CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

Spektrální stabilita relativistické kvantové částice na polopřímce

Spectral stability of a relativistic quantum particle on a half-line

Master's thesis

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Abstrakt: Uvažujeme jednorozměrný samosdružený jednoparametrický Diracův operátor $\mathcal{D}_\alpha$ na polopřímce s relativistickou hraniční podmínkou Robinova typu porušený operátorem násobení generovaným esenciálně omezenou L^1-maticovou funkcí. V závislosti na reálném parametru α odvodíme resolventu operátoru $\mathcal{D}_\alpha$ v explicitním tvaru. Pro danou poruchu sestrojíme Birman-Schwingerův operátor a pomocí principu Birmana-Schwingera odvodíme postačující podmínku stability spektra porušeného operátoru a diskutujeme optimalitu získaných výsledků.

Klíčová slova: Diracův operátor, stabilita spektra, polopřímka, Robinovská hraniční podmínka, princip Birmana-Schwingera

Title:

Spectral stability of a relativistic quantum particle on a half-line

Author: Bc. David Kramár

Abstract: We consider a one-dimensional self-adjoint one-parametric Dirac operator $\mathcal{D}_\alpha$ on a half-line perturbed by a multiplication operator generated by an essentially bounded L^1-matrix-valued function. Depending on a real parameter α we derive the resolvent of the operator $\mathcal{D}_\alpha$ in explicit form. For a given perturbation we construct the Birman-Schwinger operator and, using the Birman-Schwinger principle, we derive the sufficient condition on the stability of the spectrum of the perturbed operator and discuss the optimality of the obtained results.

Key words: Dirac operator, stability of a spectrum, half-line, Robin-type boundary condition, Birman-Schwinger principle

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Introduction

This thesis is devoted to the study of a half-line Dirac operator $\mathcal{D}_\alpha$ subjected to the relativistic Robin-type boundary conditions at zero perturbed by a (generally non-self-adjoint) matrix-valued potential. This particular operator was recently studied in [9] from a different point of view. The author set estimates for non-embedded eigenvalues of $\mathcal{D}_\alpha$ perturbed by a matrix-valued potential (not necessarily self-adjoint)

In general, the study of half-line Dirac operators is important since they represent the radial part of the 3D spherically symmetrical Dirac equation [44, 39].

Our research is a continuation of a study of the fundamental properties of Schrödinger and Dirac operators. It is well known that the spectrum of a Laplacian defined over the Euclidean space is stable under „small„ perturbations whenever the dimension of the space is three or more. If the perturbation is a real-valued function, the „smallness„ can be quantified through the famous Hardy inequality. For complex-valued potentials, the problem is much more delicate and more technical tools must be used [22]. The key tool available in both self-adjoint and non-self-adjoint settings is the famous Birman-Schwinger principle, which has recently become very popular; see [10, 25, 19, 11, 22, 37, 23, 7]. This property is usually refered to as the so-called subcriticality of the Laplacian in dimensions three or more. On the other hand, in dimensions one and two the Laplacian is so-called critial - every negative perturbation gives an eigenvalue under the threshold of the essential spectrum. However, in dimension one there is a Hardy inequality if one considers a half-line instead [29, G. H. Hardy 1920]. The half-line Schrödinger operators and their spectral properties were studied quite recently, let us mention [24] where the author established a sharp bound on eigenvalues of half-line Schrödinger operators subjected to the Dirichlet boundary condition with complex-valued potentials. This result was further extended by Enblom [18]. Nevertheless, none of the mentioned papers contained the explicit condition of the spectral stability. In 2022 Krejčiřík, Laptev and Štampach [37] studied discrete non-self-adjoint Schrödinger operators on a half-line and proved their spectral stability. Furthermore, the authors then compared their result with its continuous analogue [37, Remark 21] and established a sufficient condition for the stability of the spectrum of a half-line Laplacian subjected to Robin-type boundary conditions at zero with a general complex-valued potential.

In recent years there has been growing attention towards spectral properties of Dirac operators [11, 20, 14, 17, 41, 9, 11, 10]. However, the fundamental question regarding the stability of its spectrum remained unaswered for a long period of time. There were several partial results [21, 11, 20, 9] but the final solution has been found by D'Ancona, Fanelli, Krejčiřík and

Schiavone in 2022 [13]. The authors proved that the spectrum of a Dirac operator defined over the Euclidean space is stable under small perturbations if the dimension is greater than two and conjectured that it is not possible in dimension two. Regarding dimension one, this was already studied in [11, 12] and partially answered; see [12, Theorem 2.2]. However, the criticality of the Dirac operator in dimension two is still open.

Our goal in this thesis is to find a stability theorem for the half-line Dirac operator corresponding to the results [37, Remark 21] and [13] made for half-line Laplacian and the Dirac operator in dimensions three or more, respectively.

The thesis is structured as follows. In the first chapter we give a review of mathematical tools used later in the chapters that follow. Second chapter is devoted to the brief historical background and development of the problem of spectral stability and we present the motivation of our study. In the third chapter we introduce and properly define the studied model $\mathcal{D}_\alpha$. Its analysis and main results of the thesis follow.

Chapter 1

Mathematical background

This chapter is devoted to an overview of mathematical tools used in the later chapters and to the unification of notation and nomenclature. Most of the following content is related to functional analysis and the theory of differential equations. The results presented in this chapter are the summary of the following list of literature [36, 43, 5, 34, 15, 8, 36, 16, 28] to which we refer for more details on the contet presented. Throughout the whole chapter, $\mathcal{H}$ will denote a separable complex Hilbert space. Furthermore, in the whole thesis the complex conjugation of a complex number will be denoted by a star, i.e. $(x + iy)^* := x - iy$ for $x, y \in \mathbb{R}$.

1.1 Sobolev spaces

Sobolev spaces were introduced in the 1930s by Sergei Sobolev, and they provide a natural framework for the study of differential operators on L^p spaces. Differential operators are typically unbounded operators, and therefore it is necessary to specify the domain on which they act. Sobolev spaces, sometimes also referred to as energetic spaces, are the building blocks when constructing the extensions of such operators.

Definition 1.1.1 (Multi-index)
Let $n \in \mathbb{N}$, $\alpha_1, \cdots, \alpha_n \in \mathbb{N}_0$. Then by a multi-index we understand n-tuple $\alpha := (\alpha_1, \cdots, \alpha_n)$. Furthermore, we define the size of the multi-index α as $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Using the multi-index we define the following notation for partial derivatives as

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Definition 1.1.2 (Weak derivative)
Let $\Omega \subset \mathbb{R}^n$ be an open connected set, $\alpha \in \mathbb{N}_0^n$ a multi-index and let $\psi \in L^1_{loc}(\Omega)$. We say that $\xi \in L^1_{loc}(\Omega)$ is the weak derivative of ψ if for all $\phi \in C_0^\infty(\Omega)$ the following holds

$$\int_\Omega \xi(x)\phi(x)dx := (-1)^{|\alpha|} \int_\Omega \psi(x) D^\alpha \phi(x)dx.$$

The set $C_0^\infty(\Omega)$ is called *space of test functions* and is defined as

$$C_0^\infty(\Omega) := \{\psi \in C^\infty(\Omega) \mid \operatorname{supp}\psi \subset \Omega \text{ is compact}\}.$$

In addition, if $\psi \in C^k(\Omega)$ then for $|\alpha| \leq k$ weak and classical derivative merge - from the Divergence theorem. To simplify the notation, we will denote the weak derivative of $\psi \in L^1_{loc}(\Omega)$ of order $\alpha \in \mathbb{N}_0^n$ as $D^\alpha\psi$. In cases when it is not clear which one is being considered, it will be emphasized.

Definition 1.1.3 (Sobolev spaces)
Let $\Omega \subset \mathbb{R}^n$ be an open connected set, $p \in [1,\infty]$ and $k \in \mathbb{N}_0$. Then by the Sobolev space $W^{k,p}(\Omega)$ we understand set of functions $\psi \in L^p(\Omega)$ such that for all $|\alpha| \leq k$ their weak derivatives $D^\alpha\psi$ lies in the space $L^p(\Omega)$, that is,

$$W^{k,p}(\Omega) := \{\psi \in L^p(\Omega) \mid \forall|\alpha| \leq k : D^\alpha\psi \in L^p(\Omega)\},$$

equipped with the following norm

$$\begin{aligned} \|\psi\|_{k,p}^p = \|\psi\|_{W^{k,p}(\Omega)}^p &:= \sum_{|\alpha|\leq k} \|D^\alpha\psi\|_{L^p(\Omega)}^p, \qquad \textit{for} \qquad p \in [1,\infty)\,, \\ \|\psi\|_{k,\infty} = \|\psi\|_{W^{k,\infty}(\Omega)}^p &:= \max_{|\alpha|\leq k} \|\psi\|_{L^\infty(\Omega)}, \qquad \textit{for} \qquad p = \infty. \end{aligned}$$

In the trivial case, for $k = 0$ we indetify $W^{0,p}(\Omega) = L^p(\Omega)$.

Theorem 1.1.1 ([43, Lemma 5.2])
Let $\Omega \subset \mathbb{R}^n$ be an open connected set.

i) For all $p \in [1,\infty]$ and $k \in \mathbb{N}_0$ the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.

ii) For $p = 2$ and all $k \in \mathbb{N}$ is $W^{k,2}(\Omega)$ a Hilbert space with the inner product defined for $\psi, \phi \in W^{k,2}(\Omega)$ as

$$(\psi,\phi)_k := \sum_{|\alpha|\leq k} \int_\Omega D^\alpha\psi^*(x) D^\alpha\phi(x)dx.$$

Theorem 1.1.2 (Meyers-Serrin [43])
Let $\Omega \subset \mathbb{R}^n$ be an open set, $p \in [1,\infty]$, $k \in \mathbb{N}_0$.
Then $W^{k,p}(\Omega) = \overline{C^\infty(\Omega)}$ with respect to the norm $\|\cdot\|_{k,p}$.

Inspired by the previous theorem 1.1.2 we define the notation $W_0^{k,p}(\Omega) := \overline{C_0^\infty(\Omega)}$ with respect to the norm $\|\cdot\|_{k,p}$.

Theorem 1.1.3 ([36])
For every $n \in \mathbb{N}$ holds $W_0^{1,2}(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$.

1.2 Operator theory

We will be mostly interested in unbounded operators. Manipulation with unbounded operators has certain specifics, therefore, let us make a brief summary of the theory of general linear operators on Hilbert spaces. For more details on this content see [34, 5, 34, 15, 16].

Under *Hilbert space* we will understand a complex vector space $\mathcal{H}$ which is complete with respect to the norm $\|\cdot\|$ induced by the inner product $(\cdot,\cdot)$ on $\mathcal{H}$. The latter will be assumed to be antilinear in its first argument. As we have already outlined, we will deal with closed unbounded operators and such operators necesarilly can not be defined everywhere on $\mathcal{H}$ from the closed graph theorem [5]. By a *linear operator* we will call a two-tuple $(H, \mathrm{dom}(H))$ where $\mathrm{dom}(H)$ is subspace of $\mathcal{H}$ called (operator) domain of H and $H : \mathrm{dom}(H) \to \mathcal{H}$ is a linear map. Further in the text, we will omit the adjective „linear„. If not specified otherwise; we will always understand „linear operator„ under the word „operator„.

Since for linear maps concepts of boundedness and continuity are merging, we have a discontinuous map once the operator we consider is unbounded. To assert a broad set of characteristics and control it, we propose a less stringent requirement than continuity. The suitable concept which can substitute continuity in a certain way is *closedness*. We will define it through the concept of a *graph of a linear map*. Although it would be possible to define the closedness itself, we will take advantage of the concept of a graph later.

Definition 1.2.1 (Graph of linear map)
Let X, Y be normed vector spaces, and H be a linear mapping from X to Y. Then by a graph of the linear mapping H we understand the set

$$G(H) := \{(x, Hx) \in X \times Y \mid x \in \mathrm{dom}(H)\}$$

Furthermore, we define a norm on $G(H)$ as $\|(x,y)\|_G^2 := \|x\|_X^2 + \|y\|_Y^2$.

Proposition 1.2.1
Let $G \subset X \times Y$ then G is a graph of a linear map if and only if

$$\forall\, (x,y) \in G : x = 0 \Rightarrow y = 0.$$

Definition 1.2.2 (Closed map)
We say that H, an operator on a space $\mathcal{H}$ is closed if $G(H)$ is a closed set with respect to the graph norm.

The alternative, much more practical, definition of a closed map is that for every sequence $(\psi_n)_{n=0}^{\infty} \subset \mathrm{dom}(H)$ holds the following

$$\left(\lim_{n\to\infty} \psi_n = \psi \in \mathcal{H} \wedge \lim_{n\to\infty} H\psi_n = \phi \in \mathcal{H}\right) \Rightarrow (\psi \in \mathrm{dom}(H) \wedge H\psi = \phi)\,.$$

Compare the property above with the continuity, which can be on a general metric space stated as

$$\forall (\psi_n)_{n=0}^{\infty} \subset \mathrm{dom}(H) : \psi_n \to \psi \Rightarrow H\psi_n \to H\psi.$$

Clearly, closedness is a more general concept in the sense that every bounded operator is closed. The belonging of $(\psi_n)_{n=0}^{\infty}$ to the domain $\mathrm{dom}(H)$ is only formal here. Since every bounded operator can be extended to the whole space [5] we will further consider them as defined everywhere.

Definition 1.2.3 (Closable map and its closure)
With the same assumptions and notation as in the definition 1.2.1 we say that H is a closable map if $\overline{G(H)}$ is a graph.
If H is a closable map we define its closure $\overline{H}$ as $G(\overline{H}) := \overline{G(H)}$.

Alternatively, it is possible to reformulate the definition 1.2.3 as follows.

$$\forall (\psi_n)_{n=0}^{\infty} \subset \mathrm{dom}(H) : (\psi_n \to 0 \wedge \|H\psi_n - H\psi_m\| \to 0) \Rightarrow H\psi_n \to 0.$$

From now on, all maps will be considered closed if not specified otherwise.

Another concept which must be treated carefully when dealing with unbounded operators is an *adjoint operator* and *self-adjointness*. For a given bounded operator A the adjoint operator is characterized as an operator B such that $(\psi, A\phi) = (B\psi, \phi)$ for all $\psi, \phi \in \mathcal{H}$. Thus, self-adjoint and symmetric operators become one. For general, possibly unbounded operators, it is much more delicate, the operator domain has to be taken into account, and these two concepts have to be properly distinguished.

Definition 1.2.4 (Adjoint operator)
Let H be a densely defined, not necessary closed operator on $\mathcal{H}$. Then its adjoint operator H^* is defined as follows

$$\begin{aligned} \mathrm{dom}(H^*) &:= \{\psi \in \mathcal{H} \mid \exists \eta \in \mathcal{H} : \forall \phi \in \mathrm{dom}(H) : (\psi, H\phi) = (\eta, \phi)\} \\ H^*\psi &:= \eta. \end{aligned}$$

Definition 1.2.5 (Symmetric operator)
Let H be a densely defined but not necessarily closed operator on $\mathcal{H}$. We say that H is symmetric if $H \subset H^*$. In other words, for all $\phi, \psi \in \mathrm{dom}(H)$ we have $(\phi, H\psi) = (H\phi, \psi)$.

Definition 1.2.6 (Self-adjoint and normal operator)
Let H be a densely defined, not necessarily bounded operator on $\mathcal{H}$. We say that H is

i) self-adjoint if $H = H^*$.

ii) normal if $H^*H = HH^*$.

Definition 1.2.7 (Weak convergence)
Let $(\psi_n) \subset \mathcal{H}, \psi \in \mathcal{H}$ be a sequence and a point in a Hilbert space $\mathcal{H}$. We say that ψ_n converge weakly to ψ, denoted $\psi_n \xrightarrow{w} \psi$, if $(\psi_n, \phi) \to (\psi, \phi)$, for all $\phi \in \mathcal{H}$.

Definition 1.2.8 (Resolvent set and resolvent)
Let H be a densely defined closed operator, and $\lambda \in \mathbb{C}$ a complex number. We say that λ is from the resolvent set of the operator H ($\lambda \in \rho(H)$), if $H - \lambda\mathbb{I}$ is a bijection of $\mathrm{dom}(H)$ on $\mathcal{H}$.
In other words,

$$\rho(H) := \{\lambda \in \mathbb{C} \mid \ker(H - \lambda\mathbb{I}) = \{0\} \wedge \mathrm{Ran}(H - \lambda\mathbb{I}) = \mathcal{H}\}.$$

By a resolvent of H we understand a parametric operator $R_\lambda := (H - \lambda\mathbb{I})^{-1}$, for all $\lambda \in \rho(H)$.

Definition 1.2.9 (Spectrum and its classification)
Let H be a closed operator on a Hilbert space $\mathcal{H}$. Then the spectrum is the complement of a resolvent set to $\mathbb{C}$, i.e. $\sigma(H) := \mathbb{C}\backslash\rho(H)$. Furher, we introduce classification of its spectrum $\sigma(H)$ as follows:

i) *Point spectrum* $\sigma_p(H) := \{\lambda \in \sigma(H) | \ker(\lambda - \mathbb{I}) \neq \{0\}\}$.

ii) *Continuous spectrum* $\sigma_c(H) := \{\lambda \in \sigma(H) | \ker(H - \lambda\mathbb{I}) = \{0\} \wedge \overline{\mathrm{Ran}(H - \lambda\mathbb{I})} = \mathcal{H}\}$.

iii) *Residual spectrum* $\sigma_r(H) := \{\lambda \in \sigma(H) | \ker(H - \lambda\mathbb{I}) = \{0\} \wedge \overline{\mathrm{Ran}(H - \lambda\mathbb{I})} \neq \mathcal{H}\}$.

Nevertheless, for normal operators only point and continuous spectrum is relevant since the residual part of it is empty. To show this, let us state and prove a short serie of technical propositions.

Proposition 1.2.2
Let H be a densely defined closed operator on $\mathcal{H}$. Then $\ker(H^*) = \mathrm{Ran}(H)^\perp$.

Proof. Let $\psi \in \ker(H^*)$ be an arbitrary vector from the kernel. Then for all $\phi \in \mathrm{dom}(H)$ we have

$$0 = (H^*\psi, \phi) = (\psi, H\phi)\,,$$

and therefore $\psi \in \mathrm{Ran}(H)^\perp$. □

Proposition 1.2.3
Let H be a densely defined closed operator on $\mathcal{H}$. If H is normal, then
$\lambda \in \sigma_p(H) \Leftrightarrow \lambda^* \in \sigma_p(H^*)$.

Proof. $\lambda \in \sigma_p(H) \Leftrightarrow$ there is a non-zero $\psi \in \mathrm{dom}(H) : (H - \lambda\mathbb{I})\psi = 0$.
Then for all $\lambda \in \sigma_p(H)$ we have

$$0 = ((H - \lambda\mathbb{I})\psi, (H - \lambda\mathbb{I})\psi) = ((H^* - \lambda^*\mathbb{I})\psi, (H^* - \lambda^*\mathbb{I})\psi)\,.$$

□

Theorem 1.2.1
Let H be a densely defined closed operator on $\mathcal{H}$. If H is normal then $\sigma_r(H) = \emptyset$, i.e. its residual spectrum is empty.

Proof. Let us consider $\lambda \in \sigma_c(H) \cup \sigma_r(H)$ then according to the proposition 1.2.2 it is possible to decompose $\mathcal{H}$ as a direct sum

$$\mathcal{H} = \ker(H^* - \lambda^*\mathbb{I}) \oplus \overline{\mathrm{Ran}(H - \lambda\mathbb{I})}.$$

However, since $\lambda \in \sigma_p(H) \Leftrightarrow \lambda^* \in \sigma_p(H^*)$, $\ker(H^* - \lambda^*\mathbb{I}) = \{0\}$. Therefore $\overline{\mathrm{Ran}(H - \lambda\mathbb{I})} = \mathcal{H}$ and so $\lambda \in \sigma_c(H)$. □

Thus, for normal operators, only point and continuous part of the spectrum are relevant (possibly non-empty). For normal operators, the spectrum can be characterized as

$$\sigma(H) = \{\lambda \in \mathbb{C} \mid \exists\, (\psi_n)_{n=0}^{\infty} \subset \operatorname{dom}(H), \|\psi_n\| = 1 : \|(H - \lambda)\, \psi_n\| \xrightarrow{n\to\infty} 0\}. \tag{1.1}$$

This follows from the Weyl's criterion [5, Důsledek 7.3.6]. However, there is another possible classification of the spectrum, which can sometimes be more natural to express certain properties.

Definition 1.2.10 (Essential and discrete spectrum)
Let H be a densely defined closed operator on $\mathcal{H}$. Then the essential spectrum of H is a set $\sigma_{ess}(H) \subset \mathbb{C}$ defined as

$$\sigma_{ess}(H) := \{\lambda \in \mathbb{C} \mid \exists \textit{ non-compact}\, (\psi_n)_{n=0}^{\infty} \subset \operatorname{dom}(H), \|\psi_n\| = 1 : (H - \lambda\mathbb{I})\, \psi_n \to 0\}.$$

By the discrete spectrum of H we mean $\sigma_{disc}(H) := \sigma(H) \backslash \sigma_{ess}(H)$.

It can be easily seen that $\sigma_c(H) \subset \sigma_{ess}(H)$ and so $\sigma_{disc}(H) \subset \sigma_p(H)$. The exact relation between these two classifications is a subject of the following theorem.

Theorem 1.2.2 ([36])
Let H be a self-adjoint operator on $\mathcal{H}$, then

$$\sigma_{disc}(H) = \{\lambda \in \sigma_p \mid \nu(\lambda, H) < \infty \land \lambda \textit{ is isolated}\},$$

where $\nu(\lambda, H)$ is the multiplicity of λ as an eigenvalue of H.

Theorem 1.2.3
Let H be a self-adjoint operator on $\mathcal{H}$. Then

$$\sigma_{ess}(H) := \{\lambda \in \mathbb{C} \mid \exists\, (\psi_n)_{n=0}^{\infty} \subset \operatorname{dom}(H), \|\psi_n\| = 1, \psi_n \xrightarrow{w} 0\ : (H - \lambda\mathbb{I})\, \psi_n \to 0\}.$$

Proof. We will prove the equality of these two sets as two inclusions.

i) Let $\lambda \in \sigma_{ess}$, then there is $(\psi_n)_{n=0}^{\infty}, \|\psi_n\| = 1 : (H - \lambda\mathbb{I}) \to 0$. This means that there is a weakly convergent subsequence $(\psi_{n_k})_{n=0}^{\infty} \subset (\psi_n)_{n=0}^{\infty}$ such that $\psi_{n_k} \xrightarrow{w} \psi \in \mathcal{H}$ and $\delta > 0$ such that for all $j, k \in \mathbb{N}_0 : \|\psi_{n_j} - \psi_{n_k}\| > \delta$.

   Let us define the sequence $\phi_k := \frac{\psi_{n_{k+1}} - \psi_{n_k}}{\|\psi_{n_{k+1}} - \psi_{n_k}\|}$, for all $k \in \mathbb{N}_0$. Then $\|\phi_k\| = 1$, $\phi_k \xrightarrow{w} 0$ and $(H - \lambda\mathbb{I})\, \phi_k \to 0$.

ii) Let $\lambda \in \{\lambda \in \mathbb{C} \mid \exists\, (\psi_n)_{n=0}^{\infty} \subset \operatorname{dom}(H), \|\psi_n\| = 1, \psi_n \xrightarrow{w} 0\ : (H - \lambda\mathbb{I})\, \psi_n \to 0\}$. We show that $(\psi_n)_{n=0}^{\infty}$ is non-compact due to a contradiction.

   Let us assume that $(\psi_n)_{n=0}^{\infty}$ is compact; then there would be $(\psi_{n_k})_{n=0}^{\infty} \subset (\psi_n)_{n=0}^{\infty}$, $\psi_{n_k} \to \psi \in \mathcal{H}$ a convergent subsequence with $\|\psi\| = 1$ however, at the same time $\psi = 0$ since weak and strong limits must merge in case they exist.

□

The main reason we introduced the concept of the essential spectrum is the associated well-known theorem, which gives its stability. Before that, let us introduce the concept of (relatively) compact operators.

Definition 1.2.11 (Compact operator)
Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. We say $H \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a compact operator if the image of a bounded set is pre-compact, i.e. its closure is compact.

Definition 1.2.12 (Relatively compact operator)
Let V, H be densely defined and the latter also closed operators $\mathcal{H}$. We say that V is a relatively compact operator with respect to H if $VR_\lambda(H)$ is a compact operator for some $\lambda \in \rho(H)$.

However, for a given operator H it can be challenging to properly introduce a perturbed operator $H + V$ for some perturbation V since, as we outlined above, when dealing with unbounded operators, the domains have to be taken into account. In general, there is no universal recipe defining the perturbed operator to be self-adjoint, for example. Nevertheless, there are many tools available if the perturbation satisfies certain additional properties, for instance pseudo-Friedrichs extension, KLMN-theorem, or Kato-Rellich theorem. We will mention the latter one here.

Definition 1.2.13 (Relative boundedness)
Let H and V be densely defined operators on $\mathcal{H}$ such that $\mathrm{dom}(H) \subset \mathrm{dom}(V)$*. We say that V is H-bounded with H-bound $a > 0$ if there is $b \geq 0$ such that*

$$||V\psi|| \leq a||H\psi|| + b||\psi||,$$

for all $\psi \in \mathrm{dom}(H)$.

Theorem 1.2.4 (Kato-Rellich [5, 7.3.14])
Let H and V be self-adjoint and symmetric operators on $\mathcal{H}$, respectively. If V is H-bounded with H-bound smaller than 1 then $H + V$ is a self-adjoint operator on $\mathrm{dom}(H)$*. Moreover, if H is bounded from below, then so is $H + V$.*

Theorem 1.2.5 (Stability of the essential spectrum [5, § 10.4])
Let H be a self-adjoint operator on $\mathcal{H}$ and V be a relatively compact operator with respect to H. Then $\sigma_{ess}(H + V) = \sigma_{ess}(H)$.

1.2.1 Dirichlet Laplacian

One of the most significant differential operators is Laplacian and its different variants. It is so especially because of its application in non-relativistic quantum mechanics, where it plays a role of a "kinetic part" of the Hamiltonian of a collection of particles in space.

In the following we will properly introduce the Dirichlet Laplacian, i.e., self-adjoint extension of the formal differential operator $\tilde{H} := -\Delta$ on the domain $\mathrm{dom}(\tilde{H}) := C_0^\infty(\Omega)$ for an open connected set $\Omega \subset \mathbb{R}^n$ with piece-wise smooth boundary. Before we move on to the actual definition we will introduce the Friedrich's extension. A tool which allows us to construct self-adjoint extensions of a symmetric operator satisfying certain properties through its associated sesquilinear form. Before that, let us extend the concept of closedness for forms. In the following, all sesquilinear forms will be considered to be symmetric.

Definition 1.2.14 (h-convergence)
Let h be a sesquilinear form on a space $\mathcal{H}$ bounded from below. We say that a sequence $(\psi_n)_{n=0}^{\infty} \subset \mathrm{dom}(h)$ is h-convergent, in symbol $\psi_n \xrightarrow{h} \psi$, if

$$\psi_n \xrightarrow{n\to\infty} \psi \wedge h(\psi_n - \psi_m) \xrightarrow{n,m\to\infty} 0,$$

where we denote $h(\psi) := h(\psi, \psi)$.

Definition 1.2.15 (Closed form)
Let h be a sesquilinear form on a space $\mathcal{H}$ bounded from below. We say that h is a closed form if for every sequence $(\psi_n)_{n=0}^{\infty} \subset \mathrm{dom}(h)$

$$\psi_n \xrightarrow{h} \psi \Rightarrow \left(\psi \in \mathrm{dom}(h) \wedge h(\psi_n - \psi) \xrightarrow{n\to\infty} 0\right).$$

Definition 1.2.16 (Closable form [34, Chapter VI, Theorem 1.17])
Let h be a sesquilinear form on a space $\mathcal{H}$ bounded from below. We say that h is closable if for every sequence $(\psi_n)_{n=0}^{\infty} \subset \mathrm{dom}(h)$

$$\psi_n \xrightarrow{n\to\infty} 0 \Rightarrow h(\psi_n) \xrightarrow{n\to\infty} 0.$$

When this condition is satisfied, h has the closure (the smallest closed extension) $\overline{h}$ defined in the following way. The domain $\mathrm{dom}(\overline{h})$ is the set of all $\psi \in \mathcal{H}$ such that there exists a sequence $(\psi_n)_{n=0}^{\infty} \subset \mathrm{dom}(h)$, $\psi_n \xrightarrow{h} \psi$, and

$$\overline{h}(\psi, \phi) = \lim_{n\to\infty} h(\psi_n, \phi_n),$$

for any $(\phi_n)_{n=0}^{\infty} \subset \mathrm{dom}(h)$, $\phi_n \xrightarrow{h} \phi$.

Theorem 1.2.6 ([34, Chapter VI, Theorem 1.27])
Let h be a sesquilinear form on a space $\mathcal{H}$ such that h is densely defined and bounded from below. Then h is a closable map.

Theorem 1.2.7 ([34, Chapter VI, Theorem 2.1])
Let h be a sesquilinear form on a space $\mathcal{H}$ such that h is densely defined and bounded from below. Then there is a unique operator H on a space $\mathcal{H}$ such that it is bounded from below and satisfies

i) $\mathrm{dom}(H) \subset \mathrm{dom}(h)$.

ii) for all $\psi \in \mathrm{dom}(H)$, $\phi \in \mathrm{dom}(h)$ holds $h(\phi, \psi) = (\phi, H\psi)$.

Definition 1.2.17 (Friedrich's extension)
Let $\tilde{H}$ be an operator on $\mathcal{H}$ such that $\tilde{H}$ is densely defined and bounded from below. Let us put $\tilde{h}(\phi, \psi) := (\phi, H\psi)$ for all $\phi, \psi \in \mathrm{dom}(H)$, that is, $\mathrm{dom}(h) = \mathrm{dom}(H)$. Then according to the theorem 1.2.6 there is $h := \overline{\tilde{h}}$, a closure of $\tilde{h}$ and according to the theorem 1.2.7 there is a unique operator H such that h is its induced sesquilinear form. Operator H is called Friedrich's extension of the given operator $\tilde{H}$.

Example 1.2.1 (Definition of a Dirichlet Laplacian)
Let us consider a Hilbert space $\mathcal{H} := L^2(\Omega)$. We will start with a formal differential expression defined above $\tilde{H} = -\Delta$ with its initial domain $\mathrm{dom}(\tilde{H}) = C_0^\infty(\Omega)$. Using Friedrich's extension we will construct a self-adjoint extension of $\tilde{H}$ that corresponds to a Dirichlet boundary condition, i.e. $\psi(x) = 0$ on $\partial\Omega$.

For all $\phi, \psi \in \mathrm{dom}(\tilde{H})$ holds

$$(\phi, \tilde{H}\psi) = -\int_\Omega \phi^*(x)\Delta\psi(x)dx = -\int_\Omega \left[\nabla\left(\phi^*(x)\nabla\psi(x)\right) - \nabla\phi^*(x)\nabla\psi(x)\right] dx =$$
$$= -\int_{\partial\Omega} \phi^*(x)\frac{\partial\psi(x)}{\partial n}dS + \int_\Omega \nabla\phi^*(x)\nabla\psi(x)dx = (\nabla\phi, \nabla\psi)\,.$$

At first, we construct the induced sesquilinear form of $\tilde{H}$

$$\tilde{h} := (\phi, \tilde{H}\psi) = (\nabla\phi, \nabla\psi)\,,$$
$$\mathrm{dom}(\tilde{h}) := \mathrm{dom}(\tilde{H}).$$

Proposition 1.2.4
Form $\tilde{h}$ is closable.

Proof. $\tilde{h}$ is closable $\Leftrightarrow \forall(\psi_n)_{n=0}^\infty \subset \mathrm{dom}(\tilde{h}) : \|\psi_n\| \to 0 \wedge \|\nabla\psi_n - \nabla\psi_m\| \to 0 \Rightarrow \tilde{h}[\psi_n] \to 0$

Let $(\psi_n)_{n=0}^\infty \subset \mathrm{dom}(\tilde{h})$ be an arbitrary sequence from the domain.

Then $(\nabla\psi_n)_{n=0}^\infty \subset L^2(\Omega, \mathbb{C}^n) := \{f : \Omega \to \mathbb{C}^n \mid \int_\Omega |f|^2 < \infty\}$ is cauchy and therefore there is $f \in L^2(\Omega, \mathbb{C}^n)$ such that $\nabla\psi_n \to f$ in $L^2(\Omega, \mathbb{C}^n)$. Then for all $\phi \in C_0^\infty(\Omega, \mathbb{C}^n)$ we have

$$(\phi, f) = \lim_{n\to\infty}\int_\Omega \phi^*(x)\nabla\psi_n(x)dx = \lim_{n\to\infty}\int_{\partial\Omega}\phi^*(x)\psi_n(x)dS - \lim_{n\to\infty}\int_\Omega \mathrm{div}\,\phi^*(x)\psi_n(x)dx =$$
$$= (-\mathrm{div}\,\phi, 0) = (\phi, 0)\,.$$

Since $C_0^\infty(\Omega, \mathbb{C}^n)$ is dense in $L^2(\Omega, \mathbb{C}^n)$ it is clear that $f = 0$. □

Let us denote $h := \overline{\tilde{h}}$ the closure of $\tilde{h}$ with its domain $\mathrm{dom}(h) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_h} = W_0^{1,2}(\Omega)$.

Proposition 1.2.5
For all $\phi, \psi \in W_0^{1,2}(\Omega)$ holds, $h(\phi, \psi) = (\nabla\phi, \nabla\psi)$, where ∇ is a weak gradient.

Proof. Let $\psi \in W_0^{1,2}(\Omega)$ be an arbitrary one. Then there is $(\psi_n)_{n=0}^\infty \subset C_0^\infty(\Omega)$, $f \in L^2(\Omega, \mathbb{C}^n)$ such that

$$\lim_{n\to\infty}\psi_n = \psi \in W_0^{1,2}(\Omega), \qquad \lim_{n\to\infty}\nabla\psi_n = f \in L^2(\Omega, \mathbb{C}^n).$$

Then for all $\phi \in C_0^\infty(\Omega, \mathbb{C}^n)$ we have

$$(\phi, f) = \lim_{n\to\infty}\int_\Omega \phi^*(x)\nabla\psi_n(x)dx = \lim_{n\to\infty}\int_{\partial\Omega}\phi^*(x)\psi_n(x)dS - \lim_{n\to\infty}\int_\Omega \mathrm{div}\,\phi^*(x)\psi_n(x)dx =$$
$$= (-\mathrm{div}\,\phi, \psi)\,.$$

For all $i \in \{1, \cdots, n\}$ we then have

$$\int_\Omega \phi_i^*(x) f_i(x) dx = -\int_\Omega \frac{\partial \phi_i^*(x)}{\partial x_i} \psi(x) dx.$$

From the definition 1.1.2 is then f the weak gradient of the limit ψ, that is, $f(x) = \nabla\psi(x)$. From this we already can see that for all $\phi, \psi \in W_0^{1,2}(\Omega) : h(\phi, \psi) = (\nabla\phi, \nabla\psi)$ where ∇ is a weak gradient. □

What remains to specify is the self-adjoint operator H and its induced sesquilinear form h. According to the Theorem 1.2.7 such an operator exists since h is densely defined, closed, and bounded from below. H is self-adjoint because h is a symmetric form.

Proposition 1.2.6
The self-adjoint operator H acts on its operator domain as a weak Laplacian i.e. $H\psi = -\Delta\psi$.

Proof. We already know that H exists. Let us consider $\psi \in \mathrm{dom}(H)$ then for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \phi^*(x) H\psi(x) dx = h(\phi, \psi) = \int_\Omega \nabla\phi^*(x)\nabla\psi(x) dx = -\int_\Omega \phi^*(x)\Delta\psi(x) dx =$$
$$= \int_\Omega \phi^*(x)(-\Delta)\psi(x) dx$$

According to the definition 1.1.2 H acts as a negatively taken weak Laplacian, i.e.

$$H\psi = -\Delta\psi, \qquad \text{in the weak sense}$$
$$\mathrm{dom}(H) = \{\psi \in W_0^{1,2}(\Omega) \mid \Delta\psi \in L^2(\Omega)\}.$$

□

Remark 1.2.1
For sufficiently „nice„ $\Omega \subset \mathbb{R}^n$ (for instance, bounded Ω with the boundary $\partial\Omega \in C^2$) holds

$$\{\psi \in W_0^{1,2}(\Omega) \mid \Delta\psi \in L^2(\Omega)\} = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega).$$

Furthermore, for $\Omega := \mathbb{R}^n$ holds $W_0^{1,2}(\Omega) = W^{1,2}(\Omega)$, for all $n \in \mathbb{N}$. See [36] for the proof.

Lemma 1.2.1 ([36])
For all $n \geq 2$ holds $W_0^{1,2}(\mathbb{R}^n \backslash \{0\}) = W_0^{1,2}(\mathbb{R}^n)$.

Theorem 1.2.8 (Hardy inequality)
Let $n \geq 3$. Then, for every $\psi \in W^{1,2}(\mathbb{R}^n)$ the following holds

$$\int_{\mathbb{R}^n} |\nabla\psi(x)|^2 \, dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|\psi(x)|^2}{|x|^2} \, dx.$$

Proof. Let $a \in \mathbb{R}$ be a not yet specified real number. Then for every $\psi \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| \nabla\psi(x) - a \frac{x}{|x|^2}\, \psi(x) \right|^2 dx &= \int_{\mathbb{R}^n} |\nabla\psi(x)|^2\, dx + a^2 \int_{\mathbb{R}^n} \frac{|\psi(x)|^2}{|x|^2}\, dx \\
&\quad - 2a \int_{\mathbb{R}^n} \frac{x}{|x|^2}\, \mathrm{Re}(\psi^*(x)\nabla\psi(x))\, dx \\
&= \int_{\mathbb{R}^n} |\nabla\psi(x)|^2 + a^2 \int_{\mathbb{R}^n} \frac{|\psi(x)|^2}{|x|^2}\, dx - a \int_{\mathbb{R}^n} \frac{x}{|x|^2} \nabla|\psi(x)|^2\, dx \\
&= \int_{\mathbb{R}^n} |\nabla\psi(x)|^2 + a^2 \int_{\mathbb{R}^n} \frac{|\psi(x)|^2}{|x|^2}\, dx + a \int_{\mathbb{R}^n} \mathrm{div}\, \frac{x}{|x|^2}\, |\psi(x)|^2\, dx \\
&= \int_{\mathbb{R}^n} |\nabla\psi(x)|^2 + \left[a^2 + a\,(n-2)\right] \int_{\mathbb{R}^n} \frac{|\psi(x)|^2}{|x|^2}\, dx \geq 0.
\end{aligned}$$

With the notation $p(a) := -a^2 - a(n-2)$ we have for all $a \in \mathbb{R}$ and all $\psi \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$

$$\int_{\mathbb{R}^n} |\nabla\psi(x)|^2\, dx \geq p(a) \int_{\mathbb{R}^n} \frac{|\psi(x)|^2}{|x|^2}\, dx, \tag{1.2}$$

where for $a_0 := -\frac{n-2}{2}$ the inequality (1.2) is optimal with $p(a_0) = \frac{(n-2)^2}{4}$.

Now, let us consider an arbitrary function $\psi \in W^{1,2}(\mathbb{R}^n)$, then according to theorems 1.1.2, 1.1.3 and lemma 1.2.1, there is a sequence $(\psi_n)_{n=0}^\infty \subset C_0^\infty(\mathbb{R}^n \backslash \{0\})$ such that $\psi_n \to \psi$ in $W^{1,2}(\mathbb{R}^n)$ and therefore also in $L^2(\mathbb{R}^n)$. The latter means that there is a subsequence $(\psi_{n_k})_{k=0}^\infty \subset (\psi_n)_{n=0}^\infty$ such that $\psi_{n_k} \to \psi$ almost everywhere in $\mathbb{R}^n$. Then, from Fatou-Lebesgue theorem we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|\psi(x)|^2}{|x|^2}\, dx = \int_{\mathbb{R}^n} \lim_{k\to\infty} \frac{|\psi_{n_k}(x)|^2}{|x|^2}\, dx = \int_{\mathbb{R}^n} \liminf_{k\to\infty} \frac{|\psi_{n_k}(x)|^2}{|x|^2}\, dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} \frac{|\psi_{n_k}(x)|^2}{|x|^2}\, dx \\
\leq \liminf_{k\to\infty} \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |\nabla\psi_{n_k}(x)|^2\, dx \leq \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |\nabla\psi(x)|^2\, dx.
\end{aligned}$$

In the proof we used the following identity

$$\int_{\mathbb{R}^n} \frac{x}{|x|^2} \nabla|\psi(x)|^2\, dx = -\int_{\mathbb{R}^n} \mathrm{div}\, \frac{x}{|x|^2}\, |\psi(x)|^2\, dx + \int_{\mathbb{R}^n} \mathrm{div}\left(\frac{x}{|x|^2}|\psi(x)|^2\right) dx$$

and omitted the last term from the divergence theorem:

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathrm{div}\left(\frac{x}{|x|^2}|\psi(x)|^2\right) dx &= \lim_{r\to+\infty} \int_{\partial B(0,r)} \frac{m\cdot x}{|x|^2} |\psi(x)|^2\, dS \\
&= \lim_{r\to+\infty} -\frac{1}{r} \int_{\partial B(0,r)} |\psi(x)|^2\, dS \\
&= \lim_{r\to+\infty} -r^{n-2} \int_{\Omega} |\psi(r,\Theta)|^2\, d\Theta = 0,
\end{aligned}$$

where m is a unit vector poiting in the direction of $-x$, that is, $m \cdot x = -|x| = -r$. The limit follows from the fact that $\psi \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ and therefore has compact support. Moreover, since zero is excluded from its domain, the compactness of the support ensures that ψ is null also on a certain neighbourhood of the origin in $\mathbb{R}^n$ and so the integrals are finite. □

Remark 1.2.2 (Optimality of the Hardy inequality, [36])
The Hardy inequality is

i) never achieved:
for all $\psi \in W^{1,2}(\mathbb{R}^n)\backslash\{0\}$: $\int_{\mathbb{R}^n}|\nabla\psi(x)|^2\,dx > \frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{|\psi(x)|^2}{|x|^2}\,dx.$

ii) sharp:
there is a sequence $(\psi_n)_{k=0}^{\infty} \subset W^{1,2}(\mathbb{R}^n)$ *such that* $\lim\limits_{k\to\infty}\frac{\int_{\mathbb{R}^n}|\nabla\psi_k(x)|^2\,dx}{\int_{\mathbb{R}^n}\frac{|\psi_k(x)|^2}{|x|^2}\,dx} = \frac{(n-2)^2}{4}.$

Definition 1.2.18 ((Sub)critical operator)
We say that H, an operator on $L^2(\Omega)$ *bounded from below, is subcritical if there is a Hardy inequality, that is, if there exists* $\rho : \Omega \to [0,+\infty) \in L^1_{loc}(\Omega)$ *such that for all* $\psi \in \mathrm{dom}(h)$ *holds*

$$h(\psi) - E_1\|\psi\|^2 \geq \int_\Omega \rho(x)|\psi(x)|^2 dx,$$

where h is the associated sesquilinear form and $E_1 := \inf\sigma(H)$.
We say that H is a critical operator if it is not subcritical.

Theorem 1.2.9 (Spectrum of Dirichlet Laplacian)
For all $n \in \mathbb{N}$ *holds* $\sigma(-\Delta_D^{\mathbb{R}^n}) = [0,+\infty)$.

Proof. We will take advantage of the characterization of the spectrum by (1.1). We will construct approximate eigenfunctions using the plane waves $e^{i\lambda x}$, the formal solution of the eigenproblem

$$\left(-\Delta + |\lambda|^2\right)\psi = 0.$$

At first, we construct a support function that ensures the convergence.
Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a test function such that $\|\phi\| = 1$. Then we define a sequence $(\phi_k)_{k=0}^\infty$ as

$$\phi_k(x) := \phi\left(\frac{x}{k}\right)k^{-\frac{n}{k}}, \tag{1.3}$$

for all $k \in \mathbb{N}$. For this sequence holds the following

- $\|\phi_k\|^2 = \frac{1}{k^n}\int_{\mathbb{R}^n}\left|\phi\left(\frac{x}{k}\right)\right|^2 dx = \int_{\mathbb{R}^n}|\phi(y)|^2 dy = 1,$
- $\|\nabla\phi_k\|^2 = \frac{1}{k^n}\frac{1}{k^2}\int_{\mathbb{R}^n}\left|\phi\left(\frac{x}{k}\right)\right|^2 dx = \frac{1}{k^2}\|\nabla\phi\|^2,$
- $\|\Delta\phi_k\|^2 = \frac{1}{k^4}\|\Delta\phi\|^2.$

For a given $\lambda \in \mathbb{R}^n$ we define the approximate function $\psi_k(x)$ as

$$\psi_k(x) := \phi_k(x)e^{i\lambda x} \in \mathrm{dom}(-\Delta_D^{\mathbb{R}^n}).$$

It is clear that the norm is being preserved, i.e. $\|\psi_k\|^2 = \|\phi_k\|^2 = 1$ and by a straightforward calculation we have for all $k \in \mathbb{N}$

$$\Delta\psi_k(x) = \left[\Delta\phi_k(x) + 2i\lambda\nabla\phi(x) - |\lambda|^2\right]e^{i\lambda x}$$

and therefore

$$\left\|\left(-\Delta - |\lambda|^2\right)\psi_k\right\| \leq \|\Delta\phi_k\| + 2|\lambda|\,\|\nabla\phi_k\| \xrightarrow{k\to\infty} 0.$$

□

Theorem 1.2.10 (Rellich, [40])
For all $n \in \mathbb{N}$ holds $\sigma_p(-\Delta_D^{\mathbb{R}^n}) = \emptyset$

Combining the two last theorems, we conclude that for all $n \in \mathbb{N}$ we have

$$\sigma(-\Delta_D^{\mathbb{R}^n}) = \sigma_c(-\Delta_D^{\mathbb{R}^n}) = \sigma_{ess}(-\Delta_D^{\mathbb{R}^n}) = [0, +\infty). \tag{1.4}$$

1.3 The Birman-Schwinger principle

The last section of this chapter is devoted to the famous Birman-Schwinger principle and the proper definition of the perturbed operator $H + V$ with a given self-adjoint operator H. One of the most advantegeous aspects of Birman-Schwinger principle is its applicability to both self-adjoint and non-self-adjoint perturbations. Therefore, the Birman-Schwinger principle will be the cornerstone of the future investigation of the stability of the studied model.

Most of the results in this section are due to the work of Hansmann and Krejčiřík [28] to which we refer for a more detailed study of the present material. Before we step to the Birman-Schwinger principle itself, let us introduce several technical concepts.

For a given perturbation V, we define $B := \sqrt{|V|}U^*$ and $A := \sqrt{|V|}$, where U is an operator which maps $|V|$ onto V, that is, $V = U|V|$. Such an operator is called a polar decomposition of V and always exists. Its properties are summarized below.

Definition 1.3.1 (Partial isometry)
We say that a bounded operator U is a partial isometry if there is a closed subspace $V = \overline{V} \subset \mathcal{H}$ such that

i) $\|Ux\| = \|x\|$, *for all $x \in V$,*

ii) $Ux = 0$, *for all $x \in V^\perp$.*

Now, we will construct one. Let H be a densely defined closed operator on $\mathcal{H}$. Consider the symmetric form $h(\psi, \phi) := (H\psi, H\phi)$. Obviously h is densely defined and non-negative. In the spirit of the Theorem 1.2.7 we denote by T_h its associated operator. Since

$$(H\psi, H\phi) = (\psi, T_h\phi)$$

for all $\psi \in \mathrm{dom}(T_h)$ and $\phi \in \mathrm{dom}(h) = \mathrm{dom}(H)$, it follows that $T_h \subset H^*H$. However, since H^*H is clearly symmetric and T_h is self-adjoint, we must have $T_h = H^*H$. Further, let us denote $G = H^{\frac{1}{2}}$. Then we have

$$(H\psi, H\phi) = (G\psi, G\phi)\,, \text{ and } \|H\psi\| = \|G\psi\|, \tag{1.5}$$

for $\psi, \phi \in \mathrm{dom}(H) = \mathrm{dom}(G)$. This follows from the representation theorem proved by Kato, to which we refer for more details.

Theorem 1.3.1 (Representation theorem, [34, Chapter VI, Theorem 2.23])
Let h be a dendesly defined closed symmetric form, $h \geq 0$, and let $H = T_h$ be the associated self-adjoint operator by the Theorem 1.2.7. Then we have $\mathrm{dom}(H^{\frac{1}{2}}) = \mathrm{dom}(h)$ *and*

$$h(\psi, \phi) = \left(H^{\frac{1}{2}}\psi, H^{\frac{1}{2}}\phi\right),$$

for all $\psi, \phi \in \mathrm{dom}(h)$.

From 1.5 we can see that the assignment $G\psi \to H\psi$ defines an isometric mapping U of $\mathrm{Ran}(G)$ onto $\mathrm{Ran}(H)$ as $H\psi = UG\psi$. By continuity U can be extended to an isometric operator on $\overline{\mathrm{Ran}(G)}$ onto $\overline{\mathrm{Ran}(H)}$ and by setting $U\psi := 0$ for $\psi \in \mathrm{Ran}(G)^\perp = \ker(G)$ we can extend U to the whole space $\mathcal{H}$ which acts like

$$H = U\sqrt{H^*H} \text{ and } \mathrm{dom}(H) = \mathrm{dom}(\sqrt{H^*H}).$$

Such U is called the *polar decomposition* of H.

Definition 1.3.2 (Birman-Schwinger operator)
Let H be a self-adjoint operator on $\mathcal{H}$ and $A : \mathrm{dom}(A) \subset \mathcal{H} \to \tilde{\mathcal{H}}$, $B : \mathrm{dom}(B) \subset \mathcal{H} \to \tilde{\mathcal{H}}$ linear maps satisfying

i) $\mathrm{dom}(|H|^{\frac{1}{2}}) \subset \mathrm{dom}(A) \cap \mathrm{dom}(B)$

ii) For some (and therefore for all) $b > 0$ are $A(|H| + b)^{-\frac{1}{2}}, B(|H| + b)^{-\frac{1}{2}}$ bounded operators.

Then we define the Birman-Schwinger operator for every $z \in \rho(\mathcal{D}_\alpha)$ as

$$K(z) := \left(AG^{-\frac{1}{2}}\right)\left(G\,(H - z)^{-1}\right)\left(BG^{-\frac{1}{2}}\right)^*,$$

with $G := |H| + 1$.

Theorem 1.3.2 (Hansmann, Krejcirik [28, Theorem 5])
Let $K(z)$ be the Birman-Schwinger operator corresponding to the self-adjoint operator H and let $-1 \notin \sigma(K(z_0))$ for some $z_0 \in \rho(\mathcal{D}_\alpha)$. Then there is a unique closed extension H_V of $\tilde{H}_V := H + V$ such that $\mathrm{dom}(H_V) \subset \mathrm{dom}(|H|^{\frac{1}{2}})$ *and the following representation formula holds true*

$$(\phi, H_V\psi) := (G^{\frac{1}{2}}\phi, (HG^{-1} + [BG^{-\frac{1}{2}}]^* AG^{-\frac{1}{2}})G^{\frac{1}{2}}\psi),$$

for $\phi \in \mathrm{dom}(|H|)$ *and* $\psi \in \mathrm{dom}(H_V)$.

The extension H_V is obtained via the pseudo-Friedrichs extension [34, Chapter VI, Thm. 3.11]. Finally we are now ready to announce the Birman-Schwinger principle as we will need it it the following chapter.

Theorem 1.3.3 (Birman-Schwinger principle)
Let $K(z)$ be the Birman-Schwinger operator corresponding to the self-adjoint operator H and let the following condition be satisfied:

$$\exists c < 1 \textit{ such that } \sup_{z \in \rho(H_0)} \|K(z)\| \leq c.$$

Then the following holds:

i) $\sigma(H) = \sigma(H_V)$

ii) $\left[\sigma_p(H_V) \cup \sigma_r(H_V)\right] \subset \sigma_p(H)$ *and* $\sigma_c(H) \subset \sigma_c(H_V)$

In particular, if $\sigma(H) = \sigma_c(H)$*, then* $\sigma(H_V) = \sigma_c(H_V) = \sigma_c(H)$.

For the proof, we refer to [28, Theorem 3]. In the paper, the authors provide even finer version of the Birman-Schwinger principle regarding localization of the spectrum of the perturbed operator H_V. In its other versions, it can also be used for an investigation of the other parts of the spectrum, i.e. continuous and residual one. For our purposes, this theorem is enough.

Chapter 2

Stability of the spectrum

In this chapter we would like to introduce the reader to the study of spectral stability, its historical background, and its development. At first we start with a brief physical motivation and then deliver an overview of the results made for Schrödinger operators in this branch. Finally, we motivate study of our model by a concise discussion of relativistic quantum mechanics and its mathematical description.The content of this chapter is inspired by [5, 36, 1, 42, 26, 44, 33, 38].

2.1 Stability of Schrödinger operators

According to classical physics, electrons in atoms would have collapsed into the nucleus in a matter of nanoseconds [33, 36] which is in direct contradiction to our experience. This phenomenon is known as the problem of the stability of matter. Let us demonstrate the (in)stability on a Hydrogen atom. For more details, see [33, 38, 36].

Classical physics

From the point of view of the classical physics, atoms are described via so-called Bohr's planetary model - a point-wise charged center of mass orbited by electrons on eliptical paths. Specifically for the Hydrogen atom the Hamiltonian of the system is

$$H(p, x) = \frac{p^2}{2m} - \frac{e^2}{|x|},$$

where (p, x) are the coordinates in the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ corresponding to the system. The instability here means the unboundedness of the Hamiltonian from below, i.e.

$$\inf_{\mathbb{R}^3 \times \mathbb{R}^3} H(p, x) = -\infty. \tag{2.1}$$

As the electron orbits the nucleus it loses its kinetic energy by radiation and will eventually fall into the nucleus, which corresponds to the negative infinite energy (2.1). In other words - there is no long-term stable orbit.

Quantum physics

On the other hand, while we turn to the (non-relativistic) quantum-mechanical description via the correspondence principle, the Hamiltonian (as an operator on the Hilbert phase space $\mathcal{H} = L^2(\mathbb{R}^3)$ reads

$$H = -\frac{\hbar^2}{2m}\Delta_D^{\mathbb{R}^n} - \frac{e^2}{|x|}.$$

In this setting, there is no reachable physical interpretation in terms of particles orbiting certain area. However, we know how to quantify the stability through the energy in the same way as we did in the classical setting. The infimum of the energy is E_1, the lowest eigenvalue of the Hamiltonian H [38]:

$$E_1 := \inf_{\psi\in\mathrm{dom}(H)} \frac{(\psi, H\psi)}{||\psi||^2},$$

which is finite. Indeed, for every $\psi \in \mathrm{dom}(H) := \left\{\phi \in \mathrm{dom}\left(-\Delta_D^{\mathbb{R}^n}\right) \mid \left\|\frac{1}{|x|}\phi\right\| < \infty\right\}$, $||\psi|| = 1$ and $r > 0$ we have

$$(\psi, H\psi) = -\frac{\hbar^2}{2m}\int_{\mathbb{R}^3}|\nabla\psi(x)|\,dx - \mathrm{e}^2\int_{B(0,r)}\frac{|\psi(x)|^2}{|x|}\,dx - \mathrm{e}^2\int_{\mathbb{R}^3\setminus B(0,r)}\frac{|\psi(x)|^2}{|x|}\,dx \tag{2.2}$$

$$\geq -\frac{\hbar^2}{2m}\int_{\mathbb{R}^3}|\nabla\psi(x)|\,dx - \mathrm{e}^2 r\int_{B(0,r)}\frac{|\psi(x)|^2}{|x|^2}\,dx - \frac{\mathrm{e}^2}{r}\int_{\mathbb{R}^3\setminus B(0,r)}\frac{|\psi(x)|^2}{|x|^2}\,dx \tag{2.3}$$

$$\geq -\frac{\hbar^2}{2m}\int_{\mathbb{R}^3}|\nabla\psi(x)|\,dx - \mathrm{e}^2 r\int_{\mathbb{R}^3}\frac{|\psi(x)|^2}{|x|^2}\,dx - \frac{\mathrm{e}^2}{r}\int_{\mathbb{R}^3}|\psi(x)|^2\,dx \tag{2.4}$$

$$\geq \left(\frac{\hbar^2}{2m} - 4\mathrm{e}^2 r\right)\int_{\mathbb{R}^3}|\nabla\psi(x)|\,dx - \frac{\mathrm{e}^2}{r}\int_{\mathbb{R}^3}|\psi(x)|\,dx. \tag{2.5}$$

For the special choice $r := \frac{\hbar^2}{8me^2}$ we obtain the inequality

$$(\psi, H\psi) \geq -\frac{8m\mathrm{e}^2}{\hbar^2}, \qquad \text{for all } \psi \in \mathrm{dom}(H). \tag{2.6}$$

Where in (2.4) we estimated the middle term by the Hardy inequality. It is remarkable that the estimate (2.6) is very close to the real value of $E_1 = -\frac{me^2}{2\hbar^2}$ calculated in terms of special functions [26]. However, the Hardy inequality is a much more powerful tool and can be used to prove the spectral stability of the Laplacian in dimensions $n \geq 3$.

2.1.1 Stability of $\mathbb{R}^n$ for $n \geq 3$

Throughout this section, we will work over the Euclidean spaces and so for $n \geq 3$ we will denote $\mathcal{H} := L^2(\mathbb{R}^n)$ the Hilbert space and $H_n := -\Delta_D^{\mathbb{R}^n}$ will be the Dirichlet Laplacian on $\mathcal{H}$.

Consider $V : \mathbb{R}^n \to (-\infty, 0] \in L^1_{loc}(\mathbb{R}^n)$, a real-valued multiplication operator on $\mathcal{H}$, a potential physically speaking. Furthermore, we assume that V is relatively compact with respect to H_n that is H_n-bounded with H_n-bound smaller than 1. Then, since V is real-valued and therefore symmetric, $H_n + V$ is self-adjoint on $\mathrm{dom}(H_n)$ from Kato-Rellich - theorem 1.2.4. Due to the relative compactness, we have according to theorem 1.2.5 stability of the essential spectrum

$$\sigma_{ess}(H_n + V) = \sigma_{ess}(H_n) = \sigma(H_n) = [0, +\infty). \tag{2.7}$$

In addition, in the sense of forms, we have the following inequality

$$\begin{aligned}(\phi, (H_n + V)\,\phi) &= (\phi, H_n\phi) + (\phi, V\phi)\\ &\geq (\phi, \frac{(n-2)^2}{4|x|^2}\phi) + (\phi, V\phi)\\ &= \left(\phi, \left[\frac{(n-2)^2}{4|x|^2} + V\right]\phi\right) \overset{!}{\geq} 0,\end{aligned}$$

for all $\phi \in \mathrm{dom}(H_n)$. Thus, we conclude that if $0 \geq V \geq -\frac{(n-2)^2}{4|x|^2}$ then $H_n + V \geq 0$ and therefore $\sigma(H_n + V) \subset [0, +\infty)$. Together with the fact (2.7) we have the stability of the spectrum of the perturbed operator

$$\sigma(H_n + V) = \sigma_{ess}(H_n + V) = \sigma(H_n) = [0, +\infty).$$

This particular result can be summarized in the following theorem.

Theorem 2.1.1
Let $n \geq 3$ and $V : \mathbb{R}^n \to (-\infty, 0] \in L^1_{loc}(\mathbb{R}^n)$ a real-valued multiplication operator such that $V(H_n - \lambda)^{-1}$ is a compact operator for some $\lambda \in \rho(H_n)$ and it is H_n-bounded with H_n-bound smaller than 1. Then $\sigma(H_n + V) = \sigma_{ess}(H_n + V) = \sigma(H_n) = [0, +\infty)$ whenever $0 \geq V \geq -\frac{(n-2)^2}{4|x|^2}$.

Roughly speaking, we have $V : \mathbb{R}^n \to \mathbb{R}$ „small enough„ then $\sigma(H_n + V) = \sigma(H_n)$. This gives rise to the natural question whether this result could be generalized to complex-valued potentials. Multiplication operator associated with $V : \mathbb{R}^n \to \mathbb{C}$ is no longer possible to compare with the Hardy potenial $-\frac{(n-2)^2}{4|x|^2}$. It is obvious that another much more subtle tool has to be used. As we already outlined in the previous chapter, the proper tool here is the Birman-Schwinger principle. It was proven by Rupert L. Frank in 2011 that there is a uniform condition for, in general complex valued, potential V such that the spectrum of the perturbed operator $H_n + V$ is preserved whenever the condition is met.

Theorem 2.1.2 (R.L. Frank, [23])
Let $n \geq 3$, then there is $c_n > 0$ such that

$$\sigma_p(H_n + V) \cap \rho(H_n) = \emptyset,$$

whenever $\|V\|_{L^{n/2}(\mathbb{R}^n)} < c_n$.

In other words, the perturbed operator $H_n + V$ has no eigenvalue in $\mathbb{C}\backslash[0,+\infty)$. The proof is based on the Birman-Schwinger principle and the explicit knowledge of the corresponding resolvent kernel.

Another result reached with different methods was achieved by Fanelli, Krejčiřík and Vega in 2018. Their approach was based on extending the method of multipliers developed for self-adjoint operators in [3].

Theorem 2.1.3 (Fanelli, Krejčiřík, Vega, [22])
Let $n \geq 3$, then $\sigma_{\mathrm{p}}(H_n + V) = \emptyset$ whenever

$$\exists b < \frac{n-2}{5n-8} : \ b^2 \int_{\mathbb{R}^n} |\nabla\psi|^2 \geq \int_{\mathbb{R}^n} |x|^2 |V|^2 |\psi|^2.$$

So far we have been discussing the spectral stability of H_n in dimensions three or more. One may ask whether there are analogous results in the low dimensions and if it is possible to extend the Hardy inequality for the low dimensions. This is the subject of the following section.

2.1.2 Instability of $\mathbb{R}^n$ for low dimensions

The existence of a Hardy inequality for H_n in low dimensions is equivalent to the subriticality of H_n. Let us first answer this question.

Theorem 2.1.4 (Subcriticality of H_n)
The operator H_n acting on $\mathcal{H}$ is subcritical if and only if $n \geq 3$.

Proof. The implication $\Leftarrow$ is already proven by the Hardy inequality 1.2.8. What remains is to prove the other one. We shall proceed so in contradiction. We will show that for every $\rho \in L^1_{loc}(\mathbb{R}^n)\backslash\{0\}$, $\rho \geq 0$ holds

$$\inf_{\psi \in W^{1,2}(\mathbb{R}^n)\backslash\{0\}} \underbrace{\left(\int_{\mathbb{R}^n} |\nabla\psi(x)|^2 \, dx - \int_{\mathbb{R}^n} \rho(x)|\psi(x)|^2 \, dx \right)}_{Q(\psi)} < 0.$$

Formally, by taking $\psi(x) = 1$ we have $Q(1) = -\int_{\mathbb{R}^n} \rho(x) \, dx < 0$. Even though we cannot consider constant function, we can aproximate it. We will construct a sequence $(\psi_n)_{n=1}^{\infty} \subset W^{1,2}(\mathbb{R}^n)$ such that

- $\psi_n(x) \xrightarrow{n\to\infty} 1, \qquad$ for all $x \in \mathbb{R}^n$,
- $\|\nabla\psi(x)\| \xrightarrow{n\to\infty} 0.$

We set

$$\psi(x) = \begin{cases} 1, & \text{if } n > r, \\ \frac{\ln(n^2)-\ln(r)}{\ln(n^2)-\ln(n)}, & \text{if } n < r < n^2, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for all } n \in \mathbb{N},$$

where $r := |x|$. It is clear that the point-wise limit of $(\psi_n)_{n=1}^{\infty}$ is 1 (as a constant function). Let us now verify the other necessary properties.

$n = 1$:

- $\psi_n \in W^{1,2}(\mathbb{R})$: For all $n \in \mathbb{N}$ we have

$$\|\psi_n\|^2 = \int_{-n}^{n} 1 \, dx + 2 \int_{n}^{n^2} \frac{\ln^2(\frac{n^2}{x})}{\ln^2(n)} \, dx \leq 2n^2 < \infty,$$

  where we estimated $\frac{\ln^2(\frac{n^2}{x})}{\ln^2(n)} \leq 1$ for all $n < x < n^2$.

- $\|\psi'\| \xrightarrow{n\to\infty} 0$: For all $n \in \mathbb{N}$ we have

$$\|\psi'\|^2 = \frac{2}{\ln^2(n)} \int_{n}^{n^2} \frac{1}{x^2} \, dx = \frac{2}{\ln^2(n)} \left(\frac{1}{n^2} - \frac{1}{n} \right) \xrightarrow{n\to\infty} 0.$$

$n = 2$:

- $\psi_n \in W^{1,2}(\mathbb{R}^2)$: For all $n \in \mathbb{N}$ we have

$$\|\psi_n\|^2 = \int_{B(0,n)} 1 \, dxdy + \int_{n<r<n^2} \frac{\ln^2(\frac{n^2}{r})}{\ln^2(n)} \, dxdy \leq \pi n^4,$$

  where we estimated $\frac{\ln^2(\frac{n^2}{r})}{\ln^2(n)} \leq 1$ for all $n < r < n^2$ the same as we did above in the case $n = 1$.

- $\|\nabla\psi\| \xrightarrow{n\to\infty} 0$: For all $n \in \mathbb{N}$ we have

$$\|\nabla\psi\|^2 = \frac{1}{\ln^2(n)} \int_{n<r<n^2} \frac{1}{r^2} \, dx = \frac{2\pi}{\ln^2(n)} \int_{n}^{n^2} \frac{1}{r} \, dx = \frac{2\pi}{\ln(n)} \xrightarrow{n\to\infty} 0.$$

□

The theorem 2.1.4 finally closes the question of how it is with an analogue of the Hardy inequality in the lower dimensions; there is none. Nevertheless, as we will show further, this obstacle can be overcome by considering the half-space instead. That is, a half-line for dimension one and a half-plane for dimension two.

Theorem 2.1.5 (Half-line Hardy inequality)
For all $\psi \in W_0^{1,2}(\mathbb{R}_+)$ holds

$$\int_{\mathbb{R}_+} |\psi(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}_+} \frac{|\psi(x)|^2}{x^2} \, dx.$$

Proof. Analogically as in the proof of the Hardy inequality for $n \geq 3$ (1.2.8), for all $\psi \in W_0^{1,2}(\mathbb{R}_+)$ and $\alpha \in \mathbb{R}$ we have

$$\int_{\mathbb{R}_+} \left|\psi(x)' - a\frac{\psi(x)}{x}\right|^2 dx = \int_{\mathbb{R}_+} |\psi(x)'|^2 \, dx - 2a \int_{\mathbb{R}_+} \psi(x)' \frac{\psi^*(x)}{x} \, dx + a^2 \int_{\mathbb{R}_+} \frac{|\psi(x)|^2}{x^2} \, dx \tag{2.8}$$

$$= \int_{\mathbb{R}_+} |\psi(x)'|^2 \, dx - a \int_{\mathbb{R}_+} \frac{\left(|\psi(x)|^2\right)'}{x} \, dx + a^2 \int_{\mathbb{R}_+} \frac{|\psi(x)|^2}{x^2} \, dx \tag{2.9}$$

$$= \int_{\mathbb{R}_+} |\psi(x)'|^2 \, dx + a \int_{\mathbb{R}_+} \frac{|\psi(x)|^2}{x^2} \, dx + a^2 \int_{\mathbb{R}_+} \frac{|\psi(x)|^2}{x^2} \, dx \tag{2.10}$$

$$= \int_{\mathbb{R}_+} |\psi(x)'|^2 \, dx + \left[a + a^2\right] \int_{\mathbb{R}_+} \frac{|\psi(x)|^2}{x^2} \, dx \geq 0. \tag{2.11}$$

If we denote $p(a) := -a - a^2$, the coeffient which stands in front of $\int_{\mathbb{R}_+} \frac{|\psi(x)|^2}{|x|^2} \, dx$ and find its maximum that is achieved for $a := -\frac{1}{2}$ we arrive with the claimed statement. □

Remark 2.1.1
In (2.10) *we omit the second part of the integration by parts from the trace theorem [43].*

From the historical point of view, the half-line Hardy inequality was the original one proven by G.H. Hardy in 1920 [29] from which the higher-dimensional ones (for $n \geq 3$) can be derived by integration in spherical coordinates. However, it can be also used to prove the half-plane Hardy inequality, as we will see in the following.

Theorem 2.1.6 (Half-plane Hardy inequality)
For all $\psi \in W^{1,2}(\mathbb{R}) \otimes W_0^{1,2}(\mathbb{R}_+)$ *holds*

$$\int_{\mathbb{R}\times\mathbb{R}_+} |\nabla\psi(x,y)|^2 \, dxdy \geq \frac{1}{4} \int_{\mathbb{R}\times\mathbb{R}_+} \frac{|\psi(x,y)|^2}{y^2} \, dxdy.$$

Proof. For a given function $\phi \in W^{1,2}(\mathbb{R}) \otimes W_0^{1,2}(\mathbb{R}_+)$ we have

$$\int_{\mathbb{R}\times\mathbb{R}_+} |\nabla\psi(x,y)|^2 \, dxdy = \int_{\mathbb{R}\times\mathbb{R}_+} \left(\left|\frac{\partial\psi(x,y)}{\partial x}\right|^2 + \left|\frac{\partial\psi(x,y)}{\partial y}\right|^2\right) dxdy \geq \frac{1}{4} \int_{\mathbb{R}\times\mathbb{R}_+} \frac{|\psi(x,y)|^2}{y^2} \, dxdy,$$

where we used the half-line Hardy inequality and criticality of the Laplacian on the whole line. □

Having the Hardy inequality for half-spaces and being inspired by the construction of the Theorem 2.1.1 in dimensions $n \geq 3$ we can easily extend the idea for the half-line, resp. half-plane as follows.

Theorem 2.1.7
Let $V : \mathbb{R}_+ \to (-\infty, 0] \in L^1_{loc}(\mathbb{R}_+)$ *be a real valued multiplication operator such that* $V(H_1 - \lambda)^{-1}$ *is a compact operator for some* $\lambda \in \rho(H_1)$ *and it is* H_1*-bounded with* H_1*-bound smaller than* 1. *Then* $\sigma(H_1 + V) = \sigma_{ess}(H_1 + V) = \sigma(H_1) = [0, +\infty)$ *whenever* $0 \geq V \geq -\frac{1}{4x^2}$.

Theorem 2.1.8
Let $V : \mathbb{R} \times \mathbb{R}_+ \to (-\infty, 0] \in L^1_{loc}(\mathbb{R} \times \mathbb{R}_+)$ be a real-valued multiplication operator such that $V(H_2 - \lambda)^{-1}$ is a compact operator for some $\lambda \in \rho(H_2)$ and it is H_2-bounded with H_2-bound smaller than 1. Then $\sigma(H_2 + V) = \sigma_{ess}(H_2 + V) = \sigma(H_2) = [0, +\infty)$ whenever $0 \geq V \geq -\frac{1}{4y^2}$.

Of course, as in the case of higher dimensions, the natural question is if there is also a stability for general complex-valued potentials. It was proven by Krejčiřík, Laptev and Štampach in 2022 that the answer for the half-line is yes and it is summarized in the following theorem.

Theorem 2.1.9 (Krejčiřík, Laptev, Štampach [37])
Given by $\alpha \in \mathbb{R}$ let H_α be a Laplacian on $L^2(\mathbb{R}_+)$ subjected to the Robin boundary condition $\psi'(0) = \alpha\psi(0)$ and $V : \mathbb{R}_+ \to \mathbb{C}$ complex valued potential.
Then $\sigma(H_\alpha + V) = \sigma_c(H_\alpha + V) = \sigma(H_\alpha) = [0, +\infty)$ whenever $\int_0^\infty |V(x)| \left[1 + \left(\alpha^{-1} + x^2\right)\right] dx < 1$.

We remark that the Theorem 2.1.9 extends the Theorem 2.1.7 not only by complex-valued potentials but also allows general Robin boundary condition.

2.2 Relativistic quantum mechanics

Recently, the study of the mathematical aspects of the relativistic quantum mechanics is enjoying great popularity worldwide; see for example [11, 20, 14, 17, 41, 9, 11, 10]. In relativistic quantum mechanics the system is described by a Dirac operator instead of the Schödinger operator in non-relativistic setting [44]. Many results made in a non-relativistic setting opens a question whether there is a correspondence when one moves to the relativistic mode and the non-trivial matrix structure of the Dirac operators brings new challenges in mathematics in general.

Let us properly introduce the n-dimensional Dirac operator. Given by $n \in \mathbb{N}$ and $m > 0$ the Dirac operator in $\mathbb{R}^n$ reads

$$\mathcal{D}_{n,m} := -i \sum_{k=1}^{n} \partial_k \otimes \alpha_k + m \otimes \alpha_0,$$
$$\mathrm{dom}(\mathcal{D}_{n,m}) := W^{1,2}(\mathbb{R}^n, \mathbb{C}^N).$$

Whereby $N := 2^{\lceil \frac{n}{2} \rceil}$ and $\alpha_i \in \mathbb{C}^{N,N}$ are elements of the Clifford algebra satisfying the anti-commutation relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}\mathbb{I}, \text{ for } i, j \in \{0, \ldots, n\}.$$

It can be shown that $\mathcal{D}_{n,m}$ is self-adjoint on $W^{1,2}(\mathbb{R}^n, \mathbb{C}^N)$ [44].

The very intuitive question in our context is whether there is an analogy between Laplacian and Dirac operators in regard to stability of their spectra. It was proven by Krejřiřík, D'Ancona, Fanelli and Schiavone in 2022 that in dimensions $n \geq 3$ there is a correspondence.

Theorem 2.2.1 (Krejčiřík, D'Ancona, Fanelli, Schiavone, [13])
Let $n \geq 3$ then there are positive constants ϵ and γ such that if

$$\|(|x|^{\frac{1}{2}-\epsilon} + |x|)^2 V\|_\infty < \gamma,$$

then $\sigma(\mathcal{D}_{n,m} + V) = \sigma(\mathcal{D}_{n,m}) = (-\infty, -m] \cup [m, +\infty)$.

The authors proved that the spectrum of a Dirac operator defined over the Euclidean space is stable under small perturbations if the dimension is greater than two and conjectured that it is not possible in dimensions two and one. The question of how is it with the stability for the half-line is subject of this thesis and is answered in the next, last chapter. Our goal is to find a stability theorem for half-line the Dirac operator corresponding with the result [37, Remark 21] made for a half-line Laplacian.

Chapter 3

Model $\mathcal{D}_\alpha$

From now on we shall consider Hilbert space $\mathcal{H} := L^2(\mathbb{R}_+, \mathbb{C}^2)$. The standard norm and inner product on $\mathcal{H}$ will be denoted as $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. The latter will be assumed to be antilinear in its first argument.

Given by real parameters $\alpha \in (0, \frac{\pi}{2})$ and $m \geq 0$ we consider a Dirac operator acting on $\mathcal{H}$ as

$$\mathcal{D}_\alpha := -i\frac{d}{dx} \otimes \sigma_2 + m \otimes \sigma_3,$$

with its operator domain

$$\operatorname{dom}(\mathcal{D}_\alpha) := \left\{ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in W^{1,2}(\mathbb{R}_+, \mathbb{C}^2) \mid \phi_1(0)\cot(\alpha) = \phi_2(0) \right\},$$

where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. We remark that $\alpha := \frac{\pi}{4}$ corresponds to the so-called infinity-mass boundary condition introduced in [4], sometimes also referred to as the MIT boundary condition [2].

Half-line Dirac operators with various boundary conditions have already been studied from several aspects (see [27, 35, 20]). Especially, we refer to [9] where perturbed Dirac operators with the same boundary conditions were studied but from a different point of view. The author set estimates for non-embedded eigenvalues of $\mathcal{D}_\alpha$ perturbed by a matrix-valued (not necessarily self-adjoint) potential.

Self-adjointness

Regarding the Birman-Schwinger principle, we draw primarily from the work of Hansmann and Krejčiřík [28], whose results are stated for self-adjoint operators. We show that, indeed for all $\alpha \in (0, \frac{\pi}{2})$ the operator $\mathcal{D}_\alpha$ is self-adjoint. As we discussed above in the first chapter, the corresponding adjoint operator is by the definition

$$\operatorname{dom}(\mathcal{D}_\alpha^*) = \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in L^2(\mathbb{R}_+, \mathbb{C}^2) \mid \exists \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} : \forall \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \operatorname{dom}(\mathcal{D}_\alpha) : (\eta, \phi) = (\psi, \mathcal{D}_\alpha \phi) \right\},$$

$$\mathcal{D}_\alpha^* \psi : = \eta.$$

Without loss of generality, we will prove self-adjointness only for the case $m = 0$ since the term $m \otimes \sigma_3$ can be viewed as a bounded self-adjoint perturbation. Let us denote $(\cdot,\cdot)_2$ the standard inner product on $\mathbb{C}^2$ being antilinear in its first argument. We will show that $\mathcal{D}_\alpha \subset \mathcal{D}_\alpha^*$ and $\mathrm{dom}(\mathcal{D}_\alpha^*) \subset \mathrm{dom}(\mathcal{D}_\alpha)$. Let us start with the first. For all $\psi, \phi \in \mathrm{dom}(\mathcal{D}_\alpha)$ we have

$$
\begin{aligned}
(\psi, \mathcal{D}_\alpha\phi) = (\psi, -i\sigma_2\phi') &= \int_{\mathbb{R}_+} (\psi(x), \mathcal{D}_\alpha\phi(x))_2\, dx \\
&= \int_{\mathbb{R}_+} (-\psi_1^*(x)\phi_2' + \psi_2^*(x)\phi_1(x))\, dx \\
&= [-\psi_1^*\phi_2 + \psi_2^*\phi_1(x)]_0^\infty + \int_{\mathbb{R}_+} (-\psi_2^{*\prime}(x)\phi_1(x) + \psi_1^{*\prime}\phi_2(x))\, dx \\
&= \psi_1^*(0)\phi_2(0) - \psi_2^*(0)\phi_1(0) + (-i\sigma_2\psi', \phi) \\
&= [\psi_1(0)\cot(\alpha) - \psi_2(0)]^*\, \phi_1(0) + (-i\sigma_2\psi', \phi) \\
&= (\mathcal{D}_\alpha\psi, \phi)\,.
\end{aligned}
$$

On the other hand, let $\psi \in \mathrm{dom}(\mathcal{D}_\alpha^*)$ be an arbitrary function from the domain of the adjoint. This means that there is $\eta \in L^2(\mathbb{R}_+, \mathbb{C}^2)$ such that for all $\phi \in \mathrm{dom}(\mathcal{D}_\alpha)$ we have $(\eta, \phi) = (\psi, \mathcal{D}_\alpha\phi)$, i.e.

$$
\int_{\mathbb{R}_+} (\eta_1^*(x)\phi_1(x) + \eta_2^*\phi_2(x))\, dx = \int_{\mathbb{R}_+} (-\psi_1^*(x)\phi_2'(x) + \psi_2^*\phi_1'(x))\, dx. \tag{3.1}
$$

Since (3.1) holds for all $\phi \in \mathrm{dom}(\mathcal{D}_\alpha)$ it holds for the special choice $\phi_1 = 0$, resp. $\phi_2 = 0$. This implies that

$$
\begin{aligned}
\int_{\mathbb{R}_+} \eta_1^*(x)\phi_1(x)dx &= \int_{\mathbb{R}_+} \psi_2^*(x)\phi_1'(x)dx, \\
\int_{\mathbb{R}_+} \eta_2^*(x)\phi_2(x)dx &= \int_{\mathbb{R}_+} -\psi_1^*(x)\phi_2'(x)dx,
\end{aligned}
$$

for all $\phi_1, \phi_2 \in W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^2)$ and therefore for all $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$. From the definition 1.1.2 is then $\eta = -i\sigma_2\psi'$ in the weak sense, and hence $\psi \in W^{1,2}(\mathbb{R}_+, \mathbb{C}^2)$. Integrating by parts, one finds out that $\psi_1(0)\cot(\alpha) = \psi_2(0)$ and thus $\psi \in \mathrm{dom}(\mathcal{D}_\alpha)$. Moreover, it can be shown [46] that the spectrum of the „free„ Dirac operators $\mathcal{D}_\alpha$ is

$$
\sigma(\mathcal{D}_\alpha) = \sigma_c(\mathcal{D}_\alpha) = (-\infty, -m] \cup [m, +\infty).
$$

3.1 Resolvent $(\mathcal{D}_\alpha - z)^{-1}$

Since the Birman-Schwinger operator is constructed from the resolvent of the unperturbed operator, explicit knowledge of the resolvent $(\mathcal{D}_\alpha - z)^{-1}$ will be absolutely essential. We now derive the resolvent by the method of images from the whole-line resolvent.

Whole-line resolvent

Let us consider the whole-line Dirac operator $\mathcal{D}$ defined as

$$\mathcal{D} := -i\frac{d}{dx} \otimes \sigma_2 + m \otimes \sigma_3,$$
$$\mathrm{dom}(\mathcal{D}) := W^{1,2}(\mathbb{R}, \mathbb{C}^2).$$

Observing that

$$(\mathcal{D} - z)(\mathcal{D} + z)\,\phi = \left(-\Delta_D^{\mathbb{R}} - z^2 + m^2\right)\phi, \tag{3.2}$$

for all $\phi \in \mathrm{dom}\left((-\Delta_D^{\mathbb{R}} - z^2 + m^2) \otimes \mathbb{1}\right) = W^{2,2}(\mathbb{R}, \mathbb{C}^2)$ we can write for all values of the resolvent parameter $z \in \rho(\mathcal{D}) = (-\infty, -m] \cup [m, +\infty)$ (see [44])

$$\begin{aligned}(\mathcal{D} - z)^{-1}\,\phi &= (\mathcal{D} + z)\left(-\Delta_D^{\mathbb{R}} - z^2 + m^2\right)^{-1}\phi \\ &= \int_{\mathbb{R}} (\mathcal{D} + z)\,\mathcal{G}(x, y; z^2 - m^2)\phi(y)dy \\ &=: \int_{\mathbb{R}} \mathcal{R}(x, y; z)\phi(y)dy,\end{aligned}$$

where [28] $\mathcal{G}(x, y; z) = i\,\frac{\exp\left(i\sqrt{z}\,|x-y|\right)}{2\sqrt{z}}$ and $\sqrt{z}$ is chosen such that $\mathrm{Im}(z) > 0$. By a straight-forward calculation we have the whole-line Dirac resolvent kernel in the form

$$\mathcal{R}(x, y; z) = \frac{1}{2}\begin{pmatrix} \zeta(z) & \mathrm{sgn}(x-y) \\ -\mathrm{sgn}(x-y) & \zeta^{-1}(z)\end{pmatrix}\exp(ik(z)|x-y|), \tag{3.3}$$

with $k(z) := \sqrt{z^2 - m^2}$ and $\zeta(z) := \frac{z+m}{k(z)}$.

Half-line resolvent

We now derive the half-line Dirac resolvent using the knowledge of the resolvent of the Dirac operator on the whole line by embedding $L^2(\mathbb{R}_+, \mathbb{C}^2)$ into $L^2(\mathbb{R}, \mathbb{C}^2)$ as follows.

Let $\phi \in L^2(\mathbb{R}_+, \mathbb{C}^2)$ be a function on the half-line and $\mathbb{A} := \mathrm{diag}(\mu_1, \mu_2) \in \mathbb{C}^{2,2}$ be a diagonal matrix. For every such ϕ we define $\phi_{\mathbb{A}}(x) := \phi(|x|)\Theta(x) + \mathbb{A}\phi(|x|)\Theta(-x) \in L^2(\mathbb{R}, \mathbb{C}^2)$.
For all $\phi \in L^2(\mathbb{R}_+, \mathbb{C})$ we then have

$$\left((\mathcal{D} - z)^{-1}\,\phi_{\mathbb{A}}\right)(x) = \int_{\mathbb{R}} \mathcal{R}(x, y; z)\phi_{\mathbb{A}}(y)dy = \int_0^\infty (\mathcal{R}(x, y; z) + \mathcal{R}(x, -y; z)\mathbb{A})\,\phi(y)dy = \begin{pmatrix}\xi_1(x) \\ \xi_2(x)\end{pmatrix}.$$

For every such ϕ we demand the boundary condition of the image to be met:

$$\xi_1(0)\cot(\alpha) \stackrel{!}{=} \xi_2(0).$$

This gives us the following equation.

$$\begin{aligned}&\int_0^\infty [\,(\mathcal{R}_{11}(0, y; z) + \mu_1\mathcal{R}_{11}(0, -y; z))\cot(\alpha) - \mathcal{R}_{21}(0, y; z) - \mu_1\mathcal{R}_{21}(0, -y; z)]\,\phi_1(y)dy \\ = &\int_0^\infty [\mathcal{R}_{22}(0, y; z) + \mu_2\mathcal{R}_{22}(0, -y; z) - (\mathcal{R}_{12}(0, y; z) + \mu_2\mathcal{R}_{12}(0, -y; z))\cot(\alpha)]\,\phi_2(y)dy.\end{aligned}$$

Since this equation has to be satisfied for every $\phi \in L^2(\mathbb{R}_+, \mathbb{C}^2)$, for the special choice $\phi_1 = 0$, resp. $\phi_2 = 0$ it breaks down into two independent conditions

$$
\begin{aligned}
(\mathcal{R}_{11}(0, y; z) + \mu_1 \mathcal{R}_{11}(0, -y; z)) \cot(\alpha) &= \mathcal{R}_{21}(0, y; z) + \mu_1 \mathcal{R}_{21}(0, -y; z), \\
(\mathcal{R}_{12}(0, y; z) + \mu_2 \mathcal{R}_{12}(0, -y; z)) \cot(\alpha) &= \mathcal{R}_{22}(0, y; z) + \mu_2 \mathcal{R}_{22}(0, -y; z),
\end{aligned}
$$

for all $y \in \mathbb{R}_+$. From the expression 3.3 we obtain a unique solution for μ_1, μ_2 and the matrix $\mathbb{A}$ reads

$$
\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1 - i\zeta(z)\cot(\alpha)}{1 + i\zeta(z)\cot(\alpha)} = \sigma_3 \frac{1 - i\zeta(z)\cot(\alpha)}{1 + i\zeta(z)\cot(\alpha)}.
$$

The resolvent kernel of the resolvent $(\mathcal{D}_\alpha - z)^{-1}$ is then given as

$$
\mathcal{R}_\alpha(x, y; z) = \mathcal{R}(x, y; z) + \mathcal{R}(x, -y; z)\mathbb{A},
$$

for all $x, y \in \mathbb{R}_+$. By a straightforward calculation one finds out that the resolvent kernel has the following structure

$$
\mathcal{R}_\alpha(x, y; z) = R_\alpha(x, y; z)\Theta(x - y) + R_\alpha^T(y, x; z)\Theta(y - x).
$$

For $x < y$, the matrix $R_\alpha(x, y; z)$ component-wise read

$$
\begin{aligned}
[R_\alpha(x, y; z)]_{11} &= \frac{i\zeta(z)}{1 + i\zeta(z)\cot(\alpha)} \exp(ik(z)x) \left[\cos(k(z)y) + \zeta(z)\sin(k(z)y)\right], \\
[R_\alpha(x, y; z)]_{12} &= \frac{i\zeta(z)}{1 + i\zeta(z)\cot(\alpha)} \exp(ik(z)x) \left[\cot(\alpha)\cos(k(z)y) - \zeta(z)^{-1}\sin(k(z)y)\right], \\
[R_\alpha(x, y; z)]_{21} &= \frac{-1}{1 + i\zeta(z)\cot(\alpha)} \exp(ik(z)x) \left[\cos(k(z)y) + \zeta(z)\sin(k(z)y)\right], \\
[R_\alpha(x, y; z)]_{22} &= \frac{-1}{1 + i\zeta(z)\cot(\alpha)} \exp(ik(z)x) \left[\cot(\alpha)\cos(k(z)y) - \zeta(z)^{-1}\sin(k(z)y)\right],
\end{aligned}
$$

where $W(z) := 1 + i\zeta(z)\cot(\alpha)$. We remind that the square root is chosen such that $\mathrm{Im}[k(z)] > 0$. However, it is not difficult to see that there is an additional inner structure of $R_\alpha(x, y; z)$. Let us for all $\alpha \in \left(0, \frac{\pi}{2}\right)$ define the functions $\psi_\alpha(x; z)$, $\phi_\alpha(y; z)$ as follows

$$
\begin{aligned}
\psi_\alpha(x; z) &:= \exp(ik(z)x) \begin{pmatrix} i\zeta(z) \\ -1 \end{pmatrix}, \\
\phi_\alpha(y; z) &:= \begin{pmatrix} \cos(k(z)y) + \zeta(z)\cot(\alpha)\sin(k(z)y) \\ \zeta(z)^{-1}\sin(k(z)y) + \cot(\alpha)\cos(k(z)y) \end{pmatrix},
\end{aligned}
$$

Using this notation, it is easy to verify that the resolvent kernel can be compactly expressed in the following formula

$$
\mathcal{R}_\alpha(x, y; z) = \frac{1}{W(z)} \left[\psi_\alpha(x; z)\,(\phi_\alpha^*(y; z), \cdot)_2\,\Theta(x - y) + \phi_\alpha(x; z)\,(\psi_\alpha^*(y; z), \cdot)_2\,\Theta(y - x)\right]. \tag{3.4}
$$

3.2 Resolvent analysis

Throughout this section, we keep the notation for the inner product on $\mathbb{C}^2$ as $(\cdot,\cdot)_2$, and as customary, the euclidean norm on $\mathbb{C}^2$ is denoted by $|\cdot|_2$. The main goal of this section is to prove the following lemma.

Lemma 3.2.1
Let $\alpha \in (0, \frac{\pi}{2})$ be a real parameter. Then

$$\sup_{z\in\rho(\mathcal{D}_\alpha)} \|\mathcal{R}_\alpha(x,y;z)\|^2_{\mathcal{L}(\mathbb{C}^2)} = 1 + (q + 2m\min(x,y))^2,$$

where $q = \max(\cot(\alpha), \cot(\alpha)^{-1})$.

The norm of the resolvent kernel $\mathcal{R}_\alpha(x,y;z)$ as an operator $\mathbb{C}^2 \to \mathbb{C}^2$ reads

$$\|\mathcal{R}_\alpha(x,y;z)\|_{\mathcal{L}(\mathbb{C}^2)} = \frac{1}{|W(z)|}\left[|\psi_\alpha(x;z)|_2|\phi_\alpha(y;z)|_2\Theta(x-y) + |\psi_\alpha(y;z)|_2|\phi_\alpha(x;z)|_2\Theta(y-x)\right].$$

Without loss of generality, we will futher consider only the case $x > y$. In addition, to simplify further expressions, let us introduce the following notation.

$$\begin{aligned}
\eta_1 &:= \frac{1}{4}\left(|1 - i\zeta(z)\cot(\alpha)|^2 + |\cot(\alpha) + i\frac{1}{\zeta(z)}|^2\right),\\
\eta_2 &:= \frac{1}{4}\left(|1 + i\zeta(z)\cot(\alpha)|^2 + |\cot(\alpha) - i\frac{1}{\zeta(z)}|^2\right),\\
\eta_3 &:= \frac{1}{2}\left(1 - \cot^2(\alpha)|\zeta(z)|^2\right),\\
\eta_4 &:= \frac{1}{2}\left(2\operatorname{Re}[\zeta(z)]\cot(\alpha)\right).
\end{aligned}$$

One then finds the $\mathbb{C}^2$-norm of ψ_α and ϕ_α by a straightforward calculation as

$$\begin{aligned}
|\psi_\alpha(x;z)|_2^2 &= \exp(-2\operatorname{Im}[k(z)]x)\left(|\zeta(z)|^2 + 1\right),\\
|\phi_\alpha(y;z)|_2^2 &= \eta_1(z)\exp(-2\operatorname{Im}[k(z)]y) + \eta_2(z)\exp(2\operatorname{Im}[k(z)]y)\\
&\quad + \left(1 - \frac{1}{|\zeta(z)|^2}\right)(\eta_3(z)\cos(2\operatorname{Re}[k(z)]y) + \eta_4(z)\sin(2\operatorname{Re}[k(z)]y)).
\end{aligned}$$

Since the sums, products, and compositions of holomorphic functions are holomorphic, it is obvious that the resolvent kernel $\mathcal{R}_\alpha(x,y;z)$ is a holomorphic function of the spectral parameter z. Therefore, the supremum of its modulus can not be achieved in the resolvent set. This fact follows from the maximum modulus principle.

Theorem 3.2.1 (Maximum modulus principle, [6, Corollary 5.10])
Let $\Omega \subset \mathbb{C}$ be an open connected set, f holomorphic function on Ω. If $|f|$ attains a maximum in Ω, then f is constant.

On the other hand, the supremum itself has to exist. Therefore, either the supremum lies in the complex infinity or it is achieved somewhere in the spectrum. Now we show that the latter is true.

At first, we show that $\|\mathcal{R}_\alpha(x, y; z)\|_{\mathcal{L}(\mathbb{C}^2)}$ can be uniformly bounded as $z \to \infty$. By an estimation of the exponential functions one can see that

$$\begin{aligned}
&\|\mathcal{R}_\alpha(x, y; z)\|^2_{\mathcal{L}(\mathbb{C}^2)}\Theta(x-y) = \frac{1}{|W|}|\psi_\alpha(x; z)|_2^2|\phi_\alpha(y; z)|_2^2\Theta(x-y)\\
&\le \frac{|\zeta(z)|^2+1}{|W|}\left[\eta_1(z)+\eta_2(z)+\left(1-\frac{1}{|\zeta(z)|^2}\right)(\cos(2\operatorname{Re}[k(z)]y)\eta_3(z)+\sin(2\operatorname{Re}[k(z)]y)\eta_4(z))\right]\\
&=: \gamma(y; z),
\end{aligned}$$

for all $z \in \rho(\mathcal{D}_\alpha)$. Although the complex limit at infinity of the norm does not exist, it does exist for the function γ; $\lim\limits_{z\to\infty}\gamma(y; z) = 2$.

This follows from the fact $\lim\limits_{z\to\infty}\zeta(z) = \lim\limits_{z\to\infty}\sqrt{\frac{z+m}{z-m}} = 1$. Then for every $\epsilon > 0$ there is $M > 0$ such that for every $z \in \mathbb{C}$; $|z| > M$ we have

$$\|\mathcal{R}_\alpha(x, y; z)\|^2_{\mathcal{L}(\mathbb{C}^2)}\Theta(x-y) \le \gamma(y; z) < 2+\epsilon. \tag{3.5}$$

Now, we investigate the behavior of the restriction $\|\mathcal{R}_\alpha(x, y; z)\|^2_{\mathcal{L}(\mathbb{C}^2)}$ to the spectrum. That is $z := u \in \sigma(\mathcal{D}_\alpha) = (-\infty, -m] \cup [m, +\infty)$. The coefficients $k(u)$ and $\zeta(u)$ are now purely real, and the restriction of $\|\mathcal{R}_\alpha(x, y; z)\|^2_{\mathcal{L}(\mathbb{C}^2)}$ reads

$$\begin{aligned}
\|\mathcal{R}_\alpha(x, y; u)\|^2_{\mathcal{L}(\mathbb{C}^2)}\Theta(x-y) &= \frac{2u^2}{u^2-m^2} + \cos(2k(u)y)\frac{u-m-\cot^2(\alpha)(u+m)}{u-m+\cot^2(\alpha)(u+m)}\frac{2mu}{u^2-m^2}\\
&+ \sin(2k(u)y)\frac{\cot(\alpha)4m}{\sqrt{u^2-m^2}}\frac{1}{u-m+\cot^2(\alpha)(u+m)} =: \chi(u, y).
\end{aligned}$$

We show that the function $\chi(u, y)$ has no local extremes for $u \in \sigma(\mathcal{D}_\alpha)$. Indeed, the partial derivation of $\chi(u, y)$ with respect to u;

$$\frac{\partial\chi}{\partial u}(u, y) = \xi_1(u) + \xi_2(u)\cos(2k(u)y) + \xi_3(u)\cos(2k(u)y)y \tag{3.6}$$

$$+ \xi_4(u)\sin(2k(u)y) + \xi_5(u)\sin(2k(u)y)y = 0, \tag{3.7}$$

can be viewed as the linear combination of linearly independent functions in variable y for every fixed $u \in \sigma(\mathcal{D}_\alpha)$. Therefore, the only possible solution of equation (3.6) is $\xi_i = 0$ for every $i \in \{1, \ldots, 5\}$. By calculation of ξ_1 one finds that this can be satisfied only for $u = 0$;

$$\xi_1(u) = \frac{4mu}{u^2-m^2} = 0 \Leftrightarrow u = 0 \notin (-\infty, -m] \cup [m, +\infty).$$

The supremum of $\chi(u, y)$ (in variable u) is then the maximum of the limits at the „boundaries„. These are

$$\lim_{u\to\pm\infty} \chi(u, y) = 2,$$
$$\lim_{u\to -m} \chi(u, y) = 1 + (\cot(\alpha) + 2my)^2 ,$$
$$\lim_{u\to m} \chi(u, y) = 1 + \left(\cot(\alpha)^{-1} + 2my\right)^2 .$$

Thus, we conclude that

$$\sup_{z\in\rho(\mathcal{D}_\alpha)} \|\mathcal{R}_\alpha(x, y; u)\|^2_{\mathcal{L}(\mathbb{C}^2)} = 1 + (q + 2m \min(x, y))^2,$$

with $q := \max(\cot(\alpha), \cot(\alpha)^{-1})$.

3.3 Stability Theorem

Consider a generic potential $V : \mathbb{R}_+ \to \mathbb{C}^{2,2} \in L^1(\mathbb{R}_+, \mathbb{C}^{2,2}) \cap L^\infty(\mathbb{R}_+, \mathbb{C}^{2,2})$. We denote its L^1 norm as

$$\|V\|_1 := \int_{\mathbb{R}_+} \|V(x)\|_{\mathcal{L}(\mathbb{C}^2)}\, dx.$$

Since V is an essentially bounded funtion, the corresponding multiplication operator, denoted by the same symbol is bounded and therefore everywhere defined. Therefore, the perturbed operator $\mathcal{D}_\alpha + V$ is closed on $\mathrm{dom}(\mathcal{D}_\alpha)$.

Birman-Schwinger operator

In our concerte setting, speaking in the notation of the theorem 1.3.3, we have $\mathcal{H} = \tilde{\mathcal{H}} = L^2(\mathbb{R}_+, \mathbb{C}^2)$ and V is a multiplication operator generated by a matrix-valued function introduced above. For V we consider the polar decomposition $V = U|V|$, where $|V| := \sqrt{V^*V}$ and U is a partial isometry. We put $A := \sqrt{|V|}$, $B := \sqrt{|V|}U^*$. The corresponding multiplication operators are denoted by the same symbol; A and B, respectively. Since V is bounded, the square root is bounded as well and the assumptions of the definition 1.3.2 are met.

Note that the operator $K(z)$ from the definition 1.3.2 is defined as a bounded extension of the "formal" Birman-Schwinger operator $A(H_0 - z)^{-1}B^*$ acting on $\mathrm{dom}(B^*)$. Since in our setting B^* is defined everywhere on $\mathcal{H}$, K_z is $A(H_0 - z)^{-1}B^*$. This operator acts as an integral operator on $\mathcal{H}$ with its integral kernel

$$\mathbb{K}_\alpha(x, y; z) = |V(x)|^{\frac{1}{2}}\mathcal{R}_\alpha(x, y; z)U|V(y)|^{\frac{1}{2}}.$$

Since it is not possible to express the norm of the operator $K_\alpha(z)$ explicitly, we will estimate the norm of $K_\alpha(z)$ by its Hilbert-Schmidt norm $\|K_\alpha(z)\|_{HS}$ defined as follows.

$$\begin{aligned}
|(\phi, K_\alpha(z)\psi)| &\leq \int_0^\infty \int_0^\infty |(\phi(x), \mathbb{K}_\alpha(x,y;z)\psi(y))_2| \, dxdy \\
&\leq \int_0^\infty \int_0^\infty |\phi(x)| \, |\psi(y)| \, \|\mathbb{K}_\alpha(x,y;z)\|_{\mathcal{L}(\mathbb{C}^2)} \, dxdy \\
&\leq \left(\int_0^\infty |\phi(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^\infty |\psi(y)|^2 dy\right)^{\frac{1}{2}} \left(\int_0^\infty \int_0^\infty \|\mathbb{K}_\alpha(x,y;z)\|^2_{\mathcal{L}(\mathbb{C}^2)} dxdy\right)^{\frac{1}{2}} \\
&=: \|\phi\| \, \|\psi\| \, \|K_\alpha(z)\|_{HS},
\end{aligned}$$

for every $\phi \in \mathcal{H}$, $\psi \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$. The penultimate inequality was obtained by the Schwarz-Cauchy inequality in both x and y variables. For the supremum of $\|K_\alpha(z)\|$ we have

$$\sup_{z\in\rho(\mathcal{D}_\alpha)} \|K_\alpha(z)\|^2 \leq \sup_{z\in\rho(\mathcal{D}_\alpha)} \|K_\alpha(z)\|^2_{HS} = \sup_{z\in\rho(\mathcal{D}_\alpha)} \int_0^\infty \int_0^\infty \|\mathbb{K}_\alpha(x,y;z)\|^2_{\mathcal{L}(\mathbb{C}^2)} dxdy \tag{3.8}$$

$$\leq \sup_{z\in\rho(\mathcal{D}_\alpha)} \int_0^\infty \int_0^\infty \|V(x)\|_{\mathcal{L}(\mathbb{C}^2)} \, \|V(y)\|_{\mathcal{L}(\mathbb{C}^2)} \, \|\mathcal{R}_\alpha(x,y;z)\|^2_{\mathcal{L}(\mathbb{C}^2)} dxdy \tag{3.9}$$

$$= \int_0^\infty \int_0^\infty \|V(x)\|_{\mathcal{L}(\mathbb{C}^2)} \, \|V(y)\|_{\mathcal{L}(\mathbb{C}^2)} (1 + (q + 2m \min(x,y))^2) dxdy \tag{3.10}$$

$$\leq \|V\|_1^2 + \left(\int_0^\infty \|V(x)\|_{\mathcal{L}(\mathbb{C}^2)} \, (q + 2mx) \; dx\right)^2, \tag{3.11}$$

where we estimated the minimum as $\min(x,y) \leq x$ and $\min(x,y) \leq y$, for all $x, y \in \mathbb{R}_+$ to express the integrals explicitly. Using the Birman-Schwinger principle 1.3.3 this can be summarized in the following theorem.

Theorem 3.3.1
Let $V : \mathbb{R}^+ \to \mathbb{C}^{2,2}$ be an essentially bounded matrix-valued potential.
Then $\sigma(\mathcal{D}_\alpha + V) = \sigma_c(\mathcal{D}_\alpha + V) = \sigma(\mathcal{D}_\alpha)$ whenever the following condition holds;

$$\|V\|_1^2 + \left(\int_0^\infty \|V(x)\| \, (q + 2mx) \; dx\right)^2 < 1, \tag{3.12}$$

where $q = \max(\cot(\alpha), \cot(\alpha)^{-1})$.

3.4 Open problem

What remains an open problem with respect to the result is its optimality.

Let us briefly summarize the result that we have just proven. For a given potenial V we denote

$$C(V) := \int_0^\infty \int_0^\infty \|V(x)\| \, \|V(y)\| (1 + (q + 2m \min(x,y))^2) dxdy.$$

Our result then means that whenever $C(V) < 1$ we have $\sigma(\mathcal{D}_\alpha + V) = \sigma_c(\mathcal{D}_\alpha + V) = \sigma(\mathcal{D}_\alpha)$. The interesting question now is how much did we „waste„ by estimating the operator norm of $K(z)$ by its Hilbert-Schmidt norm in (3.8). In other words, whether the condition $C(V) < 1$ is optimal. To be more precise, what we understand under optimality here is if there is a potential V such that $C(V) > 1$ and $\sigma_p(\mathcal{D}_\alpha + V) \neq \emptyset$.

In [11] Cuenin, Laptev and Tretter were dealing with localization of the non-embedded eigenvalues of a Dirac operator $\mathcal{D}$ defined on the whole line perturbed by an L^1 potential V. The authors have proven that if $\|V\|_1 < 1$ then all non-embedded eigenvalues lie in the union of two disks, i.e.

$$\sigma_{disc}(\mathcal{D} + V) \subset B(-x_0, r_0) \cup B(x_0, r_0),$$

where the radius $r_0 := \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1-\|V\|_1^2)} - \frac{1}{2}}$ and the point $x_0 := \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1-\|V\|_1^2)} + \frac{1}{2}}$ are determined by the L^1 norm of V. Furthermore, the authors managed to prove the sharpness of their result in the sense that there is a V_{sharp}, such that $\|V_{sharp}\|_1 < 1$ and $\lambda \in \sigma_{disc}(\mathcal{D} + V_{sharp})$ such that $\lambda \in \partial B(-x_0, r_0) \cup \partial B(x_0, r_0)$. The sharpening potential V_{sharp} was chosen as a family of delta potentials - point interactions - given formally in the form

$$V_{sharp} := i\kappa\delta(x) \begin{pmatrix} e^{i\tau} & 0 \\ 0 & e^{-i\tau} \end{pmatrix},$$

for $\kappa > 0$ and $-\pi \leq \tau \leq \pi$.

Motivated by this approach, we suggest looking for a critical potential which would prove the optimality as we stated above in the form of a delta potential as well. We propose to bounce the point interactions from the boundary $x = 0$ since for such potentials is the optimality achieved for the Schrödinger operator on a half-line [24].

Let us consider a family of formal delta potentials generated by a formal expression

$$V := \delta(x - a) \otimes \mathbb{A},$$

for $a > 0$ and $\mathbb{A} \in \mathbb{C}^{2,2}$. Formally speaking, the perturbed operator $\tilde{H}_\alpha^{\mathbb{A}}$ reads

$$\tilde{H}_\alpha^{\mathbb{A}}\phi = (\mathcal{D}_\alpha + \delta(x - a) \otimes \mathbb{A})\,\phi. \tag{3.13}$$

One of the possible ways to properly define the above operator is through the boundary condition [30]. The idea is to exclude the point of the interaction out of the domain and introduce an appropriate following boundary condition.

The motivation comes from the requirement for the integrability of the image $\tilde{H}_\alpha^{\mathbb{A}}$. Indeed, in the distributional sense, $\tilde{H}_\alpha^{\mathbb{A}}$ acts as

$$\begin{aligned}
\tilde{H}_\alpha^{\mathbb{A}}\phi &= m\sigma_3\phi - i\sigma_2\phi' + \frac{\mathbb{A}\phi(a^+) + \mathbb{A}\phi(a^-)}{2} \\
&= m\sigma_3\phi - i\sigma_2\{\phi'\} - i\sigma_2\left(\phi(a^+) - \phi(a^-)\right) + \frac{\mathbb{A}\phi(a^+) + \mathbb{A}\phi(a^-)}{2} \\
&\stackrel{!}{=} m\sigma_3\phi - i\sigma_2\{\phi'\},
\end{aligned}$$

for all $\phi \in \mathrm{dom}(\mathcal{D}_\alpha)$. In order to assign to the image $\tilde{H}_\alpha^{\mathbb{A}}\phi$ a meaningful role as an element of $L^2(\mathbb{R}_+, \mathbb{C}^2)$ we employ the following boundary condition.

$$(\mathbb{A} - 2i\sigma_2)\,\phi(a^+) = -\,(\mathbb{A} + 2i\sigma_2)\,\phi(a^-), \tag{3.14}$$

where we denote $\phi(a^\pm) := \lim_{x \to a\pm} \phi(x)$. In this way, the integrability of the image $\tilde{H}_\alpha^{\mathbb{A}}\phi$ is ensured. The proper definition of the point interaction $\tilde{H}_\alpha^{\mathbb{A}}$ reads

$$\tilde{H}_\alpha^{\mathbb{A}}\phi := \mathcal{D}_\alpha \phi \tag{3.15}$$

$$\mathrm{dom}(\tilde{H}_\alpha^{\mathbb{A}}) := \left\{ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in W^{1,2}(\mathbb{R}_+ \backslash \{a\}, \mathbb{C}^2) \mid \phi_1(0)\cot(\alpha) = \phi_2(0) \wedge (3.14) \right\}. \tag{3.16}$$

It was shown by Hughes [32, 31] and then generalized by Tušek and Heriban [45, 30] that such delta potential perturbing the Dirac operator $\mathcal{D}$ defined on the whole real-line can be approximated by L^1 potentials as a limit $\epsilon \to 0$ (in the norm-resolvent sense) of the following family of operators

$$\mathcal{D}_\epsilon^{\mathbb{A}} := \mathcal{D} + h_\epsilon(x) \otimes \mathbb{A}. \tag{3.17}$$

Whereby $h_\epsilon(x) := \frac{1}{\epsilon} h\left(\frac{x}{\epsilon}\right)$, $\int_{\mathbb{R}} h(x)dx = 1$ and the matrix $\mathbb{A}$ must satisfy certain technical assumptions, see [45, 30] for details. However, for the initial setting (3.17) the norm-resolvent limit of $\mathcal{D}_\epsilon^{\mathbb{A}}$ is

$$\mathcal{D}^{\mathbb{A}}\phi := \mathcal{D}\phi \tag{3.18}$$

$$\mathrm{dom}(\mathcal{D}^{\mathbb{A}}) := \left\{ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in W^{1,2}(\mathbb{R} \backslash \{a\}, \mathbb{C}^2) \mid \phi(a^+) = \Lambda \phi(a^-) \right\}, \tag{3.19}$$

where $\Lambda = \mathbb{B}\exp(\mathbb{A})$ and $\mathbb{B} \in \mathbb{C}^{2,2}$ is only a multiplicative factor. However, it is not clear what should be the „norm„ of such a perturbation. We propose to compare the formal boundary condition (3.14) with norm resolvent boundary condition (3.19). The associated formal point interaction corresponding to (3.19) is then $W := \delta(x - a) \otimes \mathbb{D}$, where $\mathbb{D}$ is given by the equation

$$\Lambda = -\,(\mathbb{D} - 2i\sigma_2)^{-1}\,(\mathbb{D} + 2i\sigma_2)\,.$$

That is, $\mathbb{D} = -2i\sigma_2\,(\mathbb{1} - \Lambda)\,(\mathbb{1} + \Lambda)^{-1}$. In the context of this correspondence, it seems meaningful to assign $\|W\|_{L^1} := \|\mathbb{D}\|$.

As a continuation of our work we suggest to investigating the possibility of extension of the results of Tušek and Heriban [45, 30] to the half-line setting of ours. In the positive case, consider the following realization of the operator (3.15)

$$H_\alpha^{\mathbb{A}}\phi := \mathcal{D}_\alpha \phi$$

$$\mathrm{dom}(H_\alpha^{\mathbb{A}}) := \left\{ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in W^{1,2}(\mathbb{R}_+ \backslash \{a\}, \mathbb{C}^2) \mid \phi_1(0)\cot(\alpha) = \phi_2(0) \wedge \phi(a^+) = \Lambda\phi(a^-) \right\},$$

analyze the spectral properties of $H_\alpha^{\mathbb{A}}$. If we were able to find for all $\delta \in \mathbb{R}_+$ a particular matrix $\mathbb{A}$ such that $\|\mathbb{D}\| = 1 + \delta$ and $\sigma_{disc}(H_\alpha^{\mathbb{A}}) \neq \emptyset$ we know that there is a L^1 potential with the same properties and therefore the stability theorem 3.3.1 is optimal.

Conclusion

In this master's thesis we considered a one-parametric half-line Dirac operator

$$\mathcal{D}_\alpha := -i\frac{d}{dx} \otimes \sigma_2 + m \otimes \sigma_3, \tag{3.20}$$

$$\operatorname{dom}(\mathcal{D}_\alpha) := \left\{ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in W^{1,2}(\mathbb{R}_+, \mathbb{C}^2) \mid \phi_1(0)\cot(\alpha) = \phi_2(0) \right\}, \tag{3.21}$$

previously studied in [9]. Motivated by the result of Krejčiřík, Štampach and Laptev [37, Remark 21] we were interested in the existence of a spectral stability of $\mathcal{D}_\alpha$. In other words, we were interested whether there is a uniform condition for a given multiplication operator V such that whenever the condition is met one has $\sigma(\mathcal{D}_\alpha + V) = \sigma(\mathcal{D}_\alpha)$.

We showed that $\mathcal{D}_\alpha$ defined on the domain (3.21) is self-adjoint and derived an explicit formula of its resolvent $(\mathcal{D}_\alpha - z)^{-1}$ in closed form and analyzed it. We found the supremum of its norm as an operator from $\mathbb{C}^2$ to $\mathbb{C}^2$ over the resolvent set of $\mathcal{D}_\alpha$. This particular result was summarized in the lemma 3.2.1 which was essential in the proof of the stability theorem 3.3.1. The subject of this theorem is a sufficient condition for a given essentially bounded L^1 matrix-valued potential perturbing the studied model $\mathcal{D}_\alpha$ ensuring the stability of the spectrum of the perturbed operator. In general, our approach was based on two elements: the Birman-Schwinger principle and explicit knowledge of the resolvent $(\mathcal{D}_\alpha - z)^{-1}$.

It is worth mentioning that our condition (3.12) is explicit and easy to verify. Moreover, let us also point out that the condition is linearly dependent on the mass of the particle, that is, the more mass the particle has, the weaker the potential has to be to ensure stability, which is fairly counterintuitive from the classical point of view.

In the very last section, we disscused the optimality of the obtained results. It was shown [45, 30] that for the Dirac operator on the whole-line, delta-potentials can be approximated by L^1 potentials in the norm resolvent sense. We suggested to investigating a possibility of extension of this result to the half-line setting of ours and find a proper counter example when the stability condition (3.12) is broken in the form of a delta-potential. Unfortunately, we did not manage to answer the question of the optimality yet. We also expect that the essencial boundedness is not necessary and the theorem 3.3.1 could be extended for a general L^1 potentials.

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