České vysoké učení technické v Praze

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# Arithmetics in generalised Cantor base systems

## Aritmetika v zobecněných Cantorových systémech

Master's thesis

Diplomová práce

Autor: Bc. Katarína Studeničová

Vedoucí práce: Prof. Ing. Zuzana Masáková, Ph.D.

Akademický rok: 2022/2023

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# ZADÁNÍ DIPLOMOVÉ PRÁCE

| | |
|---|---|
| Student: | Bc. Katarína Studeničová |
| Studijní program: | Aplikovaná algebra a analýza |
| Název práce (česky): | Aritmetika v zobecněných Cantorových systémech |
| Název práce (anglicky): | Arithmetics in generalised Cantor base systems |
| Jazyk práce: | angličtina |

Pokyny pro vypracování:

1. V návaznosti na svůj výzkumný úkol pokračujte ve studiu systémů se zobecněnou Cantorovou bází, jak byly definovány v článcích Caalim Demeglio 2020 a Charlier Cisternino 2021.
2. Pokuste se formulovat a dokázat tvrzení o existenci a jednoznačnosti Cantorovy báze k daným hladovým reprezentacím čísla 1 jako analogii k větě W. Parryho.
3. Zkoumejte aritmetické vlastnosti systémů se zobecněnou Cantorovou bází. Zaměřte se na formulování analogie k tzv. finiteness property.
4. Využijte znalosti finiteness property k určení, která racionální čísla mají v systému se zobecněnou Cantorovou bází čistě periodický rozvoj.

Doporučená literatura:


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V Praze dne 31.10.2022

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vedoucí katedry

děkan

*Čestné prohlášení*

Prohlašuji, že jsem tuto práci vypracovala samostatně a uvedla jsem všechnu použitou literaturu.

V Praze dne 03. 05. 2023

Bc. Katarína Studeničová

*Poděkování*

Na tomto místě bych ráda poděkovala svojí školitelce, paní profesorce Masákové, za možnost být součástí skvělého týmu a pracovat na zajímavém tématu, za velké množství času a energie, které mi věnovala, za její nekonečnou trpělivost, a za kritiku, která byla vždy konstruktivní a posunovala mě vpřed. Rovněž chci poděkovat paní profesorce Pelantové, která se postupně stala neodmyslitelnou součástí mého studia a vždy přinesla do našich setkání nejen obrovský odborný přínos, ale i humor a cennou podporu. V neposlední řadě patří dík i mým rodičům, kteří mi poskytli ideální zázemí na studium a přípravu diplomové práce.

Bc. Katarína Studeničová

*Název práce:* **Aritmetika v zobecněných Cantorových systémech**

*Autor:* Bc. Katarína Studeničová

*Obor:* Aplikovaná algebra a analýza

*Druh práce:* Diplomová práce

*Vedoucí práce:* Prof. Ing. Zuzana Masáková, Ph.D.
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*Abstrakt:* Studujeme nedávno definované poziční numerační sytémy, takzvané systémy s Cantorovou reálnou bází. Mezi ně patří klasické rozvoje reálných čísel známé jako Cantorovy řady a také dobře známé Rényiho rozvoje v reálné bázi. My pak konkrétně uvažujeme alternující systémy, ve kterých roli báze hraje čístě periodická posloupnost. Studujeme aritmetiku v těchto systémech a definujeme takzvanou vlastnost konečnosti a vlastnost kladné konečnosti. Uvádíme a dokazujeme několik nutných a postačujících podmínek pro tyto vlastnosti. Mezi ostatní výsledky patří i věta o existenci a jednoznačnosti alternující báze s periodou 2, potřebná pro zobecnění Parryho věty. Nakonec se zaměřujeme na vlastnosti čistě periodických rozvojů v systémech s Cantorovou reálnou bází. Výsledky týkající se aritmetiky a čistě periodických rozvojů jsou znázorněny v několika příkladech.

*Klíčová slova:* systémy se zobecněnou Cantorovou bází, alternující báze, aritmetika, čistě periodické rozvoje

*Title:* **Arithmetics in generalised Cantor base systems**

*Author:* Bc. Katarína Studeničová

*Supervisor:* Prof. Ing. Zuzana Masáková, Ph.D.

*Abstract:* We study recently defined positional numeration systems called Cantor real base systems. They include classic expansions of real numbers known as Cantor series and also the well-known Rényi expansions in a real base beta. Particularly, we consider alternate base systems with the base being a purely periodic sequence of real numbers. We study arithmetics in these systems, and define so-called positive finiteness and finiteness property. We state and prove several necessary conditions and a sufficient condition of these properties. Other results include a proposition about existence and uniqueness of an alternate base with period 2 needed in order to generalise Parry theorem. Lastly, we study properties of purely periodic expansions in Cantor real bases. Results considering arithmetics and purely periodic expansions are illustrated and examined on numerous examples.

*Key words:* generalised Cantor base systems, alternate base, arithmetics, purely periodic expansions

# Contents

# Introduction

Various ways of representing real numbers have been studied by mathematicians for centuries. In this work, we focus on one particular construction, the so-called generalised Cantor base numeration system. Although Cantor published his work already in the 19th century, some of his ideas about numeration systems are still discussed lively even today. The definition of the system we are interested in was first given only recently in [4] and [6] by two independent groups of researchers. Our goal is to contribute to the study of these systems and to provide illustrative examples.

In the first chapter, we recall notions from combinatorics on words, matrix theory, and number theory. Another preliminary part is Chapter 2, in which we focus primarily on Rényi numeration systems. We provide the essential definitions and several theorems considering $\beta$-expansions, their periodicity and finiteness, and especially purely periodic expansions.

The third chapter is devoted to the description of generalised Cantor base systems. We at first focus on the definition of these systems and recall some of their properties as they were already presented in numerous papers, mainly [4] and [6]. Then we expand on the notion of generalised Cantor base, which so far allowed only representations of numbers in $[0, 1]$ by the one-sided string of digits, so that all real numbers can be represented as bi-infinite strings. We call such generalisation a bi-infinite Cantor real base. In addition, we provide a definition of positive finiteness $(PF)$ and finiteness property $(F)$ for these systems. These properties reflect closedness of the set of numbers with finite expansions under addition and subtraction, and are potentially crucial in order to perform the arithmetical operations in generalised Cantor base systems.

In Chapter 4 we focus on the proof of existence and uniqueness of a base suitable for the generalisation of the Parry theorem. This theorem is well-known in the context of Rényi numeration systems, and possible generalisation of this theorem is of great interest to researchers in the field of non-standard numeration systems. The proof of a proposition considering the existence of the suitable base is an important step towards an analogue of the result of Parry for generalised Cantor base systems.

Chapter 5 is devoted to one of the main results of our work – we focus on arithmetics in bi-infinite Cantor real bases. Firstly, we formulate and prove several necessary conditions of positive finiteness and finiteness property. We also inspect the connection between property $(PF)$ and $(F)$

for an alternate base and its shifts. One of the main results of our thesis is a sufficient condition of positive finiteness property. Moreover, we present a whole class of bases satisfying this condition. All properties were at first carefully verified on examples that we provide in Appendices A and B.

The last research chapter comments on the properties of numbers with purely periodic $\boldsymbol{\beta}$-expansions. This research question was strongly influenced by the previous research in Rényi numeration systems and systems with negative bases, see [2, 19, 1, 16]. The main problem considered the existence of $\Gamma(\boldsymbol{\beta}) \in (0,1]$ such that all rational numbers in $[0, \Gamma(\boldsymbol{\beta}))$ have purely periodic $\boldsymbol{\beta}$-expansion. We at first provide a necessary condition of the property $\Gamma(\boldsymbol{\beta}) > 0$, and then we focus on the initial step toward proving a sufficient condition. Alongside the theoretical results, we analysed expansions of multiple sets of fractions in several alternate bases. We hope that this numerical experiment will greatly help to develop an intuition and to formulate new propositions considering purely periodic expansions.

# Chapter 1

# Preliminaries

## 1.1 Combinatorics on words

Let us briefly introduce the basic concepts of combinatorics on words as may be found in [15]. A finite set of symbols $\mathcal{A}$ is called an *alphabet*, its elements are referred to as *letters*. However, in the context of numeration systems, we will usually call them *digits*. A *finite word* is a finite sequence of letters. We will denote it by $\boldsymbol{w} = w_1 w_2 \dots w_n$. In this notation $n$ is the *length* of the word $\boldsymbol{w}$, symbolically $|\boldsymbol{w}| = n$. The empty sequence is called the *empty word* and usually denoted by $\varepsilon$, we set $|\varepsilon| = 0$. We will denote by $\mathcal{A}^*$ the set of all finite words over the alphabet $\mathcal{A}$. This set equipped with the operation of concatenation, defined for any two words $\boldsymbol{v} = v_1 v_2 \dots v_m$, $\boldsymbol{w} = w_1 w_2 \dots w_n$ as

$$\boldsymbol{v}\boldsymbol{w} = v_1 v_2 \dots v_m w_1 w_2 \dots w_n,$$

forms a monoid. Similarly, an infinite sequence of letters indexed by positive integers will be called a *right-sided infinite word*, or sometimes, for the sake of brevity, just an *infinite word*. An infinite sequence of letters indexed by $n \in \mathbb{Z}$ will be referred to as a *both-sided infinite word*, or shortly just as a *bi-infinite word*. The set of all right-sided infinite words is usually denoted by $\mathcal{A}^{\mathbb{N}}$, the set of all both-sided infinite words is denoted by $\mathcal{A}^{\mathbb{Z}}$.

A finite word $\boldsymbol{w}$ is called a *factor* of a word $\boldsymbol{v}$ (finite of infinite), if there exists a finite word $\boldsymbol{x}$ and a word $\boldsymbol{y}$ such that $\boldsymbol{v} = \boldsymbol{x}\boldsymbol{w}\boldsymbol{y}$. The factor $\boldsymbol{x}$ is a *prefix* of $\boldsymbol{v}$, the factor $\boldsymbol{y}$ is a *suffix* of $\boldsymbol{v}$. The prefix $\boldsymbol{x}$ is *proper* if $\boldsymbol{v} \neq \boldsymbol{x}$, analogously, the suffix $\boldsymbol{y}$ is proper if $\boldsymbol{v} \neq \boldsymbol{y}$. A *language* is any subset of $\mathcal{A}^*$. The set of all factors of a word $\boldsymbol{u}$ is called the *language of* $\boldsymbol{u}$.

Let us assume we have an order on the alphabet $\mathcal{A}$ (denoted by $<$). The *lexicographic order* (denoted by $\prec$) on words in $\mathcal{A}^*$ is defined as follows. For two finite words $\boldsymbol{u}, \boldsymbol{v}$ we have $\boldsymbol{u} \prec \boldsymbol{v}$ if $\boldsymbol{u}$ is a proper prefix of $\boldsymbol{v}$ or if there exist finite words $\boldsymbol{x}, \boldsymbol{v}_2, \boldsymbol{u}_2$ and letters $a, b$ such that $\boldsymbol{u} = \boldsymbol{x}a\boldsymbol{u}_2, \boldsymbol{v} = \boldsymbol{x}b\boldsymbol{v}_2$ and $a < b$. For $\boldsymbol{u}, \boldsymbol{v}$ infinite words, the lexicographic order is defined as $\boldsymbol{u} \prec \boldsymbol{v}$ if there exists a finite word $\boldsymbol{x}$, infinite words $\boldsymbol{u}_2, \boldsymbol{v}_2$ and letters $a, b$ such that $\boldsymbol{u} = \boldsymbol{x}a\boldsymbol{u}_2, \boldsymbol{v} = \boldsymbol{x}b\boldsymbol{v}_2$ and $a < b$.

We denote $\boldsymbol{u}^n$ the factor $\boldsymbol{u}$ which is repeated $n$ times consecutively. Similarly, if a word $\boldsymbol{w}$ ends in infinitely many repetitions of some factor $\boldsymbol{u}$, we denote it by $\boldsymbol{w} = \boldsymbol{v}\boldsymbol{u}^\omega$. In that case we say that the word $\boldsymbol{w}$ is *eventually periodic* (or *periodic*). The word $\boldsymbol{v}$ is then called the *preperiod*, and $\boldsymbol{u}$ the *period* of the word $\boldsymbol{w}$. If $\boldsymbol{v} = \varepsilon$, then we say that the word $\boldsymbol{w} = \boldsymbol{v}\boldsymbol{u}^\omega = \boldsymbol{u}^\omega$ is *purely periodic.* Note that finite words may also be written in the form $\boldsymbol{w} = \boldsymbol{v}\boldsymbol{u}^\omega$ for $\boldsymbol{u} = \varepsilon$, therefore we consider finite words as periodic with period $\varepsilon$.

Let us introduce a metric on $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$. Let $\boldsymbol{u}, \boldsymbol{v}$ be two words in $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$. If one of them is finite, take some symbol which is not contained in $\mathcal{A}$, let us denote it $\alpha$, and extend the finite word by $\alpha^\omega$ to the right. If both words are finite, take two distinct symbols not contained in the alphabet, say $\alpha$ and $\beta$, and extend $\boldsymbol{u}$ by $\alpha^\omega$ to the right and $\boldsymbol{v}$ to $\beta^\omega$ to the right. Then the distance of two words (which were either infinite or both extended as described) $\boldsymbol{u} = u_1u_2\ldots$ and $\boldsymbol{v} = v_1v_2\ldots$, $\boldsymbol{u} \neq \boldsymbol{v}$, is defined as

$$d(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{\inf\{k \geq 1 \mid u_k \neq v_k\}}.$$

If $\boldsymbol{u} = \boldsymbol{v}$ we set $d(\boldsymbol{u}, \boldsymbol{v}) := 0$. It is readily seen that the above defined function $d$ is indeed a metric and therefore induces a topology on $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$.

Lastly, let us define the so-called *shift operator*. For a word $\boldsymbol{a} = a_1a_2a_3\ldots$ we define the shift operator $\sigma$ as

$$\sigma(\boldsymbol{a}) := a_2a_3\ldots.$$

Iterating this operator gives us $\sigma^m(\boldsymbol{a}) = a_{m+1}a_{m+2}\ldots$.

## 1.2 Standard representations of real numbers

Firstly, recall a numeration system where the base is a natural number. This system is surely well-known and used on a daily basis.

**Definition 1.1** ($\beta$-ary representation)**.** Let $\beta > 1$ be an integer, $x \geq 0$ be a real number. An infinite sequence $(x_i)_{i\leq k}$, where $x_i \in \{0, \ldots, \beta - 1\}$ for all $i$, satisfying

$$\sum_{i\leq k} x_i\beta^i = x$$

for some $k \in \mathbb{Z}$ is called a *$\beta$-ary representation* of $x$.

Now consider a numeration system where the base is a real number $\beta > 1$. These systems are know as *Rényi numeration systems.* We will recall their properties and other details in Chapter 2.

**Definition 1.2** ($\beta$-representation)**.** Let $\beta > 1$ and $x \geq 0$ be real numbers. A *$\beta$-representation* of $x$ is a sequence $(x_i)_{i=-\infty}^{k}$, where $x_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}$ and $x_k \neq 0$, satisfying

$$x = \sum_{i=-\infty}^{k} x_i \beta^i. \tag{1.1}$$

We denote a $\beta$-representation of $x$, as it is usual even in other numeration systems, by

$$\begin{array}{ll} x_k x_{k-1} \dots x_0 \bullet x_{-1} x_{-2} \dots & \text{for } k \geq 0, \\ 0 \bullet \underbrace{0 \dots 0}_{-(k+1) \text{ times}} x_k x_{k-1} \dots & \text{otherwise.} \end{array}$$

In the first case, the word $x_k x_{k-1} \cdots x_0$ is called the *integer part* of the $\beta$-representation of $x$, the word $x_{-1} x_{-2} \cdots$ is the *fractional part* of the $\beta$-representation of $x$. Analogically, in the second case, the integer part is zero and the fractional part consists of all the symbols to the right of the radix point. If a $\beta$-representation ends in infinitely many zeros we call the representation *finite* and we often omit repeating zeros.

Another generalisation of numeration systems with integer base was proposed by Cantor in 1869 [5]. Instead of considering just one integer, in this case the base consists of a sequence of positive integers.

**Definition 1.3** (Cantor representation)**.** Let $x \in [0, 1)$ be a real number and $(b_i)_{i \geq 1}$ be a sequence of integers greater than 1. Let $x$ be represented as

$$x = \sum_{n=1}^{+\infty} \frac{a_n}{\prod_{i=1}^{n} b_i} = \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \cdots, \tag{1.2}$$

where $a_n \in \{0, \dots, b_n - 1\}$ for all $n \in \mathbb{N}$. If $a_n a_{n+1} a_{n+2} \dots \neq (b_n - 1)(b_{n+1} - 1)(b_{n+2} - 1) \dots$ for all $n \in \mathbb{N}$, we call the series (1.2) the *Cantor series* of $x$ and the sequence $(a_n)_{n \geq 1}$ the *Cantor representation* of $x$.

Note that the condition $a_n a_{n+1} a_{n+2} \dots \neq (b_n - 1)(b_{n+1} - 1)(b_{n+2} - 1) \dots$ for all $n \in \mathbb{N}$ is necessary in order to ensure uniqueness of expansions. It can be proven that for each sequence $(b_i)_{i \geq 1}$ of integers greater than 1 and for each real number $x \in [0, 1)$ there exists a unique Cantor representation of $x$ (the digits can be obtained by an algorithm similar to the standard greedy algorithm). The sequence $(b_i)_{i \geq 1}$ from the above definition is usually called the *Cantor base*. Other properties of Cantor base systems will be presented in Chapter 3.

## 1.3 Classes of numbers

Let us now recall several definitions from number theory. An *algebraic number* $\beta$ is a root of a monic polynomial with rational coefficients. Among such polynomials the one with the smallest degree is irreducible over $\mathbb{Q}$ and is called the *minimal polynomial* of $\beta$. The *degree* of an algebraic number is defined as the degree of its minimal polynomial. Other roots of the minimal polynomial of $\beta$ are called *conjugates* of $\beta$. Note that conjugates of $\beta$ are mutually distinct. Some authors consider even $\beta$ its own conjugate. To avoid confusion, especially in newly formulated propositions in later chapters, we will always make clear if that is the case or not.

An *algebraic integer* $\beta$ is a root of a monic polynomial with integer coefficients. It can be shown that its minimal polynomial also has integer coefficients.
According to the properties of their conjugates we may define several classes of algebraic integers. An algebraic integer $\beta > 1$ is called

- a *Pisot number* if all its conjugates have modulus $< 1$,
- a *Salem number* if all its conjugates have modulus $\leq 1$ and at least one conjugate lies on the unit circle,
- a *Perron number* if all its conjugates have modulus $< \beta$.

From the definition, it is clear that both Pisot and Salem numbers are Perron numbers. Since Pisot numbers will be often discussed in later chapters, let us now give some examples of these numbers. The smallest Pisot number is the larger root of the polynomial $x^3 - x - 1$. Another well-known class is a set of quadratic Pisot numbers. They can be classified as follows

1. larger roots of polynomials of the form $x^2 - mx - n$, where $m, n \in \mathbb{N}$, $m \geq n$,
2. larger roots of polynomials of the form $x^2 - mx + n$, where $m, n \in \mathbb{N}$, $m \geq n + 2 \geq 3$.

Lastly, let us give several comments on isomorphisms of number fields we will be working with. We will denote $\mathbb{Q}(\beta)$ the minimal subfield of $\mathbb{C}$ containing $\beta$. In case that $\beta$ is an algebraic number of degree $n$, the field can be written in the following form

$$\mathbb{Q}(\beta) = \{a_0 + a_1\beta + \cdots + a_{n-1}\beta^{n-1} \mid a_i \in \mathbb{Q}\}.$$

Note that fields $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\beta_j)$, where $\beta_j$ are conjugates of $\beta$, are isomorphic. It is clear from the relation above that $\alpha \in \mathbb{Q}(\beta)$ may be written as $\alpha = g(\beta)$, where $g$ is some polynomial with rational coefficients. Then we may define an isomorphism $\phi_j : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_j)$ as

$$\phi_j(g(\beta)) = g(\beta_j).$$

Moreover, for an algebraic number $\beta$, the subfields $\mathbb{Q}(\beta_j)$ are the only subfields of $\mathbb{C}$ isomorphic to $\mathbb{Q}(\beta)$.

## 1.4 Matrix theory

We now recall two well-known theorems about matrices – Perron-Frobenius and Gershgorin circle theorem [10].

**Theorem 1.4** (Perron-Frobenius)**.** *Let $\mathbb{M}$ be an irreducible non-negative complex $p \times p$ matrix with spectral radius $\lambda$. Then the following statements hold.*

1. *The number $\lambda$ is a positive real number and it is an eigenvalue of the matrix $\mathbb{M}$.*
2. *The eigenvalue $\lambda$ has one-dimensional eigenspace.*
3. *$\mathbb{M}$ has an eigenvector with eigenvalue $\lambda$ whose components are all positive.*
4. *The only eigenvectors whose components are all positive are those associated with the eigenvalue $\lambda$.*

The number $\lambda$ from the theorem is called the *Perron–Frobenius eigenvalue.* The theorem holds for an irreducible non-negative matrix. According to the definition, the matrix is *irreducible* if and only if it can not be transformed by any simultaneous permutation of rows and columns into a block upper triangular form. Equivalently, a matrix is irreducible if and only if the oriented graph associated with the matrix is strongly connected.

**Theorem 1.5** (Gershgorin)**.** *Let $\mathbb{M}$ be a complex $p \times p$ matrix. Denote $m_{ij}$ elements of $\mathbb{M}$ and $R_i := \sum_{i \neq j} |m_{ij}|$. Denote $D_i := D(a_{ii}, R_i)$ the closed disk centered in $a_{ii}$ with radius $R_i$. Then the spectrum of $\mathbb{M}$ is a subset of $\bigcup_i D_i$.*

In particular, if the matrix $\mathbb{M}$ is diagonal, then the set $\bigcup_i D_i$ coincides with spectrum. For diagonally dominant matrix with non-negative elements the above theorem implies that the real parts of all eigenvalues are non-negative.

# Chapter 2

# Rényi numeration systems

This chapter will be devoted to properties of $\beta$-representations of real numbers. We have already given the definition of these numeration systems in Preliminaries, see Definition 1.2.

## 2.1 $\beta$-expansions

So far the only requirement that we have on $\beta$-representation is the convergence of the sum (1.1). We will now introduce a particular $\beta$-representation – the representation computed by the so-called greedy algorithm.

**Greedy algorithm**

Let $\beta > 1$ and $x \geq 0$ be real numbers. Let us denote $\lfloor x \rfloor$ the integer part of $x$ and $\{x\} := x - \lfloor x \rfloor$ the fractional part of $x$. Then the *greedy algorithm* consists of the following steps.

- Find $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$.
- Put $x_k = \lfloor \frac{x}{\beta^k} \rfloor$ and $r_k = \{ \frac{x}{\beta^k} \}$.
- For $i < k$ put $x_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$.

In this way we obtain the digits $x_i$ of the $\beta$-representation of $x$ in the form (1.1). We will denote this representation by $\langle x \rangle_\beta$ and call it the *greedy expansion* of $x$. If $\beta$ is an integer, the digits obtained by the algorithm belong to $\{0, 1, \ldots, \beta - 1\}$, otherwise the digits are in $\{0, 1, \ldots, \lfloor \beta \rfloor\}$. Note that for $\beta$ being an integer the greedy representation coincides with the $\beta$-ary representation as introduced in Definition 1.1.

**Remark 2.1** (Radix order). Let us introduce the *radix order* on $\beta$-representations. A $\beta$-representation $x_k \ldots x_0 \bullet x_{-1} \ldots$ is greater with respect to the radix order than the $\beta$-representation $y_l \ldots y_0 \bullet y_{-1} \ldots$, if either $k > l$, or $k = l$ and $x_j > y_j$ for $j = \max\{i \leq k \mid x_i \neq y_i\}$. We will denote this by

$$x_k \ldots x_0 \bullet x_{-1} \ldots \;>_{rad}\; y_l \ldots y_0 \bullet y_{-1} \ldots .$$

Note that the greedy algorithm yields the largest $\beta$-representation of a given $x$ according to the radix order. Lastly, let us recall the fact, that the radix order on greedy expansions corresponds to the ordering of real numbers, i.e. for $\beta > 1$ and $x, y \geq 0$, the following holds

$$x < y \iff \langle x \rangle_\beta <_{rad} \langle y \rangle_\beta.$$

**Remark 2.2.** Note that if $\langle x \rangle_\beta = x_k x_{k-1} \dots x_0 \bullet x_{-1} \dots$, then $x_k x_{k-1} \dots x_1 \bullet x_0 x_{-1} \dots = \langle \frac{x}{\beta} \rangle_\beta$. We see that multiplication by some power of $\beta$ is equivalent to shifting the radix point. Now let $x > 1$ be a real number. For any $\beta > 0$ there exists $k \in \mathbb{N}$ such that $\beta^k \leq x < \beta^{k+1}$, hence $0 < \frac{x}{\beta^{k+1}} < 1$. Therefore it is enough to consider just representations of numbers in $[0, 1]$, representations of all other positive numbers can be obtained simply by shifting the radix point. In the following text, when working with the numbers $x \in [0, 1)$, we will use notation $x = \sum_{i \geq 1} x_i \beta^{-i}$ instead of indexing used in (1.1), i.e. $x$ will be represented by the sequence $(x_i)_{i \geq 1}$.

The $\beta$-representation of $x \in [0, 1)$ obtained by the greedy algorithm can be defined equivalently in terms of a particular transformation and its orbits. Let us therefore introduce the following mapping.

**Transformation $T_\beta$**

In the light of Remark 2.2 we consider just $x \in [0, 1], \beta > 1$. The mapping $T_\beta : [0, 1] \longrightarrow [0, 1)$ defined by

$$T_\beta(x) = \{\beta x\}$$

is called the *$\beta$-transformation.* One particular $\beta$-representation $(x_i)_{i \geq 1}$ of $x$ can be then defined by $x_i := \lfloor \beta T_\beta^{i-1}(x) \rfloor$ for $i \geq 1$. We will denote this $\beta$-representation of $x \in [0, 1]$ as $d_\beta(x)$ and call it the *$\beta$-expansion.* Note that for $x = 1$ the transformation does not give us the same result as the greedy algorithm, however we will see later that $d_\beta(1)$, the so-called *Rényi expansion of 1*, will play an important role. Therefore, we will often use a special notation for the digits of this representation

$$d_\beta(1) = t_1 t_2 t_3 \dots .$$

On the other hand, the greedy expansion of 1 is always of the form $\langle 1 \rangle_\beta = 1\bullet$.

**Example 2.3.** The golden ratio $\tau$ is the larger root of the polynomial $x^2 - x - 1$, i.e. $\tau = \frac{1+\sqrt{5}}{2} \approx 1.618\dots$. Let us find $d_\beta(1)$ for $\beta = \tau$. The digits will be $t_1 = \lfloor \tau \rfloor = 1$, $t_2 = \lfloor \tau(\tau - 1) \rfloor = \lfloor 1 \rfloor = 1$, $0 = t_3 = t_4 = \dots$. Therefore $d_\tau(1) = 110^\omega$.

We now define the *infinite Rényi expansion of 1* in the base $\beta$ as

$$d^*_\beta(1) := \lim_{\varepsilon \to 0^+} d_\beta(1 - \varepsilon),$$

where the limit is in the metric introduced in Section 1.1. It can be shown that this expansion can be rewritten in the following form.

**Proposition 2.4.** *For $\beta > 1$ the infinite Rényi expansion of 1 is of the form*

$$d_\beta^*(1) = \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite,} \\ (t_1 t_2 \dots t_{m-1}(t_m - 1))^\omega & \text{if } d_\beta(1) = t_1 \cdots t_m 0^\omega \text{ for some } m \in \mathbb{N} \text{ and } t_m \neq 0. \end{cases}$$

**Example 2.5.** Proposition 2.4 allows us to easily calculate that in the base of the golden ratio we have $d_\tau^*(1) = (10)^\omega$.

The infinite Rényi expansion of 1 enables us to decide whether a given $\beta$-representation is even the $\beta$-expansion or not [18].

**Theorem 2.6** (Parry condition)**.** *For $\beta > 1$ let $\boldsymbol{x} = (x_i)_{i\geq 1}$ be a $\beta$-representation of $x \in [0, 1)$. Then $x_1 x_2 \dots = d_\beta(x)$ if and only if for all $j \geq 0$*

$$\sigma^j(\boldsymbol{x}) \prec d_\beta^*(1). \tag{2.1}$$

A string of non-negative integers satisfying the Parry condition (2.1) is called *admissible*. If the string of non-negative integers does not satisfy (2.1), we say it is *forbidden*.

We could ask what conditions some sequence $(t_i)_{i\geq 1}$ must satisfy in order to be a $\beta$-expansion of 1 or an infinite Rényi expansion of 1 in some base $\beta > 1$. The following theorems by Parry [18] answer this question.

**Theorem 2.7.** *Let us have $\boldsymbol{t} = t_1 t_2 \dots$, $\boldsymbol{t} \succ 10^\omega$. Then $\boldsymbol{t} = d_\beta(1)$ for some $\beta > 1$ if and only if for all $j \in \mathbb{N}$*

$$\sigma^j(\boldsymbol{t}) \prec \boldsymbol{t}. \tag{2.2}$$

**Theorem 2.8.** *Let us have $\boldsymbol{t} = t_1 t_2 \dots$, $\boldsymbol{t} \succ 10^\omega$ with infinitely many non-zero digits. Then $\boldsymbol{t} = d_\beta^*(1)$ for some $\beta > 1$ if and only if for all $j \in \mathbb{N}$*

$$\sigma^j(\boldsymbol{t}) \preceq \boldsymbol{t}. \tag{2.3}$$

## 2.2 Periodicity and finiteness

Let us now mention some notions connected with finiteness and periodicity of the greedy expansions. The well-known result considering $\beta$-expansions of 1 is that $d_\beta(1)$ is never purely periodic. However, a special class of numeration systems with $\beta$ such that $d_\beta(1)$ is eventually periodic is interesting from the number theoretical point of view, and was first examined already by Parry [18]. Therefore, such numbers $\beta$ are called *Parry numbers*. It is known that

$$\text{Pisot numbers} \;\subset\; \text{Parry numbers} \;\subset\; \text{Perron numbers}\ .$$

On the other hand, the relation between Salem and Parry numbers had not been fully described yet. Note that in the case of quadratic algebraic numbers Pisot and Parry numbers coincide. Let us give one interesting illustrative example. The larger of real roots of $x^4 - 3x^3 - 2x^2 - 3$ is a Perron number, but it is neither a Pisot nor a Salem number. Moreover, $\beta$-expansion of 1 is in this case finite: $d_\beta(1) = 3203$, therefore this number is a Parry number.

We now present several known results about sets of numbers with periodic expansions, finite expansions, and numbers with fractional part of their $\beta$-representation $0^\omega$. For $\beta > 1$ we denote

$$\begin{aligned}
\operatorname{Per}(\beta) &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta \text{ is eventually periodic}\},\\
\operatorname{Fin}(\beta) &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta \text{ has just finitely many non-zero digits }\},\\
\mathbb{Z}_\beta &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k x_{k-1} \dots x_0 \bullet 0^\omega \text{ for some } k \in \mathbb{N}_0\}.
\end{aligned}$$

One of the main results concerning the set $\operatorname{Per}(\beta)$ is a well-known theorem of K. Schmidt [19].

**Theorem 2.9** (Schmidt)**.** *Let $\beta > 1$ be a real number. If $\mathbb{Q}(\beta) = \operatorname{Per}(\beta)$, then $\beta$ is either a Pisot or a Salem number. On the other hand, if $\beta$ is a Pisot number, then $\operatorname{Per}(\beta) = \mathbb{Q}(\beta)$.*

**Remark 2.10.** Note that the greedy expansion of $x \in [0, 1)$ is finite if and only if there exists $n$ such that $T_\beta^n(x) = 0$ and is periodic if and only if the orbit of $x$ under $T_\beta$ is finite. By the orbit of $x$ under $T_\beta$ we mean the set $\{T_\beta^n(x) \mid n \geq 1\}$.

The set $\mathbb{Z}_\beta$ is also known as the set of $\beta$-integers. It is not difficult to show that for $\beta > 1$ the set $\mathbb{Z}_\beta$ is a ring if and only if $\beta \in \mathbb{N}$.

On the other hand, the situation for the set $\operatorname{Fin}(\beta)$ is far more complicated. If $\beta > 1$ is an integer, then $\operatorname{Fin}(\beta)$ is closed under addition, subtraction, and multiplication, i.e. it is a ring. However, if $\beta > 1$ is not an integer, we do not know any general algebraic description that would help us decide whether $\operatorname{Fin}(\beta)$ is a ring. Let us now introduce a so-called *finiteness property* and *positive finiteness property*, we denote $(F)$ and $(PF)$.

**Definition 2.11.** Let $\beta > 1$. We say that $\beta$ satisfies $(F)$, if $\mathrm{Fin}(\beta)$ is closed under addition. We say that $\beta$ satisfies $(PF)$, if $\mathrm{Fin}(\beta)$ is closed under addition of positive elements.

It is not difficult to show that $\beta > 1$ satisfies $(F)$ if and only if $\mathrm{Fin}(\beta)$ is a ring, i.e. closed under addition and multiplication of its elements. Equivalently, $\beta > 1$ satisfies $(F)$, if and only if $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta, \beta^{-1}]$. We now state several results considering Property $(F)$ as they were presented in [3, 11].

**Proposition 2.12.** *Let $\beta > 1$ be such that Property $(F)$ is satisfied. Then $\beta$ is a Pisot number such that none of its conjugates $\gamma$, $\gamma \neq \beta$, is a positive real number.*

**Proposition 2.13.** *Let $\beta > 1$.*

1. *If $d_\beta(1)$ is infinite, then $\beta$ does not satisfy $(F)$.*
2. *If $d_\beta(1)$ is finite, then $\beta$ satisfies $(F)$ if and only if it satisfies $(PF)$.*

Note that even if $d_\beta(1)$ is infinite, the set $\mathrm{Fin}(\beta)$ may be closed under addition of positive elements. Let us now explore the second case of the above proposition. The goal is to describe the process of addition of positive elements of $\mathrm{Fin}(\beta)$. We follow the construction presented in [3]. Note that authors were working especially with representations with non-zero digits just on the left-hand side of the radix point. They may indeed do so, because the shift of representation corresponds just to multiplication of $x$ by $\beta$, as we already mentioned. In this chapter we follow their formalism, and thus usually index digits of the given word with descending indices.

**Definition 2.14.** Let $\beta > 1$. A forbidden string $a_k a_{k-1} \ldots a_0$ of non-negative integers is called *minimal*, if

- $a_{k-1} \ldots a_0$ and $a_k \ldots a_1$ are admissible, and
- $a_i \geq 1$ implies $a_k \ldots a_{i+1}(a_i - 1)a_{i-1} \ldots a_0$ is admissible for all $i \in \{0, \ldots, k\}$.

For a $\beta > 1$ such that $d_\beta(1) = t_1 t_2 \ldots t_m$ is finite, surely a minimal forbidden string has to have one of the forms

$$(t_1 + 1), \qquad t_1(t_2 + 1), \qquad , \ldots , \qquad t_1 t_2 \ldots (t_{m-1} + 1), \qquad t_1 t_2 \ldots t_{m-1} t_m.$$

Note that not all such strings have to be minimal forbidden. In order to construct an algorithm for addition, authors of [3] use wisely chosen transcriptions of forbidden strings. Let us present their definition of transcription.

**Definition 2.15** (Transcription)**.** Let $k, p \in \mathbb{Z}$, $k \geq p$ and let $z = \sum_{i=p}^{k} z_i\beta^i$, where $z_i \in \mathbb{N}_0$ for all $i$. A finite sum $\sum_{j=l}^{n} v_j\beta^j$ such that $\sum_{j=l}^{n} v_j\beta^j = z$ and

$$k \leq n \qquad \text{and} \qquad v_n v_{n-1} \dots v_l \succ \underbrace{00\dots0}_{(n-k)\text{ times}}\; z_k \dots z_p$$

is called a *transcription* of $\sum_{i=p}^{k} z_i\beta^i$.

Now the following necessary condition on $\beta$ such that $\mathrm{Fin}(\beta)$ is closed under addition of positive elements may be stated.

**Proposition 2.16** (Property T)**.** *Let $\beta > 1$. If $\mathrm{Fin}(\beta)$ is closed under addition of positive elements, then $\beta$ satisfies Property T:*

*For every minimal forbidden string $a_k a_{k-1} \dots a_0$ there exists a transcription of* $\displaystyle\sum_{i=0}^{k} a_i\beta^i$.

If Property T is satisfied, then the transcription of a series representing number $z \geq 0$ in base $\beta > 1$ can be obtained in the following way. Again, as already mentioned before, it suffices to consider only representations with non-zero digits just on the left-hand side of the radix point. If a representation of $z$ contains a forbidden string, the series representing $z$ can be written as a sum of a value of minimal forbidden string on a suitable position $\beta^j(v_n\beta^n + \cdots + v_l\beta^l)$ and a series representing some number $u$ in base $\beta$. The new transcribed $\beta$-representation of z is obtained by digit-wise addition of the transcription $\beta^j(v_n\beta^n + \cdots + v_l\beta^l)$ and the $\beta$-representation of $u$. Repeating this process yields a lexicographically increasing sequence of transcribed $\beta$-representations of $z$. In general, this procedure may be repeated infinitely many times without obtaining the lexicographically greatest $\beta$-representation of $z$. The following propositions provide sufficient conditions for avoiding this situation.

**Proposition 2.17.** *Let $\beta > 1$. Suppose that for every minimal forbidden string $a_k a_{k-1} \dots a_0$ there exists a transcription $\sum_{j=l}^{n} v_j\beta^j$ of $\sum_{j=0}^{k} a_i\beta^i$ such that*

$$\sum_{j=l}^{n} v_j \leq \sum_{j=0}^{k} a_i.$$

*Then $\beta$ satisfies $(PF)$. Moreover, for every positive $x, y \in \mathrm{Fin}(\beta)$, the greedy expansion of $x + y$ can be obtained from any finite $\beta$-representation of $x + y$ using just finitely many transcriptions.*

The following corollary presents one class of bases satisfying the conditions of the above proposition. This result was originally presented in [11].

**Corollary 2.18.** *Let $\beta > 1$ be such that $d_\beta(1) = t_1 \dots t_m$, $t_1 \geq t_2 \geq \dots \geq t_m \geq 1$. Then $\beta$ satisfies $(PF)$.*

In the light of Proposition 2.13 we may conclude that such $\beta$ satisfies finiteness property. A result similar to the above corollary for the case when $d_\beta(1)$ is periodic with length of period 1 was shown as well in [11].

**Proposition 2.19.** *Let $\beta > 1$ be such that $d_\beta(1) = t_1 \dots t_{m-1}(t_m)^\omega$, $t_1 \geq t_2 \geq \dots \geq t_{m-1} > t_m \geq 1$. Then $\beta$ satisfies $(PF)$.*

## 2.3 Purely periodic expansions

When dealing with numbers with periodic $\beta$-expansions, we may be interested if these expansions are even purely periodic or not. Consider the decimal system and numbers in $(0,1)$. It is well known that in this case all rational numbers have periodic expansions. Among them, the numbers of the form $\frac{p}{q}$ where $p$ and $q$ are coprime and $q$ and 10 are coprime, have purely periodic expansion. This result may be straightforwardly reformulated for integer bases $\beta > 1$. However, when considering $\beta \notin \mathbb{N}$, the problem becomes much harder.

Consider the following examples. Let $\beta_1 = \tau$ be the golden ratio. In this case Schmidt [8] proved, that each rational number in $(0,1)$ indeed has purely periodic $\tau$-expansion. On the other hand, it has been shown in [13], that for $\beta_2$ the larger root of polynomial $x^2 - 3x + 1$ there are no rational numbers in $(0,1)$ with purely periodic $\beta_2$-expansion.

In the sequel we present several results from [1] about bases $\beta$ such that all sufficiently small rational numbers have purely periodic $\beta$-expansions. Denote

$$\Gamma(\beta) := \sup\{c \in [0,1) \mid \forall p/q : \ 0 \leq p/q \leq c, \ d_\beta(p/q) \text{ is purely periodic}\}.$$

In the examples above $\Gamma(\beta_1) = 1$ and $\Gamma(\beta_2) = 0$. It can be shown, that if $\Gamma(\beta) > 0$, then necessarily $\beta$ is an algebraic integer. Moreover, it was shown in [2] that $\beta$ must be an algebraic unit. Furthermore, by the result of Schmidt [19], such base $\beta$ is either a Pisot or a Salem number. Later, see the work of Akiyama [2], this result was further specified – it has been proven that $\beta$ such that $\Gamma(\beta) > 0$ cannot have a positive conjugate, thus it is not a Salem number. Therefore, let us conclude this section by results about the only class of bases which possibly can have $\Gamma(\beta) > 0$, namely the Pisot bases.

It has been proven by [2] that the property $\Gamma(\beta) > 0$, even though it is quite restrictive, holds for the whole following class of $\beta$.

**Proposition 2.20.** *If $\beta$ is a Pisot unit satisfying $(F)$, then $\Gamma(\beta) > 0$.*

On the other hand, numbers which are not Pisot units does not have the desired property [1].

**Proposition 2.21.** *Let $\beta > 1$ be a real number which is not a Pisot unit. Then* $\Gamma(\beta) = 0$.

The case when $\beta$ is either a quadratic or a cubic algebraic integer have already been characterised fully [1, 13].

**Proposition 2.22.** *Let $\beta > 1$ be a quadratic number. Then $\Gamma(\beta) > 0$ if and only if $\beta$ is a Pisot unit satisfying $(F)$. In that case,* $\Gamma(\beta) = 1$.

Furthermore, if the condition is not satisfied, i.e. $\Gamma(\beta) = 0$, then no rational number in $(0, 1)$ has purely periodic $\beta$-expansion (as we have already seen in the example above). Finally, let us present results considering cubic Pisot numbers [1].

**Proposition 2.23.** *Let $\beta > 1$ be a cubic number. Then $\Gamma(\beta) > 0$ if and only if $\beta$ is a Pisot unit satisfying $(F)$.*

Moreover, it can be shown that Pisot quadratic units satisfying $(F)$ correspond to positive roots of polynomials $x^2 - nx - 1$ for $n \in \mathbb{N}$. Similarly, it was proven in [2], that cubic Pisot units satisfying $(F)$ are precisely the largest real roots of polynomials $x^3 - ax^2 - bx - 1$, where $a, b$ are integers such that $a \geq 1$ and $-1 \leq b \leq a + 1$.

Note that for $\beta$ such as in Proposition 2.23 it does not need to be $\Gamma(\beta) = 1$. Akiyama [2] provided a counterexample – for $\alpha$ the real root of $x^3 - x - 1$ the value of $\Gamma$ is approximately $\Gamma(\alpha) \approx 0,6667$. Moreover, this example may be generalised as follows [1].

**Proposition 2.24.** *Let $\beta$ be a cubic Pisot unit satisfying $(F)$ with complex Galois conjugates. Then $\Gamma(\beta)$ is irrational. In particular,* $0 < \Gamma(\beta) < 1$.

# Chapter 3

# Cantor base

Let us continue the discussion about Cantor base systems. We have already presented these systems in Preliminaries, see Definition 1.3. Multiple articles were written about the properties of Cantor series, especially from the stochastic perspective, for example [9, 12, 14, 20, 22], or other works of A. Rényi. The authors mainly discuss digit frequencies for Cantor representations. For a systematic review of these results we refer the reader to our previous work [21].

## 3.1 Cantor real base

We now introduce a numeration system which is a generalisation of Cantor series. Instead of an integer base, we now consider real sequences. Such systems were first considered in [4] and independently in [6].

**Definition 3.1** (Generalised Cantor)**.** Let $x \in [0,1]$ be a real number, $\boldsymbol{\beta} = (\beta_i)_{i\geq 1}$ be a sequence of real numbers greater than 1 such that $\prod_{i\geq 1} \beta_i = +\infty$. Let $x$ be represented as a convergent series of the form

$$x = \sum_{n=1}^{+\infty} \frac{a_n}{\prod_{i=1}^{n} \beta_i} = \frac{a_1}{\beta_1} + \frac{a_2}{\beta_1\beta_2} + \frac{a_3}{\beta_1\beta_2\beta_3} + \cdots\,, \tag{3.1}$$

where $a_n \in \mathbb{N}_0$. We call the sequence $(a_n)_{n\geq 1}$ a *$\boldsymbol{\beta}$-representation* of $x$. The sequence $\boldsymbol{\beta}$ with given properties is called a *Cantor real base*.

In the case that $\beta_i = \beta$ for all $i \in \mathbb{N}$, the representation in base $\boldsymbol{\beta}$ coincides with the $\beta$-representation, as it was presented in Definition 1.2.

We will later need to represent numbers not only in some given base $(\beta_i)_{i\geq 1}$, but also in bases $(\beta_i)_{i\geq n}$ for $n \in \mathbb{N}$. Let us, therefore, introduce the following notation

$$\boldsymbol{\beta}^{(m)} = (\beta_{m+1}, \beta_{m+2}, \dots), \tag{3.2}$$

for $m \in \mathbb{N}_0$. Bases $\boldsymbol{\beta}^{(m)}$ are often referred to as *shifts* of base $\boldsymbol{\beta} = \boldsymbol{\beta}^{(0)}$.

Similarly as in the case of $\beta$-representations, the representation of $x$ in base $\boldsymbol{\beta}$ need not be unique. We will consider one special representation, which can be obtained by the greedy algorithm, or equivalently by the $\beta$-transformation, as shown below.

**Greedy algorithm for Cantor real base**

For $x \in [0,1]$ and the Cantor real base $\boldsymbol{\beta}$ one particular $\boldsymbol{\beta}$-representation of $x$, let us denote it by $(a_i)_{i\geq 1}$, can be found by the so-called greedy algorithm in the following way.

- Put $a_1 = \lfloor \beta_1 x \rfloor$ and $r_1(x) = \beta_1 x - a_1$.
- For $n \geq 2$ put $a_n = \lfloor \beta_n r_{n-1}(x) \rfloor$ and $r_n(x) = \beta_n r_{n-1}(x) - a_n$.

Consequently $a_n \in \{0, \dots, \lfloor \beta_n \rfloor\}$ for all $n \in \mathbb{N}$. This algorithm indeed gives us a $\boldsymbol{\beta}$-representation of $x$, because we assume that $\prod_{\geq 1} \beta_i = +\infty$. We denote this particular $\boldsymbol{\beta}$-representation of $x$ as $d_{\boldsymbol{\beta}}(x)$ and call it the *$\boldsymbol{\beta}$-expansion* of $x$.

As we mentioned before, we may obtain this representation also by the $\beta$-transformation.

**Transformation $T_\beta$ for Cantor real base**

We have already defined the mapping $T_\beta$ on the unit interval in Chapter 2. Let us recall that for $\beta > 1$ we have $T_\beta = \{\beta x\}$. Let $x \in [0,1]$ be a real number and let $\boldsymbol{\beta} = (\beta_i)_{i\geq 1}$ be a Cantor real base. Then the digits of $d_{\boldsymbol{\beta}}(x)$ and the remainders $r_n(x)$ from the greedy algorithm can be obtained as

$$a_n = \lfloor \beta_n (T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_1}(x)) \rfloor \qquad r_n(x) = T_{\beta_n} \circ \cdots \circ T_{\beta_1}(x).$$

Similarly as for representations in base $\beta > 1$, also in the case of some given Cantor base $\boldsymbol{\beta}$, the $\boldsymbol{\beta}$-representation of $x \in [0,1]$ obtained by the greedy algorithm, i.e. $d_{\boldsymbol{\beta}}(x)$, is lexicographically maximal among all $\boldsymbol{\beta}$-representations of $x$. Moreover, the lexicographical order on the $\boldsymbol{\beta}$-expansions corresponds to the ordering of real numbers – for a Cantor real base $\boldsymbol{\beta}$ and for $x, y \in [0,1]$ the following holds true

$$x < y \iff d_{\boldsymbol{\beta}}(x) \prec d_{\boldsymbol{\beta}}(y).$$

In the sequel we will be mainly interested in Cantor real bases $\boldsymbol{\beta} = (\beta_i)_{i\geq 1}$ such that the sequence $(\beta_i)_{i\geq 1}$ is purely periodic, i.e. there exists $p \in \mathbb{N}$ such that $\boldsymbol{\beta}^{(kp+i)} = \boldsymbol{\beta}^{(i)}$ for all $k \in \mathbb{N}_0$ and for all $i \in \mathbb{N}_0$, where $\boldsymbol{\beta}^{(m)}$ as in (3.2). We call such base an *alternate base*, the integer $p$ is called the *length of period* of the base $\boldsymbol{\beta}$. We usually consider $p$ minimal such, and sometimes, for the sake of brevity, we just say that the base has *period* $p$. We denote the base of this form as $\boldsymbol{\beta} = \overline{(\beta_1, \dots, \beta_p)}$ and the $\boldsymbol{\beta}$-expansion of $x$ in this base as $d_{\boldsymbol{\beta}}(x)$ as we already denoted it before for all real Cantor bases. Another notation which will be used for $d_{\boldsymbol{\beta}}(x)$ in some cases is $d_{(\beta_1,\dots,\beta_p)}(x)$, i.e. we write explicitly the components of the base and we omit the line over them for the sake of simplicity. Note that for alternate bases it always holds that $\prod_{i\geq 1} \beta_i = +\infty$.

**Remark 3.2.** According to Remark 2.10, for a real number $\beta > 1$ the $\beta$-expansion of $x \in [0,1]$ is periodic if and only if the orbit of $x$ under $T_\beta$ is finite, or equivalently, if and only if the set of remainders $r_k$ in the greedy algorithm is finite. Similarly, for an alternate base $\boldsymbol{\beta} = \overline{(\beta_1, \ldots, \beta_p)}$, the $\boldsymbol{\beta}$-expansion of $x$ is periodic if and only if there exists $m, k, j \in \mathbb{N}_0, m \neq k$ such that $r_{pm+j}(x) = r_{pk+j}(x)$, where $r_n(x)$ denotes the remainder in the greedy algorithm for a Cantor real base. Note that it is indeed necessary to consider just remainders with indices congruent modulo $p$.

**Example 3.3.** Let $\boldsymbol{\beta} = \overline{(\tau, \tau^2)}$, where $\tau$ is the golden ratio, i.e. the greater root of $x^2 - x - 1$. Let us calculate $d_{\boldsymbol{\beta}}(1) = t_1 t_2 \ldots$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = s_1 s_2 \ldots$. For base $\boldsymbol{\beta}$ the greedy algorithm yields

$$\begin{aligned} t_1 &= \lfloor \tau \rfloor = 1 & r_1 &= \tau - 1 \\ t_2 &= \lfloor \tau^2(\tau - 1) \rfloor = \lfloor \tau^2 - 1 \rfloor = \lfloor \tau \rfloor = 1 & r_2 &= \tau - 1 \\ t_3 &= \lfloor \tau(\tau - 1) \rfloor = \lfloor 1 \rfloor = 1 & r_3 &= 0. \end{aligned}$$

Similarly, for the shifted base $\boldsymbol{\beta}^{(1)}$

$$\begin{aligned} s_1 &= \lfloor \tau^2 \rfloor & r_1 &= \tau^2 - 2 \\ s_2 &= \lfloor \tau(\tau^2 - 2) \rfloor = \lfloor \tau(\tau - 1) \rfloor = \lfloor 1 \rfloor = 1 & r_2 &= 0. \end{aligned}$$

Consequently for all $n \in \mathbb{N}_0$

$$\begin{aligned} d_{(\tau,\tau^2)}(1) &= d_{\boldsymbol{\beta}}(1) = d_{\boldsymbol{\beta}^{(2n)}}(1) = 1110^\omega = 111, \\ d_{(\tau^2,\tau)}(1) &= d_{\boldsymbol{\beta}^{(1)}}(1) = d_{\boldsymbol{\beta}^{(2n+1)}}(1) = 210^\omega = 21. \end{aligned}$$

Now we would like to generalise other notions that we have already seen in Chapter 2. We may define an *infinite Rényi expansion of 1* in the Cantor real base $\boldsymbol{\beta}$ as

$$d^*_{\boldsymbol{\beta}}(1) := \lim_{\varepsilon \to 0^+} d_{\boldsymbol{\beta}}(1 - \varepsilon),$$

where the limit is considered in the metric introduced in Section 1.1. Authors of [6] call this sequence the *quasi-greedy expansion of 1.* Note that $d^*_{\boldsymbol{\beta}}(1)$ is lexicographically greatest of all representations of 1 with infinitely many non-zero digits in the given base.

We may ask if this limit can be rewritten in the form that would allow us to calculate $d^*_{\boldsymbol{\beta}}(1)$ in a simpler way. It is indeed possible – it can be shown, see [6], that $d^*_{\boldsymbol{\beta}}(1)$ for the given Cantor real base $\boldsymbol{\beta}$ is of the form

$$d^*_{\boldsymbol{\beta}}(1) = \begin{cases} d_{\boldsymbol{\beta}}(1) & \text{if } d_{\boldsymbol{\beta}}(1) \text{ has infinitely many non-zero digits,} \\ (t_1 t_2 \cdots t_{m-1})(t_m - 1) d^*_{\boldsymbol{\beta}^{(m)}}(1) & \text{if } d_{\boldsymbol{\beta}}(1) = t_1 \cdots t_m \text{ for some } m \in \mathbb{N} \text{ and } t_m \neq 0. \end{cases} \tag{3.3}$$

**Example 3.4.** Let again $\boldsymbol{\beta} = \overline{(\tau, \tau^2)}$. Then the infinite Rényi expansions of 1 in bases $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^{(1)} = \overline{(\tau^2, \tau)}$ may be derived as follows. We need to calculate both expansions simultaneously. The first step of the calculation is

$$\begin{aligned} d^*_{\boldsymbol{\beta}}(1) &= 110 d^*_{\beta^{(3)}}(1) = 110 d^*_{\beta^{(1)}}(1), \\ d^*_{\boldsymbol{\beta}^{(1)}}(1) &= 20 d^*_{\boldsymbol{\beta}^{(2)}}(1) = 20 d^*_{\boldsymbol{\beta}}(1). \end{aligned}$$

We may now concatenate in the following way

$$\begin{aligned} d^*_{\boldsymbol{\beta}}(1) &= 110 d^*_{\beta^{(1)}}(1) = 11020 d^*_{\boldsymbol{\beta}}(1), \\ d^*_{\boldsymbol{\beta}^{(1)}}(1) &= 20 d^*_{\boldsymbol{\beta}}(1) = 20110 d^*_{\beta^{(1)}}(1). \end{aligned}$$

Therefore, both infinite Rényi expansions of 1 are purely periodic

$$\begin{aligned} d^*_{\boldsymbol{\beta}}(1) &= (11020)^\omega, \\ d^*_{\boldsymbol{\beta}^{(1)}}(1) &= (20110)^\omega. \end{aligned}$$

Let us now state an important result that was proven in [6] – a generalisation of the Parry condition (Theorem 2.6).

**Theorem 3.5** (Parry for Cantor real base)**.** *Let $\boldsymbol{\beta} > 1$ be a Cantor real base and let $(a_i)_{i\geq 1}$ be a $\boldsymbol{\beta}$-representation of $x \in [0,1)$. Then $(a_i)_{i\geq 1}$ is a $\boldsymbol{\beta}$-expansion of $x$ if and only if for all $i \geq 0$ it holds that*

$$a_{i+1}a_{i+2}\cdots \prec d^*_{\boldsymbol{\beta}^{(i)}}(1).$$

We say that a sequence $\boldsymbol{x} = (x_n)_{n\geq 1}$ of integers is *admissible* in base $\boldsymbol{\beta}$, if there exists a real number $x \in [0,1)$ such that $\boldsymbol{x} = d_{\boldsymbol{\beta}}(x)$. The above theorem provides a criterion of admissibility. It can be proven, see [17], that a finite string is admissible in $\boldsymbol{\beta}$ if and only if for all $i \in \mathbb{N}_0$

$$a_{i+1}a_{i+2}\cdots \prec d_{\boldsymbol{\beta}^{(i)}}(1).$$

## 3.2 Bi-infinite Cantor real base

We now generalise the notion of Cantor real base even further. As an introductory example consider the binary numeration system as it was presented in Section 1.2. In this system, each $x \geq 0$ may be represented in the form $x = \sum_{n\in\mathbb{Z}} x_n 2^n$. A suitable sequence of digits $x_n$ may be obtained by the greedy algorithm. In that case $x_n \in \{0,1\}$ for all $n \in \mathbb{Z}$. Denote $(U_n)_{n\in\mathbb{Z}} := (2^n)_{n\in\mathbb{Z}}$. Then the ratio $\frac{U_{n+1}}{U_n} = 2$ yields the base $\beta = 2$ for all $n$. Note that we may perform the greedy algorithm, because $\lim_{n\to+\infty} U_n = +\infty$ (thus arbitrarily large $x$ may be represented) and $\lim_{n\to-\infty} U_n = 0$ (thus arbitrarily small $x \geq 0$ may be represented).

Let us now generalise several notions from the above example. Let $(U_n)_{n\in Z}$ be a real sequence such that $U_0 = 1$, $\frac{U_{n+1}}{U_n} > 1$ for all $n \in \mathbb{Z}$ and

$$\begin{aligned} \lim_{n\to+\infty} U_n &= +\infty, \\ \lim_{n\to-\infty} U_n &= 0. \end{aligned} \tag{3.4}$$

Denote $\beta_n := \frac{U_{n+1}}{U_n}$ for all $n \in \mathbb{Z}$. Then we may invert the relation between $\beta_n$ and $U_n$ as follows

$$U_n = \begin{cases} \beta_{n-1}\cdots\beta_0 & \text{for } n > 0, \\ 1 & \text{for } n = 0, \\ \frac{1}{\beta_{-1}\cdots\beta_n} & \text{for } n < 0. \end{cases} \tag{3.5}$$

Again, for every $x \geq 0$ there exists an integer sequence $(x_n)_{n\in\mathbb{Z}}$ such that $x = \sum_{n\in\mathbb{Z}} x_n U_n$. If the digits $x_n$ are obtained by the greedy algorithm, then $x_n \in \{0,\dots,\lfloor U_n \rfloor\}$ for all $n \in \mathbb{Z}$. Indeed, we may perform the algorithm because of the additional assumptions on limits (3.4). The greedy representation of $x$ in terms of $U_n$ may be rewritten in terms of $\beta_n$ as

$$x = \sum_{n\in\mathbb{Z}} x_n U_n = \sum_{k\geq 0} x_k \beta_{k-1}\cdots\beta_0 + \sum_{k<0} \frac{x_k}{\beta_{-1}\cdots\beta_k}.$$

**Definition 3.6.** Consider $(U_n)_{n\in\mathbb{Z}}$ as above. Denote $\mathcal{B} := (\beta_n)_{n\in\mathbb{Z}}$, where $\beta_n := \frac{U_{n+1}}{U_n}$ for all $n \in \mathbb{Z}$. We call such sequence $\mathcal{B}$ a *bi-infinite Cantor real base* induced by the sequence $(U_n)_{n\in\mathbb{Z}}$. We denote $\mathcal{B} = (\dots\beta_2\beta_1\beta_0 \bullet \beta_{-1}\beta_{-2}\dots)$. Take $x \geq 0$. The sequence of digits $(x_n)_{n\in\mathbb{Z}}$ obtained by the greedy algorithm as explained above is called the *greedy expansion* of $x$ in $\mathcal{B}$. We denote

$$\langle x \rangle_{\mathcal{B}} = \dots x_2 x_1 x_0 \bullet x_{-1} x_{-2} \dots.$$

Moreover, any sequence $(y_n)_{n\in\mathbb{Z}} \subset \mathbb{N}_0$ such that

$$x = \sum_{k\geq 0} y_k \beta_{k-1}\cdots\beta_0 + \sum_{k<0} \frac{y_k}{\beta_{-1}\cdots\beta_k}$$

is called a *$\mathcal{B}$-representation* of $x$.

In any $\mathcal{B}$-representation of a real number $x$, the sequence $(y_n)_{n\in\mathbb{Z}}$ has only finitely many non-zero digits at the left side of the radix point. We usually omit the leading zeros and write just

$y_m \dots y_0 \bullet y_{-1}y_{-2}\dots$. Similarly, we omit the suffix $0^\omega$, if there are only finitely many non-zero digits at the right side of the radix point.

If we consider a bi-infinite sequence of real numbers $(\beta_n)_{n\in\mathbb{Z}}$ such that $\beta_n > 1$ for all $n$, $\lim_{n\to+\infty}\prod_{k=0}^{n-1}\beta_k = +\infty$ and $\lim_{n\to-\infty}\prod_{k=-1}^{n}\beta_k = +\infty$, then the sequence $(U_n)_{n\in\mathbb{Z}}$ defined by the relations (3.5) satisfies $U_0 = 1$, $\frac{U_{n+1}}{U_n} > 1$ for all $n$, $\lim_{n\to+\infty} U_n = +\infty$, $\lim_{n\to-\infty} U_n = 0$, thus it is the suitable sequence which induces base consisting of these $\beta_n$. We see that there is a clear correspondence between the base and its inducing sequence. Therefore, we will usually mention just the base.

Note that the greedy expansion of $x \in (0,1)$ in bi-infinite $\mathcal{B} = (\beta_n)_{n\in\mathbb{Z}}$ has non-zero digits just at the right side of the radix point and these digits coincide with digits of the $\boldsymbol{\beta}$-expansion of $x$ in base $\boldsymbol{\beta} = (\beta_{-n})_{n\geq 1}$. Therefore for $x \in (0,1)$ we will sometimes call the greedy expansion of $x$ in $\mathcal{B}$ just the $\mathcal{B}$-expansion of $x$.

In the one-sided Cantor real base $\boldsymbol{\beta}$ we considered one particular representation of 1, namely the $\boldsymbol{\beta}$-expansion, denoted $d_{\boldsymbol{\beta}}(1)$. This notion may be naturally extended for the bi-infinite Cantor real base $\mathcal{B} = (\dots\beta_1\beta_0 \bullet \beta_{-1}\beta_{-2}\dots)$ as follows

$$d_{\mathcal{B}}(1) := 0 \bullet d_{\boldsymbol{\beta}}(1),$$

where $\boldsymbol{\beta} = (\beta_{-n})_{n\geq 1}$. Again, the $\mathcal{B}$-expansion of 1 is lexicographically greatest of all $\mathcal{B}$-representations of 1 such that they have non-zero digits just at the right side of the radix point. Instead of writing $d_{\mathcal{B}}(1) = 0 \bullet x_{-1}x_{-2}\dots$ we will often use the notation similar as for the case of one-sided base $d_{\mathcal{B}}(1) = t_1t_2\dots$.

Similarly as when working with one-sided Cantor real base, we define the *i-th shift* of base $\mathcal{B} = (\dots\beta_2\beta_1\beta_0 \bullet \beta_{-1}\beta_{-2}\dots)$ as

$$\mathcal{B}^{(i)} := \begin{cases} (\dots\beta_1\beta_0\beta_{-1}\dots\beta_{-i} \bullet \beta_{-i-1}\dots) & \text{for } i \leq 0, \\ (\dots\beta_i \bullet \beta_{i-1}\dots\beta_1\beta_0\beta_{-1}\dots) & \text{for } i > 0. \end{cases}$$

We will be especially interested in the case when $\mathcal{B}$ is periodic with period $p \in \mathbb{N}$, i.e.

$$\mathcal{B} = (\dots\beta_p\beta_1\beta_2\dots\beta_p \bullet \beta_1\beta_2\dots\beta_p\beta_1\dots).$$

In this case $\mathcal{B} = \mathcal{B}^{(kp)}$ for all $k \in \mathbb{Z}$. This case is a natural extension of the notion of an alternate Cantor real base $\boldsymbol{\beta} = \overline{(\beta_1,\dots,\beta_p)}$. We call such $\mathcal{B}$ a *bi-infinite periodic extension* of $\boldsymbol{\beta}$. We will also denote the periodic bi-infinite base as $\mathcal{B} = \overline{(\beta_1,\dots,\beta_p)}$ and call it an *alternate bi-infinite Cantor real base.*

We conclude this section with definitions of several notions, especially considering finiteness of representations, analogous to those in Section 2.2.

**Definition 3.7.** Let $\mathcal{B}$ be a bi-infinite Cantor real base. We denote

$$\mathrm{Fin}\,(\mathcal{B}) = \{x \in \mathbb{R} \mid \langle |x| \rangle_{\mathcal{B}} \text{ has only finitely many non-zero digits}\}.$$

**Definition 3.8.** Let $\mathcal{B}$ be a bi-infinite Cantor real base. We say that $\mathcal{B}$ satisfies *finiteness property*, denoted $(F)$, if $\mathrm{Fin}(\mathcal{B})$ is a closed under addition. We say that $\mathcal{B}$ satisfies *positive finiteness property*, denoted $(PF)$, if $\mathrm{Fin}(\mathcal{B})$ is closed under the addition of positive elements.

Note that, unlike in the case of Rényi bases, Property $(F)$ does not need to mean that $\mathrm{Fin}(\mathcal{B})$ is a ring, because it may not be closed under multiplication of its elements, as the following example illustrates.

**Example 3.9.** Consider $\mathcal{B} = \overline{(\beta_1, \beta_2)}$, where $\beta_1$ is the positive root of the polynomial $2x^2 - 7x - 3$, and $\beta_2$ is the positive root of the polynomial $3x^2 - 5x - 4$. Then $d_{\mathcal{B}}(1) = 32$, $d_{\mathcal{B}^{(1)}}(1) = 21$, and the system has Property $(F)$, as it will be shown in Section 5.4. Now consider the number $x = \frac{1}{\beta_1}$. Surely, $x \in \mathrm{Fin}(\mathcal{B})$, since $\langle x \rangle_{\mathcal{B}} = 0 \bullet 1$, but

$$\langle x * x \rangle_{\mathcal{B}} = 0 \bullet 0\ [0\ 2]^{\omega},$$

thus $\mathrm{Fin}(\mathcal{B})$ is not closed under multiplication of its elements.

# Chapter 4

# Existence and uniqueness of a suitable base for generalised Parry theorem

In this chapter we focus on the proof of existence and uniqueness of a suitable base, as it is needed in order to further generalise Parry's [18] characterisation of digit sequences which can serve as greedy expansions of 1, see Theorem 2.7. We are especially interested in a generalisation for the systems with alternate Cantor bases. The original theorem for a Rényi base $\beta$ is a simple consequence of other results of Parry, mainly of Theorem 2.6. However, the situation for an alternate Cantor base is considerably more difficult. We solve this question for the case when the length of period of the base is $p = 2$.

Firstly, let us comment on the existence of $\beta_1$ and $\beta_2$ such that given sequences would be representations of 1 in an alternate Cantor base $\overline{(\beta_1, \beta_2)}$ and $\overline{(\beta_2, \beta_1)}$ respectively. At first, we state a lemma that will be needed in the proof of existence of a suitable base.

**Lemma 4.1.** *Let $(c_n)_{n\geq 1}$ be a bounded sequence of non-negative real numbers, $c_1 \geq 1$. Then there exists a unique $\beta \geq 1$ such that*

$$1 = \sum_{k\geq 1} \frac{c_k}{\beta^k}. \tag{4.1}$$

*Proof.* For a bounded sequence $(c_n)_{n\geq 1}$ the right-hand side of (4.1) is a power series in $\frac{1}{\beta}$ with radius of convergence $\rho > 0$. Denote $\sum_{k\geq 1} \frac{c_k}{\beta^k} =: S\left(\frac{1}{\beta}\right)$. For $\frac{1}{\beta}$ approaching $\rho$ from the left $S$ goes to infinity, for $\frac{1}{\beta}$ approaching 0 from the right the value of $S$ goes to zero. Moreover, the power series $S$ is continuous on the interval $(0, \rho)$. Therefore, there has to exist $\beta_1$ such that (4.1) holds true, i.e. $S\left(\frac{1}{\beta_1}\right) = 1$. The function $S$ is strictly monotone, thus such $\beta_1$ is unique. Since $c_1 \geq 1$ and all terms are non-negative, it has to be $\beta_1 \geq c_1 \geq 1$. $\square$

Note that if the sequence above has at least two non-zero digits, then the suitable $\beta$ is strictly greater than $c_1$. We may now prove the existence and uniqueness of a suitable Cantor real base for the case of two given sequences.

**Lemma 4.2.** *Let $a_1a_2\ldots$, $b_1b_2\ldots$ be two bounded sequences of non-negative integers such that $a_1 \geq 1$, $b_1 \geq 1$ and both sequences have at least two non-zero digits. Then there exist unique $\beta_1 > 1$ and $\beta_2 > 1$ such that the given sequences are representations of 1 in the bases $\overline{(\beta_1, \beta_2)}$ and $\overline{(\beta_2, \beta_1)}$ respectively.*

*Proof.* **Existence.** Firstly, note that $\beta_1$ and $\beta_2$ have to be positive solutions of the following system of two equations

$$0 = F_a(x, y) := -1 + \frac{a_1}{x} + \frac{a_2}{xy} + \frac{a_3}{x^2y} + \cdots \tag{4.2}$$

$$0 = F_b(x, y) := -1 + \frac{b_1}{y} + \frac{b_2}{xy} + \frac{b_3}{xy^2} + \cdots . \tag{4.3}$$

Now for $F_b(x, y) = 0$ denote $y = \varphi_b(x)$ the corresponding implicit function. By Lemma 4.1 applied on equation (4.3), for each $x > 0$ there exist a unique $y \geq 1$ such that $F_b(x, y) = 0$. Moreover, since all terms of the sum in (4.3) except $-1$ are non-negative and there exist $k > 1$ such that $b_k > 0$, it has to be $y > b_1$. Thus $\varphi_b$ is a function that maps $(0, +\infty)$ into $(b_1, +\infty)$. Let us now comment on surjectivity of this functions onto $(b_1, +\infty)$. Equation (4.3) may be rewritten as

$$1 = \frac{b_1}{y} + \frac{b_2}{xy} + \frac{b_3}{xy^2} + \cdots = \frac{b_1}{y} + \sum_{i=1}^{+\infty} \frac{1}{x^i}\left(\frac{b_{2i}}{y^i} + \frac{b_{2i+1}}{y^{i+1}}\right) =: \frac{b_1}{y} + S\left(\tfrac{1}{x}\right).$$

Take $y \in (b_1, +\infty)$ fixed. The above sum may be considered a power series in variable $\frac{1}{x}$ with radius of convergence $\rho > 0$, because $(b_k)_{k\geq 1}$ is bounded and $y$ is fixed. For $\frac{1}{x}$ approaching $\rho$ from the left, the value of $S$ goes to infinity. For $\frac{1}{x}$ approaching 0 from the right, the value of $S$ goes to zero. Power series $S$ is continuous on $(0, \rho)$ and we have $\frac{b_1}{y} < 1$, thus there has to exist $x_1 > 0$ such that

$$1 = \frac{b_1}{y} + S\left(\frac{1}{x_1}\right),$$

i.e. $y = \varphi_b(x_1)$. Therefore, $\varphi_b$ is onto $(b_1, +\infty)$. Moreover, since there are at least two non-zero digits in the sequence $(b_k)_{k\geq 1}$, the function $\varphi_b$ is strictly decreasing, thus injective. The function decreases to the horizontal asymptote $y = b_1$.

Similarly, for $F_a(x, y)$ denote $x = \psi_a(y)$ the corresponding implicit function. By Lemma 4.1 applied on equation (4.2) and discussion similar as above we have that for any $y > 0$ there exists $x > a_1$ such that $F_a(x, y) = 0$. Analogously, $\psi_a : (0, +\infty) \to (a_1, +\infty)$ is strictly decreasing, thus injective and it can be shown that it is also surjective. Therefore, there exist its inverse function

$\varphi_a := (\psi_a)^{-1} : (a_1, +\infty) \to (0, +\infty)$. The vertical asymptote for $\varphi_a$ is $x = a_1$, the function is strictly decreasing and the horizontal asymptote is $y = 0$.

Finally, note that both functions $\varphi_a$ and $\varphi_b$ are continuous, therefore there has to exist an intersection of their plots $(x, y)$, such that $x > a_1$, $y > b_1$. These $x$, $y$ are solutions of both (4.2) and (4.3), thus $x$ and $y$ are a suitable choice for $\beta_1, \beta_2 > 1$. We illustrate this for one particular choice of sequences in Figure 4.1.

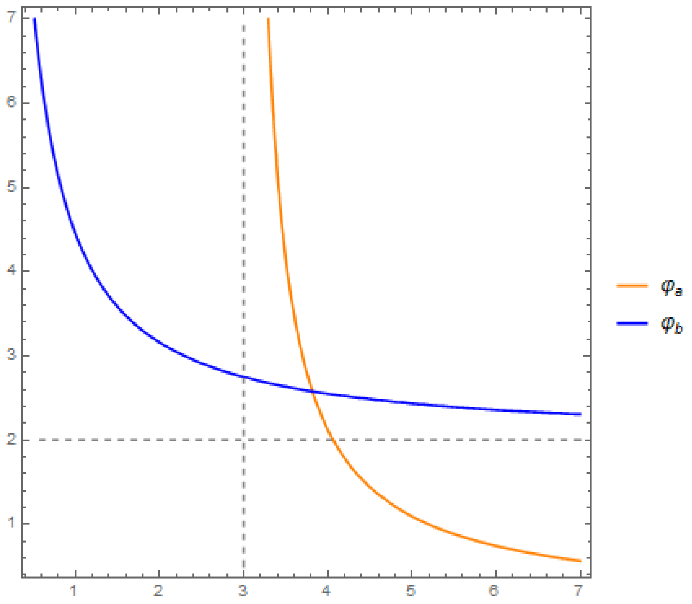


Figure 4.1: A plot of functions $\varphi_a$ in orange and $\varphi_b$ in blue for sequences 321 and 222.

**Uniqueness.** With notation as above we have $F_a(x, y) = F_a(x, \varphi_a(x)) = 0$ for all $x > a_1$. The chain rule for the derivative yields

$$\frac{\partial F_a}{\partial x} + \frac{\partial F_a}{\partial y} \cdot \varphi_a' = 0,$$

therefore

$$\varphi_a' = -\frac{\frac{\partial F_a}{\partial x}}{\frac{\partial F_a}{\partial y}}.$$

Similarly

$$\varphi_b' = -\frac{\frac{\partial F_b}{\partial x}}{\frac{\partial F_b}{\partial y}}.$$

We now show that $(\varphi_a - \varphi_b)' < 0$ at the interval $(a_1, +\infty)$. Consequently, the vertical distance between graphs of continuous functions $\varphi_a$ and $\varphi_b$ is strictly decreasing (we consider this distance negative when the graph of $\varphi_a$ is below the graph of $\varphi_b$), thus there cannot be more than one intersection of these graphs at the interval $(a_1, +\infty)$, and that proves the uniqueness of suitable $\beta_1$ and $\beta_2$.

Let us now calculate partial derivatives of $F_a$ and $F_b$. Functions $F_a$ and $F_b$ may be written as follows

$$F_a(x,y) = -1 + \sum_{k=1}^{+\infty} \left( \frac{a_{2k-1}}{x^k y^{k-1}} + \frac{a_{2k}}{x^k y^k} \right),$$
$$F_b(x,y) = -1 + \sum_{k=1}^{+\infty} \left( \frac{b_{2k-1}}{x^{k-1} y^k} + \frac{b_{2k}}{x^k y^k} \right).$$

Both functions may be considered power series in one variable for a fixed $x$ or $y$, thus we may exchange partial derivative and infinite sum on the convergence domain. Then

$$\frac{\partial F_a}{\partial x} = -\frac{a_1}{x^2} - \sum_{k=1}^{+\infty} \left( k\frac{a_{2k}}{x^{k+1} y^k} + (k+1)\frac{a_{2k+1}}{x^{k+2} y^k} \right),$$
$$\frac{\partial F_a}{\partial y} = -\sum_{k=1}^{+\infty} \left( k\frac{a_{2k}}{x^k y^{k+1}} + k\frac{a_{2k+1}}{x^{k+1} y^{k+1}} \right),$$
$$\frac{\partial F_b}{\partial x} = -\sum_{k=1}^{+\infty} \left( k\frac{b_{2k}}{y^k x^{k+1}} + k\frac{b_{2k+1}}{y^{k+1} x^{k+1}} \right),$$
$$\frac{\partial F_b}{\partial y} = -\frac{b_1}{y^2} - \sum_{k=1}^{+\infty} \left( k\frac{b_{2k}}{y^{k+1} x^k} + (k+1)\frac{b_{2k+1}}{y^{k+2} x^k} \right).$$

Proving the desired inequality $(\varphi_a - \varphi_b)' < 0$ is equivalent to verifying that $\frac{\partial F_a}{\partial y} \cdot \frac{\partial F_b}{\partial x} \overset{?}{<} \frac{\partial F_a}{\partial x} \cdot \frac{\partial F_b}{\partial y}$. The left-hand side of the inequality $\frac{\partial F_a}{\partial y} \cdot \frac{\partial F_b}{\partial x} \overset{?}{<} \frac{\partial F_a}{\partial x} \cdot \frac{\partial F_b}{\partial y}$ may be written as

$$\frac{\partial F_a}{\partial y} \cdot \frac{\partial F_b}{\partial x} = \left( -\sum_{k=1}^{+\infty} \left( k\frac{a_{2k}}{x^k y^{k+1}} + k\frac{a_{2k+1}}{x^{k+1} y^{k+1}} \right) \right) \left( -\sum_{j=1}^{+\infty} \left( j\frac{b_{2j}}{y^j x^{j+1}} + j\frac{b_{2j+1}}{y^{j+1} x^{j+1}} \right) \right),$$

and the right-hand side is

$$\frac{\partial F_a}{\partial x} \cdot \frac{\partial F_b}{\partial y} = \left( -\frac{a_1}{x^2} - \underbrace{\sum_{k=1}^{+\infty} \left( k\frac{a_{2k}}{x^{k+1} y^k} + (k+1)\frac{a_{2k+1}}{x^{k+2} y^k} \right)}_{=:S_1} \right) \left( -\frac{b_1}{y^2} - \underbrace{\sum_{j=1}^{+\infty} \left( j\frac{b_{2j}}{y^{j+1} x^j} + (j+1)\frac{b_{2j+1}}{y^{j+2} x^j} \right)}_{=:S_2} \right).$$

Multiplying both sums yields

$$\frac{\partial F_a}{\partial y} \cdot \frac{\partial F_b}{\partial x} = \sum_{j,k=1}^{+\infty} \left( jk\frac{a_{2k}b_{2j}}{x^{j+k+1} y^{j+k+1}} + jk\frac{a_{2k+1}b_{2j}}{x^{j+k+2} y^{j+k+1}} + jk\frac{a_{2k}b_{2j+1}}{x^{j+k+1} y^{j+k+2}} + jk\frac{a_{2k+1}b_{2j+1}}{x^{j+k+2} y^{j+k+2}} \right),$$
$$\frac{\partial F_a}{\partial x} \cdot \frac{\partial F_b}{\partial y} = \sum_{j,k=1}^{+\infty} \left( jk\frac{a_{2k}b_{2j}}{x^{j+k+1} y^{j+k+1}} + j(k+1)\frac{a_{2k+1}b_{2j}}{x^{j+k+2} y^{j+k+1}} + \right.$$
$$\left. + (j+1)k\frac{a_{2k}b_{2j+1}}{x^{j+k+1} y^{j+k+2}} + (j+1)(k+1)\frac{a_{2k+1}b_{2j+1}}{x^{j+k+2} y^{j+k+2}} \right) + S_2\frac{a_1}{x^2} + S_1\frac{b_1}{y^2} + \frac{a_1 b_1}{x^2 y^2}.$$

We may now compare term by term

$$\begin{aligned}
jk\frac{a_{2k}b_{2j}}{x^{j+k+1}y^{j+k+1}} &= jk\frac{a_{2k}b_{2j}}{x^{j+k+1}y^{j+k+1}} \\
jk\frac{a_{2k+1}b_{2j}}{x^{j+k+2}y^{j+k+1}} &\leq j(k+1)\frac{a_{2k+1}b_{2j}}{x^{j+k+2}y^{j+k+1}} \\
jk\frac{a_{2k}b_{2j+1}}{x^{j+k+1}y^{j+k+2}} &\leq (j+1)k\frac{a_{2k}b_{2j+1}}{x^{j+k+1}y^{j+k+2}} \\
jk\frac{a_{2k+1}b_{2j+1}}{x^{j+k+2}y^{j+k+2}} &\leq (j+1)(k+1)\frac{a_{2k+1}b_{2j+1}}{x^{j+k+2}y^{j+k+2}}
\end{aligned}$$

for all $k, j \geq 1$. Moreover, $S_2\frac{a_1}{x^2} + S_1\frac{b_1}{y^2} + \frac{a_1 b_1}{x^2 y^2} > 0$, therefore finally $\frac{\partial F_a}{\partial y} \cdot \frac{\partial F_b}{\partial x} < \frac{\partial F_a}{\partial x} \cdot \frac{\partial F_b}{\partial y}$. □

**Remark 4.3.** If all the asymptotes of functions in the proof above would be zero, i.e. $a_1 = b_1 = 0$, then the suitable solution $(x, y)$ of the system of equations

$$z = \frac{a_1}{x} + \frac{a_2}{xy} + \frac{a_3}{x^2 y} + \cdots, \tag{4.4}$$

$$z = \frac{b_1}{y} + \frac{b_2}{xy} + \frac{b_3}{xy^2} + \cdots. \tag{4.5}$$

for some fixed $z = const.$ may not exist. For instance, for sequences 0202 and 0901, there is no intersection of their graphs in the plane $z = const.$ for any choice of constant as is illustrated in Figure 4.2. Thus, especially for $z = 1$, we get that there is no suitable Cantor base in which the given sequences would be representations of 1.

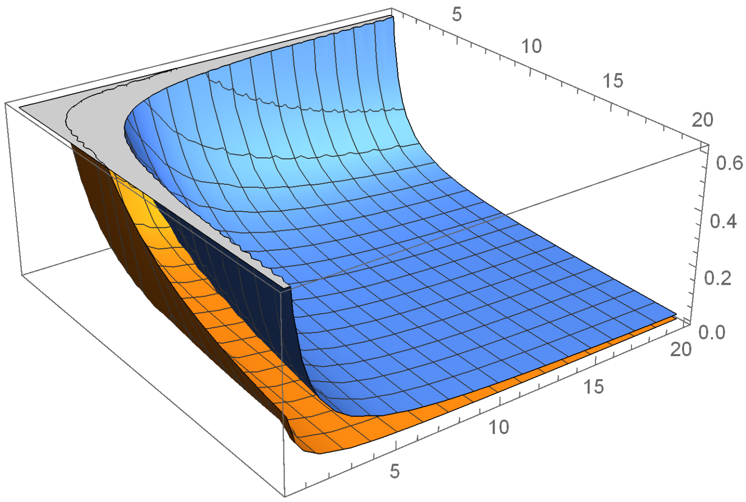


Figure 4.2: A plot of (4.4) in orange and (4.5) in blue; for sequences 0202, 0901.

In contrast with this plot, the situation with non-zero asymptotes (we consider the sequences with properties as described in Lemma 4.2 and its proof) is illustrated in Figure 4.3. The intersection of the green plane $z = 1$ with graphs for the case of sequences 321 and 222 then define the plane depicted in the proof of Lemma 4.2, see Figure 4.1.

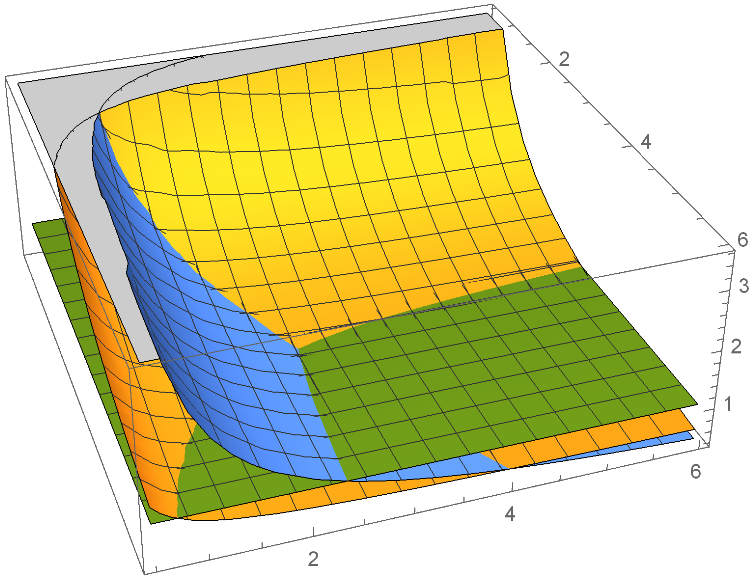


Figure 4.3: A plot of (4.4) and (4.5) for sequences 321, 222, and a green plane $z = 1$.

**Remark 4.4.** As follows from the proof, it may be easily concluded that with assumptions as in Lemma 4.2, there is no solution $x, y$ of the system of equations (4.2) and (4.3) such that $0 < x \leq a_1$ or $0 < y \leq b_1$. Indeed, let provide a short proof by contradiction. Without loss of generality for a solution $x, y > 0$ let $0 < x \leq a_1$. Since there exist at least one non-zero digit $a_k$, $k > 1$, we have the following estimate

$$F_a(x, y) + 1 = \frac{a_1}{x} + \frac{a_2}{xy} + \frac{a_3}{x^2 y} + \cdots + \frac{a_k}{x^i y^j} + \cdots \geq \frac{a_1}{a_1} + \frac{a_k}{x^i y^j} > 1,$$

which is a contradiction with (4.2). Therefore, any positive solution $x, y$ of the system of equations (4.2) and (4.3) satisfies $x > a_1$ and $y > b_1$.

**Remark 4.5.** Let us now comment on the existence of a solution of the system of equations (4.2) and (4.3) in the case when one or both sequences have only one non-zero digit. We still consider sequences such that $a_1 \geq 1$ and $b_1 \geq 1$. If both sequences have only one non-zero digit, then the system of equations is of the form

$$1 = \frac{a_1}{x} \qquad\qquad 1 = \frac{b_1}{y},$$

so clearly the only solution is $x = a_1$, $y = b_1$.

Now, without loss of generality, let $a_1 a_2 \cdots = a_1 0^\omega$ be a sequence with only one non-zero digit and let $b_1 b_2 \ldots$ have two or more positive digits and be bounded. Then $x = a_1$ and Lemma 4.1

yields that $y \geq 1$ is uniquely given by the equation

$$1 = \frac{b_1}{y} + \frac{b_2}{a_1 y} + \frac{b_3}{a_1 y^2} + \cdots .$$

Similarly, as it was already commented above, it can be shown that $y > b_1$. If we want a solution of the above system to be also a Cantor base, i.e. both $x, y > 1$, we need to assume that if the sequence is just one digit long, then this digit is strictly greater than 1.

In conclusion, together with Lemma 4.2, we have shown the existence and uniqueness of a suitable base for two bounded sequences that are both lexicographically greater than $10^\omega$.

**Proposition 4.6.** *Let $a_1 a_2 \cdots \succ 10^\omega$, $b_1 b_2 \cdots \succ 10^\omega$ be bounded sequences of non-negative integers. Then there exist unique $\beta_1 > 1$ and $\beta_2 > 1$ such that the given sequences are representations of 1 in bases $\overline{(\beta_1, \beta_2)}$ and $\overline{(\beta_2, \beta_1)}$ respectively.*

# Chapter 5

# Arithmetics in Cantor base systems

In Rényi numeration systems addition and subtraction of two numbers with finite $\beta$-expansions does not necessary yield a number with a finite $\beta$-expansion, i.e. the system does not need to have Property $(F)$. Similarly, this situation occurs in generalised Cantor base systems, as the following example illustrates.

**Example 5.1.** Let $\mathcal{B} = \overline{(2, \pi)}$. Then $d_{\mathcal{B}}(1) = 0 \bullet 20^\omega$ and $d_{\mathcal{B}^{(1)}}(1) = 0 \bullet 3001202\ldots$ is non-periodic infinite (otherwise we would be able to construct a polynomial with rational coefficients and root $\pi$). Now consider numbers 2 and $2\pi$. These numbers have finite greedy expansions in both shifts of the base $\mathcal{B}$.

$$\langle 2 \rangle_{\mathcal{B}} = 2 \bullet 0^\omega \qquad \langle 2 \rangle_{\mathcal{B}^{(1)}} = 10 \bullet 0^\omega$$

$$\langle 2\pi \rangle_{\mathcal{B}} = 100 \bullet 0^\omega \qquad \langle 2\pi \rangle_{\mathcal{B}^{(1)}} = 100 \bullet 0^\omega$$

However, the greedy expansions of $2\pi - 2$ are non-periodic infinite, since the digit string $0012021\ldots$ at the right-hand side of the radix point is a suffix of $d_{\mathcal{B}^{(1)}}(1)$.

$$\langle 2\pi - 2 \rangle_{\mathcal{B}} = 11 \bullet 0012021\ldots \qquad \langle 2\pi - 2 \rangle_{\mathcal{B}^{(1)}} = 20 \bullet 0012021\ldots$$

Therefore, none of bases $\mathcal{B}$ and $\mathcal{B}^{(1)}$ have $(F)$ property.

Let us now explore the phenomena which may occur when adding and subtracting numbers with finite greedy expansions in Cantor real base systems.

## 5.1 Necessary conditions

Firstly, we state and prove several necessary conditions of positive finiteness and finiteness property of a bi-infinite alternate Cantor real base. The results are analogous to Proposition 2.12 and Proposition 2.13 for Rényi numeration systems.

**Lemma 5.2.** *Let $p \in \mathbb{N}$ and $\mathcal{B} = \overline{(\beta_1, \dots, \beta_p)}$. If $\mathcal{B}$ satisfies $(F)$, then $d_{\mathcal{B}^{(i)}}(1)$ is finite for all $i \in \{0, \dots, p-1\}$.*

*Proof.* We proceed by contradiction. Let there exist $i \in \{1, \dots, p\}$ such that $d_{\mathcal{B}^{(i-1)}}(1) = t_1 t_2 t_3 \dots$ has infinitely many non-zero digits. The above expansion may be written as

$$1 = \frac{t_1}{\beta_i} + \frac{t_2}{\beta_i \beta_{i+1}} + \dots + \frac{t_k}{\beta_i \cdots \beta_p} + \frac{t_{k+1}}{\beta_i \cdots \beta_p \beta_1} + \dots + \frac{t_p}{\beta_1 \cdots \beta_p} + \cdots$$

for some $k \in \{1, \dots, p\}$. We multiply by $\beta_i$ and subtract $t_1$.

$$\beta_i - t_1 = \frac{t_2}{\beta_{i+1}} + \dots + \frac{t_k}{\beta_{i+1} \cdots \beta_p} + \frac{t_{k+1}}{\beta_{i+1} \cdots \beta_p \beta_1} + \dots + \frac{t_p}{\beta_1 \cdots \beta_{i-1} \beta_{i+1} \cdots \beta_p} + \cdots .$$

We repeat the process of multiplying by the denominator of the first fraction on the right-hand side and subtracting the numerator until we get

$$\beta_i \cdots \beta_p - (t_1 \beta_{i+1} \cdots \beta_p + \dots + t_k) = \frac{t_{k+1}}{\beta_1} + \dots + \frac{t_p}{\beta_1 \cdots \beta_{i-1}} + \cdots .$$

Since the string $t_1 t_2 t_3 \dots$ was obtained by the greedy algorithm, the string $t_{k+1} t_{k+2} \dots$ is the $\mathcal{B}$-expansion of a number $\beta_i \cdots \beta_p - (t_1 \beta_{i+1} \cdots \beta_p + \dots + t_k)$. Now note that the greedy expansions of the terms on the left-hand side have finitely many non-zero digits

$$\langle \beta_i \cdots \beta_p \rangle_{\mathcal{B}} = 10^i \bullet 0^\omega \qquad \langle t_1 \beta_{i+1} \cdots \beta_p + \dots + t_k \rangle_{\mathcal{B}} = t_1 \dots t_k \bullet 0^\omega,$$

thus $\beta_i \cdots \beta_p$ and $t_1 \beta_{i+1} \cdots \beta_p + \dots + t_k$ are in $\mathrm{Fin}(\mathcal{B})$. Therefore, we get that $\mathrm{Fin}(\mathcal{B})$ is not closed under subtraction of its elements and that is a contradiction. $\square$

**Remark 5.3.** Note that even if some $d_{\mathcal{B}^{(i)}}(1)$ is infinite, the above lemma does not exclude the case that the base $\mathcal{B}$ satisfies the positive finiteness property.

**Lemma 5.4.** *Let $p \in \mathbb{N}$ and $\mathcal{B} = \overline{(\beta_1, \dots, \beta_p)}$. Let $d_{\mathcal{B}^{(i)}}(1)$ be finite for all $i \in \{0, \dots, p-1\}$. Then $\delta = \prod_{i=1}^{p} \beta_i$ is an algebraic integer and $\beta_j \in \mathbb{Q}(\delta)$ for all $j \in \{1, \dots, p\}$.*

*Proof.* This property follows from the proof of the generalisation of the Schmidt's theorem that we presented in our research project, see [21]. $\square$

**Lemma 5.5.** *Let $\mathcal{B}$ be a bi-infinite Cantor real base. If $\mathcal{B}$ satisfies $(PF)$, then $\mathbb{N} \subset \mathrm{Fin}(\mathcal{B})$. Consequently, if $\mathcal{B}$ satisfies $(F)$, then $\mathbb{N} \subset \mathrm{Fin}(\mathcal{B})$.*

*Proof.* Follows straightforwardly from the definition of positive finiteness property, since $1 \in \mathrm{Fin}(\mathcal{B})$. $\square$

**Lemma 5.6.** *Let $p \in \mathbb{N}$ and $\mathcal{B} = \overline{(\beta_1, \ldots, \beta_p)}$. Let $d_{\mathcal{B}^{(i)}}(1)$ be finite for all $i \in \{0, \ldots, p-1\}$. If $\mathbb{N} \subset \mathrm{Fin}(\mathcal{B})$, then $\delta = \prod_{i=1}^{p} \beta_i$ is either a Pisot or a Salem number.*

*Proof.* According to Lemma 5.4 the number $\delta$ is an algebraic integer. It remains to show that for all $\gamma$ conjugates of $\delta$, $\gamma \neq \delta$, it holds that $|\gamma| \leq 1$. If $\delta \in \mathbb{N}$ the claim is trivial. Now consider $\delta \notin \mathbb{N}$. Take $m \in \mathbb{N}$ sufficiently large so that

$$\delta^m < \lfloor \delta^m \rfloor + 1 < \delta^m \beta_p. \tag{5.1}$$

Since obviously $\lfloor \delta^m \rfloor + 1 - \delta^m \in (0, 1)$, we obtain

$$\langle \lfloor \delta^m \rfloor + 1 \rangle_{\mathcal{B}} = 10^{pm} \bullet x_{-1} x_{-2} \ldots ,$$

where at least one digit $x_{-j}$ is non-zero, but according to the assumptions there are just finitely many non-zeros. Choose $k \in \mathbb{N}$ such that $x_{-(kp+j)}$ are zeros for all $j \in \mathbb{N}_0$. We may rewrite the above expansion in the form of sum and group terms in the following way

$$\begin{aligned} \lfloor \delta^m \rfloor + 1 &= \delta^m + \frac{x_{-1}}{\beta_1} + \frac{x_{-2}}{\beta_1 \beta_2} + \cdots + \frac{x_{-p}}{\delta} + \cdots + \frac{x_{-kp}}{\delta^k} = \\ &= \delta^m + \sum_{i=0}^{k-1} \frac{1}{\delta^{i+1}} \left( x_{-(1+ip)} \beta_2 \cdots \beta_p + x_{-(2+ip)} \beta_3 \cdots \beta_p + \cdots + x_{-(p+ip)} \right). \end{aligned}$$

We now proceed by contradiction. Let there exist $\gamma$ conjugate of $\delta$, $\gamma \neq \delta$ such that $|\gamma| > 1$. Denote $\sigma$ the isomorphism of $\mathbb{Q}(\delta)$ and $\mathbb{Q}(\gamma)$ induced by $\sigma(\delta) = \gamma$. Since $\lfloor \delta^m \rfloor + 1 \in \mathbb{N}$ we have $\sigma(\lfloor \delta^m \rfloor + 1) = \lfloor \delta^m \rfloor + 1$, i.e.

$$\gamma^m + \sum_{i=0}^{k-1} \frac{1}{\gamma^{i+1}} \left( x_{-(1+ip)} \sigma(\beta_2 \cdots \beta_p) + \ldots + x_{-(p+ip)} \right) = \delta^m + \sum_{i=0}^{k-1} \frac{1}{\delta^{i+1}} \left( x_{-(1+ip)} \beta_2 \cdots \beta_p + \ldots + x_{-(p+ip)} \right).$$

Denote $D_i = \{0, 1, \ldots, \lfloor \beta_i \rfloor\}$ and $\mathcal{D} := \{a_1 \beta_2 \cdots \beta_p + a_2 \beta_3 \cdots \beta_p + \cdots + a_p \mid a_i \in D_i\}$. Then

$$\gamma^m - \delta^m = \sum_{i=0}^{k-1} \left( \frac{1}{\delta^{i+1}} \underbrace{\left( x_{-(1+ip)} \beta_2 \cdots \beta_p + \cdots + x_{-(p+ip)} \right)}_{\in\, \mathcal{D}} - \frac{1}{\gamma^{i+1}} \underbrace{\left( x_{-(1+ip)} \sigma(\beta_2 \cdots \beta_p) + \ldots + x_{-(p+ip)} \right)}_{\in\, \sigma(\mathcal{D})} \right).$$

Denote $M := \max\{|z| \mid z \in \sigma(\mathcal{D}) \cup \mathcal{D}\}$, and $\eta := \max\left\{ \frac{1}{|\gamma|}, \frac{1}{\delta} \right\}$. Then we may estimate

$$|\delta^m - \gamma^m| \leq \sum_{i=0}^{k-1} \left( \left| \frac{M}{\delta^{i+1}} \right| + \left| \frac{M}{\gamma^{i+1}} \right| \right) \leq M \sum_{i=0}^{k-1} \frac{2}{\eta^{i+1}} = const. \tag{5.2}$$

This estimate holds true for all $m$ sufficiently large, as in (5.1). If $\gamma \in \mathbb{R}^+$, then it is readily seen that $|\delta^m - \gamma^m| \to +\infty$ as $m \to +\infty$, which is a contradiction. Now consider $\gamma \notin \mathbb{R}^+$. Then there exists a sequence $(m_i)_{i=1}^{+\infty} \subset \mathbb{N}$ such that $\mathrm{Re}(\gamma^{m_i}) \leq 0$ and all $m_i$ are sufficiently large so that (5.1) and (5.2) hold true. Then $|\delta^{m_i} - \gamma^{m_i}| \geq |\delta^{m_i}| \to +\infty$, which is a contradiction. $\square$

**Lemma 5.7.** *Let $p \in \mathbb{N}$ and $\mathcal{B} = \overline{(\beta_1, \dots, \beta_p)}$. Moreover, let $\mathbb{N} \subset \mathrm{Fin}(\mathcal{B})$ and $d_{\mathcal{B}^{(i)}}(1)$ be finite for all $i \in \{0, \dots, p-1\}$. Denote $\delta = \prod_{i=1}^p \beta_i$, let $\gamma$ be a conjugate of $\delta$, $\gamma \neq \delta$. Denote $\sigma$ the isomorphism of $\mathbb{Q}(\delta)$ and $\mathbb{Q}(\gamma)$, $\sigma(\delta) = \gamma$. Then there exists $j \in \{1, \dots, p\}$ such that $\sigma(\beta_j) \leq 0$ or $\sigma(\beta_j) \notin \mathbb{R}$.*

*Proof.* If $\delta \in \mathbb{Q}$, then there is no conjugate $\gamma \neq \delta$. Therefore consider $\delta \notin \mathbb{Q}$. All $d_{\mathcal{B}^{(i)}}(1)$ are finite, therefore there exist $n \in \mathbb{N}$ such that we may write

$$d_{\mathcal{B}^{(i)}}(1) = t_1^{(i)} t_2^{(i)} \dots t_n^{(i)}$$

for all $i \in \{0, \dots, p-1\}$. Thus we have the following relation

$$1 = \frac{t_1^{(i)}}{\beta_{i+1}} + \frac{t_2^{(i)}}{\beta_{i+1}\beta_{i+2}} + \dots + \frac{t_n^{(i)}}{\beta_{i+1} \dots \beta_{i+n}}$$

for all $i$. We now proceed by contradiction. Assume that $\sigma(\beta_j) > 0$ for all $j \in \{1, \dots, p\}$. Multiplying by $\beta_{i+1}$, subtracting $t_1^{(i)}$ and applying $\sigma$ on the above equations yields

$$\sigma(\beta_{i+1}) - t_1^{(i)} = \frac{t_2^{(i)}}{\sigma(\beta_{i+2})} + \frac{t_3^{(i)}}{\sigma(\beta_{i+2})\sigma(\beta_{i+3})} + \dots + \frac{t_n^{(i)}}{\sigma(\beta_{i+2}) \cdots \sigma(\beta_{i+n})} \geq 0,$$

i.e. for all indices $j \in \{1, \dots, p\}$ it holds that $\sigma(\beta_j) \geq t_1^{(j-1)}$. Since $\delta \notin \mathbb{Q}$, there exists at least one index $j$ such that $\beta_j \notin \mathbb{Q}$, and thus at least one of the inequalities is strict. Therefore

$$\gamma = \sigma(\delta) = \sigma(\beta_1) \cdots \sigma(\beta_p) > \prod_{i=0}^{p-1} t_1^{(i)} \geq 1.$$

This is a contradiction, because according to Lemma 5.6 the number $\delta$ is either a Pisot or a Salem number. $\square$

Statements of the above lemmata 5.2 – 5.7 can be concluded in the following theorem.

**Theorem 5.8** (Necessary conditions of $(F)$)**.** *Let $p \in \mathbb{N}$ and $\mathcal{B} = \overline{(\beta_1, \dots, \beta_p)}$. Let $\mathcal{B}$ satisfy $(F)$. Denote $\delta = \prod_{i=1}^p \beta_i$. Then*

- *$\delta$ is either a Pisot or a Salem number;*
- *$d_{\mathcal{B}^{(i)}}(1)$ is finite for all $i \in \{0, \dots, p-1\}$.*

*Moreover, for all $\beta_j$ we have $\beta_j \in \mathbb{Q}(\delta)$, and for any non-identical embedding $\sigma$ of $\mathbb{Q}(\delta)$ into $\mathbb{C}$ there exists $j \in \{1, \dots, p\}$ such that $\sigma(\beta_j) \leq 0$ or $\sigma(\beta_j) \notin \mathbb{R}$.*

## 5.2 Addition of positive elements

Consider a bi-infinite Cantor real base $\mathcal{B}$. We proceed similarly, as it was presented for a real base $\beta > 1$ in Section 2.2. At first, we describe a process of addition of two non-negative elements, say $x, y$, of $\mathrm{Fin}(\mathcal{B})$. We will in particular consider $\mathcal{B} = \overline{(\beta_1, \dots, \beta_p)}$ an alternate base. In this case, instead of working with all expansions of numbers in $\mathrm{Fin}(\mathcal{B})$, we consider just those with non-zero digits only to the right side of the radix point. It indeed suffices to do so to describe the addition of arbitrary non-negative $x, y \in \mathrm{Fin}(\mathcal{B})$, as we now explain. At first, we divide both $x$ and $y$ by a suitable power $n$ of $\delta = \prod_{i=1}^{p} \beta_i$. We choose this $n$ so that $\frac{x+y}{\delta^n} < 1$, i.e. the $\mathcal{B}$-expansion (and thus also all other $\mathcal{B}$-representations) of $\frac{x+y}{\delta^n}$ have non-zero digits only at the right side of the radix point. By the digit-wise addition of $\langle \frac{x}{\delta^n} \rangle_{\mathcal{B}}$ and $\langle \frac{y}{\delta^n} \rangle_{\mathcal{B}}$ we obtain a string representing $\frac{x+y}{\delta^n}$ in $\mathcal{B}$. We may then perform operations (for example transcriptions as described below) on this string, and afterwards convert it back to the representation of $x+y$ just multiplying by $\delta^n$, which is equivalent to shifting the radix point by $np$ digits to the right. Note that we can do this only if the Cantor real base $\mathcal{B}$ is an alternate base.

From now on we will mostly work with representations with non-zero digits at the right-hand side of the radix point, thus we define most of the following terms only for the Cantor real base $\boldsymbol{\beta}$ (not bi-infinite).

**Notation 5.9.** We denote evaluation of a word $\boldsymbol{a} = a_1 a_2 \dots$ in the base $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots)$ as

$$\mathrm{val}_{\boldsymbol{\beta}}(\boldsymbol{a}) := \sum_{n=1}^{+\infty} \frac{a_n}{\prod_{i=1}^{n} \beta_i} = \frac{a_1}{\beta_1} + \frac{a_2}{\beta_1 \beta_2} + \frac{a_3}{\beta_1 \beta_2 \beta_3} + \cdots .$$

Similarly, we denote evaluation of a bi-infinite word $\boldsymbol{b} = \dots b_1 b_0 \bullet b_{-1} b_{-2} \dots$ in the base $\mathcal{B} = (\dots \beta_2 \beta_1 \beta_0 \bullet \beta_{-1} \beta_{-2} \dots)$ as

$$\mathrm{val}_{\mathcal{B}}(\boldsymbol{b}) := \sum_{k \geq 0} b_k \beta_{k-1} \cdots \beta_0 + \sum_{k<0} \frac{b_k}{\beta_{-1} \cdots \beta_k} = \cdots + b_2 \beta_1 \beta_0 + b_1 \beta_0 + b_0 + \frac{b_{-1}}{\beta_{-1}} + \frac{b_{-2}}{\beta_{-1} \beta_{-2}} + \cdots .$$

Let us now describe the relationship considering the finiteness and the positive finiteness property for the base and its shifts. We will in particular work with alternate bases.

**Proposition 5.10.** *Let $\mathcal{B} = \overline{(\beta_1, \dots, \beta_p)}$ be an alternate base. Then for any $i \in \mathbb{N}$ the base $\mathcal{B}^{(i)}$ satisfies Property (PF) or Property (F), if and only if $\mathcal{B}$ satisfies Property (PF) or Property (F), respectively.*

*Proof.* We now show that if $\mathcal{B} = \mathcal{B}^{(0)}$ satisfies Property (PF) or Property (F), then, for every $k \geq 1$, the base $\mathcal{B}^{(k)}$ satisfies Property (PF) or Property (F), respectively. The other implication follows from periodicity of $\mathcal{B}$.

Let $k = 1$. As we already commented above, it suffices to consider $z \in [0, 1)$. From the greedy algorithm, it is not difficult to show that

$$\langle z \rangle_{\mathcal{B}^{(1)}} = 0 \bullet z_2 z_3 z_4 \cdots \iff \langle z/\beta_1 \rangle_{\mathcal{B}^{(0)}} = 0 \bullet 0 z_2 z_3 z_4 \cdots . \tag{5.3}$$

Consequently,

$$\frac{1}{\beta_1} \operatorname{Fin}\left(\mathcal{B}^{(1)}\right) \subset \operatorname{Fin}\left(\mathcal{B}^{(0)}\right).$$

Take $x, y \in \operatorname{Fin}\left(\mathcal{B}^{(1)}\right)$ such that $z := x + y < 1$. Then $\frac{x}{\beta_1}, \frac{y}{\beta_1} \in \operatorname{Fin}\left(\mathcal{B}^{(0)}\right)$, and, moreover, $\frac{z}{\beta_1} = \frac{x}{\beta_1} + \frac{y}{\beta_1} < \frac{1}{\beta_1} < 1$. By Property (PF) of the base $\mathcal{B}^{(0)}$, necessarily $\frac{z}{\beta_1} \in \operatorname{Fin}\left(\mathcal{B}^{(0)}\right)$. Moreover, since $\frac{z}{\beta_1} < \frac{1}{\beta_1}$, the greedy algorithm implies that the $\mathcal{B}^{(0)}$-expansion of $z/\beta_1$ is of the form $\langle z/\beta_1 \rangle_{\mathcal{B}^{(0)}} = 0 \bullet 0 z_2 z_3 z_4 \cdots$ with finitely many non-zero digits. Therefore $\langle z \rangle_{\mathcal{B}^{(1)}} = 0 \bullet z_2 z_3 z_4 \cdots$, and this string also has only finitely many non-zeros, which means that $x + y \in \operatorname{Fin}\left(\mathcal{B}^{(1)}\right)$. This proves that $\mathcal{B}^{(1)}$ satisfies Property (PF). The proof for subtraction is analogous. For $\mathcal{B}^{(k)}$, $k \geq 2$, we proceed by induction. ☐

We now proceed with definitions of notions needed in order to provide a construction for addition similar as was briefly presented for Rényi numeration systems in Chapter 2.

**Definition 5.11** (Transcription)**.** Let $\boldsymbol{\beta}$ be a Cantor real base. Consider words $0^j b_1 \dots b_k$, $0^l c_1 \dots c_m$ for some $j, l \in \mathbb{N}_0, k, m \in \mathbb{N}$ where $b_i, c_j \in \mathbb{N}_0$ for all $i$ and all $j$. Then the word $0^l c_1 \dots c_m$ is called a *transcription* of $0^j b_1 \dots b_k$ if

$$\operatorname{val}_{\boldsymbol{\beta}}(0^l c_1 \dots c_m) = \operatorname{val}_{\boldsymbol{\beta}}(0^j b_1 \dots b_k) \qquad \text{and} \qquad 0^l c_1 \dots c_m \succ 0^j b_1 \dots b_k. \tag{5.4}$$

**Remark 5.12.** Note that so far our only requirement is that transcriptions are lexicographically increasing, we do not require them do be admissible in the given base.

It is possible that by the digit-wise addition of two expansions with finite support we obtain a string, which is no longer admissible. We will try to construct an admissible string representing the same number by repeatedly subtracting, transcribing and digit-wisely adding certain strings, which are in some sense minimal.

**Definition 5.13** (Minimal forbidden string)**.** Let $\boldsymbol{\beta}$ be a Cantor real base. A non-admissible string of the form $0^j b_1 \dots b_k$ for some $j \in \mathbb{N}_0, k \in \mathbb{N}$ is called *minimal forbidden*,

- if both $0^j 0 b_2 \dots b_k$ and $0^j b_1 \dots b_{k-1}$ are admissible, and

- $b_i \geq 1$ implies that $0^j b_1 \dots b_{i-1}(b_i - 1)b_{i+1} \dots b_k$ is admissible for all $i \in \{1, \dots, k\}$.

Note that even minimal forbidden strings may have a non-admissible transcription (which is by definition lexicographically greater than the former string), as can be seen in Example 7, Appendix B.

**Definition 5.14** (Property T)**.** We say that a Cantor real base $\boldsymbol{\beta}$ satisfies *Property T* if for every minimal forbidden string $0^j b_1 \dots b_k$ such that $\mathrm{val}_{\boldsymbol{\beta}}(0^j b_1 \dots b_k) < 1$ there exist its transcription.

**Remark 5.15.** Let $\boldsymbol{\beta} = \overline{(\beta_1, \dots, \beta_p)}$. Denote $d_{\boldsymbol{\beta}}^{(i)}(1) = t_1^{(i)} t_2^{(i)} \dots$ for $i \in \{0, \dots, p\}$. Now consider the case that all $d_{\boldsymbol{\beta}}^{(i)}(1)$ are finite, i.e. $d_{\boldsymbol{\beta}}^{(i)}(1) = t_1^{(i)} t_2^{(i)} \dots t_{n_i}^{(i)}$ for some $n_i \in \mathbb{N}$. Then the minimal forbidden strings are either among

$$\begin{array}{llll}
0^{pj}\left(t_1^{(1)}+1\right), & 0^{pj}t_1^{(1)}\left(t_2^{(1)}+1\right), & \dots, & 0^{pj}t_1^{(1)}\dots t_{n_1-2}^{(1)}\left(t_{n_1-1}^{(1)}+1\right) \\
0^{pj+1}\left(t_1^{(2)}+1\right), & 0^{pj+1}t_1^{(2)}\left(t_2^{(2)}+1\right), & \dots, & 0^{pj+1}t_1^{(2)}\dots t_{n_2-2}^{(2)}\left(t_{n_2-1}^{(2)}+1\right) \\
\vdots & \vdots & \vdots & \vdots \\
0^{pj+p-1}\left(t_1^{(p)}+1\right), & 0^{pj+p-1}t_1^{(p)}\left(t_2^{(p)}+1\right), & \dots, & 0^{pj+p-1}t_1^{(p)}\dots t_{n_p-2}^{(p)}\left(t_{n_p-1}^{(p)}+1\right)
\end{array}$$

for some $j \in \mathbb{N}_0$, or among

$$0^{pj+k-1} t_1^{(k)} \dots t_{n_k}^{(k)}$$

for some $j \in \mathbb{N}_0$ and $k \in \{1, \dots, p\}$. Note that not all such strings have to be minimal forbidden. Also note that if we have a transcription of a string of the form $0^{pj+l} b_1 \dots b_k$ for some $j \in \mathbb{N}_0$, then transcriptions of $0^{pi+l} b_1 \dots b_k$ for all $i \in \mathbb{N}_0$ may be easily obtained just by concatenating/omitting $0^{p|i-j|}$ at the beginning of the former transcription.

**Example 5.16.** Consider base $\boldsymbol{\beta}_m = \overline{(\gamma, \gamma^2)}$, where $\gamma$ is the larger root of $x^2 - mx - 1$ for $m \in \mathbb{N}$. As presented in Appendix A the expansions of 1 are of the form $d_{\boldsymbol{\beta}}(1) = t_1 t_2 t_3 = mm1$; $d_{\boldsymbol{\beta}^{(1)}}(1) = s_1 s_2 = (m^2 + 1)m$. The minimal forbidden strings in base $\boldsymbol{\beta}_m$ have to be among

$$\begin{aligned}
0^{2k}(t_1 + 1) &= 0^{2k}(m + 1) \\
0^{2k} t_1 (t_2 + 1) &= 0^{2k} m(m + 1) \\
0^{2k} t_1 t_2 t_3 &= 0^{2k} mm1 \\
0^{2k+1}(s_1 + 1) &= 0^{2k+1}(m^2 + 2) \\
0^{2k+1} s_1 s_2 &= 0^{2k+1}(m^2 + 1)m
\end{aligned}$$

for $k \in \mathbb{N}_0$. Let us verify that $\boldsymbol{\beta}_m$ has Property T. All strings above except $(m+1)$, $m(m+1)$ and $mm1$ have evaluation in $\boldsymbol{\beta}_m$ smaller than 1. We need to find their suitable transcriptions. With the help of the greedy algorithm we obtain the following transcriptions

$$\begin{array}{lcl}
00(m+1) & \longrightarrow & 010(m^2-m+1)(m-1) \\
00m(m+1) & \longrightarrow & 0100(m-1)m1 \\
00mm1 & \longrightarrow & 01 \\
0(m^2+2) & \longrightarrow & 100m1 \\
0(m^2+1)m & \longrightarrow & 1.
\end{array}$$

It is obvious that these transcriptions satisfy the lexicographical condition (5.4). The fact that transcribed strings do really have the same evaluation as the former ones, could be verified by straightforward calculation (the form of minimal polynomial of $\gamma$ would be used during the process). Transcriptions for other $k$ may be obtained by concatenation of a suitable string of zeros, as we have already mentioned above. So indeed, our class of bases $\boldsymbol{\beta}_m$ satisfies Property T.

Consider now $\boldsymbol{\beta}$ satisfying Property T. If a $\boldsymbol{\beta}$-representation of $x+y<1$ is not admissible, then it is possible to subtract some minimal forbidden string $\boldsymbol{s}$ digit-wisely and still get a valid $\boldsymbol{\beta}$-representation (i.e. digits will be non-negative) of a number $x+y-\mathrm{val}_{\boldsymbol{\beta}}(\boldsymbol{s})$. With help of (3.3) it is not difficult to show that such minimal forbidden string indeed exists. We will not elaborate this thought here any further, however, the detailed commentary may be found in our upcoming paper [17]. Consequently, we may obtain a new $\boldsymbol{\beta}$-representation of $x+y$ by digit-wise subtraction of $\boldsymbol{s}$, followed by digit-wise addition of some transcription of $\boldsymbol{s}$. By repeating this process we obtain a lexicographically increasing sequence of transcriptions of the original representation of $x+y$. In general, this sequence does not need to be finite. If it is finite (i.e. we come to the point when it is not possible to subtract any minimal forbidden string), this process yields the lexicographically maximal $\boldsymbol{\beta}$-representation of $x+y$, i.e. its $\boldsymbol{\beta}$-expansion. Below we state and prove a theorem providing a sufficient condition so that the sequence of transcriptions terminates, see Theorem 5.19.

**Definition 5.17.** Let $p\in\mathbb{N}$. We say that $f:\mathbb{N}_0^*\cup\mathbb{N}_0^{\mathbb{N}}\to\overline{\mathbb{N}}$ is a *counting function* if for all $\boldsymbol{r},\boldsymbol{s}\in\mathbb{N}_0^*\cup\mathbb{N}_0^{\mathbb{N}}$

1. if $f(\boldsymbol{r})=0$, then $\boldsymbol{r}$ is a string of zeros;
2. $f(\boldsymbol{r}\oplus\boldsymbol{s})=f(\boldsymbol{r})+f(\boldsymbol{s})$, where $\boldsymbol{r}\oplus\boldsymbol{s}$ is a digit-wise summation (if $|\boldsymbol{r}|\neq|\boldsymbol{s}|$, we consider the shorter word concatenated from the right with sufficiently many zeros);
3. $f(\boldsymbol{t})$ is finite for all $\boldsymbol{t}\in\mathbb{N}_0^*$.

If moreover $f(\boldsymbol{r})=f(0^p\boldsymbol{r})$, we say that a counting function has period $p$.

**Example 5.18.** A simple example of a counting function is the digit sum of a string, i.e. the function defined as $f(r_1r_2r_3\dots) := \sum_{i\geq 1} r_i$ for any word $\boldsymbol{r} = r_1r_2r_3\dots \in \mathbb{N}_0^* \cup \mathbb{N}_0^{\mathbb{N}}$. We may say that this function is a counting function with period $p$ for any $p \in \mathbb{N}$. Another example is a weighted digit sum of a string with period $q$, defined as follows. Consider $q \in \mathbb{N}$, $w_1, \dots, w_{q-1} \in \mathbb{N}$. Then the function $g$ defined for any $\boldsymbol{r} = r_1r_2r_3\dots \in \mathbb{N}_0^* \cup \mathbb{N}_0^{\mathbb{N}}$ as

$$g(r_1r_2r_3\dots) := \sum_{k=0}^{q-1} w_k \sum_{i\equiv k \bmod q} r_i$$

is a counting function with period $nq$ for any $n \in \mathbb{N}$. We will usually write just $q$.

**Theorem 5.19.** *Let $\boldsymbol{\beta} = \overline{(\beta_1, \dots, \beta_p)}$ and let $f$ be a counting function. Assume that for each minimal forbidden string $0^jb_1\dots b_k$ such that $\mathrm{val}_{\boldsymbol{\beta}}(0^jb_1\dots b_k) < 1$ there exists its transcription $0^ic_1\dots c_m$ with the following property*

$$f(0^jb_1\dots b_k) \geq f(0^ic_1\dots c_m). \tag{5.5}$$

*Denote $\mathcal{B}^{(i)}$ a bi-infinite periodic extension of $\boldsymbol{\beta}^{(i)}$ for $i \in \{0, \dots, p-1\}$. Then for every positive $x, y \in \mathrm{Fin}(\mathcal{B}^{(i)})$ the $\mathcal{B}^{(i)}$-expansion of $x + y$ may be obtained from any finite $\mathcal{B}^{(i)}$-representation of $x + y$ by using finitely many transcriptions. Consequently, $\mathrm{Fin}(\mathcal{B}^{(i)})$ is closed under addition of positive elements, i.e. $\mathcal{B}^{(i)}$ satisfies $(PF)$ for all $i \in \{0, \dots, p-1\}$.*

*Proof.* Let us comment on the case when $i = 0$. For $\mathcal{B}^{(i)}$, $i \geq 1$, the claim follows from Proposition 5.10. Without loss of generality (since $\mathcal{B}$ is alternate), let $x, y \in \mathrm{Fin}(\mathcal{B})$ be positive and such that $x + y < 1$. As we have already commented on above, since this system has *Property T*, either the $\boldsymbol{\beta}$-expansion (which is the same as $\mathcal{B}$-expansion) of $x + y$ may be obtained by finite process of transcribing the digit-wise sum of $x$ and $y$, or there exists an infinite sequence of lexicographically increasing transcriptions of this sum.

In order to exclude the latter possibility, we proceed by contradiction. Both $x$ and $y$ are in $\mathrm{Fin}(\mathcal{B})$, thus there exists $m \in \mathbb{N}$ such that their $\boldsymbol{\beta}$-expansions can be written as $d_{\boldsymbol{\beta}}(x) = x_1x_2\dots x_m$, $d_{\boldsymbol{\beta}}(y) = y_1y_2\dots y_m$. We denote the digit-wise sum of $d_{\boldsymbol{\beta}}(x)$ and $d_{\boldsymbol{\beta}}(y)$ by $\boldsymbol{d} = (x_1 + y_1)(x_2 + y_2)\dots(x_m + y_m)$. Suppose we can transcribe $\boldsymbol{d}$ infinitely many times (each transcription obtained by subtracting some minimal forbidden string digit-wisely, transcribing it, and then adding it back digit-wisely). Consider $k \in \mathbb{N}$. Since $x + y < 1$, there exists $l_k \in \mathbb{N}_0$ such that the $\boldsymbol{\beta}$-representation of $x + y$ obtained after the $k$-th transcription of $\boldsymbol{d}$ is of the form

$$0^{l_k} d_{l+1}^{(k)} \dots d_{l+n_k}^{(k)}$$

for some $n_k \in \mathbb{N}$. Moreover, for every $i \in \mathbb{N}$ there exists $f_i \in \mathbb{N}$ such that

$$\mathrm{val}_{\boldsymbol{\beta}}(0^{i-1}f_i) \geq x + y,$$

thus $d_i^{(k)} \leq f_i$ for all $k$. Note that for each $r \in \mathbb{N}$ and each $l \in \mathbb{N}_0$ there are only finitely many sequences of non-negative integers $d_{l+1}d_{l+2}\dots d_{l+r}$ satisfying $0 \leq d_i \leq f_i$ for all $i \in \{l+1, \dots, l+r\}$. Since in every step $k$ the sequence $0^{l_k}d_{l+1}^{(k)}\dots d_{l+n_k}^{(k)}$ lexicographically increases, for every $r$ there exists a step $K$, so that $l_k = l = \mathit{const.}$ and digits $d_{l+1}^{(k)}, d_{l+2}^{(k)}, \dots, d_{l+r}^{(k)}$ are constant for all $k \geq K$. Now consider any $r \in \mathbb{N}$ fixed. We assume that $\boldsymbol{d}$ may be transcribed infinitely many times. Therefore, it is not possible that the digits $d_i^{(K)}$ for $i > l + r$ are all equal to 0. Denote $t$ the minimal index $t > r$ such that $d_t^{(K)} > 0$. To obtain a contradiction we repeat the above idea. For $n_1 = r$ we find $K_1$, $t_1$ as above, therefore we may write

$$x + y = \mathrm{val}_{\boldsymbol{\beta}}\left(0^l d_{l+1}^{(K_1)}\dots d_{l+n_1}^{(K_1)}\right) + \mathrm{val}_{\boldsymbol{\beta}}\left(0^{t_1-1}d_{t_1}^{(K_1)}\right) + \mathrm{val}_{\boldsymbol{\beta}}\left(0^{t_1}d_{t_1+1}^{(K_1)}d_{t_1+1}^{(K_1)}\dots\right).$$

For $k \geq K_1$ the value of $f\big(0^l d_{l+1}^{(k)}\dots d_{l+n_1}^{(k)}\big)$ remains constant and $f\big(0^{l+n_1}d_{l+n_1+1}^{(k)}\dots d_{t_1}^{(k)}\big) \geq 1$, because the sequence of digits lexicographically increases with each transcription, thus $d_{l+n_1+1}^{(k)}\dots d_{t_1}^{(k)}$ is not a string of zeros. Therefore for every $k \geq K_1$

$$f\big(0^l d_{l+1}^{(k)}\dots d_{t_1}^{(k)}\big) = f\big(0^l d_{l+1}^{(k)}\dots d_{l+n_1}^{(k)}\big) + f\big(0^{l+n_1}d_{l+n_1+1}^{(k)}\dots d_{t_1}^{(k)}\big) \geq f\big(0^l d_{l+1}^{(k)}\dots d_{l+n_1}^{(k)}\big) + 1.$$

We repeat the process for $l + r = t_1$. We find $K_2 > K_1$ and $t_2 > t_1$ such that for all $k \geq K_2$

$$f\big(0^l d_{l+1}^{(k)}\dots d_{t_2}^{(k)}\big) \geq f\big(0^l d_{l+1}^{(k)}\dots d_{t_1}^{(k)}\big) + 1.$$

By repeating infinitely many times, we obtain strictly increasing sequences $(K_j)_{j\geq 1}$ and $(t_j)_{j\geq 1}$ such that $f\big(0^l d_{l+1}^{(K_s)}\dots d_{t_s}^{(K_s)}\big)$ increases with $s$ to infinity. We have started with $\boldsymbol{d}$ finite, thus $f(\boldsymbol{d})$ is finite. Each transcription of $\boldsymbol{d}$ was obtained by subtracting minimal forbidden string, transcribing it and adding it back, and since transcriptions of minimal forbidden strings by assumptions do not increase the value of $f$ and $f(\boldsymbol{r} + \boldsymbol{s}) = f(\boldsymbol{r}) + f(\boldsymbol{s})$ for any $\boldsymbol{r}, \boldsymbol{s}$; transcriptions of $\boldsymbol{d}$ do not increase the value of $f$. Thus $f\big(0^{l_k}d_{l+1}^{(k)}\dots d_{l+n_k}^{(k)}\big) \leq f(\boldsymbol{d}) = \mathit{const.} < +\infty$ for all $k \in \mathbb{N}$, and that is a contradiction. $\square$

**Remark 5.20.** Note that if the assumptions of the above theorem hold true, then it is clear from the proof, that any finite string $\boldsymbol{s}$ may be transcribed into $\mathcal{B}$-expansion of $\mathrm{val}_{\mathcal{B}}(\boldsymbol{s})$ by finitely many transcriptions. Moreover, if the assumptions of Theorem 5.19 are satisfied, then there cannot exist an infinite sequence of transcriptions of any finite string (i.e. all possible sequences of transcriptions of all finite strings are terminating). If all sequences of transcriptions are finite, it does not matter which order of transcriptions we choose.

On the other hand, if the assumptions are not satisfied and we are transcribing some string $\boldsymbol{r}$ such that the $\mathcal{B}$-expansion of $R := \mathrm{val}_{\mathcal{B}}(\boldsymbol{r})$ is finite, it might happen that some sequence of transcriptions is finite and leads to the $\mathcal{B}$-expansion of $R$, but other sequence is infinite and the transcribed strings converge to some other infinite $\mathcal{B}$-representation of $R$. We described one such case in Appendix B, Example 5.

**Example 5.21.** Consider the base $\boldsymbol{\beta} = \boldsymbol{\beta}_1 = \overline{(\tau, \tau^2)}$ from Example 5.16. Then transcriptions of the minimal forbidden strings may be chosen as

$$\begin{array}{lcl} 0^{2k}2 & \longrightarrow & 0^{2k-1}101 \\ 0^{2k}12 & \longrightarrow & 0^{2k-1}100011 \\ 0^{2k}111 & \longrightarrow & 0^{2k-1}1 \\ 0^{2k-1}3 & \longrightarrow & 0^{2k-2}10011 \\ 0^{2k-1}21 & \longrightarrow & 0^{2k-2}1. \end{array}$$

for some $k \in \mathbb{N}$. As the counting function consider the digit sum, as we have already presented it in Example 5.18. Then all these transcriptions satisfy conditions of Theorem 5.19, thus the set $\mathrm{Fin}(\mathcal{B}^{(i)})$, where $\mathcal{B}^{(i)}$ the bi-infinite periodic extension of $\boldsymbol{\beta}^{(i)}$, is closed under addition of positive elements for $i \in \{0, 1\}$. Let us illustrate the process of addition in $\mathcal{B}$ for $x = \frac{1}{\tau^6}$ and $y = \frac{1}{\tau^3} + \frac{2}{\tau^6}$. The $\mathcal{B}$-expansions of $x$ and $y$ are $\langle x \rangle_{\mathcal{B}} = 0 \bullet 0001$, $\langle y \rangle_{\mathcal{B}} = 0 \bullet 0102$. By digit-wise addition of the digits at the right-hand side of the radix point we obtain

$$\begin{array}{r} 0001 \\ +\ 0102 \\ \hline 0103. \end{array}$$

We have $\mathrm{val}_{\mathcal{B}}(0 \bullet 0103) = x + y$, but the string 0103 is not admissible in $\boldsymbol{\beta}$. We now subtract one of the minimal forbidden strings, namely 0003, and add its transcription 0010011 digit-wisely (we align strings on the left).

$$\begin{array}{ll} & 0103 \\ - & 0003 \\ + & 0010011 \\ \hline & 0110011. \end{array}$$

The result $0 \bullet 0110011$ is the $\mathcal{B}$-expansion of $x + y$.

## 5.3 From positive finiteness to finiteness property

We have described several necessary conditions of $(PF)$ and $(F)$ property and stated a sufficient condition of $(PF)$ property. Finally, let us describe the connection between $(PF)$ and $(F)$ property for systems with finite expansions of 1. Note that, clearly, the property $(PF)$ may be reformulated as follows.

**Lemma 5.22.** *A Cantor real base $\mathcal{B}$ satisfies Property (PF) if and only if for any string $\boldsymbol{z}$ of non-negative integers with finite support we have* $\mathrm{val}_{\mathcal{B}}(\boldsymbol{z}) \in \mathrm{Fin}(\mathcal{B})$.

Let us now proceed with the statement providing a step from positive finiteness to finiteness property.

**Lemma 5.23.** *Let $p \in \mathbb{N}$, $\mathcal{B} = \overline{(\beta_1, \ldots, \beta_p)}$. Let $\mathcal{B}$ satisfy $(PF)$ and let $d_{\mathcal{B}^{(j)}}(1)$ be finite for all $j \in \{0, \ldots, p-1\}$. Then $\mathcal{B}^{(l)}$ satisfies $(F)$ for all $l \in \{0, \ldots, p-1\}$.*

We at first prove an auxiliary statement.

**Lemma 5.24.** *Let $p \in \mathbb{N}$, and $\mathcal{B} = \overline{(\beta_1, \ldots, \beta_p)}$. Let $d_{\mathcal{B}^{(i)}}(1)$ be finite for all $i \in \{0, \ldots, p-1\}$. Then for all $j, k \in \mathbb{N}_0$, $j \geq 1$, there exists a $\mathcal{B}$-representation of the number* $\mathrm{val}_{\mathcal{B}}(0 \bullet 0^{j-1}10^{\omega})$ *of the form $0 \bullet 0^{j-1}u_j u_{j+1} u_{j+2} \cdots$, such that it has finitely many non-zero digits, and $u_{j+k} \geq 1$.*

*Proof.* We proceed by induction on $k$. For $k = 0$, the statement is trivial. Let $k \geq 1$. By induction hypothesis, there exists a string $\boldsymbol{u} = 0 \bullet 0^{j-1}u_j u_{j+1} u_{j+2} \cdots$ with finitely many non-zeros, such that $\mathrm{val}_{\mathcal{B}}(0 \bullet 0^{j-1}10^{\omega}) = \mathrm{val}_{\mathcal{B}}\, \boldsymbol{u}$ and $u_{j+k} \geq 1$. Denote $d_{\mathcal{B}^{(n)}}(1) = 0 \bullet t_1^{(n)} t_2^{(n)} \cdots = 0 \bullet \boldsymbol{t}^{(n)}$. Then from the definition of $\boldsymbol{t}^{(j+k)}$, we derive that the string $0 \bullet 0^{j+k-1}(-1)\boldsymbol{t}^{(j+k)}$ of integer digits has evaluation $\mathrm{val}_{\mathcal{B}}\left(0 \bullet 0^{j+k-1}(-1)\boldsymbol{t}^{(j+k)}\right) = 0$. The digit-wise sum of the strings $\boldsymbol{u}$ and $0^{j+k-1}(-1)\boldsymbol{t}^{(j+k)}$ has all digits non-negative and has finite support (i.e. only finitely many non-zero digits). Its digit at the position $j+k+1$ is equal to $u_{j+k+1} + t_1^{(j+k)} \geq 1$, and its evaluation is equal to $\mathrm{val}_{\mathcal{B}}(0 \bullet 0^{j-1}10^{\omega})$. □

Let us now conclude the proof of Lemma 5.23.

*Proof of Lemma 5.23.* Assume that the base $\mathcal{B}$ has Property (PF). Again, since we consider alternate base, we can limit ourselves to numbers in $[0, 1)$. According to Lemma 5.22, if a number $z$ has a $\mathcal{B}$-representation with finite support, then the $\mathcal{B}$-expansion of $z$ has only finitely many non-zero digits as well. In order to show Property (F), it is thus sufficient to verify that for any $x, y \in \mathrm{Fin}(\mathcal{B}) \cap [0, 1)$ such that $z := x - y \in (0, 1)$, we can find a $\mathcal{B}$-representation of $z$ with finite support.

Let $\boldsymbol{x}$, $\boldsymbol{y}$ be the $\mathcal{B}$-expansions of numbers $x, y \in [0, 1)$ such that $x = \mathrm{val}_{\mathcal{B}}\, \boldsymbol{x} > y = \mathrm{val}_{\mathcal{B}}\, \boldsymbol{y} > 0$, and let support of $\boldsymbol{x}$ and $\boldsymbol{y}$ be finite. We proceed by induction on the sum of digits in $\boldsymbol{y}$. If the sum is 0, then $y = 0$ and the statement is trivial. Suppose the sum of digits in $\boldsymbol{y}$ is positive. Denote $\oplus$

the operation of a digit-wise addition of strings. Since $\mathrm{val}_{\mathcal{B}}\, \boldsymbol{x} > \mathrm{val}_{\mathcal{B}}\, \boldsymbol{y}$, we have $\boldsymbol{x} \succ \boldsymbol{y}$, and there exist $j,k \in \mathbb{N}_0$, $j \geq 1$, such that $\boldsymbol{x} = \boldsymbol{x}' \oplus \boldsymbol{x}''$ and $\boldsymbol{y} = \boldsymbol{y}' \oplus \boldsymbol{y}''$, where $\boldsymbol{x}'' = 0 \bullet 0^{j-1}10^{\omega}$ and $\boldsymbol{y}'' = 0 \bullet 0^{j+k-1}10^{\omega}$. It follows directly from Lemma 5.24 that $\mathrm{val}_{\mathcal{B}}\, \boldsymbol{x}'' - \mathrm{val}_{\mathcal{B}}\, \boldsymbol{y}''$ has a $\mathcal{B}$-representation, say $\boldsymbol{z}''$, with finitely many non-zero digits. Hence

$$z = \mathrm{val}_{\mathcal{B}}\, \boldsymbol{x} - \mathrm{val}_{\mathcal{B}}\, \boldsymbol{y} = \mathrm{val}_{\mathcal{B}}\, \boldsymbol{x}' + \mathrm{val}_{\mathcal{B}}\, \boldsymbol{z}'' - \mathrm{val}_{\mathcal{B}}\, \boldsymbol{y}'.$$

Property (PF) guarantees that $\mathrm{val}_{\mathcal{B}}\, \boldsymbol{x}' + \mathrm{val}_{\mathcal{B}}\, \boldsymbol{z}''$ has a finite $\mathcal{B}$-expansion, say $\boldsymbol{x}_{new}$. Thus

$$z = \mathrm{val}_{\mathcal{B}}\, \boldsymbol{x} - \mathrm{val}_{\mathcal{B}}\, \boldsymbol{y} = \mathrm{val}_{\mathcal{B}}\, \boldsymbol{x}_{new} - \mathrm{val}_{\mathcal{B}}\, \boldsymbol{y}'.$$

The sum of digits in $\boldsymbol{y}'$ is smaller by 1 than the sum of digits in $\boldsymbol{y}$, and $\boldsymbol{y}'$ is the $\mathcal{B}$-expansion of $\mathrm{val}_{\mathcal{B}}\, \boldsymbol{y}'$. Induction hypothesis implies that $z$ has a finite $\mathcal{B}$-representation, and, by Property (PF), also a finite $\mathcal{B}$-expansion. $\square$

Lemmata 5.2 and 5.23 are summarised in the following statement.

**Proposition 5.25.** *Let $p \in \mathbb{N}$ and $\mathcal{B} = \overline{(\beta_1, \ldots, \beta_p)}$. Let $\mathcal{B}$ satisfy $(PF)$. Then $\mathcal{B}^{(j)}$ satisfies $(F)$ for all $j \in \{0, \ldots, p-1\}$ if and only if $d_{\mathcal{B}^{(j)}}(1)$ is finite for all $j \in \{0, \ldots, p-1\}$.*

## 5.4 Special choice of bases

Let us now explore properties $(F)$ and $(PF)$ for some particular choice of bases. Firstly, we may be interested in arithmetic properties of the bases we have already explored in other context in our previous research [21]. We recapitulate our results concerning this topic in Appendix A. To see the complete discussion about arithmetic properties of these bases we refer the reader to Appendix B. The discussion is detailed, and thus lengthy, therefore we do not include all examples already in this chapter. However, we recommend the reader to explore the examples in this appendix first, otherwise the proofs in the sequel may seem to be exceedingly technical.

We now briefly present just one of the many examples there, see Example 7, Appendix A. In the case of base $\boldsymbol{\beta}$ with expansions of 1 of the form $a_1a_2a_3$ and $b_1b_2b_3$, we have proven that if $a_2 \geq b_3$ and $b_2 \geq a_3$, the system satisfies $(PF)$ and consequently also $(F)$. However, in other cases we have proven that it is not possible to find a suitable weighted digit sum with period 2 satisfying assumptions of Theorem 5.19, when transcriptions are chosen as admissible. Based on this observation, we now present two classes of bases where stronger assumptions on inequalities between terms of expansions of 1 ensure the existence of a suitable counting function needed to satisfy assumptions of Theorem 5.19. Note that these propositions are generalisations of Corollary 2.18 and Proposition 2.19.

**Proposition 5.26.** *Let $\boldsymbol{\beta} = \overline{(\beta_1, \beta_2)}$. Denote $\mathcal{B}^{(i)}$ a bi-infinite periodic extension of $\boldsymbol{\beta}^{(i)}$ and $d_{\boldsymbol{\beta}}(1) = a_1a_2a_3\dots$, $d_{\boldsymbol{\beta}^{(1)}}(1) = b_1b_2b_3\dots$. Moreover, let*

$$\begin{aligned} a_1 \geq b_2 \geq a_3 \geq b_4 \geq \dots \geq b_{j_0} = a_{j_0+1} = \cdots \\ b_1 \geq a_2 \geq b_3 \geq a_4 \geq \dots \geq a_{j_0} = b_{j_0+1} = \cdots, \end{aligned} \tag{5.6}$$

*for some $j_0$ in $\mathbb{N}$. Then $\mathcal{B}$ and $\mathcal{B}^{(1)}$ satisfy $(PF)$.*

*Proof.* We show that the assumptions of Theorem 5.19 are satisfied. Because of (5.6) we need to consider just two cases. Either both $d_{\boldsymbol{\beta}}(1)$ and $d_{\boldsymbol{\beta}^{(1)}}(1)$ are finite, or they both have infinitely many non-zero terms. At first, we comment on the case when both sequences have infinitely many non-zero digits.

We now find a suitable set of transcriptions of minimal forbidden strings. Consider the set of strings containing all minimal forbidden strings (similar as in Remark 5.15, but without the strings of the form $0^jt_1t_2t_3\dots$). In our case this set consists of strings of the form

$$\begin{aligned} &0^{2j}a_1\dots a_k(a_{k+1}+1) \\ &0^{2j+1}b_1\dots b_k(b_{k+1}+1) \end{aligned}$$

for $j, k \in \mathbb{N}_0$. For $k = 0$ we consider $a_1 \dots a_k = \varepsilon$ the empty word. We now derive suitable transcriptions of all these strings with evaluation in $\boldsymbol{\beta}$ smaller than 1. We consider two cases. At first let $k$ be even, i.e. $k = 2l$ for some $l$. Take $\boldsymbol{s_1} := 0^{2j}a_1\dots a_{2l}(a_{2l+1}+1)$ for some $j \in \mathbb{N}$. To construct lexicographically greater $\boldsymbol{\beta}$-representation of $\mathrm{val}_{\boldsymbol{\beta}}(s)$ we digit-wisely add strings $\boldsymbol{r} := 0^{2j-1}1(-a_1)(-a_2)\dots$ and $\boldsymbol{t} := 0^{2(j+l)}(-1)b_1b_2\dots$. Note that $\mathrm{val}_{\boldsymbol{\beta}}(\boldsymbol{r}) = \mathrm{val}_{\boldsymbol{\beta}}(\boldsymbol{t}) = 0$, thus the digit-wise sum $\boldsymbol{s} \oplus \boldsymbol{r} \oplus \boldsymbol{t}$ yields a $\boldsymbol{\beta}$-representation of $\mathrm{val}_{\boldsymbol{\beta}}(s)$. This representation is of the form

$$0^{2j-1}10^{2l+1}(b_1-a_{2l+2})(b_2-a_{2l+3})\dots,$$

and is readily seen that it is indeed lexicographically greater than the string $\boldsymbol{s_1}$. Because of (5.6), this representation has only finitely many non-zero digits. Now consider $\boldsymbol{s_2} := 0^{2j+1}b_1\dots b_{2l}(b_{2l+1}+1)$. We can proceed similarly as above and obtain a transcription of $\boldsymbol{s_2}$ in the following form

$$0^{2j}10^{2l+1}(a_1-b_{2l+2})(a_2-b_{2l+3})\dots.$$

In the case when $k$ is odd, i.e. $k = 2l-1$ for some $l$, we would proceed analogously and obtain a suitable transcription of $\boldsymbol{s_3} := 0^{2j}a_1\dots a_{2l-1}(a_{2l}+1)$ of the form

$$0^{2j-1}10^{2l}(a_1-a_{2l+1})(a_2-a_{2l+2})\dots.$$

Similarly, a transcription of $\boldsymbol{s_4} := 0^{2j+1}b_1 \dots b_{2l-1}(b_{2l}+1)$ may be chosen as

$$0^{2j}10^{2l}(b_1 - b_{2l+1})(b_2 - b_{2l+2})\dots.$$

In conclusion, we have the following rules for transcriptions

$$\begin{aligned}
\boldsymbol{s_1} &= 0^{2j}a_1 \dots a_{2l}(a_{2l+1}+1) &&\to && 0^{2j-1}10^{2l+1}(b_1 - a_{2l+2})(b_2 - a_{2l+3})\dots\\
\boldsymbol{s_2} &= 0^{2j+1}b_1 \dots b_{2l}(b_{2l+1}+1) &&\to && 0^{2j}10^{2l+1}(a_1 - b_{2l+2})(a_2 - b_{2l+3})\dots\\
\boldsymbol{s_3} &= 0^{2j}a_1 \dots a_{2l-1}(a_{2l}+1) &&\to && 0^{2j-1}10^{2l}(a_1 - a_{2l+1})(a_2 - a_{2l+2})\dots\\
\boldsymbol{s_4} &= 0^{2j+1}b_1 \dots b_{2l-1}(b_{2l}+1) &&\to && 0^{2j}10^{2l}(b_1 - b_{2l+1})(b_2 - b_{2l+2})\dots,
\end{aligned}$$

for indices as considered above. Note that all the above transcriptions are admissible and indeed consist only of non-negative digits. Moreover, only finitely many of them are non-zero. For these transcriptions we now construct a suitable counting function needed in order to satisfy the assumptions of Theorem 5.19.

Consider a weighted digit sum with period 2, denote its weights as $u, v$. The weights need to satisfy condition (5.5) for all the above transcriptions. Firstly, consider the case when $k$ is odd in the notation as in the first part of the proof. Then the condition (5.5) for strings $\boldsymbol{s_3}$ and $\boldsymbol{s_4}$ and their admissible transcriptions yields

$$\begin{aligned}
v\sum_{j=1}^{l} a_{2j-1} + u\sum_{i=1}^{l} a_{2i} + u &\geq v\sum_{j\geq 1}\big(a_{2j-1} - a_{2(l+j)-1}\big) + u\sum_{i\geq 1}\big(a_{2i} - a_{2(l+i)}\big) + u,\\
u\sum_{j=1}^{l} b_{2j-1} + v\sum_{i=1}^{l} b_{2i} + v &\geq u\sum_{j\geq 1}\big(b_{2j-1} - b_{2(l+j)-1}\big) + v\sum_{i\geq 1}\big(b_{2i} - b_{2(l+i)}\big) + v.
\end{aligned}$$

These inequalities are satisfied for any positive $u, v$. Now consider $k$ even, i.e. strings $\boldsymbol{s_1}$ and $\boldsymbol{s_2}$. The condition (5.5) for these strings and their admissible transcriptions yields

$$\begin{aligned}
v\sum_{j=1}^{l+1} a_{2j-1} + u\sum_{i=1}^{l} a_{2i} + v &\geq v\sum_{j\geq 1}\big(b_{2j} - a_{2(l+j)+1}\big) + u\sum_{i\geq 1}\big(b_{2i-1} - a_{2(l+i)}\big) + u,\\
u\sum_{j=1}^{l+1} b_{2j-1} + v\sum_{i=1}^{l} b_{2i} + u &\geq u\sum_{j\geq 1}\big(a_{2j} - b_{2(l+j)+1}\big) + v\sum_{i\geq 1}\big(a_{2i-1} - b_{2(l+i)}\big) + v.
\end{aligned}$$

Equivalently,

$$v\left(1+\sum_{j\geq 1}(a_{2j-1}-b_{2j})\right)\geq u\left(1+\sum_{j\geq 1}(b_{2j-1}-a_{2j})\right)$$
$$u\left(1+\sum_{j\geq 1}(b_{2j-1}-a_{2j})\right)\geq v\left(1+\sum_{j\geq 1}(a_{2j-1}-b_{2j})\right),$$

where all sums are finite, since the sequences in (5.6) are constant from some $j_0$ on. The suitable choice of weights satisfying these inequalities is for example

$$u := 1+\sum_{j\geq 1}(a_{2j-1}-b_{2j})$$
$$v := 1+\sum_{j\geq 1}(b_{2j-1}-a_{2i}),$$

thus we have found suitable transcriptions and a counting function satisfying assumptions of Theorem 5.19 for the case of infinite expansions of 1.

It remains to discuss the case when both $d_{\boldsymbol{\beta}}(1)$ and $d_{\boldsymbol{\beta}^{(1)}}(1)$ have just finitely many non-zero digits. Denote $d_{\boldsymbol{\beta}}(1)=a_1a_2\ldots a_n0^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1)=b_1b_2\ldots b_m0^\omega$, where $a_n,b_m\neq 0$. Consider the set of strings containing all minimal forbidden strings, as we have seen it in Remark 5.15. In our case these strings are either of the form

$$\begin{aligned}&0^{2j}a_1\ldots a_k(a_{k+1}+1)\\&0^{2j+1}b_1\ldots b_i(b_{i+1}+1)\end{aligned}\tag{5.7}$$

or $0^{2j}a_1\ldots a_n$, $0^{2j+1}b_1\ldots b_m$ for some $j\in\mathbb{N}_0$ and some $k\in\{0,\ldots,n-2\}$, $i\in\{0,\ldots,m-2\}$. The inequality (5.6) then yields $|m-n|\leq 1$.

We need to find transcriptions of the above set of minimal forbidden strings. The transcriptions of the strings of the form (5.7) may be chosen the same as in the case of infinite expansions of 1 as it was discussed above. Transcriptions of $0^{2j}a_1\ldots a_n$ and $0^{2j+1}b_1\ldots b_m$ are

$$\begin{aligned}0^{2j}a_1\ldots a_n &\quad\rightarrow\quad 0^{2j-1}1\\0^{2j+1}b_1\ldots b_m &\quad\rightarrow\quad 0^{2j}1.\end{aligned}$$

When deriving the suitable weights $u,v$ we would proceed the same way as for the case of infinite expansions of 1. The inequalities (5.5) for strings of the form (5.7) where $k,i$ is odd with transcriptions as in the infinite case, would be satisfied trivially for any $u,v\in\mathbb{N}$. Inequalities for strings where $k, i$ is even would be satisfied for example for the choice

$$u := 1 + \sum_{j=1}^{\lceil \frac{n}{2} \rceil} a_{2j-1} - \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} b_{2i}$$
$$v := 1 + \sum_{j=1}^{\lceil \frac{m}{2} \rceil} b_{2j-1} - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_{2i}.$$

Lastly, we need to check that the weighted digit-sum $f$ with these weights $u, v$ satisfies (5.5) even for transcriptions of $0^{2j}a_1 \dots a_n$ and $0^{2j+1}b_1 \dots b_m$, i.e. we need to verify

$$\begin{aligned} v(a_1 + a_3 + \cdots) + u(a_2 + a_4 + \cdots) &\geq u \\ u(b_1 + b_3 + \cdots) + v(b_2 + b_4 + \cdots) &\geq v. \end{aligned} \tag{5.8}$$

Let us verify the first inequality in (5.8). Either there exist $j$ such that $a_{2j} > 0$, then the inequality is satisfied trivially, or $0 = a_{2i}$ for all $i \geq 0$. Then also $0 = b_{2i+1}$ for all $i \geq 0$. Therefore $v = 1 + b_1$, and the first inequality may be then rewritten as

$$(1 + b_1)(a_1 + a_3 + \cdots) = (1 + b_1) \sum_{j=1}^{\lceil \frac{n}{2} \rceil} a_{2j-1} \geq 1 + \sum_{j=1}^{\lceil \frac{n}{2} \rceil} a_{2j-1} - \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} b_{2i},$$

which holds true, since $a_1 b_1 \geq 1$. The second inequality in (5.8) could be verified analogously. In conclusion, we have found the suitable counting function in the form of the weighted sum with period 2 with the above weights $u, v$ for all cases. $\square$

**Corollary 5.27.** *Let $\mathcal{B} = \overline{(\beta_1, \beta_2)}$ and let $d_{\mathcal{B}}(1) = a_1 a_2 \dots$ and $d_{\mathcal{B}^{(1)}}(1) = b_1 b_2 \dots$ be finite. Moreover, let*

$$a_1 \geq b_2 \geq a_3 \geq b_4 \geq \cdots$$
$$b_1 \geq a_2 \geq b_3 \geq a_4 \geq \cdots.$$

*Then $\mathcal{B}$ and $\mathcal{B}^{(1)}$ satisfy $(F)$.*

*Proof.* According to Proposition 5.26 both $\mathcal{B}$ and $\mathcal{B}^{(1)}$ satisfy $(PF)$. Therefore the result follows from Proposition 5.25. $\square$

Note that if both expansions of 1 Proposition 5.26 have infinitely many non-zero digits, the system cannot have $(F)$ property. Thus we have found a whole class of bases which have $(PF)$, but not $(F)$ property.

Let us now state and prove a result similar to Proposition 5.26 for an alternate base with period of the length $p = 3$.

**Proposition 5.28.** *Let $\boldsymbol{\beta} = \overline{(\beta_1, \beta_2, \beta_3)}$. Denote $\mathcal{B}^{(i)}$ the bi-infinite periodic extension of $\boldsymbol{\beta}^{(i)}$ and $d_{\boldsymbol{\beta}}(1) = a_1 a_2 \ldots$, $d_{\boldsymbol{\beta}^{(1)}}(1) = b_1 b_2 \ldots$, $d_{\boldsymbol{\beta}^{(2)}}(1) = c_1 c_2 \ldots$. Moreover, let*

$$\begin{aligned} &a_1 \geq c_2 \geq b_3 \geq a_4 \geq c_5 \geq b_6 \geq \cdots \geq a_{j_0} = c_{j_0+1} = b_{j_0+2} = \cdots \\ &b_1 \geq a_2 \geq c_3 \geq b_4 \geq a_5 \geq c_6 \geq \cdots \geq b_{j_0} = a_{j_0+1} = c_{j_0+2} = \cdots \\ &c_1 \geq b_2 \geq a_3 \geq c_4 \geq b_5 \geq a_6 \geq \cdots \geq c_{j_0} = b_{j_0+1} = a_{j_0+2} = \cdots , \end{aligned} \tag{5.9}$$

*for some $j_0 \in \mathbb{N}$. Then $\mathcal{B}$, $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ satisfy (PF).*

*Proof.* Note that we need to consider two cases - either all three expansions $d_{\boldsymbol{\beta}^{(i)}}(1)$ have infinitely many non-zero digits, or they all have just finitely many non-zero digit. Firstly consider the case when all $d_{\boldsymbol{\beta}^{(i)}}(1)$ have infinitely many non-zeros. Similarly as in the proof of Proposition 5.26, we now show that the assumptions of Theorem 5.19 are satisfied. At first, we need to find transcriptions of the following strings

$$\begin{gathered} 0^{3j} a_1 \ldots a_{k-1}(a_k + 1) \\ 0^{3j-2} b_1 \ldots b_{k-1}(b_k + 1) \\ 0^{3j-1} c_1 \ldots c_{k-1}(c_k + 1) \end{gathered}$$

for $j \in \mathbb{N}$, $k \in \mathbb{N}_0$. We need to distinguish three cases according to $k$ mod 3. At first let $k = 3l$ for some $l \in \mathbb{N}_0$. The transcriptions of the above strings for such $k$ may be chosen as follows

$$\begin{aligned} 0^{3j} a_1 \ldots a_{3l-1}(a_{3l} + 1) &\quad\rightarrow\quad 0^{3j-1} 1 0^{3l} (a_1 - a_{3l+1})(a_2 - a_{3l+2}) \ldots \\ 0^{3j-2} b_1 \ldots b_{3l-1}(b_{3l} + 1) &\quad\rightarrow\quad 0^{3j-3} 1 0^{3l} (b_1 - b_{3l+1})(b_2 - b_{3l+2}) \ldots \\ 0^{3j-1} c_1 \ldots c_{3l-1}(c_{3l} + 1) &\quad\rightarrow\quad 0^{3j-2} 1 0^{3l} (c_1 - c_{3l+1})(c_2 - c_{3l+2}) \ldots . \end{aligned}$$

Now consider $k = 3l + 1$ for some $l \in \mathbb{N}_0$. In this case suitable transcriptions are

$$\begin{aligned} 0^{3j} a_1 \ldots a_{3l}(a_{3l+1} + 1) &\quad\rightarrow\quad 0^{3j-1} 1 0^{3l+1} (b_1 - a_{3l+2})(b_2 - a_{3l+3}) \ldots \\ 0^{3j-2} b_1 \ldots b_{3l}(b_{3l+1} + 1) &\quad\rightarrow\quad 0^{3j-3} 1 0^{3l+1} (c_1 - b_{3l+2})(c_2 - b_{3l+3}) \ldots \\ 0^{3j-1} c_1 \ldots c_{3l}(c_{3l+1} + 1) &\quad\rightarrow\quad 0^{3j-2} 1 0^{3l+1} (a_1 - c_{3l+2})(a_2 - c_{3l+3}) \ldots . \end{aligned}$$

Lastly, consider $k = 3l + 2$ for some $l \in \mathbb{N}_0$. Then the transcriptions may be chosen as

$$\begin{aligned} 0^{3j} a_1 \ldots a_{3l+1}(a_{3l+2} + 1) &\quad\rightarrow\quad 0^{3j-1} 1 0^{3l+2} (c_1 - a_{3l+3})(c_2 - a_{3l+4}) \ldots \\ 0^{3j-2} b_1 \ldots b_{3l+1}(b_{3l+2} + 1) &\quad\rightarrow\quad 0^{3j-3} 1 0^{3l+2} (a_1 - b_{3l+3})(a_2 - b_{3l+4}) \ldots \\ 0^{3j-1} c_1 \ldots c_{3l+1}(c_{3l+2} + 1) &\quad\rightarrow\quad 0^{3j-2} 1 0^{3l+2} (b_1 - c_{3l+3})(b_2 - c_{3l+4}) \ldots . \end{aligned}$$

All these realtions were derived analogously as transcriptions in the proof of Proposition 5.26. Note that all the above transcriptions indeed have just finitely many non-zero digits, since sequences (5.9) are constant from some index on. Now we construct a weighted digit sum with period 3 and weights $u, v, w$ such that (5.5) holds true on all the above transcriptions. The condition (5.5) on these transcriptions yields the system of nine inequalities for $u, v, w$, out of which three are trivially satisfied, and the non-trivial ones are

$$u\left(1+\sum_{i\geq 0}(a_{3i+1}-b_{3i+3})\right)+v\left(\sum_{i\geq 0}(a_{3i+2}-b_{3i+1})\right)+w\left(\sum_{i\geq 0}(a_{3i+3}-b_{3i+2})\right)\geq w \tag{5.10}$$

$$u\left(\sum_{i\geq 0}(a_{3i+1}-c_{3i+2})\right)+v\left(1+\sum_{i\geq 0}(a_{3i+2}-c_{3i+3})\right)+w\left(\sum_{i\geq 0}(a_{3i+3}-c_{3i+1})\right)\geq w \tag{5.11}$$

$$u\left(\sum_{i\geq 0}(b_{3i+3}-c_{3i+2})\right)+v\left(1+\sum_{i\geq 0}(b_{3i+1}-c_{3i+3})\right)+w\left(\sum_{i\geq 0}(b_{3i+2}-c_{3i+1})\right)\geq u \tag{5.12}$$

$$u\left(\sum_{i\geq 0}(b_{3i+3}-a_{3i+1})\right)+v\left(\sum_{i\geq 0}(b_{3i+1}-a_{3i+2})\right)+w\left(1+\sum_{i\geq 0}(b_{3i+2}-a_{3i+3})\right)\geq u \tag{5.13}$$

$$u\left(\sum_{i\geq 0}(c_{3i+2}-a_{3i+1})\right)+v\left(\sum_{i\geq 0}(c_{3i+3}-a_{3i+2})\right)+w\left(1+\sum_{i\geq 0}(c_{3i+1}-a_{3i+3})\right)\geq v \tag{5.14}$$

$$u\left(1+\sum_{i\geq 0}(c_{3i+2}-b_{3i+3})\right)+v\left(\sum_{i\geq 0}(c_{3i+3}-b_{3i+1})\right)+w\left(\sum_{i\geq 0}(c_{3i+1}-b_{3i+2})\right)\geq v. \tag{5.15}$$

Now note that inequalities (5.10) and (5.13) may be rewritten as

$$u\left(\sum_{i\geq 0}(a_{3i+1}-b_{3i+3})\right)+v\left(\sum_{i\geq 0}(a_{3i+2}-b_{3i+1})\right)+w\left(\sum_{i\geq 0}(a_{3i+3}-b_{3i+2})\right)\geq w-u$$
$$u\left(\sum_{i\geq 0}(a_{3i+1}-b_{3i+3})\right)+v\left(\sum_{i\geq 0}(a_{3i+2}-b_{3i+1})\right)+w\left(\sum_{i\geq 0}(a_{3i+3}-b_{3i+2})\right)\leq w-u,$$

respectively, therefore there has to be equality in both of these inequalities. We may proceed similarly for inequalities (5.12) and (5.15) and for inequalities (5.14) and (5.11). In this way we obtain the following system of equations equivalent to the system of inequalities (5.10)–(5.15)

$$\begin{aligned}
u\left(\sum_{i\geq 0}(a_{3i+1}-b_{3i+3})\right)+v\left(\sum_{i\geq 0}(a_{3i+2}-b_{3i+1})\right)+w\left(\sum_{i\geq 0}(a_{3i+3}-b_{3i+2})\right)&=w-u\\
u\left(\sum_{i\geq 0}(b_{3i+3}-c_{3i+2})\right)+v\left(\sum_{i\geq 0}(b_{3i+1}-c_{3i+3})\right)+w\left(\sum_{i\geq 0}(b_{3i+2}-c_{3i+1})\right)&=u-v\\
u\left(\sum_{i\geq 0}(c_{3i+2}-a_{3i+1})\right)+v\left(\sum_{i\geq 0}(c_{3i+3}-a_{3i+2})\right)+w\left(\sum_{i\geq 0}(c_{3i+1}-a_{3i+3})\right)&=v-w.
\end{aligned}\tag{5.16}$$

Note that since we assume (5.9), all the above sums are finite. Denote $d_i$ the distance between $i$-th and $i+1$-th term of the sequence $a_1, c_2, b_3, a_4, c_5, b_6 \ldots$, i.e. $d_1 := a_1 - c_2$, $d_2 := c_2 - b_3$ etc. Similarly, let $\Delta_i$ for $i \in \mathbb{N}$ denote the distances between terms of the sequence $b_1, a_2, c_3, b_4, a_5, c_6 \ldots$ and $D_i$ the distances between the terms of $c_1, b_2, a_3, c_4, b_5, a_6 \ldots$. According to (5.9) all $d_i$, $\Delta_i$, $D_i$ are non-negative for all $i \in \mathbb{N}_0$. Especially for $i \geq j_0$ they are all equal to zero. Denote

$$\begin{aligned}
d &:= \sum_{i\geq 0} d_{3i+1} & \Delta &:= \sum_{i\geq 0}\Delta_{3i+1} & D &:= \sum_{i\geq 0} D_{3i+1}\\
\tilde{d} &:= \sum_{i\geq 0} d_{3i+2} & \tilde{\Delta} &:= \sum_{i\geq 0}\Delta_{3i+2} & \tilde{D} &:= \sum_{i\geq 0} D_{3i+2}.
\end{aligned}$$

Now note that summing up the equations (5.16) yields $0 = 0$, thus they are linearly dependent. Therefore there exist a non-zero vector $(u, v, w)^T$ satisfying the system of equations (5.16), which can be in our notation written as

$$\begin{pmatrix} 1+d+\tilde{d} & -\Delta & -1-\tilde{D}\\ -1-\tilde{d} & 1+\Delta+\tilde{\Delta} & -D\\ -d & -1-\tilde{\Delta} & 1+D+\tilde{D}\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

We need to show that $(u, v, w)$ can be chosen as a vector with positive components. We may rewrite the above system as follows

$$\begin{pmatrix} \Delta+\tilde{\Delta}+D+\tilde{D} & \Delta & 1+\tilde{D}\\ 1+\tilde{d} & d+\tilde{d}+D+\tilde{D} & D\\ d & 1+\tilde{\Delta} & d+\tilde{d}+\Delta+\tilde{\Delta}\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}=(1+d+\tilde{d}+\Delta+\tilde{\Delta}+D+\tilde{D})\begin{pmatrix}u\\ v\\ w\end{pmatrix}.$$

Denote the above matrix as $\mathbb{M}$. This matrix is non-negative, diagonally dominant and irreducible. The number $\lambda_1 := 1+d+\tilde{d}+\Delta+\tilde{\Delta}+D+\tilde{D}$ is its eigenvalue corresponding to the vector $(u, v, w)^T$. We now show that $\lambda_1$ is a dominant eigenvalue of matrix $\mathbb{M}$.

We need to consider two cases. Denote $\sigma(\mathbb{M})$ the spectrum of the matrix $\mathbb{M}$. Either $\sigma(\mathbb{M}) = \{\lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{R}$; or $\lambda_1 \in \mathbb{R}$ and $\lambda_2 = \overline{\lambda_3} \in \mathbb{C} \setminus \mathbb{R}$. Firstly, let $\sigma(\mathbb{M}) \subset \mathbb{R}$. Gershgorin theorem 1.5 implies that $\lambda_i$ are all non-negative. Since $\lambda_1+\lambda_2+\lambda_3 = \mathrm{tr}(\mathbb{M}) = 2(d+\tilde{d}+\Delta+\tilde{\Delta}+D+\tilde{D})$, we have $\lambda_1 - 2 = \lambda_2 + \lambda_3 > 0$, and thus $\lambda_1$ is the dominant eigenvalue.

Now consider the case $\lambda_2 = \overline{\lambda_3} \in \mathbb{C} \setminus \mathbb{R}$. According to Perron-Frobenius, there exist a positive eigenvalue of $\mathbb{M}$ such that it is equal to the spectral radius of $\mathbb{M}$. In this case it has to be $\lambda_1$. Thus in either case $\lambda_1$ is a Perron-Frobenius eigenvalue.

For $\lambda_1$ the dominant eigenvalue of $\mathbb{M}$, Perron-Frobenius theorem implies existence of a positive eigenvector corresponding to $\lambda_1$ with one dimensional eigenspace. Therefore $(u, v, w)^T$ is in this eigenspace and thus may be chosen positive. Moreover, since matrix $\mathbb{M}$ has integer components, and $\lambda_1$ is also an integer, the vector $(u, v, w)^T$ may be chosen as integer vector, which concludes the proof for the case when all $d_{\boldsymbol{\beta}^{(i)}}(1)$ have infinitely many non-zero digits.

Let us now comment on the case when all $d_{\boldsymbol{\beta}^{(i)}}(1)$ have just finitely many non-zero digits. Denote $k_i$ the index of the last non-zero digit in $d_{\boldsymbol{\beta}^{(i)}}(1)$. The set of minimal forbidden strings contains also strings of the form $0^{3j}a_1a_2\ldots a_{k_0}$, $0^{3j-2}b_1b_2\ldots b_{k_1}$ and $0^{3j-1}c_1c_2\ldots c_{k_2}$. Their transcriptions can be chosen as

$$\begin{aligned} 0^{3j}a_1\ldots a_{k_0} &\quad\to\quad 0^{3j-1}1 \\ 0^{3j-2}b_1\ldots b_{k_1} &\quad\to\quad 0^{3j-3}1 \\ 0^{3j-1}c_1\ldots c_{k_2} &\quad\to\quad 0^{3j-2}1. \end{aligned} \tag{5.17}$$

The transcriptions of other minimal forbidden strings and weight $u, v, w$ may be chosen the same as in the first part of the proof. Consider now any fixed positive integer vector $(u, v, w)^T$ fulfilling all the conditions in the first part of the proof. It is left to verify that the weighted sum with period 3 with weights $u, v, w$ chosen this way is non-increasing also on transcriptions (5.17). The condition (5.5) yields the following system of inequalities

$$\begin{aligned} A := u\sum_{i\ge 0} a_{3i+1} + v\sum_{i\ge 0} a_{3i+2} + w\sum_{i\ge 0} a_{3i+3} &\ge w \\ B := u\sum_{i\ge 0} b_{3i+3} + v\sum_{i\ge 0} b_{3i+1} + w\sum_{i\ge 0} b_{3i+2} &\ge u \\ C := u\sum_{i\ge 0} c_{3i+2} + v\sum_{i\ge 0} c_{3i+3} + w\sum_{i\ge 0} c_{3i+1} &\ge v. \end{aligned} \tag{5.18}$$

Let us verify that these inequalities indeed hold true. Indeed, by summing up $A+B+C$, we obtain

$$A+B+C = u\sum_{i\ge 0}(a_{3i+1}+b_{3i+3}+c_{3i+2}) + v\sum_{i\ge 0}(a_{3i+2}+b_{3i+1}+c_{3i+3}) + w\sum_{i\ge 0}(a_{3i+3}+b_{3i+2}+c_{3i+1}).$$

Note that in each sum there is at least one non-zero term, namely $a_1$, $b_1$, $c_1$ respectively. Therefore $A + B + C \ge u + w + v$. We now show that $A \ge w$, $B \ge u$ and $C \ge v$. Let us proceed by contradiction. Suppose that, for example, $A < w$. Then $B + C \geq u + v + (w - A) > u + v$. Necessarily, either $B > u$, or $C > v$. Consider $B > u$. Then $A - B < w - u$, i.e.

$$u\left(\sum_{i\geq 0}(a_{3i+1} - b_{3i+3})\right) + v\left(\sum_{i\geq 0}(a_{3i+2} - b_{3i+1})\right) + w\left(\sum_{i\geq 0}(a_{3i+3} - b_{3i+2})\right) < w - u,$$

which is a contradiction, since $u$, $v$, $w$ were chosen such that they satisfy (5.16). The other cases would be treated similarly. □

**Corollary 5.29.** *Let $\mathcal{B} = \overline{(\beta_1, \beta_2, \beta_3)}$ and let $d_{\mathcal{B}}(1) = a_1a_2\ldots$, $d_{\mathcal{B}^{(1)}}(1) = b_1b_2\ldots$ and $d_{\boldsymbol{\beta}^{(2)}}(1) = c_1c_2\ldots$ be finite. Moreover, let*

$$\begin{aligned} &a_1 \geq c_2 \geq b_3 \geq a_4 \geq c_5 \geq b_6 \geq \cdots \\ &b_1 \geq a_2 \geq c_3 \geq b_4 \geq a_5 \geq c_6 \geq \cdots \\ &c_1 \geq b_2 \geq a_3 \geq c_4 \geq b_5 \geq a_6 \geq \cdots . \end{aligned}$$

*Then $\mathcal{B}$, $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ satisfy $(F)$.*

*Proof.* According to Proposition 5.28 all three bases $\mathcal{B}$, $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ satisfy $(PF)$. Therefore the result follows from Proposition 5.25. □

**Remark 5.30.** In Proposition 5.28 we assumed that the sequences $a_1a_2\ldots$, $b_1b_2\ldots$ and $c_1c_2\ldots$ are expansions of 1, i.e. $d_{\boldsymbol{\beta}^{(i)}}(1)$. We may ask if the fact that some $\boldsymbol{\beta}^{(i)}$-representations of 1 satisfy inequalities (5.9) does not already imply that these $\boldsymbol{\beta}^{(i)}$-representations of 1 are in fact even $\boldsymbol{\beta}^{(i)}$-expansions of 1. It is not necessarily so. Consider $a_1a_2\ldots$, $b_1b_2\ldots$ and $c_1c_2\ldots$ all with infinitely many non-zero digits satisfying (5.9). It may be the case that not all three sequences are $d_{\boldsymbol{\beta}^{(i)}}(1)$, but some of them is $d^*_{\boldsymbol{\beta}^{(i)}}(1)$. For example, sequences $5(241)^\omega$, $(241)^\omega$, $(412)^\omega$ satisfy inequalities (5.9), but they are not $d_{\boldsymbol{\beta}^{(i)}}(1)$. It can be shown that for the case of finite sequences the inequalities (5.9) are indeed strong enough to ensure that the given sequences are $d_{\boldsymbol{\beta}^{(i)}}(1)$. Similar reasoning can be done for the case $p = 2$, i.e. for Proposition 5.26.

# Chapter 6

# Purely periodic expansions

This chapter will be devoted to properties of purely periodic expansions in generalised Cantor base systems. We will focus on generalisations of properties already known for Rényi numeration systems, as were briefly presented in Section 2.3.

Firstly, note that similarly as in the case of Rényi numeration systems, it can be shown that for an alternate Cantor base the $\boldsymbol{\beta}$-expansion of 1 cannot be purely periodic [6]. Therefore, in the sequel, we will focus on $\boldsymbol{\beta}$-expansions of $x \in [0,1)$.

Let $\boldsymbol{\beta}$ be a Cantor real base. We denote

$$\Gamma(\boldsymbol{\beta}) := \sup\{c \in [0,1) \mid \forall\, p/q :\ 0 \le p/q \le c,\ d_{\boldsymbol{\beta}}(p/q) \text{ is purely periodic}\}.$$

Our aim is to characterise bases $\boldsymbol{\beta}$ such that $\Gamma(\boldsymbol{\beta}) > 0$. We present some necessary and sufficient conditions for this. We then include numerical experiments with alternate Cantor bases, see Section 6.3. Based on these results, it is likely that for an alternate base with period $p$ we have $\Gamma\big(\boldsymbol{\beta}^{(i)}\big) > 0$ for all $i \in \{0,1,\ldots,p-1\}$, whenever $\Gamma(\boldsymbol{\beta}) > 0$.

## 6.1 Necessary conditions

As we have already seen in Section 2.3, if for a real base $\beta$ it holds that $\Gamma(\beta) > 0$, then $\beta$ is an algebraic integer, and, moreover, it is a Pisot number. Let us now show a similar result for alternate Cantor bases. In the proof, we will use a result proven as Theorem 7 in [7].

**Theorem 6.1** (Theorem 7 [7]). *Let $p \in \mathbb{N}$ and $\boldsymbol{\beta} = \overline{(\beta_1,\ldots,\beta_p)}$. If for every $i \in \{0,\ldots,p-1\}$ there exists a non-zero integer $q_i$ and an eventually periodic sequence $\boldsymbol{a}^{(i)} = \big(a_n^{(i)}\big)_{n\ge1}$ such that*

$$\sum_{n\ge1} \frac{a_n^{(i)}}{\prod_{k=1}^{n} \beta_{i+k}} = \frac{1}{q_i},$$

*then $\delta$ is an algebraic integer. If, moreover, these $p$ sequences have non-negative elements and for all $i \in \{0, \dots, p-1\}$ there exist $m_i$ such that $a^{(i)}_{m_i p+1} \geq 1$, then $\beta_j \in \mathbb{Q}(\delta)$ for all $j \in \{1, \dots, p\}$.*

In the sequel, we will need to use also the claim of the second part of the above theorem. Therefore we now comment on existence of such sequences with non-zero digits on certain positions. It suffices to consider the situation just in the base $\boldsymbol{\beta} = \boldsymbol{\beta}^{(0)}$ and for the sequence $\boldsymbol{a}^{(0)}$, the results for the shifted bases (i.e. other indices $i$ of $\boldsymbol{a}^{(i)}$) is straightforward.

**Lemma 6.2.** *Let $\boldsymbol{\beta} = \overline{(\beta_1, \dots, \beta_p)}$ be an alternate base. Then there exists $r \in \mathbb{N}, r \neq 1$, such that in the $\boldsymbol{\beta}$-expansion*

$$d_{\boldsymbol{\beta}}(1/r) = x_1 x_2 x_3 \dots$$

*there exists an index $l \in \mathbb{N}$, $l \equiv 1 \bmod p$, so that $x_l \neq 0$. Moreover, for any $K > 0$ the number $r$ may be chosen so that $1/r < K$.*

*Proof.* Since $\beta_1 > 1$ and $\delta > 1$, it is clear that for every sufficiently large $l = Mp + 1$ the length of the interval $I = (\beta_1 \cdots \beta_{l-1}, \beta_1 \cdots \beta_l) = (\delta^M, \delta^M \beta_1)$ is greater than 1. Therefore there exist an integer $r \in I$. For any $K > 0$, one can chose $l$ large enough, so that

$$\frac{1}{r} < \frac{1}{\delta^M} < K.$$

Now, by the greedy algorithm, the $\boldsymbol{\beta}$-expansion of $\frac{1}{r}$, i.e. $d_{\boldsymbol{\beta}}(1/r) = x_1 x_2 x_3 \dots$, is of the form $0^{l-1} x_l x_{l+1} \dots$ with $x_l \neq 0$. ☐

With these propositions at hand, let us finally state our result. Note that the proof will follow similar ideas as the proof of Theorem 4.4 in our research project [21].

**Proposition 6.3.** *Let $p \in \mathbb{N}$ and $\boldsymbol{\beta} = \overline{(\beta_1, \dots, \beta_p)}$. If $\Gamma\big(\boldsymbol{\beta}^{(i)}\big) > 0$ for all $i \in \{0, \dots, p-1\}$, then $\delta = \prod_{i=1}^{p} \beta_i$ is either a Pisot or a Salem number and $\beta_j \in \mathbb{Q}(\delta)$ for all $j \in \{1, \dots, p\}$.*

*Proof.* By theorem 6.1 combined with Lemma 6.2, the number $\delta$ is an algebraic integer and $\beta_j \in \mathbb{Q}(\delta)$ for all $j \in \{1, \dots, p\}$. Let us now show that none of conjugates of $\delta$ other than $\delta$ itself has modulus strictly greater than 1. Since $\Gamma(\boldsymbol{\beta}) > 0$ and $\delta > 1$, for all sufficiently large $k \in \mathbb{N}$ it holds that $\frac{2}{\delta^k} < \Gamma(\boldsymbol{\beta})$. Moreover, for each such $k$ and each $n \in \mathbb{N}$ there exists a rational number $\alpha_n$ in the interval $\big(\frac{1}{\delta^k}, \frac{1}{\delta^k} + \frac{1}{\delta^{k+n}}\big)$. Such $\alpha_n$ has its purely periodic $\boldsymbol{\beta}$-expansion of the form

$$\alpha_n = \frac{1}{\delta^k} + \sum_{j \geq p(k+n)+1} \frac{x_j}{\prod_{i=1}^{j} \beta_i},$$

where $x_j \in D_j = \{0, 1, \ldots, \lfloor \beta_j \rfloor\}$ are digits of the $\boldsymbol{\beta}$-expansion of $\alpha_n$, i.e. they are obtained by the greedy algorithm. We may rewrite the relation as

$$\alpha_n = \frac{1}{\delta^k} + \sum_{j \geq k+n+1} \frac{z_j}{\delta^j}, \tag{6.1}$$

where $z_j \in \mathcal{D} = \{a_1\beta_2 \cdots \beta_p + a_2\beta_3 \cdots \beta_p + \cdots + a_{p-1}\beta_p + a_p \mid a_i \in D_i\}$. We have shown that $\beta_i \in \mathbb{Q}(\delta)$ for all $i$, therefore $\mathcal{D} \subset \mathbb{Q}(\delta)$. We assume that $(x_j)_{j \geq p(k+n)+1}$ is periodic, thus also $(z_j)_{j \geq (k+n)+1}$ is a periodic sequence. Let $\gamma$ denote a conjugate of $\delta$ (other than $\delta$), and let $\sigma$ denote the isomorphism of $\mathbb{Q}(\delta)$ and $\mathbb{Q}(\gamma)$ induced by $\sigma(\delta) = \gamma$. For a contradiction let $|\gamma| > 1$. Since $\alpha_n \in \mathbb{Q}$, we know that $\alpha_n = \sigma(\alpha_n)$, i.e.

$$\frac{1}{\delta^k} + \sum_{j \geq k+n+1} \frac{z_j}{\delta^j} = \sigma\left(\frac{1}{\delta^k} + \sum_{j \geq k+n+1} \frac{z_j}{\delta^j}\right) = \frac{1}{\gamma^k} + \sum_{j \geq k+n+1} \frac{\sigma(z_j)}{\gamma^j}.$$

Note that periodicity of the sequence $(z_j)_{j \geq n+1}$ allowed us to map each member of the sum with $\sigma$ separately. Therefore

$$\left|\frac{1}{\delta^k} - \frac{1}{\gamma^k}\right| = \left|\sum_{j \geq k+n+1} \left(\frac{\sigma(z_j)}{\gamma^j} - \frac{z_j}{\delta^j}\right)\right| \leq \sum_{j \geq k+n+1} \left(\left|\frac{\sigma(z_j)}{\gamma^j}\right| + \left|\frac{z_j}{\delta^j}\right|\right) \leq 2C \sum_{j \geq k+n+1} \frac{1}{\eta^j} = 2C \frac{\eta}{\eta - 1} \cdot \frac{1}{\eta^{k+n+1}},$$

where $\eta := \min\{|\gamma|, |\delta|\} > 1$, $C := \max\{\max_{z \in \mathcal{D}} |\sigma(z)|, \max_{z \in \mathcal{D}} |z|\}$. Since the inequality holds for all $n \in \mathbb{N}$, the right-hand side may be arbitrarily small, which is a contradiction, because $k$ may be chosen so that the left-hand side is non-zero. □

## 6.2 Sufficient conditions

In this section we will in particular focus on the case of an alternate Cantor base with finiteness property. Moreover, we will consider only bases where $\delta = \prod_{i=1}^{p} \beta_i$ is either a Pisot or a Salem number, and it is an algebraic unit. Note that this setting is analogous to the assumptions of Proposition 2.20 for Rényi numeration systems. In the sequel we will follow similar steps as authors of [16] (they focused on the same problem, but for numeration systems with negative bases).

Let $\boldsymbol{\beta} = \overline{(\beta_1, \ldots, \beta_p)}$ be an alternate base and let $\mathcal{B}$ denote its bi-infinite periodic extension. At first, let us give some remarks considering the properties of $x \in (0,1)$ with purely periodic $\boldsymbol{\beta}$-expansion. Consider such $x \in (0,1)$. Then there exist some $j \in \mathbb{N}$ so that its $\boldsymbol{\beta}$-expansion can be written as

$$d_{\boldsymbol{\beta}}(x) = a_1 a_2 \ldots a_{pj} a_1 a_2 \ldots a_{pj} \ldots = (a_1 a_2 \ldots a_{pj})^\omega,$$

i.e. we may stretch the period so that its length is a multiple of $p$. Then the number $x$ itself may be written in the following form

$$x = \left(\frac{a_1}{\beta_1} + \frac{a_2}{\beta_1\beta_2} + \cdots + \frac{a_{pj}}{\delta^j}\right) \cdot \left(1 + \frac{1}{\delta} + \frac{1}{\delta^2} + \cdots\right) = \left(\frac{a_1}{\beta_1} + \frac{a_2}{\beta_1\beta_2} + \cdots + \frac{a_{pj}}{\delta^j}\right) \frac{\delta^j}{\delta^j - 1}. \tag{6.2}$$

We may now multiply each term in parenthesis on the right-hand side by $\delta^j$ and multiply the whole equation by $\delta^j - 1$

$$(\delta^j - 1)x = a_1\delta^{j-1}\beta_2 \cdots \beta_p + a_2\delta^{j-1}\beta_3 \cdots \beta_p + \cdots + a_{pj-1}\beta_p + a_{pj}. \tag{6.3}$$

With this motivation at hand, let us proceed with the first step towards a sufficient condition. Consider $\boldsymbol{\beta}$ a Cantor base, $\mathcal{B}$ a bi-infinite Cantor base, and $\delta>1$ an algebraic unit. We will use the following notation

$$\begin{aligned}
\mathbb{Z}[\boldsymbol{\beta}] &:= \{x \in \mathbb{R} \mid \exists\, \boldsymbol{a} = (a_i)_{i=1}^{+\infty}, a_i \in \mathbb{N}_0 \text{ with finitely many non-zeros so that } \operatorname{val}_{\boldsymbol{\beta}}(\boldsymbol{a}) = x\},\\
\mathbb{Z}[\mathcal{B}] &:= \{x \in \mathbb{R} \mid \exists\, \boldsymbol{b} = (b_i)_{-\infty}^{+\infty}, b_i \in \mathbb{N}_0 \text{ with finitely many non-zeros so that } \operatorname{val}_{\mathcal{B}}(\boldsymbol{b}) = x\},\\
\mathbb{Z}[\delta] &:= \{x \in \mathbb{R} \mid \exists\, \boldsymbol{c} = (c_i)_{-\infty}^{+\infty}, c_i \in \mathbb{N}_0 \text{ with finitely many non-zeros so that } \operatorname{val}_{\delta}(\boldsymbol{c}) = x\},
\end{aligned}$$

where $\operatorname{val}_{\boldsymbol{\beta}}$ for an infinite sequence $\boldsymbol{a} = a_1a_2\dots$ and $\operatorname{val}_{\mathcal{B}}$ for a bi-infinite sequence $\dots b_1b_0 \bullet b_{-1}b_{-2}\dots$ were already defined in Chapter 5, see Notation 5.9. The symbol $\operatorname{val}_\delta$ for a real number $\delta > 1$ denotes standard evaluation of the bi-infinite sequence in a base $\delta$, i.e.

$$\operatorname{val}_\delta(\boldsymbol{c}) = \cdots + c_2\delta^2 + c_1\delta + c_0 + \frac{c_{-1}}{\delta} + \frac{c_{-2}}{\delta^2} + \cdots.$$

Note that for an alternate base $\boldsymbol{\beta} = \overline{(\beta_1, \dots, \beta_p)}$, for its bi-infinite periodic extension $\mathcal{B}$, and for $\delta = \prod_{i=1}^p \beta_i$ it holds that

$$\mathbb{Z}[\boldsymbol{\beta}] \subset \mathbb{Z}[\mathcal{B}] \qquad \text{and} \qquad \mathbb{Z}[\delta] \subset \mathbb{Z}[\mathcal{B}].$$

In analogy to [16], the first property needed in order to formulate a sufficient condition is the following claim.

**Lemma 6.4.** *Let $\boldsymbol{\beta} = \overline{(\beta_1, \dots, \beta_p)}$ be an alternate base and denote $\mathcal{B}$ its bi-infinite periodic extension. Let $\delta = \prod_{i=1}^p$ be an algebraic unit. Then for every $x \in \mathbb{Q}(\delta)$ there exist infinitely many $j \in \mathbb{N}$ such that $(\delta^j - 1)x \in \mathbb{Z}[\mathcal{B}]$.*

*Proof.* It has already been shown in [16], see Lemma 2, that for an algebraic unit $\delta > 1$ and for $x \in \mathbb{Q}(\delta)$ there exist infinitely many $j \in \mathbb{N}$ such that $(\delta^j - 1)x \in \mathbb{Z}[\delta]$. The proof is concluded since $\mathbb{Z}[\delta] \subset \mathbb{Z}[\mathcal{B}]$, as we already mentioned above. $\square$

Recall now that in this section we consider in particular alternate bases with finiteness property. For such bases, it holds true that $\mathbb{Z}[\mathcal{B}] = \mathrm{Fin}(\mathcal{B})$. Therefore, with assumptions as in Lemma 6.4, for the case of an alternate base with Property $(F)$, each $(\delta^j - 1)x \in \mathbb{Z}[\mathcal{B}] = \mathrm{Fin}(\mathcal{B})$ have the greedy expansion in $\mathcal{B}$ of the form

$$\langle(\delta^j - 1)x\rangle_{\mathcal{B}} = x_n x_{n-1} \dots x_0 \bullet x_{-1} \dots x_{-m+1} x_{-m}$$

for some $m, n \in \mathbb{N}$. To proceed further, we need to comment on the size of minimal possible such indices $m$ and $n$.

Firstly, motivated by the above analysis of numbers with purely periodic $\boldsymbol{\beta}$-expansions, see Equation (6.3), we would like to have $x_{-1} \dots x_{-m+1} x_{-m} = 0^m$. Similar question for the Rényi numeration systems with base $\beta > 1$ was investigated in [23]. Authors provided a condition ensuring that the right-hand side of the considered $\beta$-expansions is $0^\omega$. The condition was formulated in terms of images of conjugates of $\beta$ in non-identical isomorphisms of $\mathbb{Q}(\beta)$. However, to generalise this result for the case of an alternate base seems to be very technical, we left this question open. From now on we will comment on the case when the condition $x_{-1} \dots x_{-m+1} x_{-m} = 0^m$ holds true.

Let us analyse the size of the index $n$. Consider $j$ fixed. Since we are working with the greedy expansions in the base with positive elements, it is clear that the index $n$ can be controlled just by the size of the number $x$ in expression $(\delta^j - 1)x$. To be precise, it has to be $\prod_{i=1}^{n+1} \beta_i < (\delta^j - 1)x$. Again, motivated by Equation (6.3), we firstly need to ensure that $n + 1 \leq pj$. Then $(\delta^j - 1)x$ may be represented in the form of a sum similar as in (6.3), and thus $x$ may be written in the form as in (6.2)

$$x = \left(\frac{0}{\beta_1} + \dots + \frac{0}{\beta_1 \dots \beta_{pj-n-1}} + \frac{x_n}{\beta_1 \dots \beta_{pj-n}} + \dots + \frac{x_0}{\delta^j}\right) \cdot \left(1 + \frac{1}{\delta} + \frac{1}{\delta^2} + \cdots\right).$$

Consequently, there exists a $\boldsymbol{\beta}$-representation of $x$ of the form $(0^{nj-p} x_n \dots x_0)^\omega$. However, it does not yet need to be the $\boldsymbol{\beta}$-expansion of $x$. In order to ensure this property, the index $n$ has to be small enough, so that the string $(0^{nj-p} x_n \dots x_0)^\omega$ is admissible in base $\boldsymbol{\beta}$. This can be achieved by controlling the size of $x$ (more zeros will be padded at the beginning of the sequence for smaller $x$), i.e. the proper choice of $\Gamma(\mathcal{B})$ will be needed.

## 6.3 Numerical experiments

In this section we present several numerical experiments considering the value of $\Gamma(\boldsymbol{\beta})$ for alternate Cantor bases. We hope that these experiments will greatly help with building an intuition much needed in order to state new conjectures, and hopefully prove theorems in the future research.

All calculations were done in Julia programming language with help of Nemo.jl package. During the experiments multiple sets of fractions with values in $(0,1)$ (details differ in each example, see below) were generated and their $\boldsymbol{\beta}$-expansions up to the certain number of digits were calculated using the greedy algorithm. Then the length of preperiod and period were determined. Note that if we want the greedy algorithm to function properly, one should choose a symbolic representations of numbers via their minimal polynomials, instead of numerical representations with fixed precision. Moreover, the periodicity of expansions is verified using comparison of certain reminders generated by the greedy algorithm, as we explained in Remark 3.2, therefore in this step it is indeed necessary to represent numbers symbolically.

Recall now one of previous results considering alternate Cantor bases – a generalisation of Schmidt's theorem 2.9, as it was proven independently by authors of [7] and by us in [21]. In order to state the result, we will need the following notation

$$\operatorname{Per}\boldsymbol{\beta} := \{x \in [0,1) \mid d_{\boldsymbol{\beta}}(x) \text{ is eventually periodic}\}.$$

**Theorem 6.5** (Schmidt for an alternate base)**.** *Let $\boldsymbol{\beta} = \overline{(\beta_1, \dots, \beta_p)}$ be an alternate base. Denote $\delta = \prod_{j=1}^{p} \beta_j$.*

1. *If $\mathbb{Q} \cap [0,1) \subset \operatorname{Per}\boldsymbol{\beta}^{(j)}$ for all $j \in \mathbb{N}_0$, then $\delta$ is either a Pisot or a Salem number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \mathbb{N}$.*
2. *If $\delta$ is a Pisot number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \mathbb{N}$, then $\mathbb{Q}(\delta) \cap [0,1) = \operatorname{Per}\boldsymbol{\beta}^{(j)}$ for all $j \in \mathbb{N}_0$.*

In majority of examples below the base will be chosen so that $\delta$ is a Pisot number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \mathbb{N}$. Therefore, we will always be able to decide whether fractions in the tested sets indeed have purely periodic expansion or not just by calculating sufficiently many digits and running an algorithm for a periodicity check. Note that in the case when $\delta$ is a Salem number, this method does not need to provide a desired answer considering periodicity.

Let us now present the results of our numerical experiments. We will focus on bases that we have already encountered in other context, see Appendices A and B. One example will also comment on the favourite base of authors of [6]. The whole generated data set is rather large (it contains all generated fractions along with length of their preperiods and periods), thus we chose to present here just the qualitative results, which we consider crucial for the further research. The script used to generate data and the final data set is available upon request.

In the examples below we will denote $\gamma_m$ the positive root of the polynomial $x^2 - mx - 1$ for $m \in \mathbb{N}$, i.e. $\gamma_i$ is a quadratic Pisot unit without positive conjugate. Note that for $m = 1$ the number $\gamma_1 = \tau$ is a Golden ratio. In all examples we denote $\delta = \prod_{i=1}^{p} \beta_i$ and $\Gamma_{est}(\boldsymbol{\beta})$ will denote our estimate of $\Gamma$ for the given base $\boldsymbol{\beta}$ based on the results of numerical experiments. We round this estimate usually to 2 or 3 decimal places.

**Example 6.6.** Denote $\boldsymbol{\alpha}_1 = \overline{(\gamma_1, \gamma_1^2)} = \overline{(\tau, \tau^2)}$. We generated three sets of fractions

$$S_1 = \left\{ \frac{1}{n} \;\middle|\; n = 2, 3, \ldots, 100 \right\},$$
$$S_2 = \left\{ \frac{n}{100} \;\middle|\; n = 1, 2, \ldots, 99 \right\},$$
$$S_3 = \left\{ \frac{n+1}{2n} \;\middle|\; n = 2, 3, \ldots, 100 \right\}.$$

For the set $S_1$ all fractions have purely periodic expansions. Note that all these numbers have value at most $\frac{1}{2}$. However, for the set $S_3$ where fractions are greater than $\frac{1}{2}$, all numbers in this set have just eventually, but not purely periodic expansions. Finally, for the set $S_2$ also all fractions with value $\leq \frac{1}{2}$ are purely periodic, but the fractions which are strictly greater than $\frac{1}{2}$ do not have purely periodic expansions. Note that in general for $\Gamma > 0$ it does not need to hold true that all fractions greater than the given bound are not purely periodic. It may be that above the value of $\Gamma$ some fractions have purely periodic expansions, and some do not. We will see such (possible) examples below. In conclusion, for the base $\boldsymbol{\alpha}_1$ we estimate $\Gamma_{est}(\boldsymbol{\alpha}_1) \approx 0.5$.

**Example 6.7.** Denote $\boldsymbol{\alpha}_j = \overline{(\gamma_j, \gamma_j^2)}$ for $j \in \{2, 3, 4, 5\}$. We at first tested sets $S_1$ and $S_2$ as in Example 6.6. For each $j \in \{2, 3, 4, 5\}$ we found multiple fractions in $S_1$ such that they do not have purely periodic expansion, i.e. it is clear, that $\Gamma$ has to be smaller than $\frac{1}{2}$ for all $j$. In order to further specify the value of $\Gamma(\boldsymbol{\alpha}_j)$ for each $j \in \{2, 3, 4, 5\}$, we generated the sets

$$S_4^j = \left\{ m_j \cdot \frac{n}{n+1} \;\middle|\; n = 2, 3, \ldots, 100 \right\},$$

where $m_j$ was chosen as the smallest fraction from $S_2$ such that its $\boldsymbol{\alpha}_j$-expansion is not purely periodic. The values of $m_j$ were thus chosen as follows

$$m_2 = \frac{23}{100} \qquad m_3 = \frac{3}{25} \qquad m_4 = \frac{7}{100} \qquad m_5 = \frac{7}{100}.$$

For the index $j = 5$ we generated and tested one more set of fractions

$$S_4 = \left\{ \frac{9n}{200(n+1)} \;\middle|\; n = 2, 3, \ldots, 100 \right\},$$

because we wanted to sample the interval close to possible $\Gamma(\boldsymbol{\alpha}_5)$ more densely. Our estimates of $\Gamma$, after calculating the length of preperiods of all these sets of fractions, may be chosen as the value of the smallest fraction out of all tested fractions for the given base, which does not have a purely periodic expansion. The resulting estimates are

$$\Gamma_{est}(\boldsymbol{\alpha}_2) \approx 0.21 \qquad \Gamma_{est}(\boldsymbol{\alpha}_3) \approx 0.11 \qquad \Gamma_{est}(\boldsymbol{\alpha}_4) \approx 0.066 \qquad \Gamma_{est}(\boldsymbol{\alpha}_5) \approx 0.042.$$

**Example 6.8.** Consider shifts of bases $\boldsymbol{\alpha}_j$ from the above examples, i.e. $\boldsymbol{\alpha}_j^{(1)} = \overline{(\gamma_j^2, \gamma_j)}$ for $j \in \{1,2,3,4,5\}$. We tested sets $S_1$ and $S_2$ as defined in Example 6.6, and, moreover, the set

$$S_6 = \left\{ \frac{n}{n+1} \;\middle|\; n = 2, 3, \ldots, 100 \right\}.$$

All tested fractions have a purely periodic expansion, thus we estimate

$$\Gamma_{est}\big(\boldsymbol{\alpha}_j^{(1)}\big) \approx 1 \qquad\qquad \text{for all } j \in \{1,2,3,4,5\}.$$

For bases in Examples 6.6, 6.7 and 6.8 we know that Property $(F)$ holds true, see Remark 6 in Appendix B. Based on the results known about Rényi expansions (i.e. alternate bases with period $p = 1$), this property might be connected with the existence of $\Gamma(\boldsymbol{\beta}) > 0$ for a given Cantor base $\boldsymbol{\beta}$. We have already seen several examples of bases with Property $(F)$, let us therefore continue with examples of bases, which either do not have finiteness property, or we cannot decide about their finiteness property based on our sufficient an necessary conditions proven in Chapter 5.

**Example 6.9.** Denote $\boldsymbol{\psi} = \overline{(\tau, \tau^2, \tau^3)}$. In this case we tested sets $S_1$ and $S_2$ for all three shifts of this base. Expansions of fractions in the set $S_1$ were all purely periodic in all three shifts of the given base. However, when testing the set $S_2$, we found some fractions which do not have purely periodic expansions (such fractions existed for each of three shifts of base $\boldsymbol{\psi}$). Thus again, we may estimate $\Gamma$ for each shift as the value of the smallest fraction in $S_2$ such that its expansion in the given base is not purely periodic. Resulting estimates are

$$\Gamma_{est}(\boldsymbol{\psi}) \approx 0.39 \qquad \Gamma_{est}\big(\boldsymbol{\psi}^{(1)}\big) \approx 0.70 \qquad \Gamma_{est}\big(\boldsymbol{\psi}^{(2)}\big) \approx 0.52.$$

Note that in all three shifts of the given base we found fractions with purely periodic expansions even above the estimated bound $\Gamma_{est}$. In our opinion, this property might be quite common and is not surprising. As we already mentioned, for this base it is not clear if the finiteness property holds true or not.

**Example 6.10.** Let $\boldsymbol{\mu} = \overline{(\tau, \tau^3, \tau^2)}$. This base is also an example of an alternate base about which we cannot decide if it has finiteness property with help of our sufficient and necessary conditions. Let us now explore properties of this base and its shift in a similar way as in the above examples. Firstly, we tested fractions in the set $S_2$ for all shifts. For $\boldsymbol{\mu}^{(1)}$ all these tested numbers had purely periodic expansions, we did not test this shift for any other fractions. In the case of the second shift and of the base itself, the set $S_2$ contained multiple fractions with expansions that were eventually, but not purely periodic. In the base $\boldsymbol{\mu}$ the minimal such fraction was $k_0 = \frac{1}{10}$. For the second shift of the base, the smallest fraction in $S_2$ with non-purely periodic expansion was $k_2 = \frac{7}{50}$. We tested few more terms of sequences convergent to values $k_0$ and $k_2$ from below. To be precise, we tested fractions of the form $k_i \cdot \frac{n}{n+1}$ for $n$ integer bigger than 2, until we found a fraction which does not have a purely periodic expansion. This is the fraction that might serve as our final estimate of the value of $\Gamma$. Therefore, the resulting values for $\boldsymbol{\mu}$ and its shifts are

$$\Gamma_{est}(\boldsymbol{\mu}) \approx 0.05 \qquad \Gamma_{est}\big(\boldsymbol{\mu}^{(1)}\big) \approx 1 \qquad \Gamma_{est}\big(\boldsymbol{\mu}^{(2)}\big) \approx 0.13.$$

**Remark 6.11.** Note that even if we consider an alternate base satisfying the second part of Schmidt's theorem 6.5, i.e. we know that all fractions in $(0,1)$ have at least eventually periodic expansions, their periods and preperiods might be considerably long and thus it may be needed to let the greedy algorithm and the algorithm for periodicity check run for even more than few thousand cycles. In the above example 6.10, the further refinement of the value $\Gamma\big(\boldsymbol{\mu}^{(2)}\big)$ could be done for instance by calculating preperiods and periods of fractions in the set

$$S_7 = \left\{ \frac{93n}{700(n+1)} \;\middle|\; n = 2, 3, \ldots, 100 \right\}.$$

However, when calculating preperiods and periods on first 5000 digits, for multiple fractions in this set we were not able to decide if their expansions are even purely periodic or not, therefore we decided to not continue with estimating the value of desired $\Gamma$ even further.

**Example 6.12.** Let us now examine an example of an alternate base without finiteness property. Let $\boldsymbol{\xi} = \overline{(\tau^2, \tau^4)}$. This base does not fulfil necessary conditions of finiteness property as stated in Theorem 5.8, namely the last condition considering images of $\beta_i$ of non-identical isomorphisms of $\mathbb{Q}(\delta)$. Moreover, neither $d_{\boldsymbol{\beta}}(1)$, nor $d_{\boldsymbol{\beta}^{(1)}}(1)$ are finite. In the numerical experiments, we tested fractions from the set

$$S_8 = \left\{ \frac{1}{n} \;\middle|\; n = 2, 3, \ldots, 500 \right\},$$

i.e. the same form of fractions as we usually test in the set $S_1$, but for more vales of $n$. We found out, that none of these fractions has a purely periodic expansion, neither in the base $\boldsymbol{\xi}$, nor in its shift $\boldsymbol{\xi}^{(1)}$. Therefore, we estimate

$$\Gamma_{est}(\boldsymbol{\xi}) \approx 0 \qquad\qquad \Gamma_{est}\big(\boldsymbol{\xi}^{(1)}\big) \approx 0.$$

**Example 6.13.** Let us examine one last example of the base consisting of powers of a Golden ratio. Consider $\boldsymbol{\eta} = \overline{(\tau, \tau^2, \tau^4)}$. We tested, as usually, the sets $S_1$ and $S_2$. The estimated values of $\Gamma$ for the given base and its shifts might be chosen as the smallest fraction from $S_1 \cup S_2$ such that its expansion is eventually, but not purely periodic. The results are as follows

$$\Gamma_{est}(\boldsymbol{\eta}) \approx 0.09 \qquad \Gamma_{est}\big(\boldsymbol{\eta}^{(1)}\big) \approx 0.37 \qquad \Gamma_{est}\big(\boldsymbol{\eta}^{(2)}\big) \approx 0.87.$$

**Example 6.14.** Consider the favourite base of authors of [6], i.e. let $\boldsymbol{\chi} = \overline{\left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)}$. In this case $\delta = \frac{1+\sqrt{13}}{2} \cdot \frac{5+\sqrt{13}}{6} = \frac{3+\sqrt{13}}{2}$ is a Pisot unit with minimal polynomial $x^2 - 3x - 1$. The expansions of 1 are of the form $d_{\boldsymbol{\chi}}(1) = 201$ and $d_{\boldsymbol{\chi}^{(1)}}(1) = 11$. Note that these expansions satisfy sufficient condition for finiteness property as stated in Corollary 5.27. Again, we tested the sets of fractions $S_1$ and $S_2$ for both shifts, as in all the above examples. In the base itself we found multiple fractions with non-purely periodic expansions, and the smallest fraction from $S_1 \cup S_2$ with just eventually, but not purely periodic $\boldsymbol{\chi}$-expansion was $\frac{5}{12}$. Therefore, in order to further specify the value of $\Gamma(\boldsymbol{\chi})$ we tested fractions of the form $\frac{5}{12} \cdot \frac{n}{n+1}$ for $n$ integers greater than 2. The smallest fraction with non-purely periodic expansion we found this way was $\frac{37}{90} \approx 0.41$, thus we set our estimate as $\Gamma_{est}(\boldsymbol{\chi}) \approx 0.41$. In the shifted base all fractions in $S_1 \cup S_2$ were purely periodic. Moreover, we tested the set of fractions of the form $\frac{n}{n+1}$ for $n \in \{2, 3, \ldots, 100\}$, i.e. the fractions in $S_6$. In this case also all tested numbers had purely periodic $\boldsymbol{\chi}^{(1)}$-expansions. In conclusion, our estimates are

$$\Gamma_{est}(\boldsymbol{\chi}) \approx 0.41 \qquad \qquad \Gamma_{est}\big(\boldsymbol{\chi}^{(1)}\big) \approx 1.$$

**Example 6.15.** Consider $\delta$ the positive root of $x^3 - 31x^2 - 12x - 3$ and $\beta_1 := \frac{5\delta+3}{\delta-4}$, $\beta_2 := \frac{5\delta+1}{\delta-2}$. Then $\beta_1$ is the positive root of the polynomial $21x^3 - 103x^2 - 108x - 39$ and $\beta_2$ is the positive root of $13x^3 - 71x^2 + 10x - 21$, i.e. neither $\beta_1$ nor $\beta_2$ are algebraic integers. According to Proposition 12 in Appendix A, we indeed have $\delta = \beta_1\beta_2$. Denote $\boldsymbol{\theta} = \overline{(\beta_1, \beta_2)}$. In this base the expansions of 1 are $d_{\boldsymbol{\theta}}(1) = 543$ and $d_{\boldsymbol{\theta}^{(1)}}(1) = 521$. In this case a sufficient condition for finiteness property Corollary 5.27 does not hold true, therefore we do not know if this system has $(F)$. Note that we also analyse this example (transcriptions of the set of minimal forbidden strings and possible weights of the weighted sum needed to Theorem 5.19) in more detail in Example 7, Case 2, Appendix B. In this base the analysis of expansions of various sets of fractions analogous to the previous examples was not possible due to computational limitations – it is much slower to run the greedy algorithm and the algorithm for periodicity check on the bases where $\delta$ is not an algebraic unit. Therefore we tested just several of the fractions of the form $\frac{1}{n}$ and tried to find some small fractions with non-purely periodical expansions. In the base $\boldsymbol{\theta}$ we found that for example $\frac{1}{207} \approx 0.0048$ has a preperiod of the length 10, in the shifted base $\boldsymbol{\theta}^{(1)}$ the fraction $\frac{1}{243} \approx 0.0041$ has a preperiod 21 digits long. These fractions are rather small and with quite long preperiods, therefore we estimate

$$\Gamma_{est}(\boldsymbol{\theta}) \approx 0 \qquad \qquad \Gamma_{est}\big(\boldsymbol{\theta}^{(1)}\big) \approx 0.$$

**Example 6.16.** Lastly, let us comment on the case when $\delta$ is a Salem number. Note that in the case of generalised Cantor bases the possibility that $\Gamma$ might be non-zero even though $\delta$ is a Salem number had not yet been disproven. Consider $\beta$ the positive root of $x^4 - x^3 - 3x^2 - x + 1$. Let $\boldsymbol{\beta} = \overline{\left(\frac{\beta}{2}, 2\right)}$. Due to the limitations mentioned in connection with generalised Schmidt's theorem above, it is needed to choose a fixed number of digits that we compute and on which we check the periodicity, because it might be the case that $\boldsymbol{\beta}$-expansions of some fractions are not even periodic. Therefore we calculated just first 1500 digits. The smallest fraction with eventually, but not purely periodic $\boldsymbol{\beta}$-expansion that we found, is $\frac{1}{250} \approx 0.004$ with preperiod 735 digits long. Similarly, for the shifted base $\boldsymbol{\beta}^{(1)}$ the smallest such fraction that we found is $\frac{1}{300} \approx 0.003$ with preperiod of the length 505 digits. Therefore we conclude

$$\Gamma_{est}(\boldsymbol{\beta}) \approx 0 \qquad\qquad \Gamma_{est}\big(\boldsymbol{\beta}^{(1)}\big) \approx 0.$$

# Conclusion

Let us now recapitulate our results. Firstly, we recalled several definitions and well-known theorems considering combinatorics on words, standard numeration systems, matrix theory and algebraic numbers. In the second chapter we focused solely on Rényi numeration systems. We presented essential definitions considering these systems, as well as results connected with finiteness property and periodicity.

Chapter 3 was devoted to generalised Cantor base systems. We recapitulated several already known results, as well as provided our own definition of bi-infinite Cantor real base. Another result of our research was presented in Chapter 4, namely a proposition considering existence and uniqueness of a suitable alternate base with period $p = 2$ much needed to further generalise the well-known result of Parry (Theorem 2.7).

The goal of Chapter 5 was to describe the arithmetics, mainly addition and subtraction, in alternate Cantor bases and their bi-infinite extensions. We formulated and proved necessary and sufficient conditions for positive finiteness and finiteness property. The proof of the sufficient condition may be considered constructive. Consequently, we provided a class of alternate bases with period $p = 2$ and $p = 3$ satisfying (positive) finiteness property. All notions were also described in detail in numerous examples in Appendices A and B.

The research project was concluded by investigating the properties of purely periodic $\boldsymbol{\beta}$-expansions. We considered in particular alternate Cantor bases $\boldsymbol{\beta} = \overline{(\beta_1, \ldots, \beta_p)}$. We show that if there exists $\Gamma(\boldsymbol{\beta}) \in (0, 1]$ such that all rational numbers in $[0, \Gamma(\boldsymbol{\beta}))$ have purely periodic $\boldsymbol{\beta}$-expansion, then $\delta := \prod_{i=1}^{p} \beta_i$ is either a Pisot or a Salem number. Next, in analogy to known results for Rényi numeration systems, we focused on a sufficient condition for $\Gamma(\boldsymbol{\beta}) > 0$ for the case when $\delta$ is a Pisot unit and the system has $(F)$. We provided an analysis of steps needed in order to formulate such proposition. We concluded this chapter by several numerical experiments considering expansions of fractions in multiple chosen bases.

We hope that both Appendices A and B provided a greater insight into the examined topics even for readers who considered some proofs exceedingly technical. We also hope that numerical results presented at the and of Chapter 6 will serve to build an intuition much needed in order to formulate new research ideas considering purely periodic expansions in generalised Cantor base systems.

To conclude, generalised Cantor base systems are a still very new and lively discussed area of research. The present work is a step towards solutions of many open problems in this field.

# Appendix A

## Expansions in bases consisting of powers of a quadratic Pisot unit

We now recall our earlier results concerning expansions in bases consisting of powers of a quadratic Pisot unit. All proofs may be found in [21].

Let $\gamma$ be the larger root of $x^2 - mx - 1$ where $m \in \mathbb{N}$. We have calculated the greedy expansions of 1 in bases $\boldsymbol{\beta} := \overline{(\gamma, \gamma^2)}$, $\boldsymbol{\gamma} := \overline{(\gamma, \gamma^2, \gamma^3)}$, $\boldsymbol{\delta} := \overline{(\gamma, \gamma^3, \gamma^2)}$ and in all their shifts.

**Proposition 1.** Let $m \in \mathbb{N}$ and let $\gamma$ be the larger root of $x^2 - mx - 1$. Let $\boldsymbol{\beta} := \overline{(\gamma, \gamma^2)}$. Then

$$\begin{aligned} d_{\boldsymbol{\beta}}(1) &= m\ m\ 1 \\ d_{\boldsymbol{\beta}^{(1)}}(1) &= (m^2+1)\ m. \end{aligned}$$

**Proposition 2.** Let $m \in \mathbb{N}$ and let $\gamma$ be the larger root of $x^2 - mx - 1$. Let $\boldsymbol{\gamma} := \overline{(\gamma, \gamma^2, \gamma^3)}$. Then

$$\begin{aligned} d_{\boldsymbol{\gamma}}(1) &= m\ m\ (m^2+1)\ m \\ d_{\boldsymbol{\gamma}^{(1)}}(1) &= \begin{cases} 2\ 2\ 1 \quad \text{for} \quad m = 1 \\ (m^2+1)\ (m^3+2m-1)\ (m-1)\ (m+1)\ (m^2+1)\ m \quad \text{for} \quad m \geq 2 \end{cases} \\ d_{\boldsymbol{\gamma}^{(2)}}(1) &= (m^3+3m)\ 0\ 1. \end{aligned}$$

**Proposition 3.** Let $m \in \mathbb{N}$ and let $\gamma$ be the larger root of $x^2 - mx - 1$. Let $\boldsymbol{\delta} := \overline{(\gamma, \gamma^3, \gamma^2)}$. Then

$$\begin{aligned} d_{\boldsymbol{\delta}}(1) &= m\ (m^2+1)\ m^2\ m \\ d_{\boldsymbol{\delta}^{(1)}}(1) &= (m^3+3m)\ 0\ 1 \\ d_{\boldsymbol{\delta}^{(2)}}(1) &= (m^2+1)\ m. \end{aligned}$$

Note that all calculated expansions of 1 were finite, which is an interesting fact – if we would for example define $\gamma$ as the larger root of the polynomial $x^2 - mx + 1$ where $m \geq 3$, some expansions of 1 in bases consisting of powers of $\gamma$ as above would be eventually periodic, but not finite.

With the above propositions at hand, the infinite Rényi expansions of 1 in studied bases could be calculated similarly as it was shown in Example 3.4. We would obtain the following results.

**Proposition 4.** Let $m \in \mathbb{N}$ and let $\gamma$ be the larger root of $x^2 - mx - 1$. Let $\boldsymbol{\beta} := \overline{(\gamma, \gamma^2)}$. Then

$$
\begin{aligned}
d^*_{\boldsymbol{\beta}}(1) &= [m \ m \ 0 \ (m^2+1) \ (m-1)]^\omega \\
d^*_{\boldsymbol{\beta}^{(1)}}(1) &= [(m^2+1) \ (m-1) \ m \ m \ 0]^\omega.
\end{aligned}
$$

**Proposition 5.** Let $m \in \mathbb{N}$ and let $\gamma$ be the larger root of $x^2 - mx - 1$. Let $\boldsymbol{\gamma} := \overline{(\gamma, \gamma^2, \gamma^3)}$. Then for $m = 1$

$$
\begin{aligned}
d^*_{\boldsymbol{\gamma}}(1) &= [1 \ 1 \ 2 \ 0 \ 2 \ 2 \ 0]^\omega \\
d^*_{\boldsymbol{\gamma}^{(1)}}(1) &= [2 \ 2 \ 0 \ 1 \ 1 \ 2 \ 0]^\omega \\
d^*_{\boldsymbol{\gamma}^{(2)}}(1) &= 4 \ 0 \ 0 \ [1 \ 1 \ 2 \ 0 \ 2 \ 2 \ 0]^\omega
\end{aligned}
$$

and for $m \geq 2$

$$
\begin{aligned}
d^*_{\boldsymbol{\beta}}(1) =& [m \ m \ (m^2+1) \ (m-1) \ (m^2+1) \ (m^3+2m-1) \ (m-1) \ (m+1) \ (m^2+1) \ (m-1)]^\omega \\
d^*_{\boldsymbol{\beta}^{(1)}}(1) =& [(m^2+1) \ (m^3+2m-1) \ (m-1) \ (m+1) \ (m^2+1) \ (m-1) \ m \ m \ (m^2+1) \ (m-1)]^\omega \\
d^*_{\boldsymbol{\beta}^{(2)}}(1) =& (m^3+3m) \ 0 \ 0 \ [m \ m \ (m^2+1) \ (m-1) \ (m^2+1) \ (m^3+2m-1) \ (m-1) \\
& (m+1) \ (m^2+1) \ (m-1)]^\omega.
\end{aligned}
$$

**Proposition 6.** Let $m \in \mathbb{N}$ and let $\gamma$ be the larger root of $x^2 - mx - 1$. Let $\boldsymbol{\delta} := \overline{(\gamma, \gamma^3, \gamma^2)}$. Then

$$
\begin{aligned}
d^*_{\boldsymbol{\delta}}(1) &= [m \ (m^2+1) \ m^2 \ (m-1) \ (m^3+3m) \ 0 \ 0]^\omega \\
d^*_{\boldsymbol{\delta}^{(1)}}(1) &= [(m^3+3m) \ 0 \ 0 \ m \ (m^2+1) \ m^2 \ (m-1)]^\omega \\
d^*_{\boldsymbol{\delta}^{(2)}}(1) &= [(m^2+1) \ (m-1)]^\omega.
\end{aligned}
$$

## Bases with short expansions of 1

Another interesting results of our research project [21] are the properties of bases with period 2 and short expansions of 1. We now recall some of our findings.

Let us have two sequences $a_1a_2a_3\ldots$, $b_1b_2b_3\ldots$ where the digits are in $\mathbb{N}_0$, both sequences lexicographically larger than $10^\omega$, and for all $i \geq 1$ the following holds

$$\begin{aligned} a_{2i+1}a_{2i+2}\ldots &\prec a_1a_2\ldots \\ b_{2i}b_{2i+1}\ldots &\prec a_1a_2\ldots \\ a_{2i}a_{2i+1}\ldots &\prec b_1b_2\ldots \\ b_{2i+1}b_{2i+2}\ldots &\prec b_1b_2\ldots. \end{aligned} \tag{7.1}$$

We are looking for $\beta_1 > 1$ and $\beta_2 > 1$ such that

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1a_2\ldots \\ d_{(\beta_2,\beta_1)}(1) &= b_1b_2\ldots. \end{aligned}$$

Note that inequalities (7.1) are necessary for existence of such $\beta_1$ and $\beta_2$. We now present the solution of a problem of finding suitable base for some cases of finite sequences up to three digits long.

**Proposition 7.** Let $a_1 > 1, b_1 > 1$ be integers. Then there exist unique $\beta_1, \beta_2 > 1$ such that

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1 \\ d_{(\beta_2,\beta_1)}(1) &= b_1. \end{aligned}$$

In particular, $\beta_1 = a_1$, $\beta_2 = b_1$ are integers.

**Proposition 8.** Let $a_1, a_2, b_1 \in \mathbb{N}$ and $a_2 \neq 0$, $b_1 > 1$ be such that $a_2 < b_1$. Then there exist unique $\beta_1, \beta_2 > 1$ such that

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1a_2 \\ d_{(\beta_2,\beta_1)}(1) &= b_1. \end{aligned}$$

In particular, $\beta_1 = a_1 + \frac{a_2}{b_1}$, $\beta_2 = b_1$ are rational numbers.

**Proposition 9.** Let $a_1, a_2, b_1, b_2 \in \mathbb{N}$ satisfy $a_2 \leq b_1$, $b_2 \leq a_1$. Then there exist unique $\beta_1, \beta_2 > 1$ such that

$$d_{(\beta_1,\beta_2)}(1) = a_1 a_2$$
$$d_{(\beta_2,\beta_1)}(1) = b_1 b_2.$$

In particular, if $\delta$ denotes the larger root of $x^2 - x(a_2 + b_2 + a_1 b_1) + a_2 b_2$, then

$$\beta_1 = \frac{\delta - b_2}{b_1} = \frac{a_1 \delta}{\delta - a_2} \qquad \beta_2 = \frac{\delta - a_2}{a_1} = \frac{b_1 \delta}{\delta - b_2}.$$

Moreover, $\delta$ is a quadratic Pisot number.

**Proposition 10.** Let $a_1, b_1, a_3 \in \mathbb{N}$, $a_2 \in \mathbb{N}_0$, $b_1 > 1$ satisfy $a_3 \leq a_1$, $a_2 < b_1$. Then there exist unique $\beta_1, \beta_2 > 1$ such that

$$d_{(\beta_1,\beta_2)}(1) = a_1 a_2 a_3$$
$$d_{(\beta_2,\beta_1)}(1) = b_1.$$

In particular, if $\delta$ denotes the larger root of $x^2 - x(a_1 b_1 + a_2) - a_3 b_1$, then

$$\beta_1 = \frac{\delta a_1 + a_3}{\delta - a_2} \qquad \beta_2 = b_1.$$

Moreover, $\delta$ is a quadratic Pisot number.

**Proposition 11.** Let $a_1, b_1, a_3, b_2 \in \mathbb{N}$, $a_2 \in \mathbb{N}_0$ satisfy

$$a_3 \leq a_1, \qquad b_2 \leq a_1, \qquad a_2 < b_1 \quad \text{or} \quad (a_2 = b_1 \text{ and } a_3 < b_2).$$

Then there exist unique $\beta_1, \beta_2 > 1$ such that

$$d_{(\beta_1,\beta_2)}(1) = a_1 a_2 a_3$$
$$d_{(\beta_2,\beta_1)}(1) = b_1 b_2.$$

In particular, if $\delta$ denotes the larger root of $x^2 - x(a_1 b_1 + a_2 + b_2) - (a_3 b_1 - a_2 b_2)$, then

$$\beta_1 = \frac{\delta a_1 + a_3}{\delta - a_2} \qquad \beta_2 = \frac{\delta b_1}{\delta - b_2}.$$

Moreover, if $a_3 b_1 \neq a_2 b_2$, then $\delta$ is a quadratic Pisot number. Otherwise, $\delta = a_1 b_1 + a_2 + b_2$ is an integer.

Lastly, recall the result for sequences both three digits long.

**Proposition 12.** Let us have $a_1, a_3, b_1, b_3 \in \mathbb{N}$, $a_2, b_2 \in \mathbb{N}_0$ such that

$$\begin{aligned} a_3 &\le a_1, &\qquad b_2 < a_1 \text{ or } (b_2 = a_1 \text{ and } b_3 \le a_2),\\ b_3 &\le b_1, &\qquad a_2 < b_1 \text{ or } (a_2 = b_1 \text{ and } a_3 \le b_2). \end{aligned}$$

Then there exist unique $\beta_1, \beta_2 > 1$ such that

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1a_2a_3\\ d_{(\beta_2,\beta_1)}(1) &= b_1b_2b_3. \end{aligned} \tag{7.2}$$

In particular, if we denote $\delta$ the root of polynomial $x^3-x^2(a_1b_1+a_2+b_2)-x(a_3b_1-a_2b_2+a_1b_3)-a_3b_3$ greater than 1, then

$$\beta_1 = \frac{\delta a_1 + a_3}{\delta - a_2} \qquad \beta_2 = \frac{\delta b_1 + b_3}{\delta - b_2}.$$

Moreover, $\delta$ is a cubic Pisot number.

## Algebraic properties of alternate bases for $p = 2$

Also other bases with finite or at least periodic expansions of 1 have interesting algebraic properties. We have already discussed them earlier in our research project [21] and we present them here without proofs. Namely, we have show that $\delta = \beta_1\beta_2$ is an algebraic integer if the given sequences $a_1b_1 \ldots$, $b_1b_2 \ldots$ are finite or at least eventually periodic. For the sake of simplicity, but without loss of generality, we consider finite sequences of even length, or in the case of eventually periodic sequences we consider even length of both preperiod and period.

**Proposition 13.** Let $\beta_1 > 1$ and $\beta_2 > 1$ be such that

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1a_2 \ldots a_{2k}\\ d_{(\beta_2,\beta_1)}(1) &= b_1b_2 \ldots b_{2m} \end{aligned}$$

for some $k, m \in \mathbb{N}$. Then $\delta := \beta_1\beta_2$ is an algebraic integer of degree less than or equal to $k + m$.

**Proposition 14.** Let $\beta_1 > 1$ and $\beta_2 > 1$ be such that

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1a_2 \ldots a_{2k} \left(a_{2k+1}a_{2k+2} \ldots a_{2(k+l)}\right)^\omega\\ d_{(\beta_2,\beta_1)}(1) &= b_1b_2 \ldots b_{2m} \left(b_{2m+1}b_{2m+2} \ldots b_{2(m+n)}\right)^\omega \end{aligned}$$

for some $k, l, m, n \in \mathbb{N}$. Then $\delta := \beta_1\beta_2$ is an algebraic integer of degree less than or equal to $(k+l)+(m+n)$.

It follows from the proof of generalisation of Schmidt's theorem that we presented in [21], that this property holds also for an alternate Cantor base with period $p \in \mathbb{N}$.

**Proposition 15.** Let $\boldsymbol{\beta} = \overline{(\beta_1, \ldots, \beta_p)}$. Let $d_{\boldsymbol{\beta}^{(i)}}(1)$ be eventually periodic for all $i \in \{0, \ldots, p-1\}$. Then $\delta := \prod_{i=1}^{p} \beta_i$ is an algebraic integer.

# Appendix B

In this section we describe arithmetic properties of several classes of alternate bases with $p = 2$ having short expansions of 1. Our aim is to illustrate on these simple examples all the results of Chapter 5 on Properties $(PF)$ and $(F)$. At first consider $\beta_1 > 1$ and $\beta_2 > 1$ such that

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1a_2\dots a_m \\ d_{(\beta_2,\beta_1)}(1) &= b_1b_2\dots b_n \end{aligned} \tag{8.1}$$

for some $m, n \in \mathbb{N}$, $a_m$ and $b_n$ are considered non-zero. Take $\boldsymbol{\beta} = \overline{(\beta_1, \beta_2)}$. For the case of sequences up to three digits long we have even presented the exact values of such $\beta_1$ and $\beta_2$ in Appendix A. We now describe transcriptions of minimal forbidden strings of these systems and find a suitable counting function for Theorem 5.19 in a form of weighted sum when it is possible. Recall that the minimal forbidden strings are of the form presented in Remark 5.15. In the following, we will consider transcriptions of all strings of such form, we will not discuss if they are indeed minimal or not for the particular case of base. Moreover, in the light of the assumptions of Theorem 5.19 we will consider just strings with evaluation in base $\boldsymbol{\beta}$ smaller than 1. Note that since we consider bases with finite expansions of 1, the property $(PF)$ implies $(F)$.

It can be shown that the expansions $a_1a_2a_3\dots$, $b_1b_2b_3\dots$ are such that they are $\succ 10^\omega$ and they satisfy the lexicographical condition (7.1). We will use this fact in the derivation of the suitable transcriptions. In all examples we denote $\mathcal{B}^{(i)}$ the bi-infinite periodic extension of $\boldsymbol{\beta}^{(i)}$.

For a better orientation in this section, let us at first summarise the main results in the following table showing which bases satisfy property $(F)$. Parameters $m, n$ and coefficients $a_i$, $b_i$ refer to (8.1).

| $m, n$ | Additional conditions | Reference | (F) |
|---|---|---|---|
| $1, 1$ | – | Example 1 | Yes |
| $2, 1$ | – | Example 2 | Yes |
| $2, 2$ | – | Example 3 | Yes |
| $3, 1$ | – | Example 4 | Yes |
| $3, 2$ | $b_2 \geq a_3$ | Example 5, Case 1 | Yes |

| | | | |
|---|---|---|---|
| $3,2$ | $b_2 < a_3$ | Example 5, Case 2 | Not known |
| $3,3$ | $a_2 \geq b_3$, $b_2 \geq a_3$ | Example 7, Case 1 | Yes |
| $3,3$ | $a_2 \geq b_3$, $b_2 < a_3$ | Example 7, Case 2 | Not known |
| $3,3$ | $a_2 < b_3$, $b_2 < a_3$ | Example 7, Case 3 | Not known |
| $3,3$ | $a_2 < b_3$, $b_2 \geq a_3$ | Example 7, Case 4 | Not known |

Table 8.1: Results considering finiteness property of bases with period $p = 2$ and expansions of 1 in the base and its shift of lengths $m, n$

**Example 1.** With notation as above let $m = n = 1$, i.e.

$$d_{(\beta_1,\beta_2)}(1) = a_1$$
$$d_{(\beta_2,\beta_1)}(1) = b_1.$$

The minimal forbidden strings are of the form $0^{2k}a_1$, $0^{2k+1}b_1$ for $k \in \mathbb{N}_0$. Suitable transcriptions of such strings with evaluation in $\boldsymbol{\beta}$ smaller than 1 are

$$\begin{aligned} 0^{2k}a_1 &\quad\longrightarrow\quad 0^{2k-1}1 \\ 0^{2k-1}b_1 &\quad\longrightarrow\quad 0^{2k-2}1 \end{aligned}$$

for all $k \in \mathbb{N}$. The digit sum of such transcriptions is non-increasing, thus may be chosen as a suitable counting function in Theorem 5.19. Consequently, all bases of such form satisfy assumptions of this theorem and $\mathrm{Fin}(\mathcal{B}^{(i)})$ is closed under addition of positive elements for $i \in \{0, 1\}$. According to Proposition 5.25 then both $\mathcal{B}$ and $\mathcal{B}^{(1)}$ satisfy finiteness property.

**Example 2.** Now consider $m = 2$, $n = 1$, i.e.

$$d_{(\beta_1,\beta_2)}(1) = a_1a_2$$
$$d_{(\beta_2,\beta_1)}(1) = b_1.$$

Transcriptions of the set of forbidden strings as in Remark 5.15 with evaluation smaller than 1, may be chosen as

$$\begin{aligned} 0^{2k}(a_1 + 1) &\quad\longrightarrow\quad 0^{2k-1}10(b_1 - a_2) \\ 0^{2k}a_1a_2 &\quad\longrightarrow\quad 0^{2k-1}1 \\ 0^{2k-1}b_1 &\quad\longrightarrow\quad 0^{2k-2}1 \end{aligned}$$

for all $k \in \mathbb{N}$, since (7.1) implies that $a_2 < b_1$. In this case, the digit sum is a suitable counting function in Theorem 5.19 only if $a_1 \geq b_1 - a_2$. This inequality is restrictive, thus it is natural to ask if there exists some counting function with period 2 that would satisfy assumptions of Theorem 5.19 for the above transcriptions without any additional conditions. Consider $f$ a weighted digit sum as in Example 5.18 with weights $w_0 = u, w_1 = v$, and with period $p = 2$, i.e.

$$f(r_1 r_2 r_3 \dots) := \sum_{k=0}^{p-1} w_k \sum_{i \equiv k \bmod p} r_i = \sum_{j \geq 1} v r_{2j-1} + u r_{2j}. \tag{8.2}$$

Then the condition (5.5) yields inequalities

$$\begin{aligned} v(a_1 + 1) &\geq u(b_1 - a_2 + 1) \\ v a_1 + u a_2 &\geq u \\ u b_1 &\geq v. \end{aligned}$$

The second inequality is trivially satisfied. To satisfy the first and the third inequality, we may choose $u = a_1 + 1$, $v = 1 + b_1 - a_2$. With this choice of weights the function $f$ fulfils the assumptions of Theorem 5.19 without any further requirements on coefficients $a_i$, $b_j$, thus also for this class of bases it holds that $\mathcal{B}$ and $\mathcal{B}^{(1)}$ satisfy $(PF)$ and since they have finite expansions of 1, these bases also satisfy $(F)$.

**Example 3.** With notation as above let $m = n = 2$, i.e.

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1 a_2 \\ d_{(\beta_2,\beta_1)}(1) &= b_1 b_2. \end{aligned}$$

Suitable transcriptions of forbidden strings as in Remark 5.15 may be chosen as

$$\begin{array}{lcl} 0^{2k}(a_1+1) & \longrightarrow & 0^{2k-1}10(b_1-a_2)b_2 \\ 0^{2k}a_1a_2 & \longrightarrow & 0^{2k-1}1 \\ 0^{2k-1}(b_1+1) & \longrightarrow & 0^{2k-2}10(a_1-b_2)a_2 \\ 0^{2k-1}b_1b_2 & \longrightarrow & 0^{2k-2}1 \end{array}$$

for all $k \in \mathbb{N}$, since (7.1) implies $a_2 \leq b_1$, $b_2 \leq a_1$. The digit sum is non-increasing on transcriptions of these strings only if $a_1 \geq b_1 - a_2 + b_2$ and $b_1 \geq a_1 - b_2 + a_2$. These conditions are again quite restrictive. Thus similarly as in the previous example, we are looking for a better counting function. Also in this case it is possible to find a suitable weighted digit sum $f$, which satisfies assumptions of Theorem 5.19 even without any further requirements on digits of expansions of 1. We want $f$

with weights $u, v$ to be non-increasing on the above transcriptions, i.e.

$$\begin{aligned}
v(a_1+1) &\geq vb_2 + u(1+b_1-a_2)\\
va_1 + ua_2 &\geq u\\
u(b_1+1) &\geq ua_2 + v(1+a_1-b_2)\\
vb_2 + ub_1 &\geq u.
\end{aligned}$$

The second and the last inequality are satisfied for all $u, v \in \mathbb{N}$. The first and the third inequality imply $u(b_1 - a_2 + 1) = v(a_1 - b_2 + 1)$. Thus a suitable choice of $u$, $v$ is for example $u = a_1 - b_2 + 1$, $v = b_1 - a_2 + 1$ and this choice is unique up to scaling by a constant. Consequently, for the case of expansions of 1 both 2 digits long, bases $\mathcal{B}$ and $\mathcal{B}^{(1)}$ satisfy $(PF)$ and also $(F)$ property.

**Example 4.** Let $m = 3$, $n = 1$, i.e.

$$\begin{aligned}
d_{(\beta_1,\beta_2)}(1) &= a_1a_2a_3\\
d_{(\beta_2,\beta_1)}(1) &= b_1.
\end{aligned}$$

Transcriptions of the set of forbidden strings as in Remark 5.15 may be chosen as the admissible transcriptions

$$\begin{array}{lcl}
0^{2k}(a_1+1) & \longrightarrow & 0^{2k-1}10(b_1-a_2-1)(a_1-a_3)a_2a_3\\
0^{2k}a_1(a_2+1) & \longrightarrow & 0^{2k-1}100(a_1-a_3)a_2a_3\\
0^{2k}a_1a_2a_3 & \longrightarrow & 0^{2k-1}1\\
0^{2k-1}b_1 & \longrightarrow & 0^{2k-2}1
\end{array}$$

for all $k \in \mathbb{N}$, since (7.1) implies $b_1 \geq a_2 - 1$, $a_1 \geq a_3$. In this case the digit sum fulfils assumptions of Theorem 5.19 only under the condition $b_1 = 1$. However, that never happens, because $b_1 = \beta_1 > 1$. Let us now find a weighted digit sum $f$ of the form (8.2), that satisfies assumptions of Theorem 5.19. We want $f$ to be non-increasing at the above transcriptions, i.e.

$$\begin{aligned}
v(a_1+1) &\geq va_1 + ub_1\\
va_1 + u(a_2+1) &\geq va_1 + u(a_2+1)\\
v(a_1+a_3) + ua_2 &\geq u\\
ub_1 &\geq v.
\end{aligned}$$

The first and the last inequality yield $v = ub_1$, the second inequality is trivially satisfied. The choice $v = b_1$, $u = 1$ satisfies all four inequalities without any further assumptions on $a_i$, $b_j$, thus the weighted sum with weights $1, b_1$ is a suitable choice of counting function $f$ for Theorem 5.19. Again, the choice of weights is unique up to scaling. In conclusion, bases with expansions of 1 of lengths 3 and 1 also have Properties $(PF)$ and $(F)$.

**Example 5.** Let $m = 3$, $n = 2$, i.e

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1a_2a_3 \\ d_{(\beta_2,\beta_1)}(1) &= b_1b_2. \end{aligned}$$

Sequences satisfying lexicographical conditions (7.1) are of two kinds: either $b_2 \geq a_3$, or $b_2 < a_3$.

**Case 1:** $b_2 \geq a_3$

Transcriptions of forbidden strings as in Remark 5.15 may be chosen as

$$\begin{array}{lcl} 0^{2k}(a_1+1) & \longrightarrow & 0^{2k-1}10(b_1-a_2)(b_2-a_3) \\ 0^{2k}a_1(a_2+1) & \longrightarrow & 0^{2k-1}100(a_1-a_3)a_2a_3 \\ 0^{2k}a_1a_2a_3 & \longrightarrow & 0^{2k-1}1 \\ 0^{2k-1}(b_1+1) & \longrightarrow & 0^{2k-2}10(a_1-b_2)a_2a_3 \\ 0^{2k-1}b_1b_2 & \longrightarrow & 0^{2k-2}1, \end{array}$$

for all $k \in \mathbb{N}$. The digit sums of the above transcriptions are non-increasing if and only if $a_1 \geq b_1 + b_2 - a_2 - a_3$ and $b_1 \geq a_1 + a_2 + a_3 - b_2$. However, similarly as in the previous examples, there exists a weighted digit sum satisfying the condition of Theorem 5.19 for Case 1 without any additional requirements on $a_i$, $b_i$. Let us now derive it. If the weighted digit sum with weights $u, v$ is to be non-increasing on the above transcriptions, $u, v$ have to satisfy

$$\begin{aligned} v(a_1+1) &\geq v(b_2-a_3) + u(b_1-a_2+1) \\ va_1 + u(a_2+1) &\geq va_1 + u(a_2+1) \\ v(a_1+a_3) + ua_2 &\geq u \\ u(b_1+1) &\geq v(a_1+a_3-b_2+1) + ua_2 \\ vb_2 + b_1u &\geq v. \end{aligned}$$

The second and the last equation are satisfied trivially. The first and the fourth equation yields $u(b_1 - a_2 + 1) = v(a_1 + a_3 - b_2 + 1)$. A suitable choice of $u, v$ is for example $v = b_1 - a_2 + 1$, $u = a_1 + a_3 - b_2 + 1$. With this choice even the third inequality is satisfied. Therefore, according to Theorem 5.19, $\mathrm{Fin}(\mathcal{B}^{(i)})$ is closed under addition of positive elements for $i \in \{0, 1\}$. According to Proposition 5.25, both $\mathcal{B}$ and $\mathcal{B}^{(1)}$ satisfy $(F)$.

**Case 2:** $b_2 < a_3$

In this case one has $a_2 < b_1$, and transcriptions may be chosen for example as

$$\begin{array}{lcl} 0^{2k}(a_1+1) & \longrightarrow & 0^{2k-1}10(b_1-a_2-1)(b_2+a_1-a_3)a_2a_3 \\ 0^{2k}a_1(a_2+1) & \longrightarrow & 0^{2k-1}100(a_1-a_3)a_2a_3 \\ 0^{2k}a_1a_2a_3 & \longrightarrow & 0^{2k-1}1 \\ 0^{2k-1}(b_1+1) & \longrightarrow & 0^{2k-2}10(a_1-b_2)a_2a_3 \\ 0^{2k-1}b_1b_2 & \longrightarrow & 0^{2k-2}1 \end{array}$$

for all $k \in \mathbb{N}$. Note that these transcriptions are all admissible. In this case the digit sum of transcriptions is non-increasing if moreover $b_1 \geq a_1 + a_2 + a_3 - b_2$ and $1 \geq b_1 + b_2$. This, however, can never happen, because we assume $b_1 \geq 1$, $b_2 \geq 1$. Similarly as in the above examples, we might hope that there exists a suitable counting function $f$ of the form of a weighted digit sum. We now show that neither that is the case. There is no suitable choice of weights such that the weighted digit sum is non-increasing on the above transcriptions, thus we are left with no good candidate for a counting function satisfying assumptions of Theorem 5.19. Indeed, if a weighted digit sum with weights $u, v$ and period 2 is non-increasing on the above transcriptions, the weights $u, v$ satisfy inequalities

$$\begin{aligned} v(a_1+1) &\geq v(b_2+a_1) \\ va_1 + u(a_2+1) &\geq va_1 + u(a_2+1) \\ v(a_1+a_3) + ua_2 &\geq u \\ u(b_1+1) &\geq v(a_1+a_3-b_2+1) + ua_2 \\ vb_2 + ub_1 &\geq v. \end{aligned}$$

The first inequality can be rewritten as $v(1-b_2) \geq ub_1$, and that cannot be satisfied with any choice of $u, v \in \mathbb{N}$.

We now show that in this case there exist a finite string, which we may transcribe infinitely many times, which is a contradiction with Remark 5.20. Thus the reason we did not succeed with our attempts of constructing a suitable counting function is, that there is no such mapping for the above transcriptions. Consider the string $\boldsymbol{s} = 00(a_1+b_2)a_2a_3$, denote $S := \mathrm{val}_{\boldsymbol{\beta}}(\boldsymbol{s})$. We now construct an infinite sequence of transcriptions of this string. We will always apply transcriptions on the minimal forbidden string $0^{2j}(a_1+1)$. In each step the digit $a_1+b_2$ is the first non-admissible digit when we search from the left side, we subtract $a_1+1$ and transcribe as follows. The symbol

$\oplus$ denotes the operation of digit-wise addition of strings. The first transcription is

$$\begin{aligned}
\boldsymbol{s} = 00(a_1+b_2)a_2a_3 &= 00(b_2-1)a_2a_3 \oplus 00(a_1+1) \\
&\to 00(b_2-1)a_2a_3 \oplus 010(b_1-a_2-1)(b_2+a_1-a_3)a_2a_3 \\
&= 01(b_2-1)(b_1-1)(b_2+a_1)a_2a_3,
\end{aligned}$$

we sum digits aligned from the left. The second transcription is

$$\begin{aligned}
01(b_2-1)(b_1-1)(b_2+a_1)a_2a_3 &= 01(b_2-1)(b_1-1)(b_2-1)a_2a_3 \oplus 0000(a_1+1) \\
&\to 01(b_2-1)(b_1-1)(b_2-1)a_2a_3 \oplus 00010(b_1-a_2-1)(b_2+a_1-a_3)a_2a_3 \\
&= 01(b_2-1)b_1(b_2-1)(b_2+a_1)a_2a_3.
\end{aligned}$$

We see that again the first non-admissible digit from the left is $b_2+a_1$, we iterate the above process and obtain strings of the form

$$01[(b_2-1)b_1]^k(b_2+a_1)a_2a_3.$$

This sequence of strings converges to the lexicographically greatest infinite representation of $S$, i.e. $01[(b_2-1)b_1]^\omega$. Remark 5.20 then yields that for the above transcriptions, the assumptions of Theorem 5.19 are not satisfied, in particular there exists no suitable counting function. Note that the $\boldsymbol{\beta}$-expansion of $S$ is equal to $d_{\boldsymbol{\beta}}(S) = 01b_2$, and can be obtained by transcribing the forbidden strings in a different order, namely

$$\begin{aligned}
\boldsymbol{s} = 00(a_1+b_2)a_2a_3 &= 00a_1a_2a_3 \oplus 00b_2 \\
&\to 01 \oplus 00b_2 \\
&= 01b_2.
\end{aligned}$$

Therefore, in this case we still cannot decide whether the set $\mathrm{Fin}(\mathcal{B})$ is closed under addition of positive elements or not. Let us therefore check if the necessary conditions of $(F)$ as stated in Theorem 5.8 are satisfied. We know that in this case the base $\overline{(\beta_1,\beta_2)}$ has the following properties, see Appendix A, Proposition 11. Firstly, $\delta = \beta_1\beta_2$ is the larger root of the polynomial

$$r(x) := x^2 - (a_1b_1 + a_2 + b_2)x - (a_3b_1 - a_2b_2),$$

and elements of the base are of the form $\beta_1 = \frac{\delta a_1 + a_3}{\delta - a_2}$, $\beta_2 = \frac{\delta b_1}{\delta - b_2}$. It may be either the case that the polynomial $r$ is reducible with root 0 (then the necessary conditions are trivially satisfied), or the polynomial $r$ has two distinct non-zero roots. Let us therefore discuss the case when $\delta$ is quadratic. We have already shown in our research project [21], as recalled in Appendix A, Proposition 11, that $\delta$ is a Pisot number. Now denote $\gamma$ the conjugate of $\delta$, $\gamma \neq \delta$. Denote $\sigma$ the isomorphism of $\mathbb{Q}(\delta)$ and $\mathbb{Q}(\gamma)$ induced by $\sigma(\delta) = \gamma$. Let us verify that there indeed exist $j \in \{0, 1\}$ such that $\sigma(\beta_j) \leq 0$. The values of the polynomial $r$ in $-1, 0$ are

$$\begin{aligned} r(-1) &= 1 + a_1b_1 + a_2 + b_2 - a_3b_1 + a_2b_2 \geq 1 + a_2 + b_2 + a_2b_2 > 0 \\ r(0) &= a_2b_2 - a_3b_1 < 0, \end{aligned}$$

where the last inequality holds true since we are considering Case 2. Therefore $\gamma \in (-1, 0)$. Then $\sigma(\beta_2) = \frac{\gamma b_1}{\gamma - b_2} > 0$. We would like to verify that $\sigma(\beta_1) = \frac{\gamma a_1 + a_3}{\gamma - a_2} < 0$. The denominator is negative, thus we need to show that $\gamma a_1 + a_3 > 0$. This may be done straightforwardly by expressing the root $\gamma$ as a function of $a_i$, $b_j$ with help of a relation for the roots of a quadratic equation, and substituting into the inequality. Inequality will hold without any additional assumptions on coefficients. To conclude, the necessary conditions of $(F)$ are satisfied in both cases, and we do not have any tool to decide whether the considered base satisfies $(F)$ or not.

**Remark 6.** Note that $\boldsymbol{\beta}_m$ as defined in Example 5.16 is a base with expansions of 1 of lengths 3 and 2, in particular

$$\begin{aligned} d_{\boldsymbol{\beta}_m}(1) &= m\ m\ 1 \\ d_{\boldsymbol{\beta}_m^{(1)}}(1) &= (m^2 + 1)\ m. \end{aligned}$$

Denote $\mathcal{B}_m$ the bi-infinite periodic extension of $\boldsymbol{\beta}_m$. We have already commented on the case $m = 1$ in Example 5.21. We have concluded that for such base a suitable counting function may be chosen simply as digit sum. On the other hand, with transcriptions chosen as in the Example 5.16, for other cases of $m$ the digit sum is not non-increasing. However, we are in Case 1 of Example 5, since for bases $\boldsymbol{\beta}_m$ we have $b_2 = m$, $a_3 = 1$. Thus $b_2 \geq a_3$ a suitable counting function (for transcriptions as in Example 5.16, which are the same as those in the above Example 5) may be chosen as a weighted digit sum with period 2 and weights $v = b_1 - a_2 + 1 = m^2 - m + 2$, $u = a_1 + a_3 - b_2 + 1 = 2$. In conclusion, all bases $\boldsymbol{\beta}_m$ satisfy assumptions of Theorem 5.19, therefore $\mathrm{Fin}(\mathcal{B}_m^{(i)})$ is closed under addition of positive elements for all $m \in \mathbb{N}$ and for $i \in \{0, 1\}$. Consequently, according to Proposition 5.25, all bases $\mathcal{B}_m^{(i)}$ have Property $(F)$.

**Example 7.** Finally, let $m = n = 3$, i.e.

$$\begin{aligned} d_{(\beta_1,\beta_2)}(1) &= a_1a_2a_3 \\ d_{(\beta_2,\beta_1)}(1) &= b_1b_2b_3. \end{aligned}$$

We need to consider four cases according to the inequality between digits $a_2$ and $b_3$, and the inequality between $b_2$ and $a_3$.

**Case 1:** $a_2 \geq b_3$ **and** $b_2 \geq a_3$

We may transcribe the forbidden strings as in Remark 5.15 in the following way

$$\begin{array}{lcl} 0^{2k}(a_1+1) & \longrightarrow & 0^{2k-1}10(b_1-a_2)(b_2-a_3)b_3 \\ 0^{2k}a_1(a_2+1) & \longrightarrow & 0^{2k-1}100(a_1-a_3)a_2a_3 \\ 0^{2k}a_1a_2a_3 & \longrightarrow & 0^{2k-1}1 \\ 0^{2k-1}(b_1+1) & \longrightarrow & 0^{2k-2}10(a_1-b_2)(a_2-b_3)a_3 \\ 0^{2k-1}b_1(b_2+1) & \longrightarrow & 0^{2k-2}100(b_1-b_3)b_2b_3 \\ 0^{2k-1}b_1b_2b_3 & \longrightarrow & 0^{2k-2}1 \end{array}$$

for all $k \in \mathbb{N}$. Note that all these transcriptions are admissible. Again, the digit sum is a suitable counting function in Theorem 5.19 only with some additional requirement, namely if $a_1+a_2+a_3 = b_1+b_2+b_3$. Let us now construct a weighted digit sum with weights $u$, $v$, that satisfies the condition in Theorem 5.19 without this additional requirement. Such weighted digit sum is non-increasing on the above transcriptions if

$$\begin{aligned} v(a_1+1) &\geq v(b_2-a_3) + u(b_1+b_3-a_2+1) \\ va_1 + u(a_2+1) &\geq va_1 + u(a_2+1) \\ v(a_1+a_3) + ua_2 &\geq u \\ u(b_1+1) &\geq v(a_1+a_3-b_2+1) + u(a_2-b_3) \\ ub_1 + v(b_2+1) &\geq v(1+b_2) + ub_1 \\ vb_2 + u(b_1+b_3) &\geq v. \end{aligned}$$

The second and the fifth inequality trivially hold true. The first and the fourth inequality yield $v(a_1+a_3-b_2+1) = u(b_1+b_3-a_2+1)$. Thus define the weights as $u = a_1+a_3-b_2+1$, $v = b_1+b_3-a_2+1$. Then also the third and the last inequality are satisfied. Therefore, we have found a suitable counting function fulfilling assumptions of Theorem 5.19, and we conclude that the base satisfies Property $(F)$.

**Case 2:** $a_2 \geq b_3$ **and** $b_2 < a_3$

Lexicographical condition (7.1) yields $a_2 < b_1$. We will try to construct a non-increasing weighted digit sum for two sets of transcriptions. However, neither of the attempts will be successful. Consider at first the following transcriptions of the set of forbidden strings as in Remark 5.15:

$$\begin{array}{lcl}
0^{2k}(a_1+1) & \longrightarrow & 0^{2k-1}10(b_1-a_2-1)(b_2-a_3+a_1)(b_3+a_2)a_3 \\
0^{2k}a_1(a_2+1) & \longrightarrow & 0^{2k-1}100(a_1-a_3)a_2a_3 \\
0^{2k}a_1a_2a_3 & \longrightarrow & 0^{2k-1}1 \\
0^{2k-1}(b_1+1) & \longrightarrow & 0^{2k-2}10(a_1-b_2)(a_2-b_3)a_3 \\
0^{2k-1}b_1(b_2+1) & \longrightarrow & 0^{2k-2}100(b_1-b_3)b_2b_3 \\
0^{2k-1}b_1b_2b_3 & \longrightarrow & 0^{2k-2}1
\end{array}$$

for all $k \in \mathbb{N}$. Note that these transcriptions are non-admissible (the first transcription has a non-admissible digit $b_3 + a_2$). We now show that for this set of transcriptions there is no suitable choice of weighted digit sum with period 2 that satisfies assumptions of Theorem 5.19. Consider a weighted sum such that it is non-increasing on the above transcriptions. Then for the weights $u, v$ the following must hold

$$\begin{aligned}
v(a_1+1) &\geq v(b_2+a_1) + u(b_1+b_3) \\
va_1 + u(a_2+1) &\geq va_1 + u(a_2+1) \\
v(a_1+a_3) + ua_2 &\geq u \\
u(b_1+1) &\geq v(a_1-b_2+a_3+1) + u(a_2-b_3) \\
v(b_2+1) + ub_1 &\geq v(b_2+1) + ub_1 \\
u(b_1+b_3) + vb_2 &\geq v.
\end{aligned}$$

The second and the fifth inequality are trivially satisfied. The first inequality can be satisfied only if $b_2 = 0$, because $u(b_1 + b_3) > 0$. Thus for other cases there is surely no suitable weighted sum with period 2. We now show that even if $b_2 = 0$, there is no suitable solution $u, v \in \mathbb{N}$ of the above system of inequalities. Consider $b_2 = 0$. Then the non-trivial inequalities of the above system are

$$\begin{aligned}
v(a_1+1) &\geq va_1 + u(b_1+b_3) \\
v(a_1+a_3) + ua_2 &\geq u \\
u(b_1+1) &\geq v(a_1+a_3+1) + u(a_2-b_3) \\
u(b_1+b_3) &\geq v.
\end{aligned}$$

The first and the last inequality yields $u(b_1 + b_3) = v$. However, when we substitute this relation into the third inequality, we get

$$u(b_1 + 1) \geq u(b_1 + b_3)(a_1 + a_3 + 1) + u(a_2 - b_3),$$

which cannot be satisfied by any $u \in \mathbb{N}$. Therefore, we have no good candidate for a counting function necessary to fulfil assumptions of Theorem 5.19.

Consider now the set of admissible transcriptions

$$\begin{array}{lcl}
0^{2k}(a_1+1) & \longrightarrow & 0^{2k-1}10(b_1-a_2-1)(b_2-a_3+a_1+1)(b_3+a_2-b_1)(a_3-b_2-1)(b_1-b_3)b_2b_3 \\
0^{2k}a_1(a_2+1) & \longrightarrow & 0^{2k-1}100(a_1-a_3)a_2a_3 \\
0^{2k}a_1a_2a_3 & \longrightarrow & 0^{2k-1}1 \\
0^{2k-1}(b_1+1) & \longrightarrow & 0^{2k-2}10(a_1-b_2)(a_2-b_3)a_3 \\
0^{2k-1}b_1(b_2+1) & \longrightarrow & 0^{2k-2}100(b_1-b_3)b_2b_3 \\
0^{2k-1}b_1b_2b_3 & \longrightarrow & 0^{2k-2}1.
\end{array}$$

It turns out that a weighted digit sum with period 2 for these transcriptions is also never non-increasing (the conditions on the weights would be the same as in the case of the above non-admissible transcriptions).

**Case 3:** $a_2 < b_3$ **and** $b_2 < a_3$
The condition (7.1) implies $b_2 < a_1$ and $a_2 < b_1$. We may transcribe

$$\begin{array}{lcl}
0^{2k}(a_1+1) & \longrightarrow & 0^{2k-1}10(b_1-a_2-1)(b_2+a_1-a_3+1)(b_3+a_2-b_1)(a_3-b_2-1)(b_1-b_3)b_2b_3 \\
0^{2k}a_1(a_2+1) & \longrightarrow & 0^{2k-1}100(a_1-a_3)a_2a_3 \\
0^{2k}a_1a_2a_3 & \longrightarrow & 0^{2k-1}1 \\
0^{2k-1}(b_1+1) & \longrightarrow & 0^{2k-2}10(a_1-b_2-1)(a_2+b_1-b_3+1)(a_3+b_2-a_1)(b_3-a_2-1)(a_1-a_3)a_2a_3 \\
0^{2k-1}b_1(b_2+1) & \longrightarrow & 0^{2k-2}100(b_1-b_3)b_2b_3 \\
0^{2k-1}b_1b_2b_3 & \longrightarrow & 0^{2k-2}1
\end{array}$$

for all $k \in \mathbb{N}$. Note that the chosen transcriptions are admissible. Similarly as in Case 2, we now show that there is no weighted digit sum with period 2 non-increasing on the above transcriptions.

The weights of such sum would need to satisfy the following system of inequalities

$$\begin{aligned} v(a_1+1) &\geq v(b_2+a_1)+u(b_1+b_3) \\ va_1+u(a_2+1) &\geq va_1+u(a_2+1) \\ v(a_1+a_3)+ua_2 &\geq u \\ u(b_1+1) &\geq v(a_1+a_3)+u(a_2+b_1) \\ v(b_1+1)+ub_1 &\geq v(b_1+1)+ub_1 \\ vb_2+u(b_1+b_3) &\geq v. \end{aligned}$$

The second and the fifth inequality are always satisfied. However, the first and the fourth inequality might be satisfied only if $a_2 = b_2 = 0$. We now show that even if $a_2 = b_2 = 0$, the system does not have any solution $u, v \in \mathbb{N}$. If $a_2 = b_2 = 0$, then the non-trivial of the above inequalities are

$$\begin{aligned} v(a_1+1) &\geq va_1+u(b_1+b_3) \\ v(a_1+a_3) &\geq u \\ u(b_1+1) &\geq v(a_1+a_3)+ub_1 \\ u(b_1+b_3) &\geq v. \end{aligned}$$

These inequalities yield $v = u(b_1+b_3)$ and $u = v(a_1+a_3)$, which cannot be satisfied for any $u, v \in \mathbb{N}$, since $a_1, b_1, a_3, b_3 \in \mathbb{N}$. Therefore no weighted digit sum with period 2 is a suitable candidate of a counting function needed to satisfy assumptions of Theorem 5.19 for the above transcriptions.

**Case 4:** $a_2 < b_3$ **and** $b_2 \geq a_3$

Then (7.1) yields $b_2 < a_1$ and suitable transcriptions are

$$\begin{aligned} &0^{2k}(a_1+1) &&\longrightarrow &&0^{2k-1}10(b_1-a_2)(b_2-a_3)b_3 \\ &0^{2k}a_1(a_2+1) &&\longrightarrow &&0^{2k-1}100(a_1-a_3)a_2a_3 \\ &0^{2k}a_1a_2a_3 &&\longrightarrow &&0^{2k-1}1 \\ &0^{2k-1}(b_1+1) &&\longrightarrow &&0^{2k-2}10(a_1-b_2-1)(a_2+b_1-b_3+1)(a_3+b_2-a_1)(b_3-a_2-1)(a_1-a_3)a_2a_3 \\ &0^{2k-1}b_1(b_2+1) &&\longrightarrow &&0^{2k-2}100(b_1-b_3)b_2b_3 \\ &0^{2k-1}b_1b_2b_3 &&\longrightarrow &&0^{2k-2}1 \end{aligned}$$

for all $k \in \mathbb{N}$. These transcriptions are as well admissible. It could be shown similarly as is Case 2, that there is no weighted sum with period 2 non-increasing on the above transcriptions (to obtain a proof just interchange roles of $a_i$, $b_j$ and $u, v$ in the inequalities in Case 2).

Let us now show that necessary conditions of $(F)$, as presented in Theorem 5.8, are satisfied in four cases of this example. As we recalled in Appendix A, $\delta$ is a Pisot number and its minimal polynomial is

$$q(x) := x^3 - (a_1b_1 + a_2 + b_2)x^2 - (a_3b_1 - a_2b_2 + a_1b_3)x - a_3b_3.$$

Denote $\gamma_1, \gamma_2$ other roots of $q$. Denote $\sigma_i$ the isomorphism of $\mathbb{Q}(\delta)$ and $\mathbb{Q}(\gamma_i)$ induced by $\sigma_i(\delta) = \gamma_i$. We may consider two cases: either $\gamma_1, \gamma_2 \in \mathbb{R}$, or $\gamma_1, \gamma_2 \in \mathbb{C} \setminus \mathbb{R}$ and $\gamma_1 = \overline{\gamma_2}$. We now show that the vector $\big(\sigma_i(\beta_1), \sigma_i(\beta_2)\big)$ for $i \in \{1, 2\}$ cannot have both components real non-negative.

**Real case.** Let $\gamma_1, \gamma_2 \in \mathbb{R}$. Note that $\gamma_i = \sigma_i(\delta) = \sigma_i(\beta_1)\sigma_i(\beta_2)$. If there exists $i \in \{1, 2\}$ such that $\gamma_i < 0$, surely it cannot be $\sigma_i(\beta_1) \geq 0$ and $\sigma_i(\beta_2) \geq 0$. It is left to discuss the case when $\gamma_1 > 0$, $\gamma_2 > 0$. Let us now discuss the signs of coefficients of the polynomial $q$. Surely $-(a_1b_1+a_2+b_2) < 0$ and $-a_3b_3 < 0$. Since we are discussing the case when all three roots of this polynomial are positive, according to Descartes' rule of signs it has to be

$$a_3b_1 - a_2b_2 + a_1b_3 < 0.$$

Thus necessarily $a_2 \geq 1$, $b_2 \geq 1$. Now recall that the elements of the base are of the form

$$\beta_1 = \frac{\delta a_1 + a_3}{\delta - a_2} \qquad \beta_2 = \frac{\delta b_1 + b_3}{\delta - b_2}.$$

Lastly, note that $\delta$ is a Pisot number, thus $\gamma_i < 1$. Therefore

$$\sigma_i(\beta_1) = \frac{\gamma_i a_1 + a_3}{\gamma_i - a_2} < 0 \qquad \sigma_i(\beta_2) = \frac{\gamma_i b_1 + b_3}{\gamma_i - b_2} < 0,$$

and the necessary condition is satisfied.

**Complex case.** Now consider $\gamma_1, \gamma_2 \in \mathbb{C} \setminus \mathbb{R}$, $\gamma_1 = \overline{\gamma_2}$. We know that $\beta_j \in \mathbb{Q}(\delta)$ and that $\delta$ is cubic. Now note that $\mathbb{Q}(\gamma_i) \cap \mathbb{R} = \mathbb{Q}$ for $i \in \{1, 2\}$. Indeed, if $\alpha \in \mathbb{Q}(\gamma_i) \cap \mathbb{R}$ is such that $\alpha \notin \mathbb{Q}$, then $\alpha$ has to be cubic. Thus $\mathbb{Q}(\alpha) = \mathbb{Q}(\gamma_i)$, because all numbers of maximal degree in $\mathbb{Q}(\gamma_i)$ are generators of this field, and that is a contradiction, because $\mathbb{Q}(\gamma_i)$ contains non-real elements, but $\mathbb{Q}(\alpha) \subset \mathbb{R}$. Thus either $\sigma_i(\beta_j) \in \mathbb{Q}$ or $\sigma_i(\beta_j) \notin \mathbb{R}$. Surely, it cannot happen that both $\sigma_i(\beta_1)$, $\sigma_i(\beta_2)$ are rational, since then $\sigma_i(\beta_1) = \beta_1$, $\sigma_i(\beta_2) = \beta_2$ and thus $\delta = \beta_1\beta_2 \in \mathbb{Q}$, which is a contradiction.

Another special case of bases with short expansions of 1 are bases consisting of powers of a quadratic Pisot unit, as we have already introduced them in Appendix A. We have commented on the case when the base is $\overline{(\gamma, \gamma^2)}$ already above, let us now do a similar analysis for the base $\overline{(\gamma, \gamma^3, \gamma^2)}$.

**Example 8.** Consider $\gamma$ the larger root of $x^2 - mx - 1$ for $m \in \mathbb{N}$ and an alternate base of the form $\boldsymbol{\delta} = \overline{(\gamma, \gamma^3, \gamma^2)}$. As we have already seen in Appendix A, the expansions of 1 in this base and its shifts are of the form

$$\begin{aligned} d_{\boldsymbol{\delta}}(1) &= m(m^2+1)m^2m \\ d_{\boldsymbol{\delta}^{(1)}}(1) &= (m^3+3m)01 \\ d_{\boldsymbol{\delta}^{(2)}}(1) &= (m^2+1)m. \end{aligned}$$

Therefore, the minimal forbidden strings have to be of the form below, and their greedy transcriptions are

$$\begin{array}{lcl}
0^{3k}(m+1) & \longrightarrow & 0^{3k-1}10(m^3-m^2+3m-2)11 \\
0^{3k}m(m^2+2) & \longrightarrow & 0^{3k-1}1001 \\
0^{3k}m(m^2+1)(m^2+1) & \longrightarrow & 0^{3k-1}10000(m^2+1)m^2m \\
0^{3k}m(m^2+1)m^2m & \longrightarrow & 0^{3k-1}1 \\
0^{3k+1}(m^3+3m+1) & \longrightarrow & 0^{3k}10(m^2+1)(m-1) \\
0^{3k+1}(m^3+3m)1 & \longrightarrow & 0^{3k}100(m-1)(m^2+1)m^2m \\
0^{3k+1}(m^3+3m)01 & \longrightarrow & 0^{3k}1 \\
0^{3k+2}(m^2+2) & \longrightarrow & 0^{3k+1}100(m^2+1)m^2m \\
0^{3k+2}(m^2+1)m & \longrightarrow & 0^{3k+1}1
\end{array}$$

for some $k \in \mathbb{N}_0$. If a weighted digit sum with period 3 satisfies assumptions of Theorem 5.19, the weights $u, v, w \in \mathbb{N}$ have to satisfy inequalities

$$\begin{aligned} u(m+1) &\geq u + v(m^3 - m^2 + 3m - 2) + 2w \\ um + v(m^2+2) &\geq 2w \\ um + v(m^2+1) + w(m^2+1) &\geq um + v(m^2+1) + w(m^2+1) \\ 2um + v(m^2+1) + wm^2 &\geq w \\ v(m^3+3m+1) &\geq um + w(m^2+1) \\ v(m^3+3m) + w &\geq 2um + v(m^2+1) + wm^2 \\ v(m^3+3m+1) + u &\geq u \\ w(m^2+2) &\geq um + v(m^2+2) + wm^2 \\ um + w(m^2+1) &\geq v. \end{aligned}$$

It is not difficult to show that there are no suitable $u, v, w \in \mathbb{N}$ fulfilling these conditions. However, it can be verified that all the necessary conditions hold true, thus do not have tools to decide whether this system has Property $(F)$.

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