}$ on $\mathcal{M}|_U$, we can consider a frame $\{\frac{\partial}{\partial x^i}\}_{i=1}^n \cup \{\#t_\mu\}_{\mu=1}^\ell$ for $\mathscr{X}_{\mathcal{B}}(\underline{\pi}^{-1}(U))$. As such, for every j there is $D_j = {D_j}^i \frac{\partial}{\partial x^i} + {D_j}^\mu \#t_\mu$. Since $\{D_i\}_{i=1}^n \cup \{\#t_\mu\}_{\mu=1}^\ell$ also form a frame for $\mathscr{X}_{\mathcal{B}}(\underline{\pi}^{-1}(U))$ we may write, for any j, $\frac{\partial}{\partial x^j} = {\alpha_j}^k D_k + {\alpha_j}^\mu \#t_\mu$ for some unique ${\alpha_j}^k, {\alpha_j}^\mu \in C^\infty_{\mathcal{P}}(\underline{\pi}^{-1}(U))$. Put together these decompositions yield

$$D_j = {D_j}^k \frac{\partial}{\partial x^k} + {D_j}^\mu \#t_\mu = {D_j}^k {\alpha_k}^i D_i + \Big({D_j}^k {\alpha_k}^\mu + {D_j}^\mu\Big) \#t_\mu \tag{4.124}$$

which in particular means that $D_j{}^k \alpha_k{}^i = \delta_j{}^i$ for every i, j. By assumption there is, for every k,

$$0 = g^j \cdot (\theta^* \circ D_j \circ \pi^*) \, x^k = g^j \, \theta^* \left(D_j{}^k \right), \tag{4.125}$$

which one may multiply by $\theta^*(\alpha_k{}^i)$ from the right to obtain

$$0 = g^j \, \theta^* \left(D_j{}^k \alpha_k{}^i \right) = g^j \, \theta^* \left(\delta_j{}^i \right) = g^i, \tag{4.126}$$

for every i, as desired. It follows that X^h is θ-invariant, since (4.120) is now seen to be equal to zero. ■

Let $\pi : \mathcal{B} \to \mathcal{M}$ be a fiber bundle with a horizontal connection $\mathscr{D}$ and let V be some finite-dimensional graded vector space. For any $U \in \mathrm{Op}(P)$ and any $\omega \in \Omega^p_{\mathcal{B}}(U, V)$ we may define the **horizontal part of** ω similarly as in the non-graded case via

$$(\mathrm{Hor}\, \omega)(X_1, \ldots, X_p) := \omega \left(\mathrm{Hor}\, X_1, \ldots, \mathrm{Hor}\, X_p \right), \tag{4.127}$$

for any $X_1, \ldots, X_p \in \mathscr{X}_{\mathcal{P}}(U)$. We say that ω **is horizontal** if $\omega = \mathrm{Hor}\, \omega$. The graded $C^\infty_{\mathcal{B}}(U)$-module of all horizontal p-forms is denoted as $\Omega^p_{\mathcal{B},\mathrm{hor}}(U, V) \subseteq \Omega^p_{\mathcal{B}}(U, V)$. From the fact that the horizontal projection is a morphism of sheaves of $C^\infty_{\mathcal{B}}$-modules $\mathrm{Hor} : \mathscr{X}_{\mathcal{B}} \to \mathscr{D}$ we can see that the assignment $\Omega^p_{\mathcal{B},\mathrm{hor}} : U \mapsto \Omega^p_{\mathcal{B},\mathrm{hor}}(U, V)$ is also a sheaf of $C^\infty_{\mathcal{B}}$-modules and a subsheaf of $\Omega^p_{\mathcal{B}}$. Furthermore, we may consider the assignment $\omega \mapsto \mathrm{Hor}\, \omega$ as a morphism of sheaves of $C^\infty_{\mathcal{B}}$-modules

$$\mathrm{Hor} \, : \Omega^p_{\mathcal{P}} \to \Omega^p_{\mathcal{B},\mathrm{hor}}, \tag{4.128}$$

which we also call the horizontal projector. With it in hand, we may define the **exterior covariant derivative** $\mathrm{D} : \Omega^p_{\mathcal{B}}(U, V) \to \Omega^{p+1}_{\mathcal{B},\mathrm{hor}}(U, V)$ by

$$\mathrm{D} := \mathrm{Hor} \, \circ d. \tag{4.129}$$

This leads to the following definition of the form of curvature.

Definition 4.20 (Form of Curvature)**.** Let $\mathcal{P}$ be a graded principal $\mathcal{G}$-bundle equipped with a principal connection $\mathscr{D}$ and denote as $A \in \Omega^1_{\mathcal{P}}(P, \mathfrak{g})$ the associated form of connection. We then say that

$$\Omega := \mathrm{D}A \ \in \Omega^2_{\mathcal{P},\mathrm{hor}}(P, \mathfrak{g}) \tag{4.130}$$

is the **form of curvature** associated to $\mathscr{D}$. We say that the connection $\mathscr{D}$ is **flat** if $\Omega = 0$.

Proposition 4.21. *Let $\pi : \mathcal{P} \to \mathcal{M}$ be a principal $\mathcal{G}$-bundle equipped with a principal connection $\mathscr{D}$. Then $\mathscr{D}$ is involutive if and only if it is flat.*

Proof. Let $A = t_\mu A^\mu$ be the form of connection on $\mathcal{P}$. The proof is a straightforward application of the Cartan relations [11, Theorem 2.8]. Consider some $X, Y \in \mathscr{X}_{\mathcal{P}}(P)$, then

$$\begin{aligned} A\left([\mathrm{Hor}\, X, \mathrm{Hor}\, Y]\right) &= i_{[\mathrm{Hor}\, X, \mathrm{Hor}\, Y]} A = -(-1)^{|X|(|Y|+1)} i_{\mathrm{Hor}\, Y} \mathcal{L}_{\mathrm{Hor}\, X} A \\ &= -(-1)^{|X|(|Y|+1)} i_{\mathrm{Hor}\, Y} i_{\mathrm{Hor}\, X} dA = -(-1)^{|X|(|Y|+1)} i_{\mathrm{Hor}\, Y} dA\left(\mathrm{Hor}\, X, \cdot\right) \\ &= -dA\left(\mathrm{Hor}\, X, \mathrm{Hor}\, Y\right) = -\mathrm{D}A(X, Y). \end{aligned} \tag{4.131}$$

The conclusion follows immediately. ■

Example 4.22 (Connections on the Trivial Bundle)**.** Consider the trivial principal $\mathcal{G}$-bundle $\mathcal{P} = \mathcal{M} \times \mathcal{G}$ with $\pi = p_1$ and $\theta = 1 \times \mu$. Let $(t_\mu)_{\mu=1}^{\ell}$ be a basis for $\mathrm{T}_e\mathcal{G}$. The fundamental vector fields $\# t_\mu$ are simply

$$\# t_\mu = 1 \otimes t_\mu^L, \tag{4.132}$$

for any μ. Note that since both left-invariant and right-invariant vector fields generate $\mathscr{X}_\mathcal{G}$, we have

$$1 \otimes t_\mu^L = \left(1 \otimes t_\nu^R\right) \cdot R^\nu{}_\mu \qquad \text{and} \qquad 1 \otimes t_\mu^R = \left(1 \otimes t_\nu^L\right) \cdot S^\nu{}_\mu, \tag{4.133}$$

for some graded functions $R^\nu{}_\mu,\ S^\nu{}_\mu \in C^\infty_\mathcal{P}(P)$ which satisfy

$$S^\varkappa{}_\nu R^\nu{}_\mu = \delta^\varkappa{}_\mu = R^\varkappa{}_\nu S^\nu{}_\mu. \tag{4.134}$$

Now, consider some principal connection $\mathscr{D}$ on $\mathcal{P}$. Without loss of generality we may consider some open cover $\{U_\alpha\}_{\alpha \in I}$ of M by coordinate patches such that for every α there exists a frame for $\mathscr{D}(U \times G)$ made up of θ-invariant vector fields. Let $U \in \{U_\alpha\}_{\alpha \in I}$ be one such set and denote the θ-invariant frame for $\mathscr{D}(U \times G)$ as $(D_1, \ldots, D_n)$. We can now consider four frames for $\mathscr{D}(U \times G)$ made up of either $\{D_i\}$ or $\frac{\partial}{\partial x^i}$ and $\{1 \otimes t_\mu^R\}$ or $\{1 \otimes t_\mu^L\}$. In particular, we have

$$D_i = \frac{\partial}{\partial x^k} \cdot D^k{}_i + \left(1 \otimes t_\mu^L\right) \cdot D^\mu{}_i, \tag{4.135}$$

$$\frac{\partial}{\partial x^k} = D_i \cdot \alpha^i{}_k + \left(1 \otimes t_\mu^L\right) \cdot \alpha^\mu{}_k, \tag{4.136}$$

for some unique functions $D^k{}_i, D^\mu{}_i, \alpha^i{}_k, \alpha^\mu{}_k \in C^\infty_\mathcal{P}(U \times G)$. Using these transformations twice in a row we find that

$$D^i{}_k \alpha^k{}_j = \delta^i{}_j \qquad \text{and} \qquad D^\mu{}_k \alpha^k{}_j = -\alpha^\mu{}_j. \tag{4.137}$$

Let now $A \in \Omega^1(\mathcal{P}, \mathfrak{g})$ be the form of connection corresponding to $\mathscr{D}$, that is, defined by $A(D_i) = 0$ and $A(1 \otimes t_\mu^L) = t_\mu$. Locally on U we have $A = t_\mu A^\mu$ where $A^\mu = A^\mu{}_k\, dx^k + A^\mu{}_\nu (\# t)^\nu$ for some unique graded functions $A^\mu{}_k, A^\mu{}_\nu \in C^\infty_\mathcal{P}(U \times G)$ and where $(\# t)^\mu$ denotes the dual section to $\# t_\mu \equiv 1 \otimes t_\mu^L$. From the definition of A we have

$$\delta^\mu{}_\nu = A^\mu (1 \otimes t_\nu^L) = A^\mu{}_\nu \tag{4.138}$$

and

$$0 = A^\mu(D_i) = A^\mu \left(\frac{\partial}{\partial x^k} D^k{}_i + \left(1 \otimes t_\nu^L\right) D^\nu{}_i, \right) = A^\mu{}_k D^k{}_i + D^\mu{}_i. \tag{4.139}$$

We may multiply the last by $\alpha^i{}_j$ from the right to obtain

$$0 = A^\mu{}_k D^k{}_i \alpha^i{}_j + D^\mu{}_i \alpha^i{}_j = A^\mu{}_j - \alpha^\mu{}_j, \tag{4.140}$$

through the use of (4.137). We see that for our particular choice of frames the local shape of a form of connection associated to any horizontal distribution $\mathscr{D}$ on $\mathcal{M} \times \mathcal{G}$ is very simple:

$$A^\mu = \alpha^\mu{}_j\, dx^j + (\# t)^\mu, \tag{4.141}$$

where the graded functions $\alpha^\mu{}_j$ come from (4.136). But how does the fact that D_i are θ-invariant manifest on A? First note that also $\frac{\partial}{\partial x^k}$ and $(1 \otimes t_\mu^R)$ are θ-invariant. Indeed, recall that with our (understandable) abuse of notation we actually have $\frac{\partial}{\partial x^k} \equiv \frac{\partial}{\partial x^k} \otimes 1 \in \mathscr{X}_{\mathcal{P}}(U \times G)$ which is $(1 \times \mu)$-invariant due to Lemma 3.2, and $1 \otimes t^R_\mu$ is $(1 \times \mu)$-invariant since $t^R_\mu \otimes 1 \sim_\mu t^R_\mu$ by definition of right-invariance. As a result,

$$\begin{aligned}(1 \times \mu)^* \circ \frac{\partial}{\partial x^k} &= (1 \times \mu)^* \circ \left(D_i \cdot \alpha^i{}_k + \left(1 \otimes t^R_\nu\right) \cdot R^\nu{}_\mu \alpha^\mu{}_k\right) \\ &= \left((1 \times \mu)^* \circ D_i\right) \cdot (1 \times \mu)^* \left(\alpha^i{}_k\right) + \left((1 \times \mu)^* \circ (1 \otimes t^R_\mu)\right) \cdot (1 \times \mu)^* \left(R^\nu{}_\mu \alpha^\mu{}_k\right) \quad (4.142)\end{aligned}$$

is equal to

$$\left(\frac{\partial}{\partial x^k} \otimes 1\right) \circ (1 \times \mu)^* = \left((D_i \otimes 1) \cdot p_1^* \left(\alpha^i{}_k\right) + \left(1 \otimes t^R_\nu \otimes 1\right) \cdot p_1^* \left(R^\nu{}_\mu \alpha^\mu{}_k\right)\right) \circ (1 \times \mu)^* \quad (4.143)$$

$$= \left((1 \times \mu)^* \circ D_i\right) \cdot p_1^* \left(\alpha^i{}_k\right) + \left((1 \times \mu)^* \circ (1 \otimes t^R_\nu)\right) \cdot p_1^* \left(R^\nu{}_\mu \alpha^\mu{}_k\right). \quad (4.144)$$

Which implies in particular that

$$p_1^* \left(R^\nu{}_\mu A^\mu{}_k\right) = (1 \times \mu)^* \left(R^\nu{}_\mu A^\mu{}_k\right), \quad (4.145)$$

for any $\mu \in \{1, \dots, \ell\}$ and $k \in \{1, \dots, n\}$, where we used that $A^\mu{}_k = \alpha^\mu{}_k$ from (4.140).

On the other hand, let now $A = t_\mu A^\mu$ be some 1-form on $\mathcal{P}$ valued in $\mathfrak{g}$ and suppose that for any coordinate patch U on $\mathcal{M}$ with coordinates $\{x^i\}$ the component 1-forms $A^\mu \in \Omega^1_{\mathcal{P}}(U \times G)$ are of the shape

$$A^\mu = A^\mu{}_k \, dx^k + (\#t)^\mu, \quad (4.146)$$

where $A^\mu{}_k$ satisfy the relation (4.145). We will show that $\mathscr{D} := \ker A$ is a principal connection on $\mathcal{P}$ by constructing a frame for $\mathscr{D}(U)$. For every k denote

$$X_k := \frac{\partial}{\partial x^k} - \left(1 \otimes t^L_\mu\right) \cdot A^\mu{}_k \equiv \frac{\partial}{\partial x^k} - \left(1 \otimes t^R_\mu\right) \cdot R^\mu{}_\nu A^\nu{}_k. \quad (4.147)$$

It is clear that $(X_1, \dots, X_n, 1 \otimes t^L_1, \dots, 1 \otimes t^L_\ell)$ is a frame for $\mathscr{X}_{\mathcal{P}}(U \times G)$ and we have

$$A^\mu(X_k) = A^\mu{}_k - A^\mu{}_k = 0, \quad (4.148)$$

which together with the fact that $A^\mu(1 \otimes t^L_\mu) = t_\mu$ means that $\{X_i\}$ form a frame for $\mathscr{D}(U)$. Finally, we find that

$$\begin{aligned}(1 \times \mu)^* \circ X_i &= (1 \times \mu)^* \circ \left(\frac{\partial}{\partial x^k} - \left(1 \otimes t^R_\mu\right) \cdot R^\mu{}_\nu A^\nu{}_k\right) \\ &= \left(\frac{\partial}{\partial x^k} \otimes 1\right) \circ (1 \times \mu)^* - \left(\left(1 \otimes t^R_\mu \otimes 1\right) \circ (1 \times \mu)^*\right) \cdot (1 \times \mu)^* \left(R^\mu{}_\nu A^\nu{}_k\right) \\ &= \left(\frac{\partial}{\partial x^k} \otimes 1\right) \circ (1 \times \mu)^* - \left(\left(1 \otimes t^R_\mu \otimes 1\right) \circ (1 \times \mu)^*\right) \cdot p_1^* \left(R^\mu{}_\nu A^\nu{}_k\right) \\ &= \left(\left(\frac{\partial}{\partial x^k} - (1 \otimes t^R_\mu) \cdot R^\mu{}_\nu A^\nu{}_k\right) \otimes 1\right) \circ (1 \times \mu)^* \\ &= (X_i \otimes 1) \circ (1 \times \mu)^*, \end{aligned} \quad (4.149)$$

where between the second and the third equality we used (4.145). We have shown that on the trivial principal $\mathcal{G}$-bundle $\mathcal{P} = \mathcal{M} \times \mathcal{G}$ there is a one-to-one correspondence between principal connections $\mathscr{D}$ and 1-forms $A = t_\mu A^\mu \in \Omega^1_{\mathcal{P}}(P, \mathfrak{g})$ locally of the shape (4.146) which satisfy (4.145). Of course, the most straightforward choice is $A = t_\mu (\#t)^\mu$ corresponding in the terms of vector bundles to $\mathscr{D} = p_1^*(\mathrm{T}\mathcal{M})$, see [10, Proposition 5.15]. In other words, the local frame for this particular $\mathscr{D}$ is simply $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$. Apparently for this choice there is $\Omega = \mathrm{D}A = 0$, i.e. the connection is flat.

Conclusion

In the first chapter of the text we managed to use the Yoneda embedding to learn something about group objects and their actions in any locally small category with finite products and a terminal object. The main results of this chapter are Corollary 1.26 and Corollary 1.28, which tell that us some useful basic facts about classical groups and their actions generalize to a broad range of categories. We may also mention Corollary 1.29 which may be potentially useful in connection with the notion of a subgroup (object). In particular, it may possibly be used to verify when a graded manifold is a Lie subgroup of some Lie group, if and when Lie subgroups are defined.

The main body of the second chapter was devoted to the construction of the graded general linear group $\mathrm{GL}((n_j),\mathbb{R})$. Using our knowledge of matrices of graded linear maps we found global coordinates in which the multiplication arrow (2.47) could be defined in a very simple manner. We found that $\mathrm{GL}((n_j),\mathbb{R})$ reduces to the ordinary general linear group $\mathrm{GL}(n,\mathbb{R})$ whenever the sequence (n_j) of non-negative integers has no more than one non-zero entry. Otherwise the pullback of coordinates by the multiplication arrow has a non-trivial purely graded part (2.53). We also found the explicit form of the inversion arrow (2.71) and (2.82). In particular, the pullback of coordinates by the inversion arrow is generally a rather non-trivial formal power series. We also investigated the canonical actions of $\mathrm{GL}((n_j),\mathbb{R})$ on itself and on the graded domain $\mathrm{g}\mathbb{R}^{(n_j)}$.

We provided two other examples of graded Lie groups: the first was a simple example where we defined a multiplication arrow on a graded domain using vector addition on the corresponding graded vector space. In the second example we encountered a graded Lie group whose underlying manifold was a discrete group. Specifically, we investigated all possible graded Lie group structures on a graded manifold of graded dimension $(\dots,0,0,1,\dots)$ whose underlying Lie group is the two-point group $\mathbb{Z}_2$, (2.133). We believe this example nicely illustrates how the underlying Lie group manifests in the structure of a graded Lie group.

In the third chapter, inspired by the definition of fundamental vector fields in [2], we found an equivalent definition of left-invariant vector fields on a Lie group that did not involve any integral curves or tangent vectors and we used it to define left-invariant vector fields on graded Lie groups. The main body of this chapter was devoted to their examination, including examples on $\mathrm{GL}((n_j),\mathbb{R})$ for a general (n_j) in Example 3.14 and for $(n_j) = (\dots,0,1,1,\dots)$ in Example 3.15. The main result is probably the correspondence between tangent vectors at the unit and left-invariant vector fields in Theorem 3.16. In Example 3.17 we used this correspondence to find that the Lie algebra of left-invariant vector fields for $\mathrm{GL}((n_j),\mathbb{R})$ is isomorphic to the Lie algebra of graded $(n_j)\times(n_j)$ matrices. Similarly as for left-invariant vector fields, we found an alternative definition of fundamental vector fields suitable for uplift to the graded setting and in Proposition 3.22 we found the graded infinitesimal generator map.

In the fourth and last chapter we began by the definition of a graded principal bundle. This required to know what it means for a graded Lie group action to act "freely and transitively along fibers", which we found once more by choosing a classical definition which does not make use of points of the smooth manifolds. A large portion of this chapter is taken up by the construction of the graded frame bundle $\mathcal{F}(\mathcal{M})$, which is a graded principal $\mathrm{GL}((n_j), \mathbb{R})$-bundle over any graded manifold with graded dimension (n_{-j}). In Proposition 4.8 we learned that the name is justified even in the graded setting as local sections of the graded frame bundle correspond to local frames for graded vector fields on $\mathcal{M}$.

We defined vertical vector fields for any fiber bundle similarly as in the ordinary setting and learned that they form a distribution. In Proposition 4.15 we found that for any principal bundle, the vertical distribution is generated by fundamental vector fields. In Definition 4.16 we defined a horizontal distribution as any complement to the vertical distribution and in Proposition 4.19 we learned how to perform horizontal lifts of vector fields. We introduced principal connections on principal bundles as horizontal distribution locally generated by invariant vector fields and used the horizontal distribution to define the form of connection, the exterior covariant derivative and the form of curvature. Finally, we examined these objects in the case of a trivial principal bundle in Example 4.22.

Our study of graded Lie theory was not complete by far: for one, we lack the graded Lie group—Lie algebra correspondence. We also do not have the graded Frobenius theorem and so we do not know whether involutive distributions coincide with integral ones. We are also missing the definition of a representation of a graded Lie group and in particular the definition of the adjoint representation. Until that is remedied, our description of graded forms of curvature will likely be very limited. We leave these questions for future investigation.

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