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DISSERTATION THESIS

REACTION DIFFUSION SYSTEMS AND PATTERN FORMATION IN BIOLOGY

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Abstract

Abstrakt:

Vznik vzorků v biologických systémech je předmětem intenzivního mezioborového výzkumu. Jedním z nejslavnějších mechanismů, které mohou vést ke vzniku vzorků v chemických, biologických a dalších systémech, je Turingova nestabilita. Ta může vést ke vzniku prostorovch struktur v systémech se dvěma komponentami. Na druhou stranu je ale nutné předpokládat, že rychlost difúze jedné chemikálie je výrazně rychlejší než rychlost difúze té druhé, což je častým terčem kritiky tohoto mechanismu. Neumožňuje ani snadno modelovat nepravidelné vzory a vysvětlit některé jevy, jako např. agregaci melaninu na hřbetu kožichu královského geparda. Některé z těchto problémů řeší přidání určitých vyrovnávacích mechanismů do rovnic, v našem případě jednostranných členů. Tato práce se proto věnuje analýze systémů dvou rovnic reakce-difúze, u kterých dochází k Turingově nestabilitě a které jsou následně doplněny tzv. jednostrannými členy. Existence bifurkačních bodů, a tedy i existence vzorků, jsou analyticky dokázány pro systémy, kde poměr difúzí obou chemikálií je blíže jedné. Jsou analyzovány jak systémy s Dirichletovými nebo smíšenými, tak s Neumannovými okrajovými podmínkami. Stacionární řešení těchto systémů jsou pro vybrané systémy numericky aproximována, což napovídá, jak by mohly vzorky vypadat. Také je použitím numerických metod aproximována poloha bifurkačních bodů vybraných systémů s jednostrannými zdroji.

Abstract:

The pattern formation in biological systems is being a subject of intensive research in interdisciplinary sciences. One of the most famous mechanism, which could lead to a formation of pattern in chemical, biological and other systems is Turing instability. It can lead to the formation of spatial structures in systems with two components. However, it supposes that the diffusivity of one chemical must be significantly different from the other one, which is often a target of criticism of such systems. It also cannot in a simple way model irregular patterns and it cannot explain e.g. an aggregation of melanin on king cheetah coat back. To bypass these problems, addition of some regulatory mechanisms (unilateral terms) to the equations has been proposed. This dissertation thesis therefore concerns with systems of two reaction-diffusion equations which exhibit Turing Instability, and are supplemented with the so-called unilateral terms. The existence of bifurcation points, and hence the existence of patterns, is proved analytically for systems, where the diffusivenesses are less distinct. The systems with Dirichlet or mixed, as well as with Neumann boundary conditions are analyzed. The stationary solutions of selected systems are numerically computed and that gives a hint, which patterns could form in such systems. Moreover, the location of bifurcation points is approximated for particular systems with unilateral terms numerically as well.

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Finally, my greatest thanks go to my family members, for their infinite patience and care.

Dedication

In memory of my close friend and great and enthusiastic mathematician prof. Teo Sturm, who passed away in 2017. Rest in peace.

Contents

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Abbreviations

Number Sets

$\mathbb{N}$	Natural numbers set
$\mathbb{N}_0$	Natural numbers set with zero
$\mathbb{Z}$	Integer numbers set
$\mathbb{R}$	Real numbers set
$\mathbb{R}_+$	Set of positive real numbers

Vector spaces and auxiliary notation

Ω	domain in $\mathbb{R}^m$ with Lipschitz boundary
$\partial\Omega$	boundary of Ω
$\Gamma_D,\ \Gamma_N$	subsets of the boundary of Ω
$W^{k,p}(\Omega)$	Sobolev space on Ω
$W_0^{k,p}(\Omega)$	Sobolev space on Ω with zero boundary values
$W_D^{1,2}(\Omega)$	$W^{1,2}(\Omega)$ space with zero boundary values on the set $\Gamma_D \subset \partial\Omega$
$\mathbb{H}$	Hilbert space
$\mathbb{X}$	Banach space
$\hookrightarrow$	embedding
$\hookrightarrow^c$	compact embedding
$\|\cdot\|$	$W^{1,2}(\Omega)$ norm/ norm on general Banach space
$\|\cdot\|_{L^p}$	$L^p(\Omega)$ norm
$\|\cdot\|_{k,p}$	norm on $W^{k,p}(\Omega)$
$\to$	(strong) convergence
$\rightharpoonup$	weak convergence
$\mathscr{L}(W_D^{1,2}(\Omega), W_D^{1,2}(\Omega))$	space of all linear operators from $W_D^{1,2}(\Omega)$ to $W_D^{1,2}(\Omega)$
∂_u	partial derivative w.r.t. u
$\mu_m(\Omega)$	m-dimensional Lebesgue measure of a set Ω

Reaction-diffusion systems

d_1, d_2	diffusion paramteres
$-\Delta$	Laplacian
κ_i	Laplacian eigenvalues
f, g	functions modelling the reaction kinetics
g_+, g_-	source, sink functions
s_+, s_-	partial derivative of g_+, g_- w.r.t. x at zero
y_k	asymptote of the hyperbola, $y_k = b_{11}/\kappa_k$

Operators

$\sigma(X)$	spectrum of the operator X
$\mathcal{A}$	linear symmetric compact operator with $\sigma(A) \subset [0,1]$
$\mathcal{S}$	symmetric linear compact operator
$\mathcal{B}$	positively homogeneous compact and continuous operator
$\mathcal{N}$	small nonlinear perturbation
$\mathcal{I}$	identity operator
$C^1(X,Y)$	space of all operators from X to Y with the continuous derivative

Chapter **1**

Introduction

1.1 Short excursion into history

Formation of patterns plays an essential role not only in biology, but also in other scientific fields like chemistry, physics and even geology, and it is an important phenomenon studied in the interdisciplinary research. However, up to the end of 19^{th} century there was not known any mechanism which could lead to the formation of such patterns. One of the essential question in this area was how the pattern is created during the development of the embryo. The effort of many researchers at the beginning of 20^{th} century was aiming to find the explanation for this process. Although the genes play here the key role, they cannot explain the mechanisms by which the ingredients mix together and assembly a coherent pattern. Nevertheless, the study of genetic influence on animal coat patterns was a predominant direction of research and this dominance was strengthen by the Watson's and Crick's discovery of the structure of DNA.

This was the reason why the pioneering paper of A. Turing [53], published in 1952, did not attach much attention during the first decades after its publication. In his paper Turing concerned with a system of two reaction-diffusion equations defined on a ring with a diameter ρ. By making the stability analysis of a linear problem

$$\frac{\partial X}{\partial t} = a(X-h) + b(Y-k) + \frac{\mu'}{\rho^2}\frac{\partial^2 X}{\partial \theta^2},$$
$$\frac{\partial Y}{\partial t} = c(X-h) + d(Y-k) + \frac{\nu'}{\rho^2}\frac{\partial^2 Y}{\partial \theta^2},$$

where θ is a polar coordinate, X, Y are concentrations, μ', ν' are diffusibilities, a–d are reaction rates and h, k are steady state concentrations, Turing was able to show that under additional assumptions the diffusion can destabilize the steady state and this instability caused by diffusion can lead to the growth of a structure at a particular wavelength. The resulting non-homogeneous structure can be interpreted as a pattern. This mechanism could explain the creation of animal coat pattern. This was an original and unconventional approach – the diffusion was considered as a stabilizing element of physical systems and the idea of the existence of heterogeneous steady-states in such system seemed to be counter-intuitive. The pattern formation in chemical systems has already been known since the mid-19^{th} century, but nobody before him gave such clear and simple explanation how a very simple mechanism can explain the formation of patterns during the embryogenesis. Unfortunately, the described mechanism can work only in systems, where the two chemicals have significantly different diffusion, which is often criticized as not realistic assumption. Despite the lukewarm reception his paper became later very famous and up to these days (2018) it has more than eleven thousand citations.

In 1972, A. Gierer and H. Meinhardt published a paper [20], where they tried to solve the same question as Turing – how it happens that almost homogeneous tissue develops into a heterogeneous tissue structure. Similarly to Turing, they considered two reacting and diffusing chemicals – activator and inhibitor and proposed an idea of short-range activation and long-range inhibition, it means that the structures can appear only in a system where the diffusion of inhibitor is much faster than the diffusion of activator. Their model had a form

$$\frac{\partial a}{\partial t} = \frac{\rho a^2}{h} - \mu a + D_a \frac{\partial^2 a}{\partial x^2},$$
$$\frac{\partial h}{\partial t} = \rho a^2 - \nu h + D_h \frac{\partial^2 a}{\partial x^2}.$$

In addition, the turnover of an inhibitor must be more rapid than the turnover of activator, i.e. $\nu > \mu$, [39].

The theoretical forecasts were verified against the experimental results of hydra regeneration. In 1994, Gierer and Meinhardt reviewed the application of this system to biological pattern formation of complex structures.

In 1979, Schnackenberg published his simple model based on a reaction

$$\mathrm{X} \underset{\mathrm{k}_{-1}}{\overset{\mathrm{k}_1}{\rightleftharpoons}} \mathrm{A}, \quad \mathrm{B} \xrightarrow{\mathrm{k}_1} \mathrm{Y}, \quad 2\,\mathrm{X} + \mathrm{Y} \xrightarrow{\mathrm{k}_3} 3\,\mathrm{X}.$$

A system with this reaction is called Schnackenberg system and due to its simplicity it is extensively explored and often used as a reference example in many papers. It will of course appear also in this dissertation thesis in Section 5.2.

The Thomas model [51] from 1976 is of a so-called type substrate depletion and was designed to model a real chemical reaction between the substrate oxygen and uric acid reacting in presence of enzyme urinase. The outcomes were experimentally verified [51].

Legyel and Epstein proposed in their paper [34] from 1990 a model to simulate the behavior of CIMA reaction (chlorine/iodicide/mallonic acid system). The theoretical forecasts were for CIMA reaction verified experimentally, the modeled and observed 1D patterns can be seen in [34]. Another example of patterns in CIMA reaction is in Fig. 1.1.

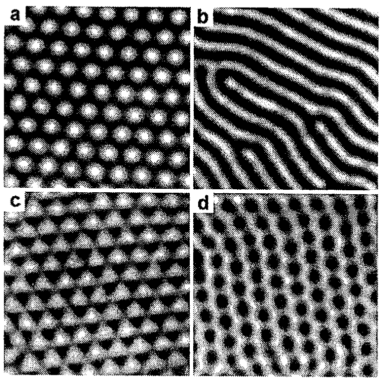


Figure 1.1: Patterns in CIMA reaction, [25]

The above mentioned papers have had a very strong influence on research concerning pattern formation in reaction-systems. There are many more interesting and excellent papers related to this topic. For example – the pattern formation on animal coat [56] has been already mentioned, other applications are in ecology [24], control of infectious diseases [50], carcinogenesis, angiogenesis [43], morphogenesis [53] and even social sciences [49]. The pattern formation has been studied for systems with receptors [37], on a growing domain [6], manifolds, etc. This shows the generality of this concept and importance for the understanding of fundamental biological processes.

1.2 Motivation, methods and recent results for reaction-diffusion systems with unilateral terms

The main interest of this dissertation thesis are reaction-diffusion equations with unilateral source and sink, further referred to as unilateral terms. As a motivation for this equations, we can use the king cheetah coat pattern. The chemical substance responsible for a black color on its coat is called melanin. The irregularity of its pattern is hard to explain using classical models like e.g. Schnackenberg. One proposal how to describe this pattern was written in [56]. The system has a certain feedback, which prohibits the decrease of level of chemicals under some level. This feedback can be modeled by unilateral terms in the equations, i.e. by the terms of a type $(u-\overline{u})^-$, where u is an actual concentration of given chemical, $\overline{u}$ is a steady-state concentration and minus is here for the negative part. In this dissertation thesis we will consider a more general models, where the feedback reacts also on deviations of chemical concentration in opposite direction. This is expressed by the terms of a type $(u-\overline{u})^+$ in the equations.

It is necessary to emphasize that even though there is a lot of chemical processes involved in the formation of coat pattern during embryogenesis, it is supposed that there are two significant reactions which in the end determine the shape of pattern.

The application of unilateral terms are not limited to king cheetah coat patterning. Another application can be in ecology, where are two dominant species and a farmer. The farmer does not affect the diffusion or interaction between these two species, but controls a part of the area of the ecosystem a level of one species. If there is a lack of it, he plants it. If there is an excess of it, he harvests it. The main question is the concentration distribution of these two species in the whole ecosystem – the results in the paper [56] suggest that the influence of unilateral terms on the final pattern is significant, and numerical results in Section 5 of this dissertation thesis this suggestion supports as well.

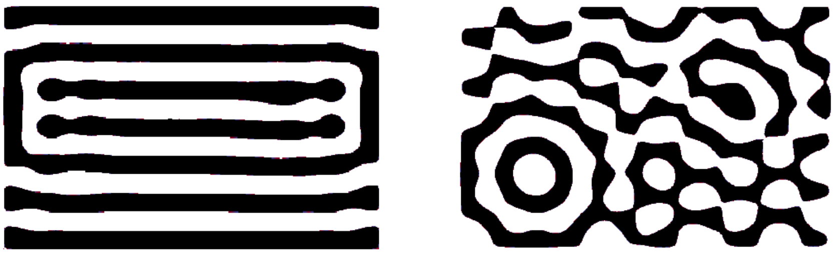


Figure 1.2: Pattern in a system with Schnackenberg reaction and without unilateral regulations on the left, with unilateral regulations on the right. Both systems had the same diffusion paramaters and Neumann b.c.

It will be seen later that the conditions on the so-called diffusion driven instability, for the first time described by Turing in [53], excludes the systems having the energy functional, and this means a use of the variational techniques. One of the tricks in a study of a bifurcation in these systems is to rewrite them as one operator equation now with symmetric linear operator perturbed by positively homogeneous potential operator and a smooth perturbation. The idea of reduction to one equation is not new and was already introduced in the paper [31].

In some particular cases this equation has a potential (energy functional), which allows to use the methods of Calculus of Variations, and Section 3.2 is devoted to abstract results in this field. The heuristic considerations can also predict the shape of pattern [38], but it is not the subject of this dissertation thesis. When the sources have an infinite strength, the whole problem can be modeled by the systems with obstacles. In particular, this was of an interest of two recent papers [14], [5].

Another method is in a use of Implicit Function Theorem, which allows to treat a very general class of reaction-diffusion systems. These methods have already been applied to standard reaction-diffusion systems (e.g. [42]). We will present a generalization of Crandall-Rabinowitz

Theorem for equations perturbed with lipschitz continuous operator, and this generalization will be consequently applied to our particular systems.

The existence global bifurcation points will be proved by using a Topological Degree theory. This is a content of Section 3.3, and Theorem 8 is a main result of this Section. There are recent papers [30], [33], [27], [14], [13], [16], concerned with the reaction-diffusion systems with obstacles. There is also a paper [10], which contains the bifurcation results for the problems with the terms of a type u^+, u^-, v^+, v^-.

The majority of the papers mentioned above is concerned with the reaction-diffusion systems with unilateral obstacles. It will be discussed later that these obstacles are not suitable for describing the regulatory mechanisms in biological systems. Therefore this dissertation thesis is purely focused on systems containing the terms with a negative and positive parts of concentrations, which are more reliable.

1.3 Structure of the dissertation thesis

The dissertation thesis is organized into 7 chapters as follows:

1. *Introduction*: Introduces the reader to the topic and gives a basic overview of reaction-diffusion systems with unilateral sources. It also explains the main contributions of this dissertation thesis. The results are less formal here.

2. *Abstract Formulation*: Introduces the reader to an abstract formulation of stationary reaction-diffusion equations and explains all necessary steps in the reduction of this system to one operator equation. The conditions under which the one operator equation has the potential are stated.

3. *General results about positively homogeneous problems*: Together with the subsequent chapter constitutes the core of this dissertation thesis. It mostly contains new results about eigenvalues and bifurcation of equations with positively homogeneous operators on an abstract Hilbert space.

   The largest eigenvalue of equation with positively homogeneous operator is characterized by a variational formula. The existence of bifurcation points of equations with positively homogeneous operators is proved by using three basic methods - variational approach, topological degree and implicit function theorem. Each method has different assumptions and gives different conclusions about the bifurcation points.

4. *Application to reaction-diffusion equations*: General results from the previous chapter are applied to reaction-diffusion systems with unilateral sources, giving the existence of bifurcation points for these problems in an area, where bifurcation points of the problem without unilateral terms are not present. Systems with Dirichlet/mixed boundary conditions and Neumann boundary conditions are studied.

5. *Numerical results for real-world system*: The achieved results are demonstrated on selected specific system from the literature, patterns are found numerically.

6. *Conclusions*: Contains the summary of the results and contributions of this dissertation thesis, and concluding remarks.

7. *Appendix*: Covers the necessary minimal theoretical background for this dissertation thesis.

1.4 Systems of two reaction-diffusion equations

The pattern formation will be studied in systems with two reacting and diffusing components. The components can be e.g. chemical compounds, populations or granular materials, depending on a specific system. The system is described by two coupled partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + f(u,v) \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + g(u,v) \end{aligned} \qquad \text{in } \Omega \times [0,\infty), \tag{1.1}$$

with the boundary and initial conditions

$$\begin{aligned} &\frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial \vec{n}} = 0 \quad \text{on } \Gamma_N, \\ &u = \overline{u}, \quad v = \overline{v} \quad \text{on } \Gamma_D, \end{aligned} \tag{1.2}$$

$$u(0) = u_0, \quad v(0) = v_0, \tag{1.3}$$

where $\Omega \subset \mathbb{R}$ is a bounded domain with Lipschitz boundary, $d_1, d_2 \in \mathbb{R}$ are diffusion coefficients, $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are nonlinear real functions, $u_0, v_0 : \Omega \to \mathbb{R}$ are real functions and $\overline{u}, \overline{v}$ are constants. The sets Γ_D and Γ_N are pairwise disjoint subsets of the boundary, Γ_N is open, and $\Gamma_D \cup \Gamma_N = \partial\Omega$. In chemical systems the variables u, v represent the concentrations of components, in population models the density of population, etc. The functions f, g represent the reaction kinetics of the system. The explicit form of these functions can be found e.g. from Law of Mass Action, by heuristic considerations, or from experiments. The kinetic functions f, g are supposed to satisfy

$$f(\overline{u}, \overline{v}) = 0, \quad g(\overline{u}, \overline{v}) = 0, \tag{1.4}$$

in order to ensure that $(\overline{u}, \overline{v})$ is a (homogeneous) solution of (1.1), (1.2).

Doing a formal Taylor series expansion of f, g at the point $(\overline{u}, \overline{v})$ and neglecting higher-order terms lead to the so-called linearized system

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + b_{11}(u - \overline{u}) + b_{12}(v - \overline{v}), \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + b_{21}(u - \overline{u}) + b_{22}(v - \overline{v}), \end{aligned} \tag{1.5}$$

where

$$b_{11} = \frac{\partial f}{\partial u}(\overline{u}, \overline{v}), \quad b_{12} = \frac{\partial f}{\partial v}(\overline{u}, \overline{v}), \quad b_{21} = \frac{\partial g}{\partial u}(\overline{u}, \overline{v}), \quad b_{22} = \frac{\partial g}{\partial v}(\overline{u}, \overline{v}). \tag{1.6}$$

An Ansatz $(u(x,t), v(x,t)) = (\exp(\lambda t)\tilde{u}(x) + \overline{u}, \exp(\lambda t)\tilde{v}(x) + \overline{v})$ gives an eigenvalue problem

$$\begin{aligned} \lambda \tilde{u} &= d_1 \Delta \tilde{u} + b_{11} \tilde{u} + b_{12} \tilde{v}, \\ \lambda \tilde{v} &= d_2 \Delta \tilde{v} + b_{21} \tilde{u} + b_{22} \tilde{v}. \end{aligned} \tag{1.7}$$

Definition 1. *If there exists $\xi < 0$ such that all eigenvalues λ of (1.7), (1.2) satisfy $Re(\lambda) < \xi$, the solution $(\overline{u}, \overline{v})$ of (1.1), (1.2) is called linearly stable. If there exists at least one eigenvalue with the real part being positive, the solution $(\overline{u}, \overline{v})$ is called linearly unstable.*

Definition 2. *The system (1.1), (1.2) exhibits the diffusion driven instability (DDI) if for $d_1 = d_2 = 0$ its solution $(\overline{u}, \overline{v})$ is linearly stable as a solution of a system of ODE's and for diffusion coefficients from a certain nonempty set $D_U \subset \mathbb{R}^2_+$ its solution $(\overline{u}, \overline{v})$ is linearly unstable.*

The detailed linear stability analysis of (1.1), (1.2) can be found in [40], Chap. 2.3. It shows that DDI is not present for an arbitrary system. The constants b_{ij} defined in (1.6) have to satisfy the sign conditions

$$\begin{aligned} &b_{11} > 0, \quad b_{22} < 0, \quad b_{12} b_{21} < 0, \\ &b_{11} b_{22} - b_{12} b_{21} > 0, \quad b_{11} + b_{22} < 0. \end{aligned} \tag{1.8}$$

If $b_{21} < 0$, the system (1.1) is of the so-called substrate-depletion type, and if $b_{12} < 0$, the system (1.1) is of the type activator-inhibitor.

The conditions (1.8) explicitly exclude the case $d_2 = d_1$ and in general the diffusion coefficients must be significantly different. This is used in a criticism of these systems, because some authors consider the assumption on significantly different diffusion coefficients as not biological. Therefore the question motivating this dissertation thesis has been whether there are models giving patterns for diffusion coefficients closer to the line $d_2 = d_1,\ d_1 \in \mathbb{R}^+$.

It is common for simplicity to shift the initial homogeneous steady state to zero, i.e. to relabel $u \equiv u - \overline{u}$, $v \equiv v - \overline{v}$. Since the pattern is a stationary solution of a reaction-diffusion system, it has to satisfy after this shift the equations

$$\begin{aligned} d_1 \Delta u + b_{11} u + b_{12} v + n_1(u,v) = 0 \\ d_2 \Delta v + b_{21} u + b_{22} v + n_2(u,v) = 0 \end{aligned} \quad \text{in } \Omega, \tag{1.9}$$

$$\begin{aligned} &\frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial \vec{n}} = 0 \ \text{ on } \Gamma_N, \\ &u = 0, \quad v = 0 \ \text{ on } \Gamma_D, \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} n_1(u,v) &:= f(u,v) - b_{11} u - b_{12} v, \\ n_2(u,v) &:= g(u,v) - b_{21} u - b_{22} v. \end{aligned}$$

The functions n_1, n_2, if smooth enough, satisfy

$$n_1(0,0) = n_2(0,0) = 0, \quad n_1'(0,0) = n_2'(0,0) = 0, \tag{1.11}$$

where the prime denotes the derivative. The respective linearized problem is

$$\begin{aligned} d_1 \Delta u + b_{11} u + b_{12} v = 0 \\ d_2 \Delta v + b_{21} u + b_{22} v = 0 \end{aligned} \quad \text{in } \Omega, \tag{1.12}$$

with the b.c. (1.10).

1.5 Systems of reaction-diffusion equations with unilateral sources

The system of reaction-diffusion equations with unilateral terms is

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + b_{11} u + b_{12} v + n_1(u,v) \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + b_{21} u + b_{22} v + n_2(u,v) + \hat{g}_-(x, v^-) - \hat{g}_+(x, v^+) \end{aligned} \quad \text{in } \Omega, \tag{1.13}$$

with the boundary conditions (1.10) and certain initial conditions. The symbols v^+ and v^- denote the positive and the negative part of the function v respectively and $\hat{g}_-, \hat{g}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ are nonlinear real functions representing the source and the sink respectively. To save some space we will often use notation $\hat{g}_\pm$ for $\hat{g}_+, \hat{g}_-$ and to emphasize the spatial dependence of $\hat{g}_\pm$ we write $\hat{g}_\pm(x, v^\pm)$. It will be supposed

$$\hat{g}_\pm(x, 0) \equiv 0, \text{ for all } x \in \Omega,$$

in order to be $(0,0)$ a solution of (1.13), (1.10).

The unilateral sources should work in a following way. If the value of v decreases below zero at a point in the area where the unilateral source $\hat{g}_-$ is present, the environment activates the mechanisms, which, in accordance with the principle of homeostasis, start to increase the value of v at this point. If the value of v reaches the steady-state level at this point, the mechanism deactivates. And vice versa for the values of v above zero in the area where the source $\hat{g}_+$ is present. Therefore it is natural to assume that $\hat{g}_\pm(x, v^\pm)$ is nonnegative at any point x and for any function v.

These sources are a weaker form of strict obstacles, which in the respective areas does not allow the value of v to increase above zero or decrease below zero. It is a very strict condition and in biology unreliable. The environment has often a limited possibility to regulate an unfavorable concentrations of chemicals, and the unilateral sources can be used to describe it more reliably, for example by using a saturation function

$$\hat{g}_\pm(x,\xi) = \frac{s_\pm(x)\xi}{1+\xi},$$

with $s_\pm : \Omega \to \mathbb{R}_+$.

The assumption of positivity of $\hat{g}_\pm$ mentioned above will be weaken in this dissertation thesis to an assumption of positivity of $s_\pm$, see (2.21).

This dissertation thesis is mainly interested in stationary solutions of the above systems, hence, we will work with the problem

$$\begin{aligned} &d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v) = 0 \\ &d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v) + \hat{g}_-(x,v^-) - \hat{g}_+(x,v^+) = 0 \end{aligned} \quad \text{in } \Omega, \tag{1.14}$$

having the b.c. (1.10).

The majority of results in the dissertation thesis is for systems having the boundary conditions (1.10), however, in Sections 2.4, 4.5 some minor results for systems with unilateral sources on the boundary will be placed.

1.6 Main contributions of this dissertation thesis

1.6.1 Preliminary

The main task of this dissertation thesis is to prove the existence of bifurcation points of the problem (1.14), (1.10). Stationary states exist for the diffusion parameters close to a bifurcation point and for this reason the location of bifurcation points is valuable information.

Before summarizing the main results of this dissertation thesisit is necessary to introduce some definitions. It is useful to remind here that Ω is the bounded domain in $\mathbb{R}^m$ with the Lipschitz boundary.

The negative part of $v \in W^{1,2}(\Omega)$ is defined by

$$v^-(x) = \begin{cases} 0 & \text{for a.a. } x \in \Omega \text{ for which } v(x) > 0 \\ -v(x) & \text{for a.a. } x \in \Omega \text{ for which } v(x) \le 0. \end{cases}$$

and the positive part by

$$v^+(x) = \frac{1}{2}(|v(x)| - v^-(x)) \text{ for a.a. } x \in \Omega.$$

It is $|v|, v^-, v^+ \in W^{1,2}(\Omega)$ for any $v \in W^{1,2}(\Omega)$, see e.g. [62].

Definition 3. *For $\mu_{m-1}(\Gamma_D) > 0$, where μ_{m-1} is the $m-1$ dimensional Lebesgue measure, a space $W^{1,2}_D(\Omega)$ is defined by*

$$W^{1,2}_D(\Omega) := \left\{ v \in W^{1,2}(\Omega) \middle| \; v|_{\Gamma_D} = 0 \right\}.$$

Definition 4. *The couple* $(u,v) \in W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ *is called a weak solution of the system (1.14), (1.10) if it satisfies*

$$\int_\Omega d_1 \nabla u \cdot \nabla \varphi - b_{11} u\varphi - b_{12} v\varphi - n_1(u,v)\varphi \; dx = 0$$
$$\int_\Omega d_2 \nabla v \cdot \nabla \varphi - b_{21} u\varphi - b_{22} v\varphi - n_2(u,v)\varphi - \hat{g}_-(x,v^-)\varphi + \hat{g}_+(x,v^+)\varphi \; dx = 0,$$
$$\text{for all } \varphi \in W_0^{1,2}(\Omega). \tag{1.15}$$

By a solution of a system (1.14), (1.10) we will always mean a weak solution.

A prominent type of (1.14) is the so-called homogenized system

$$\begin{aligned} d_1\Delta u + b_{11}u + b_{12}v &= 0 \\ d_2\Delta v + b_{21}u + b_{22}v + s_-(x)v^- - s_+(x)v^+ &= 0 \end{aligned} \qquad \text{in } \Omega, \tag{1.16}$$

with the b.c. (1.10), where

$$s_\pm(x) := \frac{\partial \hat{g}_\pm}{\partial \xi}(x,\xi)\bigg|_{\xi=0}.$$

The weak formulation of (1.16) , (1.10) is defined by taking $n_1, n_2 \equiv 0$, $\hat{g}_\pm(x,v^\pm) := s_\pm(x)v^\pm$ in (1.15). The weak formulation of (1.12), (1.10) and (1.9), (1.10) is defined by taking $\hat{g}_\pm(x,v^\pm) \equiv 0$, $n_1, n_2 \equiv 0$ and $\hat{g}_\pm(x,v^\pm) \equiv 0$ in (1.15) respectively.

The following definition (cf. Definitions 15, 16) will appear throughout this dissertation thesis.

Definition 5. *Let* $d_1 \in \mathbb{R}$ *be fixed. A point* $d_2 \in \mathbb{R}$ *is a critical point of the problem (1.16), (1.10) with fixed* d_1 *if there exists a solution* $(u,v) \in W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$, $(u,v) \neq 0$ *of this system.*

Let $d_1 \in \mathbb{R}$ *be fixed. A point* $d_2 \in \mathbb{R}$ *is a bifurcation point of (1.14), (1.10) with fixed* d_1 *if in any neighborhood of* $(d_2, 0, 0)$ *in* $\mathbb{R} \times W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ *there exists a triple* $(\tilde{d}_2, u, v)$ *such that* $(u,v) \neq 0$ *is a solution of (1.14), (1.10) with* d_2 *replaced by* $\tilde{d}_2$.

Remark 1. *Although there are two parameters in the equations (1.14), (1.16), in the applications* d_1 *will be always fixed and* d_2 *a bifurcation parameter. The presence of only one bifurcation parameter will allow us to reduce the analysis of the system to an analysis of one equation with one parameter. Since it may not be clear on a first sight how to apply Definitions 15, 16 in our systems, we decided to write it separately as Definition 5.*

Remark 2. *Since every bifurcation point of (1.14), (1.10) is simultaneously a critical point of (1.16), (1.10), as will be proved later, the existence and location of critical points is valuable information.*

1.6.2 Outcomes of this dissertation thesis

The main theoretical results concerning reaction-diffusion systems are contained in Theorems 11–21 in Chapter 4. We will always assume here that b_{ij} satisfy (1.8). However, in order to explain the main ideas of these results, we will not place here the exact assumptions and mention only the crucial ones (e.g. we will omit the growth conditions (2.4), (2.7), smoothness of the domain, etc.). The following remark contains information which is crucial for understanding of the main outcomes of this dissertation thesis.

Remark 3. *It is well-known that for any* $d_1 \in \mathbb{R}$ *all respective* $d_2 \in \mathbb{R}$ *for which the problem (1.12), (1.10) has a nontrivial solution can be expressed in a form*

$$d_2 = \frac{1}{\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right),$$

where $k \in \mathbb{N}$ and κ_k are the eigenvalues of the Laplacian with the b.c. (1.10) and $d_1 \neq b_{11}/\kappa_k$. Let us introduce a set C as

$$C = \bigcup_{k=1}^{\infty} C_k, \quad C_k = \left\{ d = (d_1, d_2) \in \mathbb{R}^2_+ \;\middle|\; d_2 := \frac{1}{\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right),\ d_1 \neq \frac{b_{11}}{\kappa_k} \text{ for all } k \in \mathbb{N} \right\}.$$

If we plot this set in $\mathbb{R}^2_+$, with d_1 on horizontal axis and d_2 on vertical axis, we would obtain infinitely many hyperbola segments with asymptotes $y_k = b_{11}/\kappa_k$. If $\kappa_k \neq \kappa_j$, the hyperbolas C_k and C_j intersect in exactly one point [15].

There are no segments to the right from the vertical line $d_1 = b_{11}/\kappa_1$. At the line $d_1 = 0$ the ends of these segments accumulates at the point $d_2 = 0$.

The sketch can be found in Fig. 1.3. The domain to the left from the envelope C_E of these hyperbolas is called domain of instability, and denoted by D_U and to the right from the envelope of these hyperbolas is called domain of stability, denoted by D_S. By definition there is no critical point of (1.12), (1.10) with fixed $d_1 > 0$ in D_S.

More details will be given in Section 2.5.

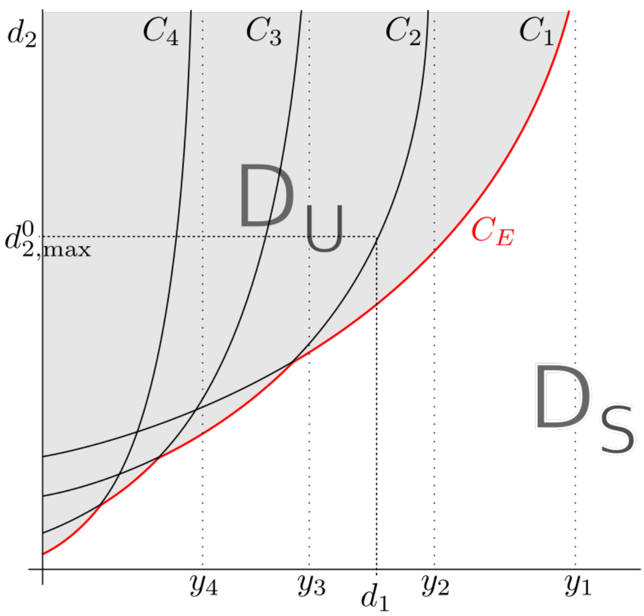


Figure 1.3: The black lines are the sets $C_1, \cdots, C_4$, vertical lines $y_1 \cdots y_4$ are asymptotes of the corresponding hyperbolas. Grey background marks the domain of instability, white background the domain of stability. Red line marks the envelope of the hyperbolas. The point (d_1, d_2) is lying on the hyperbola C_2 and therefore d_2 is a critical point of (1.12), (1.10) with fixed d_1.

Systems (1.16), (1.10) and (1.14), (1.10) with Dirichlet or mixed b.c. Assume that $\text{meas}_{m-1}(\Gamma_D) > 0$, then the systems (1.16), (1.10) and (1.14), (1.10) have Dirichlet or mixed boundary conditions. Denote $d^0_{2,\max}$ the largest critical point of (1.12), (1.10) with a fixed $d_1 \in (0, y_1) \setminus \{y_j \mid j = 2, 3, \cdots\}$. The number $d^0_{2,\max}$ is always lying on a hyperbola segment.

The situation from Theorem 11 on pg. 63 is sketched in Fig. 1.4. For any $d_1 \in (y_2, y_1)$ this theorem gives the existence of the largest critical points d_2^m of (1.16), (1.10) with fixed d_1, located in the dark gray area in D_S, provided that $s_\pm$ have nonzero supports. Furthermore, for any $d_1 \in (0, y_2) \setminus \{y_3, ...\}$ the largest critical points d_2^m of (1.16), (1.10) with fixed d_1 are bounded from above by $d^0_{2,\max}$ and from below by the r.h.s. of (4.3). These bounds are sketched by the dashed lines in Fig. 1.4. All of these critical points are characterized by the variational formula (4.4). The main assumption is that $\|s_\pm\|_{L^\infty}$ are not very large, or for some particular values of d_1 they can even have arbitrary size − this is a content of Corollary 4 and Lemma 20. As $s_\pm \to 0$, the dashed lines merge with the hyperbolas and dark gray area shrinks to the empty set.

Theorem 12 on pg. 64 gives that the critical points found in Theorem 11 can be bifurcation points under the assumption on skew-symmetry of the reaction kinetics, see also condition (2.31) on pg. 20.

Theorem 13 on pg. 65 says that for systems with $\|s_\pm\|_{L^\infty}$ sufficiently small there are global bifurcation points of (1.14), (1.10) in the dark gray area, which is a subset of D_S. Furthermore, for any $d_1 \in (0, y_2)\backslash\{y_3, \cdots\}$, under some additional assumptions, there is a global bifurcation point of (1.14), (1.10) with fixed d_1 in a neighborhood below d_2. Theorem 19 is an analogue of Theorems 11–13 for systems with unilateral terms on the boundary.

Finally, Theorem 14 gives for any d_1 the existence of two distinct critical points d_2^+, d_2^- of (1.16), (1.10) with fixed d_1 such that $(d_1, d_2^+), (d_1, d_2^-) \in D_S$. The main assumptions are that $\|s_\pm\|_{L^\infty}$ are sufficiently small, and the system (1.12), (1.10) with $(d_1, d_2) \in C_E$ must have unique solution up to multiples. Moreover, $d_2^\pm \to d_2$ as $\|s_\pm\|_{L^\infty} \to 0$. These critical points are also bifurcation points of (1.14), (1.10) with fixed d_1.

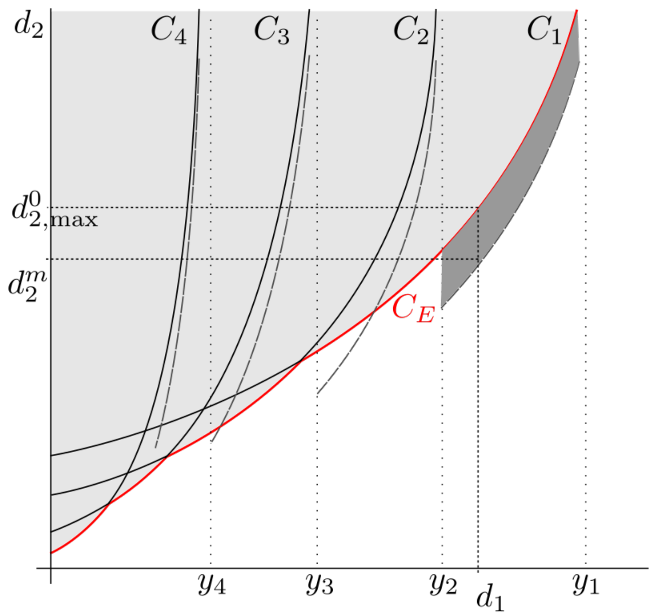


Figure 1.4: Points generating black lines are positive critical points of system (1.9), dotted lines are lower bounds on critical points of (1.16), (1.10) and the dark grey area contains the bifurcation points of (1.14), (1.10) located in D_S, cf. also Fig. 1.3

Systems (1.16) and (1.14) with Neumann b.c. The situation is more complicated for the systems (1.14) and (1.16), both with Neumann boundary conditions. This occurs when $\text{meas}_{m-1}(\Gamma_D) = 0$.

If the unilateral terms are sufficiently large, which is expressed by the condition (4.16) on pg. 4.16, and under some restrictions on the area size of the source and sink Theorem 15 gives the existence of $d_1^0 > 0$ such that for any $d_1 > d_1^0$ there exists a critical point d_2^m of (1.16) with Neumann b.c. with fixed d_1, provided that $\|s_\pm\|_{L^1}$ are sufficiently large. Clearly $(d_1, d_2^m) \in D_S$. Theorem 16 says that the branch of critical points with $d_1 > d_1^0$ is bounded from below and above by some constants C_m, C_M, independent of d_1. And finally, for skew symmetric systems, see (2.31), all of these critical points are bifurcation points of (1.14) with Neumann boundary conditions and fixed d_1, see Theorem 17. Again, similar conclusions can be done for systems with unilateral sources on the boundary, which is a content of Theorem 20 . The bifurcation results are only local, i.e. the existence of branches of bifurcating solutions is known only for a certain neighborhood of zero. The situation is sketched in the Fig. 1.5.

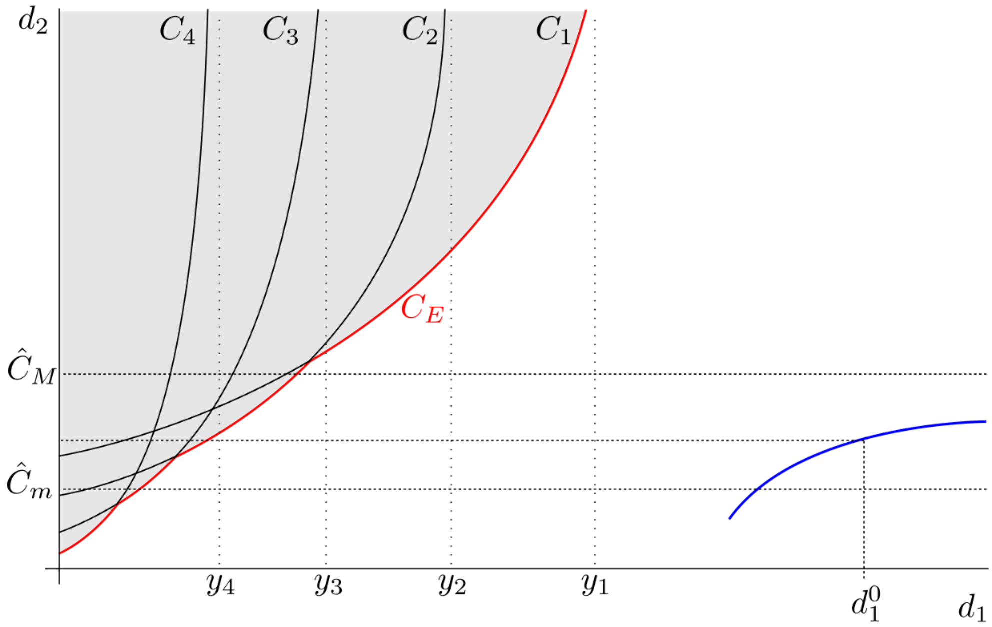


Figure 1.5: —Sketch of hyperbola segments for the problem with unilateral terms and Neumann boundary conditions. Besides the solutions in D_U which are above the hyperbolas C_k, there is a new line of critical points in D_S, bounded from above and below by constants $\hat{C}_M$ and C_m respectively. For skew-symmetric systems these points are bifurcation points.

Theorem 18 is an analogue of Theorem 14 for systems (1.14) with Neumann b.c.

is an analogue of Theorems 15–13 for systems with unilateral terms on the boundary.

Theorem 21 is significantly distinct in the assumptions in comparison to the previous ones. It assumes the $C^{1,1}$ smoothness of the domain Ω and therefore the solution of (1.14) with Neumann b.c. has a higher regularity and it allows on the other hands to relax some other assumptions (e.g. it is not necessary to assume the so-called growth conditions (2.4), (2.7)). The assertions of this Theorem are similar to the ones in Theorem 18.

Numerical results Theorems 11–21 are giving the existence and location of bifurcation points of the problem (1.14), (1.10). However, they are not giving a qualitative answer about the corresponding bifurcating solutions. In particular, it is not certain whether the solutions bifurcating from zero are attracting, whether it is possible to find them numerically and how does they look like. To partially fill this gap we did several numerical experiments on a problem (1.14) having a Schnackenberg kinetics and (homogeneous) Dirichlet b.c. and (homogeneous) Neumann b.c. To have a comparison, we did at first several numerical experiments on a system (1.9), (1.10) with Schnackenberg kinetics. The value of d_1 was fixed to a selected value. The norm of solutions was decreasing to zero as (d_1, d_2) has been approaching C_E, and the shape of solutions was and more similar to the solution of linear problem (1.12), (1.10), which suggests that we were close to a bifurcation point. The most important conclusion is - the solver, chosen numeric scheme and its implementation seems to give correct results in this case. Then we did some experiments on the system with unilateral terms. Again, we fixed the value of d_1 and varied the value of d_2. The norm of these solutions was decreasing to zero in observed experiments as the parameter d_2 was approaching certain value, so we feel confident enough to conclude that we were able to numerically find these bifurcating solutions. The shape of solutions is influenced by unilateral terms. See also Fig. 5.4.

Unfortunately, we have not succeed in finding the solutions with $d_1 > d_1^0$ from Theorem 17.

In conclusion, we were able to find solutions in D_S for selected values $d_1 < y_1$ and locate a bifurcation point – as the value of d_2 was closer to bifurcation point, the $W^{1,2}(\Omega)$ norm of the solutions was mostly decreasing, which suggest that it should be bifurcating solutions predicted by Theorem 14 for a problem with Dirichlet b.c., and by Theorem 18 for a problem with Neumann b.c. and the resulting shape of solutions, and consequently patterns, are strongly affected by the presence of unilateral terms.

Chapter **2**

Abstract formulation of reaction-diffusion systems with unilateral terms

It will be necessary to rigorously formulate the problem before doing the analysis of reaction-diffusion systems with unilateral terms. For this purpose the weak formulation of the problem (1.14), (1.10) has been introduced in Definition 4. A usual procedure is to rewrite the weak formulation of the problem (1.14), (1.10) as a system of two operator equations on $W^{1,2}(\Omega)$. This will be done in Section 2.2. This formulation significantly simplifies the analysis of this system. However, variational methods used for an analysis of nonlinear problems require certain symmetry of the problem. Therefore the next step is a fixing of d_1 and reducing the system of two coupled operator equation to a single equation with a symmetric linear compact operator. This will be done in the Sections 2.2.2, 2.3.2.

Although it is possible to do an analysis of the system (1.14) with Dirichlet, mixed and Neumann boundary conditions at once, for clarity the systems with Dirichlet and mixed b.c., and Neumann b.c. will be handled separately. However, before doing all of the described steps it will be necessary to place some assumptions and definitions.

2.1 Basic definitions and assumptions

As a first step, we will fix some assumptions and notations. We will be using a universal symbol C for various constants. The set $\Omega \subset \mathbb{R}^m$ is a bounded domain with a Lipschitz boundary, $\Gamma_N, \Gamma_D \subset \partial\Omega$ are relatively open and disjoint, and $\partial\Omega \backslash (\Gamma_N \cup \Gamma_D)$ is having the $m-1$ dimensional Lebesgue measure zero. The elements $b_{ij} \in \mathbb{R}$ of the matrix $B \in \mathbb{R}^{2,2}$ satisfy

$$b_{11} > 0, \quad b_{22} < 0, \quad b_{12}b_{21} < 0, \tag{2.1}$$

$$b_{11}b_{22} - b_{12}b_{21} = \det B > 0, \quad b_{11} + b_{22} = \operatorname{Tr} B < 0. \tag{2.2}$$

The nonlinear functions $n_1, n_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy

$$n_i(0,0) = 0, \quad \partial_{x_i} n_j(0,0) = 0 \quad \text{for all } i,j \in \{1,2\}, \tag{2.3}$$

and the following growth conditions:

$$\begin{aligned} &\text{there exists } C > 0 \text{ such that:} \\ &|n_1(\chi,\xi)| + |n_2(\chi,\xi)| \le C(1 + |\xi|^{p-1} + |\chi|^{p-1}) \quad \text{for all } \xi, \chi \in \mathbb{R}, \end{aligned} \tag{2.4}$$

for some p bounded by

$$p > 2 \text{ for } m \le 2 \quad \text{or} \quad 2 < p < \frac{2m}{m-2} \text{ for } m > 2. \tag{2.5}$$

The functions $\hat{g}_-, \hat{g}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy

$$\hat{g}_+(x,0) = \hat{g}_-(x,0) = 0 \ \text{ for a.a. } x \in \Omega, \tag{2.6}$$

Carathéodory conditions, the growth conditions

$$\text{there exists } C > 0 \text{ such that:}$$
$$|\hat{g}_-(x,\xi)| + |\hat{g}_+(x,\xi)| \le C(1+|\xi|^{p-1}) \quad \text{for a.a. } x \in \Omega, \quad \text{for all } \xi \in \mathbb{R}, \text{ for some } p \text{ from (2.5)}, \tag{2.7}$$

and have the derivative at zero w.r.t. real variable. Let us define the functions $s_+, s_- : \Omega \to \mathbb{R}$ as

$$s_\pm(x) := \frac{\partial \hat{g}_\pm}{\partial \xi}(x,\xi)\bigg|_{\xi=0}. \tag{2.8}$$

Moreover, we will assume

$$s_\pm(x) \in L^\infty(\Omega). \tag{2.9}$$

All of these assumptions will appear throughout this dissertation thesis.

2.2 System with Dirichlet or mixed boundary conditions

This section concerns with the weak formulation of the reaction-diffusion systems with the unilateral terms and with Dirichlet/mixed or Neumann b.c. Among other things, it contains several simple lemmas which will be used to prove the main theorems in Section 4.

2.2.1 Weak and operator formulation of the problem

Let

$$\mu_{m-1}(\Gamma_D) > 0, \tag{2.10}$$

in the whole Section (2.2). If $\mu_{m-1}(\Gamma_N) = 0$, then (1.14), (1.10) is a system with Dirichlet boundary conditions, otherwise it is a system with mixed boundary conditions. We will extensively use here the space $W^{1,2}_D(\Omega)$ from Definition 3 on pg. 7.

To simplify some calculations, we will consider a different scalar product on $W^{1,2}_D(\Omega)$.

Definition 6. *The space $W^{1,2}_D(\Omega)$ will be equipped with the scalar product and norm*

$$\langle v, \varphi \rangle = \int_\Omega \nabla v \cdot \nabla \varphi, \quad \|v\|_{1,2} = \left(\int_\Omega |\nabla v|^2\right)^{\frac{1}{2}} \quad \text{for all } v, \varphi \in W^{1,2}_D(\Omega). \tag{2.11}$$

Under the assumption (2.10) the norm (2.11) is equivalent to the standard norm on $W^{1,2}(\Omega)$, see also formula (7.1) in Appendix. Let us remind here Definition 19 from Appendix of an (algebraic) multiplicity of an eigenvalue of a compact operator. As usual, if the multiplicity of an eigenvalue is equal to one, we will call the eigenvalue simple. Since we are going to work with symmetric linear operators, the algebraic multiplicity of an eigenvalue is equal to its geometric multiplicity, and therefore it is not necessary to distinguish between them.

Notation 1. *The eigenvalues of the Laplacian with Dirichlet/mixed b.c. will be denoted by κ_k, and will be ordered as*

$$0 < \kappa_1 < \kappa_2 \le \cdots \to \infty, \tag{2.12}$$

see also Remark 28 and formula (7.5) in Appendix.

An orthonormal base $\{e_k\}_{k\in\mathbb{N}}$ is chosen in a way that for any $k \in \mathbb{N}$, e_k is an eigenfunction corresponding to κ_k. The eigenvalue κ_1 is simple and the eigenfunction e_1 does not change its sign in Ω, see [22].

Lemma 1. *The operator* $A : W_D^{1,2}(\Omega) \to W_D^{1,2}(\Omega)$ *defined by*

$$\langle Av, \varphi \rangle = \int_\Omega v\varphi \ dx, \qquad \text{for all } v, \varphi \in W_D^{1,2}(\Omega),$$

is a symmetric linear compact operator. Moreover

$$\sigma(A) = \left\{ \kappa_k^{-1} \,\middle|\, k \in \mathbb{N} \right\} \cup \{0\}, \tag{2.13}$$

and v *is an eigenfunction of the Laplacian corresponding to the eigenvalue* κ_k *if and only if* v *is an eigenfunction of* A *corresponding to the eigenvalue* κ_k^{-1}*. The largest eigenvalue of the operator* A *is simple and a corresponding eigenfunction does not change sign in* Ω*. Furthermore,* $\{\kappa_k^{-1}\}_{k\in\mathbb{N}}$ *is a decreasing sequence with* $\lim_{k\to\infty} \kappa_k^{-1} = 0$ *(see (2.12)) and zero is the only accumulation point of* $\sigma(A)$*.*

Proof. First step is to show that A is well-defined. We use Hölder inequality to find

$$\langle Av, \varphi \rangle = \int_\Omega v\varphi \ \mathrm{dx} \le \|v\|_2 \|\varphi\|_2 < \infty,$$

and according to Riesz Representation Theorem, $\varphi \to \langle Av, \varphi \rangle$ is for any $v \in W_D^{1,2}(\Omega)$ well-defined linear functional and can be represented by a vector $w \in \mathbb{H}, w = Av$. Thus A is well-defined. The linearity follows directly from the definition. Let $v_n \rightharpoonup v$. Since $W_D^{1,2}(\Omega) \hookrightarrow^c L^2(\Omega)$, the sequence v_n converges strongly to v in $L^2(\Omega)$. Then

$$\|Av_n - Av\| = \sup_{\substack{\varphi \in W_D^{1,2}(\Omega) \\ \|\varphi\| \le 1}} \langle Av_n - Av, \varphi \rangle = \sup_{\substack{\varphi \in W_D^{1,2}(\Omega) \\ \|\varphi\| \le 1}} \int_\Omega (v_n - v)\varphi \le \|v_n - v\|_2 \to 0,$$

it means A is compact. Let κ be an eigenvalue of Laplacian and v be a corresponding eigenfunction. Then

$$\langle Av, \varphi \rangle = \int_\Omega v\varphi \ \mathrm{dx} = \kappa^{-1} \int_\Omega \nabla v \cdot \nabla \varphi \ \mathrm{dx} = \kappa^{-1} \langle v, \varphi \rangle \quad \text{for all } \varphi \in W_D^{1,2}(\Omega), \tag{2.14}$$

which means κ^{-1} is an eigenvalue of A and v is a corresponding eigenfunction. Conversely, if κ^{-1} is an eigenvalue of A with an eigenvector v, then clearly from (2.14) a number κ is an eigenvalue of Laplacian and v is a corresponding eigenvector.

The largest eigenvalue of A is κ_k^{-1}, which is simple and the corresponding eigenvector with unit norm is e_1, which does not change its sign in Ω. Since κ_k is increasing sequence, κ_k^{-1} is a decreasing sequence. Operator A is compact therefore 0 is an accumulation point of the set of all eigenvalues. □

Now we formally define the operators

$$\langle N_1(u,v), \varphi \rangle = \int_\Omega n_1(u,v)\varphi \ \mathrm{dx} \quad \text{for all } u, v, \varphi \in W_D^{1,2}(\Omega), \tag{2.15}$$

$$\langle N_2(u,v), \varphi \rangle = \int_\Omega n_2(u,v)\varphi \ \mathrm{dx} \quad \text{for all } u, v, \varphi \in W_D^{1,2}(\Omega). \tag{2.16}$$

The basic properties of these operators are subject of the following lemma, let us remind that (2.3), (2.4) are supposed for n_1, n_2.

Lemma 2. *The operators* N_1, N_2 *are well-defined, continuous and compact operators from* $W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ *to* $W_D^{1,2}(\Omega)$*. Furthermore,*

$$\lim_{u,v\to 0} \frac{N_1(u,v)}{\|u\| + \|v\|} = 0, \qquad \lim_{u,v\to 0} \frac{N_2(u,v)}{\|u\| + \|v\|} = 0. \tag{2.17}$$

Proof. The first step is to show that $N_1 : W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega) \to W_D^{1,2}(\Omega)$ is a well defined operator. The growth condition (2.4), embedding of $W_D^{1,2}(\Omega)$ into $L^p(\Omega)$ for p given by (2.5) and Continuity of Nemyckii operator give that n_1 is a continuous operator from $W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ to $L^{p'}$. The Hölder inequality and the growth condition (2.4) give an estimate

$$\begin{aligned}\langle N_1(u,v),\varphi\rangle = \int_\Omega n_1(u,v)\varphi \text{ dx} \le \left(\int_\Omega |n_1(u,v)\ |^{p'}\right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} \le \\ \le C\,(1+\|u\|_{L^p}^p + \|v\|_{L^p}^p)^{\frac{1}{p'}} \le C(1+\|u\|+\|v\|)^{p-1}, \\ \text{for all } u,v,\varphi \in W_D^{1,2}(\Omega),\end{aligned} \tag{2.18}$$

where $p' = p/(p-1)$ and C is used for various constants. Riesz Representation Theorem and Continuity of Nemyckii operator gives that N_1 is a well-defined continuous operator form $W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega)$ to $W_D^{1,2}(\Omega)$.

Let $(u_n, v_n) \rightharpoonup (u,v)$ in $W_D^{1,2}(\Omega)$. The compact embedding $W_D^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for p given by (2.5) and continuity of Nemyckii operator n_1 give

$$\begin{aligned}\|N_1(u_n,v_n) - N_1(u,v)\| \le \max_{\substack{\varphi\in W^{1,2}(\Omega)\\ \|\varphi\|\le 1}} \left(\int_\Omega |n_1(u,v) - n_1(u_0,v_0)\|^{p'}\right)^{\frac{1}{p'}} \\ \le \|n_1(u_n,v_n) - n_1(u,v)\|_{L^{p'}} \to 0,\end{aligned} \tag{2.19}$$

which means that N_1 is compact.

The proof for N_2 is analogous. The formula (2.17) can be obtained by using (2.3), (2.4), the proof can be found e.g. in Appendix A.1. in [30]. □

The operators β^+, β^- will be formally defined by

$$\begin{aligned}\langle\beta^-(v),\varphi\rangle &= -\int_\Omega s_-(x)v^-\varphi \text{ dx} \quad \text{for all } v,\varphi \in W_D^{1,2}(\Omega), \\ \langle\beta^+(v),\varphi\rangle &= \int_\Omega s_+(x)v^+\varphi \text{ dx} \quad \text{for all } v,\varphi \in W_D^{1,2}(\Omega),\end{aligned} \tag{2.20}$$

and the operators $\hat{G}_+, \hat{G}_-$ by

$$\begin{aligned}\langle\hat{G}_-(v),\varphi\rangle &= -\int_\Omega \hat{g}_-(x,v^-)\varphi \text{ dx} \quad \text{for all } v,\varphi \in W_D^{1,2}(\Omega), \\ \langle\hat{G}_+(v),\varphi\rangle &= \int_\Omega \hat{g}_-(x,v^+)\varphi \text{ dx} \quad \text{for all } v,\varphi \in W_D^{1,2}(\Omega).\end{aligned}$$

Before writing the following lemma, let us remind here that (2.7) and (2.9) are assumed for $\hat{g}_\pm$ and $s_\pm$ respectively.

Lemma 3. *The operators $\hat{G}_+, \hat{G}_-$ are well-defined operators from $W_D^{1,2}(\Omega)$ to $W_D^{1,2}(\Omega)$. If $v_n \rightharpoonup v$ in $W_D^{1,2}(\Omega)$, then $\hat{G}_\pm(v_n) \to \hat{G}_\pm(v)$. In particular, β^+, β^- are well-defined operators from $W_D^{1,2}(\Omega)$ to $W_D^{1,2}(\Omega)$ and if $v_n \rightharpoonup v$ in $W_D^{1,2}(\Omega)$, then $\beta^\pm(v_n) \to \beta^\pm(v)$. If $v_n \to 0$, $v_n/\|v_n\| \rightharpoonup w$ in $W_D^{1,2}(\Omega)$, then*

$$\frac{\hat{G}_\pm(v_n)}{\|v_n\|} \to \beta^\pm(w).$$

The operators $\beta^\pm$ are Lipschitz continuous. If

$$s_+(x) \ge 0 \textit{ for a.a. } x\in\Omega \quad \textit{and} \quad s_-(x) \ge 0 \textit{ for a.a. } x \in \Omega \tag{2.21}$$

then

$$\langle\beta^+(v),v\rangle \ge 0 \quad \textit{for all } v \in W_D^{1,2}(\Omega) \quad \textit{and} \quad \langle\beta^-(v),v\rangle \ge 0 \quad \textit{for all } v\in W_D^{1,2}(\Omega), \tag{2.22}$$

respectively.

Proof. Using the conditions (2.7) and Hölder inequality yield

$$\sup_{\varphi\in W_D^{1,2}(\Omega),\|\varphi\|=1} \langle \hat{G}_-(v),\varphi\rangle = \sup_{\varphi\in W_D^{1,2}(\Omega),\|\varphi\|=1} -\int_\Omega \hat{g}_-(x,v)\varphi \le C\|\hat{g}_-(x,v^-)\|_{L^{p'}}\|\varphi\|_{L^p} \le$$
$$\le C(1+\|v\|_{L^p})^{p-1}\|\varphi\|_{L^p} < \infty, \text{ for all } v\in W_D^{1,2}(\Omega),$$

where $p' = p/(p-1)$. The application of Riesz Theorem gives the well definition of $\hat{G}_-$ as an operator from $W_D^{1,2}(\Omega)$ to $W_D^{1,2}(\Omega)$. Let $v_n \rightharpoonup v$. The growth condition (2.7), the compact embedding $W_D^{1,2}(\Omega)$ into $L^p(\Omega)$, for p given by (2.5) and continuity of Nemyckii operator give

$$\left\|\hat{G}^-(v_n) - \hat{G}^-(v)\right\| = \sup_{\varphi\in W_D^{1,2}(\Omega),\|\varphi\|=1} -\int_\Omega \left(\hat{g}_-(x,{v_n}^-) - \hat{g}_-(x,v^-)\right)\varphi \,\mathrm{dx} \le$$
$$\le C\|\hat{g}_-(x,{v_n}^-) - \hat{g}_-(x,v^-)\|_{L^{p'}} \to 0. \tag{2.23}$$

Hence $\hat{G}^-(v_n) \to \hat{G}^-(v)$. Since β^- is a special case of $\hat{G}_-$ with $\hat{g}_-(x,v^-) = s_-(x)v^-$, it also gives the well-definition of β^- and $\beta^-(v_n) \to \beta^-(v)$.

Let us define the operator $G : W_D^{1,2}(\Omega) \to W_D^{1,2}(\Omega)$ by

$$\langle G(v),\varphi\rangle = -\int_\Omega (\hat{g}_-(x,v) - s_-(x)v)\varphi \,\mathrm{dx} \text{ for all } v,\varphi\in W_D^{1,2}(\Omega).$$

The definition (2.8) and the assumption (2.6) implies

$$\lim_{\xi\to 0} \frac{\hat{g}_-(x,\xi) - s_-(x)\xi}{\xi} = 0 \text{ for a.a. } x\in\Omega,$$

and this together with (2.7) lead to

$$\lim_{v\to 0} \frac{G(v)}{\|v\|} = 0,$$

see Proposition 3.2 from [11]. If $v_n \to 0$, then also $v_n^- \to 0$, and the choice $v := v_n^-$ yields

$$\lim_{n\to\infty} \frac{\|\hat{G}_-(v_n) - \beta^-(v_n)\|}{\|v_n\|} = \lim_{n\to\infty} \frac{\|G(v_n^-)\|}{\|v_n\|} \le \lim_{n\to\infty} \frac{\|G(v_n^-)\|}{\|v_n^-\|} = 0. \tag{2.24}$$

If $v_n/\|v_n\| \rightharpoonup w$, then this together with the positive homogeneity of β^- and (2.23) with v_n replaced by $v_n/\|v_n\|$ and v replaced by w give

$$\frac{\hat{G}_-(v_n)}{\|v_n\|} \to \beta^-(w).$$

The Lipschitz continuity follows from

$$\|\beta^-(u) - \beta^-(v)\| = \sup_{\varphi\in\mathbb{H},\|\varphi\|=1} \langle \beta^-(u) - \beta^-(v),\varphi\rangle = \int_\Omega s_-(x)(u^- - v^-)\varphi \,\mathrm{dx} \le$$
$$\le \|s_-\|_{L^\infty}\|u^- - v^-\|_{L^2} \le \|s_-\|_{L^\infty}\|u-v\|_{L^2} \le \|s_-\|_{L^\infty}\kappa_1^{-1}\|u-v\|,$$

where in the last step we used the relation

$$\min_{v\in\mathbb{H},v\neq 0} \frac{\|v\|}{\|v\|_{L^2}} = \min_{v\in\mathbb{H},v\neq 0} \frac{\langle v,v\rangle}{\langle Av,v\rangle} = \kappa_1,$$

see Lemma 1 on pg. 15 and Remark 31 on pg. 112 in Appendix.

Clearly

$$\langle \beta^-(v),v\rangle = \int_\Omega s_-(x)\left(v^-\right)^2 \,\mathrm{dx},$$

and (2.21) gives (2.22). The proof for β^+, $\hat{G}_+$ is analogous. □

Remark 4. *Using the definitions of the operators A, N_1, N_2 and $\hat{G}_-$, $\hat{G}_+$ the system (1.15) can be rewritten as*

$$\langle d_1 u - b_{11}Au - b_{12}Av - N_1(u,v), \varphi\rangle = 0,$$
$$\langle d_2 v - b_{21}Au - b_{22}Av - N_2(u,v) + \hat{G}_-(v) + \hat{G}_+(v), \varphi\rangle = 0 \quad \text{for all } \varphi \in W^{1,2}_D(\Omega),$$

which is equivalent to a system of operator equations

$$\begin{aligned} &d_1 u - b_{11}Au - b_{12}Av - N_1(u,v) = 0, \\ &d_2 v - b_{21}Au - b_{22}Av - N_2(u,v) + \hat{G}_-(v) + \hat{G}_+(v) = 0. \end{aligned} \tag{2.25}$$

For further purposes let us also consider the homogenized problem

$$\begin{aligned} &d_1 u - b_{11}Au - b_{12}Av = 0, \\ &d_2 v - b_{21}Au - b_{22}Av + \beta^-(v) + \beta^+(v) = 0, \end{aligned} \tag{2.26}$$

which is equivalent to the weak formulation of (1.16), (1.10).

To sum up, the couple $(u,v) \in W^{1,2}_D(\Omega)$ is a (weak) solution of (1.14), (1.10) if and only it is a solution of (2.25). And the couple $(u,v) \in W^{1,2}_D(\Omega)$ is a (weak) solution of (1.16), (1.10) if and only if it is a solution of (2.26).

Potentiality of $\hat{G}_\pm$, N_1, N_2 This paragraph contains auxiliary results about operators having potential. To prove them, it will be necessary to strengthen the assumptions on the functions n_1, n_2. The following lemma will be essential for the proof of Lemma 5 and Theorem 1.

Lemma 4. *Let n_1 satisfy in addition to basic assumptions given in Section 2.1 also that*

$$\begin{aligned} &n_1 \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R}) \text{ and} \\ &\text{there exists } C>0: \ |\partial_\xi n_1(\chi,\xi)| + |\partial_\chi n_1(\chi,\xi)| \le C(1+|\xi|^{p-2}+|\chi|^{p-2}) \quad \text{for all } \xi,\chi \in \mathbb{R}, \end{aligned} \tag{2.27}$$

with some p from (2.5). Then the operator N_1 defined in (2.15) satisfies $N_1 \in C^1(W^{1,2}_D(\Omega)\times W^{1,2}_D(\Omega), W^{1,2}_D(\Omega))$ and its Fréchet derivative is given by

$$\langle N_1'(u,v)(h_1,h_2),\varphi\rangle = \int_\Omega n_1'(u,v)(h_1,h_2)\cdot\varphi \ dx = \int_\Omega (\partial_u n_1(u,v)h_1 + \partial_v n_1(u,v)h_2)\varphi \ dx \tag{2.28}$$
$$\text{for all } u,v,\varphi,h_1,h_2 \in W^{1,2}_D(\Omega).$$

And analogously, if n_2 satisfies the basic assumptions and

$$\begin{aligned} &n_2 \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R}) \text{ and} \\ &\text{there exists } C>0: \ |\partial_\xi n_2(\chi,\xi)| + |\partial_\chi n_2(\chi,\xi)| \le C(1+|\xi|^{p-2}+|\chi|^{p-2}) \quad \text{for all } \xi,\chi \in \mathbb{R}, \end{aligned} \tag{2.29}$$

with some p from (2.5), then N_2 from (2.16) satisfies $N_2 \in C^1(W^{1,2}_D(\Omega)\times W^{1,2}_D(\Omega), W^{1,2}_D(\Omega))$ and its Fréchet derivative is given by

$$\langle N_2'(u,v)(h_1,h_2),\varphi\rangle = \int_\Omega n_2'(u,v)(h_1,h_2)\cdot\varphi \ dx = \int_\Omega (\partial_u n_2(u,v)h_1 + \partial_v n_2(u,v)h_2)\varphi \ dx$$
$$\text{for all } u,v,\varphi,h_1,h_2 \in W^{1,2}_D(\Omega).$$

Proof. Under the assumptions (2.4), (2.27), Nemyckii operators $(u,v) \to \partial_u n_1(u,v)$, $(u,v) \to \partial_v n_1(u,v)$ map $L^p(\Omega)\times L^p(\Omega)$ into $L^{\frac{p}{p-2}}$. Hence, using the embedding $W^{1,2}_D(\Omega) \hookrightarrow L^p(\Omega)$, for any $u,v,h_1,h_2 \in W^{1,2}_D(\Omega)$ we can define $N_1'(u,v)(h_1,h_2) \in W^{1,2}_D(\Omega)$ by (2.28). We will show that

$N_1'(u,v)(h_1,h_2)$ is a directional derivative of N_1 at the point (u,v) and in the direction (h_1,h_2). Let $B_1 \subset W_D^{1,2}(\Omega)$ be the unit ball centered at the origin. Using Hölder inequality we get

$$\lim_{t\to 0}\left\| \frac{N_1((u,v)+t(h_1,h_2)) - N_1(u,v)}{t} - N_1'(u,v)(h_1,h_2)\right\| =$$
$$= \lim_{t\to 0}\sup_{\varphi\in B_1}\int_\Omega \left(\frac{n_1((u,v)+t(h_1,h_2)) - n_1(u,v)}{t} - n_1'(u,v)(h_1,h_2)\right)\varphi \,\mathrm{dx} \le$$
$$\le C\lim_{t\to 0}\left(\int_\Omega \left|\frac{n_1((u,v)+t(h_1,h_2)) - n_1(u,v)}{t} - n_1'(u,v)(h_1,h_2)\right|^{p'} \mathrm{dx}\right)^{\frac{1}{p'}},$$

where $p' = p/(p-1)$, p is from (2.4). We want to apply Dominated Convergence Theorem to exchange limit and integral, hence, we have to find an integrable majorant. We use Mean Value Theorem to get

$$\left|\frac{n_1((u,v)+t(h_1,h_2)) - n_1(u,v)}{t} - n_1'(u,v)(h_1,h_2)\right| = |(n_1'((u,v)+t\theta(h_1,h_2)) - n_1'(u,v))(h_1,h_2)|$$
$$\text{for a.a. } x\in\Omega,$$

where $\theta(x) \in [0,1]$ for a.a. $x\in\Omega$. From now we will use one universal symbol C for various constants. We use the triangle inequality and condition (2.27) to get the existence of $C>0$ such that

$$|n_1'((u,v)+\theta t(h_1,h_2))(h_1,h_2)| \le \left|\frac{\partial n_1}{\partial u}(u+\theta th_1, v+\theta th_2)\right||h_1| + \left|\frac{\partial n_1}{\partial v}(u+\theta th_1, v+\theta th_2)\right||h_2| \le$$
$$\le C\left(1 + |u+\theta th_1|^{p-2} + |v+\theta th_1|^{p-2}\right)(|h_1|+|h_2|). \tag{2.30}$$

The Young inequality with $(p-1)/(p-2)$ and $(p-1)$ implies

$$|u+\theta th_1|^{p-2}|h_1| \le C(|u|^{p-2}|h_1| + (\theta t)^{p-2}|h_1|^{p-1}) \le C(|u|^{p-1} + (1+(\theta t)^{p-2})|h_1|^{p-1}).$$

Analogous estimates can be done for the other terms in (2.30). Using all these estimates together with the embedding $W_D^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ we get for sufficiently small t that

$$|(n_1'((u,v)+t\theta(h_1,h_2)) - n_1'(u,v))(h_1,h_2)| \le$$
$$\le C\left(|u|^{p-1} + |v|^{p-1} + |h_1|^{p-1} + |h_2|^{p-1}\right) \in L^{p'} \quad \text{for any } u,v,h_1,h_2 \in W_D^{1,2}(\Omega).$$

Summarizing, we obtain

$$\left|\frac{n_1((u,v)+t(h_1,h_2)) - n_1(u,v)}{t} - n_1'(u,v)(h_1,h_2)\right|^{p'} \le$$
$$\le C\left(|u|^{p-1} + |v|^{p-1} + |h_1|^{p-1} + |h_2|^{p-1}\right)^{p'} \in L^1,$$

and Dominated Convergence Theorem gives

$$\lim_{t\to 0}\left\|\frac{N_1((u,v)+t(h_1,h_2)) - N_1(u,v)}{t} - N_1'(u,v)(h_1,h_2)\right\| = 0.$$

Hence, $N_1'(u,v)(h_1,h_2)$ is a directional derivative of $N_1(u,v)$ in an arbitrary direction (h_1,h_2).

Let $(u,v) \in W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ be arbitrary fixed. It is clear that the operator $N_1'(u,v) : (h_1,h_2) \mapsto N_1'(u,v)(h_1,h_2)$ from (2.28) is linear. Using the generalized Hölder inequality and (2.27) we get

$$\|N_1'(u,v)(h_1,h_2)\| = \sup_{\varphi\in B_1}\int_\Omega n_1'(u,v)(h_1,h_2)\varphi\,\mathrm{dx} \le$$
$$\le C(1 + \|u\|_{L^p} + \|v\|_{L^p})(\|h_1\|_{L^p} + \|h_2\|_{L^p}) \le C(\|h_1\| + \|h_2\|).$$

Hence, the linear operator $N_1'(u,v)$ is bounded and therefore it is a Gâteaux derivative.

Let $(u_0,v_0) \in W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ be arbitrary. Then

$$\lim_{(u,v)\to(u_0,v_0)} \|N_1'(u,v) - N_1'(u_0,v_0)\|_{\mathscr{L}(W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega), W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega))} =$$

$$= \lim_{(u,v)\to(u_0,v_0)} \sup_{\varphi\in B_1} \sup_{(h_1,h_2)\in B_1\times B_1} \int_\Omega \left(n_1'(u,v) - n_1'(u_0,v_0)\right)(h_1,h_2)\cdot\varphi \text{ dx}.$$

The growth conditions (2.27) and the generalized Hölder inequality lead to

$$\int_\Omega \left(n_1'(u,v) - n_1'(u_0,v_0)\right)(h_1,h_2)\cdot\varphi \le$$

$$\le \left\|\frac{\partial n_1}{\partial u}(u,v) - \frac{\partial n_1}{\partial u}(u_0,v_0)\right\|_{L^{\frac{p}{p-2}}} \|h_1\|_{L^p}\|\varphi\|_{L^p} + \left\|\frac{\partial n_1}{\partial v}(u,v) - \frac{\partial n_1}{\partial v}(u_0,v_0)\right\|_{L^{\frac{p}{p-2}}} \|h_2\|_{L^p}\|\varphi\|_{L^p}.$$

Nemyckii operators $(u,v) \to \partial_u n_1(u,v)$, $(u,v) \to \partial_v n_1(u,v)$ are under the conditions (2.27) continuous from $L^p(\Omega)\times L^p(\Omega)$ into $L^{\frac{p}{p-2}}(\Omega)$. Hence,

$$\lim_{(u,v)\to(u_0,v_0)} \left\|\frac{\partial n_1}{\partial u}(u,v) - \frac{\partial n_1}{\partial u}(u_0,v_0)\right\|_{L^{\frac{p}{p-2}}} = 0, \quad \lim_{(u,v)\to(u_0,v_0)} \left\|\frac{\partial n_1}{\partial v}(u,v) - \frac{\partial n_1}{\partial v}(u_0,v_0)\right\|_{L^{\frac{p}{p-2}}} = 0,$$

and

$$\lim_{(u,v)\to(u_0,v_0)} \|N_1'(u,v) - N_1'(u_0,v_0)\|_{\mathscr{L}(W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega), W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega))} = 0,$$

i.e. the map $(u,v) \to N'(u,v)$ from $W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega)$ into $\mathscr{L}(W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega))$ is continuous and therefore it is a Fréchet derivative, see e.g. Proposition 3.2.15 in [9]. The proof for N_2 is analogous. □

Remark 5. *The proof of the Lemma 4 has been inspired by [9], Exercise 3.2.41.*

Remark 6. *We note that for $m = 2$ there is $p < \infty$ and for $m = 3$ there is $p < 6$ in (2.27), (2.29).*

In Chapter 3.2.2 we will prove one bifurcation theorem for skew-symmetric problems with potentials. For this reason will be useful to have some result about potentiality of the operators $\hat{G}_\pm$, N_1, N_2.

Lemma 5. *The operators $\hat{G}_\pm$ have potentials $\Phi_{\hat{G}_\pm} : W_D^{1,2}(\Omega) \to \mathbb{R}$ in a form*

$$\Phi_{\hat{G}_+}(v) := \int_\Omega \int_0^{v(x)} \hat{g}_+(x,\xi^+) d\xi \ dx,$$

$$\Phi_{\hat{G}_-}(v) := -\int_\Omega \int_0^{v(x)} \hat{g}_-(x,\xi^-) d\xi \ dx.$$

In particular, the operators $\beta^\pm$ have potentials. If in addition to the basic assumptions from Section 2.1 also (2.27), (2.29) and

$$\partial_\chi n_1(\xi,\chi) = -\partial_\xi n_2(\xi,\chi), \text{ for all } \xi,\chi \in \mathbb{R}, \tag{2.31}$$

are true, the operator $\mathbf{N} := (-N_1, N_2)$ has a potential $\Phi_{\mathrm{N}} : W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega) \to \mathbb{R}$ in a form

$$\Phi_{\mathrm{N}}(u,v) = \int_0^1 \left(\int_\Omega -n_1(tu(x),tv(x))u(x) + n_2(tu(x),tv(x))v(x) \ dx\right) dt. \tag{2.32}$$

Proof. Since Ω is a bounded domain and $\hat{g}_\pm$ satisfy the growth condition (2.7), the first assertion follows directly from [61], Proposition 41.10.

The assumptions (2.4) and (2.27), (2.29) guarantee that

$$N_1, N_2 \in C^1(W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega), W_D^{1,2}(\Omega)),$$

see Lemma 4. According to Corollary 3.2.20 from [9] the operator $\mathbf{N} = (-N_1, N_2)$ has the Fréchet derivative. The assumption (2.31) gives

$$\langle \mathbf{N}'(u,v)\mathbf{w}, \mathbf{z}\rangle_{W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega)} = \langle \mathbf{w}, \mathbf{N}'(u,v)\mathbf{z}\rangle_{W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega)}$$
$$\text{for all } u, v \in W_D^{1,2}(\Omega), \mathbf{w}, \mathbf{z} \in W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega),$$

where prime denotes as usual Fréchet derivative and therefore $\mathbf{N}$ has the potential

$$\Phi_{\mathrm{N}}(\mathbf{w}) = \int_0^1 \langle \mathbf{N}(\mathbf{tw}), \mathbf{w}\rangle_{W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega)} \mathrm{dt} \qquad \text{for all } \mathbf{w} \in W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega),$$

see Proposition 41.5 in [61]. □

Remark 7. *A potential to $(-N_1, N_2)$ can be written in more practical form*

$$\Phi_{\mathrm{N}}(u,v) = \int_\Omega \int_0^{u(x)} -n_1(\xi, v(x)) d\xi \ dx + \int_\Omega \int_0^{v(x)} n_2(0,\xi) d\xi \ dx, \tag{2.33}$$

cf. with $\Phi_{\hat{G}_\pm}$. We will not prove it here rigorously, but a formal differentiation of (2.33) gives

$$\begin{aligned}
\Phi_{\mathrm{N}}'(u,v)(h_1,h_2) &= \int_\Omega -n_1(u(x),v(x))h_1(x) \ dx - \int_\Omega \int_0^{u(x)} \partial_v n_1(\xi, v(x))h_2(x) \ d\xi \\
&+ \int_\Omega n_2(0,v(x))h_2(x) \ dx = \\
&= \int_\Omega -n_1(u(x),v(x))h_1(x) \ dx + \int_\Omega \int_0^{u(x)} \partial_u n_2(\xi, v(x))h_2(x) d\xi \ dx \\
&+ \int_\Omega n_2(0,v(x))h_2(x) \ dx = \\
&= \int_\Omega -n_1(u(x),v(x))h_1(x) \ dx + \int_\Omega n_2(u(x),v(x))h_2(x) \ d\xi \ dx - \int_\Omega n_2(0,v(x))h_2(x) \ dx \\
&+ \int_\Omega n_2(0,v(x))h_2(x) \ dx = \\
&= \int_\Omega (-n_1(u(x),v(x)), n_2(u(x),v(x))) \cdot (h_1(x), h_2(x)) \ dx = \langle \mathbf{N}(u,v), (h_1,h_2)\rangle_{W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega)}
\end{aligned}$$

for all $u, v, h_1, h_2 \in W_D^{1,2}(\Omega)$, which suggest that (2.33) is indeed the potential to $(-N_1, N_2)$. The rigorous proof is quite long, as well as finding a transformation between (2.32) and (2.33) and therefore we will not do it here.

Nice examples are unilateral terms with a saturation

$$\hat{g}_+(x, v^+) = s_+(x)\frac{v^+}{1+(v^+)^2}, \quad \hat{g}_-(x, v^-) = s_-(x)\frac{v^-}{1+(v^-)^2},$$

where the respective $\hat{G}_\pm$ are having the potentials

$$\Phi_{\hat{G}_+}(v) = \frac{1}{2}\int_\Omega s_+(x) \left(\ln(1+(v^+)^2)\right) \ \mathrm{dx}.$$
$$\Phi_{\hat{G}_-}(v) = -\frac{1}{2}\int_\Omega s_-(x) \left(\ln(1+(v^-)^2)\right) \ \mathrm{dx}.$$

The potentials for β^+, β^- are

$$\Phi_{\beta^+}(v) := \frac{1}{2}\int_\Omega s_+(v^+)^2 \text{ dx}, \quad \Phi_{\beta^-}(v) := \frac{1}{2}\int_\Omega s_-(v^-)^2 \text{ dx}.$$

In general, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Carathéodory conditions and the growth conditions (2.4) with $\hat{g}_\pm$ replaced by f. The potential

$$\Phi_f(v) := \|v\|^2 \int_\Omega f\left(x, \frac{v^2}{\|v\|^2}\right) \text{ dx}$$

generates a positively homogeneous operator. For the operators $\beta^\pm$ the generating function is $f(x,\xi) := s_\pm(x)(\xi^-)$.

2.2.2 Reduction of the Dirichlet/mixed problem to one equation

From now we will always consider that $d_1 > 0$ is arbitrary fixed. The aim of this section is to reduce the system (2.25) to one operator equation and then show its basic properties.

Let us remind here that κ_k denotes the eigenvalue of the Laplacian with the Dirichlet or mixed b.c., and e_k the corresponding eigenvector. For further purposes let us define $y_k := b_{11}/\kappa_k$, for any $k \in \mathbb{N}$.

2.2.2.1 Linear reaction-diffusion system

Let $\hat{g}_+, \hat{g}_-, n_1, n_2 \equiv 0$, then

$$\hat{G}_-, \hat{G}_+, N_1, N_2 \equiv 0,$$

and (2.25) has the form

$$\begin{aligned} d_1 u - b_{11}Au - b_{12}Av &= 0, \\ d_2 v - b_{21}Au - b_{22}Av &= 0. \end{aligned} \tag{2.34}$$

This is equivalent to the weak formulation of the problem (1.12), (1.10). The first equation of (2.34) can be rewritten as

$$(d_1 I - b_{11}A)u = b_{12}Av.$$

Under the assumption $d_1 \neq y_j$ for all $j \in \mathbb{N}$, the operator $(d_1 I - b_{11}A)$ is invertible, as follows from Fredholm Alternative. Therefore it is possible to multiply this equation by $(d_1 I - b_{11}A)^{-1}$ and insert u into the second equation of (2.34) to obtain the system

$$\begin{aligned} u &= b_{12}(d_1 I - b_{11}A)^{-1}Av, \\ d_2 v &- b_{12}b_{21}A(d_1 I - b_{11}A)^{-1}Av - b_{22}Av = 0. \end{aligned} \tag{2.35}$$

An operator $S : W_D^{1,2}(\Omega) \to W_D^{1,2}(\Omega)$ will be defined for any fixed $d_1 > 0,\ d_1 \neq y_j$ for all $j \in \mathbb{N}$ by

$$S := b_{12}b_{21}A(d_1 I - b_{11}A)^{-1}A + b_{22}A, \tag{2.36}$$

and the second equation in (2.35) has the form

$$d_2 v = Sv. \tag{2.37}$$

Since (2.35) is equivalent to the weak formulation of (1.12), (1.10) it is possible to make conclusions summarized in the following remark.

Remark 8. *Let $d_1 \in (0, y_1)\setminus\{y_j \mid j = 2, 3, \cdots\}$ be fixed. A point $d_2 \in \mathbb{R}$ is a critical point of (1.12), (1.10) with fixed d_1 (and simultaneously of (2.34) with fixed d_1) if it is an eigenvalue of the operator S.*

A pair $(b_{12}(d_1 I - b_{11}A)^{-1}v_0, v_0)$ is a solution of (1.12), (1.10) with fixed d_1 (and simultaneously of (2.34) with fixed d_1) if d_2 is an eigenvalue of the operator S with the corresponding eigenvector v_0.

Lemma 6. *The operator S is linear, compact and symmteric.*

Proof. The operator S is linear because it is a composition of linear operators. The operators A and $(d_1 I - b_{11}A)$ commute and are symmetric which means that S is symmetric. The operator A is compact and $(d_1 I - b_{11}A)$ is continuous, therefore the operator S is compact. □

Due to the compactness, the spectrum of S is discrete, countable and with the only accumulation point at zero. Let e_k be an eigenvector of A. Then

$$\begin{aligned} Se_k = b_{12}b_{21}A(d_1 I - b_{11}A)^{-1}Ae_k + b_{22}Ae_k = \frac{b_{12}b_{21}}{\kappa_k^2\left(d_1 - b_{11}\dfrac{1}{\kappa_k}\right)}\, e_k + \frac{b_{22}}{\kappa_k}e_k = \\ = \frac{1}{\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right) e_k = \lambda_k^S e_k, \end{aligned} \tag{2.38}$$

it means e_k is an eigenvector of S with the corresponding eigenvalue λ_k^S. Since e_k is an orthonormal base in $W_D^{1,2}(\Omega)$, all eigenvalues λ_k^S of S can be expressed in a form

$$\lambda_k^S = \frac{1}{\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right), \quad k = 1, 2, 3, \cdots, \tag{2.39}$$

and if e_k is an eigenfunction of the Laplacian respective to the eigenvalue κ_k then e_k is an eigenfunction of S respective to λ_k^S. The largest eigenvalue of S will be denoted by $\lambda_{\max}^S$. Since every eigenvalue of the operator S is simultaneously a critical point of the problem (1.12), (1.10) with fixed d_1, see Remark 8, we will introduce a notation

$$d_{2,k}^0 := \frac{1}{\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right), \quad k = 1, 2, 3, \cdots, \tag{2.40}$$

for these critical points. The largest critical point of this problem will be denoted by $d_{2,\max}^0$ and is equal to $\lambda_{\max}^S$. Let us note that in systems with Neumann b.c., $d_{2,k}^0$ and λ_k^S are different in general, see (2.60) and (2.61) on pg. 30.

The operator S is in general not positive and can be even negative for some values of d_1, as will be proved in following two lemmas.

Lemma 7. *The operator S is negative for any $d_1 > y_1$.*

Proof. Since $d_1\kappa_k > d_1\kappa_1 > b_{11}$ for any $k \in \mathbb{N}, k \geq 2$, see (2.12), the expression $d_1\kappa_k - b_{11}$ is positive, and because $b_{12}b_{21} < 0$ and $b_{22} < 0$, see (2.1), (2.2), it is true that

$$\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right) = \frac{-\det\ B + \kappa_k b_{22} d_1}{d_1\kappa_k - b_{11}} < 0 \quad \text{for all } k \in \mathbb{N},$$

i.e. all eigenvalues of the operator S are negative and therefore S is negative operator. □

The situation for $d_1 \in (0, y_1)$ is more complicated.

Lemma 8. *Let $y_j \neq y_{j+1}$ and let $d_1 \in (y_{j+1}, y_j)$ for given $j \in \mathbb{N}$. Then $\lambda_k^S > 0$ for any $k \leq j$ and $\lambda_k^S < 0$ for any $k > j$.*

Proof. First we rewrite

$$\lambda_k^S \kappa_k = \left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right) = \frac{b_{22}d_1\kappa_k - \det B}{d_1\kappa_k - b_{11}}. \tag{2.41}$$

Due to (2.2) there is $b_{22}d_1\kappa_k - \det B < 0$ for all $d_1 > 0$. Since $d_1 \in (y_{j+1}, y_j)$ we get by using (2.12) that $d_1\kappa_k - b_{11} < 0$ for any $k \leq j$ and $d_1\kappa_k - b_{11} > 0$ for any $k > j$. The assertion now follows from (2.41). □

Remark 9. *Especially for $d_1 \in (y_1, y_2)$ the operator S has only one positive eigenvalue λ_1^S. The eigenvalue is simple, because κ_1 is simple, and the corresponding eigenfunction is e_1. The eigenfunction has the constant sign in Ω.*

Let us also note that the eigenvalues λ_j^S are not monotone w.r.t. index j.

We will be interested in diffusion constants which are positive therefore the case $d_1 > y_1$ will not be of our interest in the Dirichlet case. However, the case $d_1 > y_1$ will be relevant for systems with Neumann boundary conditions.

If $d_1 \in (0, y_1)\backslash\{y_j \mid j \geq 2\}$, then $\lambda_{\max}^S$ is positive, see Lemma 8, and it is possible to express it through Rayleigh quotient by

$$\lambda_{\max}^S = \max_{v \in W_D^{1,2}(\Omega), v \neq 0} \frac{\langle Sv, v\rangle}{\|v\|^2} = \max_{v \in W_D^{1,2}(\Omega), \|v\|=1} \langle Sv, v\rangle > 0. \tag{2.42}$$

The other *positive* eigenvalues can be obtained recursively using Rayleigh quotient over complements of eigenspaces, see (7.19) in Appendix.

2.2.2.2 System with nonlinear operators $N_1(u,v),\ N_2(u,v), \hat{G}_{\pm}(v)$

This section concerns with the reduction of the system (2.25) with nontrivial nonlinear operators. We will prove that there is a neighborhood of the point $0 \in W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ in which it is possible to express the variable u as a function of the variable v. In contrast to linear system, the reduction is in general not true in the whole $W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$. The main tools here will be Implicit Function Theorem and Mean Value Theorem. Also the existence of potential for the reduced problem will be discussed.

Theorem 1. *Let $d_1 > 0$ be fixed such that $d_1 \neq y_j$ for all $j \in \mathbb{N}$, let $n_1 \in C^1(\mathbb{R} \times \mathbb{R})$ and (2.27) be true. Then there exists a neighborhood $U \times V \subset W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ of zero such that $(u,v) \in U \times V$ satisfy (2.25) if and only if*

$$\begin{aligned} v \in V :\ & d_2 v - Sv - N(v) + \beta^+(v) + \beta^-(v) = 0, \\ & u = F(v), \end{aligned} \tag{2.43}$$

where $F : V \to U$ is a C^1-continuous map, $S := b_{12}b_{21}A(d_1 I - b_{11}A)^{-1}A + b_{22}A$ is a linear, compact and symmetric operator and $N : V \to \mathbb{H}$ is a compact and continuous nonlinear operator satisfying

$$\lim_{v \to 0} \frac{N(v)}{\|v\|} = 0. \tag{2.44}$$

Proof. We will show that assumption of Implicit Function Theorem, see pg. 107 in Appendix, are fulfilled for a map

$$T_1(u,v) := d_1 u - b_{11}Au - b_{12}Av - N_1(u,v). \tag{2.45}$$

As $N_1(0,0) = 0$ it apparently holds $T_1(0,0) = 0$. The operator N_1 is continuously differentiable due to (2.27) and Lemma 4, and the operator A is linear, hence, $T_1 \in C^1(W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega), W_D^{1,2}(\Omega))$. Because $N_1'(0,0) = 0$ and because of the assumption $d_1 \neq b_{11}/\kappa_j$ for all $j \in \mathbb{N}$, the partial derivative

$$\partial_u T_1(0,0) = d_1 I - b_{11}A - (\partial_u N_1)(0,0) = d_1 I - b_{11}A$$

is an isomorphism of the space $W_D^{1,2}(\Omega)$. According to Implicit Function Theorem neighborhoods U, V of 0 in $W_D^{1,2}(\Omega)$ and a map $F : V \to U$ exist, so that

$$\begin{aligned} & T_1(F(v), v) = 0 \text{ for all } v \in V \\ & T_1(u,v) = 0 \text{ if and only if } u = F(v) \text{ for all } (u,v) \in U \times V. \end{aligned}$$

Moreover, $F(0) = 0$ and $F \in C^1(V)$. In particular, $F(v) \leq C\|v\|$ for all $v \in V$ and for some constant C. We calculate the partial derivative

$$\partial_v T_1(0,0) = -b_{12}A - (\partial_v N_1)(0,0) = -b_{12}A,$$

and the derivative of F at zero can be found as

$$F'(0) = b_{12}A\left(d_1 I - b_{11}A\right)^{-1},$$

see again Implicit Function Theorem an in particular (7.8). By using the relation $u = F(v)$ the second equation in (2.25) can be rewritten as

$$d_2 v - Sv - N(v) + \beta^-(v) + \beta^+(v) = 0,$$

with

$$\begin{aligned} S &:= b_{21}AF'(0) + b_{22}A = b_{12}b_{21}A\left(d_1 I - b_{11}A\right)^{-1}A + b_{22}A, \\ N(v) &:= b_{21}A(F(v) - F'(0)v) + N_2(F(v), v) - \hat{G}_+(v) + \beta^+(v) - \hat{G}_-(v) + \beta^-(v), \quad \text{for all } v \in V. \end{aligned} \tag{2.46}$$

The assertions concerning S were proven in Lemma 6 and therefore it remains to prove that N is compact and satisfies (2.44). The operators $A, N_1, N_2, \hat{G}_\pm, \beta^\pm$ are continuous and compact, therefore N is compact. It remains to prove (2.44).

Mean Value Theorem, see pg. 108 in Appendix, gives

$$\|F(v) - F'(0)v\| \leq \sup_{t\in[0,1]} \|F'((1-t)v)(v) - F'(0)(v)\|.$$

The r.h.s satisfies

$$\begin{aligned} \lim_{v\to 0} \frac{\sup_{t\in[0,1]} \|F'(tv)(v) - F'(0)(v)\|}{\|v\|} &= \\ &= \lim_{v\to 0} \sup_{t\in[0,1]} \left\| F'(tv)\left(\frac{v}{\|v\|}\right) - F'(0)\left(\frac{v}{\|v\|}\right) \right\|. \end{aligned}$$

Suppose that

$$\lim_{v\to 0} \sup_{t\in[0,1]} \left\| F'(tv)\left(\frac{v}{\|v\|}\right) - F'(0)\left(\frac{v}{\|v\|}\right) \right\| \neq 0.$$

Then there exist $\varepsilon > 0$ and sequences $t_n \in [0,1]$ and $v_n \in W_D^{1,2}(\Omega)$ such that $v_n \to 0$ and

$$\text{for all } n \in \mathbb{N}: \quad \sup_{t\in[0,1]} \left\| F'(tv)\left(\frac{v_n}{\|v_n\|}\right) - F'(0)\left(\frac{v_n}{\|v_n\|}\right) \right\| > \varepsilon.$$

However, using the continuity of F' gives

$$\lim_{t\to 0} \left\| F'(tv_n)\left(\frac{v_n}{\|v_n\|}\right) - F'(0)\left(\frac{v_n}{\|v_n\|}\right) \right\| = 0,$$

which is a contradiction and therefore

$$\lim_{v\to 0} \frac{\sup_{t\in[0,1]} \|F'(tv)(v) - F'(0)(v)\|}{\|v\|} = 0. \tag{2.47}$$

The formula (2.17) implies

$$\lim_{v\to 0} \frac{N_2(F(v), v)}{\|v\|} = \lim_{v\to 0} \frac{N_2(F(v), v)}{\|F(v)\| + \|v\|} \frac{\|F(v)\| + \|v\|}{\|v\|} = 0. \tag{2.48}$$

By using (2.24) which holds for any $v_n \to 0$, its analogue for $\hat{G}_+, \beta^+$, (2.47) and (2.48) we get (2.44). □

It is clear from the previous theorem that we will be mostly concerned with the single equation

$$v \in V: \quad d_2 v - Sv - N(v) + \beta^-(v) + \beta^+(v) = 0. \tag{2.49}$$

A necessary part of this problem will be also an analysis of the equation

$$d_2 v - Sv + \beta^-(v) + \beta^+(v) = 0, \tag{2.50}$$

as will be suggested by the following remark. Let us point out here Definition 15 of critical point and Definition 16 of bifurcation point in Appendix.

Remark 10. *Let $d_1 > 0, d_1 \neq y_j$ for all $j \in \mathbb{N}$. In Chapter 3 we will prove one Lemma which applied to the problems (2.49) and (2.50) will give the following conclusion:*
Any bifurcation point of (2.49) is simultaneously a critical point of the problem (2.50).

Since the equation (2.50) is a generalization of eigenvalue problem, it is usual to call the critical points of problems of a type (2.50) as eigenvalues. In Section 3 we will be using this concept, see Definition 3.6 and Remark 2.

Corollary 1 (Corollary of Theorem 1 and Remarks 4, 10)**.** *Let $d_1 \in (0, y_1)$ be fixed.*

A number $d_2 > 0$ is a critical point of (1.16), (1.10) with fixed d_1 if d_2 is a critical point of the problem (2.50). A number $d_2 > 0$ is a bifurcation point of the system (1.14), (1.10) with fixed d_1 if and only if d_2 is a bifurcation point of the equation (2.49).

In particular, $d_2 > 0$ is a critical point of (1.12), (1.10) with fixed d_1 if d_2 is an eigenvalue of the operator S. And $d_2 > 0$ is a bifurcation point of the system (1.9), (1.10) with fixed d_1, if and only if d_2 is a bifurcation point of (2.49) with $\beta^\pm \equiv 0$ and $N = b_{21}A(F(v) - F'(0)) + N_2(F(v), v)$ (see (2.46)), respectively.

And finally, any bifurcation point of (1.14), (1.10) with fixed d_1 is simultaneously a critical point of (1.16), (1.10) with fixed d_1.

A special class of the reaction-diffusion systems are the so-called skew symmetric systems. These systems have a potential, and we will prove now that consequently also the operator N in (2.49) has the potential. It will be crucial for the proof of Theorems 12, 17 in Section 4.

Lemma 9. *Let V be the set from Theorem 1, let $b_{12} = -b_{21}$. If (2.27), (2.29), (2.31) are true, the operator N from Theorem 1 has on the set V a potential.*

Proof. The proof has been inspired by [26] and is based on Lyapunov-Schmidt reduction.

According to Theorem 1 the system (2.25) can on V reduced to the problem (2.43). The operator $\hat{N} := (-N_1, N_2)$ has the potential $\Phi_{\hat{N}}$, according to Lemma 5. Let us define a map

$$\begin{aligned}\Phi(u,v) :=& \frac{1}{2}(d_1\|u\|^2 - b_{11}\langle Au, u\rangle - b_{12}\langle Av, u\rangle) + \Phi_{\hat{N}}(u,v) + \\ &+ \frac{1}{2}(b_{21}\langle Au, v\rangle + b_{22}\langle Av, v\rangle) - \Phi_{\hat{G}_-}(v) - \Phi_{\hat{G}_+}(v), \quad \text{for all } u, v \in W^{1,2}_D(\Omega),\end{aligned}$$

where $\Phi_{\hat{G}_\pm}$ are from Lemma 5. To simplify calculations, let us define two operators $T_1, T_2 : W^{1,2}_D(\Omega) \times W^{1,2}_D(\Omega) \to W^{1,2}_D(\Omega)$ by

$$\begin{aligned}T_1(u,v) &:= d_1 u - b_{11}Au - b_{12}Av - N_1(u,v),\\ T_2(u,v) &:= b_{21}Au + b_{22}Av + N_2(u,v) - \hat{G}_-(v) - \hat{G}_+(v).\end{aligned}$$

It is possible to verify by direct calculation with using $b_{21} = -b_{12}$ that $\partial_u\Phi = T_1,\ \partial_v\Phi = T_2$. Moreover, Theorem 1 gives $T_1(F(v), v) = 0$ for all $v \in V$.

Let S, N be from Theorem 1. The goal will be to show that a functional $\Phi_{S+N-\beta^+-\beta^-} : V \to \mathbb{R}$ defined by

$$\Phi_{S+N-\beta^+-\beta^-}(v) = \Phi(F(v), v) \quad \text{for all } u, v \in V,$$

is a potential of the operator $S + N - \beta^+ - \beta^-$. The potential to N then will be equal to $\Phi_{S+N-\beta^+-\beta^-} - \Phi_S - \Phi_{\beta^+} - \Phi_{\beta^-}$, where $\Phi_S, \Phi_{\beta^+}, \Phi_{\beta^-}$ are the potentials to S, β^+, β^-, respectively.

Lemma 5 gives $\Phi \in C^1(W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega), \mathbb{R})$. The derivative of $\Phi_{S+N-\beta^+-\beta^-}$ is

$$\Phi'_{S+N-\beta^+-\beta^-}(v) = \left(\frac{\partial}{\partial F(v)}\Phi(F(v), v)\right) F'(v) + \frac{\partial}{\partial v}\Phi(F(v), v) =$$
$$= T_1(F(v), v)F'(v) + T_2(F(v), v) = (Sv + N(v) - \beta^+(v) - \beta^-(v)) \quad \text{for all } v \in V,$$

where we used the definitions of S, N from (2.46). Hence, $\Phi_{S+N-\beta^+-\beta^-}$ is the potential to $S + N - \beta^+ - \beta^-$ and consequently N has the potential. □

Remark 11. *The violation of the assumption (2.31) does not necessarily mean that the system (2.25) and consequently the operator N does not have potential. Let us demonstrate it on a particular example. Let us consider a system*

$$\begin{aligned} &d_1\Delta u + u - 2v + 4vu^2 = 0 \\ &d_2\Delta v + \frac{1}{2}u - \frac{3}{8}v - \frac{1}{3}u^3 + \hat{g}_-(x, v^-) - \hat{g}_+(x, v^+) = 0. \end{aligned} \qquad \textit{in } \Omega \times [0, \infty),$$

This system is neither skew-symmetric, nor satisfying the assumptions (2.2). However, if we multiply the first equation by $b_{21} = 1/2$ and the second equation by $-b_{12} = 2$, we get the formulation

$$\begin{aligned} &\frac{d_1}{2}\Delta u + \frac{1}{2}u - v + 2vu^2 = 0 \\ &2d_2\Delta v + u - \frac{3}{4}v - \frac{2}{3}u^3 + 2\hat{g}_-(x, v^-) - 2\hat{g}_+(x, v^+) = 0 \end{aligned} \qquad \textit{in } \Omega \times [0, \infty),$$

which is skew-symmetric and satisfy (2.1), (2.2).

Remark 12. *As examples of skew-symmetric systems can serve the Fitz-Hugh Nagumo model, having the form*

$$\begin{aligned} u_t &= d_1\Delta u + f(u) - v \\ v_t &= d_2\Delta v + \varepsilon(u - \gamma v) + \hat{g}_-(x, v^-) - \hat{g}_+(x, v^+) \end{aligned} \qquad \textit{in } \Omega \times [0, \infty), \tag{2.51}$$

where $\varepsilon > 0, f'(0) > 0$ and $\gamma \geq 0$, and regularized Gierer-Meinhardt model

$$\begin{aligned} u_t &= d_1\Delta u + u + \frac{u^p}{q(\varepsilon + v)^q} + \sigma \\ v_t &= d_2\Delta v - v + \frac{u^{p+1}}{(p+1)(\varepsilon + v)^{q+1}} - \hat{g}_-(x, v^-) - \hat{g}_+(x, v^+) \end{aligned} \qquad \textit{in } \Omega \times [0, \infty), \tag{2.52}$$

where $p > 1, \varepsilon > 0, q > 0, \sigma \geq 0$ and u, v denote the absolute values of concentrations, cf. also [59].

2.3 Systems with Neumann boundary conditions

2.3.1 Weak and operator formulation of the problem

The rewriting of the reaction diffusion system (1.14) with Neumann boundary conditions as operator equations is similar to the Dirichlet/mixed case. However, boundary conditions have a significant impact on the behavior of the problem, as will be seen later.

Suppose now

$$\mu_{m-1}(\Gamma_D) = 0.$$

The space $W_D^{1,2}(\Omega) = W^{1,2}(\Omega)$ is equipped with the scalar product and norm

$$\begin{aligned} \langle v, \varphi \rangle &= \int_\Omega \nabla v \cdot \nabla \varphi + v\varphi \ \mathrm{dx}, \\ \|v\|_{1,2} &= \left(\int_\Omega |\nabla v|^2 + v^2 \ \mathrm{dx} \right)^{\frac{1}{2}} \quad \text{for all } v, \varphi \in W^{1,2}(\Omega). \end{aligned} \tag{2.53}$$

See also the definitions (7.3), (7.1). We will not use the symbol $W_D^{1,2}(\Omega)$ for $W^{1,2}(\Omega)$ here, in order to better distinguish Neumann and Dirichlet/mixed case.

Notation 2. *The eigenvalues of the Laplacian with Neumann b.c. will be denoted by κ_k, and will be ordered as*

$$0 = \kappa_0 < \kappa_1 \le \kappa_2 \le \cdots \to \infty,$$

see also Remark 28 and formula (7.6) in Appendix.

An orthonormal base $\{e_k\}_{k \in \mathbb{N}_0}$ is chosen in a way that for any $k \in \mathbb{N}_0$, e_k is an eigenfunction corresponding to κ_k. The eigenvalue $\kappa_0 = 0$ is simple and an eigenfunction e_0 is constant in Ω. We will fix it to be positive with $\|e_0\| = 1$. Since e_0 is orthogonal to all e_k, $k \in \mathbb{N}$, there is no other eigenfunction having constant sign in Ω.

Lemma 10. *The operator $A : W_D^{1,2}(\Omega) \to W_D^{1,2}(\Omega)$ defined by*

$$\langle Av, \varphi \rangle = \int_\Omega v\varphi \ dx, \qquad \text{for all } v, \varphi \in W_D^{1,2}(\Omega),$$

is a symmetric linear compact operator. Moreover

$$\sigma(A) = \left\{ \frac{1}{1+\kappa_k} \,\middle|\, k \in \mathbb{N}_0 \right\} \cup \{0\}, \tag{2.54}$$

and v is an eigenfunction of the Laplacian corresponding to the eigenvalue κ_k if and only if v is an eigenfunction of A corresponding to the eigenvalue $(1+\kappa_k)^{-1}$. The largest eigenvalue of the operator A is equal to one, it is simple and a corresponding eigenfunction does not change sign in Ω. Furthermore, $\{(1+\kappa_k)^{-1}\}_{k \in \mathbb{N}_0}$ is a decreasing sequence with $\lim_{k\to\infty} \kappa_k^{-1} = 0$ (see (2.12)) and zero is the only accumulation point of $\sigma(A)$.

Proof. The assertions are either analogous to the ones in Lemma 1 or follows directly from (2.54) and Notation 2 and therefore the proof will be skipped. □

It is necessary to emphasize that zero is an eigenvalue of Laplacian with Neumann boundary condition, which means $1 \in \sigma(A)$. The respective eigenfunction for the eigenvalue one is any nonzero function constant on Ω.

Observation 1. *From (2.54) and Notation 2 follows immediately that $\sigma(A) \subset [0,1]$ and since A is symmetric and compact, it satisfies*

$$\langle (I-A)v, v \rangle \in [0,1) \ \ \text{for all } v \in W^{1,2}(\Omega), \ \|v\| = 1.$$

Because $(I-A)$ is symmetric and positive definite, $\langle (I-A)v, v\rangle = 0$ if and only if $v \in Span\{e_0\}$.

Using the conditions (2.4), (2.7) the operators $N_1, N_2 : (W^{1,2}(\Omega))^2 \to W^{1,2}(\Omega)$ and $\hat{G}_+, \hat{G}_-$, $\beta^-, \beta^+ : W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ can be defined by the relations

$$
\begin{aligned}
\langle N_1(u,v), \varphi\rangle &= \int_\Omega n_1(u,v)\varphi \,\mathrm{dx} \quad \text{for all } u,v,\varphi \in W^{1,2}(\Omega),\\
\langle N_2(u,v), \varphi\rangle &= \int_\Omega n_2(u,v)\varphi \,\mathrm{dx} \quad \text{for all } u,v,\varphi \in W^{1,2}(\Omega),\\
\langle \hat{G}_-(v), \varphi\rangle &= -\int_\Omega \hat{g}_-(x,v^-)\varphi \,\mathrm{dx} \quad \text{for all } v,\varphi \in W^{1,2}(\Omega),\\
\langle \hat{G}_+(v), \varphi\rangle &= \int_\Omega \hat{g}_+(x,v^+)\varphi \,\mathrm{dx} \quad \text{for all } v,\varphi \in W^{1,2}(\Omega),\\
\langle \beta^-(v), \varphi\rangle &= -\int_\Omega s_-(x)v^-\varphi \,\mathrm{dx} \quad \text{for all } v,\varphi \in W^{1,2}(\Omega),\\
\langle \beta^+(v), \varphi\rangle &= \int_\Omega s_+(x)v^+\varphi \,\mathrm{dx} \quad \text{for all } v,\varphi \in W^{1,2}(\Omega).
\end{aligned}
\tag{2.55}
$$

The formulas defining the operators have the same form as for the system with Dirichlet/mixed b.c., however, the operators are defined here on the whole $W^{1,2}(\Omega)$. It is possible to prove an analogue of Lemmas 2, 3.

Remark 13. *The weak formulation (1.15) of (1.14) with Neumann b.c. is equivalent to operator equations*

$$
\begin{aligned}
&d_1(I-A)u - b_{11}Au - b_{12}Av - N_1(u,v) = 0, \\
&d_2(I-A)v - b_{21}Au - b_{22}Av - N_2(u,v) + \hat{G}_-(v) + \hat{G}_+(v) = 0.
\end{aligned}
\tag{2.56}
$$

The weak formulation of (1.16) with Neumann b.c. is equivalent to

$$
\begin{aligned}
&d_1(I-A)u - b_{11}Au - b_{12}Av = 0, \\
&d_2(I-A)v - b_{21}Au - b_{22}Av + \beta^-(v) + \beta^+(v) = 0.
\end{aligned}
\tag{2.57}
$$

The main difference against Dirichlet/mixed problem is the presence of the non-invertible operator $I - A$ next to d_1, d_2. This will cause complications in the application of variational methods.

2.3.2 Reduction of the Neumann problem to one equation

The main ideas of the reduction of the system (2.56) to one equation is similar to the problem with Dirichlet boundary conditions discussed in Section 2.2.2 and for this reason the reduction will be not discussed in detail. Let us remind that κ_k will denote an eigenvalue of the Laplacian with (homogeneous) Neumann b.c., e_k will be the corresponding eigenfunction. For further purposes we will again introduce a notation $y_k := b_{11}/\kappa_k$ for all $k \in \mathbb{N}$.

2.3.2.1 Linear reaction-diffusion system with Neumann b.c.

Let $d_1 > 0$ be fixed. The first studied problem will be a simple linear equation

$$
\begin{aligned}
&d_1(I-A)u - b_{11}Au - b_{12}Av = 0, \\
&d_2(I-A)v - b_{21}Au - b_{22}Av = 0.
\end{aligned}
\tag{2.58}
$$

This is equivalent the problem (1.12) with (homogeneous) Neumann b.c. Under the assumption $d_1 \neq y_k := b_{11}/\kappa_k$ for all $k \in \mathbb{N}$, the operator $d_1(I-A) - b_{11}A$ is invertible due to Fredholm Alternative, and it is possible to rewrite (2.58) as

$$
\begin{aligned}
&u = b_{12}(d_1 I - (d_1 + b_{11})A)^{-1}Av, \\
&d_2(I-A)v - b_{21}b_{12}A(d_1 I - (d_1 + b_{11})A)^{-1}Av + b_{22}Av = 0.
\end{aligned}
$$

We define an operator $S : W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ by

$$S := b_{12}b_{21}A(d_1 I - (d_1 + b_{11})A)^{-1}A + b_{22}A,$$

and the second equation then has a form

$$d_2(I - A)v - Sv = 0. \tag{2.59}$$

The eigenvalues of the operator S can be calculated as

$$\lambda_k^S = \frac{1}{1+\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right) \quad \text{for } k \in \mathbb{N}_0. \tag{2.60}$$

The zeroth eigenvalue is $\lambda_0^S = -b_{12}b_{21}/b_{11} + b_{22} = \det B/b_{11} > 0$. However, $d_2(I - A)e_0 = 0$ and therefore the function e_0 is not a solution of (2.59). On the other hand, for any $k \in \mathbb{N}$ the function e_k is a solution of (2.59) respective to the critical point

$$d_{2,k}^0 := \frac{1}{\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right). \tag{2.61}$$

This is formally the same formula as (2.39). The largest critical point will be again denoted as $d_{2,\max}^0$.

It is possible to characterize $d_{2,k}^0$ by using a variational formula. However, mindless modification of the Dirichlet case will fail here, because $\langle Se_0, e_0\rangle = \lambda_0^S > 0$, $\langle (I-A)e_0, e_0\rangle = 0$ and consequently

$$\sup_{v\in\mathbb{H}, \|v\|=1} \frac{\langle Sv, v\rangle}{\langle (I - A)v, v\rangle} = \infty,$$

see also Remark 31 at the end of Appendix. It is necessary to get rid of the constant function. For $d_1 \in (0, y_1)\backslash \left\{y_j \middle| \, j = 2, 3, \cdots\right\}$ the largest eigenvalue of (2.59) is characterized by the following formula:

$$d_{2,\max}^0 = \max_{v\in\{e_0\}^\perp} \frac{\langle Sv, v\rangle}{\langle (I - A)v, v\rangle},$$

the orthogonal complement is w.r.t. $W^{1,2}(\Omega)$. The value $d_{2,\max}^0$ is now finite and positive. The other positive critical points can be again found through iterative formula, see (7.20) in Appendix.

It is easy to modify Lemmas 7, 8 to this particular situation.

Lemma 11. *Let $d_1 > y_1$. Then all critical points of (2.59) are negative.*

Lemma 12. *Let $y_j := b_{11}/\kappa_j$ and let $d_1 \in (y_{j+1}, y_j)$, $j \in \mathbb{N}$. Then $d_{2,k}^0 > 0$ for any $k \le j$ and $d_{2,k}^0 < 0$ for any $k > j$.*

2.3.2.2 Nonlinear reaction-diffusion system with Neumann b.c.

An analogue of Theorem 1 for the case of Neumann boundary conditions can be proved.

Theorem 2. *Let d_1 be fixed such that $d_1 \neq y_j$ for all $j \in \mathbb{N}$. Then it exists a neighborhood $U \times V \subset W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$ of zero such that the system (2.57) is on $U \times V$ equivalent to the problem*

$$\begin{aligned} v \in V : d_2(I - A)v - Sv - N(v) + \beta^+(v) + \beta^-(v) = 0,\\ u = F(v), \end{aligned} \tag{2.62}$$

where $F : V \to U$ is C^1-continuous map, $S := b_{12}b_{21}A(d_1(I - A) - b_{11}A)^{-1}A + b_{22}A$ is a linear, compact and symmetric operator and $N : W_D^{1,2}(\Omega) \to W_D^{1,2}(\Omega)$ is a compact and continuous nonlinear operator satisfying

$$\lim_{v\to 0} \frac{N(v)}{\|v\|} = 0. \tag{2.63}$$

Proof. Define the operator $T_1 : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ by

$$T_1(u, v) := d_1(I - A)u - b_{11}Au - b_{12}Av - N_1(u, v).$$

The assumption $d_1 \neq y_j$ for all $j \in \mathbb{N}$ excludes that d_1 is an eigenvalue of the operator

$$\partial_u T_1(0, 0) = d_1(I - A) - b_{11}A,$$

and according to Fredholm Alternative the operator $\partial_u T_1(0, 0)$ is invertible. The rest of the proof is an analogue of the proof of Theorem 1. □

Similarly to Dirichlet case, our main interest will be the equation

$$d_2(I - A)v - Sv - N(v) + \beta^+(v) + \beta^-(v) = 0. \tag{2.64}$$

It will be necessary to also analyze the homogenized problem

$$d_2(I - A)v - Sv + \beta^+(v) + \beta^-(v) = 0. \tag{2.65}$$

as the following remark explains.

Remark 14. *Let $d_1 > 0, d_1 \neq y_j$ for all $j \in \mathbb{N}$. In Chapter 3 we will prove one lemma which applied to the problems (2.64) and (2.65) will give a following conclusion:*
Any bifurcation point of (2.64) is simultaneously a critical point of the problem (2.65).

The analogous version of Corollary 1 applies here as well.

Corollary 2 (Corollary of Theorem 2 and Remark 13)**.** *Let $d_1 > 0$ be fixed and $d_1 \neq y_j$ for all $j \in \mathbb{N}$.*

A number $d_2 > 0$ is a critical point of (1.16) with Neumann b.c. and with fixed d_1 if and only if d_2 is a critical point of the problem (2.65). A number $d_2 > 0$ is a bifurcation point of the system (1.14) with Neumann b.c. and with fixed d_1 if and only if d_2 is a bifurcation point of the equation (2.64).

In particular, $d_2 > 0$ is a critical point of (1.12) with Neumann b.c. and with fixed d_1 if and only if d_2 is a critical point of the equation (2.59). And $d_2 > 0$ is a bifurcation point of the system (1.9), with Neumann b.c. and with fixed d_1, if and only if d_2 is a bifurcation point of the equation (2.64) with $\beta^{\pm} \equiv 0$ and $N = b_{21}A(F(v) - F'(0)) + N_2(F(v), v)$ (cf. (2.46)), respectively.

And also it is possible to find an analogue of Lemma 9.

Lemma 13. *Let V be the set form Theorem 2. If (2.27), (2.29), (2.31) are true, the operator $S + N - \beta^+ - \beta^-$ from the problem (2.62) has on the set V a potential.*

Proof. The proof is analogous to the proof of Lemma 9 and therefore will be skipped. □

2.4 Systems with unilateral terms on the boundary

At the end of this section the problem with unilateral sources on the boundary will be discussed. The abstract formulation is analogous to previous cases and therefore will be only briefly commented.

Since the problem will change, we will assume here only (2.1)–(2.5) and drop the assumption (2.6)–(2.9).

Mixed problem with unilateral sources on the boundary Suppose (2.10). Consider the system

$$\begin{aligned} d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v) &= 0 \\ d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v) &= 0 \end{aligned} \quad \text{in } \Omega, \tag{2.66}$$

with boundary conditions

$$\begin{aligned} &u = v = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma_N, \\ &\frac{\partial v}{\partial \vec{n}} = s_-(x)v^- - s_+(x)v^+ \quad \text{on } \Gamma_N, \end{aligned} \tag{2.67}$$

where

$$s_\pm \in L^\infty(\Gamma_N), \tag{2.68}$$

and

$$\text{ess supp}(s_-) \cap \text{ess supp}(s_+) = \emptyset. \tag{2.69}$$

The boundary conditions are considered in a sense of traces. Since $s_-(x)v^- - s_+(x)v^+ \in L^2(\Gamma_N)$, the boundary conditions are well-defined, see e.g. [2]. The system (2.66), (2.67) has for v three types of boundary conditions: Dirichlet b.c. on the set Γ_D, the unilateral conditions on the set $\text{ess supp}(s_-(x)) \cup \text{ess supp}(s_+(x))$ and Neumann b.c. on the set $\Gamma_N \setminus (\text{ess supp}(s_-) \cup \text{ess supp}(s_+))$. The space of solutions will be $W^{1,2}_D(\Omega)$.

We define the operators $\beta^\pm_U : W^{1,2}_D(\Omega) \to W^{1,2}_D(\Omega)$ for unilateral terms by

$$\begin{aligned} \langle \beta^-_U(v), \varphi \rangle &= -\int_{\Gamma_N} s_-(x)v^-\varphi \text{ dS} \quad \text{for all } v, \varphi \in W^{1,2}_D(\Omega), \\ \langle \beta^+_U(v), \varphi \rangle &= \int_{\Gamma_N} s_+(x)v^+\varphi \text{ dS} \quad \text{for all } v, \varphi \in W^{1,2}_D(\Omega). \end{aligned}$$

The derivation of the operator equations from (2.66), (2.67) is similar to Section 2.2.1 giving

$$\begin{aligned} &d_1 u - b_{11}Au - b_{12}Av - N_1(u,v) = 0, \\ &d_2 v - b_{21}Au - b_{22}Av - N_2(u,v) + \beta^-_U(v) + \beta^+_U(v) = 0, \end{aligned} \tag{2.70}$$

The weak form of the linear problem

$$\begin{aligned} d_1\Delta u + b_{11}u + b_{12}v &= 0 \\ d_2\Delta v + b_{21}u + b_{22}v &= 0 \end{aligned} \quad \text{in } \Omega, \tag{2.71}$$

with the b.c. (2.67) leads to a system of operator equations

$$\begin{aligned} &d_1 u - b_{11}Au - b_{12}Av = 0, \\ &d_2 v - b_{21}Au - b_{22}Av + \beta^-_U(v) + \beta^+_U(v) = 0. \end{aligned} \tag{2.72}$$

An analogue of Theorem 1 can be proved to get that a system in a form

$$\begin{aligned} d_2 v - Sv - N(v) + \beta^-_U(v) + \beta^+_U(v) &= 0, \\ u &= F(v) \end{aligned} \tag{2.73}$$

is equivalent on a neighborhood of the origin with (2.70). Since the upper equation in (2.70) is the same as (2.25), the operators F, S in (2.73) are also the same as F, S in Theorem 1 and N is a nonlinear compact perturbation satisfying (2.44). This leads us to the following conclusions (cf. Corollary 1).

Any critical point of

$$d_2 v - Sv + \beta_U^-(v) + \beta_U^+(v) = 0,$$

is simultaneously a critical point of (2.71), (2.67) with fixed d_1 and vice versa. Any bifurcation point of (2.70) is a bifurcation point of (2.66), (2.67) with fixed d_1 and vice versa.

To avoid some technical complications arising from the presence of the set Γ_D, we have not included general nonlinear terms $\hat{g}_\pm$ in the boundary conditions like in Neumann case below. However, it is technically feasible to extend the results also for this systems with these functions in boundary condition.

Neumann problem with unilateral sources on the boundary Now consider a system (2.66) with b.c.

$$\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega, \quad \frac{\partial v}{\partial \vec{n}} = \hat{g}_-(x, v^-) - \hat{g}_+(x, v^+) \text{ on } \partial\Omega, \tag{2.74}$$

where $\hat{g}_\pm : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory conditions and $v^\pm$ in the boundary conditions are considered in the sense of traces. Moreover, suppose that $\hat{g}_\pm(x, 0) = 0$ for a.a $x \in \partial\Omega$ and $\hat{g}^\pm$ satisfy the growth conditions

$$\text{there exists } C > 0 \text{ such that:} \tag{2.75}$$

$$|\hat{g}_-(x,\xi)| + |\hat{g}_+(x,\xi)| \leq C(1 + |\xi|^{p-1}), \quad \text{for a.a. } x \in \partial\Omega, \text{ for all } \xi \in \mathbb{R}, \tag{2.76}$$

for some p satisfying

$$p > 2 \text{ for } m \leq 2 \quad \text{or} \quad 2 < p < \frac{2(m-1)}{m-2} \text{ for } m > 2, \tag{2.77}$$

cf. (2.5). Denote

$$s_\pm(x) := \frac{\partial \hat{g}_\pm}{\partial \xi}(x, \xi)\big|_{\xi=0},$$

and assume that $s_\pm \in L^\infty(\partial\Omega)$, $s_\pm(x) \geq 0$ for a.a. $x \in \partial\Omega$, and

$$\text{ess supp}(s_-) \cap \text{ess supp}(s_+) = \emptyset.$$

The abstract formulation of Neumann problem is similar to the mixed case. We define the operators $\hat{G}_U^\pm : W_D^{1,2}(\Omega) \to W_D^{1,2}(\Omega)$ and $\beta_U^\pm : W_D^{1,2}(\Omega) \to W_D^{1,2}(\Omega)$ by

$$\begin{aligned}
\langle \hat{G}_U^-(v), \varphi \rangle &= -\int_{\partial\Omega} \hat{g}_-(x, v^-)\varphi \text{ dS} \;\; \text{for all } v, \varphi \in W_D^{1,2}(\Omega), \\
\langle \hat{G}_U^+(v), \varphi \rangle &= \int_{\partial\Omega} \hat{g}_+(x, v^+)\varphi \text{ dS} \;\; \text{for all } v, \varphi \in W_D^{1,2}(\Omega), \\
\langle \beta_U^-(v), \varphi \rangle &= -\int_{\partial\Omega} s_-(x) v^- \varphi \text{ dS} \;\; \text{for all } v, \varphi \in W_D^{1,2}(\Omega), \\
\langle \beta_U^+(v), \varphi \rangle &= \int_{\partial\Omega} s_+(x) v^+ \varphi \text{ dS} \;\; \text{for all } v, \varphi \in W_D^{1,2}(\Omega).
\end{aligned}$$

The well-definition of $\hat{G}_U^\pm$ is maybe not clear at the first sight. However, one has to employ the assumption (2.76) and embedding $W^{\frac{1}{2},2}(\partial\Omega) \hookrightarrow L^{p_0}(\partial\Omega)$ with $p_0 \in [1, (2m-2)/(m-2))$, see Theorem 23. Furthermore, the operators $\beta_U^\pm$ are homogenizations of $\hat{G}_U^\pm$, in the sense of Lemma 3. The proofs are analogous to the proof of Lemma 3. The homogenized boundary conditions are then

$$\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega, \qquad \frac{\partial v}{\partial \vec{n}} = s_-(x) v^- - s_+(x) v^+ \text{ on } \partial\Omega. \tag{2.78}$$

Procedure analogous to the one from Section 2.3 gives the weak formulation of (2.66), (2.74) as

$$\begin{aligned} &d_1(I-A)u - b_{11}Au - b_{12}Av - N_1(u,v) = 0, \\ &d_2(I-A)v - b_{21}Au - b_{22}Av - N_2(u,v) + \hat{G}_U^-(v) + \hat{G}_U^+(v) = 0, \end{aligned} \tag{2.79}$$

and the weak formulation of (2.71), (2.78) as

$$\begin{aligned} &d_1(I-A)u - b_{11}Au - b_{12}Av = 0, \\ &d_2(I-A)v - b_{21}Au - b_{22}Av + \beta_U^-(v) + \beta_U^+(v) = 0. \end{aligned} \tag{2.80}$$

An analogue of Theorem 1 can be proved to get that a system in the form

$$\begin{aligned} d_2(I-A)v - Sv - N(v) + \beta_U^-(v) + \beta_U^+(v) = 0, \\ u = F(v), \end{aligned} \tag{2.81}$$

is equivalent in a neighborhood of the origin with (2.79). For both problems the operators F, S are again the same as in Theorem 1, and N is a nonlinear compact perturbation satisfying (2.44).

The conclusions analogue to Corollary 1 can be found here.

Any critical point the equation

$$d_2(I-A)v - Sv + \beta_U^-(v) + \beta_U^+(v) = 0,$$

is simultaneously a critical point of (2.71), (2.78) with fixed d_1 and vice versa. Any bifurcation point of (2.79) is a bifurcation point of (2.66), (2.74) with fixed d_1 and vice versa.

2.5 Sets of critical points of reaction diffusion systems

This section is intended as an extension of Remark 3 from Section 1.6. The explicit formulae (2.39) and (2.61) for the eigenvalues λ_k^S of (2.37) and the critical points $d_{2,k}^0$ of (2.59), respectively, are the same. Therefore the discussion about their interpretation can be done for both problems at once. Assume $d_1, d_2 \in \mathbb{R}$, and besides the assumptions from Section 2.1, suppose also (2.21), (2.27).

Since all eigenvalues of (2.37) and critical points of (2.59) are simultaneously the critical points of (1.12) with Dirichlet/mixed and Neumann boundary conditions respectively and with fixed d_1, the set of all couples (d_1, d_2) for which these equations have solutions can be written as

$$\tilde{C} = \bigcup_{k=1}^{\infty} \tilde{C}_k$$

where the sets $\tilde{C}_k$ are

$$\tilde{C}_k = \left\{ d = (d_1, d_2) \in \mathbb{R}^2 \;\middle|\; d_2 := \frac{1}{\kappa_k} \left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22} \right) \right\}.$$

Because the diffusion coefficients are supposed to be positive, it is useful to introduce a sets C_k which are the parts of $\tilde{C}_k$ lying in the positive quadrant of $\mathbb{R}^2$, i.e.

$$C_k = \left\{ d = (d_1, d_2) \in \mathbb{R}_+^2 \;\middle|\; d_2 := \frac{1}{\kappa_k} \left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22} \right) \right\}.$$

The envelope of the sets C_j will be denoted by

$$C_E = \left\{ \left(d_1^E, d_2^E\right) \in \mathbb{R}_+^2 \mid \left(d_1^E, d_2^E\right) = \min_k \{(d_1, d_2(\kappa_k)) \in C_k\} \right\}.$$

The sets $\tilde{C}_k$, C_k and C_E have a nice geometrical interpretation. If one plots $\tilde{C}_k$ in the space $\mathbb{R}^2$, he finds that each of them consists of of two hyperbolas with an asymptote

$$y_k := \frac{b_{11}}{\kappa_k}. \tag{2.82}$$

First hyperbola is in the quadrant $(0, \infty) \times (-\infty, 0)$, the second one lies in the remaining quadrants. The part of the second hyperbola being in the quadrant $\mathbb{R}^2_+$ is the set C_k. The envelope of all hyperbolas C_k is the set C_E. The whole situation is sketched at the Fig. 2.1

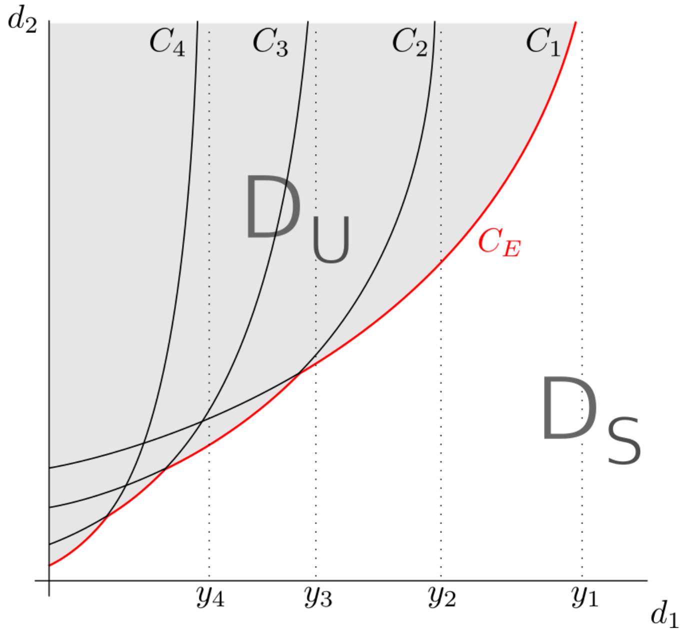


Figure 2.1: Space of parameters, dotted lines are asymptotes y_i

It is common to define two sets of parameters D_U and D_S

$$D_U = \left\{(d_1, d_2) \in \mathbb{R}^2_+ \mid \ d \text{ lies to the left from at least one } C_j,\ j \in \mathbb{N}\right\},$$

$$D_S = \left\{(d_1, d_2) \in \mathbb{R}^2_+ \mid \ d \text{ lies to the right from all } C_j,\ j \in \mathbb{N}\right\}.$$

The sets D_S resp. D_U are called the domain of stability resp. domain of instability, the origin of this name is explained by the following remark.

Remark 15. *Let us consider an eigenvalue problem*

$$\begin{aligned} d_1 \triangle u + b_{11} u + b_{12} v &= \lambda u, \\ d_2 \triangle v + b_{21} u + b_{22} v &= \lambda v. \end{aligned}$$

If $d \in D_S$ then there exists $\varepsilon > 0$ such that Re $\lambda < -\varepsilon < 0$ for all eigenvalues of the system, or if $d \in D_U$ there exists at least one positive eigenvalue $\lambda > 0$ of the system.

Chapter **3**

Abstract results about positively homogeneous problems

This part of dissertation thesis contains results about equations with positively homogeneous operators. It is designed as a stand-alone section therefore it is possible to read it independently on other sections. For this reason the notation introduced here is valid only for this chapter and has no relation to objects stated before. The studied equations are however formulated in a way which allows a straightforward application of achieved results to reaction-diffusion systems with unilateral terms.

This chapter will be divided into four main sections. The notation is introduced in the first one, and also some introductory remarks and general assumptions are given there. The second one contains the results concerning eigenvalues of positively homogeneous operators, and also a bifurcation result. All results therein are achieved by a use of the variational methods. The third section contains the results obtained by the topological degree methods and the last one is containing two bifurcation theorems proved by application of the Implicit Function Theorem. The second theorem, very general, stated without proof, is a result of Lutz Recke and Martin Väth [47].

All of these results will be used in the forthcoming sections in order to study the existence of critical and bifurcation points of reaction-diffusion systems with unilateral terms.

3.1 Basic notation and assumptions

The following basic notation and assumptions will be used throughout this chapter.

(i) $\mathbb{H}$ will denote a real Hilbert space with a scalar product $\langle\cdot,\cdot\rangle$ and a norm $\|\cdot\|$ induced by it, $\mathcal{I}$ will be the identity operator on $\mathbb{H}$.

(ii) $\mathcal{B}:\mathbb{H}\to\mathbb{H}$ will be a positively homogeneous operator of the degree one, i.e.

$$\mathcal{B}(tv)=t\mathcal{B}(v) \text{ for all } t\geq 0,$$

and will satisfy

$$v_n\rightharpoonup v \quad\Rightarrow\quad \mathcal{B}(v_n)\to\mathcal{B}(v), \tag{3.1}$$

and

$$\langle\mathcal{B}(v),v\rangle\geq 0 \quad \text{for all } v\in\mathbb{H}. \tag{3.2}$$

(iii) Let us denote

$$|\mathcal{B}|:=\max_{v\in\mathbb{H},\|v\|=1}\|\mathcal{B}(v)\|. \tag{3.3}$$

The assumption (3.1) guarantees the existence of the maximum in (3.3).

(iv) $\lambda \in \mathbb{R}$ will be a parameter.

(v) $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ will be a linear positive compact operator with $\sigma(\mathcal{A}) \subset [0,1]$, and having the largest eigenvalue simple. If $1 \in \sigma(\mathcal{A})$, then $e_0 \in \mathbb{H}, \|e_0\| = 1$ will denote the corresponding eigenvector.

(vi) $\mathcal{S} : \mathbb{H} \to \mathbb{H}$ will be a linear symmetric compact operator. Its maximal eigenvalue, if it exists, will be denoted by $\lambda^S_{\max}$.

(vii) If $1 \in \sigma(\mathcal{A})$, we will assume

$$\langle \mathcal{S}e_0 - \mathcal{B}(e_0), e_0 \rangle \neq 0, \ \langle \mathcal{S}(-e_0) - \mathcal{B}(-e_0), -e_0 \rangle \neq 0. \tag{3.4}$$

(viii) $\mathcal{N} : \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ will be a small compact nonlinear perturbation, i.e. $\mathcal{N}$ is a continuous compact nonlinear operator satisfying

$$\lim_{v \to 0} \frac{\mathcal{N}(\lambda, v)}{\|v\|} = 0, \quad \text{uniformly on } \lambda \text{ - compact intervals.} \tag{3.5}$$

Definition 7. *If there exists $v \in \mathbb{H}, \ v \neq 0$ such that*

$$\lambda(\mathcal{I} - \mathcal{A})v - \mathcal{S}v + \mathcal{B}(v) = 0, \tag{3.6}$$

for some $\lambda \in \mathbb{R}$, then v is called an eigenvector and λ the eigenvalue of (3.6).

Observation 2. *If v is an eigenvector of (3.6) corresponding to an eigenvalue λ, then for any $\alpha > 0$ the vector αv is also an eigenvector of this equation corresponding to λ. This is a consequence of the positive homogeneity of $\mathcal{B}$.*

Besides eigenvalues of (3.6) we are going to study bifurcation points of a related nonlinear problem.

Definition 8. *A number $\lambda_b \in \mathbb{R}$ is a (local) bifurcation point of*

$$\lambda(\mathcal{I} - \mathcal{A})v - \mathcal{S}v - \mathcal{N}(\lambda, v) + \mathcal{B}(v) = 0. \tag{3.7}$$

if in any neighborhood of $(\lambda_b, 0)$ in $\mathbb{R} \times \mathbb{X}$ a nontrivial solution $(\lambda, v) \in \mathbb{R} \times \mathbb{X}$ of (3.7) exists.

In the forthcoming chapters we will work with the equation (3.6) and also with a special case of this equation with $\mathcal{A} \equiv 0$. In such a case the equation (3.6) becomes an eigenvalue problem for positively homogeneous operator $\mathcal{S} - \mathcal{B}$:

$$\lambda v - \mathcal{S}v + \mathcal{B}(v) = 0. \tag{3.8}$$

The nonlinear problem (3.7) with $\mathcal{A} \equiv 0$ has the form

$$\lambda v - \mathcal{S}v - \mathcal{N}(\lambda, v) + \mathcal{B}(v) = 0. \tag{3.9}$$

Remark 16. *The equations (3.6) and (3.8) are generalizations of the abstract formulations (2.65) and (2.50) respectively, to a general Hilbert space. The problems (3.7) and (3.9) are generalizations of (2.62) and (2.49), respectively.*

Observation 3. *It follows immediately from the symmetry of $\mathcal{A}$ and $\sigma(\mathcal{A}) \subset [0,1]$ that*

$$\langle (\mathcal{I} - \mathcal{A})v, v \rangle \in [0,1] \quad \textit{for all } v \in \mathbb{H}, \ \|v\| = 1. \tag{3.10}$$

If $1 \notin \sigma(\mathcal{A})$, then

$$\inf_{v \in \mathbb{H}, \|v\|=1} \langle (\mathcal{I} - \mathcal{A})v, v \rangle > 0. \tag{3.11}$$

If $1 \in \sigma(\mathcal{A})$, then $Ker(\mathcal{I} - \mathcal{A}) = Span\{e_0\}$ and

$$\langle \mathcal{A}v, v \rangle = \|v\|^2 \quad \textit{if and only if} \ \ v \in Ker(\mathcal{I} - \mathcal{A}). \tag{3.12}$$

3.2 Variational methods

This section will be divided into two parts. The first one contains abstract results concerning the eigenvalues of a positively homogeneous equations. The second one is devoted to existence of bifurcation from the largest eigenvalue of a positively homogeneous equation.

3.2.1 Eigenvalues of equations with positively homogeneous operators

Notation 3. *The basic notation from Section 3.1 will be supplemented with the following one:*

(I) $\mathscr{K} \subset \mathbb{H}$ *will denote a closed convex cone defined by*

$$\mathscr{K} := \{v \in \mathbb{H} \mid \mathcal{B}(v) = 0\}.$$

(II) The second largest eigenvalue of $\mathcal{A}$ *will be denoted as* λ_2^A.

In addition, we will often use in this section the following assumptions:

$$\langle \mathcal{B}(v), v) \rangle > 0 \text{ for all } v \notin \mathscr{K}, \tag{3.13}$$

and

$$\text{if } 1 \in \sigma(\mathcal{A}), \text{ the vector } e_0 \text{ satisfies: } e_0 \notin \mathscr{K} \cup (-\mathscr{K}). \tag{3.14}$$

The assumptions (3.13), (3.14) imply $\langle \mathcal{B}(e_0), e_0 \rangle > 0$, $\langle \mathcal{B}(-e_0), -e_0 \rangle > 0$. The assumption (3.4) excludes e_0 and $-e_0$ as eigenvectors of (3.6). According to Observation 3 any eigenvector v of (3.6) satisfies $\langle (I - A)v, v \rangle > 0$.

Since this subsection is about eigenvalues, the operator $\mathcal{N}$ will not appear here.

Theorem 3. *Let* $v_0 \in \mathbb{H}$ *satisfy*

$$\lim_{t \to 0} \frac{1}{t} \left(\frac{1}{2} \langle \mathcal{B}(v_0 + th), v_0 + th \rangle - \frac{1}{2} \langle \mathcal{B}(v_0), v_0 \rangle \right) = \langle \mathcal{B}(v_0), h \rangle, \quad \textit{for all } h \in \mathbb{H}, \tag{3.15}$$

and let the maximum

$$\lambda_{max} := \max_{\substack{v \in \mathbb{H} \\ v \notin Ker(\mathcal{I} - \mathcal{A})}} \frac{\langle \mathcal{S}v - \mathcal{B}(v), v \rangle}{\langle (\mathcal{I} - \mathcal{A})v, v \rangle} \tag{3.16}$$

exist with v_0 *being its maximizer. Then* $\lambda_{\max}$ *is the largest eigenvalue of the problem (3.6) and* v_0 *is a corresponding eigenvector.*

Let (3.15) be fulfilled for any $v \in \mathbb{H}$ *and let the largest eigenvalue of (3.6) exist with* $v_0 \in \mathbb{H}$ *being an eigenvector respective to it. Then the largest eigenvalue is equal to* $\lambda_{\max}$ *and* v_0 *is a maximizer of (3.16).*

It is easy to see that the maximum (3.16) does not always exist – for example if $\mathcal{B}$ is the zero operator and S is a negative operator with an infinite-dimensional range, the assumption (3.15) is fulfilled trivially and the supremum over the argument in (3.16) is equal to zero. If the maximum existed, the number zero would be the largest eigenvalue of the compact operator S with the infinite dimensional range, which is not possible and therefore the maximum does not exists. We are going to give some criteria for the existence of maximum in Theorem 4 on pg. 40 below.

Notation 4. *We will be using the notation* $\lambda_{\max}$ *for the largest eigenvalue of (3.6), whenever the maximum (3.16) exists and (3.15) is true.*

Proof of Theorem 3. Let (3.15) be fulfilled. We have to prove that $\lambda_{\max}$ is the largest eigenvalue of the equation (3.6). Since $v_0 \notin \text{Ker}(\mathcal{I} - \mathcal{A})$, there exists $t_0 > 0$ such that $\langle (\mathcal{I} - \mathcal{A})(v_0 + th), (v_0 + th) \rangle >$

0 for all $t \in (0, t_0)$ and for all $h \in \mathbb{H}$, $\|h\| = 1$. The vector v_0 is a maximizer of (3.16), hence, any such $v_0 + th$ satisfies

$$\frac{\langle \mathcal{S}(v_0 + th) - \mathcal{B}(v_0 + th), v_0 + th\rangle}{\langle (\mathcal{I} - \mathcal{A})(v_0 + th), v_0 + th\rangle} \leq \frac{\langle \mathcal{S}v_0 - \mathcal{B}(v_0), v_0\rangle}{\langle (\mathcal{I} - \mathcal{A})v_0, v_0\rangle}. \tag{3.17}$$

We will write the expression (3.17) as

$$\begin{aligned}&\left(\langle \mathcal{S}v_0, v_0\rangle + 2t\langle \mathcal{S}v_0, h\rangle + t^2\langle \mathcal{S}h, h\rangle - \langle \mathcal{B}(v_0 + th), v_0 + th\rangle\right)\langle (\mathcal{I} - \mathcal{A})v_0, v_0\rangle \leq \\ &\quad \leq \langle \mathcal{S}v_0 - \mathcal{B}(v_0), v_0\rangle \left(\langle (\mathcal{I} - \mathcal{A})v_0, v_0\rangle + 2t\langle (\mathcal{I} - \mathcal{A})v_0, h\rangle + t^2\langle (\mathcal{I} - \mathcal{A})h, h\rangle\right).\end{aligned}$$

Dividing by $\langle (\mathcal{I} - \mathcal{A})v_0, v_0\rangle$ and using the notation

$$\lambda_{\max} = \frac{\langle \mathcal{S}v_0 - \mathcal{B}(v_0), v_0\rangle}{\langle (\mathcal{I} - \mathcal{A})v_0, v_0\rangle},$$

yields

$$\begin{aligned}&\left(2t\langle \mathcal{S}v_0, h\rangle + t^2\langle \mathcal{S}h, h\rangle - \langle \mathcal{B}(v_0 + th), v_0 + th\rangle + \mathcal{B}(v_0), v_0\rangle\right) \leq \\ &\quad \leq \langle \lambda_{\max}\left(\langle 2t\langle (\mathcal{I} - \mathcal{A})v_0, h\rangle + t^2\langle (\mathcal{I} - \mathcal{A})h, h\rangle\right).\end{aligned}$$

After dividing by $2t$ and evaluating the limits $t \to 0+, t \to 0-$ we obtain by using (3.15) two inequalities:

$$\begin{aligned}\langle \mathcal{S}v_0, h\rangle - \langle \mathcal{B}(v_0), h\rangle &\leq \lambda_{\max}\langle (\mathcal{I} - \mathcal{A})v_0, h\rangle, \\ \langle \mathcal{S}v_0, h\rangle - \langle \mathcal{B}(v_0), h\rangle &\geq \lambda_{\max}\langle (\mathcal{I} - \mathcal{A})v_0, h\rangle.\end{aligned}$$

This implies

$$\langle \mathcal{S}v_0, h\rangle - \langle \mathcal{B}(v_0), h\rangle = \lambda_{\max}\langle (\mathcal{I} - \mathcal{A})v_0, h\rangle \quad \text{for all } h \in \mathbb{H},\ \|h\| = 1,$$

and finally

$$\mathcal{S}v_0 - \mathcal{B}(v_0) = \lambda_{\max}(\mathcal{I} - \mathcal{A})v_0,$$

which means that $\lambda_{\max}$ is an eigenvalue of the equation (3.6) with the corresponding nontrivial solution v_0. Let $\lambda_1 > 0$ be another eigenvalue of this equation and v_1 be a corresponding nontrivial solution. Then

$$\lambda_1(\mathcal{I} - \mathcal{A})v_1 = \mathcal{S}v_1 - \mathcal{B}(v_1). \tag{3.18}$$

Because e_0 cannot be the eigenvector, see (3.4), multiplication of this equation by v_1 and dividing by $\langle (\mathcal{I} - \mathcal{A})v_1, v_1\rangle$ gives

$$\lambda_1 = \frac{\langle \mathcal{S}v_1 - \mathcal{B}(v_1), v_1\rangle}{\langle (\mathcal{I} - \mathcal{A})v_1, v_1\rangle} \leq \frac{\langle \mathcal{S}v_0 - \mathcal{B}(v_0), v_0\rangle}{\langle (\mathcal{I} - \mathcal{A})v_0, v_0\rangle} = \lambda_{\max}. \tag{3.19}$$

Hence, $\lambda_{\max}$ is the largest eigenvalue of the equation (3.6).

Let $v_0 \in \mathbb{H}$ be an eigenvector to the largest eigenvalue of (3.6) and assume that v_0 is not maximizing (3.16). Let us denote here the largest eigenvalue as λ_m. Multiplying the equation

$$\lambda_m(I - A)v_0 - Sv_0 + B(v_0) = 0$$

by v_0 and dividing it by $\langle (I - A)v_0, v_0\rangle$ give

$$\lambda_m = \frac{\langle \mathcal{S}v_0 - \mathcal{B}(v_0), v_0\rangle}{\langle (\mathcal{I} - \mathcal{A})v_0, v_0\rangle} < \lambda_{\max}.$$

Since (3.15) is assumed for all $v \in \mathbb{H}$, any maximizer of (3.16) satisfies this condition and according to the first statement of this theorem it is an eigenvector to (3.6) corresponding to $\lambda_{\max}$. So $\lambda_{\max}$ is an eigenvalue larger than the largest eigenvalues λ_m which is a contradiction. Hence, v_0 must the maximizer of (3.16) and $\lambda_m = \lambda_{\max}$.

□

Remark 17. *If (3.15) is fulfilled for any $v_0 \in \mathbb{H}$, the operator $\mathcal{B}$ has a potential (see Definition 20 in Appendix) defied by $\Phi_B(v) := 2^{-1}\langle \mathcal{B}(v), v\rangle$ for all $v \in \mathbb{H}$.*

Observation 4. *Let $\mathcal{A} \equiv 0$. It is easy to see from (3.3), (3.2) that if $\lambda_{\max}$ exists, then*

$$\lambda^S_{\max} = \max_{v\in\mathbb{H},\|v\|=1} \langle \mathcal{S}v, v\rangle \geq \max_{v\in\mathbb{H},\|v\|=1} \langle \mathcal{S}v, v\rangle - \langle \mathcal{B}(v), v\rangle) = \lambda_{\max} \geq$$
$$\geq \max_{v\in\mathbb{H},\|v\|=1} \langle \mathcal{S}v, v\rangle - \max_{v\in\mathbb{H},\|v\|=1} \langle \mathcal{B}(v), v\rangle \geq \lambda^S_{\max} - |\mathcal{B}|.$$

These inequalities will play an essential role in estimating the critical points of the reaction-diffusion systems.

The following example is inspired by the Example 1 in [32]. It demonstrates that violation of (3.15) can lead to the non-existence of the largest eigenvalue of (3.8).

Remark 18. *Let $\mathbb{H} := \mathbb{R}^3$, $\mathcal{A} = 0$ and let $\mathcal{B}, \mathcal{S}$ be defined by matrices*

$$\mathcal{B} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{a}{2} & a \\ 0 & -a & \frac{a}{2} \end{pmatrix}, \qquad \mathcal{S} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3a}{2} & 0 \\ 0 & 0 & \frac{3a}{2} \end{pmatrix},$$

with $a > 0$ being a parameter. Then

$$\mathcal{S} - \mathcal{B} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -a \\ 0 & a & a \end{pmatrix}.$$

Since

$$\langle \mathcal{B}(v), v\rangle = \frac{a}{2}(\|v_2\|^2 + \|v_3\|^2) \geq 0 \textit{ for all } v := (v_1, v_2, v_3) \in \mathbb{R}^3,$$

condition (3.2) is fulfilled. Due to the finite dimension of the problem the operators $\mathcal{S}$ and $\mathcal{B}$ are compact, $\mathcal{S}$ is clearly symmetric. The eigenvalues of (3.8) are $\lambda_{\max} = 1$, $\lambda_2 = a(1 + i)$, $\lambda_3 = a(1 - i)$. If $a = 1$, then

$$\langle \mathcal{S}v - \mathcal{B}v, v\rangle = \|v\|^2, \quad \textit{for all } v \in \mathbb{R}^3,$$

and the maximum in (3.16) is 1. However, only the maximizer $v := (1, 0, 0)$ and its multiples satisfy the condition (3.15) and $v, -v$ are the only eigenvectors with unit norm corresponding to the eigenvalue $\lambda_{\max}$. If $a = 2$, then the maximum in (3.16) is 2, no maximizer satisfies (3.15) and the largest eigenvalue of the equation is not characterized by the formula (3.16).

Two assumptions will play a crucial role in the following theorem. The first one is that

$$\text{there exists } v \in \mathbb{H} \text{ such that } \frac{\langle \mathcal{S}v, v\rangle - \langle \mathcal{B}(v), v\rangle}{\langle (\mathcal{I} - \mathcal{A})v, v\rangle} \in (0, \infty). \tag{3.20}$$

This assumption can be difficult to verify (because it contains a nonlinear operator) therefore in practice we will be sometimes checking the following stronger assumption:

$$\text{there exists } \varphi \in \mathscr{K} \text{ satisfying } \frac{\langle \mathcal{S}\varphi, \varphi\rangle}{\langle (\mathcal{I} - \mathcal{A})\varphi, \varphi\rangle} \in (0, \infty). \tag{3.21}$$

Theorem 4. *Let $1 \notin \sigma(\mathcal{A})$ and (3.20) be true. Then the maximum in (3.16) exists and is positive.*

Let $1 \in \sigma(\mathcal{A})$ and (3.13), (3.14), (3.20) be true. Under the condition

$$\max \left\{ \frac{\langle \mathcal{S}e_0, e_0\rangle}{\langle \mathcal{B}(e_0), e_0\rangle}, \frac{\langle \mathcal{S}e_0, e_0\rangle}{\langle \mathcal{B}(-e_0), -e_0\rangle}, 0 \right\} < 1, \tag{3.22}$$

the maximum in (3.16) exists and is positive.

Proof. The proof of first statement follows directly from (3.20), (3.11), (3.1) and compactness of $\mathcal{S}$ and $\mathcal{A}$.

Let $1 \in \sigma(\mathcal{A})$. Due to the assumptions (3.1), (3.2), (3.22) and because $\mathcal{S}$ is continuous, there exists a neighborhood V of e_0 such that

$$\langle \mathcal{S}v, v \rangle - \langle \mathcal{B}(v), v \rangle < 0, \quad \text{for all } v \in V \cup (-V). \tag{3.23}$$

There also exists a sequence v_n, $\|v_n\| = 1$ such that

$$\lim_{n\to\infty} \frac{\langle \mathcal{S}v_n, v_n \rangle - \langle \mathcal{B}(v_n), v_n \rangle}{\langle (\mathcal{I} - \mathcal{A})v_n, v_n \rangle} = \sup_{v\in\mathbb{H}, v\neq 0} \frac{\langle \mathcal{S}v, v \rangle - \langle \mathcal{B}(v), v \rangle}{\langle (\mathcal{I} - \mathcal{A})v, v \rangle}.$$

Due to the assumption (3.20) this supremum is positive. Because $\mathbb{H}$ is a Hilbert space, we can assume without loss of generality that $v_n \rightharpoonup v_0$. Let us suppose that

$$\lim_{n\to\infty} \frac{\langle \mathcal{S}v_n, v_n \rangle - \langle \mathcal{B}(v_n), v_n \rangle}{\langle (\mathcal{I} - \mathcal{A})v_n, v_n \rangle} = +\infty. \tag{3.24}$$

Because of (3.2) we have $\langle \mathcal{B}(v_n), v_n \rangle \geq 0$ for all $n \in \mathbb{N}$ and consequently

$$\langle \mathcal{S}v_n, v_n \rangle - \langle \mathcal{B}(v_n), v_n \rangle \leq \langle \mathcal{S}v_n, v_n \rangle \leq |\mathcal{S}|,$$

thus the l.h.s of this inequality is uniformly bounded from above and it means that (3.24) can be satisfied only if

$$\langle (\mathcal{I} - \mathcal{A}), v_n, v_n \rangle = 1 - \langle \mathcal{A}v_n, v_n \rangle \to 0.$$

Because the operator $\mathcal{A}$ is linear and compact we have

$$\lim_{n\to\infty} \langle \mathcal{A}v_n, v_n \rangle = \langle \mathcal{A}v_0, v_0 \rangle,$$

which implies

$$\langle \mathcal{A}v_0, v_0 \rangle = 1.$$

We will prove that $\|v_0\| = 1$ which will due to the (3.12) imply that $v = \pm e_0$. The norm is weakly lower semicontinous, i.e. $\|v_0\| \leq 1$ and obviously $\|v_0\| > 0$. If it were $0 < \|v_0\| < 1$, then we would have

$$\left\langle \mathcal{A}\left(\frac{v_0}{\|v_0\|}\right), \frac{v_0}{\|v_0\|} \right\rangle = \frac{1}{\|v_0\|^2} > 1,$$

which contradicts $\sigma(\mathcal{A}) \subset (0, 1]$. Hence, $\|v_0\| = 1$ and we conclude that $v_0 = \pm e_0$. As $\mathcal{S}$ and $\mathcal{B}$ are compact and continuous the formula (3.23) gives the existence of n_0 such that

$$\langle \mathcal{S}v_n, v_n \rangle - \langle \mathcal{B}(v_n), v_n \rangle < 0, \quad \text{for all } n > n_0.$$

This together with (3.10) gives

$$\lim_{n\to\infty} \frac{\langle \mathcal{S}v_n, v_n \rangle - \langle \mathcal{B}(v_n), v_n \rangle}{\langle (\mathcal{I} - \mathcal{A})v_n, v_n \rangle} = -\infty,$$

which is a contradiction with the assumption (3.24). Hence,

$$\sup_{v\in\mathbb{H}, v\neq 0} \frac{\langle \mathcal{S}v, v \rangle - \langle \mathcal{B}(v), v \rangle}{\langle (\mathcal{I} - \mathcal{A})v, v \rangle} < \infty.$$

Use of $v_n \rightharpoonup v_0$, (3.1) and the compactness of $\mathcal{S}$, $\mathcal{A}$ give

$$\mathcal{S}v_n \to \mathcal{S}v_0, \quad \mathcal{A}v_n \to \mathcal{A}v_0, \quad \mathcal{B}(v_n) \to \mathcal{B}(v_0),$$

and this together with (3.20) lead to

$$\sup_{v\in\mathbb{H}, v\neq 0} \frac{\langle \mathcal{S}v, v \rangle - \langle \mathcal{B}(v), v \rangle}{\langle (\mathcal{I} - \mathcal{A})v, v \rangle} = \frac{\langle \mathcal{S}v_0, v_0 \rangle - \langle \mathcal{B}(v_0), v_0 \rangle}{\langle (\mathcal{I} - \mathcal{A})v_0, v_0 \rangle} \in (0, \infty).$$

□

Corollary 3. *Assume (3.15), (3.20). If either* $1 \notin \sigma(A)$ *or alternatively* $1 \in \sigma(A)$*, (3.13), (3.14), (3.22), then the number* $\lambda_{\max}$ *from Theorem 3 is the largest eigenvalue of the problem (3.6).*

Remark 19. *The assumption (3.22) is fulfilled if either* $\langle Se_0, e_0\rangle < 0$ *or*

$$\max\{\langle \mathcal{B}(e_0), e_0\rangle, \langle \mathcal{B}(-e_0), -e_0\rangle\}$$

is sufficiently large.

In terms of applications, the latter case will mean that the sources and sinks in reaction-diffusion system, represented by the operators $\beta^{\pm}$ *defined in Section 2.2, are sufficiently strong. See also Theorem 15 on pg. 70.*

It is crucial to mention that the number $\lambda_{\max}$ can diverge to infinity as the l.h.s of (3.22) approaches one. The violation of (3.22) gives the non-existence of maximum on the r.h.s. of (3.16), and the supremum of its argument is then equal to plus infinity.

The next lemma gives a behavior of the largest eigenvalue of the equation with positively homogeneous operator $\tau\mathcal{B}$ as $\tau \to \infty$, where τ is a real parameter.

Lemma 14. *Under the assumptions (3.13), (3.14), (3.21) the maxima satisfy*

$$\lim_{\tau\to\infty} \max_{\substack{v\in\mathbb{H}\\ v\notin Ker(\mathcal{I}-\mathcal{A})}} \frac{\langle \mathcal{S}v, v\rangle - \tau\langle \mathcal{B}(v), v\rangle}{\langle (\mathcal{I}-\mathcal{A})v, v\rangle} = \max_{v\in\mathscr{K}, v\neq 0} \frac{\langle \mathcal{S}v, v\rangle}{\langle (\mathcal{I}-\mathcal{A})v, v\rangle} > 0.$$

Proof. We will prove this Lemma by a contradiction. Let us suppose that there exist $\varepsilon > 0$ and sequences $\tau_n \to \infty$, $v_n \subset \mathbb{H}\backslash\mathrm{Ker}(\mathcal{I}-\mathcal{A})$, $\|v_n\| = 1$ satisfying

$$\frac{\langle \mathcal{S}v_n - \tau_n\mathcal{B}(v_n), v_n\rangle}{\langle (\mathcal{I}-\mathcal{A})v_n, v_n\rangle} \geq \max_{\varphi\in\mathscr{K}, \varphi\neq 0} \frac{\langle \mathcal{S}\varphi, \varphi\rangle}{\langle (\mathcal{I}-\mathcal{A})\varphi, \varphi\rangle} + \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{3.25}$$

Let us note that the maximum over the cone is due to the assumption (3.21) positive. The space $\mathbb{H}$ is reflexive, hence, without loss of generality $v_n \rightharpoonup v_0$, $v_0 \in \mathbb{H}$. Let us suppose that $v_0 = 0$. Then using (3.2) gives

$$\lim_{n\to\infty} \frac{\langle \mathcal{S}v_n, v_n\rangle - \tau_n\langle \mathcal{B}(v_n), v_n\rangle}{\langle (\mathcal{I}-\mathcal{A})v_n, v_n\rangle} \leq \lim_{n\to\infty} \frac{\langle \mathcal{S}v_n, v_n\rangle}{1 - \langle \mathcal{A}v_n, v_n\rangle} = 0,$$

which contradicts (3.25), thus $v_0 \neq 0$.

Suppose that $v_0 \notin \mathscr{K}$. Then the assumption (3.13) gives $\langle \mathcal{B}(v_0), v_0\rangle > 0$ and (3.1) gives the existence of $n_0 \in \mathbb{N}$ such that

$$\langle \mathcal{B}(v_n), v_n\rangle > \frac{\langle \mathcal{B}(v_0), v_0\rangle}{2} \quad \text{for all } \; n > n_0.$$

Moreover, the operator $\mathcal{S}$ is bounded and $\tau_n \to \infty$ therefore

$$\langle \mathcal{S}v_n, v_n\rangle - \tau_n\langle \mathcal{B}(v_n), v_n\rangle < \langle \mathcal{S}v_n, v_n\rangle - \tau_n\frac{\langle \mathcal{B}(v_0), v_0\rangle}{2} \leq \|\mathcal{S}\| - \tau_n\frac{\langle \mathcal{B}(v_0), v_0\rangle}{2} \to -\infty.$$

By using the property (3.10) we obtain

$$\lim_{n\to\infty} \frac{\langle \mathcal{S}v_n, v_n\rangle - \tau_n\langle \mathcal{B}(v_n), v_n\rangle}{\langle (\mathcal{I}-\mathcal{A})v_n, v_n\rangle} = -\infty,$$

which is in a contradiction with (3.25). Hence, $v_0 \in \mathscr{K}$. As $v_0 \in \mathscr{K}\backslash\{0\}$ and because of (3.11) if $1 \notin \sigma(\mathcal{A})$ and (3.10), (3.13), (3.14) if $1 \in \sigma(\mathcal{A})$, we have that

$$\langle (\mathcal{I}-\mathcal{A})v_0, v_0\rangle > 0.$$

Use of (3.2), (3.10) leads to

$$\limsup_{n\to\infty} \frac{\langle \mathcal{S}v_n, v_n\rangle - \tau_n\langle \mathcal{B}(v_n), v_n\rangle}{\langle (\mathcal{I}-\mathcal{A})v_n, v_n\rangle} \leq \lim_{n\to\infty} \frac{\langle \mathcal{S}v_n, v_n\rangle}{\langle (\mathcal{I}-\mathcal{A})v_n, v_n\rangle} = \\ = \frac{\langle \mathcal{S}v_0, v_0\rangle}{\langle (\mathcal{I}-\mathcal{A})v_0, v_0\rangle} \leq \max_{\varphi\in\mathscr{K},\varphi\neq 0} \frac{\langle \mathcal{S}\varphi, \varphi\rangle}{\langle (\mathcal{I}-\mathcal{A})\varphi, \varphi\rangle},$$

which contradicts (3.25) and

$$\lim_{\tau\to\infty} \max_{\substack{v\in\mathbb{H}\\ v\notin \mathrm{Ker}(\mathcal{I}-\mathcal{A})}} \frac{\langle \mathcal{S}v, v\rangle - \tau\langle \mathcal{B}(v), v\rangle}{\langle (\mathcal{I}-\mathcal{A})v, v\rangle} \leq \max_{\varphi\in\mathscr{K},\varphi\neq 0} \frac{\langle \mathcal{S}\varphi, \varphi\rangle}{\langle (\mathcal{I}-\mathcal{A})\varphi, \varphi\rangle}.$$

But also

$$\max_{\varphi\in\mathscr{K},\varphi\neq 0} \frac{\langle \mathcal{S}\varphi, \varphi\rangle}{\langle (\mathcal{I}-\mathcal{A})\varphi, \varphi\rangle} \leq \max_{\substack{v\in\mathbb{H}\\ v\notin \mathrm{Ker}(\mathcal{I}-\mathcal{A})}} \frac{\langle \mathcal{S}w - \tau\mathcal{B}(w), w\rangle}{\langle (\mathcal{I}-\mathcal{A})w, w\rangle} \quad \text{for all} \quad \tau > \tau_0, \tag{3.26}$$

which gives that

$$\lim_{\tau\to\infty} \max_{\substack{v\in\mathbb{H}\\ v\notin \mathrm{Ker}(\mathcal{I}-\mathcal{A})}} \frac{\langle \mathcal{S}w - \tau\mathcal{B}(w), w\rangle}{\langle (\mathcal{I}-\mathcal{A})w, w\rangle} = \max_{\varphi\in\mathscr{K},\varphi\neq 0} \frac{\langle \mathcal{S}\varphi, \varphi\rangle}{\langle (\mathcal{I}-\mathcal{A})\varphi, \varphi\rangle} < \infty.$$

□

The following theorem gives an estimate of the largest eigenvalue of (3.6).

Theorem 5. *Let $1 \in \sigma(A)$, let (3.13), (3.14), (3.20), (3.22) be true, let v_0 be a maximizer from Theorem 3, let v_0 satisfy (3.15). Finally, let there exist a constant $\widehat{C} > 0$ such that*

$$\left| \frac{1}{2}\langle \mathcal{B}(e_0+h), e_0+h\rangle - \frac{1}{2}\langle \mathcal{B}(e_0), e_0\rangle - \langle \mathcal{B}(e_0), h\rangle \right| \leq \widehat{C}\|h\|^2 \quad \textit{for all } h \in \mathbb{H}. \tag{3.27}$$

Then

$$\lambda_{\max} = \max_{\substack{v\in\mathbb{H}\\ v\notin Ker(\mathcal{I}-\mathcal{A})}} \frac{\langle \mathcal{S}v, v\rangle - \langle \mathcal{B}(v), v\rangle}{\langle (\mathcal{I}-\mathcal{A})v, v\rangle} \leq \left(1 + \frac{1}{\varepsilon_0^2}\right) \frac{\lambda^S_{\max}}{1-\lambda^A_2}, \tag{3.28}$$

where

$$\varepsilon_0 = \frac{-(|\mathcal{S}| + |\mathcal{B}|) + \left((|\mathcal{S}| + |\mathcal{B}|)^2 + C(\widehat{C} + |\mathcal{S}|)\right)^{\frac{1}{2}}}{(\widehat{C} + |\mathcal{S}|)},$$

where C is defined as

$$C = -\max\left\{\langle \mathcal{S}e_0, e_0\rangle - \langle \mathcal{B}(e_0), e_0\rangle, \langle \mathcal{S}e_0, e_0\rangle - \langle \mathcal{B}(-e_0), -e_0\rangle\right\}, \tag{3.29}$$

and $|\mathcal{S}|$ is the norm of the operator $\mathcal{S}$.

Let us give some comments to the theorem before we prove it. If $C \to 0$, then $\varepsilon \to 0$. The assumption (3.27) implies (3.15).

The theorem will be used to estimate $\lambda_{\max}$ in situation when $\mathcal{S}$ is depending on some parameter and $\mathcal{A}, \mathcal{B}$ are fixed. It changes the task to find an estimate independent of parameter for a maximum depending on parameter to find an estimate independent of the parameter for the largest eigenvalue of $\mathcal{S}$.

To be more specific, the application of it will be in Chapter 4. The constants C and $\widehat{C}$ will be explicitly found for the reaction-diffusion systems (1.16) with the Neumann boundary conditions. The theorem will be afterwards applied to the proof of Theorem 16, giving an upper bound for a certain set of critical points.

This theorem is concerned only with the case $1 \in \sigma(\mathcal{A})$. The case $1 \notin \sigma(\mathcal{A})$ is much easier, and if $\lambda^S_{\max}$ exists and is positive, it is possible to find an estimate analogous to (3.28):

$$\lambda_{\max} \leq \frac{\lambda^S_{\max}}{1 - \lambda^A_{\max}},$$

where $\lambda^A_{\max} \in [0, 1)$ is the largest eigenvalue of the operator $\mathcal{A}$.

Proof of Theorem 5. Suppose that $\mathcal{B}$, $\mathcal{S}$ and e_0 are satisfying

$$\langle \mathcal{S}e_0, e_0\rangle - \langle \mathcal{B}(-e_0), -e_0\rangle < \langle \mathcal{S}e_0, e_0\rangle - \langle \mathcal{B}(e_0), e_0\rangle = -C, \tag{3.30}$$

and a maximzer v_0 of l.h.s. of (3.28) satisfies $v_0 \notin \{e_0\}^\perp$. The situation of the opposite inequality can be treated by interchanging e_0 with $-e_0$. The case $v_0 \in \{e_0\}^\perp$ will be discussed at the end of the proof.

The key step is to find $\varepsilon_0 > 0$ such that for any $v_1 \in \{e_0\}^\perp$ satisfying $\|v_1\| < \varepsilon_0$ the inequality

$$\langle \mathcal{S}(e_0 + v_1) - \mathcal{B}(e_0 + v_1), e_0 + v_1\rangle \leq 0 \tag{3.31}$$

is fulfilled. Such ε_0 can be always found, see (3.23). Because of (3.20), (3.10), the vector $e_0 + v_1$ with $\|v_1\| < \varepsilon_0$ cannot be a maximizer of (3.16).

The l.h.s. of (3.31) can be estimated as

$$\begin{aligned}
&\langle \mathcal{S}(e_0 + v_1) - \mathcal{B}(e_0 + v_1), e_0 + v_1\rangle = \\
&\quad = \langle \mathcal{S}e_0, e_0\rangle + 2\langle \mathcal{S}v_1, e_0\rangle + \langle \mathcal{S}v_1, v_1\rangle - \langle \mathcal{B}(e_0 + v_1), e_0 + v_1\rangle = \\
&\quad = \langle \mathcal{S}e_0, e_0\rangle - \langle \mathcal{B}(e_0), e_0\rangle + \langle \mathcal{B}(e_0), e_0\rangle + 2\langle \mathcal{S}v_1, e_0\rangle + \langle \mathcal{S}v_1, v_1\rangle - \langle \mathcal{B}(e_0 + v_1), e_0 + v_1\rangle \leq \\
&\quad \leq -C + \langle \mathcal{B}(e_0), e_0\rangle - \langle \mathcal{B}(e_0 + v_1), e_0 + v_1\rangle + 2|\mathcal{S}|\|e_0\|\|v_1\| + |\mathcal{S}|\|v_1\|^2.
\end{aligned} \tag{3.32}$$

The assumption (3.27) with $h := v_1$ inserted in (3.32) gives

$$\begin{aligned}
&\langle \mathcal{S}(e_0 + v_1) - \mathcal{B}(e_0 + v_1), e_0 + v_1\rangle \leq \\
&\qquad \leq -C + \widehat{C}\|v_1\|^2 + 2\|\mathcal{B}(e_0)\|\|v_1\| + 2|\mathcal{S}|\|v_1\| + |\mathcal{S}|\|v_1\|^2 = \\
&\qquad \leq -C + (2|\mathcal{S}| + 2\|\mathcal{B}(e_0)\|)\|v_1\| + (|\mathcal{S}| + \widehat{C})\|v_1\|^2.
\end{aligned}$$

Use of (3.3) gives

$$\|\mathcal{B}(e_0)\| \leq |\mathcal{B}|\|e_0\| \leq |\mathcal{B}|.$$

Now we assume $\|v_1\| < \varepsilon_0$, which leads to an inequality

$$\langle \mathcal{S}(e_0 + v_1) - \mathcal{B}(e_0 + v_1), e_0 + v_1\rangle < -C + (2|\mathcal{S}| + 2|\mathcal{B}|)\varepsilon_0 + (|\mathcal{S}| + \widehat{C})\varepsilon_0^2$$

The r.h.s of this inequality has two roots

$$\varepsilon_{1,2} = \frac{-(2|\mathcal{S}| + 2|\mathcal{B}|) \pm \Big((2|\mathcal{S}| + 2|\mathcal{B}|)^2 + 4C(\widehat{C} + |\mathcal{S}|)\Big)^{\frac{1}{2}}}{2(\widehat{C} + |\mathcal{S}|)},$$

but only the positive one is relevant. Thus the inequality (3.31) is true when

$$\|v_1\| \leq \varepsilon_0 := \frac{-(|\mathcal{S}| + |\mathcal{B}|) + \Big((|\mathcal{S}| + |\mathcal{B}|)^2 + C(\widehat{C} + |\mathcal{S}|)\Big)^{\frac{1}{2}}}{(\widehat{C} + |\mathcal{S}|)}.$$

The conclusion after all these calculation is that the maximizer of (3.16) is of a form $v_0 = e_0 + v_1$, where $\|v_1\| > \varepsilon_0$ or of a form $v_0 = -e_0 + v_1$, with $\|v_1\| > \varepsilon_0$.

Now we will prove the estimate (3.28). Assume from now that $\|v_1\| > \varepsilon_0$. First, let us focus on the denominator in (3.16). Since $(\mathcal{I}-\mathcal{A})e_0 = 0$ we have

$$\langle (\mathcal{I}-\mathcal{A})(e_0+v_1), e_0+v_1\rangle = \langle (\mathcal{I}-\mathcal{A})v_1, v_1\rangle = \|v_1\|^2 - \langle \mathcal{A}v_1, v_1\rangle.$$

The formula

$$\max_{v\in\{e_0\}^\perp} \frac{\langle Av, v\rangle}{\|v\|^2} = \lambda_2^A,$$

gives

$$\langle \mathcal{A}v_1, v_1\rangle \le \lambda_2^A \|v_1\|^2.$$

Hence,

$$\|v_1\|^2 - \langle \mathcal{A}v_1, v_1\rangle \ge (1-\lambda_2^A)\|v_1\|^2,$$

and

$$\frac{\langle (\mathcal{I}-\mathcal{A})v_1, v_1\rangle}{\|e_0+v_1\|^2} = \frac{\|v_1\|^2 - \langle \mathcal{A}v_1, v_1\rangle}{\|e_0+v_1\|^2} = \frac{\|v_1\|^2 - \langle \mathcal{A}v_1, v_1\rangle}{1+\|v_1\|^2} \ge \frac{(1-\lambda_2^A)}{1+\dfrac{1}{\|v_1\|^2}} \ge \frac{1}{1+\dfrac{1}{\varepsilon_0^2}}(1-\lambda_2^A).$$

Using the last inequality we obtain

$$0 < \max_{v\in\mathbb{H}, v\neq 0} \frac{\langle \mathcal{S}v, v\rangle - \langle \mathcal{B}(v), v\rangle}{\langle (\mathcal{I}-\mathcal{A})v, v\rangle} \le \max_{v\in\mathbb{H}, v\neq 0} \frac{\left\langle \mathcal{S}\left(\dfrac{v}{\|v\|}\right), \dfrac{v}{\|v\|}\right\rangle}{\dfrac{\langle (\mathcal{I}-\mathcal{A})v_1, v_1\rangle}{\|e_0+v_1\|^2}} \le$$

$$\le \max_{v\in\mathbb{H}, \|v\|=1} \left(1+\frac{1}{\varepsilon_0^2}\right)\frac{\langle \mathcal{S}v, v\rangle}{(1-\lambda_2^A)} = \left(1+\frac{1}{\varepsilon_0^2}\right)\frac{\lambda_{\max}^S}{(1-\lambda_2^A)},$$

and it is the estimate (3.28).

The last case to discuss is $v_0 \perp e_0$. Here we have even a better inequality

$$\max_{v\in\mathbb{H}} \frac{\langle \mathcal{S}v, v\rangle - \langle \mathcal{B}(v), v\rangle}{\langle (\mathcal{I}-\mathcal{A})v, v\rangle} = \frac{\langle \mathcal{S}v_0, v_0\rangle - \langle \mathcal{B}(v_0), v_0\rangle}{\langle (\mathcal{I}-\mathcal{A})v_0, v_0\rangle} \le \frac{\lambda_{\max}^S}{1-\lambda_2^A} < \left(1+\frac{1}{\varepsilon_0^2}\right)\frac{\lambda_{\max}^S}{(1-\lambda_2^A)}.$$

□

Equivalence of (3.6) and (3.8) In some cases it is possible to find a simple transformation between (3.6) and (3.8), which conserves the properties of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$, as we will show now.

The operator $(\mathcal{I}-\mathcal{A})$ is symmetric, positive operator and isomorphism on $\mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp$, therefore

$$(\mathcal{I}-\mathcal{A})^{-\frac{1}{2}} : \mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp \to \mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp$$

is an isomorphism which is symmetric and positive. Assume that $\mathcal{S}(\mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp) \subseteq \mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp$, i.e. $\mathrm{Ker}(\mathcal{I}-\mathcal{A})$ is an invariant subspace of $\mathcal{S}$ and $\langle \mathcal{S}v, v\rangle > 0$ for some $v \in \mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp$. Then

$$\begin{aligned}\max_{v\in\mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp} \frac{\langle \mathcal{S}v, v\rangle}{\langle (\mathcal{I}-\mathcal{A})v, v\rangle} &= \max_{v\in\mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp} \frac{\langle \mathcal{S}v, v\rangle}{\langle (\mathcal{I}-\mathcal{A})^{\frac{1}{2}}v, (\mathcal{I}-\mathcal{A})^{\frac{1}{2}}v\rangle} = \\ &= \max_{v\in\mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp} \frac{\langle \mathcal{S}(\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}v, (\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}v\rangle}{\langle v, v\rangle} = \\ &= \max_{v\in\mathrm{Ker}(\mathcal{I}-\mathcal{A})^\perp} \frac{\langle (\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}\mathcal{S}(\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}v, v\rangle}{\langle v, v\rangle}.\end{aligned} \tag{3.33}$$

Therefore the number

$$\tilde{\lambda}^{S}_{\max} := \max_{v\in \mathrm{Ker}(\mathcal{I}-\mathcal{A})^{\perp}} \frac{\langle \mathcal{S}v, v\rangle}{\langle (\mathcal{I}-\mathcal{A})v, v\rangle} \tag{3.34}$$

is the largest eigenvalue of a symmetric compact operator

$$(\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}\mathcal{S}(\mathcal{I}-\mathcal{A})^{-\frac{1}{2}} : \mathrm{Ker}(\mathcal{I}-\mathcal{A})^{\perp} \to \mathrm{Ker}(\mathcal{I}-\mathcal{A})^{\perp},$$

and that means it is the largest eigenvalue of the equation

$$v \in \mathrm{Ker}(\mathcal{I}-\mathcal{A})^{\perp} : \quad \lambda v - (\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}\mathcal{S}(\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}v = 0,$$

as follows from (3.33) and Remark 31 on pg. 31.

If $1 \notin \sigma(\mathcal{A})$, then $\mathrm{Ker}(\mathcal{I}-\mathcal{A})^{\perp} = \{0\}^{\perp} = \mathbb{H}$ and the equation (3.6) can be reduced by defining

$$\tilde{\lambda} := \lambda,\ \tilde{\mathcal{S}} := (\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}\mathcal{S}(\mathcal{I}-\mathcal{A})^{-\frac{1}{2}},\ \tilde{\mathcal{B}} := (\mathcal{I}-\mathcal{A})^{-\frac{1}{2}}\mathcal{B}(\mathcal{I}-\mathcal{A})^{-\frac{1}{2}},\ \tilde{v} := (\mathcal{I}-\mathcal{A})^{\frac{1}{2}}v,$$

to a simple form

$$\tilde{\lambda}\tilde{v} - \tilde{\mathcal{S}}\tilde{v} + \tilde{\mathcal{B}}(\tilde{v}) = 0. \tag{3.35}$$

This equation is the same as (3.8). It is possible to show that all of the properties of the operators $\mathcal{S}, \mathcal{B}$ from Section 3.1 are valid even for $\tilde{\mathcal{S}}, \tilde{\mathcal{B}}$. If $\mathcal{S}$, $\mathcal{B}$, $\mathcal{A}$ fulfill (3.20), then $\langle \tilde{\mathcal{S}}\tilde{v}, \tilde{v}\rangle - \langle \tilde{\mathcal{B}}(\tilde{v}), \tilde{v}\rangle > 0$, where $\tilde{v} := (\mathcal{I}-\mathcal{A})^{\frac{1}{2}}v$, with v from (3.20). If v_0 is a maximizer of (3.16) then $\tilde{v}_0 := (\mathcal{I}-\mathcal{A})^{\frac{1}{2}}v_0$ is a maximizer of

$$\tilde{\lambda}_{\max} := \max_{\tilde{v}\in\mathbb{H}, v\neq 0} \frac{\langle \tilde{\mathcal{S}}\tilde{v} - \tilde{\mathcal{B}}(\tilde{v}), \tilde{v}\rangle}{\|\tilde{v}\|^2}, \tag{3.36}$$

and vice versa. Finally, it is possible to verify that $\tilde{\mathcal{B}}$ and $\tilde{v}_0$ satisfy (3.15), provided that $\mathcal{B}$ and v_0 satisfy this condition. Therefore Theorem 3 with $\tilde{\mathcal{S}}, \tilde{\mathcal{B}}$ and with $\tilde{\mathcal{A}} \equiv 0$ can be applied to (3.35) to get that its largest eigenvalue $\tilde{\lambda}_{\max}$ is characterized by the formula (3.36).

Let us note that if $1 \notin \sigma(A)$, then $\mathrm{Ker}(\mathcal{I}-\mathcal{A}) = \{0\}$ and the assumption $\mathcal{S}(\mathrm{Ker}(\mathcal{I}-\mathcal{A})^{\perp}) \subseteq \mathrm{Ker}(\mathcal{I}-\mathcal{A})^{\perp}$ is fulfilled.

3.2.2 Bifurcation Theorem

The aim of this Section is to prove a bifurcation theorem for the problem of a type

$$\lambda(\mathcal{I}-\mathcal{A})v - \mathcal{S}v + \mathcal{B}(v) - \mathcal{N}(v) = 0. \tag{3.37}$$

Theorem 6. *Let $\mathcal{B}$ have a potential on $\mathbb{H}$ and $\mathcal{N}$ have a potential on a neighborhood of zero and let the largest eigenvalue $\lambda_{\max}$ of the problem (3.6) exists and be positive. Then the number $\lambda_{\max}$ is the largest bifurcation point of the problem (3.37).*

Remark 20. *Theorem 6 uses variational methods to get the bifurcation, like Krasnoselskii Potential Bifurcation Theorem, see Appendix. The main difference between the equation in Theorem 6 and the equation in Krasnoselskii Theorem is in the presence of positively homogeneous perturbation $\mathcal{B}$, which is not differentiable. In contrast to Krasnoselskii Theorem, Theorem 6 is not solving the question if eigenvalues of (3.6) which are not the largest one, are bifurcation points.*

Proof of Theorem 6. Let $\Phi_B, \Phi_S, \Phi_N : \mathbb{H} \to \mathbb{R}$ be potentials to $\mathcal{B}, \mathcal{S}, \mathcal{N}$ respectively and let us define a functional $\Phi : \mathbb{H} \to \mathbb{R}$ by

$$\Phi(v) := \Phi_B(v) + \Phi_S + \Phi_N(v). \tag{3.38}$$

Outline: The goal is to construct a suitable set of sets $\{B_r\}_{r>0}$ and show that solutions of the problem

$$\text{for any sufficiently small } r>0 \text{ find } v_r \text{ such that:} \quad \lambda(r) := \frac{1}{r^2}\max_{v\in B_r}\Phi(v) = \frac{1}{r^2}\Phi(v_r), \quad \lim_{r\to 0}\lambda(r) = \lambda_{\max},$$

satisfy

$$\lambda(r)(\mathcal{I}-\mathcal{A})v_r - \mathcal{S}v_r + \mathcal{B}(v_r) - \mathcal{N}(v_r) = 0$$
$$\langle(\mathcal{I}-\mathcal{A})v_r, v_r\rangle = r^2.$$

The sets $\{B_r\}_{r>0}$ will be constructed in a way that $v_r \to 0$ as $r \to 0$ and consequently $\lambda_{\max}$ will be the largest bifurcation point of (3.37), with v_r being bifurcating solutions.

The first step will be a construction of the sets B_r. Since $v_0 \notin \operatorname{Ker}(\mathcal{I}-\mathcal{A})$, as follows from (3.4), well define a class of sets

$$\mathscr{H}_\alpha := \{v \in \mathbb{H} | \|v - \alpha v_0\| \le \alpha\eta\} \quad \text{for any } \alpha \ge 0,$$

where $0 < \eta < \operatorname{dist}(\operatorname{Ker}(\mathcal{I}-\mathcal{A}), v_0)$ is fixed and we also define a set

$$\mathscr{H} := \bigcup_{\alpha\ge 0}\mathscr{H}_\alpha.$$

In particular, $0 \in \mathscr{H}$. The sketch of the set $\mathscr{H}$ is in the Fig. 3.1. If $v \in \mathscr{H} \cap \operatorname{Ker}(\mathcal{I}-\mathcal{A})$ and $v \ne 0$, then $v \in \mathscr{H}_\alpha$ for some $\alpha > 0$, $v/\alpha \in \operatorname{Ker}(\mathcal{I}-\mathcal{A})$ and

$$\left\|\frac{v}{\alpha} - v_0\right\| \le \eta < \operatorname{dist}(\operatorname{Ker}(\mathcal{I}-\mathcal{A}), v_0),$$

which is contradiction. Therefore $\mathscr{H} \cap \operatorname{Ker}(\mathcal{I}-\mathcal{A}) = \{0\}$.

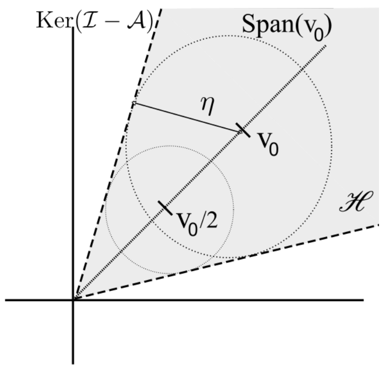


Figure 3.1: Sketch of the set $\mathscr{H}$, delimited by dashed lines and of the sets $\mathscr{H}_{1/2}$, $\mathscr{H}_1$ (dotted lines).

Proposition 1. *The set $\mathscr{H}$ is closed, convex and there exists $c, C > 0$ such that*

$$c\|v\|^2 \le \langle(\mathcal{I}-\mathcal{A})v, v\rangle \le C\|v\|^2 \quad \text{for all } v \in \mathscr{H}. \tag{3.39}$$

Proof. We will show that the set $\mathscr{H}$ is closed in $\mathbb{H}$. Let $v_n \subset \mathscr{H}$ be a sequence, $v_n \to v \in \mathbb{H}$. The goal is to show that $v \in \mathscr{H}$. If $v = 0$ then it is true. Let $v \ne 0$. Then for any v_n with n sufficiently large it can be found α_n for which $v_n \in \mathscr{H}_{\alpha_n}$. Let $\alpha_n \to \infty$. Then

$$\|v_0\| - \left\|\frac{v_n}{\alpha_n}\right\| \le \left\|\frac{v_n}{\alpha_n} - v_0\right\| \le \eta,$$

and it is possible only if $\|v_n\| \to \infty$, which contradicts the convergence of v_n in $\mathbb{H}$. Therefore the sequence α_n is bounded and it is possible to find a subsequence $\alpha_{n_k} \to \alpha$. Then $\|v_{n_k} - \alpha_{n_k} v_0\| \leq \alpha_{n_k}\eta \to d\alpha$ and $v \in \mathscr{H}_\alpha$ which means that $v \in \mathscr{H}$. Therefore $\mathscr{H}$ is closed.

Now the convexity. Let $u, w \in \mathscr{H}$. The situation when at least one of the vectors u, w is zero is easy to handle. Therefore let $u, w \neq 0$. There exists $\alpha_u, \alpha_w > 0$ for which $u \in \mathscr{H}_{\alpha_u}, w \in \mathscr{H}_{\alpha_w}$. Let $\xi_t := t\alpha_u + (1-t)\alpha_w,\ t \in [0,1]$. Then

$$\|tu + (1-t)w - \xi_t v_0\| = \|t(u - \alpha_u v_0) + (1-t)(w - \alpha_y w_0)\| \leq \\ \leq t\|(u - \alpha_u v_0)\| + (1-t)\|(w - \alpha_w v_0)\| \leq \eta(t\alpha_u + (1-t)\alpha_w) = \eta\xi_t,$$

therefore $tu + (1-t)w \in H_{\xi(t)}$ for any $t \in [0,1]$ and $\mathscr{H}$ is convex. In particular, $\mathscr{H}$ is weakly closed.

Since $\mathcal{A}$ is bounded, the choice $C := \sup_{v \in \mathbb{H}, \|v\|=1} \|(\mathcal{I} - \mathcal{A})v\|$, gives the upper inequality. The lower inequality will be proved by a contradiction. Let there exist $v_n \in \mathscr{H},\ v_n \rightharpoonup w$ with $\|v_n\| = 1$ for which

$$\langle (\mathcal{I} - \mathcal{A})v_n, v_n \rangle \to 0.$$

Since $\mathscr{H}$ is weakly closed, $w \in \mathscr{H}$. The operator $\mathcal{A}$ is compact therefore

$$0 = \lim_{n\to\infty} \langle (\mathcal{I} - \mathcal{A})v_n, v_n \rangle = 1 - \lim_{n\to\infty} \langle \mathcal{A}v_n, v_n \rangle = 1 - \langle \mathcal{A}w, w \rangle.$$

This means $\langle \mathcal{A}w, w \rangle = 1$. Clearly $w \neq 0$ and if it were $0 < \|w\| < 1$, then we would have

$$\left\langle \mathcal{A}\left(\frac{w}{\|w\|}\right), \frac{w}{\|w\|} \right\rangle = \frac{1}{\|w\|^2} > 1, \tag{3.40}$$

which contradicts $\sigma(\mathcal{A}) \subset (0,1]$ therefore $\|w\| = 1$ and in particular, $w = \pm e_0 \notin \mathscr{H}$, see (3.12). This contradicts $w \in \mathscr{H}$. □

Let $r > 0$. The sets B_r will be defined as

$$B_r := \{v \in \mathscr{H} | \langle (\mathcal{I} - \mathcal{A})v, v \rangle \leq r^2\}.$$

The Proposition 1 gives that any set B_r is closed, convex and bounded.

Proposition 2. *The functional Φ defined by (3.38) is on a neighborhood of zero weakly sequentially continuous.*

Proof. The assumptions (3.1) gives that Φ_B is weakly sequentially continuous on $\mathbb{H}$. The operator Φ_N has the compact Fréchet derivative on a neighborhood of zero and according to [61][Theorem 41.9] it is weakly sequentially continuous. The functional $\Phi_S(v) = 2^{-1}\langle \mathcal{S}v, v \rangle$ is weakly continuous because $\mathcal{S}$ is compact. Therefore the functional Φ is weakly continuous on B_r for any sufficiently small r. □

Proposition 3. *For any sufficiently small $r > 0$ the functional Φ attains on B_r its maximum.*

Proof. The statement follows directly from Theorem 38.A in [61] with the map $F := -\Phi$. However, we will write it for the sake of completeness in a detail. Let $r > 0$ be so small that $\mathcal{N}$ have the potential on B_r and

$$\lambda_r = \sup_{v \in B_r} \Phi(v).$$

There exists a sequence v_n such that $\Phi(v_n) \to \lambda_r$. According to Proposition 1 the set B_r is convex, closed and bounded and this means that B_r is weakly compact. Therefore there exists $v_r \in B_r$ and a subsequence $v_{n_k} \rightharpoonup v_r$. The functional Φ is according to Proposition 2 weakly continuous, which leads to

$$\lambda_r = \lim_{k\to\infty} \Phi\left(v_{n_k}\right) = \Phi(v_r) \leq \lambda_r.$$

Hence, $\lambda_r = \Phi(v_r)$ for sufficiently small $r > 0$ and the maximum exists. □

Now let $r_0 > 0$ be sufficiently small and define a map $\lambda : (0, r_0) \to \mathbb{R}$ by $\lambda(r) := \lambda_r$.

Proposition 4. *There exists $r_0 > 0$ such that for any $r \in (0, r_0)$ any maximizer from Proposition 3 satisfies $v_r \in S_r$, where $S_r := \{v \in \mathscr{H} | \langle (\mathcal{I} - \mathcal{A})v, v\rangle = r^2\}$.*

Proof. Let us define $\Phi_0 := \Phi_S + \Phi_B$ and put

$$\lambda(0) := \frac{1}{r^2} \max_{v \in B_r} \Phi_0(v) = \max_{v \in B_1} \Phi_0(v).$$

Therefore $\lambda(0) \geq \lambda_{\max} = \Phi_0(v_0) > 0$. Since

$$\lim_{r \to 0} \sup_{v \in B_r} \frac{|\Phi_N(v)|}{r^2} = 0,$$

as follows from (3.5), (3.39), $\Phi_N(0) = 0$ and Mean Value Theorem, there must be

$$\lim_{r \to 0} \lambda(r) = \lambda(0) > 0. \tag{3.41}$$

In particular, this means $r^{-2}\Phi_0(v_r) > 0$ for any sufficiently small $r > 0$. Let $v_r \notin S_r$ for infinitely many $r_n \to 0$. Then for any $r = r_n$ there exist $t > 0$ such that $\Phi(v_r(1 + t)) \leq \Phi(v_r)$. Dividing by t gives

$$\frac{\Phi(v_r(1 + t)) - \Phi(v_r)}{t} \leq 0,$$

which after the limit $t \to 0$ and using Riesz Theorem yields $\langle \Phi'(v_r), v_r\rangle \leq 0$. Since

$$\lim_{r \to 0} \sup_{v \in B_r} \frac{\langle N(v), v\rangle}{r^2} = 0,$$

as follows from (3.5) and (3.39), we get

$$\begin{aligned} \lim_{r \to 0} \frac{1}{r^2} \langle \Phi'(v_r), v_r\rangle &= \lim_{r \to 0} \frac{1}{r^2} \langle S(v_r) - B(v_r) - N(v_r), v_r\rangle = \\ &= \lim_{r \to 0} \frac{1}{r^2} \langle S(v_r) - B(v_r), v_r\rangle - \lim_{r \to 0} \frac{\langle N(v_r), v_r\rangle}{r^2} = \\ &= \lim_{r \to 0} \frac{1}{r^2} \langle S(v_r) - B(v_r), v_r\rangle \leq 0, \end{aligned}$$

which contradicts (3.41). Hence, $v_r \in S_r$ for any sufficiently small r. □

Proposition 5. *For any $r \in (0, r_0)$ the number $\lambda(r)$ and any maximizer v_r satisfy*

$$\lambda(r)(\mathcal{I} - \mathcal{A})v_r - \mathcal{S}v_r - \mathcal{B}(v_r) - \mathcal{N}(v_r) = 0.$$

Proof. The proof is based on Lagrange Multiplier method. For any $r \in (0, r_0)$ set $U(v_0) := \mathscr{H}$,

$$G(v) := \langle (\mathcal{I} - \mathcal{A})v, v\rangle - r^2,$$

and $F(v) := \Phi(v)$. Since F, G have continuous derivatives and $G'(v_0) = (\mathcal{I} - \mathcal{A})v_0 \neq 0$, Proposition 43.6 in [61] gives the claim, see also text below the proof. □

The last step is show that $\lim_{r \to 0} \lambda(r) = \lambda_{\max}$. It was mentioned above that $\lambda(0) > \lambda_{\max}$. Applying the Proposition 5 on the problem with $\mathcal{N} \equiv 0$ gives that $\lambda(0)$ is a critical point of the problem (3.6). However, $\lambda_{\max}$ is supposed to be the largest critical point of the equation (3.6), which implies

$$\max_{v \in B_1} \Phi_0(v) = \lambda_{\max}.$$

The formula (3.41) now gives the claim.

It remains to prove the non-existence of a bifurcation point larger than $\lambda_{\max}$.

Lemma 15. *Let $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n \to \lambda \neq 0$, let $\{v_n\}$ be a sequence in $\mathbb{H}$ satisfying $v_n \to 0,\ v_n/\|v_n\| \rightharpoonup w$ and*

$$\lambda_n(I - A)v_n - \mathcal{S}v_n + \mathcal{B}(v_n) - \mathcal{N}(\lambda_n, v_n) = 0. \tag{3.42}$$

Then

$$\frac{v_n}{\|v_n\|} \to w,\ \|w\| = 1 \quad \text{and} \quad \lambda w - \mathcal{S}w + \mathcal{B}(w) = 0.$$

Proof. Dividing (3.42) by $\|v_n\|$ gives

$$\lambda_n(I - A)\frac{v_n}{\|v_n\|} = \mathcal{S}\left(\frac{v_n}{\|v_n\|}\right) - \mathcal{B}\left(\frac{v_n}{\|v_n\|}\right) + \frac{\mathcal{N}(\lambda_n, v_n)}{\|v_n\|}. \tag{3.43}$$

The operators $\mathcal{S}$, $\mathcal{A}$ are compact and linear, the operator $\mathcal{B}$ satisfies (3.1) and the nonlinear operator $\mathcal{N}$ satisfies (3.5), therefore the r.h.s. of the equation (3.43) converges strongly. Since $\lambda_n \to \lambda \neq 0$, it implies that $v_n/\|v_n\|$ converges strongly and the only possible limit is the vector w, $\|w\| = 1$. Providing the limit in the equation (3.43) and using (3.5) yields

$$\lambda(I - A)w = \mathcal{S}w - \mathcal{B}(w).$$

□

Applied to our problem, every bifurcation point of (3.37) is also the critical point of (3.6) and therefore $\lambda_{\max}$ is the largest bifurcation point. □

The crucial step of the proof of this theorem was a use of the Lagrange Multiplier Method. We used the formulation from Proposition 43.6 in [61] which could be for our purposes rewritten as follows. Let $u_0 \in \mathbb{H}$ and let $U(u_0)$ be its neighborhood. If $F, G : U(u_0) \to \mathbb{R}$ are C^1 functionals then there exists a real number λ_0 such that the equation

$$F'(u_0) - \lambda_0 G'(u_0) = 0 \tag{3.44}$$

holds, when the two following conditions are satisfied:

1. F has at u_0 a local maximum w.r.t. the side condition $M := \{u \in U(u_0) |\ G(u) = 0\}$,
2. $G'(u_0) \neq 0$.

The non-emptiness of the set M is guaranteed in the proof by Proposition 4.

Remark 21. *By changing the signs and respective inequalities it is straightforward to modify the previous results to getting the smallest eigenvalue of the problem (3.6). It is possible to see from the proof that the assumption on the simplicity of $1 \in \mathcal{A}$ is here superfluous.*

3.3 Topological Methods

3.3.1 Abstract Global Bifurcation Theorem

Exceptionally, we will be concerned with a problem different from (3.9), (3.7).

Notation 5. *The space $\mathbb{X}$ will be a real Banach space with a norm $\|\cdot\|$, $\lambda \in \mathbb{R}$ will be a bifurcation parameter, $\mathcal{B} : \mathbb{X} \to \mathbb{X}$ will be a positively homogeneous operator satisfying (3.1), $\mathcal{L} : \mathbb{X} \to \mathbb{X}$ will be a linear compact operator, and $\mathcal{N} : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ will be a compact and continuous nonlinear operator satisfying (3.5), where the limit is considered in $\mathbb{X}$. The symbol deg will denote Leray-Schauder degree, see Appendix.*

The following global bifurcation theorem is a special case of abstract Theorem 7 in [55].

Theorem 7. *Consider the problem*

$$\lambda v - \mathcal{L}v + \mathcal{B}(v) - \mathcal{N}(\lambda, v) = 0, \tag{3.45}$$

on $\mathbb{X}$. For any $\mathcal{B}$, let $\mathscr{S}$ be defined by

$$\mathscr{S}(\mathcal{B}) = \overline{\{(\lambda, v) \in (0,\infty) \times \mathbb{X} \mid v \neq 0,\ (\lambda, v) \text{ satisfies (3.45)}\}}.$$

Let us assume that positive $\lambda_1 < \lambda_2$ are not eigenvalues of the operator $\mathcal{L} - \mathcal{B}$ and

$$deg\left(\mathcal{I} - \frac{1}{\lambda_1}(\mathcal{L} - \mathcal{B}), B_r, 0\right) \neq deg\left(\mathcal{I} - \frac{1}{\lambda_2}(\mathcal{L} - \mathcal{B}), B_r, 0\right) \text{ for all } r > 0. \tag{3.46}$$

Then there exists $\lambda_b \in [\lambda_1, \lambda_2]$ such that the connected component $\mathscr{S}_{\lambda_b}$ of the set $\mathscr{S}(\mathcal{B})$ containing the point $(\lambda_b, 0)$ satisfies at least one of the following conditions:

1. *$\mathscr{S}_{\lambda_b}$ is unbounded,*
2. *there exists $v \in X, v \neq 0$ such that $(0, v) \in \mathscr{S}_{\lambda_b}$,*
3. *there exists an eigenvalue $\lambda_c \in (0, +\infty) \setminus [\lambda_1, \lambda_2]$ of the operator $\mathcal{L} - \mathcal{B}$ such that $(\lambda_c, 0) \in \mathscr{S}_{\lambda_b}$.*

Proof. To obtain the assertion we use the abstract Theorem 7 in [55], where we set

$$\Lambda = (0,\infty),\ \Omega = \mathbb{X},\ \Omega_0 = B_r,\ r > 0 \text{ small enough},$$
$$F = \mathcal{I},\ \phi(\lambda, v) = \lambda^{-1}(\mathcal{L}v - \mathcal{B}(v) + \mathcal{N}(\lambda, v)),\ x_0 = 0$$

and $\mathscr{B} = \mathscr{B}_0$ can be the system of all bounded subsets of $\Lambda \times \Omega$, see also remarks below Lemma 8 in [55]. The assumptions (7), (8) in that theorem and (a),(b) on the top of the p. 217 can be written in our particular situation as the following conditions:

$$\text{zero is an isolated solution of (3.45) for any } \lambda \text{ in a neighbourhood of } \lambda_1 \text{ and } \lambda_2, \tag{3.47}$$

$$\deg\left(\mathcal{I} - \frac{1}{\lambda_1}(\mathcal{L} - \mathcal{B} + \mathcal{N}), B_r, 0\right) \neq \deg\left(\mathcal{I} - \frac{1}{\lambda_2}(\mathcal{L} - \mathcal{B} + \mathcal{N}), B_r, 0\right) \text{ for } r > 0 \text{ small enough}, \tag{3.48}$$

$$\text{the set of all } (\lambda, v) \text{ satisfying (3.45) is closed in } (0,\infty) \times \mathbb{X}, \tag{3.49}$$

$$\text{any closed and bounded set of } (\lambda, v) \text{ satisfying (3.45) is compact.} \tag{3.50}$$

Let us verify these conditions. If (3.47) were not true then λ_n, v_n satisfying (3.45) would exist such that $\lambda_n \to \lambda_j$, $j = 1$ or $j = 2$, $v_n \to 0$. Dividing (3.45) by $\|v_n\|$ and using the compactness of $\mathcal{L}$ and $\mathcal{B}$ and the condition (3.5) we get a subsequence of v_n satisfying $v_{n_k}/\|v_{n_k}\| \to w$ with some $w \in X$ and $\lambda_j w = \mathcal{L}w - \mathcal{B}(w)$. Therefore λ_j is an eigenvalue of the operator $\mathcal{L} - \mathcal{B}$, and it is a contradiction with the assumptions. The condition (3.48) for sufficiently small $r > 0$ follows easily from (3.46) by the homotopy invariance of the degree (see also remark above Proposition

14) by using the homotopy $\mathcal{H}(t,v) = v - \frac{1}{\lambda_j}(\mathcal{L}v - \mathcal{B}v + t\mathcal{N}(\lambda_j, v))$, $t \in [0,1]$ and the assumption (3.5). The condition (3.49) is clearly fulfilled due to continuity of our maps. The condition (3.50) follows from the compactness of the operator $\mathcal{L} - \mathcal{B} + \mathcal{N}$.

Now, the assertion of Theorem 7 in [55] translated to our particular situation gives the assertion of our Theorem 7. Let us only recall that we have chosen $\mathscr{B}_0$ as the system of all bounded subsets of $(0,\infty) \times \mathbb{X}$ and therefore our case (3) coincides with the condition (i) in Theorem from [55] stating that $\mathscr{S}_b$ is not contained in a set from $\mathscr{B}_0$.

□

3.3.2 Global bifurcation theorem for the problem (3.9)

Now we return back to the notation from Section 3.1. We are going to prove a global bifurcation result for the problem (3.9), which has a form

$$\lambda v - \mathcal{S}v + \mathcal{B}(v) - \mathcal{N}(\lambda, v) = 0, \tag{3.51}$$

using the theorem proved in the previous section.

Theorem 8. *Let $\lambda^S_{\max}$ be positive, let its multiplicity be odd, let λ^S_2 denote the second largest eigenvalue of $\mathcal{S}$. Then for any $\varepsilon \in (0, \min\{(\lambda^S_{\max} - \lambda^S_2)/2, \lambda^S_{\max}\})$ there exists $\tau_0 > 0$ such that the following assertion is true. If $\mathcal{B}$ satisfies $|\mathcal{B}| \le \tau_0$ and $\mathcal{S} - \mathcal{B}$ fulfills (3.15) with v_0 being a maximizer of (3.16) with $\mathcal{A} \equiv 0$, then $\lambda^S_{\max} - \varepsilon < \lambda_{\max}$ and there is a global bifurcation point $\lambda_b \in [\lambda^S_{\max} - \varepsilon, \lambda_{\max}]$ of the problem (3.51) in the following sense. The connected component $\mathscr{S}_{\lambda_b}$ of $\mathscr{S}(B)$ containing the point $(\lambda_b, 0) \in \mathbb{R} \times \mathbb{H}$ satisfies at least one of the following conditions:*

(i) $\mathscr{S}_{\lambda_b}$ is unbounded,

(ii) there exists $v \in \mathbb{H}, v \neq 0$ such that $(0, v) \in \mathscr{S}_{\lambda_b}$,

(iii) there exists an eigenvalue $\lambda_c \notin [\lambda^S_{\max} - \varepsilon, \lambda_{\max}]$ of the operator $\mathcal{S} - \mathcal{B}$ such that $(\lambda_c, 0) \in \mathscr{S}_{\lambda_b}$.

Remark 22. *Consider now $\mathcal{B} \equiv 0$ and assume that all assumptions of Theorem 8 are fulfilled. Then it is well-known that $\lambda_b = \lambda^S_{\max}$, as follows from Rabinowitz Theorem. However, Rabinowitz Theorem cannot be applied to the problem (3.51), because of the presence of the positively homogeneous operator $\mathcal{B}$. For this reason Theorem 8 can be considered as a modification of Rabinowitz Theorem for the problems (3.51), but in contrast to it, our theorem does not give any assertion for any other eigenvalues than the largest one.*

For the proof of this Theorem we will need an auxiliary lemma.

Lemma 16. *For any $\varepsilon > 0$ there exists $\tau_0 > 0$ such that*

$$deg\left(\mathcal{I} - \frac{1}{\lambda}(\mathcal{S} - \mathcal{B}), B_r, 0\right) = deg\left(\mathcal{I} - \frac{1}{\lambda}\mathcal{S}, B_r, 0\right)$$

for any $\lambda \in \mathbb{R}\backslash\{0\}$ satisfying $dist(\lambda, \sigma(\mathcal{S})) > \varepsilon$, any $\mathcal{B}$ with $|\mathcal{B}| \le \tau_0$, and all $r > 0$.

Proof. Due to a homotopy invariance of the degree it suffices to prove that for any $\varepsilon > 0$ there is $\tau_0 > 0$ such that

$$v - \frac{1}{\lambda}(\mathcal{S}v - t\mathcal{B}(v)) \neq 0 \text{ for all } \lambda, \mathcal{B} \text{ from the assumptions, } t \in [0,1], \ \|v\| = 1.$$

Let us suppose that it is not true. Then there exist $\varepsilon > 0$, $t_n \in [0,1]$, λ_n with $\text{dist}(\lambda_n, \sigma(\mathcal{S})) > \varepsilon$, $\mathcal{B}_n$ and v_n with $\|v_n\| = 1$ for all $n \in \mathbb{N}$, satisfying

$$\lambda_n \to \lambda, \quad v_n \rightharpoonup v, \quad \|\mathcal{B}_n(v_n)\| \to 0, \tag{3.52}$$

and

$$\lambda_n v_n - \mathcal{S} v_n + t_n \mathcal{B}_n(v_n) = 0. \tag{3.53}$$

Since $\operatorname{dist}(\lambda_n, \sigma(\mathcal{S})) > \varepsilon$ we have $|\lambda_n| > \varepsilon$ for all $n \in \mathbb{N}$. The compactness of $\mathcal{S}$ together with (3.52), (3.53) gives $v_n \to v$, $\|v\| = 1$. Providing the limit in the equation (3.53) leads to

$$\lambda v - \mathcal{S} v = 0,$$

but simultaneously $\operatorname{dist}(\lambda, \sigma(\mathcal{S})) > \varepsilon$ which is a contradiction. □

Proof of Theorem 8. Due to Leray-Schauder Index Formula, see pg. 109 in Appendix, and the assumed odd multiplicity of $\lambda^S_{\max}$ we have

$$\deg\left(I - \frac{1}{\lambda}\mathcal{S}, B_r, 0\right) = 1 \ \text{ for all } \lambda > \lambda^S_{\max},\ r > 0$$
$$\deg\left(I - \frac{1}{\lambda}\mathcal{S}, B_r, 0\right) = -1 \ \text{ for all } \lambda \in (\lambda^S_2, \lambda^S_{\max}),\ r > 0,$$

where λ^S_2 is the second largest eigenvalue of $\mathcal{S}$. Lemma 16 gives that for any ε from the assumptions there exists $\tau_0 > 0$ such that for $\mathcal{B}$ satisfying $|\mathcal{B}| \le \tau_0$

$$\deg\left(I - \frac{1}{\lambda}(\mathcal{S} - \mathcal{B}), B_r, 0\right) = \deg\left(I - \frac{1}{\lambda}\mathcal{S}, B_r, 0\right) = 1 \ \text{ for all } \lambda > \lambda^S_{\max} + \varepsilon,\ r > 0$$
$$\deg\left(I - \frac{1}{\lambda}(\mathcal{S} - \mathcal{B}), B_r, 0\right) = \deg\left(I - \frac{1}{\lambda}\mathcal{S}, B_r, 0\right) = -1 \ \text{ for all } \lambda \in (\lambda^S_2 + \varepsilon, \lambda^S_{\max} - \varepsilon),\ r > 0.$$

We take τ_0 smaller, if necessary, to ensure that $\lambda_{\max}$ exists and is positive, see the estimate in Observation 4. It follows from Theorem 7 that there exists $\lambda_b \in [\lambda^S_{\max} - \varepsilon, \lambda^S_{\max} + \varepsilon]$ such that the closure of a connected component $\mathscr{S}_{\lambda_b}$ of $\mathscr{S}(\mathcal{B})$ containing the point $(\lambda_b, 0)$ satisfies at least one of the alternatives (i)–(iii) of Theorem 7.

According to Theorem 4 with $\mathcal{A} \equiv 0$ the largest eigenvalue $\lambda_{\max}$ of $\mathcal{S} - \mathcal{B}$ exists. Observation 4 gives that $\lambda^S_{\max} - \varepsilon < \lambda_{\max}$, if τ_0 is small enough.

According to Lemma 15, there is no bifurcation point larger than the largest critical point $\lambda_{\max}$ and the number λ_b must be in the interval $[\lambda^S_{\max} - \varepsilon, \lambda_{\max}]$ and at least one of the conditions (i)–(iii) of Theorem 7 must be fulfilled. □

3.4 Results based on Implicit Function Theorem

Notation 6. *The notation from Section 3.1 will be supplemented with the following three assumptions.*

(A) $\mathcal{B}$ is Lipschitz continuous with a Lipschitz constant L.

(B) $\mathcal{N}$ has the continuous Fréchet derivative $\mathcal{N}'$ and the derivative of $\mathcal{N}$ w.r.t. v satisfy $\partial_v \mathcal{N}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

(C) If $1 \in \sigma(A)$, the operator $\mathcal{S}$ satisfy $Se_0 \neq 0$.

The aim of this section is to study Crandall-Rabinowitz type bifurcation, see Theorem 25 in Appendix, for nonlinear equations of a type

$$\lambda(\mathcal{I} - \mathcal{A})v - \mathcal{S}v + \mathcal{B}(v) - \mathcal{N}(\lambda, v) = 0, \tag{3.54}$$

from eigenvalues of positively homogeneous equations

$$\lambda(\mathcal{I} - \mathcal{A})v - \mathcal{S}v + \mathcal{B}(v) = 0. \tag{3.55}$$

A special role will play here also a linear equation

$$\lambda(\mathcal{I} - \mathcal{A})v - \mathcal{S}v = 0. \tag{3.56}$$

Observation 5. *Let $1 \in \sigma(A)$. Since 0 is not an eigenvalue of $\mathcal{S}$ by the assumption (C), e_0 cannot be an eigenvector of (3.56). For this reason the scalar product $\langle(\mathcal{I} - \mathcal{A})v, v\rangle$ is positive for any eigenvector of (3.56). And even more, since $\mathcal{A}$ is compact, zero is an isolated eigenvalue of $(\mathcal{I}-\mathcal{A})$, and therefore there does not exist a sequence of eigenvectors of (3.56) satisfying $\langle(\mathcal{I}-\mathcal{A})v_n, v_n\rangle \to 0$. If dim Im$(\mathcal{S}) = +\infty$, the assumption (C) follows directly from the compactness of $\mathcal{S}$.*

Theorem 9. *Let λ^S be a simple eigenvalue of the problem (3.56), not necessarily the largest one, and v_s be the respective eigenvector. There exist $R > 0, L_0 > 0, \delta > 0$ such that for any $\mathcal{B}$ with the Lipschitz constant $L < L_0$ there exist four Lipschitz continuous maps $v_+, v_- : [0, R] \to \mathbb{H}$, $\lambda_+, \lambda_- : [0, R] \to \mathbb{R}$ for which the following is true:*

(a) a pair $(\lambda, v) \in \mathbb{R} \times \mathbb{H}$ with $\|v\| + |\lambda - \lambda^S| \leq \delta$ is a solution of (3.54) if and only if there exists $r \in (0, R]$ such that either $(\lambda, v) = (\lambda_+(r), rv_+(r))$ or $(\lambda, v) = (\lambda_-(r), rv_-(r))$. A pair (λ, v) is a solution of (3.55) with $|\lambda - \lambda^S| < \delta$ if and only if either $(\lambda, v) = (\lambda_+(0), v_+(0)$ or $(\lambda, v) = (\lambda_-(0), v_-(0))$, up to positive multiples of $v_\pm(0)$.

(b) if $\mathcal{B}(v_s) \neq -\mathcal{B}(-v_s)$, then $(\lambda_+(0), v_+(0)) \neq (\lambda_-(0), v_-(0))$,

(c) for any $\varepsilon > 0$ a constant $L_1 \in (0, L_0)$ exists such that for any $\mathcal{B}$ with the Lipschitz constant $L < L_1$ it is true that

$$|\lambda_+(0) - \lambda^S| < \varepsilon, \quad |\lambda_-(0) - \lambda^S| < \varepsilon, \quad |v_+(0) - v_s| < \varepsilon, \quad |v_-(0) + v_s| < \varepsilon,$$

(d) if $\langle\mathcal{B}(v_s), v_s\rangle > 0$, then $\lambda_+(0) < \lambda^S$, if $\langle\mathcal{B}(-v_s), -v_s\rangle > 0$, then $\lambda_-(0) < \lambda^S$.

Proof. The pair (λ^S, v_s) satisfies

$$\lambda^S(\mathcal{I} - \mathcal{A})v_s = \mathcal{S}v_s. \tag{3.57}$$

We let $\|v_s\| = 1$ without loss of generality. Since λ^S is a simple eigenvalue of (3.56), its eigenspace is Span$\{v_s\}$. We write λ and v in (3.54) as

$$\lambda = \lambda^S + \hat{\lambda}, \tag{3.58}$$

$$v = r(v_s + \hat{v}), \tag{3.59}$$

with $r > 0$, $\hat{v} \in \{v_s\}^\perp$, $\hat{\lambda} \in \mathbb{R}$. Then the equation (3.54) has a form

$$(\lambda^S + \hat{\lambda})(\mathcal{I} - \mathcal{A})(r(v_s + \hat{v})) - \mathcal{S}(r(v_s + \hat{v})) + \mathcal{B}(r(v_s + \hat{v})) - \\ - \mathcal{N}(\lambda^S + \hat{\lambda}, r(v_s + \hat{v})) = 0.$$

Use of (3.57) gives

$$\lambda^S(\mathcal{I} - \mathcal{A})(r\hat{v}) + \hat{\lambda}(\mathcal{I} - \mathcal{A})(rv_s) - \mathcal{S}(r\hat{v}) = -\hat{\lambda}(\mathcal{I} - \mathcal{A})(r\hat{v}) - \mathcal{B}(r(v_s + \hat{v})) + \mathcal{N}(\lambda^S + \hat{\lambda}, r(v_s + \hat{v})).$$

We define a map $\mathcal{J} : \mathbb{R} \times \{v_s\}^\perp \to \mathbb{H}$ by

$$\mathcal{J}(\mu, \hat{w}) := \lambda^S(\mathcal{I} - \mathcal{A})\hat{w} + \mu(\mathcal{I} - \mathcal{A})v_s - \mathcal{S}\hat{w}, \quad \text{for all } \mu \in \mathbb{R}, \hat{w} \in \{v_s\}^\perp,$$

and rewrite the last equation as

$$\mathcal{J}(r\hat{\lambda}, r\hat{v}) = -\hat{\lambda}(\mathcal{I} - \mathcal{A})r\hat{v} - \mathcal{B}(r(v_s + \hat{v})) + \mathcal{N}(\lambda^S + \hat{\lambda}, r(v_s + \hat{v})). \tag{3.60}$$

Lemma 17. *The map $\mathcal{J}$ is an isomorphism.*

Proof. Any $w \in \mathbb{H}$ can be uniquely written as $w = \mu(\mathcal{I} - \mathcal{A})v_s + \hat{w}$, where $\mu \in \mathbb{R}$, $\hat{w} \in \{v_s\}^\perp$; if $1 \notin \sigma(A)$ this is clear from the invertibility of $(\mathcal{I} - \mathcal{A})$, if $1 \in \sigma(A)$, this follows from the fact that $v_s \neq \pm e_0$, see also Observation 5. According to Fredholm Alternative, the operator $\lambda^S(\mathcal{I} - \mathcal{A}) - \mathcal{S}$ is injective on $\{v_s\}^\perp$. As the inverse operator $\mathcal{J}^{-1}(w) := (\mu, (\lambda^S(\mathcal{I} - \mathcal{A}) - \mathcal{S})^{-1}\hat{w})$ is defined on $\mathbb{R} \times \{v_s\}^\perp$, the operator $\mathcal{J}$ is a bijection. The linearity of $\mathcal{J}$ follows directly from the definition. □

For given $r > 0$ we define a nonlinear map $\mathcal{Q}_r^+ : \mathbb{R} \times \{v_s\}^\perp \to \mathbb{R} \times \{v_s\}^\perp$ by

$$\mathcal{Q}_r^+(\hat{\lambda}, \hat{v}) := \frac{1}{r}\mathcal{J}^{-1}\left(-\hat{\lambda}(\mathcal{I} - \mathcal{A})(r\hat{v}) - \mathcal{B}(r(v_s + \hat{v})) + \mathcal{N}(\lambda^S + \hat{\lambda}, r(v_s + \hat{v}))\right) \tag{3.61}$$

and rewrite (3.60) as

$$(\hat{\lambda}, \hat{v}) = \mathcal{Q}_r^+(\hat{\lambda}, \hat{v}). \tag{3.62}$$

The equation (3.54) is now rewritten as a fixed-point problem. As $\mathcal{N}$ satisfies (3.5), it is suitable to define the map $\mathcal{Q}_0^+$ by

$$\mathcal{Q}_0^+(\hat{\lambda}, \hat{v}) = \mathcal{J}^{-1}\left(-\hat{\lambda}(\mathcal{I} - \mathcal{A})\hat{v} - \mathcal{B}(v_s + \hat{v})\right).$$

The space $\mathbb{R} \times \{v_s\}^\perp$ will be equipped with the norm

$$\|(\mu, v)\|_2 := \left(|\mu|^2 + \|v\|^2\right)^{\frac{1}{2}} \qquad \text{for all } (\mu, v) \in \mathbb{R} \times \{v_s\}^\perp.$$

The norm of $\mathcal{J}^{-1}$ will be defined as

$$|\mathcal{J}^{-1}| := \sup_{v \in \mathbb{H}, \|v\|=1} \|\mathcal{J}^{-1}(v)\|_2$$

For further purposes we also define a norm of operator $\mathcal{I} - \mathcal{A}$ by

$$|\mathcal{I} - \mathcal{A}| = \sup_{v \in \mathbb{H}, \|v\|=1} \|(\mathcal{I} - \mathcal{A})v\|.$$

Lemma 18. *There exists $L_b > 0$, $R_b > 0$, $C_b > 0$ such that for any operator $\mathcal{B}$ with $L < L_b$ and any $r \in [0, R_b]$ the map $\mathcal{Q}_r^+$ maps the ball $\overline{\mathscr{B}_{C_b}}(0) \subset \mathbb{R} \times \{v_s\}^\perp$ into itself.*

Proof. Let $\hat{\lambda} \in \mathbb{R}, \hat{v} \in \{v_s\}^{\perp}$. Then

$$\begin{aligned}
\|\mathcal{Q}_r^+(\hat{\lambda},\hat{v})\|_2^2 &= \left\| \mathcal{J}^{-1}\left(-\hat{\lambda}(\mathcal{I}-\mathcal{A})\hat{v} - \mathcal{B}(v_s+\hat{v}) + \frac{\mathcal{N}(\lambda^S+\hat{\lambda}, r(v_s+\hat{v}))}{r}\right)\right\|_2^2 \leq \\
&\leq |\mathcal{J}^{-1}|^2 \left(|\hat{\lambda}||\mathcal{I}-\mathcal{A}|\|\hat{v}\| + \|\mathcal{B}((v_s+\hat{v})) - \mathcal{B}(0)\| + \left\| \frac{\mathcal{N}(\lambda^S+\hat{\lambda}, r(v_s+\hat{v}))}{r} \right\| \right)^2 \leq \\
&\leq |\mathcal{J}^{-1}|^2 \left(|\hat{\lambda}||\mathcal{I}-\mathcal{A}|\|\hat{v}\| + L_b(1+\|\hat{v}\|) + C(r)(1+\|\hat{v}\|) \right)^2 \leq \\
&\leq |\mathcal{J}^{-1}|^2 \left(\Lambda_b|\mathcal{I}-\mathcal{A}|n_b + \left(C(R_b)+L_b\right)(1+n_b)\right)^2,
\end{aligned}$$

where $C(r) \to 0$ as $r \to 0$, see also (3.5), and n_b, Λ_b, R_b are yet unknown bounds on $\|\hat{v}\|, \hat{\lambda}, r$. We also used that $\|v_s\| = 1$. Now we use the Young inequality to get

$$\begin{aligned}
&|\mathcal{J}^{-1}|^2 \left(\left(C(R_b)+L_b\right)(1+n_b) + \Lambda_b|\mathcal{I}-\mathcal{A}|n_b\right)^2 \leq \\
&\qquad \leq \frac{3}{2}|\mathcal{J}^{-1}|^2\Lambda_b^2|\mathcal{I}-\mathcal{A}|^2 n_b^2 + \frac{3}{2}|\mathcal{J}^{-1}|^2\left(L_b+C(R_b)\right)^2(1+n_b)^2.
\end{aligned}$$

It remains to find Λ_b, n_b, L_b, R_b for which

$$\frac{3}{2}|\mathcal{J}^{-1}|^2\Lambda_b^2|\mathcal{I}-\mathcal{A}|^2 n_b^2 + \frac{3}{2}|\mathcal{J}^{-1}|^2\left(L_b+C(R_b)\right)^2(1+n_b)^2 \leq n_b^2 + \Lambda_b^2. \tag{3.63}$$

Dividing (3.63) by Λ_b^2, n_b^2 gives

$$\frac{3}{2}\frac{|\mathcal{J}^{-1}|^2\left(C(R_b)+L_b\right)^2(1+n_b)^2}{\Lambda_b^2 n_b^2} + |\mathcal{J}^{-1}|^2|\mathcal{I}-\mathcal{A}|^2 \leq \frac{1}{\Lambda_b^2} + \frac{1}{n_b^2}.$$

The r.h.s. can be done larger than $|\mathcal{J}^{-1}|^2|\mathcal{I}-\mathcal{A}|^2$ by a choice of Λ_b, n_b, and then L_b, R_b must be found accordingly small in order to fulfill this inequality.

In sum, it is possible to find Λ_b, n_b, R_b and L_b so small that if $L < L_b,\ r \in (0, R_b]$ then

$$\|\mathcal{Q}_r^+(\hat{\lambda},\hat{v})\|_2 \leq \sqrt{n_b^2+\Lambda_b^2}, \quad \text{for any } \ \hat{\lambda},\hat{v} \text{ with } |\hat{\lambda}| < \Lambda_b,\ \|\hat{v}\| < n_b.$$

Define $C_b := \sqrt{n_b^2+\Lambda_b^2}$. Then for any $\hat{\lambda}, \hat{v},\ \|(\hat{\lambda},\hat{v})\|_2 \leq C_b$ we have

$$\|\mathcal{Q}_r^+(\hat{\lambda},\hat{v})\|_2 \leq C_b.$$

and therefore $\mathcal{Q}_r^+$ maps the ball $\overline{\mathscr{B}_{C_b}}$ into itself. □

Lemma 19. *There exists $L_c > 0$, $R_c > 0$, $C_c > 0$ so that for any $\mathcal{B}$ with $L < L_c$ and any $r \in [0, R_c]$, the map $\mathcal{Q}_r^+$ is a contraction on $\overline{\mathscr{B}_{C_c}}$.*

Proof. Let $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathbb{R}$ and $\hat{v}_1, \hat{v}_2 \in \{v_s\}^{\perp}$. Use Lemma 18 to get L_b, R_b, C_b. Since $\mathcal{N} \in C^1(\mathbb{R}\times\mathbb{H})$, the Mean Value Theorem gives

$$\begin{aligned}
&\left\| \frac{\mathcal{N}(\lambda^S+\hat{\lambda}_1, r(v_s+\hat{v}_1))}{r} - \frac{\mathcal{N}(\lambda^S+\hat{\lambda}_2, r(v_s+\hat{v}_2))}{r} \right\|^2 \leq \\
&\qquad\qquad \leq \sup_{(\mu,w)\in B_{R_c}} \|\mathcal{N}'(\mu,w)\|^2 \left(|\hat{\lambda}_2-\hat{\lambda}_1|^2 + \|\hat{v}_1-\hat{v}_2\|^2\right).
\end{aligned}$$

Then

$$
\begin{aligned}
&\|\mathcal{Q}_r^+(\hat\lambda_1,\hat v_1)-\mathcal{Q}_r^+(\hat\lambda_2,\hat v_2)\|_2^2 = \\
&\quad= \Big\|\mathcal{J}^{-1}\Big(-\hat\lambda_1(\mathcal{I}-\mathcal{A})\hat v_1-\mathcal{B}(v_s+\hat v_1)+\frac{\mathcal{N}(\lambda^S+\hat\lambda_1,r(v_s+\hat v_1))}{r}+ \\
&\qquad\quad+\hat\lambda_2(\mathcal{I}-\mathcal{A})\hat v_2+\mathcal{B}(v_s+\hat v_2)-\frac{\mathcal{N}(\lambda^S+\hat\lambda_2,r(v_s+\hat v_2))}{r}\Big)\Big\|_2^2\le \\
&\quad\le|\mathcal{J}^{-1}|^2\Big\|(\hat\lambda_2-\hat\lambda_1)(\mathcal{I}-\mathcal{A})\hat v_1-\hat\lambda_2(\mathcal{I}-\mathcal{A})(\hat v_1-\hat v_2)- \\
&\qquad\quad-\mathcal{B}(v_s+\hat v_1)+\mathcal{B}(v_s+\hat v_2)+\frac{\mathcal{N}(\lambda^S+\hat\lambda_1,r(v_s+\hat v_1))}{r}-\frac{\mathcal{N}(\lambda^S+\hat\lambda_2,r(v_s+\hat v_2))}{r}\Big\|^2\le \\
&\quad\le|\mathcal{J}^{-1}|^2\Big(|\mathcal{I}-\mathcal{A}|^2|\hat\lambda_2-\hat\lambda_1|^2\|v_1\|^2+|\hat\lambda_2|^2\|\mathcal{I}-\mathcal{A}\|^2\|\hat v_1-\hat v_2\|^2+L_c^2\|\hat v_1-\hat v_2\|^2+ \\
&\qquad\quad+\sup_{(\mu,w)\in B_{R_c}}\|\mathcal{N}'(\mu,w)\|^2(|\hat\lambda_2-\hat\lambda_1|^2+\|\hat v_1-\hat v_2\|^2)\Big)\le \\
&\quad\le K\left(|(\hat\lambda_1-\hat\lambda_2)|^2+\|\hat v_1-\hat v_2\|^2\right)=K\|(\hat\lambda_1,v_1)-(\hat\lambda_2,v_2)\|_2^2,\quad\text{for all }\|\hat v_{1,2}\|\le n_c,\hat\lambda_{1,2}<\Lambda_c
\end{aligned}
$$

where $\mathscr{B}_{R_c}$ is a the smallest ball with the center at $(\lambda^S+\hat\lambda_1,r(v_s+\hat v_1))$ containing the point $(\lambda^S+\hat\lambda_2,r(v_s+\hat v_2))$ and L_c,n_c,Λ_c are bounds to be determined and

$$
\begin{aligned}
K := |\mathcal{J}^{-1}|^2\max\Big\{&|\mathcal{I}-\mathcal{A}|^2n_c^2+\sup_{(\mu,w)\in B_{R_c}}\|\mathcal{N}'(\mu,w)\|^2, \\
&\Lambda_c^2|\mathcal{I}-\mathcal{A}|^2+L_c^2+\sup_{(\mu,w)\in B_{R_c}}\|\mathcal{N}'(\mu,w)\|^2\Big\}.
\end{aligned}\tag{3.64}
$$

It is possible to choose $L_c\le L_b, C_c\le\min\{\Lambda_c,n_c,C_b\}, R_c\le R_b$ so small that for any $\mathcal{B}$ with $L<L_c$, for any $r\le R_c$ and any $\hat\lambda_1,\hat\lambda_2,\hat v_1,\hat v_2\in\overline{\mathscr{B}_{C_c}}$ the constant K satisfy $K\in(0,1)$, and Lemma 18 gives the assertion. □

Alternatively, we can choose in (3.58) vector $-v_s$ and do the whole procedure again to get a problem

$$(\hat\lambda,\hat v)=\mathcal{Q}_r^-(\hat\lambda,\hat v),\tag{3.65}$$

where

$$\mathcal{Q}_r^-(\hat\lambda,\hat v):=\frac{1}{r}\mathcal{J}^{-1}\left(-\hat\lambda(\mathcal{I}-\mathcal{A})(r\hat v)-\mathcal{B}(r(-v_s+\hat v))+\mathcal{N}(\lambda^S+\hat\lambda,r(-v_s+\hat v))\right).\tag{3.66}$$

Analogously, the map $\mathcal{Q}_0^-$ will be defined by

$$\mathcal{Q}_0^-(\hat\lambda,\hat v)=\mathcal{J}^{-1}\left(-\hat\lambda(\mathcal{I}-\mathcal{A})\hat v-\mathcal{B}((-v_s+\hat v))\right).$$

Application of Lemma 19 on Q_r^- yield a set of constants $\tilde L_c,\tilde R_c,\tilde C_c$ for $\mathcal{Q}_r^-$.

Let the Lipschitz constant L of $\mathcal{B}$ satisfy $L<L_0:=\min\{L_c,\tilde L_c\}$ and put $R:=\min\{R_c,\tilde R_c\}$. Define the closed balls

$$B_+:=\overline{\mathscr{B}_{C_c}},\qquad B_-:=\overline{\mathscr{B}_{\tilde C_c}(0)}.$$

Since $(B_+,\|\cdot\|)$ is a complete metric space, the equation (3.62) has for any $r\in(0.R_c)$ a solution according to the Fixed Point Theorem and thus also (3.54) has a solution. Moreover, this solution is unique in the closed ball B_+. Define the maps

$$\lambda_+(r):=\lambda^S+\hat\lambda_r,\quad v_+(r):=\hat v_r,$$

where for any $r \in [0, R]$, $(\hat{\lambda}_r, \hat{v}_r)$ is a fixed point of (3.62). Similar procedure will be done for the negative case – there is the unique fixed point of (3.65) in the set B_-, and therefore the maps $\lambda_-(r), \hat{v}_-(r)$ and consequently the maps $(\lambda_-(r), v_-(r)$ can be defined.

The continuity and linearity of $\mathcal{J}$ together with (3.5) give

$$\lim_{r\to 0} \mathcal{Q}_r^+(\lambda, v) = \lim_{r\to 0} \mathcal{J}^{-1}\left(-\hat{\lambda}(\mathcal{I}-\mathcal{A})(\hat{v}) - \mathcal{B}((v_s+\hat{v})) + \frac{\mathcal{N}(\lambda^S+\hat{\lambda}, r(v_s+\hat{v}))}{r\|v_s+\hat{v}\|}\|v_s+\hat{v}\|\right) =$$
$$= \mathcal{Q}_0^+(\lambda, v), \quad \text{for all} \quad (\lambda, v) \in \mathbb{R}\times\{v_s\}^{\perp},$$

and therefore the fixed point problem

$$(\hat{\lambda}, \hat{v}) = \mathcal{Q}_0^+(\hat{\lambda}, \hat{v}), \tag{3.67}$$

is equivalent to (3.55). There is a unique fixed point $(\hat{\lambda}_0, \hat{v}_0)$ of (3.67) in B_+. Define

$$\lambda_+(0) := \lambda^S + \hat{\lambda}_0, \;\; v_+(0) = \hat{v}_0.$$

Similarly for the case with the negative sign. The values $\lambda_\pm(0)$ are eigenvalues of (3.55) with the respective eigenvectors $v_\pm(0)$. Since these points are unique, there are no other eigenvalues and eigenvectors in the sets $B_\pm$. As the problem (3.55) is positively homogeneous, any positive multiple of $v_\pm(0)$ is also a solution of this equation with $\lambda = \lambda_\pm(0)$.

The map $\mathcal{Q}_r$ is Lipschitz continuous w.r.t. (λ, v), see (A) and C^1-continuous w.r.t parameter r, see (B) in Notation 6 and (3.5), and therefore the fixed points determining maps $v_\pm(r), \lambda_\pm(r)$ are Lipschitz continuous in $[0, R]$ w.r.t r as well, see [60][§ 1.2].

Now it remains to choose $\delta \leq \min\{n_c, \tilde{n}_c, \Lambda_c, \tilde{\Lambda}_c\}$ sufficiently small such that (a) is true.

Assume $\mathcal{B}(v_s) \neq -\mathcal{B}(-v_s)$, then the mappings $\mathcal{Q}_0^+$ and $\mathcal{Q}_0^-$ satisfy $\mathcal{Q}_0^+(0,0) \neq \mathcal{Q}_0^-(0,0)$. As $Q_0^\pm$ are continuous w.r.t. $\hat{\lambda}, \hat{v}$, it is possible to take $L, R, n_c, \tilde{n}_c, \Lambda_c, \tilde{\Lambda}_c$ smaller if necessary such that $Q_0^+(B_+) \cap Q_0^-(B_-) = \emptyset$. Then the fixed points of the problems (3.62), (3.65) with $\mathcal{Q}_0^+$ and $\mathcal{Q}_0^-$ are different, which leads to $(\lambda_+(0), v_+(0)) \neq (\lambda_-(0), v_-(0))$. This gives (b). See also Fig 3.2.

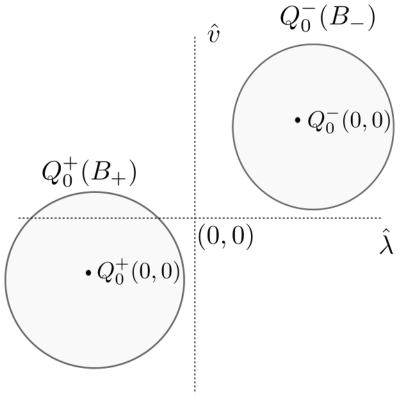


Figure 3.2: The sketch of the sets $\mathcal{Q}_0^+(B_+)$, $\mathcal{Q}_0^-(B_-)$. Since $Q_0^+(0,0) \neq Q_0^-(0,0)$, and $Q_0^\pm$ are continuous, the sketched sets are disjoint for small $B_\pm$.

If $|\mathcal{B}| \to 0$, then $\mathcal{Q}_r^+(\hat{\lambda}, \hat{v})$ and $\mathcal{Q}_r^-(\hat{\lambda}, \hat{v})$ converge to

$$\mathcal{Q}_r^+(\hat{\lambda}, \hat{v}) = \mathcal{J}^{-1}\left(-\hat{\lambda}(\mathcal{I}-\mathcal{A})\hat{v} + \frac{\mathcal{N}(\hat{\lambda}, r(v_s+\hat{v}))}{r}\right),$$
$$\mathcal{Q}_r^-(\hat{\lambda}, \hat{v}) = \mathcal{J}^{-1}\left(-\hat{\lambda}(\mathcal{I}-\mathcal{A})\hat{v} + \frac{\mathcal{N}(\hat{\lambda}, r(-v_s+\hat{v}))}{r}\right),$$

for all $(\hat{\lambda}, \hat{v}) \in \mathbb{R} \times \{v_s\}^\perp$. The maps $\mathcal{Q}_0^+, \mathcal{Q}_0^-$ are

$$\mathcal{Q}_0^+(\hat{\lambda}, \hat{v}) = \mathcal{J}^{-1}\left(-\hat{\lambda}(\mathcal{I} - \mathcal{A})\hat{v}\right),$$
$$\mathcal{Q}_0^-(\hat{\lambda}, \hat{v}) = \mathcal{J}^{-1}\left(-\hat{\lambda}(\mathcal{I} - \mathcal{A})\hat{v}\right),$$

and since λ^S is, as an eigenvalue of compact operator, isolated and the fixed point problem is equivalent to (3.56), the only fixed points of both of them in sufficiently small ball are zeros, i.e. $\lambda_+(0) = \lambda_-(0) = \lambda^S$, $\hat{v} = 0$, therefore $v_+(0) = v_s$, $v_-(0) = -v_s$. Since $L \to 0$ implies $|B| \to 0$, (c) is proven.

It remains to prove (d). A couple $(\lambda_+(0), v_+(0))$ satisfy

$$\lambda_+(0)(\mathcal{I} - \mathcal{A})v_+(0) - \mathcal{S}v_+(0) + \mathcal{B}(v_+(0)) = 0.$$

Multiplying it by v_s, multiplying (3.57) by $v_+(0)$, subtracting the results and using the symmetry of $\mathcal{S}$ give

$$(\lambda_+(0) - \lambda^S(0))\langle(\mathcal{I} - \mathcal{A})v_+(0), v_s\rangle + \langle\mathcal{B}(v_+(0), v_s\rangle = 0.$$

Since $\langle(\mathcal{I} - \mathcal{A})v_+(0), v_s\rangle > 0$ for $\mathcal{B}$ with sufficiently small Lipschitz constant L, see (c) and Observations 3, 5, this can be rewritten as

$$\lambda_+(0) - \lambda^S(0) = -\frac{\langle\mathcal{B}(v_+(0)), v_s\rangle}{\langle(\mathcal{I} - \mathcal{A})v_+(0), v_s\rangle}.$$

The r.h.s. is negative due to the assumption $\langle\mathcal{B}(v_s), v_s\rangle > 0$, (c) and continuity of $\mathcal{B}$. Hence, $\lambda_+(0) - \lambda^S(0) < 0$ for $\mathcal{B}$ with sufficiently small Lipschitz constant L. The proof for $\langle\mathcal{B}(-v_s), -v_s\rangle > 0$ is similar. □

Remark 23. *Let us give a few comments to the previous theorem.*

The assertion (c) can be roughly formulated that $\lambda_+(0), \lambda_-(0) \to \lambda^S$, $v_+(0) \to v_s$, $v_-(0) \to -v_s$ *as the Lipschitz constant of* $\mathcal{B}$ *converges to zero.*

If one considers the operator $\mathcal{B} := \tau\tilde{\mathcal{B}}$, *where* $\tau \in \mathbb{R}$ *and* $\tilde{\mathcal{B}}$ *is an operator satisfying assumptions on* $\mathcal{B}$ *independent of* τ, *then for any sufficiently small* τ *the assertion of Theorem 9 is true. In this case, the maps* $\lambda_\pm, v_\pm$ *are depending on* τ, *and it is possible to prove that there exists* τ_0 *such that for any* $\tau < \tau_0$ *the maps* $v_\pm, \lambda_\pm$ *are Lipschitz continuous w.r.t.* τ. *To prove this it suffices to realize that the mappings* $\mathcal{Q}^\pm$ *defined in (3.61), (3.66) are then Lipschitz continuous w.r.t.* τ. *Similar argument as in the proof of Lipschitz continuity of* $\lambda_\pm, v_\pm$ *gives the Lipschitz continuity of fixed points w.r.t.* τ.

However, the dependence $\mathcal{B} := \tau\tilde{\mathcal{B}}$ *is in general not true for the problems (1.14), (1.10), cf. also Remark 26 on pg. 65.*

The next paragraph contains a bifurcation result which is similar to conclusions of Theorem 9. This one, developed by Lutz Recke and Martin Väth, in [47] is applicable for much more general problems, as can be immediately seen from the equation (3.68). On the other hand, the operator $\mathcal{G}$ is depending on a real parameter τ, which is not present in the problem (1.14), (1.10), i.e. this result is in general not applicable to our problems, see also Remark 23.

General result concerning homogenizable equations For further purposes we will place here the general result from the paper [47], which has been developed by Lutz Recke and Martin Väth.

The studied problem will be now more general:

$$\mathcal{F}(\lambda, u) = \tau\mathcal{G}(\tau, \lambda, u), \tag{3.68}$$

where $\mathcal{F} : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ and $\mathcal{G} : \mathbb{R}^2 \times \mathbb{X} \to \mathbb{Y}$ are maps, $\mathbb{X}, \mathbb{Y}$ are Banach spaces, and

$$\mathcal{F}(\lambda, 0) = \mathcal{G}(\tau, \lambda, 0) = 0 \quad \text{for all } \tau, \lambda \in \mathbb{R}. \tag{3.69}$$

Hence, for all τ and λ there exists the so-called trivial solution $u = 0$ to (3.68), and we are going to describe local bifurcation of nontrivial solutions to (3.68) from the trivial solution. Let us suppose that the map $\mathcal{F}$ is C^2-smooth and that

$$\begin{aligned} &\partial_u \mathcal{F}(0,0) \text{ is a Fredholm operator of index zero from } \mathbb{X} \text{ into } \mathbb{Y}, \\ &\text{Ker } \partial_u \mathcal{F}(0,0) = \text{Span}\{u_0\}, \quad \partial_\lambda \partial_u \mathcal{F}(0,0) u_0 \notin \text{Im } \partial_u \mathcal{F}(0,0). \end{aligned} \tag{3.70}$$

The assumption (3.70) gives that $\partial_u \mathcal{F}(0,0)$ is a subspace of codimension one in $\mathbb{Y}$ and hence, there exists a functional $v_0^\star \in \mathbb{Y}*$, such that

$$\text{Im } \partial_u \mathcal{F}(0,0) = \{u \in \mathbb{H} | \langle v_0^\star, u \rangle = 0\}.$$

Then clearly

$$\kappa := \langle v_0^*, \partial_u \partial_\lambda \mathcal{F}(0,0) u_0 \rangle \neq 0. \tag{3.71}$$

Unlike to $\mathcal{F}$ there will be not assumed that $\mathcal{G}$ in (3.68) is differentiable. Instead, we assume that

$$\begin{aligned} &\text{there exists a map } \mathcal{G}_0 : \mathbb{R}^2 \times \mathbb{X} \to \mathbb{Y} \text{ such that} \\ &\lim_{r \searrow 0} \frac{1}{r} \mathcal{G}(\tau, \lambda, ru) = \mathcal{G}_0(\tau, \lambda, u) \quad \text{for all } \tau, \lambda \in \mathbb{R} \text{ and } u \in \mathbb{X}. \end{aligned} \tag{3.72}$$

Moreover, we suppose that the map $\mathcal{G}_1 : [0, \infty) \times \mathbb{R}^2 \times \mathbb{X} \to \mathbb{Y}$, defined by

$$\mathcal{G}_1(r, \tau, \lambda, u) := \begin{cases} \frac{1}{r} \mathcal{G}(\tau, \lambda, ru) & \text{for } r > 0, \\ \mathcal{G}_0(\tau, \lambda, u) & \text{for } r = 0, \end{cases} \tag{3.73}$$

is Lipschitz continuous on sufficiently small bounded sets, i.e. there exist $c > 0$ and $L > 0$ such that

$$\begin{aligned} &\|\mathcal{G}_1(r_1, \tau_1, \lambda_1, u_1) - \mathcal{G}_1(r_2, \tau_2, \lambda_2, u_2)\| \leq L\left(|r_1 - r_2| + |\tau_1 - \tau_2| + |\lambda_1 - \lambda_2| + \|u_2 - u_2\|\right) \\ &\qquad \text{for all } r_j \in [0, c],\ \tau_j, \lambda_j \in [-c, c],\ u_j \in \mathbb{X},\ \|u_j\| \leq c,\ j = 1, 2. \end{aligned} \tag{3.74}$$

Theorem 10. *Suppose (3.69), (3.70), (3.72) and (3.74). Under above given assumptions there exist $\varepsilon > 0$, $\delta > 0$ and Lipschitz continuous maps $\hat{\lambda}_+, \hat{\lambda}_- : [0, \varepsilon] \times [-\varepsilon, \varepsilon] \to \mathbb{R}$ and $\hat{u}_+, \hat{u}_- : [0, \varepsilon] \times [-\varepsilon, \varepsilon] \to \mathbb{X}$ such that the following is true:*

(i) (τ, λ, u) is a solution to (3.68) with $|\tau| + |\lambda| + \|u\| \leq \delta$ and $u \neq 0$ if and only if for certain $r \in (0, \varepsilon]$ it holds $\lambda = \hat{\lambda}_+(r, \tau)$, $u = r\hat{u}_+(r, \tau)$ or $\lambda = \hat{\lambda}_-(r, \tau)$, $u = r\hat{u}_-(r, \tau)$.

(ii) $\hat{\lambda}_+(0,0) = \hat{\lambda}_-(0,0) = 0$, $\hat{u}_+(0,0) = u_0$, $\hat{u}_-(0,0) = -u_0$ and

$$\lim_{\tau \to 0} \frac{\hat{\lambda}_+(0, \tau)}{\tau} = \frac{\langle v_0^*, \mathcal{G}_0(0, 0, u_0) \rangle}{\langle v_0^*, \partial_\lambda \partial_u \mathcal{F}(0,0) u_0 \rangle}, \tag{3.75}$$

$$\lim_{\tau \to 0} \frac{\hat{\lambda}_-(0, \tau)}{\tau} = -\frac{\langle v_0^*, \mathcal{G}_0(0, 0, -u_0) \rangle}{\langle v_0^*, \partial_\lambda \partial_u \mathcal{F}(0,0) u_0 \rangle}. \tag{3.76}$$

(iii) Suppose

$$\rho := -\frac{\langle v_0^*, \partial_u^2 \mathcal{F}(0,0)(u_0, u_0) \rangle}{2 \langle v_0^*, \partial_\lambda \partial_u \mathcal{F}(0,0) u_0 \rangle} \neq 0. \tag{3.77}$$

Then for all $r \in [0, \varepsilon]$ and $\tau \in [-\varepsilon, \varepsilon]$ we have

$$\left. \begin{aligned} \hat{\lambda}_+(r, \tau) - \hat{\lambda}_+(0, \tau) \geq \rho r, \\ \hat{\lambda}_-(r, \tau) - \hat{\lambda}_-(0, \tau) \leq -\rho r, \end{aligned} \right\} \textit{ if } \rho > 0$$

and

$$\left.\begin{aligned}\hat{\lambda}_+(r,\tau) - \hat{\lambda}_+(0,\tau) \leq \rho r, \\ \hat{\lambda}_-(r,\tau) - \hat{\lambda}_-(0,\tau) \geq -\rho r,\end{aligned}\right\} \text{ if } \rho < 0.$$

Proof. For the proof see [47]. □

Observation 6. *Putting $\mathcal{F} := \lambda(\mathcal{I} - \mathcal{A}) - \mathcal{S}$, $\mathcal{G} := -\tau\tilde{\mathcal{B}}$ leads to an equation*

$$\lambda(\mathcal{I} - \mathcal{A}) - \mathcal{S} + \tau\tilde{\mathcal{B}} = 0,$$

which is a special case of (3.54), with the operator $\mathcal{B} := \tau\tilde{\mathcal{B}}$. However, as mentioned in Remark 23, this abstract equation is in general not applicable on the systems (1.14), (1.10).

Remark 24. *If $\mathcal{G} \equiv 0$ in (3.68), it can be seen that the assumptions and conclusions of Theorem 10 are even stronger that the one of Crandall-Rabinowitz Theorem. Therefore Theorem 10 is a generalization of Crandall-Rabinowitz Theorem for more general problems (3.68).*

CHAPTER **4**

Application to reaction-diffusion systems

4.1 Introductory remarks

The following chapter is the essential part of the dissertation thesis, because it contains the results about the existence of stationary solutions of reaction-diffusion systems with unilateral terms. The goal of this chapter will be to apply theorems from Chapter 3 to the first equations in (2.49) and (2.64), and consequently get the critical and bifurcation points of the problems (1.14) with (homogeneous) Dirichlet/mixed b.c. and (1.14) with the (homogeneous) Neumann b.c., respectively. We will also present some results for the problems (2.66), (2.67) and (2.66), (2.74) with the unilateral sources on the boundary and one result for a problem with nonlinear functions n_1, n_2 satisfying (2.31).

4.2 Systems with Dirichlet or mixed boundary conditions

Before going to theorems about critical and bifurcation points of reaction-diffusion systems with unilateral sources, it will be suitable to summarize the content of the previous chapters. In Section 2.2.1 the reaction-diffusion system (1.14) having the unilateral terms and Dirichlet/mixed boundary conditions was rewritten as a system of two operator equations (2.25) on the Hilbert space $W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$. Afterwards, in Section 2.2.2 this system was rewritten to the form (2.43), where the first equation with the symmetric compact operator S is depending only on one variable and two parameters, with the first one, denoted as d_1, being fixed and the second one, denoted as d_2, being a bifurcation parameter. The eigenvalues of S are simultaneously critical points of the linear problem (1.12) with fixed d_1, Dirichlet/mixed b.c.

In Section 2.4 the problem with unilateral terms on the boundary was formulated.

Notation 7. *We will use in this section the notation and the assumptions from Section 2.1. The assumptions (2.1) – (2.9) are supposed to be true in the whole chapter. Under the term solution we will always mean the weak solution. The numbers κ_k are eigenvalues of the Laplacian with (homogeneous) Dirichlet/mixed boundary conditions ordered in a growing sequence, see (2.12). The eigenvalues of the operator S are denoted by λ_k^S and were found explicitly in the formula (2.39). The critical points of (1.12) with fixed d_1 and Dirichlet/mixed b.c. are denoted by $d_{2,k}^0$, see (2.40), the largest critical point is denoted by $d_{2,\max}^0$. For the definitions of critical and bifurcation points see Definition 5 on page 8.*

Finally, let us emphasize that $s_\pm$ denote the derivatives of $\hat{g}_\pm$ w.r.t. ξ at zero, see (2.8).

For further purposes we define here a set

$$K := \{v \in \mathbb{H} \mid \ \beta^+(v) + \beta^-(v) = 0\},$$

cf. also Notation 3 on pg 38.

In some of the theorems in this section we will use the following assumption:

$$\text{ess supp}(s_+) \cap \text{ess supp}(s_-) = \emptyset. \tag{4.1}$$

The empty intersection of the essential supports of $s_\pm$ will guarantee that K is a closed convex cone.

Theorem 11. *Assume (2.10), (2.21), (4.1) and fixed $d_1 \in (0, y_1) \setminus \{y_j \,|\, j = 2, 3, \cdots\}$. The number d_2^m defined by*

$$d_2^m := \sup_{\{\xi_j\} \in \ell^2 \setminus \{0\}} \frac{\sum\limits_{j=1}^{\infty} \lambda_j^S \xi_j^2 + \sum\limits_{j=1}^{\infty} \xi_j \int_\Omega \left(\left(\sum\limits_{k=1}^{\infty} \xi_k e_k \right)^- s_-(x) - \left(\sum\limits_{k=1}^{\infty} \xi_k e_k \right)^+ s_+(x) \right) e_j \; dx}{\sum\limits_{j=1}^{\infty} \xi_j^2} \tag{4.2}$$

can be estimated by

$$d_{2,\max}^0 \geq d_2^m \geq \max \left\{ \sup_{j \in \mathbb{N}} \left(\lambda_j^S - \frac{\|s_-\|_{L^\infty} + \|s_+\|_{L^\infty}}{\kappa_j} \right), \sup_{\substack{\{\xi_j\} \in \ell^2 \setminus \{0\} \\ \sum \xi_j e_j \in K}} \frac{\sum\limits_{j=1}^{\infty} \lambda_j^S \xi_j^2}{\sum_{i=1}^{\infty} \xi_i^2} \right\}. \tag{4.3}$$

If d_2^m is positive, then the supremum in (4.2) is maximum, i.e.

$$d_2^m = \max_{\{\xi_j\} \in \ell^2 \setminus \{0\}} \frac{\sum\limits_{j=1}^{\infty} \lambda_j^S \xi_j^2 + \sum\limits_{j=1}^{\infty} \xi_j \int_\Omega \left(\left(\sum\limits_{k=1}^{\infty} \xi_k e_k \right)^- s_-(x) - \left(\sum\limits_{k=1}^{\infty} \xi_k e_k \right)^+ s_+(x) \right) e_j \; dx}{\sum\limits_{j=1}^{\infty} \xi_j^2}, \tag{4.4}$$

and it is the largest critical point of the system (1.16), (1.10) with fixed d_1. If $d_1 \in (y_2, y_1)$, $\|s_-\|_{L^\infty} > 0$, $\|s_+\|_{L^\infty} > 0$ and $d_2^m > 0$, then $(d_1, d_2^m) \in D_S$.

The proof is postponed to the next section. Let us note that if the first supremum in (4.3) is positive, then it is the maximum. Similarly, if the supremum over K in (4.3) is positive, then it is the maximum, see Theorem 3.2 in [5].

Corollary 4. *If $\|s_\pm\|_{L^\infty}$ are sufficiently small, then d_2^m from (4.2) is positive. If, $d_1 \in (y_2, y_1)$ and $\|s_\pm\|_{L^\infty}$ are both positive and sufficiently small, then d_2^m is a critical point of (1.16), (1.10) with fixed d_1 and $(d_1, d_2^m) \in D_S$.*

This first assertion of the corollary follows directly from the fact that $\lambda_{\max}^S > 0$ for $d_1 \in (0, y_1) \setminus \{y_j \,|\, j = 2, 3, \cdots\}$ and from (4.3). The second statement is a consequence of two last assertions of Theorem 11.

The following lemma is about the maximum over the cone K in (4.3).

Lemma 20. *Let ess sup(s_+), ess sup(s_-) have nonempty interiors. There exists $\varepsilon > 0$ such that for any $d_1 \in [y_1 - \varepsilon, y_1)$ the supremum over K in (4.3) is positive. Hence, d_2^m in (4.2) is positive and $d_1 \in [y_1 - \varepsilon, y_1)$ then $(d_1, d_2^m) \in D_S$.*

Proof. The assumptions (2.10), (2.21), (4.1) guarantee that K is a closed convex cone. The only assumption for application of Remark 3.4 in [5] is an existence of $v \in K$ with $\langle v, e_1 \rangle \neq 0$. Since essential supports of $s_\pm$ have nonempty interiors, any v positive on Ω_+ or on Ω_- and zero elsewhere satisfies this condition because

$$\langle v, e_1 \rangle = \int_\Omega \nabla v \nabla e_1 \; \mathrm{dx} = \kappa_1^{-1} \int_\Omega v e_1 \; \mathrm{dx} \neq 0,$$

and because of (2.21). Remark 3.4 in [5] gives the claim of the first assertion. The second statement follows from the last assertion of Theorem 11 □

Analogous lemma can be proved for $d_1 \in (y_{k+1}, y_k),\ k \geq 2$. In such a case however $(d_1, d_2^m) \in D_U$, which is not of our interest.

Although for the Dirichlet or mixed problem it is only considered $d_1 \in (0, y_1)$, Theorem 11 can be easily modified for the cases $d_1 < 0$ and $d_1 > y_1$. However, in these cases it is not fulfilled $(d_1, d_2^m) \in \mathbb{R}_+^2$ and therefore are not interesting for us.

It should be emphasized that the bounds in (4.3) can be found explicitly for particular systems.

Remark 25. *In the proof of the last assertion of Theorem 11 we will use the fact that the first eigenvalue κ_1 of Laplacian is simple and the eigenfunction e_1 does not change its sign in Ω. Under a more general assumption*

$$e \notin K \quad \textit{for all eigenfunctions } e \textit{ corresponding to the eigenvalues } \kappa_{j_0}, \tag{4.5}$$

where j_0 is such an index that $d_{2,\max}^0 = \lambda_{j_0}^S$, the proof of Theorem 11 can be modified to get the statement $d_2^m < d_{2,\max}^0$. However, in the case $k > 1$ it does not imply $(d_1, d_2^m) \in D_S$ because the point $(d_1, d_{2,\max}^0)$ can lie above the hyperbolas C_j with $j \leq j_0$, see Lemma 8 on pg. 23. Therefore the case $j_0 > 1$ is not included in the last statements of Theorems 11, 13.

The situation for small $\|s_\pm\|_{L^\infty}$ is sketched in Fig. 4.1. Full black lines are the curves C_j.

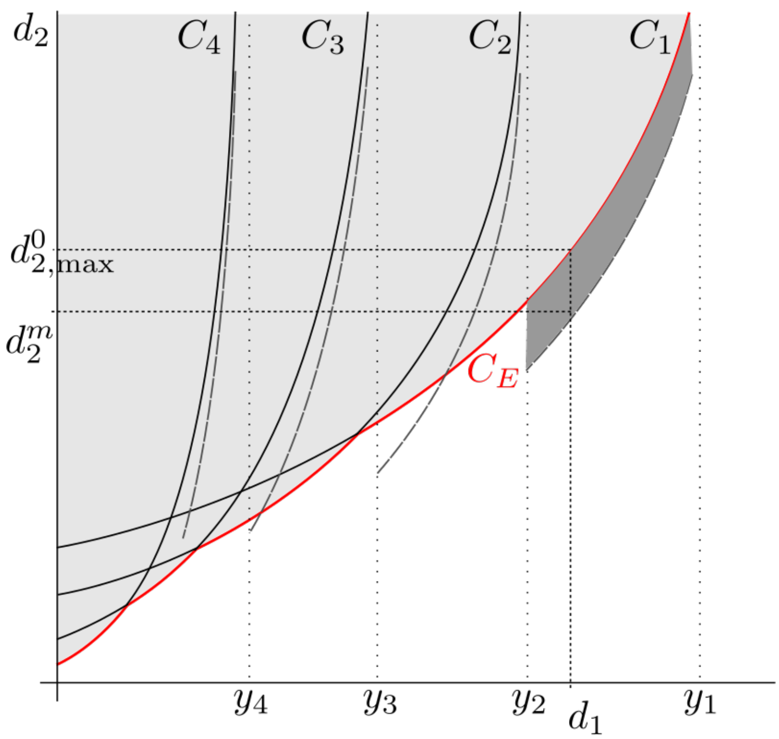


Figure 4.1: Sketch of hyperbolas for Dirichlet/mixed problem with unilateral terms.

Dashed black lines are lower bounds to the largest critical points of the system (1.16), (1.10) given by the expressions in (4.3), which depend continuous on d_1 except the points $d_1 = y_j$, for all $j \in \mathbb{N}$. Grey filling marks an area in D_S containing critical and bifurcation points of the problem (1.16), (1.10) and (1.14), (1.10), respectively, see Corollary 4 Lemma 20 and Theorem 13 below.

Theorem 12. *Let $d_1 \in (0, y_1) \setminus \{ y_j \big| \ j = 2, 3, \cdots \}$ and let (2.10), (2.21), (2.27), (2.29), (2.31), (4.1) be fulfilled. If the point d_2^m from Theorem 11 is positive, then it is the largest bifurcation point of the problem (1.14), (1.10).*

Let use denote

$$\mathscr{S} = \overline{\{(d_2,u,v) \in \mathbb{R} \times W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega) \mid (u,v) \neq 0 \text{ is solution of (1.14), (1.10) with fixed } d_1\}}.$$

Theorem 13. *Let $d_1 \in (0,y_1)\setminus\{y_j \mid j=2,3,\cdots\}$, let (2.10), (2.21), (2.27), (4.1) be true and let the multiplicity of the critical point $d^0_{2,\max}$ be odd. Then for any sufficiently small $\varepsilon > 0$ there exists $\tau_s > 0$ such that if $\|s_\pm\|_{L^\infty} \in [0,\tau_s)$ then $d^0_{2,\max} - \varepsilon < d_2^m$ and there is a global bifurcation point $d_2^b \in [d^0_{2,\max} - \varepsilon, d_2^m]$ of the system (1.14), (1.10) with fixed d_1 in the following sense. The connected component $\mathscr{S}_{d_2^b}$ of $\mathscr{S}$ containing the point $(d_2^b, 0)$ satisfies at least one of the following conditions:*

1. *$\mathscr{S}_{d_2^b}$ is unbounded,*
2. *there exists $(u,v) \in W_D^{1,2}(\Omega) \times W_D^{1,2}(\Omega)$, $(u,v) \neq (0,0)$ such that $(0,u,v) \in \mathscr{S}_{d_2^b}$,*
3. *there exists a critical point $d_2^c \notin [d^0_{2,\max} - \varepsilon, d_2^m]$ of (1.16), (1.10) with fixed d_1 such that $(d_2^c, 0, 0) \in \mathscr{S}_{d_2^b}$.*

If, moreover, $d_1 \in (y_2,y_1)$, $\|s_+\|_{L^\infty} > 0, \|s_-\|_{L^\infty} > 0$ then $(d_1, d_2^b) \in D_S$.

Proof is postponed to next section. It will be seen from it that "sufficiently small ε" means $\varepsilon \in (0, \min(d^0_{2,\max}, (d^0_{2,\max} - d^0_{2,k_0})/2))$, where k_0 is such an index, that d^0_{2,k_0} is the second largest critical point of the system (1.12), (1.10) with fixed d_1. Especially if $d_1 \in (y_2, y_1)$, then $d^0_{2,k_0} < 0$ and therefore $(d^0_{2,\max} - d^0_{2,k_0})/2 > d^0_{2,\max}/2$. Thus ε can be taken from the interval $(0, d^0_{2,k_0}/2)$. See also Fig. 4.1 and the comment below the formula (2.39).

If $s_\pm \equiv 0$, then it is known that the global bifurcation is exactly at the point $d_2^m = d^0_{2,\max}$, as can be proved by using Rabinowitz Theorem (see Appendix).

To summarize, Theorem 12 gives a local bifurcation from the point d_2^m for skew-symmetric systems. Theorem 13 gives for systems with general nonlinearities with small $\|s_\pm\|_{L^\infty}$ a global bifurcation from a bifurcation point located in the interval $[d^0_{2,\max} - \varepsilon, d_2^m]$, where ε is sufficiently small and it is neither excluded nor guaranteed that the bifurcation is in the point d_2^m. Let us note that there is no explicit assumption of the size of $\|s_\pm\|_{L^\infty}$ in Theorem 12.

The following Theorem is an application of Theorem 9 to the system (1.14), (1.10).

Theorem 14. *Let $d_1 \in (0,y_1)$, let (2.27), (2.29) be fulfilled. Assume that $\hat{g}_\pm(x,v^\pm) \equiv s_\pm(x)v^\pm(x)$ with some $s_\pm \in L^\infty(\Omega)$. Let $d_2^s > 0$ be a simple critical point of (1.12), (1.10) with fixed d_1. There exist $\tau_0 > 0$, $R > 0, \delta > 0$, neighborhoods $U, V \subset W_D^{1,2}(\Omega)$ of zero and a map $F : V \to U$ such that for any $s_\pm$ with $\|s_\pm\|_{L^\infty} < \tau_0$, $\|s_+\|_{L^\infty} + \|s_-\|_{L^\infty} > 0$ the following assertions are true:*

(a) *There exist four Lipschitz continuous maps $d_2^+, d_2^- : [0,R] \to \mathbb{R}_+, v_+, v_- : [0,R] \to V$, for which the following holds:*

A pair $(u,v) \in U \times V$ is a solution of (1.14), (1.10) with $d_2 \in (d_2^s - \delta, d_2^s]$ and with fixed d_1 if and only if $u = F(v)$ and $v = rv_+(r)$, $d_2 = d_2^+(r)$ or $v = rv_-(r)$, $d_2 = d_2^-(r)$ for some $r \in (0,R]$. The numbers $d_2^+(0), d_2^-(0)$ are the only critical points of (1.16), (1.10) with fixed d_1 in $(d_2^s - \delta, d_2^s]$, the respective eigenvectors are $v_+(0), v_-(0)$. Moreover, $(d_2^+(0), v_+(0)) \neq (d_2^-(0), v_-(0))$.

(b) *$d_2^+(0), d_2^-(0)$ converge to d_2^s as $\|s_+\|_{L^\infty} + \|s_-\|_{L^\infty} \to 0$.*

(c) *Let $(d_1, d_2^s) \in C_E$ and let (2.21) be true. If $\|s_+\|_{L^\infty} > 0$, then $(d_1, d_2^+) \in D_S$, if $\|s_-\|_{L^\infty} > 0$ then $(d_1, d_2^-) \in D_S$.*

Remark 26. *This theorem gives an existence of bifurcation points for the system (1.14), (1.10) with unilateral terms $s_\pm$ having sufficiently small $L^\infty(\Omega)$ norm and with the parameters d_1, d_2 close to a simple critical point (d_1, d_2^s) of a system (1.12), (1.10). For given nontrivial $s_\pm$, the branches of parameters and bifurcating solutions are Lipschitz continuous, isolated and separated* *(the separation follows from the assertion (a) of this theorem). If one assumes the unilateral terms $s_\pm(\tau)(x) := \tau\tilde{s}_\pm(x)$, with sufficiently small positive parameter τ and with $\tilde{s}_\pm$ given functions independent of τ, then it would be possible to prove even the Lipschitz continuity of the points $d_2^\pm$ w.r.t. parameter τ, see Remark 23.*

The main distinction against previous two theorems is in the assumption of simplicity of (d_1, d_2^s), and on the other hand there is no requirement on the maximality of d_2^s. Therefore the theorem can be applied also on the simple critical points of (1.12), (1.10) with fixed $d_1 \in (0, y_2)\backslash\{y_3, \cdots\}$ lying on C_E, giving the critical points of (1.16), (1.10) in D_S. The existence of stationary solutions in D_S will be verified by numerical computation in Section 5.

Note also that the assumption $d_1 \in (0, y_1)\backslash \{y_j \big| \; j = 2, 3, \cdots\}$ is here weaken to $d_1 \in (0, y_1)$.

4.2.1 Proofs of Theorems 11 – 14

Let us begin the proofs with the essential lemma about the operators $\beta^\pm$ defined in (2.20).

Lemma 21. *The operator $\mathcal{B} := \beta^+ + \beta^-$ is positively homogeneous, Lipschitz continuous and satisfies (3.1). If (2.21) is true, the operator $\beta^+ + \beta^-$ satifies (3.2). The operator $\beta^+ + \beta^-$ satisfies (3.15) for any $v_0 \in W^{1,2}_D(\Omega), \|v_0\| = 1$.*

Proof. The positive homogeneity follows from the definition and the condition (3.2) for $\beta^\pm$ follows directly from the definition and (2.21). The condition (3.1) and Lipschitz continuity of $\beta^\pm$ were proven in Lemma 3. The first assertion of this Lemma now follows from it.

We will show (3.15) for the operator β^- and for any $v \in \mathbb{H}$. The proof for the operator β^+ can be done similarly.

Let $t \in \mathbb{R}$ and $v, h \in W^{1,2}_D(\Omega)$ be arbitrary. We introduce the sets $\Omega_{th}, \Omega^+_{th}, \Omega^-_{th}$, such that $\Omega_{th} \cup \Omega^+_{th} \cup \Omega^-_{th} = \Omega$ and

$$
\begin{aligned}
&|v(x)| \le |th(x)| \text{ for a.a. } x \in \Omega_{th},\\
&|th(x)| \ \le v(x) \text{ for a.a. } x \in \Omega^+_{th},\\
&v(x) \le -|th(x)| \text{ for a.a. } x \in \Omega^-_{th}.
\end{aligned}
\tag{4.6}
$$

Then

$$
\begin{aligned}
v^-(x) = 0 \ \text{ for a.a. } x \in \Omega^+_{th}, \quad & v^-(x) = -v(x) \ \text{ for a.a. } x \in \Omega^-_{th},\\
(v+th)^-(x) = 0 \ \text{ for a.a. } x \in \Omega^+_{th}, \quad & (v+th)^-(x) = -(v(x)+th(x)) \ \text{ for a.a. } x \in \Omega^-_{th},\\
\left[(v(x)+th(x))^-\right]^2 \le (v(x)+th(x))^2 \le 4t^2h(x)^2 \quad & \text{for a.a. } x \in \Omega_{th}.
\end{aligned}
\tag{4.7}
$$

Hence,

$$
\langle \beta^-(v), v\rangle = -\int_\Omega s_-(x) v^- v \ \mathrm{dx} = \int_{\Omega^-_{th}} s_-(x) v^2 \ \mathrm{dx} + \int_{\Omega_{th}} s_-(x)(v^-)^2 \ \mathrm{dx}.
$$

and

$$
\begin{aligned}
&\langle \beta^-(v+th), v+th\rangle = \int_\Omega s_-(x)\left[(v+th)^-\right]^2 \ \mathrm{dx} =\\
&= \int_{\Omega^-_{th}} s_-(x)(v+th)^2 \ \mathrm{dx} + \int_{\Omega_{th}} s_-(x)\left[(v+th)^-\right]^2 \ \mathrm{dx} =\\
&= \int_{\Omega^-_{th}} s_-(x)v^2 \ \mathrm{dx} + 2t\int_{\Omega^-_{th}} s_-(x)vh \ \mathrm{dx} + t^2\int_{\Omega^-_{th}} s_-(x)h^2 \ \mathrm{dx} + \int_{\Omega_{th}} s_-(x)\left[(v+th)^-\right]^2 \ \mathrm{dx}.
\end{aligned}
$$

This means

$$\langle\beta^-(v+th), v+th\rangle - \langle\beta^-(v), v\rangle =$$
$$= 2t\int_{\Omega_{th}^-} s_-(x)vh \text{ dx} + t^2\int_{\Omega_{th}^-} s_-(x)h^2 \text{ dx} + \int_{\Omega_{th}} s_-(x)\left[(v+th)^-\right]^2 \text{ dx} - \int_{\Omega_{th}} s_-(x)(v^-)^2 \text{ dx} =$$
$$= -2t\int_{\Omega} s_-(x)v^-h \text{ dx} - \int_{\Omega_{th}} s_-(x)(v^-)^2 \text{ dx} + t^2\int_{\Omega_{th}^-} s_-(x)h^2 \text{ dx}+$$
$$+\int_{\Omega_{th}} s_-(x)\left[(v+th)^-\right]^2 \text{ dx} + 2t\int_{\Omega_{th}} s_-(x)v^-h \text{ dx} = 2t\langle\beta(v), h\rangle + \Phi_0(h,t),$$

where

$$\Phi_0(h,t) := t^2\int_{\Omega_{th}^-} s_-(x)h^2 \text{ dx} + \int_{\Omega_{th}} s_-(x)\left(\left[(v+th)^-\right]^2 - (v^-)^2 + 2tv^-h\right) \text{ dx}. \tag{4.8}$$

To finish the proof we will show that

$$-3C_H^2\|s_-\|_{L^\infty}\|h\|^2t^2 \le \Phi_0(h,t) \le 7C_H^2\|s_-\|_{L^\infty}\|h\|^2t^2, \quad \text{for all } t\in\mathbb{R}, \tag{4.9}$$

where C_H is a constant from embedding $W_D^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$, from this follows $\lim_{t\to 0}\Phi_0(h,t)/t = 0$. Using (4.6) and (4.7), the individual terms in Φ_0 can be estimated

$$0 \le t^2\int_{\Omega_{th}^-} s_-(x)h^2 \text{ dx} \le C_H^2t^2\|s_-\|_{L^\infty}\|h\|^2,$$
$$0 \le \int_{\Omega_{th}} s_-(x)\left[(v+th)^-\right]^2 \text{ dx} \le 4t^2\|s_-\|_{L^\infty}\int_{\Omega_{th}} h^2 \text{ dx} \le 4C_H^2t^2\|s_-\|_{L^\infty}\|h\|^2,$$
$$-t^2C_H^2\|s_-\|_{L^\infty}\|h\|^2 \le -\int_{\Omega_{th}} s_-(x)(v)^2 \text{ dx} \le 0,$$
$$-2t^2C_H^2\|s_-\|_{L^\infty}\|h\|^2 \le 2t\int_{\Omega_{th}} s_-(x)v^-h \text{ dx} \le 2t^2C_H^2\|s_-\|_{L^\infty}\|h\|^2.$$

Addition of these inequalities gives (4.9). Hence, β^- fulfill (3.15).

□

Proof of Theorem 11. Since $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal base in $W_D^{1,2}(\Omega)$, see Notation 1, for any $v\in W_D^{1,2}(\Omega)$ there exists $\{\xi_i\}\in\ell^2$ such that

$$v = \sum_{i=1}^{\infty}\xi_i e_i.$$

As $Se_k = \lambda_k^S e_k$ by (2.38) we get

$$\sup_{v\in W_D^{1,2}(\Omega), \|v\|=1} \langle Sv - \beta^-(v) - \beta^+(v), v\rangle =$$
$$= \sup_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\left\langle S\left(\sum_{k=1}^{\infty}\xi_k e_k\right), \sum_{j=1}^{\infty}\xi_j e_j\right\rangle - \left\langle \beta^-\left(\sum_{k=1}^{\infty}\xi_k e_k\right) + \beta^+\left(\sum_{k=1}^{\infty}\xi_k e_k\right), \sum_{j=1}^{\infty}\xi_j e_j\right\rangle}{\left\|\sum_{j=1}^{\infty}\xi_j e_j\right\|^2} =$$
$$= \sup_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\sum_{j=1}^{\infty}\lambda_j^S\xi_j^2 + \sum_{j=1}^{\infty}\xi_j\int_{\Omega}\left(\left(\sum_{k=1}^{\infty}\xi_k e_k\right)^- s_- - \left(\sum_{k=1}^{\infty}\xi_k e_k\right)^+ s_+\right)e_j \text{ dx}}{\sum_{j=1}^{\infty}\xi_j^2} = d_2^m \tag{4.10}$$

Now we will prove the estimate (4.3). Due to (2.21) we have

$$\begin{aligned}
-\langle\beta^-(v),v\rangle &= \int_\Omega s_-(x)v^-v \geq -\|s_-\|_{L^\infty}\int_\Omega v^2 = -\|s_-\|_{L^\infty}\langle Av,v\rangle,\\
-\langle\beta^+(v),v\rangle &= -\int_\Omega s_+(x)(v^+)^2 \geq -\|s_+\|_{L^\infty}\int_\Omega v^2 = -\|s_+\|_{L^\infty}\langle Av,v\rangle
\end{aligned} \tag{4.11}$$

for all $v \in W^{1,2}_D(\Omega)$. The eigenvalues of the operator $S - \|s_-\|_{L^\infty}A - \|s_+\|_{L^\infty}A$ are

$$\lambda^S_k - \frac{\|s_+\|_{L^\infty} + \|s_-\|_{L^\infty}}{\kappa_k},$$

cf. (2.38) and Lemma 1. By use of (7.18) with $\mathcal{S} := S - \|s_-\|_{L^\infty}A - \|s_+\|_{L^\infty}A$ we get

$$\sup_{v\in W^{1,2}_D(\Omega),\|v\|=1} \langle Sv - \|s_-\|_{L^\infty}Av - \|s_+\|_{L^\infty}Av, v\rangle = \sup_{j\in\mathbb{N}}\left(\lambda^S_j - \frac{\|s_-\|_{L^\infty} + \|s_+\|_{L^\infty}}{\kappa_j}\right). \tag{4.12}$$

If the last supremum is positive, then it is the maximum. If it is equal to zero, then no maximizer exists, cf. Remark 31. The first statement of Lemma 21 and the formulae (4.10)– (4.12) give

$$\begin{aligned}
d^0_{2,\max} &= \max_{j\in\mathbb{N}}\lambda^S_j = \lambda^S_{\max} = \max_{v\in W^{1,2}_D(\Omega),\|w\|=1}\langle Sv,v\rangle \geq \sup_{v\in W^{1,2}_D(\Omega),\|v\|=1}\langle Sv - \beta^-(v) - \beta^+(v), v\rangle = \\
&= d^m_2 \geq \sup_{v\in W^{1,2}_D(\Omega),\|v\|=1}\langle Sv - (\|s_-\|_{L^\infty} + \|s_+\|_{L^\infty})Av, v\rangle = \\
&= \sup_{j\in\mathbb{N}}\left(\lambda^S_j - \frac{\|s_-\|_{L^\infty} + \|s_+\|_{L^\infty}}{\kappa_j}\right).
\end{aligned} \tag{4.13}$$

Hence, the upper estimate of d^m_2 and a part of the lower estimate in (4.3) is proved. Due to the definition of K and (4.10) we see that

$$\begin{aligned}
d^m_2 &= \sup_{v\in W^{1,2}_D(\Omega),\|v\|=1}\langle Sv - \beta^-(v) - \beta^+(v), v\rangle \geq \sup_{v\in K,\|w\|=1}\langle Sv - \beta^-(v) - \beta^+(v), v\rangle = \\
&= \sup_{v\in K,\|w\|=1}\langle Sv,v\rangle = \sup_{\substack{\xi_j\in\ell^2\setminus\{0\}\\ \sum\xi_j e_j\in K}} \frac{\sum\limits_{j=1}^{\infty}\lambda^S_j\xi^2_j}{\sum\limits_{i=1}^{\infty}\xi^2_i},
\end{aligned}$$

which finishes the proof of (4.3).

We will verify that the assumptions of Theorems 3, 4 are fulfilled the positively homogeneous operator $\mathcal{B} := \beta^- + \beta^+$ and the operators $\mathcal{S} := S$, $\mathcal{A} \equiv 0$, $\mathcal{N} \equiv 0$. The assumptions (i)–(vi) from Section 3.1 were already verified, see Lemmas 3, 6, 21. The assumption (viii) is fulfilled trivially. The equality (4.10) together with assumed positiveness of d^m_2 yield

$$\sup_{v\in W^{1,2}_D(\Omega),\|v\|=1}\langle Sv - \beta^-(v) - \beta^+(v), v\rangle > 0,$$

i.e. 3.20 is fulfilled and according to Theorem 4 the maximum in (3.16) exists and is positive. Lemma 21 guarantees that (3.15) is fulfilled for any $v_0 \in W^{1,2}_D(\Omega)$. Theorem 3 gives the existence of v_0, $\|v_0\| = 1$ such that

$$\lambda^{S-\beta^--\beta^+}_{\max} = \max_{v\in W^{1,2}_D(\Omega),\|v\|=1}\langle Sv - \beta^-(v) - \beta^+(v), v\rangle = \langle Sv_0 - \beta^-(v_0) - \beta^+(v_0), v_0\rangle > 0 \tag{4.14}$$

is the largest eigenvalue of the operator $S - \beta^+ - \beta^-$. Now it follows from (4.10) and (4.14) that the supremum in (4.2) is maximum, i.e. (4.4) is proved. Due to Corollary 1 from pg. 26, the point $d^m_2 = \lambda^{S-\beta^--\beta^+}_{\max}$ is the largest critical point of the system (1.16), (1.10) with fixed d_1.

If $d_1 \in (y_2, y_1)$ then $\lambda^S_{\max} = \lambda^S_1$, $\lambda^S_{\max}$ is simple and the corresponding eigenfunction of S is e_1. Since e_1 has a constant sign in Ω, see Lemma 1 on pg. 15, we get under the assumption $\|s_+\|_{L^\infty} > 0$ and $\|s_-\|_{L^\infty} > 0$ that

$$\langle \beta^+(e_1) + \beta^-(e_1), e_1 \rangle = \int_\Omega s_+(x)(e_1^+)^2 + s_-(x)(e_1^-)^2 \,\mathrm{dx} > 0.$$

Let v_0 be from (4.14). If $v_0 \neq e_1$ then $\langle Sv_0, v_0 \rangle < \lambda^S_{\max}$ by Remark 31 and

$$\langle Sv_0, v_0 \rangle - \langle \beta^-(v_0) + \beta^+(v_0), v_0 \rangle < \lambda^S_{\max}.$$

If $v_0 = e_1$ then $\langle Sv_0, v_0 \rangle = \lambda^S_{\max}$, $\langle \beta^-(v_0) + \beta^+(v_0), v_0 \rangle > 0$ and therefore

$$\langle Sv_0, v_0 \rangle - \langle \beta^-(v_0) + \beta^+(v_0), v_0 \rangle < \lambda^S_{\max}.$$

Summarizing, we get

$$d_2^m = \max_{v \in \mathbb{H}, \|v\|=1} \langle Sv - \beta^-(v) - \beta^+(v), v \rangle < \lambda^S_{\max} = d_2^0,$$

which together with the assumption $d_2^m > 0$ implies $(d_1, d_2^m) \in D_S$. □

Proof of Theorem 13. The assumptions (i)–(viii) from Section 3.1 were already verified, see Lemmas 2, 3, 6, 21, Theorem 1.

We have

$$\langle \beta^-(v), \varphi \rangle = -\int_\Omega s_-(x) v^- \varphi \,\mathrm{dx} \leq \|s_-\|_{L^\infty} \|v^-\|_{L^2} \|\varphi\|_{L^2} \leq \frac{1}{\kappa_1} \|s_-\|_{L^\infty} \|v\| \|\varphi\|.$$

This implies

$$\|\beta^-(v)\| = \sup_{\varphi \in W^{1,2}_D(\Omega), \|\varphi\|=1} \langle \beta^-(v), \varphi \rangle \leq \frac{1}{\kappa_1} \|s_-\|_{L^\infty} \|v\|.$$

Similarly for β^+ and therefore

$$\| \left(\beta^- + \beta^+\right)(v)\| \leq \frac{1}{\kappa_1} \left(\|s_-\|_{L^\infty} + \|s_+\|_{L^\infty}\right) \|v\|.$$

We assume that $d^0_{2,\max}$ has an odd multiplicity. Let us remind here the definition of $\lambda^{S-\beta^--\beta^+}_{\max}$ from (4.14). The eigenvalue $\lambda^S_{\max} = d^0_{2,\max}$ has an odd multiplicity, see Remark 8, and therefore it follows from Theorem 8 and Observation 4 with $\mathcal{B} := \beta^- + \beta^+, \mathcal{N} := N, \mathcal{S} := S$ that for any $\varepsilon \in (0, \min\{\lambda^S_{\max}, (\lambda^S_{\max} - \lambda^S_2)/2)$ there exists $\tau_0 > 0$ such that if $\|s_-\|_{L^\infty}, \|s_+\|_{L^\infty} < \tau_s := \tau_0/(2\kappa_1)$, then $\lambda^S_{\max} - \varepsilon < \lambda^{S-\beta^--\beta^+}_{\max}$ and there is a global bifurcation point $\lambda_b \in [\lambda^S_{\max} - \varepsilon, \lambda^{S-\beta^--\beta^+}_{\max}]$ of the equation

$$\lambda v - Sv - N(v) + \beta^+(v) + \beta^-(v) = 0$$

in the sense of Theorem 8. The formulae (4.10) and (4.14) imply that $\lambda^{S-\beta^--\beta^+}_{\max} = d_2^m$. Due to Theorem 1 and Corollary 1, $d_2^b = \lambda_b \in [d^0_{2,\max} - \varepsilon, d_2^m] = [\lambda^S_{\max} - \varepsilon, \lambda^{S-\beta^--\beta^+}_{\max}]$ is simultaneously a global bifurcation point of the system (1.16), (1.10) with fixed d_1 in the sense of Theorem 13. □

Proof of Theorem 12. The proof of this Theorem is an analogue of the proof of Theorem 17 with $A = 0$ on pg. 72 and therefore it will be skipped. □

Proof of Theorem 14. The proof of this Theorem is based on application of Theorem 9 to the equation

$$d_2 v - Sv + \beta^+(v) + \beta^-(v) - N(v) = 0. \tag{4.15}$$

Then, according to Theorem 1, the conclusions for this equation will be true also for the problem (1.14), (1.10).

Let $d_1 \in (0, y_1) \setminus \{y_j \big| \; j = 2, 3, \cdots\}$ be fixed. To apply Theorem 9 it is necessary to verify its assumptions. The assumptions (i)–(viii) from Section 3.1 were already verified, see Lemmas 2, 3, 6, 21. According to Lemma 21 and Theorem 1 the operator $\beta := \beta^- + \beta^+$ is Lipschitz continuous with a Lipschitz constant $L \leq \kappa_1^{-1}(\|s_+\|_{L^\infty} + \|s_-\|_{L^\infty})$, which gives (A) from Notation 6 and according to Lemma 4, N is C^1 operator with $N'(0) = 0$, which gives (B) from Notation 6. Since $1 \notin \sigma(A)$, the assumption (C) from Notation 6 is not applicable here.

The point d_2^s is simultaneously a simple eigenvalue of S, see Remark 8. The assertion (a) of Theorem 9 now gives the maps $d_2^\pm, v_\pm$, which are Lipschitz continuous.

The C^1-continuous map F and sets U, V can be obtained from Theorem 1 and if necessary, we will take the sets U, V smaller. According to Theorem 1, pairs $(F(rv_\pm(r)), rv_\pm(r))$ are the only solutions of (1.14), (1.10) with $d_2 = d_2^\pm(r)$ and with fixed d_1 in $U \times V$.

If $\|s_-\|_{L^\infty} \neq 0$, then clearly $\beta^-(v_s) \neq \beta^-(-v_s)$ and if $\|s_+\|_{L^\infty} \neq 0$, then $\beta^+(v_s) \neq \beta^+(-v_s)$. In conclusion, because at least one of $s_\pm$ has a positive L^∞ norm, we get $\beta(v_s) \neq \beta(-v_s)$, and in particular, $(d_2^+(0), v_+(0) \neq (d_2^-(0), v_-(0))$. This finishes the proof of (a).

Since $|\beta| \to 0$ as $\|s_+\|_{L^\infty} + \|s_-\|_{L^\infty} \to 0$ and norm of β is smaller than its Lipschitz constant L, the point (c) from Theorem 9 now gives (b). The assertion (c) is a consequence of (d) of Theorem 9. □

4.3 Systems with Neumann boundary conditions

Analogously to Dirichlet case the problem (1.14) with Neumann b.c. was rewritten as a system of two operator equations (2.56) on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, see Section 2.3, and in Section 2.3.2 this system was reduced to (2.62), where the first equation with the symmetric linear compact operator S has again only on one variable, one fixed parameter d_1 and one bifurcation parameter d_2. The eigenvalues of the linear equation (2.59) were found explicitly in (2.61), and these eigenvalues are simultaneously critical points of (1.12) with fixed d_1 and Neumann b.c. The situation is more complicated here, because the operator $(I - A)$ at the bifurcation parameter d_2 in (2.59) is not isomorphism.

Notation 8. *In this section we will use the notation and assumption from Section 2.1. In addition, κ_k are the eigenvalues of the Laplacian with Neumann boundary conditions. The eigenvalues of the operator S are denoted by λ_k^S, and were found explicitly in the formula (2.60). The critical points of (1.12) with fixed d_1 and Neumann b.c., are denoted by $d_{2,k}^0$, see (2.61), the largest critical point is denoted by $d_{2,\max}^0$. For the definitions of critical and bifurcation points see again Definition 5 on page 8. We will assume in the whole section that $\Gamma_D = \emptyset$, i.e. the system (1.14) has Neumann b.c. Under the term solution we will always mean the weak solution.*

By analogy with Dirichlet case, we define here a set

$$K := \{v \in \mathbb{H} \mid \; \beta^+(v) + \beta^-(v) = 0\}.$$

Theorem 15. *Let $d_1 \in (0, y_1) \setminus \{y_j \big| \; j = 2, 3, \cdots\}$ and assume (2.21), (4.1) and*

$$1 > \max\left\{\frac{\mu_m(\Omega)\det B}{b_{11}\|s_-\|_{L^1}}, \frac{\mu_m(\Omega)\det B}{b_{11}\|s_+\|_{L^1}}\right\}. \tag{4.16}$$

Then the maximum

$$d_2^m := \max_{\{\xi_j\} \in \ell^2 \setminus \{0\}} \frac{\sum\limits_{j=0}^{\infty} \lambda_j^S \xi_j^2 + \int_\Omega \left(\left(\sum\limits_{k=0}^{\infty} \xi_k e_k\right)^- s_- - \left(\sum\limits_{k=0}^{\infty} \xi_k e_k\right)^+ s_+\right) e_j \; dx}{\sum\limits_{j=0}^{\infty} \frac{\kappa_j}{1+\kappa_j} \xi_j^2} \in (0, \infty), \tag{4.17}$$

exists and is positive, and it is the largest critical point of (1.16) with Neumann b.c. and with fixed d_1*.*

If $d_1 > y_1$*, (4.16) holds true and*

$$\frac{\min(\mu_m(ess\ supp(s_+)), \mu_m(ess\ supp(s_-)))}{\mu_m(\Omega)} < \left(1 + \left|\frac{b_{11}b_{22}}{\det B}\right|\right)^{-1}, \tag{4.18}$$

then there exists $d_1^0 > y_1$ *such that (4.17) is true also for* $d_1 > d_1^0$ *and* d_2^m *is the largest critical point of (1.16) with Neumann b.c. and with fixed* d_1*. Moreover,* $(d_1, d_2^m) \in D_S$ *and* d_2^m *depends continuously on* d_1*.*

Let us shortly comment this theorem. The operator S has for all $d_1 \in \mathbb{R}$ the eigenvalue

$$\lambda_0^S = \langle Se_0, e_0\rangle = -\frac{b_{12}b_{21}}{b_{11}} + b_{22} = \frac{b_{11}b_{22} - b_{12}b_{21}}{b_{11}} = \frac{\det B}{b_{11}} > 0, \tag{4.19}$$

as can be seen from (2.1), (2.2), (2.60). This is in a contrast to Dirichlet/mixed problem, where for $d_1 > y_1$ the operator S has no positive eigenvalue. The respective eigenfunction to λ_0^S is e_0.

If the largest critical point d_2^m of (1.16) with Neumann b.c. exists and is positive, any corresponding eigenvector v_0 satisfies

$$d_2^m = \frac{\langle Sv_0 - \beta^-(v_0) - \beta^+(v_0), v_0\rangle}{\langle (I-A)v_0, v_0\rangle}. \tag{4.20}$$

Let

$$u_0 := b_{12}A(d_1 I - (d_1 + b_{11})A)^{-1}v_0.$$

Then it is easy to see that (u_0, v_0) is a solution of (1.16) with Neumann b.c. and parameters (d_1, d_2^m).

It should be emphasized that for the linear problem (1.12) with Neumann b.c., without unilateral sources and with $d_1 > y_1$ there is always $d_2^m = +\infty$ and all other critical points are negative and therefore there is no critical point in a set $(y_1, \infty) \times (0, \infty)$, see 7 on pg. 7 which is true also for Neumann problem. However, as the theorem demonstrated, it is no more true for a problem with unilateral sources and there are systems (1.16) with Neumann b.c. having positive critical points even for $d_1 > y_1$.

The proofs of some assertions of this and the next theorem are based on results of paper [14]. This paper concerns with the variational inequalities. The authors work with the convex cone

$$K := \{v \in W^{1,2}(\Omega)|\ v = 0 \text{ a.e. in } \Omega_+ \subset \Omega,\quad v = 0 \text{ a.e. in } \Omega_- \subset \Omega\}.$$

Under the assumption $e_0 \notin K \cup (-K)$ and $d_1 \in (0, y_1)\backslash \{y_j \big|\ j = 2, 3, \cdots\}$ the number

$$d_2^{m,K} = \max_{v \in K, v \neq 0} \frac{\langle Sv, v\rangle}{\langle (I-A)v, v\rangle},$$

where the operators S, A are the same as in our text, is the largest bifurcation point of the following variational inequality with fixed d_1:

$$\begin{aligned}
&\text{find } u \in \mathbb{H}, v \in K: \\
&\quad d_1 \int_\Omega \nabla u \cdot \nabla \phi - b_{11}u\phi - b_{12}v\phi = 0, \\
&\quad d_2 \int_\Omega \nabla v \cdot \nabla(\varphi - v) - b_{21}u(\varphi - v) - b_{22}v(\varphi - v) - n(v)(\varphi - v) \geq 0 \quad \text{for all } \varphi \in K, \phi \in \mathbb{H},
\end{aligned} \tag{4.21}$$

with $n(v)$ satisfying the growth condition (2.4). The assumption (4.16) is there formally fulfilled, since $\|s_\pm\|_{L^1}$ are formally equal to plus infinity.

The assumption (4.16) is a requirement on sufficiently large unilateral terms, more precisely it is expressed by the $\|s_\pm\|_{L^1}$ in the denominator. This goes against the assumptions of Theorems 13, 14 from Dirichlet/mixed problem, which require unilateral terms with small $\|s_\pm\|_{L^\infty}$ norm and for this reason some of the results for Dirichlet/mixed problems are significantly different compared to Neumann problem.

The assumption (4.18) is taken also from the paper [14], it is a requirement on $s_\pm$, which should have sufficiently small essential supports. A theorem about the existence of a branch of bifurcation points for $d_1 > y_1$ is proved there and the statement in following Theorem 15 is an analogy of it for our systems. However, in the discussed paper [14] a monotonicity of the branch w.r.t. d_1 is proved, but the attempts to modify it to our problems have been not successful, due to the nonlinear terms represented by $\beta^\pm$.

Theorem 16. *Let (2.21), (4.1), (4.16), (4.18) be true. There exist $d_1^0 > y_1$ and constants $C_m, C_M > 0$ independent of d_1 such that for any $d_1 > d_1^0$ the point d_2^m from Theorem 15 satisfies*

$$C_m < d_2^m < C_M. \tag{4.22}$$

Theorem 17. *If (2.27), (2.29), (2.31) are true and d_2^m from Theorem 15 exists, then it is the largest bifurcation point of (1.14) with Neumann boundary conditions and with fixed d_1.*

The situation from Theorem 17 is visualized in the Fig. 4.2. There exists a continuous curve of bifurcation points to the right from all hyperbolas deep in the set D_S.

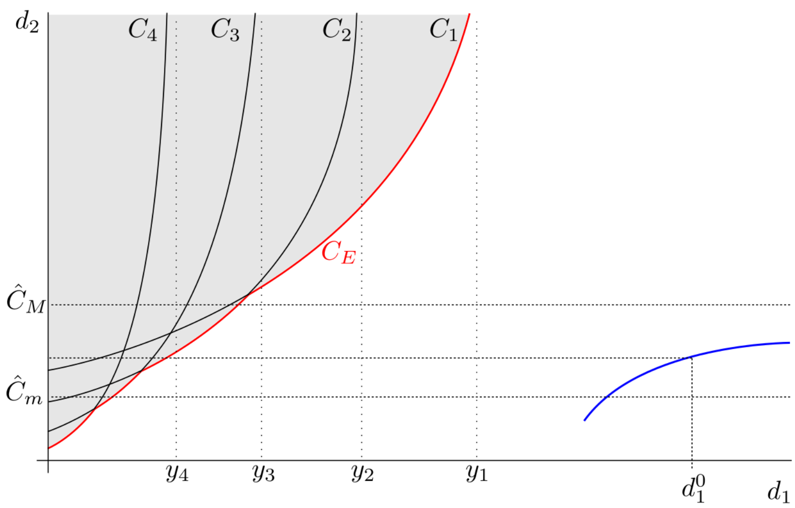


Figure 4.2: A sketch of first four hyperbola segments in $\mathbb{R}^2_+$ for Neumann problem and sketch of the branch of bifurcation points from Theorem 17 and bounds C_m, C_M from Theorem 16.The critical and bifurcation points in D_S from Theorems 16, 17 are painted by the blue line.

The following theorem is an analogue of Theorem 14.

Theorem 18. *Let $d_1 \in (0, y_1)$, let (2.27) and (2.29). Assume that $\hat{g}_\pm(x, v^\pm) \equiv s_\pm(x)v^\pm(x)$ for some $s_\pm \in L^\infty(\Omega)$. Let $d_2^s > 0$ be a simple critical point of (1.12), (1.10) with fixed d_1. There exist $\tau_0 > 0$, $R > 0, \delta > 0$, neighborhoods $U, V \subset W_D^{1,2}(\Omega)$ of zero and a map $F : V \to U$ such that for any $s_\pm$ with $\|s_\pm\|_{L^\infty} < \tau_0$, $\|s_+\|_{L^\infty} + \|s_-\|_{L^\infty} > 0$ the following assertions are true:*

(a) There exist four Lipschitz continuous maps $d_2^+, d_2^- : [0, R] \to \mathbb{R}_+, v_+, v_- : [0, R] \to V$, for which the following holds:

A pair $(u, v) \in U \times V$ is a solution of (1.14), (1.10) with $d_2 \in (d_2^s - \delta, d_2^s]$ and with fixed d_1 if and only if $u = F(v)$ and $v = rv_+(r)$, $d_2 = d_2^+(r)$ or $v = rv_-(r)$, $d_2 = d_2^-(r)$ for some

$r \in (0, R]$. The numbers $d_2^+(0), d_2^-(0)$ are the only critical points of (1.16), (1.10) with fixed d_1 in $(d_2^s - \delta, d_2^s]$, the respective eigenvectors are $v_+(0), v_-(0)$. Moreover, $(d_2^+(0), v_+(0)) \neq (d_2^-(0), v_-(0))$.

(b) $d_2^+(0), d_2^-(0)$ converge to d_2^s as $\|s_+\|_{L^\infty} + \|s_-\|_{L^\infty} \to 0$.

(c) Let $(d_1, d_2^s) \in C_E$ and let (2.21) be true. If $\|s_+\|_{L^\infty} > 0$, then $(d_1, d_2^+) \in D_S$, if $\|s_-\|_{L^\infty} > 0$ then $(d_1, d_2^-) \in D_S$.

4.4 Proof of Theorems 15–18

This section will begin with the analogue of Lemma 21 for the operators $\beta^\pm$ defined in (2.55).

Lemma 22. *The operator $\mathcal{B} := \beta^+ + \beta^-$ is positively homogeneous, Lipschitz continuous and satisfies (3.1). If (2.21) is true, the operator $\beta^+ + \beta^-$ satifies (3.2). The operator $\mathcal{B}$ satisfies satisfies (3.15) for any $v_0 \in W_D^{1,2}(\Omega), \|v_0\| = 1$. The operator S satisfy (3.4).*

Proof. The proof is the same as the proof of Lemma 21 therefore will be skipped. □

Proof of Theorem 15. To apply Theorems 3, 4 it is necessary to check their assumptions. The assumptions (i)–(iv) from Section 3.1 were already verified, see Lemmas 2, 3, 6 (which can be easily modified for the operators on $W^{1,2}(\Omega)$) and Lemma 22, Theorem 2.

The operator A is a linear symmetric compact operator with $\sigma(A) \subset [0,1]$ and $1 \in \sigma(A)$ is a simple eigenvalue with the eigenfunction e_0, which is constant. Since we have fixed $e_0 > 0$, $\|e_0\| = 1$, value of e_0 is $(\mu_m(\Omega))^{-1/2}$ in the whole Ω. Therefore A is compliant with (v) from Section 3.1. Set $\mathcal{B} \equiv \beta := \beta^+ + \beta^-$. Use of the definition of e_0 leads to

$$\langle \beta(e_0), e_0 \rangle = \frac{1}{\mu_m(\Omega)} \int_\Omega s_+(x) \,\mathrm{d}x = \frac{\|s_+\|_1}{\mu_m(\Omega)}, \ \langle \beta(-e_0), -e_0 \rangle = \frac{1}{\mu_m(\Omega)} \int_\Omega s_-(x) \,\mathrm{d}x = \frac{\|s_-\|_1}{\mu_m(\Omega)}, \tag{4.23}$$

which implies $e_0 \notin K \cup (-K)$, otherwise (4.23) would not be true. The assumption (3.4) will be verified later in the proof.

Now we will check the assumptions of Theorem 4. Since

$$\langle Se_0, e_0 \rangle = \frac{\det B}{b_{11}},$$

see (4.19), the assumption (3.22) is fulfilled when

$$1 > \max\left\{ \frac{\det B}{b_{11}\langle \beta(e_0), e_0 \rangle}, \frac{\det B}{b_{11}\langle \beta(-e_0), -e_0 \rangle} \right\}. \tag{4.24}$$

By using the (4.23) the assumption (4.24) can be rewritten as

$$1 > \max\left\{ \frac{\mu_m(\Omega) \det B}{b_{11}\|s_-\|_1}, \frac{\mu_m(\Omega) \det B}{b_{11}\|s_+\|_1} \right\},$$

which is (4.16). As (4.16) is equivalent to (3.22), the assumption (3.4) is true. Similarly to the proof of Theorem 11

$$d_2^m = \sup_{v \in \mathbb{H}, v \neq 0} \frac{\langle Sv - \beta(v), v \rangle}{\langle (I - A)v, v \rangle} =$$

$$= \sup_{\{\xi_j\} \in \ell^2 \setminus \{0\}} \frac{\sum\limits_{j=0}^\infty \lambda_j^S \xi_j + \int_\Omega \left(\left(\sum\limits_{k=0}^\infty \xi_k e_k \right)^- s_- - \left(\sum\limits_{k=0}^\infty \xi_k e_k \right)^+ s_+ \right) e_j \,\mathrm{d}x}{\sum\limits_{j=0}^\infty \frac{\kappa_j}{1 + \kappa_j} \xi_j^2}.$$

When $d_1 \in (0, y_1) \setminus \{ y_j \mid j = 2, 3, \cdots \}$, Lemma 4.1 [14] gives the existence of $\varphi \in K$ for which the assumption (3.21) is fulfilled, and therefore (3.20) is fulfilled as well. Let $d_1 > y_1$. It follows from [14], Section 4.2, that under the assumption (4.18) there exists d_1^0 such that for any $d_1 > d_1^0$ the assumption (3.21) is fulfilled and therefore (3.20) is fulfilled.

The operator β satisfies (3.15); see Lemma 22. Theorems 3, 4 now give the claim.

The proof of continuous dependence of d_2^m on d_1 is analogous to the proof of Proposition 5.3 in [14]. □

Proof of Theorem 16. The assumptions (4.16), (4.18) guarantee the existence and positivity of d_2^m for any $d_1 > d_1^0$. The proof of the existence of C_M is based on the Theorem 5. As a first step it is necessary to find the constant $C, \widehat{C}$ for the operator $\mathcal{B} \equiv \beta := \beta^+ + \beta^-$. Since the proof of Lemma 22 is the same as the proof of Lemma 21, the constant $\widehat{C}$ can be chosen

$$\widehat{C} := 14 C_H (\|s_-\|_{L^\infty} + \|s_+\|_{L^\infty}),$$

where C_H is a constant of the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$, as follows from (4.9) applied to $\beta = \beta^+ + \beta^-$. The constant C can be computed from its definition (3.29) as

$$C = -\max \left\{ \frac{\det B}{b_{11}} - \frac{\|s_-\|_1}{\mu_m(\Omega)}, \frac{\det B}{b_{11}} - \frac{\|s_+\|_1}{\mu_m(\Omega)} \right\}.$$

The constants $C, \widehat{C}$ are independent of d_1. It should be reminded that (4.16) is supposed to be fulfilled and therefore d_2^m is finite. Let us use a symbol $S(d_1)$ to emphasize the dependence of the operator S on the parameter d_1 and make a limit $d_1 \to \infty$. Then

$$\begin{aligned} \lim_{d_1 \to +\infty} |S(d_1)| &= \lim_{d_1 \to +\infty} \max_{k \in \mathbb{N}_0} \left| \lambda_k^S(d_1) \right| = \lim_{d_1 \to +\infty} \max_{k \in \mathbb{N}_0} \left| \frac{1}{1+\kappa_k} \left(\frac{b_{12} b_{21}}{d_1 \kappa_k - b_{11}} + b_{22} \right) \right| = \\ &= \lim_{d_1 \to +\infty} \max_{k \in \mathbb{N}_0} \frac{1}{1+\kappa_k} \left| \frac{-\det B}{d_1 \kappa_k - b_{11}} + \frac{b_{22} d_1 \kappa_k}{d_1 \kappa_k - b_{11}} \right| \leq \\ &\leq \frac{\det B}{b_{11}} + |b_{22}| = \lambda_0^S + |b_{22}| =: C_0 > 0. \end{aligned} \tag{4.25}$$

Now we have to find a constant $\tilde{C} > 0$ such that for the function

$$f(d_1) := 1 + \left(\frac{-(|S(d_1)| + |\beta|) + \left((|S(d_1)| + |\beta|)^2 + C(\widehat{C} + |S(d_1)|) \right)^{\frac{1}{2}}}{(\widehat{C} + |S(d_1)|)} \right)^{-2}, \tag{4.26}$$

where $|\beta|, \widehat{C}, C$ are independent of d_1, holds

$$\lim_{d_1 \to \infty} f(d_1) \leq \tilde{C}.$$

Since f is continuous, it will be possible to find suitable $\tilde{C}_0$ and d_1^0 such that $f(d_1) \leq \tilde{C}_0$ for all $d_1 > d_1^0$. The function $1 + x^{-2}$ is decreasing on the set $[0, \infty)$. Hence, it suffices to find a suitable lower bound for ε_0. Using (4.25) it can be estimated

$$\begin{aligned} \varepsilon_0 &= \frac{-(|S(d_1)| + |\beta|) + \left((|S(d_1)| + |\beta|)^2 + C(\widehat{C} + |S(d_1)|) \right)^{\frac{1}{2}}}{\widehat{C} + |S(d_1)|} = \\ &= \frac{|S(d_1)| + |\beta|}{\widehat{C} + |S(d_1)|} \left(-1 + \left(1 + \frac{C(\widehat{C} + |S(d_1)|)}{(|S(d_1)| + |\beta|)^2} \right)^{\frac{1}{2}} \right) \geq \frac{|\beta|}{\widehat{C} + C_0} \left(-1 + \left(1 + \frac{C \widehat{C}}{C_0 + |\beta|} \right)^{\frac{1}{2}} \right) = \\ &=: \tilde{C}_1 > 0. \end{aligned}$$

This estimate does not depend on d_1, hence, (4.26) is fulfilled for $\tilde{C} := 1 + \tilde{C}_1^{-2}$. Since the second largest eigenvalue of A is $1 - (1+\kappa_2)^{-1} = \kappa_2/(1+\kappa_2)$, the final form of the estimate is

$$\lim_{d_1\to\infty} \max_{v\in\mathbb{H}\setminus\mathrm{Span}\{e_0\}} \frac{\langle S(d_1)v, v\rangle - \langle \beta(v), v\rangle}{\langle (I-A)v, v\rangle} \le \tilde{C}\frac{\kappa_2}{1+\kappa_2}(\lambda_0^S + |b_{22}|).$$

Since the r.h.s. of the last expression is independent of d_1, it is possible to find $d_1^0 > y_1$ and constant C_M such that for any $d_1 > d_1^0$ the number d_2^m satisfy $d_2^m < C_M$. This is the upper bound for d_2^m. Lower bound can be obtained under the assumption (4.18) by choosing a suitable $v \in K$, see formula (4.7) in Section 4.2 in [14].

□

Proof of Theorem 17. In order to use Theorem 8 it is necessary to verify all of its assumptions. The points (i)–(v) from Section 3.1 were verified already in the proof of Theorem 15, the point (viii) follows from (2.3), (2.4), cf. Lemma 2.

The reaction-diffusion system (2.56) can be reduced to an equation

$$d_2(I-A)v - Sv + \beta(v) - N(v) = 0,$$

and since (2.27), (2.29) and (2.31) are supposed, the operator N has the potential Φ_N, see Lemma 13. The operator B has potential due to Lemma 5. The operator S has the potential because it is symmetric. Since N, S are compact and β satisfy (3.1), the functional $\Phi(v) = \frac{1}{2}(\langle Sv, v\rangle + \langle \beta(v), v\rangle) + \Phi_N(v)$ is weakly continuous. Since $\langle (I-A)v_0, v_0\rangle > 0$, where v_0 is an eigenvector to d_2^m, see (4.20), clearly $v_0 \notin \mathrm{Ker}(I-A)$. The statement that d_2^m is the largest bifurcation point of (1.14) with Neumann b.c. and with fixed d_1 follows now from Theorem 8. □

Proof of Theorem 18. First step is to rewrite the weak formulation of the system (1.14), (1.10) as one operator equation

$$d_2(I-A)v - Sv + \beta^+(v) + \beta^-(v) - N(v) = 0, \tag{4.27}$$

using Theorem 2.

Since $Se_0 = \lambda_0^S e_0$, where $\lambda_0^S > 0$, see (4.19), the assumption (C) from Section 6 is fulfilled. Now the rest of the proof is the same as the proof of Theorem 14. □

4.5 Problems with unilateral terms on the boundary

In this section, we will always assume (2.1)–(2.4), (2.68), (2.69). We will also use the operators $\beta_U^\pm, \hat{G}_U^\pm$ from Section 2.4 and because the essential supports of $s_\pm$ are disjoint, we can define the following cone.

Definition 9. *A closed convex cone K_U will be defined by*

$$K_U = \{\varphi \in \mathbb{H} \mid \beta_U^+(\varphi) + \beta_U^-(\varphi) = 0\}.$$

The following theorem is an analogue of Theorems 11, 12, 13.

Theorem 19. *Let $d_1 \in (0, y_1)\setminus\{y_j \mid j = 2, 3, \cdots\}$ and (2.10), (2.21) be true. The estimate (4.3) with d_2^m, K and $\|s_\pm\|_{L^\infty}$ replaced by*

$$\tilde{d}_2^m := \sup_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\sum_{j=1}^\infty \lambda_j^S \xi_j^2 + \sum_{j=1}^\infty \xi_j \int_{\Gamma_U} \left(\left(\sum_{k=1}^\infty \xi_k e_k\right)^- s_- - \left(\sum_{k=1}^\infty \xi_k e_k\right)^+ s_+\right) e_j \, dS}{\sum_{j=1}^\infty \xi_j^2}. \tag{4.28}$$

and K_U *and* $C_T\|s_\pm\|_{L^\infty(\Gamma_U)}$*, respectively, is valid, where* C_T *is a constant from the embedding* $W_D^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$.

If $\tilde{d}_2^m$ *is positive, then the supremum in (4.28) is maximum and* $\tilde{d}_2^m$ *is the largest critical point of the system (2.71), (2.67) with fixed* d_1.

If (2.27), (2.29), (2.31) is true, then $\tilde{d}_2^m$ *is the largest bifurcation point of (2.66), (2.67).*

Let (2.27) be true and let the multiplicity of d_2^0 *be odd. Then for any sufficiently small* $\varepsilon > 0$ *there exists* $\tau_s > 0$ *such that if* $s_-, s_+ \in L^\infty(\Gamma_N)$, $\|s_\pm\|_{L^\infty(\Gamma_N)} \in [0, \tau_s)$ *then* $d_2^0 - \varepsilon < \tilde{d}_2^m$ *and there is a global bifurcation point* $\tilde{d}_2^b \in [d_2^0 - \varepsilon, \tilde{d}_2^m]$ *of the system (2.66), (2.67) in the sense of Theorem 13. If* $d_1 \in (y_2, y_1)$, $\|s_-\|_{L^\infty(\Gamma_N)}$, $\|s_+\|_{L^\infty(\Gamma_N)} \in (0, \tau_0)$ *then* $(d_1, \tilde{d}_2^b) \in D_S$.

Theorem 20. *Let* $d_1 \in (0, y_1) \setminus \{y_j \big| \ j = 2, 3, \cdots\} \cup (y_1, \infty)$*, let* $\mu_{m-1}(\Gamma_D) = 0$ *and let*

$$1 > \max\left\{\frac{\mu_{m-1}(\partial\Omega)\det \mathbf{B}}{b_{11}\|s_-\|_{L^1(\partial\Omega)}}, \frac{\mu_{m-1}(\partial\Omega)\det \mathbf{B}}{b_{11}\|s_+\|_{L^1(\partial\Omega)}}\right\}. \tag{4.29}$$

The point

$$d_2^m := \max_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\sum_{j=1}^\infty \lambda_j^S \xi_j^2 + \sum_{j=1}^\infty \xi_j \int_{\partial\Omega}\left(\left(\sum_{k=1}^\infty \xi_k e_k\right)^- s_- - \left(\sum_{k=1}^\infty \xi_k e_k\right)^+ s_+\right) e_j \ dS}{\sum_{j=1}^\infty \xi_j^2}. \tag{4.30}$$

is positive and it is the largest critical point of the problem (2.71), (2.78). Assume (2.76), (2.77). If (2.31) is true, then it is the largest bifurcation point of the problem (2.66), (2.74).

Moreover, there exist $C_m, C_M, d_1^0 > 0$ *such that for any* $d_1 > d_1^0$ *the point* d_2^m *defined formally by (4.30) satisfy*

$$C_m < d_2^m < C_M,$$

and if (2.27), (2.29) are (2.31) is true, it is the largest bifurcation point of (2.66), (2.74).

An analogue of Remark 4 applies here as well. Let us remind the Section 2.4 which contains a guide to abstract formulation of these type of problems.

4.5.1 Proofs of Theorems 19,20

Proof of Theorem 19. An analogue of Lemma 21 can be proved. The operator $\mathcal{B} := \beta_U^+ + \beta_U^-$ fulfills (3.1) due to the compact embedding $W_D^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\partial\Omega)$. For the proof of (3.15) with $\mathcal{B} := \beta_U^-$ we introduce sets $\Gamma_{th}^+, \Gamma_{th}^-, \Gamma_0^+, \Gamma_0^-$ such that $\Gamma_N = \Gamma_{th}^+ \cup \Gamma_{th}^- = \Gamma_0^+ \cup \Gamma_0^-$,

$$\begin{aligned} &(v_0 + th)(x) < 0 \ \text{ for a.a. } x \in \Gamma_{th}^-, \quad &&(v_0 + th)(x) \geq 0 \ \text{ for a.a. } x \in \Gamma_{th}^+, \\ &v_0(x) < 0 \ \text{ for a.a. } x \in \Gamma_0^-, &&v_0(x) \geq 0 \ \text{ for a.a. } x \in \Gamma_0^+, \end{aligned}$$

and $\Gamma_{th1}, \Gamma_{th2}, \Gamma_{th3}$ such that $\Gamma_{th}^- = \Gamma_{th1} \cup \Gamma_{th2}$, $\Gamma_0^- = \Gamma_{th1} \cup \Gamma_{th3}$,

$$\begin{aligned} &v_0(x) < -th(x) \text{ and } v_0(x) < 0 \text{ for a.a. } x \in \Gamma_{th1}, \\ &v_0(x) < -th(x) \text{ and } v_0(x) \geq 0 \text{ for a.a. } x \in \Gamma_{th2}, \\ &v_0(x) \geq -th(x) \text{ and } v_0(x) < 0 \text{ for a.a. } x \in \Gamma_{th3}. \end{aligned}$$

Similarly for the operator β_U^+. Then we can follow the proof of Lemma 21. The proof of the first part of Theorem 19 is now almost the same as the proof of Theorem 11. To prove the second part of Theorem 19, we will use Theorem 8 in the same way as in the proof of Theorem 13. □

Proof of Theorem 20. The proof is an analogue to the proof of Theorem 15. It can be verified that the condition (4.29) is equivalent to (4.16). According to [14], Lemma 4.1, the assumption (3.20) is true for any $d_1 \in (0, y_1) \setminus \{y_j \big| \ j = 2, 3, \cdots\} \cup (y_1, \infty)$. The remaining assertions can be checked in a way similar to the proof of Theorem 15.

□

4.6 Systems with Neumann boundary conditions on $C^{1,1}$ domain

At the end of this chapter we are going to give a practical application of the Theorem 10. Since the assumptions and notation are significantly different from the standard ones, this application is considered as a stand-alone Section.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$. We will study stationary states of a system

$$\begin{aligned} &d_1\Delta u_1 + b_{11}u_1 + b_{12}u_2 + n_1(u_1,u_2) = 0 \quad \text{in } \Omega, \\ &d_2\Delta u_2 + b_{21}u_1 + b_{22}u_2 + n_2(u_1,u_2) = \tau\left([g_+(x,u_2)u_2]^+ - [g_-(x,u_2)u_2]^-\right) \quad \text{in } \Omega, \end{aligned} \tag{4.31}$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{4.32}$$

where $\tau \geq 0$ is a parameter. We assume that

(i) $n_1, n_2 \colon \mathbb{R}^2 \to \mathbb{R}$ are C^2 functions and $n_1(0,0) = n_2(0,0) = 0$, $\partial_i n_j(0,0) = 0$ for all $i,j \in \{1,2\}$,

(ii) $g_\pm(\cdot,u)$ are measurable for all $u \in \mathbb{R}$, $g_\pm(\cdot,0) \in L^\infty(\Omega)$, and for any $c > 0$ there exists $L > 0$ such that $|g_\pm(x,u) - g_\pm(x,v)| \leq L|u-v|$ for all $x \in \Omega$ and $u,v \in [-c,c]$ and $g_\pm(x,u) \geq 0$ for all $x \in \Omega$ and for all $u \in \mathbb{R}$.

Let us fix some $p > n$. Recalling that $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, see Theorem 22 in Appendix, we can define Banach spaces

$$U \coloneqq \{(u_1,u_2) \in [W^{2,p}(\Omega)]^2 \mid (4.32) \text{ holds}\}$$

and

$$V \coloneqq [L^p(\Omega)]^2.$$

We say that $\mathbf{U} = (u_1,u_2)$ is a solution of the problem (4.31) with the boundary conditions (4.32) if and only if $\mathbf{U} \in U$ satisfies the equations (4.31) almost everywhere in Ω. We will suppose that

(iii) $(d_1^0, d_2^0) \in \mathbb{R}^2$ are such parameters that the linear problem

$$\begin{aligned} d_1\Delta u_1 + b_{11}u_1 + b_{12}u_2 &= 0, \\ d_2\Delta u_2 + b_{21}u_1 + b_{22}u_2 &= 0 \end{aligned} \tag{4.33}$$

with the boundary conditions (4.32) and $(d_1,d_2) = (d_1^0,d_2^0)$ has up to scalar multiples unique nontrivial solution $\mathbf{U}_0 = (u_{10},u_{20}) \in U$.

Remark 27. *For a good physical interpretation, let us assume again (2.1), (2.2). Then hypothesis* (iii) *holds if and only if there is a unique j such that (d_1^0,d_2^0) belong to the hyperbola*

$$C_j = \{(d_1,d_2) \in \mathbb{R}^2 \mid (\kappa_j d_1 - b_{11})(\kappa_j d_2 - b_{22}) = b_{12}b_{21}\}.$$

This means that (d_1^0,d_2^0) does not lie on an intersection point of two different hyperbolas C_j, and the eigenvalue κ_j of $-\Delta$ is simple. In this case, u_{20} is a corresponding eigenfunction and $u_{10} = \frac{b_{12}}{d_1^0\kappa_j - b_{11}}u_{20}$. In fact, the last assertion hold also if we relax the assumption (2.1), (2.2) to

$$b_{12}b_{21} \neq 0, \quad \det B = b_{11}b_{22} - b_{12}b_{21} \neq 0. \tag{4.34}$$

The proof is almost the same. Consequently, also in Theorem 21 it is possible to replace (2.1), (2.2) by (4.34).

Theorem 21. *Let $d_1 \in (0,y_1)\setminus\left\{y_j \,\middle|\, j = 2,3,\cdots\right\}$ be fixed. Under the assumptions* (i)–(iii) *there exist $\varepsilon > 0$, $\delta > 0$ and Lipschitz continous maps $\hat{d}_2^+, \hat{d}_2^- \colon [0,\varepsilon] \to \mathbb{R}$ and $\hat{\boldsymbol{U}}_+, \hat{\boldsymbol{U}}_- \colon [0,\varepsilon] \to U$ such that the following is true.*

1. *$(\tau, d_2, (u_1, u_2))$ is a solution to (4.31) with $|\tau| + |d_2 - d_2^0| + \|(u_1, u_2)\|_{[W^{2,p}(\Omega)]^2} \le \delta$ and $u \neq 0$ if and only if $d_2 = d_2^0 + \hat{d}_2^+(r, \tau)$, $u = r\hat{u}_+(r, \tau)$ or $d_2 = d_2^0 + \hat{d}_2^-(r, \tau)$, $u = r\hat{u}_-(r, \tau)$ for some $r \in (0, \varepsilon]$.*

2. *$\hat{d}_2^+(0,0) = \hat{d}_2^-(0,0) = 0$, $\hat{\boldsymbol{U}}_+(0,0) = \boldsymbol{U}_0$, $\hat{\boldsymbol{U}}_-(0,0) = -\boldsymbol{U}_0$.*

3. *Assume (2.1), (2.2). If*

$$\mu_m\{x \in \Omega \mid u_{20}(x) > 0,\ g_+(x,0) > 0\} > 0 \ \textit{or}\ \mu_m\{x \in \Omega \mid u_{20}(x) < 0,\ g_-(x,0) > 0\} > 0 \tag{4.35}$$

then

$$\lim_{\tau \to 0} \frac{\hat{d}_2^+(0,\tau)}{\tau} < 0$$

and if

$$\mu_m\{x \in \Omega \mid u_{20}(x) > 0,\ g_-(x,0) > 0\} > 0 \ \textit{or}\ \mu_m\{x \in \Omega \mid u_{20}(x) < 0,\ g_+(x,0) > 0\} > 0 \tag{4.36}$$

then

$$\lim_{\tau \to 0} \frac{\hat{d}_2^-(0,\tau)}{\tau} < 0.$$

4. *Assuming (2.1), (2.2) let us introduce the numbers $\alpha_1 = b_{12}(d_1^0\kappa_j - b_{11})^{-1}$, $\alpha_2 = 1$, $\beta_1 = b_{21}(d_1^0\kappa_j - b_{11})^{-1}$, and $\beta_2 = 1$ with κ_j from Remark 27. If at least one of (4.35) or (4.36) is true, then ρ from (3.77) satisfies*

$$sgn(\rho) = sgn\Big(\int_\Omega u_{20}(x)^3 \ dx \sum_{i,j,k=1}^{2} \beta_k \alpha_i \alpha_j \partial_{u_i}\partial_{u_j} n_k(0,0)\Big).$$

Proof of Theorem 21. Let us introduce the operators $F\colon U \to V$ and $G\colon U \to V$ as

$$[F(\hat{d}_2, \mathbf{U})](x) = \hat{d}_2 J\Delta\mathbf{U}(x) + D\Delta\mathbf{U}(x) + B\mathbf{U}(x) + N(\mathbf{U}(x)),$$
$$[G(\mathbf{U})](x) = [g_+(x,u_2)u_2]^+ - [g_-(x,u_2)u_2]^-,$$

with

$$J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad D = \begin{pmatrix} d_1^0 & 0 \\ 0 & d_2^0 \end{pmatrix}$$

and $N\colon U \to V$ defined by $N(\mathbf{U})(x) = (n_1(\mathbf{U}(x)), n_2(\mathbf{U}(x)))$, the problem 4.31 can be written as

$$F(\hat{d}_2, \mathbf{U}) = \tau G(\mathbf{U})$$

which corresponds to (3.68). Clearly, we have $F \in C^2(U, V)$. The operator G is Lipschitz continuous on bounded sets. We recall that linear compact perturbations of isomorphisms are Fredholm operators of index zero. If we choose $\mu \notin \{\kappa_0, \kappa_1, \dots\}$ then the map $\mathbf{U} \mapsto D\Delta\mathbf{U} + \mu D\mathbf{U}$ is an isomorphism of U onto V due to [23, Theorem 2.4.2.7]. We write $\partial_u F(0,0) = D\Delta + B = (D\Delta + \mu D) + (-\mu D + B)$. The operator in the second parenthesis is compact, and therefore $\partial_u F(0,0)$ is a Fredholm operator of index zero. It is easy to see that $\mathbf{V}_0^* = (v_{10}^*, v_{20}^*)$ is a solution of the formally adjoint problem

$$d_1^0\Delta v_1^* + b_{11}v_1^* + b_{21}v_2^* = 0,$$
$$d_2^0\Delta v_2^* + b_{12}v_1^* + b_{22}v_2^* = 0$$

if and only if $\mathbf{V}_0^* = (v_{10}^*, v_{20}^*) = (b_{12}^{-1}b_{21}u_{10}, u_{20})$, where u_{10}, u_{20} is a solution of (4.33), (4.32). We have $\partial_{\hat{d}_2}\partial_{\mathbf{U}}F(0,0) = J$ and if we interpret $\mathbf{V}_0^*$ as the linear functional $\langle v_0^*, u\rangle = \int_\Omega (v_{10}^* u_1 + v_{20}^* u_2)\ \mathrm{dx}$ for all $u \in U$, then 3.71 has the form

$$\kappa = \langle \mathbf{V}_0^*, J\Delta\mathbf{U}_0\rangle = -\int_\Omega \nabla v_{20}^*(x)\nabla u_{20}(x)\ \mathrm{dx} = -\int_\Omega (\nabla u_{20}(x))^2\ \mathrm{dx} < 0,$$

which implies that (3.70) holds. We have

$$\begin{aligned}\lim_{r\to 0}\frac{1}{r}\big([g_-(x,ru_2)ru_2]^- - [g_+(x,ru_2)ru_2]^+\big) &= \lim_{r\to 0}\tau\big([g_-(x,ru_2)u_2]^- - [g_+(x,ru_2)u_2]^+\big)\\ &= [g_-(x,0)u_2]^- - [g_+(x,0)u_2]^+ =: G_0(\tau,\mathbf{U})\end{aligned}$$

and therefore the assumption (3.72) is satisfied. The assumption (3.74) is fulfilled due to ii. Hence, all of the assumptions of Theorem 10 are verified. The relations (3.75) and (3.76) take the form

$$\begin{aligned}\lim_{\tau\to 0}\frac{\hat{d}_2^+(0,\tau)}{\tau} &= \frac{-1}{\kappa}\int_\Omega\Big(g_-(x,0)u_{20}^-(x)v_{20}^*(x) - g_+(x,0)u_{20}^+(x)v_{20}^*(x)\Big)\ \mathrm{dx}\\ &= \frac{1}{\kappa}\int_\Omega\Big(g_-(x,0)\big([u_{20}(x)]^-\big)^2 + g_+(x,0)\big([u_{20}(x)]^+\big)^2\Big)\ \mathrm{dx},\\ \lim_{\tau\to 0}\frac{\hat{d}_2^-(0,\tau)}{\tau} &= \frac{1}{\kappa}\int_\Omega\Big(g_-(x,0)(-u_{20}(x))^-v_{20}^*(x) - g_+(x,0)(-u_{20}(x))^-v_{20}^*(x)\Big)\ \mathrm{dx}\\ &= \frac{1}{\kappa}\int_\Omega\Big(g_-(x,0)\big([u_{20}(x)]^+\big)^2 + g_+(x,0)\big([u_{20}(x)]^-\big)^2\Big)\ \mathrm{dx}.\end{aligned}$$

From these relations 3. follows. To prove 4. we use the C^2-smoothness of N. Let us denote

$$\sigma = \int_\Omega (v_{10}^*(x), v_{20}^*(x))\partial_{\mathbf{U}}^2 N(0,0)\left[\begin{pmatrix}u_{10}(x)\\ u_{20}(x)\end{pmatrix},\begin{pmatrix}u_{10}(x)\\ u_{20}(x)\end{pmatrix}\right]\ \mathrm{dx},$$

with $\partial_{\mathbf{U}}^2 N(0,0)$ denoting the bilinear map corresponding to the second derivative. Using the relation between u_0 and v_0^* find that

$$\begin{aligned}\sigma &= \int_\Omega (\beta_1 u_{20}(x), \beta_2 u_{20}(x))\sum_{i,j}^{2}\left(\alpha_i\alpha_j\partial_{u_i}\partial_{u_j}\begin{pmatrix}n_1(0,0)\\ n_2(0,0)\end{pmatrix}\right)u_{20}(x)^2\ \mathrm{dx}\\ &= \int_\Omega\sum_{i,j,k=1}^{2}\beta_k\alpha_i\alpha_j(\partial_{u_i}\partial_{u_j}n_k(0,0))u_{20}(x)^3\ \mathrm{dx}.\end{aligned}$$

Now we note that $\rho = -\sigma/(2\kappa)$ and $\kappa < 0$. ☐

Chapter **5**

Analysis of Schnackenberg system

The last step in the study of reaction-diffusion systems with unilateral terms is an analysis of one specific system with two types of boundary conditions - either Dirichlet or Neumann boundary conditions. The aim of this section is to show how to apply the theorems from Section 4 to a specific problem, namely to Schnackenberg system with homogeneous Dirichlet/Neumann b.c., and how to qualitatively study the patterns which result from the problem. The information of pattern shape and size is significant for deciding whether the patterns have a biological meaning. In particular, the knowledge of how the patterns are affected by the presence of the unilateral terms could be helpful for investigation which systems in nature could potentially contain feedback mechanisms that can be described by these unilateral terms. Since there are not rigorous assertions about evolution and shape of patterns, the only possibility to study them is to approximate the solutions of a specific problem by a suitable numerical method. This chapter is organized as follows. The first section concerns with the critical points for linearized/homogenized Schackenberg problem without/with unilateral terms and in the first subsection with homogeneous Dirichlet boundary conditions, in the second subsection with homogeneous Neumann boundary conditions. The conditions on Turing instability, plot of hyperbolas and explicit solutions for the problem without unilateral terms are presented. The next section contains numerical solutions of the nonlinear problem (5.1) with (d_1, d_2) close to C_E. Since the set C_E is depending on the choice of the boundary conditions, some of the critical points are chosen to capture this distinction. More precisely, the point $(d_1, d_2) \in C_E$ with given d_1 has a different value of d_2 regarding to whether we have chosen Dirichlet or Neumann b.c., see Tabs. 5.3–5.14. Also the influence of unilateral sources on the shape of patterns is studied. All results are discussed at the end of the chapter and conclusions are made.

5.1 Setting the problem

In this section a problem which to be studied in the forthcoming sections will be introduced. We have chosen the set $\Omega := (0, L_1) \times (0, L_2)$ with $L_1 = 15, L_2 = 10$ for our experiments. The set Ω is a bounded domain with the Lipschitz boundary. Schnackenberg model with unilateral sources is

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + a - u + u^2 v, \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + b - u^2 v + s_-(x)(v - \overline{v})^- - s_+(x)(v - \overline{v})^+, \end{aligned} \tag{5.1}$$

with a, b being positive parameters and

$$(\overline{u}, \overline{v}) = \left(a + b, \frac{b}{(a+b)^2} \right) \tag{5.2}$$

being the steady state of this system. The parameters have been set to $a = 0.1, b = 0.85$. For this particular choice the constant steady state (5.2) is approx. $(\overline{u}, \overline{v}) \approx (0.95, 0.94)$. The source functions $s_\pm$ are chosen as

$$\begin{aligned} s_-(x_1,x_2) &= \tau(\exp\left(-0.25[(x_1-12)^2+(x_2-7)^2]\right) - \exp(-0.25\times 2.25))\chi_{B_{1.5}(12,7)}(x_1,x_2) \\ s_+(x_1,x_2) &= \tau(\exp\left(-0.25[(x_1-3)^2+(x_2-3)^2]\right) - \exp(-0.25\times 2.25))\chi_{B_{1.5}(3,3)}(x_1,x_2) \end{aligned}, \tag{5.3}$$

where $\tau \in \mathbb{R}^+$ is a parameter. The symbol χ denotes the characteristic function of the set in the subscript and τ measures the strength of the source. The reason for such choice is practical - the supports of $s_\pm$ are not active the entire domain and the source and sink have a symmetry w.r.t. point reflections. Moreover, this function is continuous in Ω. The interpolated plot of the functions $s_\pm$ can be found in Fig. 5.1. The source functions are continuous in $\mathbb{R}^2$.

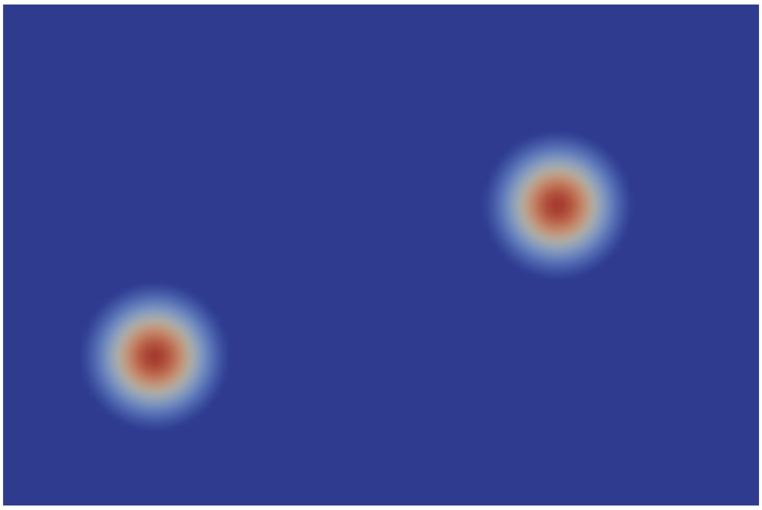


Figure 5.1: Plot of function $s_+ + s_-$ with $\tau = 1$.

We shift the constant steady state to zero. Then the stationary solutions of (5.1) have to satisfy

$$\begin{aligned} &d_1\Delta u + \left(-1+\frac{2b}{a+b}\right)u + (a+b)^2 v + 2(a+b)uv + \left(\frac{b}{(a+b)^2}+v\right)u^2 = 0, \\ &d_2\Delta v - \frac{2b}{a+b}u - (a+b)^2 v - 2(a+b)uv - \left(\frac{b}{(a+b)^2}+v\right)u^2 + s_-(x)v^- - s_+(x)v^+ = 0, \end{aligned} \tag{5.4}$$

supplemented with the zero Dirichlet/Neumann b.c. and random initial condition (u_0, v_0) satisfying $\|(u_0-\overline{u}, v_0-\overline{v})\|_{L^\infty(\Omega)\times L^\infty(\Omega)} \le 0.2$. The homogenization of (5.4) results in the problem

$$\begin{aligned} &d_1\Delta u + \left(-1+\frac{2b}{a+b}\right)u + (a+b)^2 v = 0, \\ &d_2\Delta v - \frac{2b}{a+b}u - (a+b)^2 v + s_-(x)(v)^- - s_+(x)(v)^+ = 0, \end{aligned} \tag{5.5}$$

again with the homogeneous Dirichlet/Neumann b.c. Everything is prepared to verify that the assumptions form Section 2.1 are fulfilled for our particular system.

Proposition 6. *Under the assumption $b-a < (b+a)^3$ the system (5.5) satisfies (2.1), (2.2).*

Proof. The conditions

$$b_{11} = \left(\frac{b-a}{a+b}\right) > 0, \quad b_{22} = -(a+b)^2 < 0, \quad b_{12}b_{21} = -2b(a+b) < 0,$$

$$\text{Tr } \mathbf{B} = b_{11} + b_{22} = \frac{b-a-(a+b)^3}{(a+b)} < 0, \quad \det\ \mathbf{B} = (a+b)^2 > 0,$$

are fulfilled if and only if $b - a < (b + a)^3$. □

For the particular choice $a = 0.1, b = 0.85$ the assumption $b - a < (b + a)^3$ means $0.75 < 0.857$, which is true. The nonlinear part of the kinetics is

$$\begin{aligned} n_1(u, v) &= 1.9uv + \left(\frac{0.85}{0.95^2} + v\right) u^2, \\ n_2(u, v) &= -1.9uv - \left(\frac{0.85}{0.95^2} + v\right) u^2. \end{aligned} \tag{5.6}$$

Proposition 7. *The functions n_1, n_2 defined by (5.6) satisfy the conditions (2.3) and (2.4).*

Proof. The condition (2.3) follows directly form the definition. The conditions (2.4) are satisfied by choosing $p = 4$. □

Proposition 8. *The functions $s_\pm$ from (5.3) satisfy assumptions (2.21) and (4.1).*

Proof. The first statement follows from the fact that supports of $s_\pm$ are open circles which do not intersect. The second statement follows from $\|s_\pm\| \in L^\infty(\Omega)$ and the positivity of the exponential function. □

5.2 Critical points and asymptotes of hyperbolas

5.2.1 Dirichlet boundary conditions

We are going to study a problem (5.4) with homogeneous Dirichlet b.c., altogether it is

$$\begin{aligned} &\left.\begin{aligned} &d_1\Delta u + \left(-1 + \tfrac{2b}{a+b}\right) u + (a+b)^2 v + 2(a+b)uv + \left(\tfrac{b}{(a+b)^2} + v\right) u^2 = 0, \\ &d_2\Delta v - \tfrac{2b}{a+b} u - (a+b)^2 v - 2(a+b)uv - \left(\tfrac{b}{(a+b)^2} + v\right) u^2 + s_-(x)v^- - s_+(x)v^+ = 0, \end{aligned}\right\} \text{ in } \Omega, \\ &\quad u = v = 0 \text{ on } \partial\Omega. \end{aligned} \tag{5.7}$$

For this problem the assumption (2.10) is fulfilled. The eigenvalues of the Laplacian in our set $\Omega = (0, 15) \times (0, 10)$ with homogeneous Dirichlet b.c. are

$$\kappa_{m,n} = \left(\frac{\pi m}{15}\right)^2 + \left(\frac{\pi n}{10}\right)^2, \quad m, n \in \mathbb{N},$$

and the corresponding eigenfunctions are

$$\phi_{m,n} = \sin\left(\frac{m\pi x}{15}\right) \sin\left(\frac{m\pi x}{10}\right), \quad m, n \in \mathbb{N}.$$

We rearrange the eigenvalues $\kappa_{m,n}$ of the Laplacian to a growing sequence with $\kappa_1 < \kappa_2 < \kappa_3 \cdots \to \infty$, see (7.6). The critical points of the system (5.5) with the homogeneous Dirichlet b.c. and with $\tau = 0$ have the form

$$d_{2,k}(d_1) = \lambda_k^S(d_1) = \frac{1}{\kappa_k}\left(\frac{-\,2b}{d_1\kappa_k(a+b) + a - b} - 1\right)(a+b)^2, \quad k = 1, 2, 3, \cdots$$

where k counts the eigenvalues from the smallest w.r.t. their size, see (2.39) and Remark 8 and $d_1 \neq b_{11}/\kappa_k$ for all $k \in \mathbb{N}$. Let us remind that $a = 0.1, b = 0.85$. The function $d_{2,k}^0$ is for any $k \in \mathbb{N}$ a hyperbola with the domain of definition $(-\infty, y_k) \cup (y_k, +\infty)$.

The asymptotes of the hyperbolas are

$$y_k = \frac{1}{\kappa_k}\left(\frac{b-a}{a+b}\right), \tag{5.8}$$

indexing κ_i	indexing $\kappa_{m,n}$	value	y_i	multiplicity
κ_1	$\kappa_{1,1}$	0.1426	5.62	1
κ_2	$\kappa_{1,2}$	0.2742	2.92	1
κ_3	$\kappa_{2,1}$	0.4386	1.82	1
κ_4	$\kappa_{1,3}$	0.4935	1.62	1
κ_5	$\kappa_{2,2}$	0.5702	1.40	1
κ_6	$\kappa_{3,1}$	0.9321	0.86	1

Table 5.1: First six eigenvalues of the Laplacian with zero Dirichlet b.c. and first six asymptotes y_i

see (2.82).

If $d_1 \neq y_k$ for all $k \in \mathbb{N}$, all assumptions of Theorem 11 are fulfilled. First six eigenvalues κ_i, the corresponding counterparts $\kappa_{m,n}$ and the corresponding asymptotes are in Tab. 5.1.

The following proposition has been proved by numerical computation, therefore we are not giving the proof here.

Proposition 9. *For any $d_1 \in \{0.02, 0.12, 0.5, 1.1, 1.6, 4.0\}$ there exists unique $d_2 > 0$ for which $(d_1, d_2) \in C_E$. Moreover, the point d_2 of (5.5) with homogeneous Dirichlet b.c. and fixed d_1 is simple.*

According to Theorem 11 the largest critical point of the system (5.5) with fixed d_1, homogeneous Dirichlet b.c. and with the sources (5.3) can be estimated by

$$\max_{k\in\mathbb{N}} \frac{1}{\kappa_k}\left(\frac{-1.8}{d_1\kappa_k - 0.8} - 1\right) \geq d_2^m \geq \max_{k\in\mathbb{N}} \frac{1}{\kappa_k}\left(\frac{-1.8}{d_1\kappa_k - 0.8} - (1+2\tau)\right). \tag{5.9}$$

The first six hyperbola segments in $\mathbb{R}^2_+$ are plotted in Fig. 5.2. Full lines represent upper bounds from (5.9), i.e. the couples (d_1, d_2) for which our system with $s_\pm \equiv 0$ has nontrivial solution, dotted lines are lower bounds from (5.9) with $\tau = 1$. It can be computed that for $d_1 \in (y_2, y_1)$ and $\tau < 0.625$ the lower bound in (4.3) is positive and as a consequence d_2^m is positive, and $(d_1, d_2^m) \in D_S$.

Since each κ_k is associated with a given $\kappa_{m,n}$, we can also introduce a notation $C_{m,n}$ for the hyperbola C_k. We will use this notation later in Tabs. 5.3–5.8, because it is not so simple to find for a couple (m, n) the respective k.

5.2.2 Neumann boundary conditions

Now let the system (5.4) in $\Omega := (0, 15) \times (0, 10)$ be supplemented with (homogeneous) Neumann b.c. The eigenvalues of the Laplacian on Ω with homogeneous Neumann b.c. are

$$\kappa_{m,n} = \left(\frac{\pi m}{15}\right)^2 + \left(\frac{\pi n}{10}\right)^2, \quad m, n \in \mathbb{N}_0,$$

and the eigenfunctions are

$$\phi_{m,n} = \cos\left(\frac{m\pi x}{15}\right)\cos\left(\frac{m\pi x}{10}\right), \quad m, n \in \mathbb{N}_0.$$

Let us emphasize that the indexes m, n attain a value zero, which is a difference against Dirichlet problem. Again, the eigenvalues will be rearranged to a growing sequence with $0 = \kappa_0 < \kappa_1 < \kappa_2 < \kappa_3 \cdots \to \infty$. The eigenvalues $\kappa_{i,j}$ for $i, j \in \mathbb{N}$ are the same as for the Dirichlet case. However, there are additional eigenvalues - $\kappa_0, \kappa_{0,j}, \kappa_{j,0},\ j \in \mathbb{N}$. The critical points of the system (5.5) with $s_\pm \equiv 0$ and fixed d_1 have the form

$$d_{2,k}(d_1) = \frac{1}{\kappa_k}\left(\frac{-\,2b}{d_1\kappa_k(a+b) + a - b} - 1\right)(a+b)^2, \quad k = 1, 2, 3, \cdots.$$

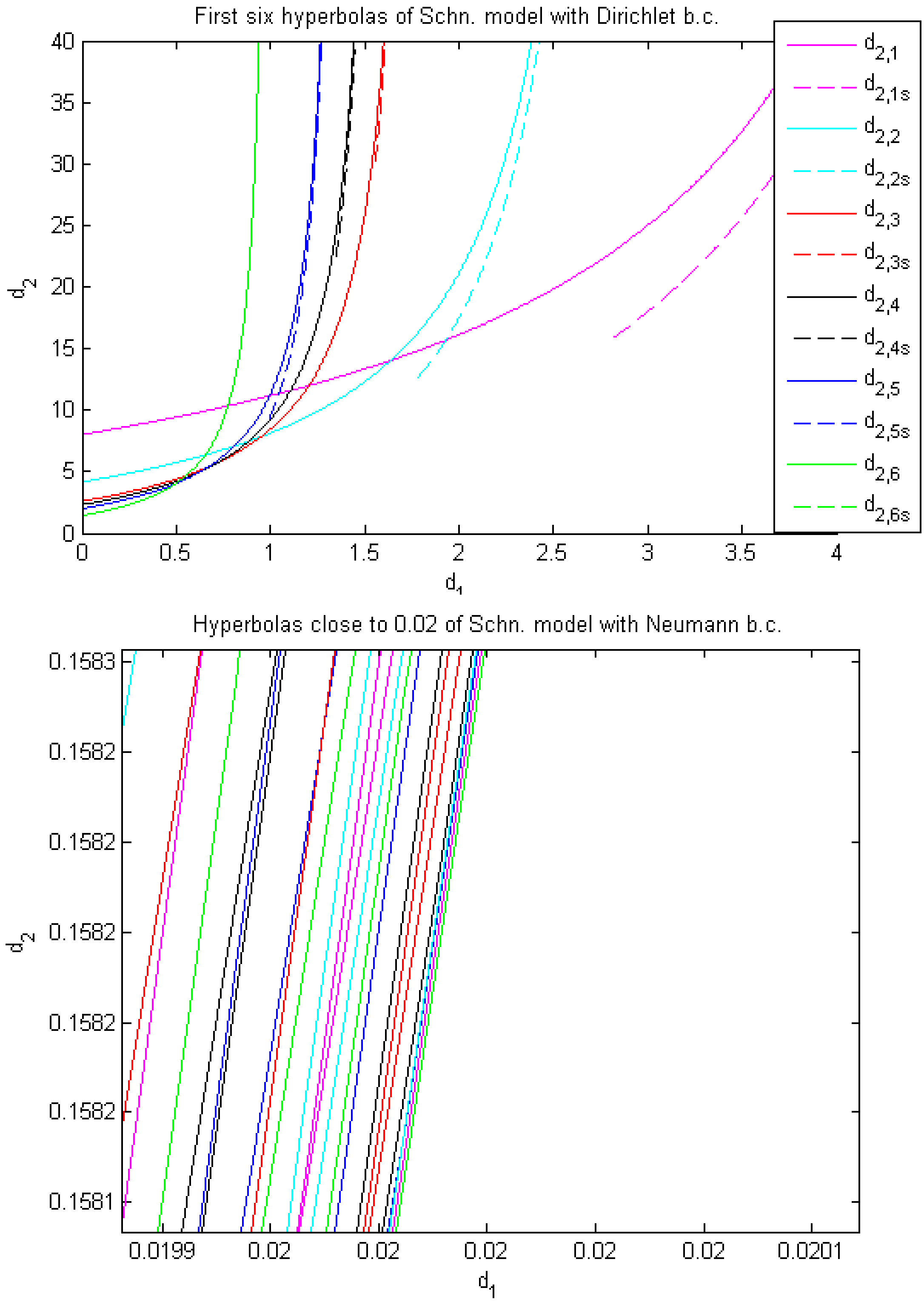


Figure 5.2: The upper figure contains plot of first six hyperbola segments (full lines) and lower (dashed lines) bounds from (4.3), with $\tau := 1$ for Schnackenberg system with homogeneous Dirichlet b.c. The lower figure contains plots of hyperbolas segments around 0.02. The hyperbola segments are very dense in the selection, which explains why we were not successful in finding solutions bifurcating from $(d_1, d_2) \in C_E$ of system (5.4) with fixed $d_1 = 0.02$, see Fig. 5.8.

Let us emphasize that $d_{2,0}$ is not defined. The function $d_{2,k}$ is a hyperbola with the asymptote (5.8). The following assertion is analogous to problem with zero Dirichlet b.c. and has been again proved by using numerical computation.

Proposition 10. *For any $d_1 \in \{0.02, 0.12, 0.5, 1.1, 1.6, 4.0\}$ there exists unique $d_2 > 0$ for which $(d_1, d_2) \in C_E$. Moreover, the point d_2 of (5.5) with homogeneous Neumann b.c. and fixed d_1 is simple.*

The hyperbolas C_1, C_2, C_4, C_6 are specific for Neumann problem, the hyperbolas C_3, C_5, C_7 coincide with hyperbolas C_1, C_2, C_3 from the Dirichlet case, respectively. The plot of hyperbolas can be found in Fig. 5.3. A hyperbola C_0 is not defined. The first seven eigenvalues of the Laplacian and first six asymptotes of C_k are summarized in Tab. 5.2. Similarly to Dirichlet case, we will introduce a notation $C_{m,n}$ for the hyperbola C_k. We will use this notation later in Tabs. 5.9–5.14. In systems with $\tau = 0$ there is no positive critical point d_2 of (5.5) with fixed $d_1 > y_1$. On the other hand, in a system with unilateral terms with sufficiently large norm it is possible to apply Theorem 15 to get the positive critical points d_2 even for systems with $d_1 > y_1$.

Proposition 11. *If $d_1 \in (0, y_1) \setminus \{y_j \big| \ j = 2, 3, \cdots\}$ and*

$$1 > \max\left\{\frac{300}{0.8\ \|s_-\|_{L^1}}, \frac{300}{0.8\ \|s_+\|_{L^1}}\right\} = \left\{\frac{306.8}{\tau}, \frac{306.8}{\tau}\right\}, \tag{5.10}$$

then the largest critical point of (5.5) with homogeneous Neumann b.c. is finite. Let (5.10) be fulfilled and $d_1 > y_1$ be fixed. For the system (5.5) with $a = 0.1, b = 0.85$ and Neumann b.c. the assumption (4.18) is fulfilled as well and the point d_2^m from (4.17) exists and it is the largest critical point of (5.5) with Neumann b.c. and fixed d_1.

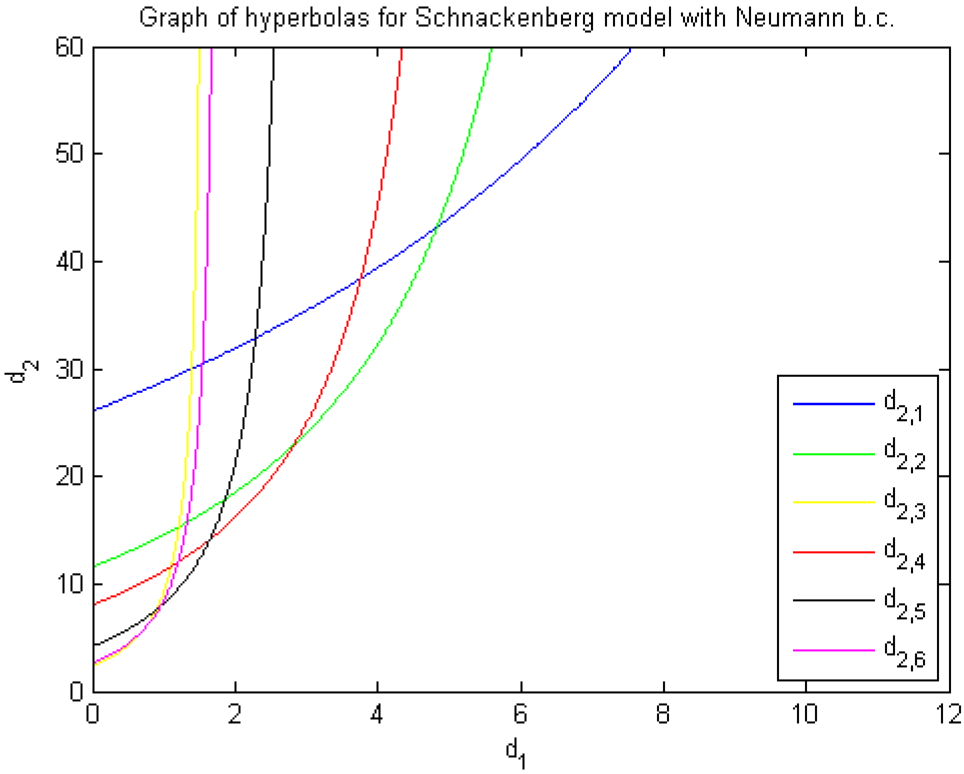
Proof. The condition (5.10) is clearly equivalent to (4.16). It can be computed that this condition is true for $\tau > 306.9$. The l.h.s. of (4.18) is equal to 1.256, the r.h.s is equal to 1.8 and the inequality is true. The assumptions of Theorem 15 are fulfilled, and from this Theorem directly follows the assertions of this Proposition. □

The bound C_M can be found for $\tau = 1000$ explicitly as 2970.5. Since the area of the sources is relatively small, they have to be very strong in order to fulfill (4.18). It appears that although the inequality (3.28) had a crucial role in the proof of Theorem 16 on pg. 72, the practical computations shows that the inequality is not optimal.

The hyperbolas are plot in Fig. 5.3.

index κ_i	index $\kappa_{m,n}$	num. value	y_i	multiplicity
κ_0	$\kappa_{0,0}$	0	—	1
κ_1	$\kappa_{0,1}$	0.04386	18.24	1
κ_2	$\kappa_{1,0}$	0.09870	8.11	1
κ_3	$\kappa_{1,1}$	0.1426	1.62	1
κ_4	$\kappa_{1,3}$	0.2742	5.61	1
κ_5	$\kappa_{2,2}$	0.4386	2.81	1
κ_6	$\kappa_{3,1}$	0.4935	1.82	1

Table 5.2: First seven eigenvalues of the Laplacian with Neumann b.c. and first six asymptotes $y_i = b_{11}/\kappa_i$



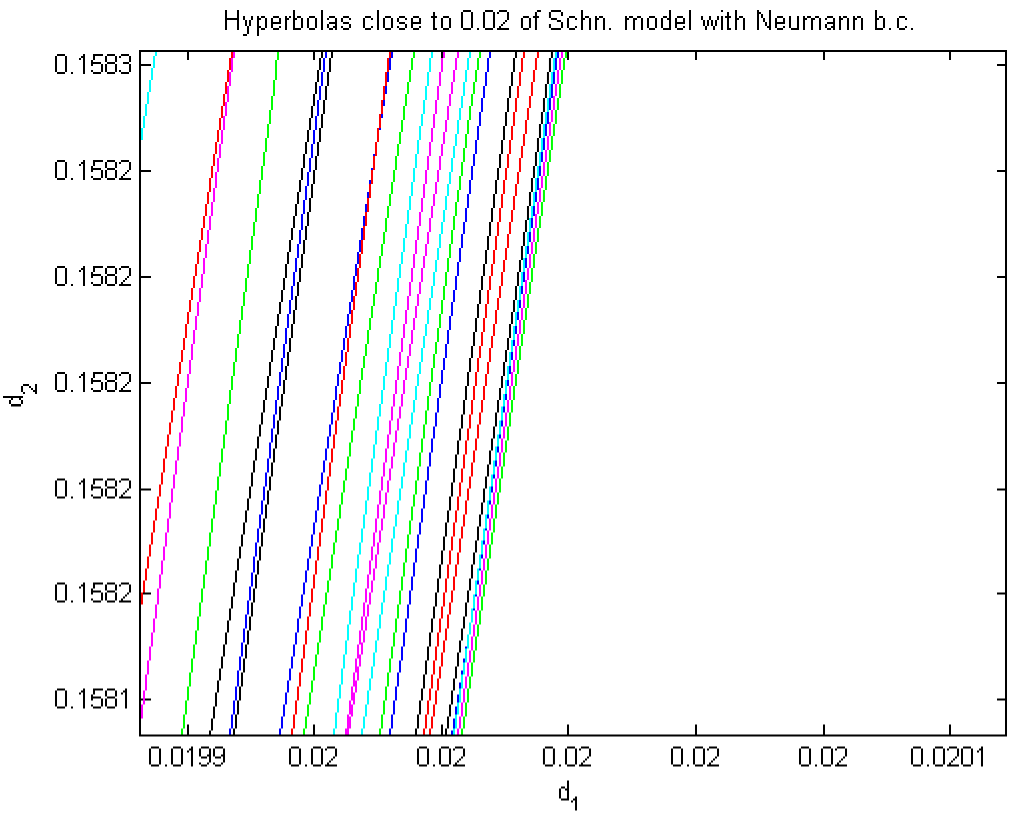


Figure 5.3: The upper figure contains plot of first six hyperbola segments $d_{2,k}$ (full lines) and lower (dashed lines) bounds from (4.3), with $\tau := 1$ for Schnackenberg system with homogeneous Neumann b.c. The lower figure contains plots of hyperbolas segments around 0.02. The hyperbola segments are very dense in the selection, which explains why we were not successful in finding solutions bifurcating from $(d_1, d_2) \in C_E$ of system (5.4) with fixed $d_1 = 0.02$, see Fig. 5.14.

5.3 Numerical solutions of Schnackenberg system with Dirichlet and Neumann b.c.

The nonlinear problem (5.4) is

$$\begin{aligned} &d_1\Delta u + 0.8u + v + 1.9uv + \left(\frac{0.85}{0.95^2} + v\right)u^2 = 0,\\ &d_2\Delta v - 1.8u - v - 1.9uv - \left(\frac{0.85}{0.95^2} + v\right)u^2 + s_-(x)v^- - s_+(x)v^+ = 0, \end{aligned}$$

and we supplement it with the homogeneous Dirichlet b.c. and homogeneous Neumann b.c.

Proposition 12. *Let τ be sufficiently small. For any $d_1 \in \{0.02, 0.12, 0.5, 1.1, 1.6, 4.0\}$ there exists a bifurcation point d_2^b of (5.4) with Dirichlet b.c. and fixed d_1 such that $(d_1, d_2^b) \in D_S$.*

Proof. Since $s_\pm$ from (5.3) have $\|s_\pm\|_{L^\infty} = \tau$, the assertion follows for sufficiently small τ from Theorem 14 and Proposition 9. □

Analogous statement can be proved also for Neumann problem.

Proposition 13. *Let τ be sufficiently small. For any $d_1 \in \{0.02, 0.12, 0.5, 1.1, 1.6, 4.0\}$ there exists a bifurcation point d_2^b of (5.4) with Neumann b.c. and fixed d_1 such that $(d_1, d_2^b) \in D_S$.*

Previous propositions prove the existence of stationary solutions of (5.1) with either Dirichlet or Neumann boundary conditions in the domain of stability.

The aim of the numerical experiments was to answer the three questions:

- Are the solutions of the system (5.4) with Dirichlet/Neumann b.c. and with (d_1, d_2) close to some bifurcation points numerically stable?
- How is the shape of the solutions influenced by the presence of the unilateral terms?
- How the solution of the homogenized system approximately look like?

To find numerical steady states (patterns) of the chosen problem, the problem has been discretized in time by using the Crank-Nicolson scheme with the time-step adaptation and in space using the first-order Lagrange elements. The mesh adaptation has been not used in order to make the computation faster.

The starting point is a weak formulation

$$\begin{aligned} &\int_\Omega \frac{\partial u}{\partial t}\varphi \,\mathrm{dx} + \int_\Omega \frac{\partial v}{\partial t}\psi \,\mathrm{dx} = \int_\Omega -d_1\nabla u\cdot\nabla\varphi + b_{11}u\varphi + b_{12}v\varphi + n_1(u,v)\varphi \,\mathrm{dx} +\\ &\int_\Omega -d_2\nabla u\cdot\nabla\psi + b_{21}u\psi + b_{22}v\psi + n_2(u,v)\psi + s_-(x,v^-)\psi - s_+(x,v^+)\psi \,\mathrm{dx}, \qquad (5.11)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \varphi,\psi \in W_D^{1,2}(\Omega), \end{aligned}$$

of the system (1.14), (1.10). This can be written as

$$\left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle_{W_D^{1,2}(\Omega)} + \left\langle \frac{\partial v}{\partial t}, \psi \right\rangle_{W_D^{1,2}(\Omega)} = F_1(d_1, x, u, v, \varphi) + F_2(d_2, x, u, v, \psi), \quad \text{for all } \varphi \in W^{1,2}(\Omega)$$

with the functionals F_1, F_2 derived from (5.11) and $\langle \cdot,\cdot \rangle_{W_D^{1,2}(\Omega)}$ being the duality pairing between $W_D^{1,2}(\Omega)$ and $(W_D^{1,2}(\Omega))^\star$. Denote $u_n(\cdot,t) := u(\cdot, n\Delta t)$, $\Delta t \in \mathbb{R}^+ n \in \mathbb{N}$. Now the Crank-Nicolson scheme [52] for a semi-discretized problem is

$$\begin{aligned} &\int_\Omega \frac{u_{n+1}-u_n}{\Delta t}\varphi + \int_\Omega \frac{v_{n+1}-v_n}{\Delta t}\psi =\\ &\quad = \frac{1}{2}\left[F_1^{n+1}(d_1,x,u,v,\varphi) + F_2^{n+1}(d_2,x,u,v,\psi) + F_1^n(d_1,x,u,v,\varphi) + F_2^{n+1}(d_1,x,u,v,\psi)\right]. \end{aligned}$$

The essential part is a suitable choice of Δt. This has been done by the following algorithm [52]:

1. in the first step set integer m and floats TOL, $\text{TOL}_\star$, TOL_+,
2. having approximation $u_n(\cdot, t_n)$ and time step Δt_n and step m find $u_{n+1}(\cdot, t_n + m\Delta t_n)$,
3. do the m small time steps and find $u_{m+n}(\cdot, t_n + m\Delta t_n)$,
4. compute the error $e = \|u_{m+n}(\cdot, t_n + m\Delta t_n) - u_{n+1}(\cdot, t_n + m\Delta t_n)\|$,
5. compute the value

$$\Delta t_\star = \Delta t_n \sqrt{\frac{\text{TOL}(m^2 - 1)}{\|u_{m+n}(\cdot, t + m\Delta t_n) - u_{n+1}(\cdot, t + m\Delta t_n)\|}},$$

6. if $\Delta t_\star < 10^{-2}\Delta t_n$, take $m - 1$ and repeat the previous step,
7. if $n > 21$ and $\|u_{n-20}(\cdot, t_{n-20}) - u_n(\cdot, t_n)\| < \text{TOL}_\star$ and $\Delta t_n > \text{TOL}_+$ break and save results.

This scheme was implemented in FENICS using the Python API[1]. The solver had started up with a random initial condition and with prescribed values m, TOL, $\text{TOL}_\star$, TOL_+. For some diffusion coefficients, the computation has been run again on finer mesh, in order to realize whether the result is not significantly changed by to choice of the mesh.

It has happened that the system with given diffusion parameters has produced various patterns for different initial conditions, see [58]. Although we have observed this phenomenon as well, we have been not concerned with it.

We have run several test with several parameters. The first set of tests was aiming on verify numerically the existence of solutions of (5.4) with parameter $(d_1, d_2) \in D_S$, which is predicted by Theorems 11, 14, 18, and discuss how the presence of unilateral terms influences a shape of the solutions.

The results are summarized in Tabs. 5.3–5.15. The values of d_1 have been at first chosen in the intervals, where the envelope C_E is different for problems with Neumann and Dirichlet b.c. More precisely, it has been the values $d_1 = 4.0, 1.6, 0.5, 0.12$. These values of d_1 were supplemented with the values $d_1 = 1.1, 0.5, 0.02$. For these values of d_1, the critical points of (1.12), (1.10) with fixed d_1, for which $(d_1, d_2) \in C_E$, are simple.

For the problem without unilateral terms, i.e. with $s_\pm \equiv 0$, and with given fixed d_1 the values of d_2 has been chosen in a following way. The couple (d_1, d_2) is in D_U, the second one is very close to C_E. For the problem with sources with $\tau = 1.0$ the values of d_2 has been chosen in a slightly different way. The largest one is the same as for the system without unilateral terms, then the second one is in the bifurcation point of the problem without sources, and the last one is the lowest one for which we have been able to find nontrivial solutions of our problem. All of this together with $L^\infty(\Omega)$ and $W^{1,2}_D(\Omega)$ norm of the v component of the solution is summarized in Tabs. 5.3–5.14.

The numerical computations are in accordance with the conclusions of Theorem 14 for the Dirichlet problem and similar conclusions for Neumann problem that there are critical and bifurcation points of (1.12), (1.10) and (1.16), (1.10) respectively, in D_S. The shape of patterns is influenced by unilateral sources and the influence grows with d_1 being smaller. The $L^\infty(\Omega)$ and $W^{1,2}_D(\Omega)$ norms of the solutions are usually getting smaller as the diffusion coefficients approach C_E, which is again in accordance with theoretical predictions. And finally, it is possible to see that the shapes of solutions are quite similar to solutions of systems with (d_1, d_2) very close to C_E. The only exception is the value $d_1 = 0.02$. It is probably caused by presence of many hyperbolas around $d_1 = 0.02$, see Fig. 5.2, 5.3, and therefore the solution is probably attracted to critical point which is not on C_E.

There is one interesting point - for the problem with Dirichlet boundary conditions and $d_1 = 4.0$ there are patterns in the system without sources even below C_E. According to Crandall-Rabinowitz Theorem there are two bifurcation branches originating from one bifurcation point.

[1]The source code of the implementation for the particular system used here is freely available on page http://github.com/Josef-Navratil/RD-system

However, since the v-component of solution has constant negative sign for values $(4.0, 48)$ which are above C_E and values $(4.0, 38)$ which are below C_E, it (probably) cannot be solutions on different bifurcation branches. The reason for this behavior is therefore not known but since the value of bifurcation point is changing with mesh refinement, there is probably some issue connected with the discretization.

The influence of unilateral terms on the shape of patterns was tested on one particular choice of diffusion coefficients $d_1 = 0.02,\ d_2 = 0.2$. The value of τ has been increased from 0.0 up to 4.0. The computed solutions are found in Tab. 5.15.

5.3.1 Results of numerical experiments with system having Dirichlet and Neumann boundary conditions

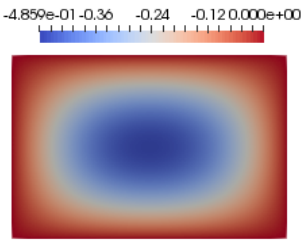
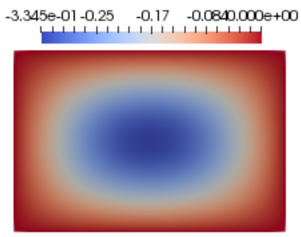
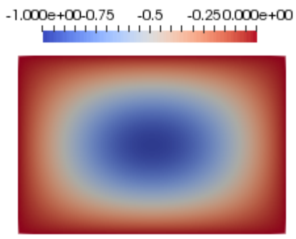
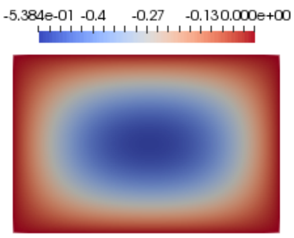
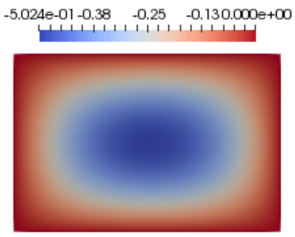
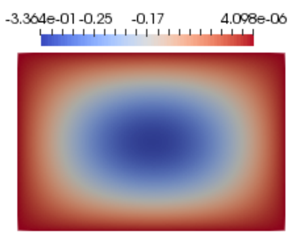
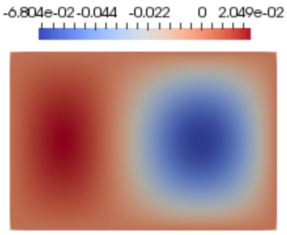
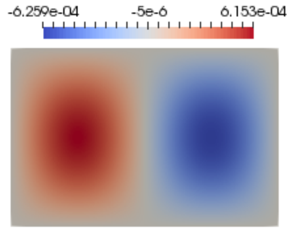
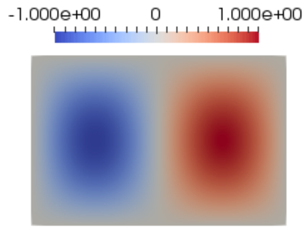
Experiment Nr. 1	Experiment Nr. 2	Experiment Nr. 3*
$d_1 = 4.0$	$d_1 = 4.0$	$d_1 = 4.0$
$d_2 = 48.0$	$d_2 = 38.61$	$d_2 = 45.34$
$\|v\|_{L^\infty(\Omega)} = 0.525$	$\|v\|_{L^\infty(\Omega)} = 0.335$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 16.619$	$\|v\| = 8.104$	$\|v\| = 2.312$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$
		
Experiment Nr. 4	Experiment Nr. 5	Experiment Nr. 6
$d_1 = 4.0$	$d_1 = 4.0$	$d_1 = 4.0$
$d_2 = 48.0$	$d_2 = 45.34$	$d_2 = 37.69$
$\|v\|_{L^\infty(\Omega)} = 0.538$	$\|v\|_{L^\infty(\Omega)} = 0.502$	$\|v\|_{L^\infty(\Omega)} = 0.336$
$\|v\| = 17.487$	$\|v\| = 15.198$	$\|v\| = 8.144$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$
		

Table 5.3: Solution of the Schnackenberg system with homogeneous Dirichlet b.c., having fixed $d_1 = 4.0$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 3$*$ shows a plot of the explicit solution $(d_1, d_2) = (4.0, 45.34) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{1,1}$. System had zero Dirichlet b.c.

Experiment Nr. 7	Experiment Nr. 8	Experiment Nr. 9*
$d_1 = 1.6$	$d_1 = 1.6$	$d_1 = 1.6$
$d_2 = 13.90$	$d_2 = 13.50$	$d_2 = 13.499$
$\|v\|_{L^\infty(\Omega)} = 0.0680$	$\|v\|_{L^\infty(\Omega)} = 6.26\text{e} - 4$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 0.936$	$\|v\| = 0.0118$	$\|v\| = 3.206$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$
		

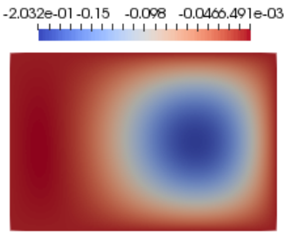
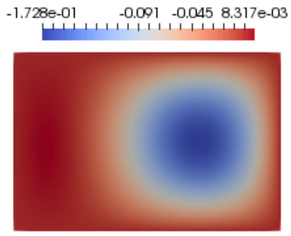
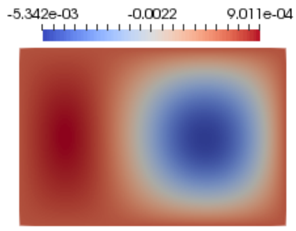
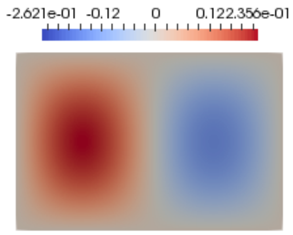
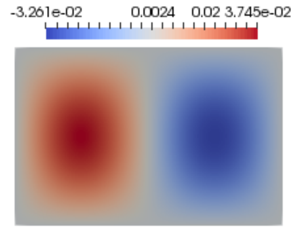
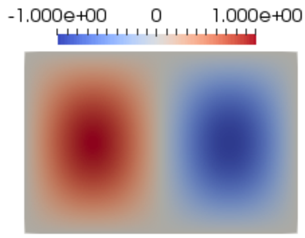
Experiment Nr. 10	Experiment Nr. 11	Experiment Nr. 12
$d_1 = 1.6$	$d_1 = 1.6$	$d_1 = 1.6$
$d_2 = 13.90$	$d_2 = 13.499$	$d_2 = 12.13$
$\|v\|_{L^\infty(\Omega)} = 0.203$	$\|v\|_{L^\infty(\Omega)} = 0.173$	$\|v\|_{L^\infty(\Omega)} = 0.00534$
$\|v\| = 2.843$	$\|v\| = 2.367$	$\|v\| = 0.069$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$
		

Table 5.4: Solution of the Schnackenberg system with homogeneous Dirichlet b.c., having fixed $d_1 = 1.6$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure $9*$ shows a plot of the explicit solution $(d_1, d_2) = (1.6, 13.499) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{2,1}$. System had zero Dirichlet b.c.

Experiment Nr. 13	Experiment Nr. 14	Experiment Nr. 15*
$d_1 = 1.1$	$d_1 = 1.1$	$d_1 = 1.1$
$d_2 = 9.50$	$d_2 = 8.78$	$d_2 = 8.78$
$\|v\|_{L^\infty(\Omega)} = 0.262$	$\|v\|_{L^\infty(\Omega)} = 0.375$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 3.260$	$\|v\| = 0.511$	$\|v\| = 3.206$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$
		

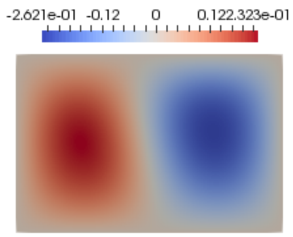
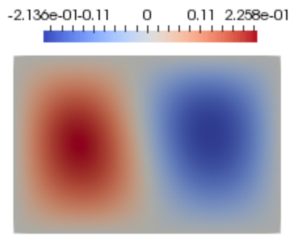
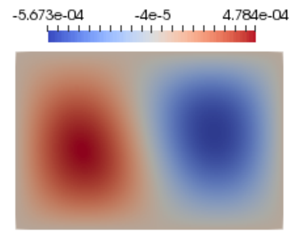
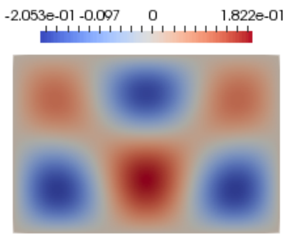
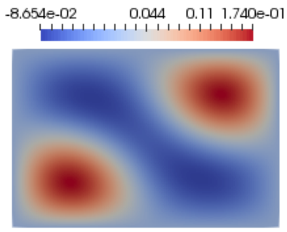
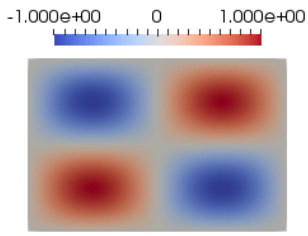
Experiment Nr. 16	Experiment Nr. 17	Experiment Nr. 18
$d_1 = 1.1$	$d_1 = 1.1$	$d_1 = 1.1$
$d_2 = 9.50$	$d_2 = 8.78$	$d_2 = 7.52$
$\|v\|_{L^\infty(\Omega)} 0.262$	$\|v\|_{L^\infty(\Omega)} = 0.226$	$\|v\|_{L^\infty(\Omega)} = 5.67e-4$
$\|v\| = 4.439$	$\|v\| = 3.685$	$\|v\| = 0.00785$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$
		

Table 5.5: Solution of the Schnackenberg system with homogeneous Dirichlet b.c., having fixed $d_1 = 1.1$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 15$*$ shows a plot of the explicit solution $(d_1, d_2) = (1.1, 8.782) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{2,1}$. System had zero Dirichlet b.c.

Experiment Nr. 19	Experiment Nr. 20	Experiment Nr. 21*
$d_1 = 0.5$	$d_1 = 0.5$	$d_1 = 0.5$
$d_2 = 4.50$	$d_2 = 4.03$	$d_2 = 4.03$
$\|v\|_{L^\infty(\Omega)} = 0.253$	$\|v\|_{L^\infty(\Omega)} = 0.174$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 3.916$	$\|v\| = 2.195$	$\|v\| = 4.624$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$
		

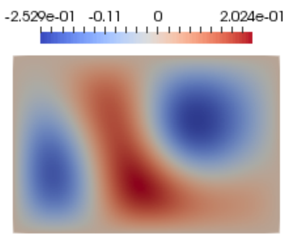
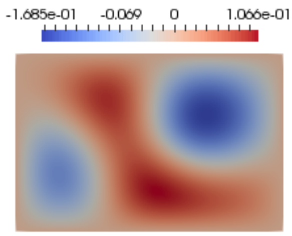
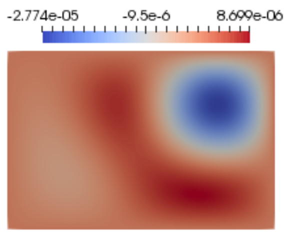
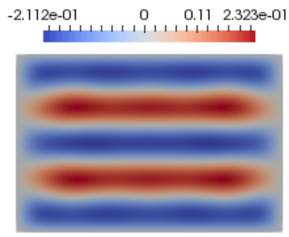
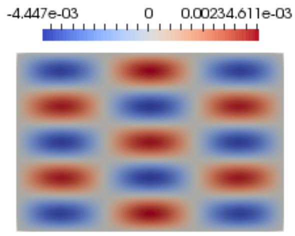
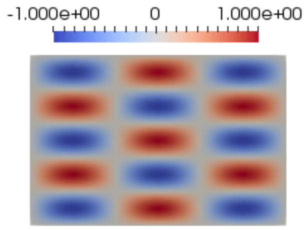
Experiment Nr. 22	Experiment Nr. 23	Experiment Nr. 24
$d_1 = 0.5$	$d_1 = 0.5$	$d_1 = 0.5$
$d_2 = 4.50$	$d_2 = 4.03$	$d_2 = 3.53$
$\|v\|_{L^\infty(\Omega)} = 0.238$	$\|v\|_{L^\infty(\Omega)} = 0.169$	$\|v\|_{L^\infty(\Omega)} = 2.77e-5$
$\|v\| = 3.991$	$\|v\| = 1.168$	$\|v\| = 0.000239$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$
		

Table 5.6: Solution of the Schnackenberg system with homogeneous Dirichlet b.c., having fixed $d_1 = 0.5$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 21$*$ shows a plot of the explicit solution $(d_1, d_2) = (0.5, 4.03) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{2,2}$. System had zero Dirichlet b.c.

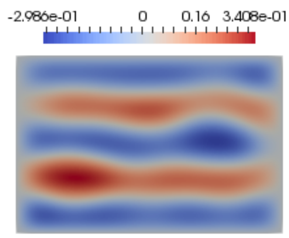
Experiment Nr. 25	Experiment Nr. 26	Experiment Nr. 27*
$d_1 = 0.12$	$d_1 = 0.12$	$d_1 = 0.12$
$d_2 = 1.1$	$d_2 = 0.95$	$d_2 = 0.95$
$\|v\|_{L^\infty(\Omega)} = 0.232$	$\|v\|_{L^\infty(\Omega)} = 0.164$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 7.302$	$\|v\| = 0.00445$	$\|v\| = 10.360$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$

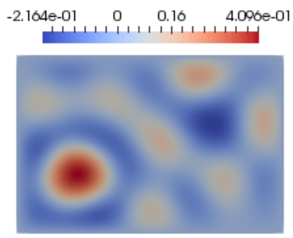






Experiment Nr. 28	Experiment Nr. 29	Experiment Nr. 30
$d_1 = 0.12$	$d_1 = 0.12$	$d_1 = 0.12$
$d_2 = 1.1$	$d_2 = 0.95$	$d_2 = 0.67$
$\|v\|_{L^\infty(\Omega)} = 0.340$	$\|v\|_{L^\infty(\Omega)} = 0.215$	$\|v\|_{L^\infty(\Omega)} = 0.374$
$\|v\| = 8.144$	$\|v\| = 4.614$	$\|v\| = 2.178$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$





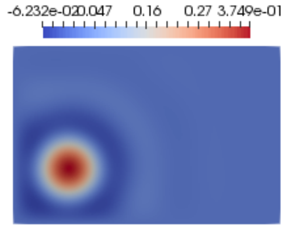
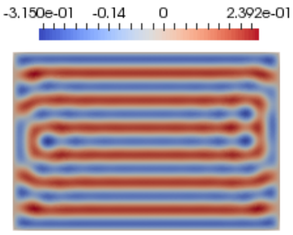
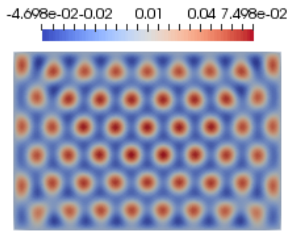
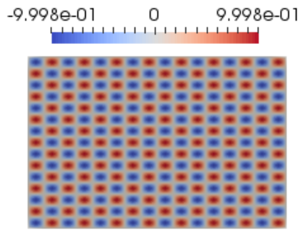


Table 5.7: Solution of the Schnackenberg system with homogeneous Dirichlet b.c., having fixed $d_1 = 0.12$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 30$*$ shows a plot of the explicit solution $(d_1, d_2) = (0.12, 0.95) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{3,5}$. System had zero Dirichlet b.c.

Experiment Nr. 31	Experiment Nr. 32	Experiment Nr. 33*
$d_1 = 0.02$	$d_1 = 0.02$	$d_1 = 0.02$
$d_2 = 0.20$	$d_2 = 0.16$	$d_2 = 0.16$
$\|v\|_{L^\infty(\Omega)} = 0.315$	$\|v\|_{L^\infty(\Omega)} = 0.0470$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 20.019$	$\|v\| = 2.949$	$\|v\| = 35.410$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$
		

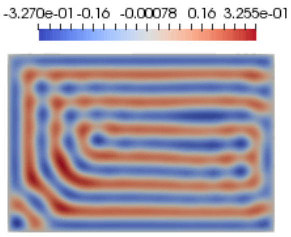
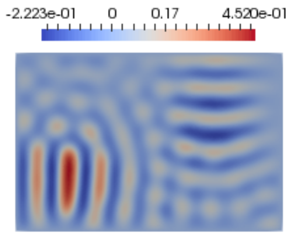
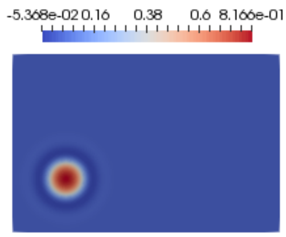
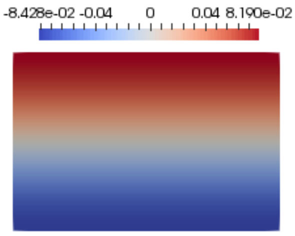
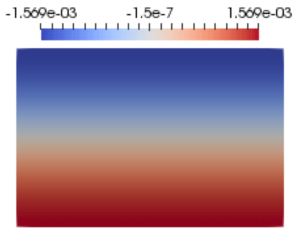
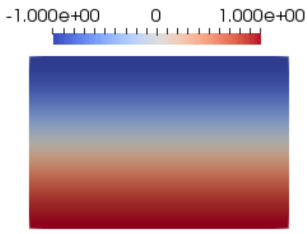
Experiment Nr. 34	Experiment Nr. 35	Experiment Nr. 36
$d_1 = 0.02$	$d_1 = 0.02$	$d_1 = 0.02$
$d_2 = 0.20$	$d_2 = 0.16$	$d_2 = 0.08$
$\|v\|_{L^\infty(\Omega)} = 0.315$	$\|v\|_{L^\infty(\Omega)} = 0.0470$	$\|v\|_{L^\infty(\Omega)} = 0.374$
$\|v\| = 0.327$	$\|v\| = 0.222$	$\|v\| = 0.812$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$
		

Table 5.8: Solution of the Schnackenberg system with homogeneous Dirichlet b.c., having fixed $d_1 = 0.02$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure $33*$ shows a plot of the explicit solution $(d_1, d_2) = (0.02, 0.16) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{16,15}$. System had zero Dirichlet b.c.

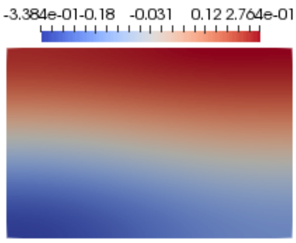
Experiment Nr. 1	Experiment Nr. 2	Experiment Nr. 3*
$d_1 = 4.0$	$d_1 = 4.0$	$d_1 = 4.0$
$d_2 = 33.5$	$d_2 = 32.32$	$d_2 = 32.31$
$\|v\|_{L^\infty(\Omega)} = 0.0843$	$\|v\|_{L^\infty(\Omega)} = 0.00157$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 1.941$	$\|v\| = 0.0355$	$\|v\| = 6.387$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$

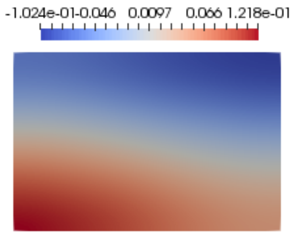






Experiment Nr. 4	Experiment Nr. 5	Experiment Nr. 6
$d_1 = 4.0$	$d_1 = 4.0$	$d_1 = 4.0$
$d_2 = 33.50$	$d_2 = 32.31$	$d_2 = 30.93$
$\|v\|_{L^\infty(\Omega)} = 0.338$	$\|v\|_{L^\infty(\Omega)} = 0.102$	$\|v\|_{L^\infty(\Omega)} = 0.0118$
$\|v\| = 2.652$	$\|v\| = --$	$\|v\| = 0.204$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$





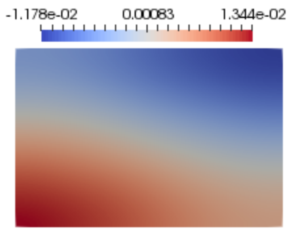
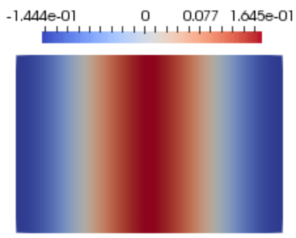
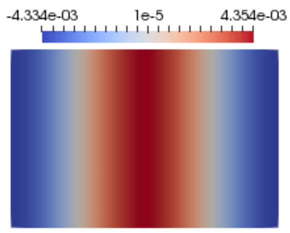
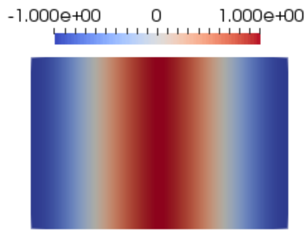


Table 5.9: Solution of the Schnackenberg system with homogeneous Neumann b.c., having fixed $d_1 = 4.0$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 3$*$ shows a plot of the explicit solution $(d_1, d_2) = (4.0, 32.31) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{0,1}$. System had zero Neumann b.c. Note that the bifurcation point has a different value compared to the Dirichlet case

Experiment Nr. 7	Experiment Nr. 8	Experiment Nr. 9*
$d_1 = 1.6$	$d_1 = 1.6$	$d_1 = 1.6$
$d_2 = 13.50$	$d_2 = 12.95$	$d_2 = 12.95$
$\|v\|_{L^\infty(\Omega)} = 0.144$	$\|v\|_{L^\infty(\Omega)} = 0.00433$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 3.059$	$\|v\| = 0.0830$	$\|v\| = 7.118$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$
		

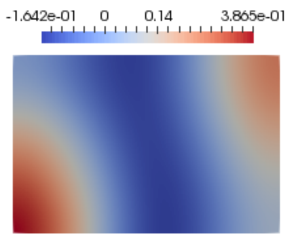
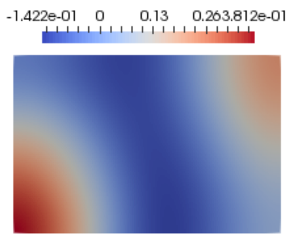
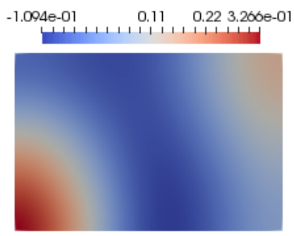
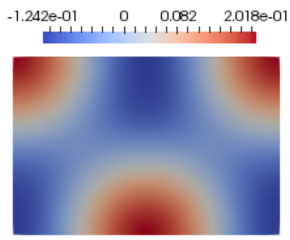
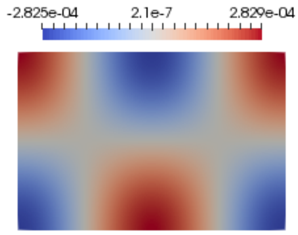
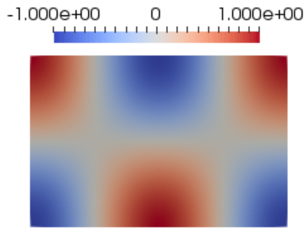
Experiment Nr. 10	Experiment Nr. 11	Experiment Nr. 12
$d_1 = 1.6$	$d_1 = 1.6$	$d_1 = 1.6$
$d_2 = 13.50$	$d_2 = 12.95$	$d_2 = 12.32$
$\|v\|_{L^\infty(\Omega)} = 0.203$	$\|v\|_{L^\infty(\Omega)} = 0.173$	$\|v\|_{L^\infty(\Omega)} = 0.00534$
$\|v\| = 3.923$	$\|v\| = 3.434$	$\|v\| = 2.581$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$
		

Table 5.10: Solution of the Schnackenberg system with homogeneous Neumann b.c., having fixed $d_1 = 1.6$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 9$*$ shows a plot of the explicit solution $(d_1, d_2) = (1.6, 12.95) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{2,0}$. System had zero Neumann b.c.

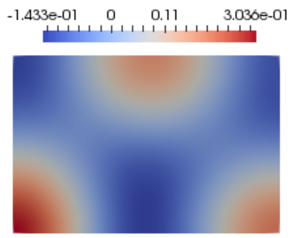
Experiment Nr. 13	Experiment Nr. 14	Experiment Nr. 15*
$d_1 = 1.1$	$d_1 = 1.1$	$d_1 = 1.1$
$d_2 = 9.50$	$d_2 = 8.78$	$d_2 = 8.78$
$\|v\|_{L^\infty(\Omega)} = 0.202$	$\|v\|_{L^\infty(\Omega)} = 2.85\text{e} - 4$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 3.260$	$\|v\| = 0.511$	$\|v\| = 3.206$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$

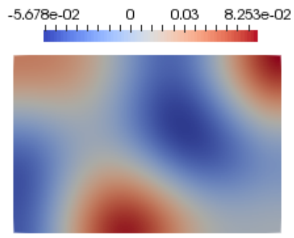






Experiment Nr. 16	Experiment Nr. 17	Experiment Nr. 18
$d_1 = 1.1$	$d_1 = 1.1$	$d_1 = 1.1$
$d_2 = 9.50$	$d_2 = 8.78$	$d_2 = 8.66$
$\|v\|_{L^\infty(\Omega)} = 0.304$	$\|v\|_{L^\infty(\Omega)} = 0.0568$	$\|v\|_{L^\infty(\Omega)} = 0.0124$
$\|v\| = 3.125$	$\|v\| = 1.124$	$\|v\| = 0.149$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$





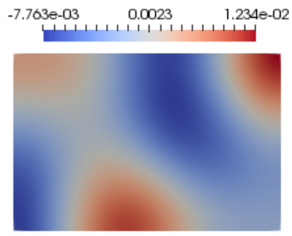
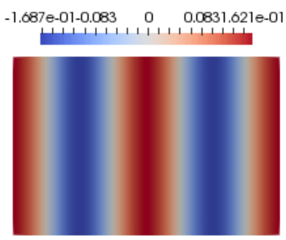
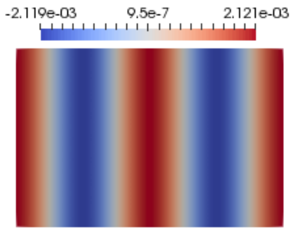
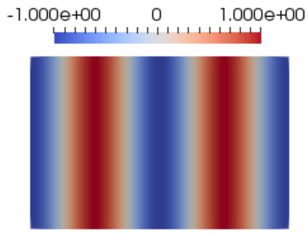


Table 5.11: Solution of the Schnackenberg system with homogeneous Neumann b.c., having fixed $d_1 = 1.1$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 15$*$ shows a plot of the explicit solution $(d_1, d_2) = (1.1, 8.782) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{2,1}$. System had zero Neumann b.c. Here the bifurcation point is the same as in the Dirichlet case.

Experiment Nr. 19	Experiment Nr. 20	Experiment Nr. 21*
$d_1 = 0.5$	$d_1 = 0.5$	$d_1 = 0.5$
$d_2 = 4.50$	$d_2 = 3.96$	$d_2 = 3.96$
$\|v\|_{L^\infty(\Omega)} = 0.169$	$\|v\|_{L^\infty(\Omega)} = 0.00212$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 4.757$	$\|v\| = 0.0548$	$\|v\| = 9.494$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$
		

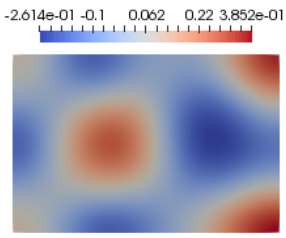
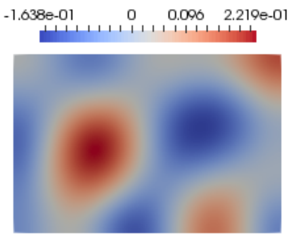
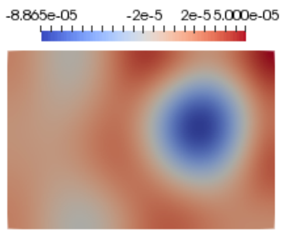
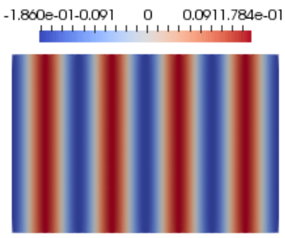
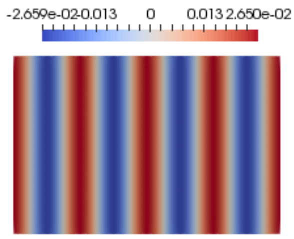
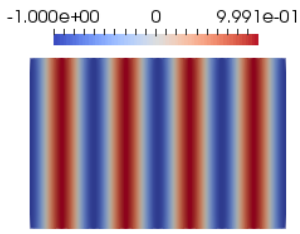
Experiment Nr. 22	Experiment Nr. 23	Experiment Nr. 24
$d_1 = 0.5$	$d_1 = 0.5$	$d_1 = 0.5$
$d_2 = 4.50$	$d_2 = 3.96$	$d_2 = 3.52$
$\|v\|_{L^\infty(\Omega)} = 0.385$	$\|v\|_{L^\infty(\Omega)} = 0.222$	$\|v\|_{L^\infty(\Omega)} = 8.877\text{e}-5$
$\|v\| = 4.801$	$\|v\| = 2.960$	$\|v\| = 0.000856$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$
		

Table 5.12: Solution of the Schnackenberg system with homogeneous Neumann b.c., having fixed $d_1 = 0.5$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 21$*$ shows a plot of the explicit solution $(d_1, d_2) = (0.5, 3.96) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{4,0}$. System had zero Neumann b.c.

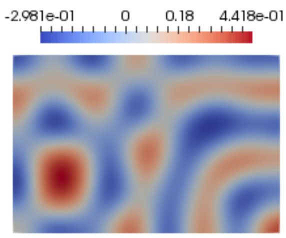
Experiment Nr. 25	Experiment Nr. 26	Experiment Nr. 27*
$d_1 = 0.12$	$d_1 = 0.12$	$d_1 = 0.12$
$d_2 = 1.1$	$d_2 = 0.95$	$d_2 = 0.95$
$\|v\|_{L^\infty(\Omega)} = 0.186$	$\|v\|_{L^\infty(\Omega)} = 0.00192$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 7.783$	$\|v\| = 0.0720$	$\|v\| = 15.750$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$

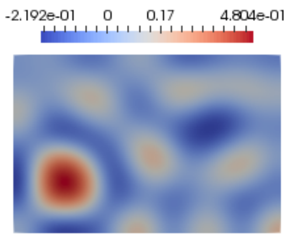






Experiment Nr. 28	Experiment Nr. 29	Experiment Nr. 30
$d_1 = 0.12$	$d_1 = 0.12$	$d_1 = 0.12$
$d_2 = 1.1$	$d_2 = 0.95$	$d_2 = 0.68$
$\|v\|_{L^\infty(\Omega)} = 0.442$	$\|v\|_{L^\infty(\Omega)} = 0.480$	$\|v\|_{L^\infty(\Omega)} = 0.458$
$\|v\| = 7.799$	$\|v\| = 4.926$	$\|v\| = 2.635$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$





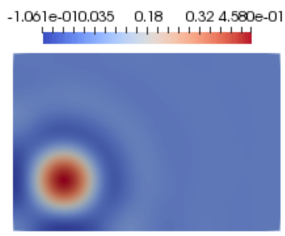
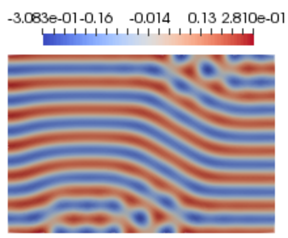
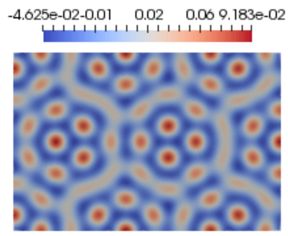
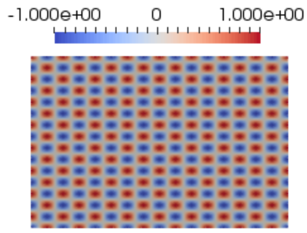


Table 5.13: Solution of the Schnackenberg system with homogeneous Neumann b.c., having fixed $d_1 = 0.12$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 27$*$ shows a plot of the explicit solution $(d_1, d_2) = (0.12, 0.95) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{8,0}$. System had zero Neumann b.c.

Experiment Nr. 31	Experiment Nr. 32	Experiment Nr. 33*
$d_1 = 0.02$	$d_1 = 0.02$	$d_1 = 0.02$
$d_2 = 0.20$	$d_2 = 0.16$	$d_2 = 0.16$
$\|v\|_{L^\infty(\Omega)} = 0.308$	$\|v\|_{L^\infty(\Omega)} = 0.0918$	$\|v\|_{L^\infty(\Omega)} = 1.0$
$\|v\| = 19.886$	$\|v\| = 3.021$	$\|v\| = 35.935$
$\tau = 0.0$	$\tau = 0.0$	$\tau = 0.0$
		

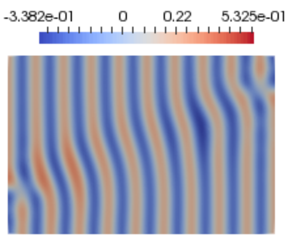
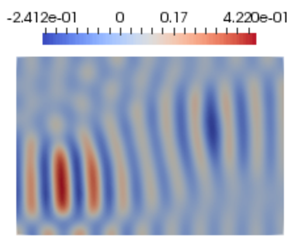
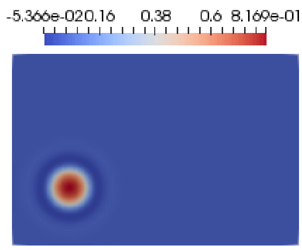
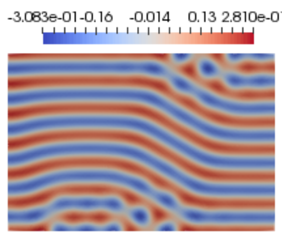
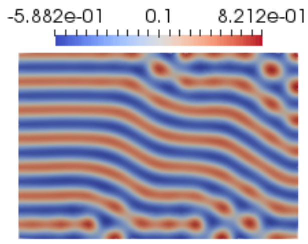
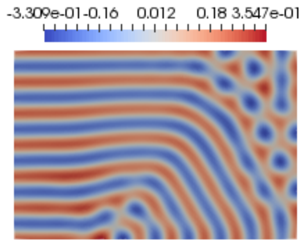
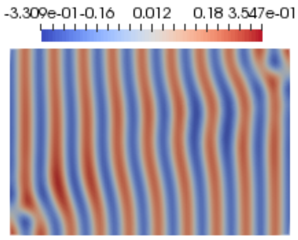
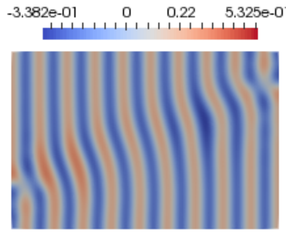
Experiment Nr. 34	Experiment Nr. 35	Experiment Nr. 36
$d_1 = 0.02$	$d_1 = 0.02$	$d_1 = 0.02$
$d_2 = 0.20$	$d_2 = 0.16$	$d_2 = 0.08$
$\|v\|_{L^\infty(\Omega)} = 0.533$	$\|v\|_{L^\infty(\Omega)} = 0.422$	$\|v\|_{L^\infty(\Omega)} = 0.817$
$\|v\| = 20.616$	$\|v\| = 9.094$	$\|v\| = 3.677$
$\tau = 1.0$	$\tau = 1.0$	$\tau = 1.0$
		

Table 5.14: Solution of the Schnackenberg system with homogeneous Neumann b.c., having fixed $d_1 = 0.02$ and selected values of d_2. The upper images are for systems with $\tau = 0$, the lower ones are for systems with $\tau = 1.0$. The figure 33$*$ shows a plot of the explicit solution $(d_1, d_2) = (0.10, 0.16) \in C_E$ of the system (5.5) with $\tau = 0.0$. This point is a bifurcation point of the Schnackenberg system and lies on the hyperbola $C_{16,15}$. System had zero Neumann b.c.

Experiment Nr. 1	Experiment Nr. 2	Experiment Nr. 3
$d_1 = 0.02$	$d_1 = 0.02$	$d_1 = 0.02$
$d_2 = 0.20$	$d_2 = 0.20$	$d_2 = 0.20$
$\lVert v \rVert_{L^\infty(\Omega)} = 0.308$	$\lVert v \rVert_{L^\infty(\Omega)} = 0.821$	$\lVert v \rVert_{L^\infty(\Omega)} = 0.355$
$\lVert v \rVert = 8.124$	$\lVert v \rVert = 19.864$	$\lVert v \rVert = 20.124$
$\tau = 0.00$	$\tau = 0.25$	$\tau = 0.50$
		

Experiment Nr. 4	Experiment Nr. 5
$d_1 = 0.02$	$d_1 = 0.02$
$d_2 = 0.20$	$d_2 = 0.20$
$\lVert v \rVert_{L^\infty(\Omega)} = 0.381$	$\lVert v \rVert_{L^\infty(\Omega)} = 0.533$
$\lVert v \rVert = 20.578$	$\lVert v \rVert = 19.754$
$\tau = 0.75$	$\tau = 1.00$
	

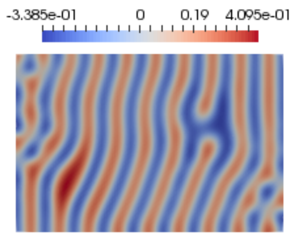
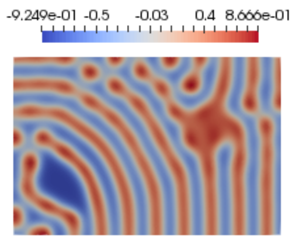
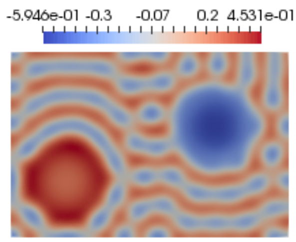
Experiment Nr. 6	Experiment Nr. 7	Experiment Nr. 8
$d_1 = 0.02$	$d_1 = 0.02$	$d_1 = 0.02$
$d_2 = 0.20$	$d_2 = 0.16$	$d_2 = 0.08$
$\lVert v \rVert_{L^\infty(\Omega)} = 0.401$	$\lVert v \rVert_{L^\infty(\Omega)} = 0.925$	$\lVert v \rVert_{L^\infty(\Omega)} = 0.595$
$\lVert v \rVert = 8.078$	$\lVert v \rVert = 20.259$	$\lVert v \rVert = 7.546$
$\tau = 1.50$	$\tau = 2.00$	$\tau = 4.00$
		

Table 5.15: Solution of the Schnackenberg system with homogeneous Neumann b. c., having fixed $d_1 = 0.02$ $d_2 = 0.2$ and selected values of τ, which increases from 0.0 for 4.0.

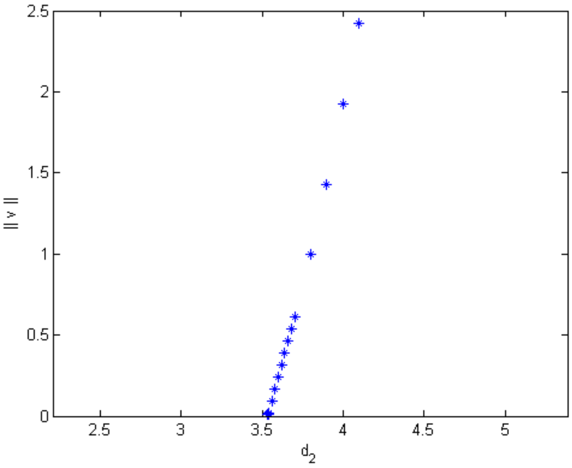


Figure 5.4: The norm of v (of numerically stationary solutions) w.r.t. parameter d_2. The parameter $d_1 = 0.5$ is fixed.

Chapter 6

Conclusions

The main purpose of this dissertation thesis, to study the systems of reaction-diffusion equations with unilateral terms, was fulfilled. The analysis was divided into two categories – systems with Dirichlet/mixed b.c. and with Neumann boundary conditions, because the boundary conditions affect the existence and location of critical and bifurcation points of the problem. The main tools were Variational methods, Topological Degree methods and methods based on Implicit Function Theorem.

A short introduction to history and motivation to the research of pattern formation, together with summary of contributions to it achieved in this dissertation thesis was given in the first section.

The technique how to rewrite the reaction diffusion systems with unilateral terms to an operator equation with positively homogeneous operator on the space $W^{1,2}(\Omega)$ is described in Section 2. All necessary details are given there, including some supplementary results about skew-symmetric systems which lead to the problem that has a potential. The conservation of potential under the Lyapunov-Schmidt reduction is proved as well. The abstract formulation of the problem with the unilateral terms on the boundary is also introduced.

The general results concerning eigenvalues of positively homogeneous operators and bifurcation in equations with positively homogeneous operators perturbed by a small nonlinear perturbation are contained in Theorems 3–10 in Section 3. These results are discussed and compared with well-known theorems for equations with differentiable operators – Krasnoselskii, Rabinowitz and Crandall-Rabinowitz. Of course all theorems are supplemented with explanatory remarks and useful hints. Also some examples are given there. One very general theorem about the bifurcation in the equations with positively homogeneous operators, which was developed by Lutz Recke and Martin Väth, is stated there without the proof.

Afterwards these general theorems are applied to reaction-diffusion systems with unilateral terms, which is a content of Theorems 11–21. Reaction-diffusion systems with Dirichlet/mixed conditions and unilateral terms behave differently than the systems with Neumann b.c. and therefore the results for them are in general different. However, one theorem has same conclusions for both problems. The main contribution is the proof of the existence of bifurcation points of our systems with unilateral terms having diffusion parameters, for which no bifurcation point of reaction diffusion systems without unilateral terms exists. Also it contains two theorems about systems with skew-symmetric reaction kinetics, which is a class of reactions that has not been studied before for systems with unilateral terms. Two theorems for systems with unilateral terms on the boundary are also formulated, giving again bifurcation points for parameters, for which no bifurcation point of the systems without unilateral terms exists. Finally, one theorem for systems defined on the domain with $C^{1,1}$-continuous boundary is proved. The domain with the smoother boundary leads to a better regularity of the solution.

In Section 5 these conclusions are verified on a particular problem – Schnackenberg system with homogeneous Dirichlet boundary conditions and Schnackenberg system with homogeneous Neumann boundary conditions. The patterns in the systems with unilateral terms are found for diffusion rates, for which in the systems without unilateral terms no patterns formed. Moreover, the location of bifurcation points is approximated as well, which gives a hint how the solutions of the so-called homogenized system should look like. The source code of the implementation of the numerical scheme is freely available on GitHub, the address can be found in Section 5.

Finally, the necessary theoretical background is summarized in Appendix. It is of course not complete, but contains all important theorems and concepts that from the point of view of the author of this dissertation thesis could help the reader to understand the content of the thesis.

To sum up, the presence of unilateral sources has a significant impact on the location of critical and bifurcation points of reaction-diffusion systems, which can be found even for diffusion rates, for which no critical or bifurcation point in systems without unilateral terms exists. The presence of the unilateral terms has also an impact on the shape of patterns.

Chapter **7**

Appendix

7.1 Sobolev spaces

Sobolev spaces The first part of the appendix is devoted to a summary of basic properties of Sobolev spaces. Proofs of two theorems mentioned here can be found in classical books like [17] or [48].

For $k \in \mathbb{N}$ and $p \in [1, \infty]$ the Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) := \left\{ v \in L^p(\Omega) \middle| \; D^\alpha v \in L^p(\Omega) \text{ for all } |\alpha| \le k \right\},$$

where the derivatives are considered in the weak sense and $L^p(\Omega)$ is, as usual, the Lebesgue space. For $p \in [1, \infty]$ the Lebesgue norm on $L^p(\Omega)$ will be denoted by $\|\cdot\|_{L^p}$. The Sobolev space $W^{k,p}(\Omega)$ equipped with the norm

$$\|v\|_{k,p} = \left(\sum_{|\alpha| \le k} \|D^\alpha v\|_{L^p}^p \right)^{\frac{1}{p}} \quad \text{for all } v \in W^{k,p}(\Omega) \;\; \text{if } p < \infty, \tag{7.1}$$

$$\|v\|_{k,\infty} = \max_{|\alpha| \le k} \|D^\alpha v\|_{L^\infty} \quad \text{for all } v \in W^{k,\infty}(\Omega), \tag{7.2}$$

is a Banach space. For $p \in (1, \infty)$ these spaces are separable and reflexive. The space $W^{k,2}(\Omega)$ with a scalar product

$$\langle u, v \rangle_{k,2} = \sum_{n=1}^{k} \int_\Omega \nabla^n u \cdot \nabla^n v \; \mathrm{dx} \quad \text{for all } u, v \in W^{k,2}(\Omega), \tag{7.3}$$

is a Hilbert space for any $k \ge 1$ The space $W^{1,2}(\Omega)$ will be the most often used space in this dissertation thesis. However, in Section 4.6 the space $W^{2,p}$ with $p > 2$ is used.

The significant properties of $W^{k,p}(\Omega)$ are continuous and compact embeddings into $L^q(\Omega)$ and $C^{1,\alpha}(\Omega)$ spaces.

Theorem 22. *Let $\Omega \in C^{0,1}$, i.e. a bounded domain in $\mathbb{R}^m$ with a Lipschitz boundary. Let $k, l \in \mathbb{N}_0$, $k \ge l$ and let $1 \le p < q < \infty$ be two real numbers such that*

$$k - \frac{m}{p} \ge l - \frac{m}{q}. \tag{7.4}$$

Then

$$W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega).$$

If the inequality (7.4) is strict, the embedding is completely continuous (compact).

Let $m < p$, let $\alpha \in (0,1]$ and k, r be integers satisfying

$$\frac{(k - r - \alpha)}{m} = \frac{1}{p}.$$

Then

$$W^{k,p}(\Omega) \hookrightarrow C^{r,\alpha}(\Omega).$$

For $m = 2$ it is $W^{1,2}(\Omega) \hookrightarrow^c L^q(\Omega)$ for any $1 \leq q < \infty$. When $m > 2$, then $W^{1,2}_D(\Omega) \hookrightarrow L^p(\Omega)$ with

$$p = \frac{2m}{m-2}.$$

For $1 \leq p < 2m/(m-2)$ the embedding is compact. Another important special case is $m = 2,\ k = 2,\ p > 2$, for which $W^{2,p}(\Omega) \hookrightarrow^c C^{1,\alpha}(\Omega)$, with $\alpha = 1 - m/p$. All of these embeddings will be used in this dissertation thesis.

Theorem 23 (Trace Theorem)**.** *Let $\Omega \in C^{0,1}$ and $p \in [1, \infty)$. Let $p_0 = (mp - p)/(m - p)$. Then there exists a unique bounded linear operator $T : W^{1,p}(\Omega) \to L^q(\partial\Omega)$, where $q \in [1, p_0]$, satisfying*

$$Tv = v|_{\partial\Omega}, \quad \textit{for all } v \in C^\infty(\overline{\Omega}).$$

The operator T is compact for $q \in [1, p_0)$. Moreover, if $p = m$ then T compact from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$ for any $q \in [1, \infty)$. If $p > m$, then T is compact from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$ for any $q \in [1, \infty]$.

It is common to drop the symbol for trace operator and write simply $v|_{\partial\Omega}$. A prominent subspace of $W^{1,p}(\Omega)$ is a space of functions with the zero trace.

Definition 10. *Let $\Omega \in C^{0,1}$. The space $W^{1,p}_0(\Omega) \subset W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}_0(\Omega) := \{v \in W^{1,p} \mid v|_{\partial\Omega} = 0\}.$$

For our purposes we define a space of functions having the zero trace only on a part of the boundary.

Definition 11. *Let $\Omega \in C^{0,1}$ and $\Gamma_D \subset \partial\Omega$. The space $W^{1,2}_D(\Omega) \subset W^{1,2}(\Omega)$ is defined by*

$$W^{1,2}_D(\Omega) = \left\{ v \in W^{1,2}(\Omega) \middle| \ v|_{\Gamma_D} = 0 \right\}.$$

Since Ω is a set with a Lipschitz boundary, the space $W^{1,2}_D(\Omega)$ is a Hilbert space, as can be proved using Trace Theorem. If $\mu_{m-1}(\Gamma_D) = 0$, then $W^{1,2}_D(\Omega) = W^{1,2}(\Omega)$ and if $\Gamma_D = \partial\Omega$, then $W^{1,2}_D(\Omega) = W^{1,2}_0(\Omega)$.

Remark 28. *Let us consider an eigenvalue problem for the Laplacian*

$$\Delta v + \kappa v = 0 \textit{ in } \Omega$$
$$v\big|_{\Gamma_D} = 0, \quad \frac{\partial v}{\partial \vec{n}}\bigg|_{\Gamma_N} = 0.$$

The weak formulation of this equation is

$$\int_\Omega \nabla v \cdot \nabla \varphi - \kappa v \varphi \ dx = 0, \quad \textit{for all } v \in W^{1,2}_D(\Omega).$$

If the problem has a Dirichlet or mixed boundary conditions, then smallest eigenvalue is positive. The problem with Neumann boundary conditions has the smallest eigenvalue equal to zero.

For the Dirichlet/mixed case it is common to order the eigenvalues of the Laplacian in a monotonous sequence

$$0 < \kappa_1 < \kappa_2 \leq \dots \to \infty, \tag{7.5}$$

counted according to their multiplicity.

The eigenvalues of the Laplacian with Neumann b.c. can be ordered in a sequence

$$0 = \kappa_0 < \kappa_1 \leq \kappa_2 \leq \cdots \to \infty. \tag{7.6}$$

It will be always assumed in this dissertation thesis that the eigenvalues of the Laplacian are ordered either as (7.5), or (7.6), depending on the prescribed b.c.

7.2 Nonlinear analysis

The second part of this section contains a necessary minimum from the nonlinear analysis. The symbols $\mathbb{X}$ and $\mathbb{H}$ will denote here a real Banach and a real Hilbert space respectively. The proofs of the theorems can be found in books [9] and [1].

Definition 12. *Let $(\mathbb{M}, \rho)$ be a complete metric space. A map $\mathcal{Q} : \mathbb{M} \to \mathbb{M}$ is called contraction if there exists a constant $K \in (0,1)$ such that*

$$\rho(\mathcal{Q}(x), \mathcal{Q}(y)) \leq K\rho(x,y) \quad \textit{for all } x, y \in \mathbb{M}.$$

Fixed Point Theorem. *Let $(\mathbb{M}, \rho)$ be a nonempty complete metric space and $\mathcal{Q} : \mathbb{M} \to \mathbb{M}$ be a contraction. Then there exists the unique $w \in \mathbb{M}$ satisfying $\mathcal{Q}(w) = w$.*

Definition 13. *Let $\mathcal{N} : \mathbb{X} \to \mathbb{X}$ be a nonlinear operator. The operator $\mathcal{N}$ is called compact if the image of every bounded set in $\mathbb{X}$ is a relatively compact set in $\mathbb{X}$.*

Definition 14. *Let $\Omega \subset \mathbb{R}^m$ be a domain and let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a real function. We say that f satisfies Carathéodory conditions if*

1. *$f(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^m$,*
2. *$f(x, \cdot)$ is continuous for a.a. $x \in \Omega$.*

Continuity of Nemyckii operator. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, let $p, q < \infty$ and $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfy Caratheodory conditions and*

$$|f(x,v)| \leq g_1(x) + c(x) \sum_{i=1}^{m} |v_i|^{\frac{p}{q}} \quad \textit{for all } v \in \mathbb{R}^m, \ \textit{for a.a. } x \in \Omega, \tag{7.7}$$

where v_i are components of v, $g_1(x) \in L^q(\Omega)$ and $c \in L^\infty(\Omega)$. Then the Nemyckii operator

$$[\mathcal{F}(v)](x) := f(x, v(x)) \quad \textit{for a.a. } x \in \Omega \ \ \textit{for all } v \in L^p(\Omega),$$

is a well defined and continuous from $L^p(\Omega)$ to $L^q(\Omega)$.

Implicit Function Theorem. *Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ be Banach spaces, $\Phi : \mathbb{X} \times \mathbb{Y} \to \mathbb{Z}$. Let $(a,b) \in \mathbb{X} \times \mathbb{Y}$ be such a point that*

$$\Phi(a,b) = 0.$$

Let $\mathscr{V}$ be an open set in $\mathbb{X} \times \mathbb{Y}$ containing the point (a,b). Let $\Phi \in C^1(\mathscr{V})$ and let the partial derivative $\partial_x \Phi(x,y)|_{x=a,y=b}$ be an isomorphism of $\mathbb{X}$ onto $\mathbb{Z}$. Then there are neighbourhoods U of a and V of b such that for any $v \in V$ there exists a unique $u \in U$ for which

$$\Phi(u,v) = 0.$$

Denote this u by $\mathcal{F}(v)$. Then $\mathcal{F} \in C^1(V, U)$. Moreover,

$$\mathcal{F}'(v) = -\left[\partial_x \Phi(x,y)|_{x=\mathcal{F}(v),y=v}\right]^{-1} \partial_y \Phi(x,y)_{x=\mathcal{F}(v),y=v}, \quad \textit{for all } v \in V. \tag{7.8}$$

Let us note that the uniqueness of u in U implies $\Phi(u,v) = 0$ if and only if $u = \mathcal{F}(v)$.

Mean Value Theorem. *Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces and let $\mathcal{F} : \mathbb{X} \to \mathbb{Y}$. If the Găteaux derivative $D\mathcal{F}(u_1 + t(u_2 - u_1)) - D\mathcal{F}(u_1)(u_2 - u_1)$ exists for given u_1, u_2 and for all $t \in [0,1]$, then*

$$\|\mathcal{F}(u_2) - \mathcal{F}(u_1) - D\mathcal{F}(u_1)(u_2 - u_1)\|_{\mathbb{Y}} \leq \sup_{t\in[0,1]} \|D\mathcal{F}(u_1 + t(u_2 - u_1))(u_2 - u_1) - D\mathcal{F}(u_1)(u_2 - u_1)\|_{\mathbb{Y}}.$$

Let us consider an abstract problem

$$\mathcal{F}(\lambda, v) = 0, \tag{7.9}$$

where $\mathbb{X}$ is a real Banach space and $\mathcal{F} : \Lambda \times \mathbb{X} \to \mathbb{X}$, with $\Lambda \subset \mathbb{R}$ being an open set. The homogenization of (7.9) is

$$\mathcal{F}_0(\lambda, v) = 0, \tag{7.10}$$

where

$$\mathcal{F}_0(\lambda, v) = \lim_{r\to 0} \frac{\mathcal{F}(\lambda, rv)}{r},$$

assuming that the limit exists. It is clear from the definition that $\mathcal{F}_0$ is positively homogeneous of the degree one in the vriable v, i.e. $\mathcal{F}(\lambda, tv) = t\mathcal{F}(\lambda, v)$ for all $t \geq 0, v \in \mathbb{X}, \lambda \in \Lambda$. If $\mathcal{F} \in C^1(\Lambda \times \mathbb{X}, \mathbb{X})$, then $\mathcal{F}_0$ is a linear operator.

Definition 15. *A critical point of the problem (7.10) is a number $\lambda_0 \in \Lambda$ for which there exists $v \in \mathbb{H},\ v \neq 0$ satisfying*

$$\mathcal{F}_0(\lambda_0, v) = 0. \tag{7.11}$$

Let use define a set

$$\mathscr{S} := \overline{\{(\lambda, v) \in \Lambda \times \mathbb{X} \mid v \neq 0,\ (\lambda, v) \text{ solves } (7.9)\}}. \tag{7.12}$$

Definition 16. *A number $\lambda_b \in \Lambda$ is a (local) bifurcation point of (7.9) if it exists in any neighborhood of $(\lambda_b, 0)$ in $\Lambda \times \mathbb{X}$ a solution $(\lambda, v) \in \Lambda \times \mathbb{X}$ of (7.9) with $v \neq 0$.*

A parameter $\lambda_b \in \Lambda$ is called a global bifurcation point of the problem (7.9) if at least one of the following cases occurs:

- *a connected component $\mathscr{S}_0$ of $\mathscr{S}$ containing $(\lambda_b, 0, 0)$ is unbounded*
- *there exists a critical point λ_c of (7.11) so that $(\lambda_c, 0, 0) \in \mathscr{S}_0$ and $\lambda_b \neq \lambda_c$*
- *there exists an element $\lambda \in \partial\Lambda$ and $v \in \mathbb{X},\ v \neq 0$ such that $(\lambda, v) \in \mathscr{S}_0$.*

In our applications the set Λ will be mostly the interval $(0, \infty)$.

Definition 17 (Brouwer degree)**.** *Let $\Omega \subset \mathbb{R}^m$ be open and bounded, let $F \in C(\overline{\Omega}, \mathbb{R}^m) \cap C^1(\Omega, \mathbb{R}^m)$. Assume that $y_0 \in \mathbb{R}^m \backslash F(\partial\Omega)$ and y_0 is a regular value of F. Then the Brouwer degree of F w.r.t. Ω and y_0 is defined as*

$$deg(F, \Omega, y_0) = \sum_{x \in F_{-1}(y_0)\cap\Omega} sgn\ J_F(x),$$

where J_F denotes the Jacobi matrix of F at the point x.

One of the significant properties of the degree is the homotopy invariance: if $F : [0,1] \times \overline{\Omega} \to \mathbb{R}^m$ is continuous and $y_0 \notin \cup_{t\in[0,1]} F(t, \partial\Omega)$, then $\deg(F, \Omega, y_0)$ does not depend on t.

It is well-known that any compact operator on an infinite-dimensional Banach space can be approximated by a sequence of operators with the finite-dimensional range. The symbol $\mathcal{C}_f(\overline{V}, \mathbb{X})$ will denote a set of compact operators from $\overline{V}$ to $\mathbb{X}$ with a finite dimensional range.

Definition 18 (Leray-Schauder degree)**.** *Let $\mathbb{X}$ be a real Banach space, V be an open bounded set in $\mathbb{X}$, $\mathcal{F}:\mathbb{X}\to\mathbb{X}$ be a compact operator, $0\in\mathbb{X}\backslash(I-\mathcal{F})(\partial V)$ and $\mathcal{F}_n\in\mathcal{C}_f(\overline{V},\mathbb{X})$ be a sequence which converges uniformly to $\mathcal{F}$ in V. Denote*

$$\mathbb{X}_n = span\ \mathcal{F}_n(\overline{V}),\quad V_n = V\cap\mathbb{X}_n,$$
$$G_n(x) = x-\mathcal{F}_n(x)\ \text{ for all }\ x\in\overline{V_n}.$$

Then the Leray-Schauder degree of $\mathcal{I}-\mathcal{F}$ with respect to V and 0 is defined by

$$deg(\mathcal{I}-\mathcal{F},V,0) := \lim_{n\to\infty} deg(G_n,V_n,0).$$

The justification of this definition (e.g. independence of the choice of sequence, existence of $\deg(G_n,V_n,0)$, etc.) can be found in [9], Chapter 5.2. Similarly to Brouwer degree, Leray-Schauder degree is invariant w.r.t. the homotopy. By the term "degree" we will be always meaning the Leray-Schauder degree.

Proposition 14. *Let $\mathbb{X}$ be a real Banach space, let Λ be an interval in $\mathbb{R}$ and let $\mathcal{F}:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ be a compact operator in the neighborhood V of 0 for all $\lambda\in\Lambda$. Let zero be an isolated solution of*

$$v-\mathcal{F}(\lambda,v)=0\ \text{ in } V,\ \text{for all } \lambda\in\Lambda\backslash\{\lambda_0\}. \tag{7.13}$$

Let

$$\lim_{\lambda\to\lambda_{0-}} deg\left(\mathcal{I}-\mathcal{F}(\lambda,\cdot),V,0\right) \neq \lim_{\lambda\to\lambda_{0+}} deg\left(\mathcal{I}-\mathcal{F}(\lambda,\cdot),V,0\right).$$

Then $(\lambda_0,0)$ is a bifurcation point of (7.13).

Remark 29. *Let $\mathcal{F}:\mathbb{X}\to\mathbb{X}$ be a linear compact operator and let $\lambda\in\sigma(\mathcal{F})$, $\lambda\neq 0$. There exists $k\in\mathbb{N}$ such that*

$$\mathbb{X} = Ker(I-\lambda^{-1}\mathcal{F})^k \oplus Im(I-\lambda^{-1}\mathcal{F})^k.$$

Moreover, dim $Ker(I-\lambda^{-1}\mathcal{F})^k<\infty$.

Definition 19. *Let $\mathcal{F}$ be from the previous remark. The number dim $Ker(I-\lambda^{-1}\mathcal{F})^k<\infty$ will be called the (algebraic) multiplicity of λ.*

When talking about multiplicity of an eigenvalue, we will be always meaning its algebraic multiplicity. It is also possible to define so-called geometric multiplicity, which is equal to dim $\text{Ker}(I-\lambda^{-1}\mathcal{F})$. In the Chapters 2–5 we worked with symmetric linear operators, where the algebraic and geometric multiplicity are the same therefore we will not distinguish between them.

Leray-Schauder Index Formula. *Let V be an open bounded set in a real Banach space $\mathbb{X}$ and let $\mathcal{F}\in C^1(\overline{V},\mathbb{X})$ be compact. Let $v_0\in V$ be a unique solution in $\overline{V}$ of the equation*

$$v=\mathcal{F}(v).$$

Assume that $I-\mathcal{F}'(v_0)$ is continously invertible. Then

$$deg(\mathcal{I}-\mathcal{F},V,v_0) = (-1)^\alpha,\qquad \alpha = \sum_{\lambda\in\sigma(\mathcal{F}'(v_0))\cap\mathbb{R},\lambda>1} m(\lambda), \tag{7.14}$$

where $m(\lambda)$ is the multiplicity of the eigenvalue λ of the operator $\mathcal{F}'(v_0)$.

Remark 30. *Let $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ be a continuously differentiable and compact operator on a real Banach space $\mathbb{X}$, assume that $\lambda_1 > 0$ is the largest eigenvalue of $\mathcal{F}'(0)$ with odd multiplicity. The Leray-Schauder index formula gives*

$$deg\left(\mathcal{I} - \frac{1}{\lambda}\mathcal{F}, B_r(0), 0\right) = (-1)^\alpha, \quad \text{for all } \lambda \neq 0, \text{ for } \alpha \text{ from (7.14)},$$

and for any sufficiently small $r > 0$. There are no eigenvalues of the operator $\mathcal{F}'(0)$ larger than λ_1. Hence, for all $\lambda > \lambda_1$ we have $\alpha = 0$ and consequently

$$deg\left(\mathcal{I} - \frac{1}{\lambda}\mathcal{F}, B_r(0), 0\right) = 1, \tag{7.15}$$

for any sufficiently small $r > 0$. Assume that $\lambda_2 > 0$ is the second largest eigenvalue of the operator $\mathcal{F}'(0)$. Because the eigenvalue λ_1 has an odd multiplicity, then for all $\lambda \in (\lambda_2, \lambda_1)$, $\lambda \neq 0$ the number α from (7.14) is odd and $(-1)^\alpha = -1$, which implies

$$deg\left(\mathcal{I} - \frac{1}{\lambda}\mathcal{F}, B_r(0), 0\right) = -1. \tag{7.16}$$

for any sufficiently small $r > 0$. In sum, there is

$$deg\left(\mathcal{I} - \frac{1}{\lambda_a}\mathcal{F}, B_r(0), 0\right) \neq deg\left(\mathcal{I} - \frac{1}{\lambda_b}\mathcal{F}, B_r(0), 0\right),$$

for any $\lambda_a > \lambda_1$ and $\lambda_b \in (\lambda_1, \lambda_2)$ and sufficiently small $r > 0$. According to Proposition 14 the point λ_1 is a bifurcation point of the equation $\lambda v = \mathcal{F}(v)$. This procedure can be simply modified for other eigenvalues. We will use it in the proof of Theorem 8 on pg. 52.

Definition 20. *Let $\mathbb{H}$ be a real Hilbert space. We say that an operator $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ has a potential on the set $V \subset \mathbb{H}$ if there exists a functional $\Phi_F : V \to \mathbb{R}$ such that*

$$\langle \Phi_F'(v), u \rangle = \langle \mathcal{F}(v), u \rangle \quad \text{for all } v \in V, u \in \mathbb{H},$$

where prime denotes the Fréchet derivative. If an operator has a potential on the whole $\mathbb{H}$, we will call it potential operator.

It will be assumed without loss of generality that $\Phi_F(0) = 0$.

Observation 7. *Any symmetric linear operator $\mathcal{S} : \mathbb{H} \to \mathbb{H}$ has the potential defined by*

$$\Phi_{\mathcal{S}}(v) := \frac{1}{2}\langle \mathcal{S}v, v \rangle \quad \text{for all } v \in \mathbb{H}.$$

7.3 Three famous bifurcation theorems

Theorem 24 (Rabinowitz Global Bifurcation Theorem)**.** *Let $\mathscr{M} \subset \mathbb{R}$ be an open set in $\mathbb{R} \times \mathbb{X}$, let $(\lambda_0, 0) \in \mathscr{M}$ and $\lambda_0 \neq 0$, let $\mathcal{L} : \mathbb{X} \to \mathbb{X}$ be a linear compact operator having $1/\lambda_0$ as an eigenvalue of an odd multiplicity and let $\mathcal{N} : \mathscr{M} \to \mathbb{X}$ be a small nonlinear compact perturbation, i.e. $\mathcal{N}$ is compact and*

$$\text{for any bounded set } \Lambda \subset \{\lambda \in \mathbb{R} \mid (\lambda, 0) \in \mathscr{M}\} :$$
$$\lim_{v \to 0} \frac{\mathcal{N}(\lambda, v)}{\|v\|} = 0, \quad \text{uniformly for any } \lambda \in \Lambda.$$

Then the set

$$\mathscr{S} = \overline{\{(\lambda, v) \in \mathscr{M} \mid v \neq 0,\ (\lambda, v) \ \text{ and } \ v - \lambda\mathcal{L}v - \mathcal{N}(\lambda, v) = 0\}}.$$

contains the point $(\lambda_0, 0)$. Let $\mathscr{S}_0$ be a component of $\mathscr{S}$ which contains $(\lambda_0, 0)$. Then at least one of the following holds:

(i) $\mathscr{S}_0$ *is not a compact set in* $\mathscr{M}$.

(ii) $\mathscr{S}_0$ *contains an even number of points* $(\lambda, 0)$, *where* $1/\lambda$ *is an eigenvalue of* $\mathcal{L}$ *of odd multiplicity.*

Then λ_0 *is a global bifurcation point of the equation*

$$\lambda v - \mathcal{L}v - \mathcal{N}(\lambda, v) = 0,$$

in the sense of Definition 16.

Later we will use a modification of this theorem developed by Martin Väth, see Theorem 7 on pg. 51, to prove a bifurcation theorem suitable for our systems.

Theorem 25 (Crandall-Rabinowitz Theorem). *Let* $\mathbb{X}, \mathbb{Y}$ *be real Banach spaces and let* $\mathcal{F} : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ *be* C^2 *in a neighborhood of* $(0, 0)$. *Let* $\mathcal{F}$ *satisfy the assumptions*

(i) $\mathcal{F}(\lambda, 0) = 0$ *for all* $\lambda \in (-\delta, \delta)$ *for some* $\delta > 0$,

(ii) *dim Ker* $\partial_v \mathcal{F}(0, 0) =$ *codim Im* $\partial_v \mathcal{F}(0, 0) = 1$,

(iii) *if* $\partial_v \mathcal{F}(0, 0)v_0 = 0$, $v_0 \neq 0$, *then* $\partial_\lambda \partial_v \mathcal{F}(0, 0) \notin$ *Im* $\partial_v \mathcal{F}(0, 0)$.

Denote by $\mathbb{X}_1$ *a topological complement of Ker* $\partial_v \mathcal{F}(0, 0)$ *in* $\mathbb{X}$. *Then there is a* C^1*-curve* $(\hat{\lambda}, \hat{v}) : (-\varepsilon, \varepsilon) \to \mathbb{R} \times \mathbb{X}_1$ *(for some* $\varepsilon > 0$*) such that*

$$\hat{\lambda}(0) = 0, \ \hat{v}(0) = 0, \ \mathcal{F}(\hat{\lambda}(t), t(v_0 + \hat{v}(t))) = 0.$$

Moreover, there is a neighborhood V *of* $(0, 0)$ *in* $\mathbb{R} \times \mathbb{X}$ *such that*

$$\mathcal{F}(\lambda, v) = 0, \quad \textit{for} \ \ (\lambda, v) \in V$$

if and only if either $v = 0$ *or* $\lambda = \hat{\lambda}(t)$, $v = t(v_0 + \hat{v}(t))$ *for a certain* t.

An analogue of this Theorem will be proved in the Section 3.4.

Theorem 26 (Krasnoselskii Potential Bifurcation Theorem). *Denote* $\mathscr{L}(\mathbb{H})$ *the set of all bounded linear operators on* $\mathbb{H}$. *Let* Φ_{F} *be a (nonlinear) functional on* $\mathbb{H}$. *Assume that* Φ_{F} *is twice differentiable in a certain neighborhood* V *of* 0, $\mathcal{F} = \Phi'_{\mathrm{F}} : V \to \mathbb{H}$ *is compact on* V *and* $\mathcal{S} = \Phi''_{\mathrm{F}} : V \to \mathscr{L}(\mathbb{H})$ *is continuous at* 0. *Then* $(\lambda^S, 0)$, *where* $\lambda^S \neq 0$, *is a bifurcation point of*

$$\lambda v - \mathcal{F}(v) = 0, \tag{7.17}$$

if and only if λ^S *is an eigenvalue of the operator* $\mathcal{S}$.

One theorem giving a bifurcation in systems containing positively homogeneous perturbation and having potential is proved in Section 3.2.2.

The common assumption of the previous theorems is the Fréchet differentiability of the operators $\mathcal{F}$ and $\mathcal{N}$. The aim of the Section 3 is to relax this assumption so that it will be possible to find their analogues for the equations of a type (3.6), where the positively homogeneous operator $\mathcal{B}$ is in general not differentiable.

7.4 Eigenvalues of symmetric linear operators

Remark 31. *Let $\mathcal{S} : \mathbb{H} \to \mathbb{H}$ be a linear symmetric compact operator. Assume that $\mathcal{S}$ has infinitely many eigenvalues. The largest eigenvalue of $\mathcal{S}$ can be found, under the assumption that $\mathcal{S}$ is not negative, by maximizing the so-called Rayleigh quotient:*

$$\lambda^S_{\max} := \max_{v \in \mathbb{H}, \|v\|=1} \langle \mathcal{S}v, v\rangle = \max_{v \in \mathbb{H}, v\neq 0} \frac{\langle \mathcal{S}v, v\rangle}{\|v\|^2}. \tag{7.18}$$

A vector $e_{\max}$ with $\|e_{\max}\| = 1$ is a maximizer of $\langle \mathcal{S}v, v\rangle$ if and only if it is an eigenvector of $\mathcal{S}$ respective to $\lambda^S_{\max}$.

Denote by $\lambda^S_{\max,i}$ the i-th largest eigenvalue of $\mathcal{S}$. Assume that $\mathcal{S}$ has $n \geq 2$ positive eigenvalues, and denote $\mathbb{H}_i$ the space generated by the eigenvectors corresponding to the eigenvalues $\lambda^S_{\max}, ..., \lambda^S_{\max,i}$. For $i \in \{1, \cdots, n-1\}$ the eigenvalue λ^S_{i+1} can be found by iterations

$$\lambda^S_{i+1} := \max_{v \in \mathbb{H}_i^{\perp}, v\neq 0} \frac{\langle \mathcal{S}v, v\rangle}{\|v\|^2}. \tag{7.19}$$

However, it is not always possible to get all eigenvalues by using this iterative formula. Let $\mathcal{S}$ have infinite dimensional range and n positive eigenvalues. If $i = n$, then

$$\sup_{v \in \mathbb{H}_n^{\perp}, v\neq 0} \frac{\langle \mathcal{S}v, v\rangle}{\|v\|^2} = 0.$$

It is well known that 0 is in the continuous spectrum of $\mathcal{S}$ and therefore the iterative formula (7.19) is not true for $i \geq n$.

Similar approach can be used to find the eigenvalues of a generalized eigenvalue problem

$$\lambda(\mathcal{I} - \mathcal{A})v - \mathcal{S}v = 0,$$

where $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ is a linear, symmetric, compact operator with $\sigma(\mathcal{A}) \subset [0, 1]$. If $1 \notin \sigma(\mathcal{A})$, this problem is equivalent to an eigenvalue problem for a symmetric operator $(\mathcal{I} - \mathcal{A})^{-\frac{1}{2}}\mathcal{S}(\mathcal{I} - \mathcal{A})^{-\frac{1}{2}}$, and its positive eigenvalues can be characterized by (7.18), (7.19), which are equivalent to

$$\begin{aligned} \lambda^S_{\max} &:= \max_{v \in \mathbb{H}, v\neq 0} \frac{\langle (\mathcal{I} - \mathcal{A})^{-\frac{1}{2}}\mathcal{S}(\mathcal{I} - \mathcal{A})^{-\frac{1}{2}}v, v\rangle}{\|v\|^2} = \max_{v \in \mathbb{H}, v\neq 0} \frac{\langle \mathcal{S}v, v\rangle}{\langle (\mathcal{I} - \mathcal{A})v, v\rangle}, \\ \lambda^S_{\max,i+1} &= \max_{v \in \mathbb{H}_i^{\perp}, v\neq 0} \frac{\langle (\mathcal{I} - \mathcal{A})^{-\frac{1}{2}}\mathcal{S}(\mathcal{I} - \mathcal{A})^{-\frac{1}{2}}v, v\rangle}{\|v\|^2} = \max_{v \in \mathbb{H}_i^{\perp}, v\neq 0} \frac{\langle \mathcal{S}v, v\rangle}{\langle (\mathcal{I} - \mathcal{A})v, v\rangle}. \end{aligned} \tag{7.20}$$

Let $1 \in \sigma(A)$ and e_0 be a corresponding eigenvector. If $\langle Se_0, e_0\rangle > 0$, then

$$\sup_{v \in \in \mathbb{H}, v\neq 0} \frac{\langle \mathcal{S}v, v\rangle}{\langle (\mathcal{I} - \mathcal{A})v, v\rangle} = +\infty.$$

Denote $\mathbb{H}_0$ the complement to the eigenspace of $1 \in \sigma(\mathcal{A})$. On this space, $1 \notin \sigma(\mathcal{A})$, and positive eigenvalues can be found through the iterative formula (7.20). More information can be found in e.g. [28], paragraph 14.8.8.

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Reviewed Publications of the Author Relevant to the Thesis

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[1] Navrátil, J. (2019) Skew-symmetric reaction-diffusion systems with unilateral terms and Neumann boundary conditions, preprint submitted to J. Math. Anal. Appl. (Elsevier), in review

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