No additional tournaments are quasirandom-forcing[*]

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Abstract

A tournament H is quasirandom-forcing if the following holds for every sequence $(G_n)_{n\in\mathbb{N}}$ of tournaments of growing orders: if the density of H in G_n converges to the expected density of H in a random tournament, then $(G_n)_{n\in\mathbb{N}}$ is quasirandom. Every transitive tournament with at least 4 vertices is quasirandom-forcing, and Coregliano et al. [Electron. J. Combin. 26 (2019), P1.44] showed that there is also a non-transitive 5-vertex tournament with the property. We show that no additional tournament has this property. This extends the result of Bucić et al. [Combinatorica 41 (2021), 175–208] that the non-transitive tournaments with seven or more vertices do not have this property.

1   Introduction

A combinatorial structure is said to be *quasirandom* if it has properties that a random structure would have asymptotically almost surely. The notion of *quasirandom graphs* goes back to the works of Rödl [26], Thomason [31,32] and Chung, Graham and Wilson [8] from the 1980s. There is a long series of results concerning quasirandomness of many other kinds of combinatorial structures, for example groups [17], hypergraphs [4,5,15,16,20,21,25,27], permutations [3,10,22], subsets of integers [7], etc. In the present short paper, we consider quasirandomness of
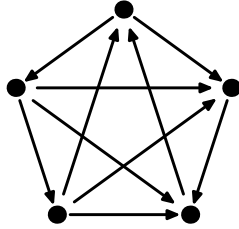

Figure 1: The non-transitive tournament F_5 on 5 vertices that is quasirandom-forcing.

tournaments as studied in [2, 6, 12]; several equivalent definitions of this notion can be found in [6].

One of the classical results on quasirandom graphs [8, 26, 31] asserts that an n-vertex graph with edge density p is quasirandom if it has $3\binom{n}{4}p^4 + o(n^4)$ cycles of length four, i.e., if the number of 4-cycles is close to its expected value in a random graph with edge density p. Skokan and Thoma [29] showed that any complete bipartite graph $K_{a,b}$ with $a, b \geq 2$ has the analogous property, i.e., a graph is quasirandom if the number of copies of $K_{a,b}$ is close to its expected value in a random graph with the same edge density. One of the major open problems in extremal combinatorics is the Forcing Conjecture by Conlon, Fox and Sudakov [9] asserting that all bipartite graphs with a cycle have this property.

We are interested in the same phenomenon for tournaments: a tournament H is *quasirandom-forcing* if the density of H in $(G_n)_{n\in\mathbb{N}}$ converging to the expected density of H in a random tournament is sufficient to guarantee the quasirandomness of the sequence. In particular, if the density of H converges to its expected density, then the density of every tournament converges to its expected density in a random tournament. Every transitive tournament T_k with $k \geq 4$ vertices is known to be quasirandom-forcing, see [12] and [23, Exercise 10.44], and Bucić, Long, Shapira and Sudakov [2] observed that every quasirandom-forcing tournament with seven or more vertices is transitive. On the other hand, Coregliano, Parente and Sato [11] showed that there is a non-transitive 5-vertex tournament F_5 that is quasirandom-forcing; the tournament F_5, which is called T_5^8 in [11], is depicted in Figure 1. Our main result asserts that there is no quasirandom-forcing tournament in addition to T_k, for $k \geq 4$, and F_5.

The paper is structured as follows. In Section 2, we recall from [1] classical results on the maximum numbers of cycles of length three and four in a tournament, which rule out the existence of a strongly connected quasirandom-forcing tournament with at most 4 vertices. In Section 3, we first show that every non-transitive quasirandom-forcing tournament must be strongly connected, hence we may focus on tournaments with 5 and 6 vertices only. We next show that every quasirandom-forcing 6-vertex tournament must be rigid and twin-free, which together with the results of Coregliano et al. [11] leaves a single 5-vertex tourna-

ment and exactly 14 tournaments with 6 vertices that are strongly connected and may be quasirandom-forcing. We analyze all these 15 tournaments in Section 4.

2 Preliminaries

In this section, we introduce notation and basic results used in the paper. We write $[k]$ for the set $\{1, \ldots, k\}$. A *tournament* is a graph G where each pair of vertices is joined by an edge oriented in one or the other direction; we write $|G|$ for the number of vertices of G. The *adjacency matrix* of a tournament is the matrix A with rows and columns indexed by the vertices of G such that its diagonal entries are zero, and $A_{uv} = 1$ and $A_{vu} = 0$ for every edge uv. A tournament is *rigid* if it has no non-trivial automorphism. Two vertices u and v in a tournament are referred to as *twins* if every out-neighbor of u possibly except for v is an out-neighbor of v and every out-neighbor of v possibly except for u is an out-neighbor of u. A tournament with no twins is said to be *twin-free*.

If G and H are tournaments, the *density* of H in G, which is denoted by $d(H, G)$, is the probability that $|H|$ randomly chosen vertices of G induce H; if $|H| > |G|$, we set $d(H, G) = 0$. A sequence $(G_n)_{n \in \mathbb{N}}$ of tournaments is *quasirandom* if

$$\lim_{n \to \infty} d(H, G_n) = \frac{k!}{|\mathrm{Aut}(H)|} \cdot 2^{-\binom{k}{2}}$$

for every tournament H, where $\mathrm{Aut}(H)$ is the group of automorphisms of H (note that the right side of the expression is the expected density of H in a random tournament with $n \geq |H|$ vertices). Finally, we say that a tournament H is *quasirandom-forcing* if every sequence $(G_n)_{n \in \mathbb{N}}$ of tournaments satisfying

$$\lim_{n \to \infty} d(H, G_n) = \frac{k!}{|\mathrm{Aut}(H)|} \cdot 2^{-\binom{k}{2}}$$

is quasirandom (only sequences satisfying $|G_n| \to \infty$ as $n \to \infty$ are considered). As we mentioned in Section 1, every k-vertex transitive tournament T_k, for $k \geq 4$, is quasirandom-forcing, and there is also a 5-vertex strongly connected tournament that is quasirandom-forcing (this is the tournament F_5 depicted in Figure 1).

We treat quasirandomness of tournaments in the language of theory of combinatorial limits, which associates (convergent) sequences of combinatorial structures with analytic limit objects. We refer the reader to the monograph by Lovász [24] for the treatment of the most studied case of graph limits, which readily translates to the setting of tournament limits (see [13, 33, 34]).

We say that a sequence $(G_n)_{n \in \mathbb{N}}$ of tournaments with $|G_n|$ tending to infinity is *convergent* if $d(H, G_n)$ converges for every tournament H. A *tournamenton* W is a measurable function $[0, 1]^2 \to [0, 1]$ such that $W(x, y) + W(y, x) = 1$ for all

$(x, y) \in [0, 1]^2$. The *density* of a k-vertex tournament H with vertices $v_1, \ldots, v_k$ in a tournamenton W, which is denoted by $d(H, W)$, is

$$d(H, W) = \frac{k!}{|\mathrm{Aut}(H)|} \int_{[0,1]^k} \prod_{\overrightarrow{v_i v_j} \in E(H)} W(x_i, x_j) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k, \tag{1}$$

where $E(H)$ is the set of (oriented) edges of H. For every convergent sequence $(G_n)_{n \in \mathbb{N}}$ of tournaments, there exists a tournamenton W such that the limit density of each tournament H in the sequence is equal to the density of H in W; we say that such W is a *limit* of the sequence $(G_n)_{n \in \mathbb{N}}$ and that the sequence $(G_n)_{n \in \mathbb{N}}$ *converges* to W. Conversely, for every tournamenton W, there exists a sequence of tournaments that converges to W.

The definition of a quasirandom-forcing tournament translates to the limit setting as follows.

Proposition 1. *A tournament H is quasirandom-forcing if every tournamenton W satisfying*

$$d(H, W) = \frac{k!}{|\mathrm{Aut}(H)|} \cdot 2^{-\binom{k}{2}}$$

is equal to $1/2$ almost everywhere.

Proposition 1 yields the following, which was also noted at the end of Section 2 in [2]. We state the proposition in the language of combinatorial limits.

Proposition 2. *Let H be a tournament that is not transitive. If there exists a tournamenton W such that W is not equal to $1/2$ almost everywhere and*

$$d(H, W) \geq \frac{k!}{|\mathrm{Aut}(H)|} \cdot 2^{-\binom{k}{2}},$$

then H is not quasirandom-forcing.

Proof. Let W be the tournamenton given by the statement. Let T be the following tournamenton, which is a limit of a sequence of transitive tournaments:

$$T(x, y) = \begin{cases} 1, & \text{if } x > y, \\ 1/2, & \text{if } x = y, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Further, we define a U_α for $\alpha \in [0, 1]$ as

$$U_\alpha(x, y) = \begin{cases} W(x, y), & \text{if } (x, y) \in [0, \alpha]^2, \text{ and} \\ T(x, y), & \text{otherwise.} \end{cases}$$

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Observe that, for any $\alpha \in [0, 1]$, U_α is not equal to $1/2$ almost everywhere. Since the tournament H is not transitive, we have $d(H, U_0) = d(H, T) = 0$. On the other hand, the assumption of the proposition yields that

$$d(H, U_1) = d(H, W) \geq \frac{k!}{|\mathrm{Aut}(H)|} \cdot 2^{-\binom{k}{2}}.$$

Since $d(H, U_\alpha)$ is a continuous function of $\alpha \in [0, 1]$, there exists $\alpha \in (0, 1]$ such that

$$d(H, U_\alpha) = \frac{k!}{|\mathrm{Aut}(H)|} \cdot 2^{-\binom{k}{2}}.$$

$\square$

A classical result on tournaments of Beineke and Harary [1] on Turán density of a cycle C_3 of length three translates to the language of tournament limits as follows: $d(C_3, W) \leq 1/4$ and the equality holds if and only if

$$\int_{[0,1]} W(x, y) \, \mathrm{d}y = \frac{1}{2}$$

for almost every $x \in [0, 1]$. Hence, the cycle C_3 is not quasirandom-forcing by Proposition 1. Let C_4 be the 4-vertex tournament obtained from the cycle of length four by adding two edges (note that all tournaments obtained in this way are isomorphic). The result of [1] on the Turán density of C_4, in the language of tournament limits, asserts $d(C_4, W) \leq 1/2$ and the equality can be attained. Hence, the tournament C_4 is not quasirandom-forcing by Proposition 2.

We next define a notion of a (weighted) step tournamenton, which is analogous to the notion of a step graphon. Informally speaking, a step tournamenton represents a large tournament such that its vertices can be split into a finite number of parts such that the tournament is quasirandom within each part and between the parts. The formal definition goes as follows. A matrix A is a *tournament matrix* if it is a square matrix, say of order k, with non-negative entries such that $A_{ij} + A_{ji} = 1$ for all $i, j \in [k]$. A vector w is *stochastic* if all its entries are non-negative and they sum to one. Let A be a $k \times k$ tournament matrix and w a k-dimensional stochastic vector. Further, let $V_1, \ldots, V_k$ be a partition of $[0, 1]$ into disjoint measurable sets such that the measure of V_i is w_i, $i \in [k]$. We define a tournamenton $W[A, w]$ as

$$W[A, w](x, y) = A_{i,j}$$

for every $(x, y) \in (V_i, V_j)$. A tournamenton W such that there exists a tournament matrix A and a (positive) stochastic vector w such that $W = W[A, w]$ is called a *weighted step tournamenton*. If $w_i = 1/k$ for all $i \in [k]$, we simply write $W[A]$ instead of $W[A, w]$. Finally, if H is a tournament, then the *blow-up* of H is

the tournamenton $W[A]$ where A is the adjacency matrix of H with $1/2$ on its diagonal.

Observe that the following formula holds for the density of H in $W[A, w]$:

$$d(H, W[A, w]) = \frac{1}{|\mathrm{Aut}(H)|} \sum_{f:V(H)\to[k]} \prod_{i\in V(H)} w_{f(i)} \prod_{\overrightarrow{v_i v_j}\in E(H)} A_{f(i),f(j)}, \qquad (2)$$

where k is the order of the matrix A. The identity (2) leads us to define $d^*(H, A, w)$ as follows.

$$d^*(H, A, w) = \sum_{f:V(H)\to[k]} \prod_{i\in V(H)} w_{f(i)} \prod_{\overrightarrow{v_i v_j}\in E(H)} A_{f(i),f(j)}. \qquad (3)$$

Again, if each entry of w is equal to $1/k$, we will simply write $d^*(H, A)$ instead of $d^*(H, A, w)$.

By combining Proposition 2, the definition of $d^*(H, A, w)$, and the identities (1) and (2), we obtain the following.

Proposition 3. *Let H be a k-vertex non-transitive tournament. If there exists an $\ell \times \ell$ tournament matrix A and an ℓ-dimensional positive stochastic vector w such that not all entries of A are equal to $1/2$ and*

$$d^*(H, A, w) \geq 2^{-\binom{k}{2}},$$

then H is not quasirandom-forcing.

3 General arguments

The purpose of this section is to establish the following two statements and use them to show that most 6-vertex tournaments are not quasirandom-forcing.

Proposition 4. *Let H be a non-transitive tournament. If H is not strongly connected, then H is not quasirandom-forcing.*

Proposition 5. *Let H be a non-transitive 6-vertex tournament. If H contains twins or has a non-trivial automorphism, then H is not quasirandom-forcing.*

Proof of Proposition 4. Let k be the number of vertices of H. Note that $k \geq 4$. For simplicity, we will write ρ for $2^{-\binom{k}{2}}$. Since the tournament H is not strongly connected, its vertices can be split into non-empty sets X_1 and X_2 such that all edges are oriented from X_1 to X_2; let k_1 and k_2 be the sizes of X_1 and X_2, respectively. For each $\alpha \in [0,1]$, consider the following tournament matrix and stochastic vector

$$A = \begin{pmatrix} 1/2 & 1 \\ 0 & 1/2 \end{pmatrix} \text{ and } w = (\alpha, 1-\alpha),$$

and set $W_\alpha = W[A, w]$. Our aim is to find an appropriate $\alpha \in (0, 1)$ so that we can apply Proposition 3 to W_α. Observe that

$$d^*(H, A, w) \geq \alpha^k \cdot \rho + \alpha^{k_1}(1 - \alpha)^{k_2} \cdot 2^{k_1 k_2} \cdot \rho + (1 - \alpha)^k \cdot \rho. \qquad (4)$$

Note that the inequality is strict if H has more than two strongly connected components. We show that $d^*(H, A, w) > \rho$ for some $\alpha \in (0, 1)$ in each of the following cases.

If $k_1 = 1$, we use the second and third term of (4) to lower bound $d^*(H, A, w)$ as follows:

$$d^*(H, A, w) > \alpha(1 - \alpha)^{k_2} \cdot \rho \cdot 2^{k_2} + (1 - \alpha)^k \cdot \rho$$
$$= \rho + \alpha \cdot (2^{k_2} - k)\rho + O(\alpha^2).$$

Since $k \geq 4$, it holds that $2^{k_2} - k > 0$, and we conclude that $d^*(H, A, w) > \rho$ for some positive α that is sufficiently small. The case $k_1 = k - 1$ is symmetric to the case $k_1 = 1$. Hence, it remains to analyze the case when $2 \leq k_1 \leq k - 2$.

If $2 \leq k_1 \leq k - 2$, we set $\alpha = 1/2$. It follows from (4) that

$$d^*(H, A, w) \geq 2^{-k} \cdot \rho + 2^{-k_1} \cdot 2^{-k_2} \cdot 2^{k_1 k_2} \cdot \rho + 2^{-k} \cdot \rho \geq \left(1 + 2^{1-k}\right) \cdot \rho,$$

where the last inequality holds since $k_1 k_2 \geq k_1 + k_2$. This concludes the proof. $\square$

We prove Proposition 5 by an argument similar to that used in [2] to observe that every quasirandom-forcing tournament with seven or more vertices is transitive.

Proof of Proposition 5. Let A be the adjacency matrix of H with $1/2$ on its diagonal. If H has a non-trivial automorphism, then $d^*(H, A) \geq 2 \cdot 6^{-6} > 2^{-15}$ as there are at least two choices of f in the sum in (3) for which the expression in the definition is non-zero. It follows that H is not quasirandom-forcing by Proposition 3.

We now consider the case that H has twins. Let $v_1, \ldots, v_6$ be the vertices of H and assume by symmetry that v_1 and v_2 are the twins. As in the previous case, it is enough to show that $d^*(H, A) \geq 2 \cdot 6^{-6}$. This time, observe that the innermost product in (3) is equal to one for the map f with $f(v_i) = i$ for all $i \in [6]$, and it is equal to $1/2$ for the two maps f satisfying $f(v_1) \in \{1, 2\}$, $f(v_1) = f(v_2)$ and $f(v_i) = i$, where $i \in \{3, 4, 5, 6\}$. $\square$

Proposition 4 implies that every quasirandom-forcing non-transitive tournament H is strongly connected. The classical results on the Turán density of C_3 and C_4 (see the discussion of these results in Section 2) yield that there is no such tournament H with three or four vertices, and the observation of Bucić et al. [2] yield that there are no such tournaments H with seven or more vertices. Hence,

we are left to analyze tournaments with five and six vertices. In the case of 5-vertex tournaments, the results of Coregliano et al. [11] imply that all 5-vertex strongly connected tournaments with the possible exception of two tournaments are not quasirandom-forcing. The two exceptional tournaments are F_5, which is depicted in Figure 1 and is quasirandom-forcing, and H_5, which is depicted in Figure 3 and is shown to be not quasirandom-forcing in the next section.

There are 55 non-transitive tournaments on 6 vertices, out of which 20 are not strongly connected, 29 contain twins, and 15 have a non-trivial automorphism (some tournaments have more than one of these properties); see Table 1. A SageMath [30] script that verifies the entries of Table 1 is available as an ancillary file on arXiv associated with the arXiv version of this manuscript [19].

By the discussion in the previous paragraph, Propositions 4 and 5 yield that 41 non-transitive 6-vertex tournaments are not quasirandom-forcing. We will analyze the remaining 14 tournaments, which are depicted in Figure 2, in the next section.

4 Specific constructions

In this section we provide two different types of arguments to rule out the remaining 15 tournaments from being quasirandom-forcing. Tournaments that we consider will be described by the upper-triangle part of their adjacency matrix, i.e., if A is the adjaceny matrix of a k-vertex tournament, then the tournament is described by

$$[A_{1,2} \cdots A_{1,k}, A_{2,3} \cdots A_{2,k}, \ldots, A_{k-2,k-1}A_{k-2,k}, A_{k-1,k}].$$

The remaining 5-vertex tournament, which is depicted in Figure 3, is described by $[0010, 001, 00, 0]$, We denote this tournament H_5 (this tournament is called T_5^{10} in [11]). The 14 remaining 6-vertex tournaments, which can also be found in Figure 2, are the following:

$$\begin{aligned}
&H_6^1 : [00010, 0000, 001, 00, 0], && H_6^2 : [00110, 0001, 000, 01, 0], \\
&H_6^3 : [00101, 0010, 000, 00, 0], && H_6^4 : [00100, 0010, 001, 00, 0], \\
&H_6^5 : [00100, 0010, 000, 01, 0], && H_6^6 : [00100, 0010, 000, 00, 1], \\
&H_6^7 : [00100, 0011, 001, 00, 0], && H_6^8 : [00100, 0011, 000, 01, 0], \\
&H_6^9 : [00111, 0010, 000, 00, 0], && H_6^{10} : [00111, 0010, 001, 00, 0], \\
&H_6^{11} : [00010, 0101, 000, 00, 0], && H_6^{12} : [01010, 0001, 000, 00, 0], \\
&H_6^{13} : [01010, 0000, 001, 00, 0], && H_6^{14} : [01010, 0000, 000, 01, 0].
\end{aligned}$$

A	B	C	D	E	Tournament
•	•	•			$00000,0000,000,01,0$
•		•			$00010,0000,000,00,0$
		•			$00011,0000,000,00,0$
		•			$00010,0001,000,00,0$
			•		$00010,0000,001,00,0$ H_6^1
		•			$00010,0000,000,01,0$
		•			$00010,0000,000,00,1$
•		•			$00000,0010,000,00,0$
		•			$00001,0010,000,00,0$
•		•			$00000,0011,000,00,0$
•					$00000,0010,001,00,0$
•					$00000,0010,000,01,0$
•		•			$00000,0010,000,00,1$
•	•				$00000,0011,001,00,0$
•	•	•			$00000,0000,010,00,0$
		•			$00001,0000,010,00,0$
•	•				$00000,0001,010,00,0$
•		•			$00000,0000,011,00,0$
•		•			$00100,0000,000,00,0$
•		•			$00110,0000,000,00,0$
		•			$00111,0000,000,00,0$
		•			$00110,0001,000,00,0$
		•			$00110,0000,001,00,0$
		•			$00110,0000,000,01,0$
		•			$00110,0000,000,00,1$
		•			$00111,0000,001,00,0$
	•	•			$00110,0001,001,00,0$
	•	•			$00111,0000,000,01,0$

A	B	C	D	E	Tournament
				•	$00110,0001,000,01,0$ H_6^2
•					$00100,0010,000,00,0$
				•	$00101,0010,000,00,0$ H_6^3
			•		$00100,0011,000,00,0$
				•	$00100,0010,001,00,0$ H_6^4
				•	$00100,0010,000,01,0$ H_6^5
				•	$00100,0010,000,00,1$ H_6^6
	•				$00101,0010,001,00,0$
				•	$00100,0011,001,00,0$ H_6^7
				•	$00100,0011,000,01,0$ H_6^8
•	•				$00110,0010,000,00,0$
				•	$00111,0010,000,00,0$ H_6^9
	•				$00111,0011,000,00,0$
				•	$00111,0010,001,00,0$ H_6^{10}
•	•	•			$00000,0100,000,00,0$
•	•				$00010,0100,000,00,0$
•	•				$00011,0100,000,00,0$
				•	$00010,0101,000,00,0$ H_6^{11}
	•				$00010,0100,000,00,1$
•	•	•			$01000,0000,000,00,0$
•	•				$01000,0000,000,01,0$
•					$01010,0000,000,00,0$
			•		$01011,0000,000,00,0$
				•	$01010,0001,000,00,0$ H_6^{12}
				•	$01010,0000,001,00,0$ H_6^{13}
				•	$01010,0000,000,01,0$ H_6^{14}
				•	$01010,0000,000,00,1$

Table 1: The table indicates for each 6-vertex non-transitive tournament the way in which it was shown to be not quasirandom-forcing as follows. A: by Proposition 4 because it is not strongly connected, B: by Proposition 5 because it has a non-trivial automorphism, C: by Proposition 5 because it has twins, D: Subsection 4.1, and E: Subsection 4.2. The tournaments are described by the upper-triangle part of their adjacency matrix, see the beginning of Section 4, and by the notation used for the tournament if a specific notation has been introduced.
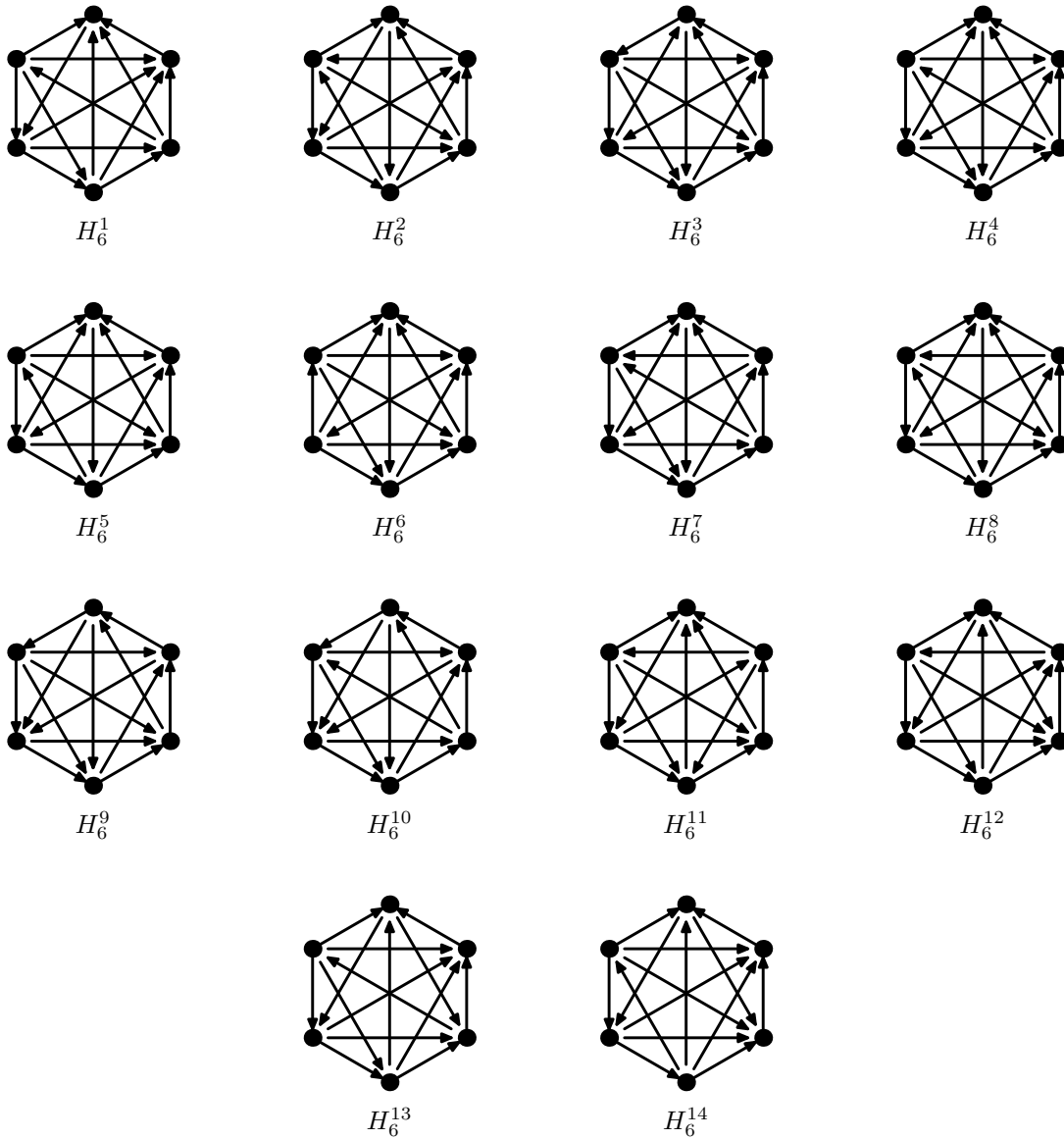
H_6^1 H_6^2 H_6^3 H_6^4

H_6^5 H_6^6 H_6^7 H_6^8

H_6^9 H_6^{10} H_6^{11} H_6^{12}

H_6^{13} H_6^{14}
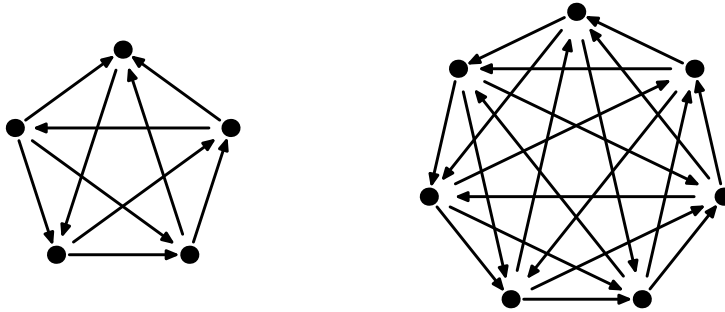
Figure 2: The tournaments $H_6^1, \ldots, H_6^{14}$.

Figure 3: The tournaments H_5 and S_7.

4.1 Blow-ups

We start this subsection with the following statement, which can also be found in [2]. Let $n(H, S)$ be the number of copies of a tournament H in a tournament S, i.e., $n(H, S) = d(H, S) \cdot \binom{|H|}{|S|}$.

Proposition 6. *Let H be a non-transitive k-vertex tournament. If there exists an s-vertex tournament S, $s > k$, such that $n(H, S) \geq s^k \cdot 2^{-\binom{k}{2}}$, then H is not quasirandom-forcing.*

Proof. Let A be the adjacency matrix of H with $1/2$ on its diagonal. Note that $d^*(H, A) \geq n(H, S) \cdot s^{-k}$. Since $n(H, S) \geq s^k \cdot 2^{-\binom{k}{2}}$, Proposition 3 yields that H is not quasirandom-forcing. □

We consider tournaments S_7, S_{11} and S_{15} with 7, 11 and 15 vertices, respectively; we remark that the tournaments S_{11} and S_{15} have been identified by a heuristic computer search maximizing the number of copies of tournaments H_6^i.

S_7 :[001011, 00101, 0010, 001, 00, 0],

S_{11} :[1100110001, 101001011, 11010101, 0001101, 100011, 00110, 1000, 100, 10, 0],

S_{15} :[01010100100110, 0011110000001, 010001001101, 10011000010, 1011101010,
110110010, 11101001, 1110001, 010110, 11110, 0101, 001, 10, 0].

The tournament S_7 is depicted in Figure 3. It is interesting to note that $n(H_5, S_7) = 21$, i.e., every 5-tuple of vertices of S_7 induces H_5, and the tournaments S_7 and S_{11} are Paley tournaments [14, 18, 28]. In particular, the adjacency matrix of S_7 is the incidence matrix of the points and lines of the Fano plane. Since $n(H_5, S_7) = 21$, Proposition 6 implies that H_5 is not quasirandom-forcing. It also holds that $n(H_6^i, S_{11}) = 55$ for $i \in \{2, 3, 4, 8, 10, 11, 13\}$ and $n(H_6^i, S_{15}) = 357$ for $i \in \{5, 12\}$. Proposition 6 implies that none of the tournaments H_6^i, $i \in \{2, 3, 4, 5, 8, 10, 11, 12, 13\}$, are quasirandom-forcing.

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4.2 Step tournamentons with variable weights

It remains to analyze the tournaments H_6^i for $i \in \{1, 6, 7, 9, 14\}$. We consider the following three tournament matrices, each of which is a function of $x \in [-1/2, 1/2]$, and show that there exists $x \neq 0$ such that Proposition 3 can be applied.

$$A_x = \begin{pmatrix} 1/2 & 1/2 - x \\ 1/2 + x & 1/2 \end{pmatrix},$$

$$B_x = \begin{pmatrix} 1/2 & 1/2 - x & 1/2 + x \\ 1/2 + x & 1/2 & 1/2 - x \\ 1/2 - x & 1/2 + x & 1/2 \end{pmatrix},$$

$$C_x = \begin{pmatrix} 1/2 & 1/2 - x & 1/2 + x & 1/2 - x \\ 1/2 + x & 1/2 & 1/2 - x & 1/2 - x \\ 1/2 - x & 1/2 + x & 1/2 & 1/2 - x \\ 1/2 + x & 1/2 + x & 1/2 + x & 1/2 \end{pmatrix}.$$

We next compute the densities of H_6^{14}, H_6^9 and H_6^6.

$$d^*(H_6^{14}, A_x) = \frac{1}{32768} + \frac{x^2}{8192} - \frac{5x^4}{16384} - \frac{9x^6}{4096} - \frac{7x^8}{4096},$$

$$d^*(H_6^9, B_x) = \frac{1}{32768} + \frac{x^4}{3072} - \frac{x^6}{216} - \frac{5x^8}{5184} + \frac{13x^{10}}{486} - \frac{x^{12}}{324},$$

$$d^*(H_6^6, C_x) = \frac{1}{32768} + \frac{3x^3}{32768} - \frac{81x^4}{131072} - \frac{3x^5}{8192} + \frac{27x^6}{65536} - \frac{63x^8}{131072} + \frac{15x^{12}}{1024}.$$

The maximum of each of the three polynomials above is larger than $2^{-15} \approx 0.000030518$. In particular, the first one is larger than 0.000037337 for $x = 0.30721$, the second is larger than 0.000030757 for $x = 0.21740$, and the third is larger than 0.000030544 for $x = 0.10418$. Hence, Proposition 3 yields that none of the tournaments H_6^{14}, H_6^9 and H_6^6 are quasirandom-forcing. Since the tournament H_6^7 can be obtained from H_6^9 by reversing the orientation of all its edges, it follows that $d^*(H_6^9, B_x) = d^*(H_6^7, B_{-x})$. Similarly, the tournament H_6^1 can be obtained from H_6^6 by reversing the orientation of all its edges and $d^*(H_6^6, C_x) = d^*(H_6^1, C_{-x})$. Hence, $d^*(H_6^7, B_{-x}) > 2^{-15}$ for $x = 0.21740$ and $d^*(H_6^1, C_{-x}) > 2^{15}$ for $x = 0.10418$, and neither H_6^7 nor H_6^1 is quasirandom-forcing by Proposition 3.

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