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Asymptotic Properties of Quantum Markov Processess

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Bibliographic entry

Abstrakt

Kvantové Markovovské procesy hrají důležitou roli ve fyzice, zejména v teorii otevřených kvantových systémů. Protože je velice obtížné nalézt analytické řešení časového vývoje obecného Markovovského procesu, většina studií se uchyluje k numerickým simulacím. Podstatná část vlastností kvantových Markovovských procesů však může být studována v asymptotickém režimu daného procesu. Hlavním cílem této práce je důkladné prozkoumání asymptotických vlastností konečnědimenzionálních stopu nezvětšujících homogenních kvantových Markovovských procesů (obou poižívaných tříd, diskrétních kvantových Markovovských řetízků a spojitých kvantových Markovovských dynamických semigrup) jak ve Schrödingerově, tak v Heisenbergově obrazu. Práce ustanovuje fundamentální teorém specifikující asymptotický prostor (tzv. atraktorový prostor) a definuje bohatou třídu transformací mezi atraktory kvantových Markovovských procesů v obou obrazech. Dále je v práci odvozen strukturní teorém ukazující jak interní struktura generátorů kvantových Markovovských procesů určuje tvar atraktorů v obou obrazech. Na základě odvozených algebraických vlastností atraktorových prostorů lze odvodit dvě charakterizace všech asymptotických a stacionárních vztahů, stejně jako charakterizaci všech asymptotických trajektorií. Všechny prezentované charakterizace připomínají svojí formou dobře známé Gibbsovské stavy používané ve statistické mechanice. Tento výsledek poté umožňuje formulaci několika verzí Jaynesova principu, platného pro všechny uvažované stopu zachovávající kvantové Markovovské procesy. Všechny popsané verze Jaynesova principu jsou odvozeny z dynamiky daného procesu, což ukazuje, že Jaynesův princip je dynamickým procesem a nikoliv důsledkem teorie informace. Získané znalosti jsou pak použity ke studiu tzv. EET sítě, pro demostraci síly vybudované teorie.

Abstract

Quantum Markov processes play an important role in physics and the theory of open quantum systems in particular. As it is very challenging to find an analytical solution to the evolution of a general Markov process, most of the studies limit themselves to numerical simulations. A significant number of important properties of quantum Markov processes can be studied in the asymptotic regime of the given process. The purpose of this thesis is to thoroughly investigate the asymptotic properties of finite dimensional trace-nonincreasing homogenous quantum Markov processes (both types, discrete quantum Markov chains and continuous quantum Markov dynamical semigroups) in both the Schrödinger and the Heisenberg picture. The thesis provides a fundamental theorem specifying the structure of the asymptotic space (so-called attractor space) and uncover a rich set of transformations between attractors of quantum Markov processes in both pictures. Next, it gives the structure theorem, showing how the internal structure of generators of quantum Markov processes determines attractors in both pictures. Furthermore, based on algebraic properties of attractor spaces, it gives two characterizations of all asymptotic and stationary states as well the characterization of all asymptotic trajectories. All presented representations strongly remind in form the well-known Gibbs states of statistical mechanics. This strong result then enables formulation of several versions of Jaynes principle, valid for all considered trace-preserving quantum Markov processes. All herein described versions of Jaynes principles are derived from the underlying Markovian dynamics, showing that the Jaynes principle for all considered quantum Markov processes is not driven by information theory, but by the dynamics of the system. Finally, gained knowledge is used to study so-called energy excitation transfer network, showcasing the power of the described theory.

Acknowledgement

The attractor method described within this thesis started to be developed more than 10 years ago by my supervisors Prof. Igor Jex and Ing. Jaroslav Novotný PhD. Back then, I was still an undergraduate student, working on my bachelor thesis. The problem of investigation of asymptotics of quantum Markov processes (back then limited to random unitary operations) was introduced to me at the start of my master studies and it continued to be developed through my PhD. studies, eventually resulting in a theory involving a general finite dimensional quantum Markov process. Tracing back the development through the years, I can only thank to my supervisors to let me be involved in development of such a coherent and compact theory which has a big potential in basic and applied research in many areas.

My involvement would of course be not possible without moral and material support of my parents, who must have shown a lot of patience not only during my university years, but mainly through elementary and high school studies as I have shown much less interest in my education than I should have.

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Contents

Chapter 1

Introduction

Physics originated from the human desire to understand causes of the phenomena manifested by Nature – translated from the Ancient Greek, the word physics literally means 'knowledge of the Nature'. There are many branches of physics, each focusing on certain set of phenomena sharing the common origin. The development of each branch was a long process of argument between theoretical hypotheses and experimental results – hypotheses provided predictions, which were afterwards either confirmed, or disproved by a carefully prepared experiments. If a hypothesis was disproved, either some modifications were made (e.g. model of atom) or it was completely renounced (e.g. existence of aether). This feedback process eventually led to a well established theory describing a large number of phenomena, supported by many experiments. However, in the beginning of the 20th century the few remaining unsolved physical problems (blackbody radiation, photoelectric effect etc.) led to the development of a completely new viewpoint on the laws of Nature. As a result, quantum physics was established. Describing the behavior of matter and radiation in atomic-scale, quantum physics often leads to counterintuitive results and disagreement with our everyday experience. Nevertheless, supported by countless experiments, it is widely regarded as the most accurate description of the most fundamental laws of Nature, considering the experimental resolution available to the present-day science.

Beside the urge for comprehension of the underlying physical laws, research is also motivated by possible applications. Similarly to a plethora of previously studied and thoroughly described physical phenomena, the understanding of quantum processes combined with the overall technological progress made it possible to utilize the gained knowledge for practical purposes. Quantum effects allowed to reduce the size of transistor down to nanometer scale, allowing to pack a large number of transistors in a small space, resulting in computational devices far more powerful than their predecessors. The laser, found in basically any contemporary computer device as a recording and reading device, used in medicine, robotics and metallurgy, the atomic clock, responsible for the most precise time measurement available, magnetic resonance imaging, these are just

some examples of technologies based on quantum effects.

One of the main goals of contemporary quantum science is to build an operational quantum computer - a device, which performs computational tasks exploiting quantum effects [1]. Compared to classical computers, the quantum computer will significantly reduce the computational time for many tasks, such as performing various search algorithms, simulation of complex systems in physics, chemistry, social sciences etc. [2]. Some of the possible applications also include factoring large integers into prime numbers in polynomial time. As the assumed difficulty of this task forms a basis of security protocols used in many present-day devices, quantum computers and the associated branch of quantum computing and information attracts also non-scientific community. The development of the quantum computer must deal with several issues, e.g. decoherence times, error correction, qubit control etc. To overcome these obstacles, it is vital to have a high understanding of underlying quantum processes.

1.1 Quantum systems

After developing the mathematical formalism of quantum physics, undergraduate courses devote a substantial amount of time to study properties of a simple quantum systems, such as free particle, or a particle in a simple external field, such as the linear harmonic oscillator, or a particle in the Coulomb potential. Such systems are called closed quantum systems. The commonly used formalism of wave mechanics allows to calculate all characteristics of closed quantum systems, e.g. possible energy levels or probabilities of measurement outcomes. It also makes it possible to calculate their time evolution, tracking the change of properties of such a system over the time. The time evolution of closed quantum systems is determined by a corresponding generator, i.e. the Hamiltonian of the system, which locally describes the time evolution of the system through the Schrödinger equation (or the equivalent von Neumann equation) [3]. As a result, the time evolution of closed quantum systems is reversible, i.e. the time evolution of the system can be always traced back to the initial state of the system.

However, real systems are in most cases in contact with their environment. The environment is typically another quantum system, often much larger than the system under investigation [4]. The mutual interaction typically introduces irreversible effects in the time evolution of the system of interest. These effects may be due to creation of correlations between the system and its environment, due to the destruction of correlations between the constituent parts of the system or because of other reasons. Additionally, the system may influence its environment, whose effect on the system can change with time. Due to this feedback process, the time evolution of the system generally depends on its whole previous history and as a result it is often impossible to describe the time evolution locally through some generator.

Irreversibility often induces a constrained set of allowed states compared to the initial one. This can be exploited in a number of useful applications

in fields of quantum optics, quantum information, quantum computation and other areas. However, the above mentioned features of the system-environment interaction often make the analysis of properties of such systems challenging. To overcome these difficulties, various approximations are typically made, allowing to describe properties of the system at least in certain, but quite general arrangements [5, 6]. An example of such approximation occurs, when it is possible to steer the system and the environment into a regime, in which the subsequent evolution of the system depends solely on its present state, e.g. if the effect of the system on the environment rapidly dissipates through the environment compared to the change of the state of the system due to its time evolution. Quantum systems, which undergo the time evolution in such a regime are called quantum Markov processes (QMPs) [2, 4, 5, 6, 7]. Since for QMPs the environment is unaffected by the system of interest, QMP can be associated with a certain generator, locally describing its time evolution.

QMPs have a strong presence in modern science, being often employed both in the investigation of the fundamental concepts of the quantum theory and in practical applications. They are often used to model the interaction of matter with electromagnetic radiation [4, 6], effects imposed by noisy environments such as energy dissipation, dephasing, population decay and decoherence [8, 35] as well as accompanying processes of these effects, i.e. relaxation, equilibration or thermalization of quantum systems [36, 37]. On the other hand, carefully engineered QMPs are able of achieving many practical goals in areas concerning quantum optics, quantum information, quantum computation and quantum communication [29, 30]. The study of information-preserving structures generated by QMPs [33, 34] such as decoherence-free subspaces (DFSs) [31, 32] resulted in a number of applications including state preparation, manipulation and storage [38, 39, 40], controllability of quantum systems [41, 42, 43], environment assisted quantum transport [44, 45], synchronization of clocks of quantum systems [46], circuit QED [47, 48], liquid state NMR [49, 50, 51], trapped ions [52, 53, 54, 55, 56, 57], engineered phase transitions [58, 59], quantum error correction [60], matrix product states [61, 62], entanglement renormalization [63, 64] etc. It is also worth to mention that the scope of QMPs reaches beyond the field of contemporary physics, influencing related fields such as biology [65] and others.

Most of the above mentioned applications employ QMPs with a generator which does not depend on time. The time evolution between times t_1 and t_2 then depends only on the difference $t = t_2 - t_1$ and such QMPs, called homogeneous QMPs will be the focus of this thesis. Homogeneous QMPs can be divided into two classes - discrete quantum Markov chains (QMCHs) and continuous quantum Markov dynamical semigroups (QMDSs). Historically, the class of QMDS received more attention, following the breakthrough result of Lindblad and Gorran, Kossakowski, Sudarshan [20, 21], who independently provided a representation of generators of QMDS. Due to this effort, the class of QMDS has applications over the whole width of fields listed above. The importance of QMCHs became obvious with the expansion of the field of quantum information and quantum computation in recent years, in which the evolution is often

described as an effect of some black box device. QMCHs utilize the Kraus form [22, 23] of the generator.

1.2 The asymptotic evolution

For closed quantum systems, one of standard procedures of investigation of the time evolution consists of the diagonalization of the corresponding generator, i.e. the Hamiltonian and subsequent solution of the evolution in terms of its eigenvectors and their dual vectors. Due to hermicity of the Hamiltonian, there is a clear algebraic relation between its eigenvectors and their corresponding dual vectors, which further simplifies the investigation of the time evolution. This procedure cannot be generally applied for QMPs, as the relevant generators are often not normal and thus the diagonalization in some orthonormal basis is not guaranteed. In addition, the algebraic relation of the eigenvectors of the generator with their dual vectors is often obscure and therefore the complete understanding of evolution under QMPs additionally requires their investigation in the Heisenberg picture.

Due to difficulties in solving the time evolution, most of the applied research concerning QMPs limits itself to numerical results and/or partial solution, consisting of time evolution of expectation values of certain observable quantities. Although these results give a valuable insight into properties of QMPs, information obtained in such a way cannot compete with having a complete picture provided by the time evolution in a closed form. Fortunatelly, in many of the mentioned applications of QMPs, it is sufficient to investigate their asymptotic regime, i.e. the time evolution for sufficiently long times. This restriction to the asymptotic regime lacks some of the above mentioned difficulties and it is therefore easier to analyze. As a result, there exist quite a general reviews concerning (not only) asymptotic properties of QMPs [24], the asymptotics is fully described in special case of unital QMCHs [66], the basic building blocks of asymptotic quantum states of QMDSs are described [67, 68] and the special case of nonfaithful QMPs were studied [69, 70]. Nevertheless, the complete picture is still unavailable.

The asymptotic evolution is important as it still holds an answers to many questions listed above, e.g. decoherence protected states, equilibration, transport efficiency, synchronization of subsystems etc. The asymptotic regime is fully determined by the peripheral spectrum of the QMP and corresponding eigenvectors (called attractors) and their dual eigenvectors and thus the knowledge of the whole spectrum and eingenspace associated with the generator of the QMP is not needed. It is therefore important to understand what structures attractors form and how do they depend on the details of dynamics, i.e. the precise form of the corresponding generator.

The goal of this thesis is to provide a procedure to obtain solutions of the evolution of finite-dimensional QMPs in the asymptotic regime, analogous to the procedure used for solution of the time evolution of closed quantum systems. Properties of resulting structures then allow to cast obtained asymptotic

states of QMPs into an alternative form, using real observables on the system – constants of motion. The resulting representation, which is an extension of commonly used generalized Gibbs states, is elegant and it delivers an interesting link between the quantum physics and statistical physics [9], provided by a derivation of the Jaynes principle for QMPs from the underlying dynamics.

The presented formalism of the asymptotic evolution through attractor spaces possesses many strengths and advantages. It exhibits a lot of freedom in the definition of the system, allowing to study many interesting situations, such as loss of control in quantum systems, asymptotic properties of subsystems etc., it offers a straightforward application to scalable systems, such as quantum networks and quantum walks. Furthermore, it also results in a neat expression of physically relevant quantities such as equilibrium states and constants of motion, hinting a deep connection with the statistical physics.

1.3   The outline of the thesis

The thesis is structured as follows. Chapters 2 and 3 provide mostly technical supplementary material. In the chapter 2, mathematical structures and physical concepts used in the thesis are summarized. The formal introduction of the two discussed classes of QMPs, their generators and spectral properties is made in the chapter 3.

Chapter 4 is devoted to the construction of the attractor formalism of the asymptotic evolution of QMPs. Focusing on a so-called faithful QMPs, first it is shown that the asymptotic evolution takes places exclusively in the attractor space and the dependence of the asymptotic state on the initial state in terms of attractors and their duals is studied. Next, a set of algebraic equations, called the attractor equations is derived. These equations can be advantageously used for the construction of attractor spaces. Basic relation between attractors in the Schrödinger picture and their duals, i.e. attractors in the Heisenberg picture is introduced. The description of the asymptotic evolution is concluded by proving that the evolution in the asymptotic regime is reversible in a certain sense. Finally, all results are extended for a general QMP.

Chapter 5 builds on results of the previous chapter. First, the relation between attractors in the Schrödinger picture and attractors in the Heisenberg picture is generalized, resulting in a whole family of relations between these two sets of attractors. Furthermore, the set of constants of motion is identified within the attractor space in the Heisenberg picture. Combining these outcomes results in a new, elegant expression of the asymptotic states in terms of constants of motion, extending the commonly used class of generalized Gibbs states. Finally, the connection between QMPs and statistical physics is made by a formulation of several distinct versions of Jaynes principle for all QMPs, which differ in an input information about the system.

In chapter 6 an example in form of the so-called EET network is studied in several interesting regimes, to showcase the power of the presented theory. Summary and outlook are left to chapter 7.

Chapter 2

Basic concepts

Before diving into the main topic of the thesis, it is important to take some time to introduce the mathematical and physical concept used through the thesis. From a practical point of view, some base level of knowledge, from which the theory is developed, needs to be taken for granted. Nevertheless, even some of the most basic concepts deserve an introduction, especially if they are directly used within the main body of the thesis. For this reason, some of elementary concepts (e.g. orthogonality) are established, while others (e.g. linear independence) are used without an introduction, to keep the amount of the supportive material at the necessary minimum.

The first two sections of this chapter serve to establish the notation and basic mathematical structures used through the thesis - finite dimensional Hilbert spaces and operators on finite dimensional Hilbert spaces, without putting much stress on the context with the physical concepts they represent.

The third section is devoted to the description of the time evolution in the scope of quantum physics from a very general and abstract point of view as maps fulfilling certain mathematical properties. In the following chapters, this approach allows to develop a very versatile theory of asymptotic evolution for QMPs, suitable for a wide spectrum of further theoretical developments and practical applications.

The fourth section then introduces the Jaynes principle [71, 72] and its more general variations. The importance of the Jaynes principle stands out mainly within the scope of the statistical physics, where its reasoning serves as a justification for the form of equilibrium distributions of Gibbs and Boltzmann. The Jaynes principle found its applications also in quantum physics, resulting in the so-called generalized Gibbs states, describing equilibrium states of systems with limited amount of information available.

Last section introduces some of the fundamental concepts of graph theory, with focus put on the directed graphs. These are often employed in description of quantum networks with asymmetric interaction, such as the incoherent EET network studied in the chapter 6.

2.1   Hilbert spaces

Each quantum system is associated with a certain **Hilbert space** $\mathcal{H}$ - a closed vector space equipped with an **inner product** $\langle \cdot | \cdot \rangle$. Elements of a Hilbert space $\mathcal{H}$ are called **vectors** or **kets** and they are denoted as $|x\rangle \in \mathcal{H}$. Each Hilbert space is associated with a dual space $\mathcal{H}^*$, whose elements, called **dual vectors** or **bras** are denoted as $\langle x| \in \mathcal{H}^*$. Any finite dimensional Hilbert space $\mathcal{H}$ fulfils $\mathcal{H}^* \simeq \mathcal{H}$. Denoting $N = \mathsf{Dim}\ \mathcal{H}$ the dimension of the Hilbert space $\mathcal{H}$, any set $\mathcal{V} = \{\ |x_j\rangle\ \}_{j=1}^N$ of linearly independent vectors is called a **basis** of the Hilbert space $\mathcal{H}$. Having a fixed basis $\mathcal{V}$ of the Hilbert space $\mathcal{H}$, any element $|x\rangle \in \mathcal{H}$ can be represented by a ordered sequence of numbers $a_n \in \mathbb{C}$ as $|x\rangle = (a_1, \ldots, a_N)^T$. The corresponding dual vector $\langle x| \in \mathcal{H}^*$ then reads $\langle x| = (\bar{a}_1, \ldots, \bar{a}_N)$, with $\bar{a}$ denoting complex conjugation of number $a \in \mathbb{C}$.

The inner product of two kets $|x\rangle, |y\rangle \in \mathcal{H}$ is denoted as $\langle x|y\rangle$. Having a basis $\mathcal{V} = \{\ |x_j\rangle\ \}_{j=1}^N$ of the Hilbert space $\mathcal{H}$, in which these kets are represented as $|x\rangle = (a_1, \ldots, a_N)^T$ and $|y\rangle = (b_1, \ldots, b_N)^T$, their inner product $\langle x|y\rangle$ reads

$$\langle x|y\rangle = \sum_{j=1}^N \bar{a}_j b_j.$$

If $\langle x|y\rangle = 0$, vectors $|x\rangle$ and $|y\rangle$ are called **orthogonal**. If $\langle x|x\rangle = 1$, the vector $|x\rangle$ is called **normalized** and the real number $||\,|x\rangle\,|| = \sqrt{\langle x|x\rangle}$ is called the **norm** of the vector $|x\rangle$ induced by the inner product $\langle \cdot | \cdot \rangle$. Two normalized orthogonal vectors are called **orthonormal**. A set $\mathcal{V} \subset \mathcal{H}$ is called orthonormal, if each pair of elements from $\mathcal{V}$ is orthonormal. If the basis $\mathcal{V}$ of the Hilbert space $\mathcal{H}$ forms an orthonormal set, it is called the **orthonormal basis**.

A **dual basis** to the basis $\mathcal{V} = \{\ |x_j\rangle\ \}$ is a basis $\mathcal{V}^* = \{\ |x^j\rangle\ \}$ of the Hilbert space $\mathcal{H}$ such that

$$\langle x^k|x_j\rangle = \delta_{jk}.$$

For $|x\rangle \in \mathcal{H}$, which reads $|x\rangle = (a_1, \ldots, a_N)^T$, the j-th coefficient a_j can be expressed via the dual basis as

$$a_j = \langle x^j|x\rangle.$$

If $\mathcal{V}$ is an orthonormal basis of the Hilbert space $\mathcal{H}$, the dual basis fulfils $\mathcal{V}^* = \mathcal{V}$.

Consider a quantum system comprised of two parts A and B, each associated with its own Hilbert space $\mathcal{H}_A$, resp. $\mathcal{H}_B$, the combined system AB is associated with the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. If $|x_A\rangle \in \mathcal{H}_A$ and $|x_B\rangle \in \mathcal{H}_B$, their product is denoted as $|x_A\rangle \otimes |x_B\rangle \equiv |x_A\rangle\,|x_B\rangle \in \mathcal{H}_{AB}$. Having an orthonormal basis $\mathcal{V} = \{\ |x_j\rangle\ \}_{j=1}^N$ of a Hilbert space $\mathcal{H}$, the so-called maximally entangled state on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ is the ket $|\Omega\rangle$ defined as

$$|\Omega\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N |x_j\rangle\,|x_j\rangle. \tag{2.1}$$

There are many applications of ket $|\Omega\rangle$. For purpose of this thesis, it will be important for definition of conditionally completely positive maps later in this chapter.

2.2 Linear operators on Hilbert spaces

A Hilbert space $\mathcal{H}$ is equipped with a space of linear operators, denoted as $\mathsf{B}(\mathcal{H})$. This space consists of all linear maps $A : \mathcal{H} \to \mathcal{H}$. Having a basis $\mathcal{V} = \{\, |\, x_j\rangle \,\}$ of the Hilbert space $\mathcal{H}$, any linear operator $A \in \mathsf{B}(\mathcal{H})$ can be written in the basis $\mathcal{V}$ as

$$A = \sum_{j,k} A_{jk} |x_j\rangle \langle x^k| , \ \ A_{jk} = \langle x_j|A|x^k\rangle , \qquad (2.2)$$

with $\mathcal{V}^* = \{\, |\, x^j\rangle \,\}$ being the dual basis of the basis $\mathcal{V}$. Using the basis $\mathcal{V}$, the operator A can be written as $N \times N$ matrix as

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \ldots & A_{NN} \end{pmatrix} .$$

Each operator $A \in \mathsf{B}(\mathcal{H})$ is associated with its **adjoint operator** with respect to the standard inner product $A^\dagger$ defined as

$$A^\dagger = \sum_{j,k} \overline{A}_{kj} |x_j\rangle \langle x^k| . \qquad (2.3)$$

The **trace** $\mathsf{Tr}[A]$ of an operator $A \in \mathsf{B}(\mathcal{H})$ given by (2.2) is defined as

$$\mathsf{Tr}[A] = \sum_j A_{jj}.$$

The trace of an operator $A \in \mathsf{B}(\mathcal{H})$ is an important characteristics of the operator A, as it is invariant with respect to change from one orthonormal basis $\mathcal{V}_1$ to another orthonormal basis $\mathcal{V}_2$.

An important linear operator is the **identity operator** I, defined as

$$I = \sum_j |x_j\rangle \langle x^j| ,$$

for any basis $\mathcal{V} = \{\, |\, x_j\rangle \,\}$. This operator leaves any ket $|x\rangle \in \mathcal{H}$ unchanged. An operator $A \in \mathsf{B}(\mathcal{H})$ is termed **invertible**, if there is an operator $A^{-1} \in \mathsf{B}(\mathcal{H})$ such that $AA^{-1} = A^{-1}A = I$. The operator A^{-1} is called the **inverse** of the operator A.

The **commutator** of two operators $A, B \in \mathsf{B}(\mathcal{H})$ is an operator $[A, B] \in \mathsf{B}(\mathcal{H})$ defined as

$$[A, B] = AB - BA.$$

If two operators $A, B \in \mathsf{B}(\mathcal{H})$ fulfil $[A, B] = 0$, i.e. if $AB = BA$, it is said that A and B **commute**.

The space $\mathsf{B}(\mathcal{H})$ forms a Hilbert space on its own, with the standard choice of the inner product being the **Hilbert-Schmidt** inner product $(\cdot, \cdot)_{\mathrm{HS}}$, defined as

$$(A, B)_{\mathrm{HS}} = \mathsf{Tr}\left[A^\dagger B\right], \ \forall A, B \in \mathsf{B}(\mathcal{H}).$$

The induced Hilbert-Schmidt norm $||\cdot||_{\mathrm{HS}}$ reads $||A||_{\mathrm{HS}} = \sqrt{\mathsf{Tr}\left[A^\dagger A\right]}$, $A \in \mathsf{B}(\mathcal{H})$. Another often used norm on the Hilbert space $\mathsf{B}(\mathcal{H})$ is the **operator norm** $||\cdot||$, defined as

$$||A|| = \sup_{|x\rangle \in \mathcal{H}} \frac{||A\,|x\rangle\,||}{||\,|x\rangle\,||}, \ A \in \mathsf{B}(\mathcal{H}).$$

An **eigenvalue** of an operator $A \in \mathsf{B}(\mathcal{H})$ is a number $\lambda \in \mathbb{C}$, which can be associated with $|x_\lambda\rangle \in \mathcal{H}$ such that

$$A\,|x_\lambda\rangle = \lambda\,|x_\lambda\rangle.$$

The vector $|x_\lambda\rangle$ is then called the **eigenvector** (resp. **right-eigenvector**) of the operator A corresponding to the eigenvalue λ. The set of all eigenvalues of the operator A is the **spectrum** of the operator and it is denoted as $\sigma(A)$. The real number $\mathsf{R}(A) = \max\limits_{\lambda \in \sigma(A)} \{\,|\lambda|\,\}$ is called the spectral radius of the operator A. All eigenvalues $\lambda \in \sigma(A)$, such that $|\lambda| = \mathsf{R}(A)$ form a subset of the spectrum labeled the **peripheral spectrum** of the operator A.

Eigenvectors of the operator A corresponding to the eigenvalue λ form a subspace $\mathsf{Ker}(A - \lambda I) \subset \mathcal{H}$ of dimension equal to $\mathsf{d}(\lambda)$, with a basis denoted as $\mathcal{V}_\lambda = \{\,|\,x_{\lambda,j}\rangle\,\}_{j=1}^{\mathsf{d}(\lambda)}$. This subspace is called the **eigenspace**, or the **kernel** corresponding to the eigenvalue λ. The kernel $\mathsf{Ker}(A)$ is also named the **null space** of the operator A. The complement of the null space of the operator $A \in \mathsf{B}(\mathcal{H})$ to the Hilbert space $\mathcal{H}$ is called the **support** of the operator A and it is denoted as $\mathsf{Supp}(A)$. The dimension of the support of the operator $A \in \mathsf{B}(\mathcal{H})$, denoted as $\mathsf{Rank}\,A$ is called the **rank** of the operator A.

Operator $A \in \mathsf{B}(\mathcal{H})$ is **diagonalizable**, if the set of all linearly independent eigenvectors $\mathcal{V} = \bigcup\limits_{\lambda \in \sigma(A)} \mathcal{V}_\lambda$ of the operator A forms a basis of the Hilbert space $\mathcal{H}$. In the basis $\mathcal{V}$, the operator A takes the form

$$A = \sum_{\lambda \in \sigma(A)} \sum_{j=1}^{\mathsf{d}(\lambda)} \lambda\,|x_{\lambda,j}\rangle\,\langle x^{\lambda,j}|.$$

The elements $|x^{\lambda,j}\rangle$ of the dual basis $\mathcal{V}^*$ are the eigenvectors of the operator $A^\dagger$ corresponding to the eigenvalue $\overline{\lambda} \in \sigma(A^\dagger)$. They are also called the **left-eigenvectors** of the operator A, as

$$A^\dagger \left| x^{\lambda,j} \right\rangle = \overline{\lambda} \left| x^{\lambda,j} \right\rangle \Leftrightarrow \left\langle x^{\lambda,j} \right| A = \lambda \left\langle x^{\lambda,j} \right|.$$

If the operator $A \in \mathsf{B}(\mathcal{H})$ is not diagonalizable, it can be written in the **Jordan normal form** – there exists a basis $\mathcal{V} = \{ |x_{\lambda,k,j}\rangle$ such that A, written in the basis $\mathcal{V}$ takes the form

$$A = \bigoplus_{\lambda \in \sigma(A)} \sum_{k=1}^{\mathsf{d}(\lambda)} \mathcal{J}_k(\lambda),$$

with $\mathcal{J}_k(\lambda)$ being a **Jordan block** corresponding to the eigenvalue λ, defined as

$$\mathcal{J}_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

Each Jordan block $\mathcal{J}_k(\lambda)$ is associated with a single eigenvector $|x_{\lambda,k,1}\rangle$, with the remaining elements of the Jordan basis $|x_{\lambda,k,j}\rangle$, $j \in \{ 2, \dots, \mathsf{Dim}(\mathcal{J}_k(\lambda)) \}$ being so-called generalized eigenvectors, fulfilling

$$(A - \lambda I)^j \left| x_{\lambda,k,j} \right\rangle = 0.$$

Consider the operator $G \in \mathsf{B}(\mathcal{H})$, whose elements are defined as

$$G^{\mu l j}_{\lambda k i} = \left\langle x_{\lambda,k,i} | x_{\mu,l,j} \right\rangle.$$

The operator G is invertible and its inverse can be used to find the dual vectors $|x^{\lambda,k,i}\rangle$ [10] as

$$\left| x^{\lambda,k,i} \right\rangle = \sum_{\mu,l,j} (G^{-1})^{\mu l j}_{\lambda k i} \left| x_{\mu,l,j} \right\rangle. \tag{2.4}$$

Operator $A \in \mathsf{B}(\mathcal{H})$ is called **normal**, if it commutes with its adjoint, i.e.

$$\left[A, A^\dagger \right] = 0.$$

Any normal operator A can be diagonalized in an orthonormal basis, i.e.

$$A = \sum_{\lambda,j} \lambda \left| x_{\lambda,j} \right\rangle \left\langle x_{\lambda,j} \right|, \quad \left\langle x_{\lambda,j} | x_{\mu,k} \right\rangle = \delta_{\lambda\mu} \delta_{jk}. \tag{2.5}$$

The set of all normal operators contains a several important subclasses.

A normal operator A is called **Hermitian**, if it fulfils $A = A^\dagger$. The set of all Hermitian operators on the Hilbert space $\mathcal{H}$ forms a real subspace $\mathsf{A}(\mathcal{H})$ of the space $\mathsf{B}(\mathcal{H})$:

$$\mathsf{A}(\mathcal{H}) = \{\, A \in \mathsf{B}(\mathcal{H}) \mid A = A^\dagger \,\}.$$

The subspace $\mathsf{A}(\mathcal{H})$ forms a C^*-algebra over $\mathbb{R}$ [83]. Using an orhonormal basis $\mathcal{V} = \{\, | \, x_j \rangle \,\}$ relations (2.2) and (2.3) implies $A_{kj} = \overline{A}_{jk}$. The spectrum of any Hermitian operator A fulfils $\sigma(A) \subset \mathbb{R}$. Furthermore, if a Hermitian operator A fulfils $\sigma(A) \subset [0, \infty]$ (resp. $\sigma(A) \subset (0, \infty)$), it is called **positive** (resp. **strictly positive**) and it is denoted as $A \geq 0$ (resp. $A > 0$). Hermitian operators $A \in \mathsf{A}(\mathcal{H})$ represent physically observable quantities and they are therefore also called **observables**.

An observable $\rho \in \mathsf{A}(\mathcal{H})$ is called a **quantum state**, if it is positive and with unit trace. Quantum states on the Hilbert space $\mathcal{H}$ form a convex subset of the subspace $\mathsf{A}(\mathcal{H})$, denoted $\mathsf{S}(\mathcal{H})$, i.e.

$$\mathsf{S}(\mathcal{H}) = \{\, \rho \in \mathsf{A}(\mathcal{H}) \mid \rho \geq 0, \mathsf{Tr}[\rho] = 1 \,\}.$$

Expressed in an arbitrary orthonormal basis basis $\mathcal{V} = \{\, | \, x_j \rangle \,\}$ as

$$\rho = \sum_{jk} \rho_{jk} \, | x_j \rangle \, \langle x_k | \, ,$$

the coefficients ρ_{jk} of the quantum state thus fulfil

$$|\rho_{jk}| \leq 1, \ \rho_{kj} = \overline{\rho}_{jk}, \ 0 \leq \ \rho_{jj} \leq 1, \ \text{and} \ \sum_j \rho_{jj} = 1.$$

The diagonal coefficients ρ_{jj} are called **populations**, the off-diagonal coefficients ρ_{jk}, $j \neq k$ are called **coherences**. The spectrum of any quantum state ρ fulfils $\sigma(\rho) \subset [0, 1]$.

There are several important quantum states worth to highlight. The **maximally mixed state** ρ_I on the Hilbert space $\mathcal{H}$ of dimension $\mathsf{Dim}\,\mathcal{H} = N$, is a quantum state defined as

$$\rho_I = \frac{1}{N} I.$$

The **generalized Gibbs state** $\rho \in \mathsf{S}(\mathcal{H})$ is another significant quantum state. Given a set of observables $\{\, A_j \,\} \subset \mathsf{A}(\mathcal{H})$ and the set of real parameters $\{\, \beta_j \,\} \subset \mathbb{R}$, a **generalized Gibbs state** $\rho \in \mathsf{S}(\mathcal{H})$ is defined as

$$
\begin{aligned}
\rho &= \frac{1}{Z} \exp\left[-\sum_j \beta_j A_j \right], \\
Z &= \mathsf{Tr}\left[\exp\left[-\sum_j \beta_j A_j \right] \right].
\end{aligned}
\tag{2.6}
$$

A generalized Gibbs state is an extension of the Gibbs states, also called the **thermal states**, for which the set of observables $\{\, A_j \,\}$ reduces to a single

observable H, called the **Hamiltonian**, representing the energy operator associated with the system. The corresponding parameter β is called the relative temperature.

Another important class of states are the **pure states**. A state $\rho \in \mathsf{S}(\mathcal{H})$ is called pure, if there is a ket $|x\rangle \in \mathcal{H}$ such that

$$\rho = |x\rangle \langle x|.$$

Any such state ρ can be faithfully represented by the ket $|x\rangle$ instead of the operator ρ.

The **expectation value** of the observable $A \in \mathsf{A}(\mathcal{H})$ in the state $\rho \in \mathsf{S}(\mathcal{H})$ is a real number $\langle A \rangle_\rho$ defined as

$$\langle A \rangle_\rho = \mathsf{Tr}[A\rho].$$

A normal operator $U \in \mathsf{B}(\mathcal{H})$ is called **unitary**, if its adjoint operator is simultaneously its inverse, i.e. $U^\dagger = U^{-1}$. The class of unitary operators is denoted as $\mathsf{U}(\mathcal{H})$:

$$\mathsf{U}(\mathcal{H}) = \{\, U \in \mathsf{B}(\mathcal{H}) \mid U^\dagger = U^{-1} \,\}.$$

The class of unitary operators forms a group [86]. The spectrum of any unitary operator $U \in \mathsf{U}(\mathcal{H})$ fulfils $\lambda \in \sigma(U) \Rightarrow |\lambda| = 1$. According to Stone's theorem [12], any unitary operator $U \in \mathsf{U}(\mathcal{H})$ can be associated with an certain observable $A \in \mathsf{A}(\mathcal{H})$ to fulfil

$$U = \exp[-iA]. \tag{2.7}$$

An observable $P \in \mathsf{A}(\mathcal{H})$ is called an **orthogonal projector**, if $P^2 = P$. The class of all orthogonal projectors on the Hilbert space $\mathcal{H}$ is denoted as $\mathsf{P}(\mathcal{H})$, i.e.

$$\mathsf{P}(\mathcal{H}) = \{\, P \in \mathsf{A}(\mathcal{H}) \mid P^2 = P \,\}.$$

Any orthogonal projector fulfils $\sigma(P) \subset \{\, 0, 1 \,\}$. To any projector P, the complementary projector $Q \in \mathsf{P}(\mathcal{H})$ can be defined as $Q = I - P$. The projector P splits the Hilbert space $\mathcal{H}$ into two subspaces $P\mathcal{H}$ and $Q\mathcal{H}$. The null space and the support of projectors P and Q fulfil

$$\mathsf{Ker}(P) = \mathsf{Supp}(Q), \ \mathsf{Supp}(P) = \mathsf{Ker}(Q).$$

Since the space $\mathsf{B}(\mathcal{H})$ is a Hilbert space on its own, one can consider a space of linear operators on the space $\mathsf{B}(\mathcal{H})$, i.e. the space of all linear maps $\mathcal{T} : \mathsf{B}(\mathcal{H}) \to \mathsf{B}(\mathcal{H})$, formally denoted as $\mathsf{B}(\mathsf{B}(\mathcal{H}))$. The elements of the space $\mathsf{B}(\mathsf{B}(\mathcal{H}))$ are called **superoperators**. All the previously defined concepts of e.g. basis, eigenvalues, Jordan normal form etc. can be defined analogously to the space $\mathsf{B}(\mathcal{H})$. The **identity superoperator** will be denoted as $\mathcal{I}$.

Through the thesis, several auxiliary superoperators are used. These are the **left multiplication operator** L_X and the **right multiplication operator** R_X, defined as

$$L_X(Y) = XY, \ R_X(Y) = YX, \ X, Y \in \mathsf{B}(\mathcal{H}). \tag{2.8}$$

These superoperators are invertible if and only if the operator $X \in \mathsf{B}(\mathcal{H})$ is invertible, with the inverse operators fulfiling

$$L_X^{-1} = L_{X^{-1}}, \ R_X^{-1} = R_{X^{-1}}.$$

Another important superoperator is the so-called **relative modular operator** [84, 85] Δ_{X_1, X_2}, defined as

$$\Delta_{X_1, X_2}(Y) = X_1 Y X_2^{-1}, \ X_1, X_2, Y \in \mathsf{B}(\mathcal{H}), \tag{2.9}$$

with X_2 being an invertible operator.

A superoperator $\mathcal{T} \in \mathsf{B}(\mathsf{B}(\mathcal{H}))$ is called **completely positive** (CP) [2, 6], if the superoperator $\mathcal{T} \otimes \mathcal{I} \in \mathsf{B}(\mathsf{B}(\mathcal{H} \otimes \mathcal{H}))$ is positive, i.e. if

$$\mathcal{T} \otimes \mathcal{I} \geq 0.$$

The action of any CP superoperator can be written for any $X \in \mathsf{B}(\mathcal{H})$ as

$$\mathcal{T}(X) = \sum_j K_j X K_j^\dagger, \ K_j \in \mathsf{B}(\mathcal{H}) \tag{2.10}$$

where K_j are called the **Kraus operators**. The set of Kraus operators $\{ K_j \}$ for a given CP superoperator $\mathcal{T}$ is not unique - two sets of operators $\{ K_j \}_{j=1}^n$ and $\{ \tilde{K}_j \}_{j=1}^m$ describe the same CP superoperator (2.10), if there is a unitary operator $U \in \mathsf{U} \left(\mathbb{C}^{\max\{ m,n \}} \right)$ such that

$$\tilde{K}_i = \sum_{j=1}^{\max\{ m,n \}} U_{ij} K_j,$$

with the smaller set of operators being completed by zero operators. The adjoint superoperator to the superoperator (2.10) with respect to the Hilbert-Schmidt inner product $\mathcal{T}^\dagger$ takes the form

$$\mathcal{T}^\dagger(X) = \sum_j K_j^\dagger X K_j, \ \forall X \in \mathsf{B}(\mathcal{H}).$$

A superoperator $\mathcal{T} \in \mathsf{B}(\mathsf{B}(\mathcal{H}))$ is called **conditionally completely positive** (CCP) [87] if

$$(I - |\Omega\rangle \langle\Omega|)(\mathcal{T} \otimes \mathcal{I})(I - |\Omega\rangle \langle\Omega|) \geq 0,$$

with $|\Omega\rangle \in \mathcal{H} \otimes \mathcal{H}$ defined by (2.1). Any CCP superoperator $\mathcal{T}$ can be decomposed using the set of operators $L_j \in \mathsf{B}(\mathcal{H})$ and a positive operator $K \geq 0$ as

$$\mathcal{T}(X) = \sum_j L_j X L_j^\dagger - KX - XK^\dagger, X \in \mathsf{B}(\mathcal{H}). \tag{2.11}$$

Similarly to CP superoperators, the adjoint superoperator to a CCP superoperator (2.11) is the superoperator

$$\mathcal{T}^\dagger(X) = \sum_j L_j^\dagger X L_j - KX - XK^\dagger, X \in \mathsf{B}(\mathcal{H})$$

Furthermore, if a superoperator $\mathcal{T}$ is CCP, then the superoperator $\exp[\mathcal{T}]$ is CP [24].

A superoperator $\mathcal{T} \in \mathsf{B}(\mathsf{B}(\mathcal{H}))$ is called **trace-preserving** (TP), if it fulfils

$$\mathsf{Tr}[\mathcal{T}(A)] = \mathsf{Tr}[A], \ \forall A \in \mathsf{A}(\mathcal{H}).$$

If $\mathcal{T} \in \mathsf{B}(\mathsf{B}(\mathcal{H}))$ fulfils

$$\mathsf{Tr}[\mathcal{T}(A)] \leq \mathsf{Tr}[A], \forall A \in \mathsf{A}(\mathcal{H})$$

instead, it is called **trace-nonincreasing** (TNI).

2.3 Evolution of quantum systems

The class of CP superoperators is important for the characterization of general physically realizable changes of quantum states. Any CPTP superoperator maps the elements of the set $\mathsf{S}(\mathcal{H})$ to other elements of the set $\mathsf{S}(\mathcal{H})$. During some real processes, a part of the quantum state may "leak" outside the Hilbert space of the system, resulting in a quantum state with a subunit trace. Such situation may occur e.g. for selective measurements, or quantum systems with sinks [93, 94] and it can be described by a CPTNI superoperator. Describing the most general physically realizable change of quantum state, we call a CPTNI superoperator a **quantum operation**.

Any **time evolution** of a quantum system between two times t_1 and t_2 can be described as a quantum operation $\mathcal{T}_{t_1,t_2}$. Fixing the initial time t_1, the family of quantum operations $\{\,\mathcal{T}_{t_1,t_2} \mid t_2 \geq 0\,\}$, describing the evolution from the initial time to the time t_2, is called a **quantum process**. Beside continuous quantum processes, one may also study discrete quantum processes, in which the time evolution follows discrete steps denoted by a parameter $n \in \mathbb{N}$. Some of presented statements will be true for both discrete and continuous processes, in such case, the time parameter is denoted as τ, representing either discrete time n or the continuous time t.

This thesis deals with so-called **homogeneous quantum Markov processes** (QMPs). QMPs are quantum processes associated with a single time-independent generator $\mathcal{G}$, locally describing the time evolution of the given QMP. For such quantum processes $\{\,\mathcal{T}_{\tau_2,\tau_1} \mid \tau_2 \geq 0\,\}$, the time evolution is independent of the particular values τ_1, τ_2, i.e. for any reference time τ_R one obtains

$\mathcal{T}_{\tau_2+\tau_R,\tau_1+\tau_R} = \mathcal{T}_{\tau_2,\tau_1}$. The resulting quantum process depends only on the difference $\tau = \tau_2 - \tau_1$ between the final time and the initial time. Consequently, the QMP is characterized by a family of quantum operations $\{\,\mathcal{T}_\tau \mid \tau \geq 0\,\}$ [5, 6].

During the time evolution, the state of the quantum system follows a trajectory in the set of quantum states $\{\,\rho(\tau) \mid \tau \geq 0\,\} \subset \mathsf{S}(\mathcal{H})$ called the **quantum trajectory**, with $\rho(\tau) = \mathcal{T}_\tau(\rho(0))$. A particularly important kind of quantum trajectories are **invariant states**. If the initial state $\rho(0)$ fulfils $\mathcal{T}_t(\rho(0)) = \rho(0)$ for some vicinity of $\tau = 0$, then the whole quantum trajectory reduces to a single quantum state $\rho = \rho(0)$, called the invariant state of the quantum process $\mathcal{T}_t$.

Beside the time evolution of quantum states, called the **Schrödinger picture**, one can also study the time evolution in terms of the time evolution of observables. Such an approach is called the **Heisenberg picture**. In the Schrödinger picture, quantum states undergo the time evolution, while the observables remain fixed, unless they are explicitly time-dependent. In the Heisenberg picture, the time evolution leads to changes in observables, while the quantum states remain fixed. As the Schrödinger picture, represented by a quantum operation $\mathcal{T}_\tau^{(S)}$ and the Heisenberg picture, represented by a quantum operation $\mathcal{T}_\tau^{(H)}$ represent the same quantum process, they should lead to the same predictions, i.e. resulting expectation values of observables should be the same for both pictures. Starting from the quantum state $\rho \in \mathsf{S}(\mathcal{H})$ and the observable $A \in \mathsf{A}(\mathcal{H})$ which is not explicitly dependent on time, in the Schrödinger picture, the quantum state evolves as $\rho_S(\tau)$ with $\rho_S(0) = \rho$ due to the quantum operation $\mathcal{T}_\tau^{(S)} \equiv \mathcal{T}_\tau$, while the observable A_S in the Schrödinger picture remains fixed to its initial value $A_S = A$. In the Heisenberg picture, the quantum state remains fixed as $\rho_H = \rho$, while the observable $A_H(\tau)$ undergoes the time evolution due to the quantum operation $\mathcal{T}_\tau^{(H)}$, with the initial condition $A_H(0) = A$. As expectation values must be the same in both pictures, one obtains

$$
\begin{aligned}
\langle A \rangle_\rho (\tau) &= \mathsf{Tr}\left[A_S \mathcal{T}_\tau^{(S)}(\rho_S(0)) \right] = \mathsf{Tr}\left[A\mathcal{T}_\tau(\rho) \right] = \mathsf{Tr}\left[\mathcal{T}_\tau^\dagger(A)\rho \right], \\
\langle A \rangle_\rho (\tau) &= \mathsf{Tr}\left[\mathcal{T}_\tau^{(H)}(A_H(0))\rho_H \right] = \mathsf{Tr}\left[\mathcal{T}_\tau^{(H)}(A)\rho \right],
\end{aligned}
\tag{2.12}
$$

and consequently $\mathcal{T}_\tau^{(H)} = \mathcal{T}_\tau^\dagger$. In the Heisenberg picture, the time evolution is therefore governed by the adjoint superoperator $\mathcal{T}_\tau^\dagger$.

A significant type of a quantum process is a **homogeneous unitary quantum process** given by one-parameter family of quantum operations $\mathcal{U}_\tau$ with each superoperator $\mathcal{U}_\tau$. Each superoperator $\mathcal{U}_\tau$ is associated with a single unitary Kraus operator $U_\tau \in \mathsf{U}(\mathcal{H})$ describing the time evolution over the period of time of length τ. The time evolution of the quantum state $\rho \in \mathsf{S}(\mathcal{H})$ thus takes the form

$$
\rho(\tau) = \mathcal{U}_\tau(\rho(0)) = U_\tau \rho(0) U_\tau^\dagger.
\tag{2.13}
$$

Using (2.7) and homogeneity of the time evolution, the operator $U_t \in \mathsf{U}(\mathcal{H})$ can be written as

$$U_\tau = \exp[-iH\tau], \ H \in \mathsf{A}(\mathcal{H}). \tag{2.14}$$

The observable $H \in \mathsf{A}(\mathcal{H})$ in the expression (2.14) is called the Hamiltonian. The Hamiltonian locally describes the time evolution generated by the quantum process $\mathcal{U}_\tau$. In the discrete case, this local evolution is given by the operator $U_1 = \exp[-iH]$ and in continuous case through von Neumann equation, which takes the form

$$
\begin{aligned}
\frac{d\rho}{dt} &= -i[\rho, H], \ \rho \in \mathsf{S}(\mathcal{H}), \\
\frac{dA}{dt} &= i[A, H], \ A \in \mathsf{A}(\mathcal{H}),
\end{aligned}
\tag{2.15}
$$

in the Schrödinger picture and the Heisenberg picture respectively. The Hamiltonian H is thus the generator of the quantum process $\mathcal{U}_\tau$. The unitary process (2.13) is therefore an example of a QMP. Additionally, such a process fulfils

$$
\begin{aligned}
\mathcal{U}_{\tau_2} \circ \mathcal{U}_{\tau_1} &= \mathcal{U}_{\tau_1 + \tau_2}, \ \forall \tau_1, \tau_2 \in \mathbb{R}, \\
\mathcal{U}_0 &= I.
\end{aligned}
\tag{2.16}
$$

Putting $\tau_1 = \tau$ and $\tau_2 = -\tau$, the evolution from the initial time to the time τ can be reversed – a quantum state $\rho(\tau)$ evolving due to a reversible quantum process $\mathcal{U}_\tau$ in the Schödinger picture can be always traced back to its initial state $\rho(0)$ by application of the quantum operation $\mathcal{U}_{-\tau}$. The QMP $\mathcal{U}_t$ therefore describes a **reversible QMP**. The converse is also true – any reversible QMP $\mathcal{U}_t$ can be characterized by a single observable H, called the Hamiltonian as (2.13) with (2.14). Any reversible QMP is consequently a homogeneous unitary quantum process. Any QMP $\mathcal{T}_\tau$ which is not of the form (2.13) therefore describes an **irreversible QMP**.

An observable $C(\tau) \in \mathsf{A}(\mathcal{H})$ is called the **constant of motion** associated with the quantum process $\mathcal{T}_\tau$ in the Schrödinger picture, if its expectation value is conserved during the time evolution. As was previously mentioned, irreversible processes tend to restrict the set of all quantum states to some smaller subset. Consequently, some degrees of freedom may become unavailable during the time evolution. To avoid problems with this redundant part of the Hilbert space, it is thus advantageous to define constants of motion with respect to a reference time τ_R. For any quantum trajectory $\rho(\tau) \subset \mathsf{S}(\mathcal{H})$ one thus obtains

$$\langle C(\tau + \tau_R) \rangle_{\rho(\tau)} = c(\tau_R) \in \mathbb{R}, \tag{2.17}$$

i.e. the expectation value of the observable $C(\tau)$ remains unchanged during the time evolution and it is only a function of the reference time τ_R. A time-independent constant of motion $C(\tau) \equiv C \in \mathsf{A}(\mathcal{H})$ is called an **integral of motion**.

Any reversible QMP can be diagonalized in an orthonormal basis consisting of eigenvectors of the corresponding Hamiltonian H and consequently, one can always find $N^2 = (\mathsf{Dim} \ \mathcal{H})^2$ linearly independent constants of motion

$$C_{j,k}(\tau) \sim a(\tau) |x_j\rangle \langle x_k| \pm \overline{a}(\tau) |x_k\rangle \langle x_j|, \; j,k \in \{1, \dots, N\}$$

with $|k_j\rangle$ and $|x_k\rangle$ being the eigenvectors of the Hamiltonian H and $a(\tau)$ being a complex function. Fixing the initial state $\rho(0)$ fixes expectation values of constants of motion and these values do not change during the evolution. Consequently, all the information about the initial state is preserved during the time evolution under reversible QMP.

For irreversible QMP, the number of available constants of motion is lower than N^2 and consequently, some information about the initial state is lost during the time evolution. All the information which is preserved is encoded in expectation values of available constants of motion which are therefore an important characteristics of any irreversible QMP.

2.4 Jaynes principle

In practice, it is often hard to obtain the exact time evolution of the quantum system at hand. This may be e.g. a consequence of the unknown form of the quantum process $\mathcal{T}_t$, or due its overly complicated form. Frequently, the only available information about the system and its state is given in form of the set of numbers $\{a_j\}$, $j \in \mathsf{K}$, representing expectation values of certain observables, typically integrals of motion I_j corresponding to the system. A way how to deal with situation with only partial information available was suggested by Jaynes in 1957 [71]. The work of Jaynes builds on the concepts of Shannon [73] who introduced a measure of uncertainty $H(\{p_\gamma\})$, called the **Shannon entropy**. This measure describes a lack of knowledge about the result of a process with possible outcomes $\Omega = \{\gamma\}$, each outcome associated with probability p_γ. It is defined as

$$H(\{p_\gamma\}) = -k \sum_{\gamma \in \Omega} p_\gamma \log p_\gamma, \; k > 0. \tag{2.18}$$

The base of the logarithm is arbitrary – for the purpose of this thesis, the natural logarithm will be used.

To overcome the problem with insufficient information, Jaynes formulated the following principle:

Proposition 2.4.1. *When encountered a problem of partial knowledge about the system, one must among all possible solutions choose the most unbiased one, consistent with the provided information about the system. This solution is given by maximizing the Shannon entropy under constraints given by provided information.*

When applied in statistical physics, the probability distribution obtained from the Jaynes principle coincides with the equilibrium distribution of Gibbs and Boltzmann [9]. The Jaynes principle thus provides a connection between the statistical physics and theory of information. More specifically, assume

a statistical system, with the expectation values $\langle I_j \rangle = a_j$, $j \in \mathsf{K}$, of some integrals of motion I_j being the only available information about the system. The probability distribution consistent with this information, which maximizes the Shannon entropy, takes the form

$$p_\gamma = \frac{1}{Z} \exp\left[-\sum_{j \in \mathsf{K}} \beta_j I_j^{(\gamma)} \right], \ Z = \sum_{\gamma \in \Omega} \exp\left[-\sum_{j \in \mathsf{K}} \beta_j I_j^{(\gamma)} \right], \tag{2.19}$$

with $I_j^{(\gamma)}$ being value of the integral of motion I_j for outcome γ. The quantity Z is called the partition sum and it can be used to determine the expectation values as

$$-\left(\frac{\partial \ln Z}{\partial \beta_j} \right)_{\beta_k \neq \beta_j} = \langle I_j \rangle. \tag{2.20}$$

Jaynes principle is used in a number of areas including statistical mechanics, fenomenological thermodynamics, economics, election models, image processing, biology, medicine and others [11].

Jaynes principle is also used in context of quantum mechanics, to obtain to describe equilibrium states of quantum systems [36, 74, 75]. Instead of the Shannon entropy, the quantity maximized is usually the **von Neumann entropy** [2, 5, 6, 7] $S(\rho)$ defined as

$$S(\rho) = -\mathsf{Tr}[\rho \ln \rho], \ \rho \in \mathsf{S}(\mathcal{H}). \tag{2.21}$$

Assuming the knowledge of expectation values of integrals of motion $\{ I_j \}$, $j \in \mathsf{K}$, the maximization of the von Neumann entropy yields the generalized Gibbs state (2.6), i.e. the equilibrium quantum state ρ_{eq} is

$$\rho_{\mathrm{eq}} = \frac{1}{Z} \exp\left[-\sum_{j \in \mathsf{K}} \beta_j I_j \right], \ Z = \mathsf{Tr}\left[\exp\left[-\sum_{j \in \mathsf{K}} \beta_j I_j \right] \right]. \tag{2.22}$$

There are also suggestions for a generalization of the Jaynes principle to incorporate also non-equilibrium processes. The proposed principle is called the **maximum Caliber principle** [76, 77, 78]. The maximum Caliber principle uses the concept of the **relative entropy** $H(\{ p_\gamma \} | \{ q_\gamma \})$ [2, 6], defined as

$$H(\{ p_\gamma \} | \{ q_\gamma \}) = \begin{cases} \sum_{\gamma \in \Omega} p_\gamma (\ln p_\gamma - \ln q_\gamma) & \text{if } q_\gamma = 0 \Rightarrow p_\gamma = 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.23}$$

To obtain the non-equilibrium probability distribution $\{ p_\gamma(t) \}$, one therefore minimizes the relative entropy with respect to the probability distribution $\{ q_\gamma \}$, called the prior probability distribution, under the constraints

$$\langle C_j(t)\rangle = \sum_{\gamma\in\Omega} p_\gamma(t)C_j(t), \tag{2.24}$$

with $C_j(t)$ being a subset of the set of constant of motions, the maximum Caliber principle yields

$$
\begin{aligned}
p_\gamma(t) &= \frac{1}{Z}\exp\left[\ln q_\gamma - \sum_{j\in\mathsf{K}}\beta_j(t)C_j^{(\gamma)}(t)\right],\\[2mm]
Z &= \sum_{\gamma\in\Omega}\exp\left[\ln q_\gamma - \sum_{j\in\mathsf{K}}\beta_j(t)C_j^{(\gamma)}(t)\right].
\end{aligned}
\tag{2.25}
$$

The non-equilibrium probability distribution $\{\,p_\gamma(t)\,\}$ is thus a function of the prior probability distribution $\{\,q_\gamma\,\}$. For certain dynamical system, it was proposed to choose the prior probability distribution as the equilibrium probability distribution resulting from the same process with ignoring all time-dependent constraints [76], however, the extent of this proposition is unknown and therefore, there is no known general way to determine the prior probability distribution $\{\,q_\gamma\,\}$, which must be consequently regarded as an additional information about the system.

Although the Jaynes principle and its generalizations prove to be valuable from both theoretical and experimental points of view, they both also attract a certain amount of scepticism. The first main point of the critique of the Jaynes principle is the fact that there is no fundamental reason to believe that the Shannon entropy (2.18) and its quantum version the von Neumann entropy (2.21) represent the only correct measure of lack of information. Sometimes, relative entropy (2.23) is used instead of the Shannon entropy in the Jaynes principle, minimizing the relative entropy, while keeping the probability distribution $\{\,q_\gamma\,\}$, called the prior distribution, fixed. In the quantum domain an analogous principle can be formulated utilizing the quantum version of relative entropy, the **relative quantum entropy** $S(\rho|\sigma)$ [2, 6], defined as

$$S(\rho|\sigma) = \begin{cases}\mathsf{Tr}[\rho(\ln\rho - \ln\sigma) & \text{if } \mathsf{Supp}(\sigma)\subset\mathsf{Supp}(\rho)\\ +\infty & \text{otherwise.}\end{cases} \tag{2.26}$$

However, both the prior distribution $\{\,q_\gamma\,\}$ and the quantum state σ represent extra information, which needs to be obtained in a different way. Furthermore, there are many other functions, which can play an analogous role to the Shannon entropy [80], each with its corresponding quantum counterpart. Similarly, there are whole families of quantum relative entropies [81]. The von Neuman, resp. the quantum relative entropy have an advantage as their maximization, resp. minimization results in an exponential, and therefore positive probability distribution, however it is unclear, whether other entropies should be preferred in certain scenarios, or whether in some situations some entropies should be avoided. For instance, it was shown that some of the generalized entropies can

not be used within the Jaynes principle to obtain the equilibrium distribution of a system subjected to linear constraints (e.g. expectation values) [82]. The problem of the true form and the scope of the Jaynes principle is connected with the final and arguably the most important point of the criticism - Jaynes principle should be consistent with the underlying dynamics of the system and one should be therefore be able to use the equations describing the dynamics of the system for derivation of the Jaynes principle, however at this moment such derivation is still missing.

2.5 Graph theory

Graphs are often used to represent networks of interacting subjects in many scientific areas, e.g. physics, biology, sociology etc. Their basic function is to visually represent the studied system and the interactions between its elements, however, their properties may often crucially influence the behavior of the studied system. They are often employed in description of quantum networks - quantum systems comprised of several subsystems, which are connected with each other via prescribed interaction. The interaction between the systems may be symmetric (i.e. two subsystems influence each other in the same way) or antisymmetric (i.e. influence of the first subsystem on the second subsystem differs from the influence of the second subsystem on the first subsystem).

A **graph** $G(V, E)$ consists of set of vertices V and the set of edges E, connecting the vertices. Any edge $e \in E$ can be written as pair of vertices (v, w), $v, w \in V$. In an **undirected graph**, the edges have no orientation and pairs of vertices (v, w) and (w, v) represent the same edge $e \in E$. If the edges are oriented, pairs of vertices (v, w) and (w, v) represent edges with opposite orientation. A graph with oriented edges is called the **directed graph**. Due to nature of quantum networks studied within the thesis, only directed graphs will be used. For any directed edge $e = (v, w) \in E$, the vertex v is called the **tail** of the edge e and the vertex w is called the **head** of the vertex e. The number of edges $\mathsf{D}(v)$, whose tail is the vertex v is called the **degree of vertex** $v \in V$.

A sequence of edges $\mathsf{P}(v, w) = (e_1, \ldots, e_n)$, such that the tail of the edge e_1 is the vertex $v \in V$, the head of the edge e_n is the vertex $w \in V$ and the head of any edge e_j is the same vertex as a tail of edge e_{j+1} is called the **path** connecting vertices v and w. If $v = w$ and the tail of each edge e_j, denoted as $v_j \in V$ fulfils $d(v_j) = 1$, then the path is called a **cycle**. A graph $G(V, E)$ without cycles is called **acyclic**.

Consider a graph $G(V, E)$. A graph $g(W, F)$, with $W \subset V$ and $F \subset E \cap W \times W$ is called a **subgraph** of graph $G(V, E)$. An important type of graph is a **tree** - an acyclic graph $T(V, E)$ such that all pairs of vertices $v, w \in V$ such that $v \neq w$ are connected by a single unique path, disrespecting the orientation of the edges. Consider an acyclic subgraph $g(W, F)$ of a graph $G(V, E)$, such that any $v \in W$ fulfils
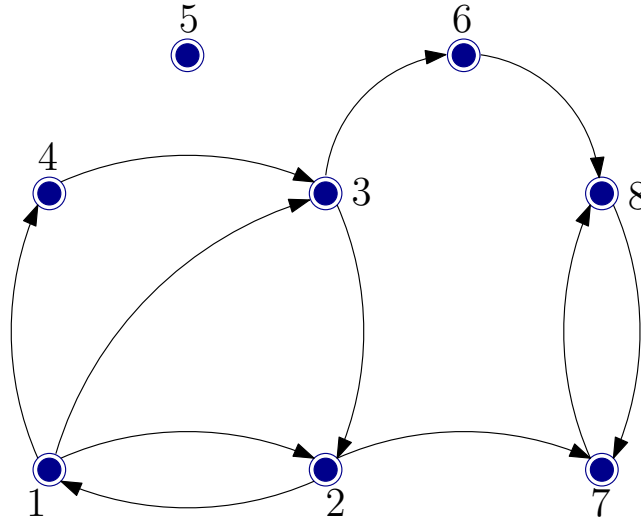
Figure 2.1: An example of a directed graph $G(V, E)$. The set V contains eight vertices.

$$D(v) = \begin{cases} 1 & \text{if } v \in V \backslash W \\ 0 & \text{if } v \in W. \end{cases} \tag{2.27}$$

Such subgraph $g(W, F)$ is called a W-tree and it is denoted as $T_W(G)$. Specially, if $W = \{v_1, \ldots, v_n\}$, the corresponding W-tree is called a **maximal tree of vertices** $\{v_1, \ldots, v_n\}$ and it is denoted as a $T_{v_1 \ldots v_n}(G)$. An example of the maximal trees is provided on the figure 2.2.

An important property of any graph is its connectivity. A graph $G(V, E)$ whose any pair of vertices $v, w \in V$ can be connected by a path is called **strongly connected**. Any graph $G(V, E)$ can be divided into n strongly connected subgraphs $G_k(V_k, E_k)$ with $V_k \cap V_l = \emptyset$ for $k \neq l$, called the **components** of the graph $G(V, E)$. Components G_k of the graph G can be then used to construct the **graph of components** $C(G)$ associated with the graph $G(V, E)$. In the graph of components, each component G_k of the graph G is represented by a single vertex G_k. Vertices G_j and G_k are then connected by an edge (G_j, G_k), if there are edges in G connecting some vertices in the component G_j with some vertices in the component G_k.

A graph $G(V, E)$ is therefore strongly connected, if its graph of components $C(G)$ consists of a single vertex. In the opposite case, the graph of components may include some vertices G_k which are not connected to other vertices in either directions. Corresponding components $G_k(V_k, E_k)$ are then called **disconnected**. If the graph of components $C(G)$ contains more than one vertex and none of the components $G_k(V_k, E_k)$ is disconnected, the graph $G(V, E)$ is called **weakly connected**. In a graph $G(V, E)$ any component G_k such that
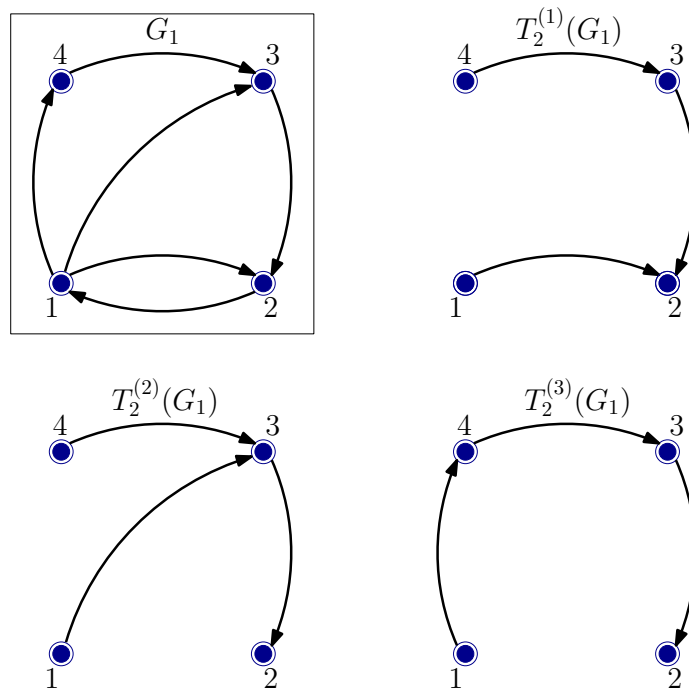
Figure 2.2: Component G_1 of the graph $G(V, E)$ from figure 2.1 and its all corresponding maximal trees $T_2^{(k)}(G_1)$.
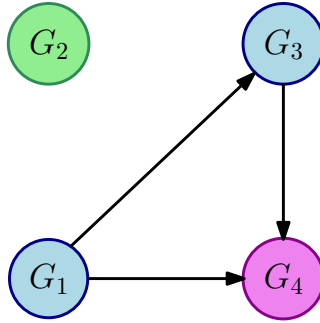
Figure 2.3: The graph of components $\mathsf{C}(G)$ corresponding to the graph $G(V, E)$ from figure 2.1. The graph $G(V, E)$ has four components, the component G_1 contains vertices $\{1, 2, 3, 4\}$, the component G_2 contains vertex $\{5\}$, the component G_3 contains vertex $\{6\}$ and the component G_4 contains vertices $\{7, 8\}$. The component G_2 (green) is disconnected from the rest of the graph and the component G_4 (violet) is a terminal component.

the corresponding vertex G_k in the graph of components $\mathsf{C}(G)$ fulfils $\mathsf{D}(G_k) = 0$ and the component $G_k(V_k, E_k)$ is not disconnected is called the **terminal component**. An example of a graph of components with a disconnected component and a terminal component is depicted on the figure 2.3.

Some problems call for the introduction of so-called **weighted graphs** - a graph $G(V, E, \Gamma)$, in which each edge $e \in E$ is associated with a nonnegative number $0 \leq \gamma_e \in \Gamma$ called **weight**. Having a weighted graph $G(V, E, \Gamma)$, weight of the graph $\Omega(G(V, E, \Gamma))$ is defined as

$$\Omega(G(V, E, \Gamma)) = \prod_{e \in E} \gamma_e.$$

g

Chapter 3

Quantum Markov processes

A quantum state $\rho(t)$ evolving due to a general quantum process $\{\,\mathcal{T}_{\tau_2,\tau_1} \mid \tau_2 \geq \tau_1\,\}$ satisfies a complicated integro-differential equations [6, 25]

$$\frac{d}{dt}\rho(t) = (Z + A)\rho(t) + \int_0^t G(t-s)\rho(s)ds,$$

with Z being superoperator describing free evolution, A being superoperator describing the nonunitary effects and $G(t)$ being so-called integral kernel [6]. In general such description does not allow an analytic study of properties of the system, e.g. its invariant states, constants of motion etc. For that reason, numerical methods are often employed to obtain at least partial information about the evolution $\{\,\mathcal{T}_{\tau_2,\tau_1} \mid \tau_2 \geq \tau_1\,\}$. One of the main obstacles in analysing a general quantum process $\{\,\mathcal{T}_{\tau_2,\tau_1} \mid \tau_2 \geq \tau_1\,\}$ is that it does not need to follow the composition law, as one typically obtains

$$\mathcal{T}_{\tau_2,\tau_1} \circ \mathcal{T}_{\tau_1,\tau_0} \neq \mathcal{T}_{\tau_2,\tau_0}, \ \tau_2 \geq \tau_1 \geq \tau_0. \tag{3.1}$$

This is due to the fact that the evolution of the quantum state $\rho(\tau)$ generally depends on the previous history of the evolution, i.e. on previous quantum states $\rho(\tau_P)$, $\tau_P < \tau$ and consequently, the time evolution cannot be locally described by some generator $\mathcal{G}(\tau)$.

A quantum process $\{\,\mathcal{T}_\tau \mid \tau \geq 0\,\}$, for which the time evolution is locally described by a time independent generator $\mathcal{G}$ is called the **homogeneous quantum Markov process** (QMP). Any QMP $\{\,\mathcal{T}_\tau \mid \tau \geq 0\,\}$ fulfil the composition law in the form

$$\begin{aligned} \mathcal{T}_{\tau_2} \circ \mathcal{T}_{\tau_1} &= \mathcal{T}_{\tau_2+\tau_1}, \ \tau_1, \tau_2 \geq 0 \\ \mathcal{T}_0 &= I. \end{aligned} \tag{3.2}$$

The composition law, expressed by the first equation, is called the **Markov property**. It guarantees that the further evolution of the quantum state $\rho(\tau)$ depends only on the quantum state $\rho(\tau)$ itself. It is fulfiled (for any real τ_1, τ_2) for e.g. homogeneous unitary quantum processes, which were introduced in the

section 2.3, however the range of quantum processes which meet the condition (3.1) is much broader.

For irreversible quantum processes, the Markov property is typically not valid, however, in several important scenarios, the real evolution can be approximated by a simplified one, which fulfils the Markov property and the given system can be thus to some extent treated as Markovian. Such scenarios include the weak coupling limit [26], the low density limit [27], the singular coupling limit [28] etc. [4, 5, 6, 25].

The QMPs under investigation consist of the class of quantum Markov chains (QMCHs) and quantum Markov dynamical semigroups (QMDSs). Similarly to homogeneous reversible quantum processes, any QMP can be characterized by a single time-independent generator $\mathcal{G}$. Generators of both mentioned classes take a different form, which reflects in some of their mathematical properties, e.g. representation of the generator by elements from the space $\mathsf{B}(\mathcal{H})$, spectral properties etc. This chapter serves as an introduction to the mathematical properties of generators of both classes of QMPs. Furthermore, it introduces an essential ingredient for the description of the asymptotics of any QMP, so-called $\mathcal{T}$-states.

3.1   Quantum Markov chains

A quantum Markov chain is a discrete set of quantum operations $\{\, \mathcal{T}_n \mid n \in \mathbb{N}_0 \,\}$, which describes the evolution of an initial quantum state $\rho \in \mathsf{S}(\mathcal{H})$ as

$$\rho(n) = \mathcal{T}_n(\rho(0)). \tag{3.3}$$

The QMCH $\mathcal{T}_n$ can be described via its generator, denoted as $\mathcal{T}$ as $\mathcal{T}_n = \mathcal{T}^n$. A single step in the quantum evolution thus yields

$$\rho(n+1) = \mathcal{T}_1(\rho(n)) = \mathcal{T}(\rho(n)). \tag{3.4}$$

As the generator $\mathcal{T}$ must be a CP superoperator, it is usually written in the Kraus form [22, 23] as

$$\mathcal{T}(X) = \sum_j K_j X K_j^\dagger, \tag{3.5}$$

with Kraus operators $K_j \in \mathsf{B}(\mathcal{H})$. The Heisenberg picture $\mathcal{T}_n^\dagger$ of QMCH $\mathcal{T}_n$ is then generated by the generator $\mathcal{T}^\dagger$, which is the adjoint superoperator to the generator $\mathcal{T}$ with respect to the Hilbert-Schmidt inner product. It reads

$$\mathcal{T}^\dagger(X) = \sum_j K_j^\dagger X K_j.$$

To represent a physical process, QMCH must be either TP, or TNI. For TP QMCHs, one obtains

$$\mathcal{T}^{\dagger}(I) = \sum_j K_j^{\dagger} K_j = I.$$

If the QMCH is TNI, the Kraus operators fulfil

$$\mathcal{T}^{\dagger}(I) = \sum_j K_j^{\dagger} K_j \leq I.$$

A significant subclass of QMCHs is formed by so-called random unitary operations (RUOs). For RUOs [66], Kraus operators are proportional to unitary operators, i.e.

$$K_j = \sqrt{p_j} U_j, \ U_j \in \mathsf{U}(\mathcal{H}), \ 0 \leq p_j \leq 1, \ \sum_j p_j = 1.$$

RUOs play an important role in the description of quantum systems with incorporated loss of control – in each step of evolution, the quantum system undergoes the change described by one of the unitary operators U_j, however due to loss of control, the only information available is the probability p_j, with which the operator U_j represents the evolution. This loss of control may occur due to a variety of reasons, e.g. presence of variables which are not traced by the observer. As an example, consider a diluted gas in a box, whose particles undergo collisions. The collision of particles j and k are described by an unitary operator U_{jk}. Furthermore, suppose that it is not possible to track positions of particles and it is therefore possible to determine only the probability p_{jk} of the collision of particles j and k. Mathematically, such a system can be then modeled as a RUO [95, 96] and it can be shown that the resulting asymptotic evolution is independent of values of probabilities p_{jk} [66]. Among other applications of RUOs are random processes, such as discrete percolated quantum walks [45].

3.2   Quantum Markov dynamical semigroups

The continuous quantum Markov processes are described by so-called quantum Markov dynamical semigroups (QMDSs) $\mathcal{T}_t$ with $t \geq 0$. Besides the time homogeneity and the Markov property, they are required to be also uniformly continuous, i.e. in the operator norm it must be be fulfilled

$$\lim_{s \to 0} ||\mathcal{T}_{t+s} - \mathcal{T}_t|| = 0, \ \forall t \in [0, +\infty).$$

Then, the time evolution of any QMDS can be then described via the Lindblad equation as

$$\frac{d}{dt} \mathcal{T}_t = \mathcal{L} \mathcal{T}_t. \tag{3.6}$$

The formal solution then takes the form

$$\mathcal{T}_t = \exp[\mathcal{L} t].$$

As the resulting QMDS $\mathcal{T}_t$ must be CP, according to chapter 2 the superoperator $\mathcal{L}$, called the **Lindbladian** must be CCP. According to (3.6) the Lindbladian $\mathcal{L}$ fully characterizes the local time evolution of QMDS $\mathcal{T}_t$ and it is therefore its generator. The general form of any CCP operator was already established in section 2.2. It reads

$$\mathcal{L}(X) = \sum_k L_k X L_k^\dagger - KX - XK^\dagger, \tag{3.7}$$

with $L_k \in \mathsf{B}(\mathcal{H})$ being called **Lindblad operators** (or noise operators) and $K \in \mathsf{B}(\mathcal{H})$. The Lindbladian (3.7) is invariant under gauge transformations

$$L_j \to L_j + c_j I,$$
$$K \to K + \sum_j \left(\frac{|c_j|^2}{2} I + \overline{c}_j L_j \right).$$

for an arbitrary $c_j \in \mathbb{C}$.

Alternatively, the Lindbladian (3.7) can be cast into more often employed form

$$\mathcal{L}(X) = -i[H, X] + \sum_k \left(L_k X L_k^\dagger - \frac{1}{2} \left\{ L_k^\dagger L_k, X \right\} \right) - GX - XG, \tag{3.8}$$

with $\mathcal{H} \in \mathsf{A}(\mathcal{H})$ and $G \geq 0$ being called the optical potential [6]. Any gauge transformation

$$L_j \to L_j + c_j I,$$
$$H \to H - \frac{i}{2} \sum_j \left(\overline{c}_j L_j - c_j L_j^\dagger \right).$$

leaves the generator $\mathcal{L}$ unchanged for an arbitrary $c_j \in \mathbb{C}$. Due to this freedom of the gauge transformation, H does not have to be the real Hamiltonian corresponding to the system. Both expressions (3.7) and (3.8) can be interchanged through relation

$$K = iH + \frac{1}{2} \sum_k L_k^\dagger L_k + G.$$

The adjoint generator $\mathcal{L}^\dagger$ with respect to the Hilbert-Schmidt scalar product describes the time evolution in the Heisenberg picture. Its generator can be written in the following forms:

$$\mathcal{L}^\dagger(X) = \sum_k L_k^\dagger X L_k - K^\dagger X - XK,$$

$$\mathcal{L}^\dagger(X) = i[H, X] + \sum_k \left(L_k^\dagger X L_k - \frac{1}{2} \left\{ L_k^\dagger L_k, X \right\} \right) - GX - XG.$$

Any map $\mathcal{T}_t = \exp[\mathcal{L}t]$ with $\mathcal{L}$ given by relations (3.7) or (3.8) is a CPTNI map, which can be expressed by equation

$$\mathcal{L}^\dagger(I) = -2G \leq 0.$$

Putting the optical potential $G = 0$ results in a subclass of the TP QMDS for which one obtains

$$\mathcal{L}^\dagger(I) = 0.$$

3.3 T-state

An essential element for the description of the asymptotics of QMPs is a $\mathcal{T}$-state. First of all, $\mathcal{T}$-states define a subspace of $\mathcal{H}$ on which the asymptotics take place. Second, $\mathcal{T}$-states transform the eigenvectors of the generator $\mathcal{G}$ in the Schrödinger to the eigenvectors of the generator $\mathcal{G}^\dagger$ in the Heisenberg picture and they are directly inscribed into attractor equations in the Schrödinger picture. Furthermore, $\mathcal{T}$-states are used to define asymptotic states in terms of constants of motion and they are essential for the formulation of the Jaynes principle for QMPs.

Definition 3.3.1. *Let $\mathcal{T}_\tau$ be a QMP on a Hilbert space H. A quantum state $\sigma \in \mathsf{S}(\mathcal{H})$ is called a $\mathcal{T}$-state, if it fulfils*

$$\mathcal{T}_\tau(\sigma) \leq \sigma, \text{ for some right vicinity of } \tau = 0,$$
$$\mathsf{Supp}(\rho) \subset \mathsf{Supp}(\sigma), \ \forall \rho \in \mathsf{S}(\mathcal{H}), \ \mathcal{T}(\rho) \leq \rho.$$

*In the case of QMCHs, the right vicinity is given by a single step of the QMCH. If a $\mathcal{T}$-state σ fulfils $\mathsf{Rank}\ \sigma = \mathsf{Dim}\ \mathcal{H}$, σ is called a **faithful $\mathcal{T}$-state** and the corresponding QMP $\mathcal{T}_\tau$ is called a **faithful QMP**. The minimal projector $P \in \mathsf{P}(\mathcal{H})$ onto the support of a $\mathcal{T}$-state is called a $\mathcal{T}$-projector.*

Plainly speaking, a $\mathcal{T}$-state is a subinvariant state of the QMP $\mathcal{T}_\tau$ with maximal possible rank. Furthermore, for QMDS $\mathcal{T}_t$ associated with a Lindbladian $\mathcal{L}$, any $\sigma \in \mathsf{S}(\mathcal{H})$ such that $\mathcal{L}(\sigma) \leq 0$ which has maximal possible rank is a $\mathcal{T}$-state.

Generally, a $\mathcal{T}$-state is not unique, however if there exist several $\mathcal{T}$-states, for most applications, one is free to choose from them. If the given QMP $\mathcal{T}_\tau$ is not faithful, the $\mathcal{T}$-projector fulfils

$$P < I$$

and the support of an arbitrary asymptotic state is confined to the subspace $P\mathcal{H}$, known as the recurrent subspace [89]. The remaining part of the Hilbert space $Q\mathcal{H}$, $Q = I - P$, cannot support any state in the asymtptotics and is therefore known as the decaying subspace. QMPs with nontrivial decaying part require more tools to analyze and therefore the next chapter is devoted to the analysis of the asymptotics of faithful QMPs, after which it is possible to discuss

changes needed to be made for the analysis of the QMP with nontrivial decaying subspace.

The knowledge of a $\mathcal{T}$-state must be regarded as an additional information about the QMP under investigation. Alternatively, the QMP $\mathcal{T}_\tau$ is engineered to produce at least one specific $\mathcal{T}$-state. If the information about a $\mathcal{T}$-state is not provided, there is no known systematic way to obtain $\mathcal{T}$-state other than direct solution of inequality $\mathcal{T}_\tau(\sigma) \leq \sigma$, resp. $\mathcal{L}(\sigma) \leq 0$ for some QMDS.

3.4 Spectral properties of quantum Markov processes

The main obstacle in investigating algebraic properties of the generators arises from the fact that the generators are generally not normal, i.e having a QMP with a generator $\mathcal{G}$, the commutator of the generator with its adjoint reads

$$\left[\mathcal{G}, \mathcal{G}^\dagger \right] \neq 0.$$

Consequently, the diagonalization of generators $\mathcal{G}$ or $\mathcal{G}^\dagger$ in some orthogonal basis is not guaranteed. Furthermore, its eigenvectors and their dual vectors are in a generally nontrivial relation, making the investigation of the algebraic properties even more complicated.

The first step on the road towards unveiling the asymptotic evolution of QMPs is to uncover the crucial information about the spectrum and the eigenvectors of the generator $\mathcal{G}$. Since the diagonal form of $\mathcal{G}$ is generally not available, one is forced to use the Jordan decomposition:

$$\mathcal{G} = R \left(\bigoplus_{\substack{k=1 \\ \lambda \in \sigma(\mathcal{G})}}^{\mathsf{d}(\lambda)} \mathcal{J}_k(\lambda) \right) R^{-1},$$

with $\mathcal{J}_k(\lambda)$ being the k-th Jordan block corresponding to the eigenvalue λ, i.e

$$\mathcal{J}_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix} = \lambda I + N_k(\lambda),$$

where $N_k(\lambda)$ is a nilpotent matrix with index equal to $\mathsf{Dim}(\mathcal{J}_k(\lambda))$, i.e. $N_k(\lambda)^{\mathsf{Dim}(\mathcal{J}_k(\lambda))} = 0$. The columns of the transformation matrix R form the Jordan basis. Since $\mathcal{G}$ is not normal, elements of the Jordan basis, i.e. operators $X_{\lambda,k,j}$ fulfil

$$(\mathcal{G} - \lambda\mathcal{I})^j (X_{\lambda,k,j}) = 0, \ \forall j \in \{ 1, \dots, \mathsf{Dim}(\mathcal{J}_k(\lambda)) \},$$

are not mutually orthogonal and one generally has

$$\mathsf{Tr}[X_{\lambda,k,j}^{\dagger} X_{\mu,m,l}] \neq 0.$$

Imposing the Jordan form of the generator $\mathcal{T}$, the time evolution of the QMCH $\mathcal{T}_n$ reads

$$\mathcal{T}_n = \left(R \left(\bigoplus_{\substack{k=1 \\ \lambda \in \sigma(\mathcal{T})}}^{\mathsf{d}(\lambda)} \mathcal{J}_k(\lambda) \right) R^{-1} \right)^n = R \left(\bigoplus_{\substack{k=1 \\ \lambda \in \sigma(\mathcal{T})}}^{\mathsf{d}(\lambda)} \mathcal{J}_k^n(\lambda) \right) R^{-1} =$$

$$R \left(\bigoplus_{\substack{k=1 \\ \lambda \in \sigma(\mathcal{T})}}^{\mathsf{d}(\lambda)} \sum_{m=0}^{n} \binom{n}{m} \lambda^{n-m} N_k(\lambda)^m \right) R^{-1}. \tag{3.9}$$

Since the matrix N_k is nilpotent, it is easy to show that matrix elements of n-th power of the Jordan block $\mathcal{J}_k(\lambda)$ fulfils

$$|(\mathcal{J}_k^n(\lambda))|_{ij} = |\lambda|^{n-(j-i)} \binom{n}{j-i}.$$

To ensure that the resulting QMCH $\mathcal{T}^n$ is TNI, any Jordan block $\mathcal{J}_k(\lambda)$ with $\mathsf{Dim}(\mathcal{J}_k(\lambda)) > 1$ must consequently fulfil $|\lambda| < 1$. For $\mathsf{Dim}(\mathcal{J}_k(\lambda)) = 1$, the nilpotent matrix N_k is equal to zero and (3.9) thus becomes

$$\mathcal{T}_n = R \left(\bigoplus_{\substack{k=1 \\ \lambda \in \sigma(\mathcal{T})}}^{\mathsf{d}(\lambda)} \lambda^{n-k} \right) R^{-1}$$

and to obtain a TNI QMCH $\mathcal{T}_n$ such Jordan blocks must correspond to λ satisfying $|\lambda| \leq 1$.

Similarly, using the Jordan form, the time evolution of the QMDS $\mathcal{T}_t$ associated with a generator $\mathcal{L}$ reads

$$\mathcal{T}_t = \exp[\mathcal{L}t] = \exp \left[R \left(\bigoplus_{\substack{k=1 \\ \lambda \in \sigma(\mathcal{L})}}^{\mathsf{d}(\lambda)} \mathcal{J}_k(\lambda) \right) R^{-1} t \right] =$$

$$R \left(\bigoplus_{\substack{k=1 \\ \lambda \in \sigma(\mathcal{L})}}^{\mathsf{d}(\lambda)} \exp\left[\mathcal{J}_k(\lambda)t\right] \right) R^{-1} = \tag{3.10}$$

$$R \left(\bigoplus_{\substack{k=1 \\ \lambda \in \sigma(\mathcal{L})}}^{\mathsf{d}(\lambda)} \exp[\lambda t] D_k(t) \right) R^{-1},$$

with

$$D_k(t) = \sum_{n=0}^{\mathsf{Dim}(\mathcal{J}_k(\lambda))-1} \frac{t^n N_k^n}{n!}.$$

For $\mathsf{Dim}(\mathcal{J}_k(\lambda)) > 1$, the norm of the operator $D_k(t)$ is polynomially increasing function in time. To ensure that the map $\mathcal{T}_t$ is bounded, it must hold that $\mathsf{Re}[\lambda] < 0$ apart from the case $\mathsf{Dim}(\mathcal{J}_k(\lambda)) = 1$, in which the norm of the operator $D_k(t)$ is constant in time and thus it suffices that $\mathsf{Re}[\lambda] \leq 0$. In conclusion, eigenvalues of the generator $\mathcal{L}$ have non-positive real parts and eigenvalues, which are purely imaginary, correspond to the one-dimensional Jordan blocks.

The correspondence between eigenvalues of the generator $\mathcal{T}$ and the QMCH $\mathcal{T}_n$ is $\lambda \in \sigma(\mathcal{T}) \Rightarrow \lambda^n \in \sigma(\mathcal{T}_n)$. For QMDS $\mathcal{T}_t$, the correspondence with the Lindbladian $\mathcal{L}$ is $\lambda \in \sigma(\mathcal{L}) \Rightarrow \exp[\lambda t] \in \sigma(\mathcal{T}_t)$. In both cases, relations (3.9), resp. (3.10) and the subsequent discussion show that the spectral radius $\mathsf{R}(\mathcal{T}_\tau)$ of any finite-dimensional QMP $\mathcal{T}_\tau$ fulfils $\mathsf{R}(\mathcal{T}_\tau) \leq 1$. Jordan blocks corresponding to the eigenvalues from the peripheral spectrum, i.e. $\mu \in \sigma(\mathcal{T}_\tau)$ such that $|\mu| = 1$ are one-dimensional. All other Jordan blocks correspond to the eigenvalues $\mu \in \sigma(\mathcal{T}_\tau)$ fulfil $|\mu| < 1$ and consequently, these blocks vanish in the limit $\tau \to \infty$.

Chapter 4

Asymptotics of quantum Markov processes

Relations (3.9) and (3.10) suggests that contrary to a general evolution given by a finite-dimensional QMP $\mathcal{T}_\tau$, the asymptotic regime allows to employ a more analytical approach. This regime is characterized by vanishing of all Jordan blocks corresponding to eigenvalues $\lambda \in \sigma(\mathcal{T}_\tau)$ out of the peripheral spectrum due to the limit procedure $\tau \to \infty$. As the remaining Jordan blocks are all one-dimensional, it is always possible to diagonalize the part of the QMP $\mathcal{T}_\tau$ responsible for the asymptotic evolution. As a result, there exists a representation of the asymptotic evolution in terms of eigenvectors of $\mathcal{T}_\tau$ corresponding to the eigenvalues from the peripheral spectrum. These eigenvectors $X \in \mathsf{B}(\mathcal{H})$ and their arbitrary linear combinations are called attractors.

The goal of this chapter is to provide a thorough description of the dependence of the asymptotic state of any finite-dimensional QMP on the initial state in terms of its attractors. Since this problem is more straightforward for faithful QMPs, for most of the chapter the focus will be put on QMPs equipped with a faithful $\mathcal{T}$-state and at the end of the chapter, the generalization for QMPs without a faithful $\mathcal{T}$-state is made. The outline, following the work [97] can be summarized as follows. First, the so-called attractor space $\mathsf{Atr}(\mathcal{G}) \subset \mathsf{B}(\mathcal{H})$ on which the asymptotic dynamics takes place must be identified. After derivation of basic algebraic relations between the attractors and their duals, the attractors are used to describe the asymptotic evolution for any initial state in a closed form. Next, to simplify the derivation of the attractor space, the set of algebraic equations, called the attractor equations is introduced both in the Schrödinger and the Heisenberg picture. Afterwards, to obtain a more complete picture about properties of attractor spaces, a more general relations between attractors and their duals are introduced. A special case of these relations is then used to study the evolution within the attractor space, revealing that it is in a certain sense unitary. Finally, a generalization for QMPs without faithful $\mathcal{T}$-state is made.

4.1 Attractor spaces

Instead of investigating the QMP $\mathcal{T}_\tau$, it is more natural to study properties of its generator $\mathcal{G}$. The peripheral spectrum of the QMP $\mathcal{T}_\tau$ can be mapped to a certain part of the spectrum of the generator $\sigma_{\mathrm{atr}} \subset \sigma(\mathcal{G})$, called the attractor spectrum. Since the generator $\mathcal{G}$ has a different form for QMCHs and QMDSs, the attractor spectrum needs to be defined separately for each class of QMPs. For QMCH with a generator $\mathcal{T}$, one thus has

$$\sigma_{\mathrm{atr}} = \{\lambda | \lambda \in \sigma(\mathcal{T}) \wedge |\lambda| = 1)\},$$

whereas for the case of QMDS generated by the Lindbladian $\mathcal{L}$ the attractor spectrum reads

$$\sigma_{\mathrm{atr}} = \{\lambda | \lambda \in \sigma(\mathcal{L}) \wedge \mathsf{Re}[\lambda] = 0\}.$$

The attractor spectrum is always nonempty, as $1 = \lambda \in \sigma_{\mathrm{atr}}$ for QMCH and $0 = \lambda \in \sigma_{\mathrm{atr}}$ for QMDS and there is always at least one invariant state of a given QMP, given by

$$\bar{\sigma} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{T}_n(\rho), \ \rho \in \mathsf{S}(\mathcal{H}) \tag{4.1}$$

for QMCH, resp.

$$\bar{\sigma} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \mathcal{T}_t(\rho), \ \rho \in \mathsf{S}(\mathcal{H}) \tag{4.2}$$

for QMDS. The state $\bar{\sigma}$ clearly fulfils $\mathcal{T}_\tau(\bar{\sigma}) = \bar{\sigma}$ and it is thus an eigenvector of the QMP $\mathcal{T}_\tau$ corresponding to the eigenvalue $\lambda = 1$, which can be associated with an eigenvalue of the corresponding generator $\mathcal{G}$ as $\lambda = 1$ for QMCH and $\lambda = 0$ for QMDS.

Each eigenvalue $\lambda \in \sigma_{\mathrm{atr}}$ yields one or several linearly independent eigenvectors $X_{\lambda,i}$, called attractors. The set of all attractors forms a subspace $\mathsf{Atr}(\mathcal{G})$ of the space $\mathsf{B}(\mathcal{H})$ called the attractor space, i.e.

$$\mathsf{Atr}(\mathcal{G}) = \bigoplus_{\lambda \in \sigma_{\mathrm{atr}}} \mathsf{Ker}(\mathcal{G} - \lambda \mathcal{I})$$

represents the attractor space of the QMP $\mathcal{T}_\tau$ associated with the generator $\mathcal{G}$ in the Schrödinger picture. The same procedure can be carried out in the Heisenberg picture, giving rise to the dual attractor space

$$\mathsf{Atr}(\mathcal{G}^\dagger) = \bigoplus_{\lambda \in \sigma_{\mathrm{atr}}} \mathsf{Ker}(\mathcal{G}^\dagger - \bar{\lambda} I).$$

The elements $X^{\lambda,i} \in \mathsf{Ker}(\mathcal{G}^\dagger - \bar{\lambda} I)$ are dual vectors to the attractors $X_{\lambda,i}$ in the Schrödinger picture, i.e.

$$\mathsf{Tr}\left[X^{\dagger}_{\lambda,i}X^{\mu,j}\right]=\delta_{\lambda\mu}\delta_{ij}$$

with $X_{\lambda,i}$ and $X^{\lambda,i}$ linked through the relation (2.4). Using explicit formulas (3.5) resp. (3.7) for the generator $\mathcal{G}$, one obtains some basic properties of the attractor spectrum σ_{atr} and the attractor spaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Atr}(\mathcal{G}^{\dagger})$

$$\mathcal{G}(X_{\lambda,i})=\lambda X_{\lambda,i}\Leftrightarrow\mathcal{G}(X^{\dagger}_{\lambda,i})=\overline{\lambda}X^{\dagger}_{\lambda,i},$$
$$\mathcal{G}^{\dagger}(X^{\lambda,i})=\overline{\lambda}X^{\lambda,i}\Leftrightarrow\mathcal{G}^{\dagger}((X^{\lambda,i})^{\dagger})=\lambda(X^{\lambda,i})^{\dagger}$$

and thus $\lambda\in\sigma_{\mathrm{atr}}\Leftrightarrow\overline{\lambda}\in\sigma_{\mathrm{atr}}$ with the attractor in the Schrödinger picture corresponding to the conjugated eigenvalue $\overline{\lambda}$ given by the adjoint operator $X^{\dagger}_{\lambda,i}$ and similarly for the attractor in the Heisenberg picture.

One of the cornerstones in the development of the asymptotic theory for QMPs is the relation between attractors in the Schrödinger and in the Heisenberg picture, given by formula (2.4). However, this formula is highly unpractical as it requires the knowledge of all generalized eigenvectors $X_{\lambda,l,j}$, even those outside the attractor space and it is desirable to find a more convenient algebraic relation between attractors in the Schrödinger and the Heisenberg picture. Such a relation can be derived using a faithful $\mathcal{T}$-state σ, by application of the right multiplication operator, resp. the left multiplication operator, via the following theorem giving basic bijections between spaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Atr}(\mathcal{G}^{\dagger})$:

Theorem 4.1.1. *Let $\mathcal{T}_{\tau}$ be a faithful QMP on the Hilbert space $\mathcal{H}$ associated with a generator $\mathcal{G}$, let $0<\sigma_1,\sigma_2\in\mathsf{S}(\mathcal{H})$ be two, not necessarily different faithful $\mathcal{T}$-states. Then we have*

$$\begin{aligned}X\in\mathsf{Ker}(\mathcal{G}-\lambda\mathcal{I})&\Leftrightarrow R_{\sigma_1^{-1}}(X)\in\mathsf{Ker}(\mathcal{G}^{\dagger}-\overline{\lambda}I),\\ X\in\mathsf{Ker}(\mathcal{G}-\lambda\mathcal{I})&\Leftrightarrow L_{\sigma_1^{-1}}(X)\in\mathsf{Ker}(\mathcal{G}^{\dagger}-\overline{\lambda}I),\\ X\in\mathsf{Ker}(\mathcal{G}-\lambda\mathcal{I})&\Leftrightarrow\Delta_{\sigma_1,\sigma_2}(X)\in\mathsf{Ker}(\mathcal{G}-\lambda\mathcal{I}),\\ X\in\mathsf{Ker}(\mathcal{G}-\lambda\mathcal{I})&\Leftrightarrow\Delta^{-1}_{\sigma_1,\sigma_2}(X)\in\mathsf{Ker}(\mathcal{G}-\lambda\mathcal{I}).\end{aligned}\tag{4.3}$$

Proof. Consider a QMP $\mathcal{T}_{\tau}$ associated with a generator $\mathcal{G}$ and a faithful $\mathcal{T}$-state σ. The first equivalence relation (4.3) can be proved using a special superoperator $\mathcal{V}:\mathsf{B}(\mathcal{H})\to\mathsf{B}(\mathcal{H})$ defined for any $X\in\mathsf{B}(\mathcal{H})$ as

$$\mathcal{V}(X):=\mathcal{T}_1^{\dagger}(X\sigma^{-\frac{1}{2}})\sigma^{\frac{1}{2}}.$$

Consider arbitrary $X,Y\in\mathsf{B}(\mathcal{H})$, then

$$\begin{aligned}(X,\mathcal{V}(Y))_{\mathrm{HS}}=&\mathsf{Tr}\left[X^{\dagger}\mathcal{T}_1^{\dagger}(Y\sigma^{-\frac{1}{2}})\sigma^{\frac{1}{2}}\right]=\mathsf{Tr}\left[\sigma^{-\frac{1}{2}}\mathcal{T}(\sigma^{\frac{1}{2}}X^{\dagger})Y\right]=\\ &\mathsf{Tr}\left[\left(\mathcal{T}(X\sigma^{\frac{1}{2}})\sigma^{-\frac{1}{2}}\right)^{\dagger}Y\right]=(\mathcal{V}^{\dagger}(X),Y)_{\mathrm{HS}}.\end{aligned}$$

The adjoint superoperator $\mathcal{V}^{\dagger}$ takes the following form:

$$\mathcal{V}^\dagger(X) = \mathcal{T}_1(X\sigma^{\frac{1}{2}})\sigma^{-\frac{1}{2}}.$$

Superoperators $\mathcal{V}$ and $\mathcal{V}^\dagger$ are in fact contractions as for arbitrary $X \in \mathsf{B}(\mathcal{H})$ one obtains

$$
\begin{aligned}
||\mathcal{V}(X)||^2 =& \mathsf{Tr}\left[\mathcal{V}(X)^\dagger \mathcal{V}(X)\right] = \mathsf{Tr}\left[\mathcal{T}_1^\dagger(\sigma^{-\frac{1}{2}}X)\mathcal{T}_1^\dagger(X\sigma^{-\frac{1}{2}})\sigma\right] \leq \\
& \mathsf{Tr}\left[\mathcal{T}_1^\dagger(\sigma^{-\frac{1}{2}}X^\dagger X\sigma^{-\frac{1}{2}})\sigma\right] = \mathsf{Tr}\left[\sigma^{-\frac{1}{2}}X^\dagger X\sigma^{-\frac{1}{2}}\mathcal{T}_1(\sigma)\right] \leq \\
& \mathsf{Tr}\left[\sigma^{-\frac{1}{2}}X^\dagger X\sigma^{-\frac{1}{2}}\sigma\right] = ||X||^2
\end{aligned}
$$

and consequently $||\mathcal{V}|| = ||\mathcal{V}^\dagger|| \leq 1$. For $X \in \mathsf{Ker}(\mathcal{T}_1 - \lambda\mathcal{I})$, one can calculate the eigenvector corresponding to $\mathcal{V}$ as

$$\mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}}) = \mathcal{T}_1(X)\sigma^{-\frac{1}{2}} = \lambda X\sigma^{-\frac{1}{2}}. \tag{4.4}$$

This can be also reversed and thus

$$X \in \mathsf{Ker}(\mathcal{T}_1 - \lambda\mathcal{I}) \Leftrightarrow X\sigma^{-\frac{1}{2}} \in \mathsf{Ker}(\mathcal{V}^\dagger - \lambda I).$$

Furthermore, for any $X \in \mathsf{B}(\mathcal{H})$ one obtains

$$
\begin{aligned}
||\mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}})||^2 =& \mathsf{Tr}\left[\left(\mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}})\right)^\dagger \mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}})\right] = \\
& \mathsf{Tr}\left[\left(X\sigma^{-\frac{1}{2}}\right)^\dagger \mathcal{V}\mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}})\right] \leq \\
& ||X\sigma^{-\frac{1}{2}}|| \cdot ||\mathcal{V}\mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}})|| \leq \\
& ||\mathcal{V}|| \cdot ||\mathcal{V}^\dagger|| \cdot ||X\sigma^{-\frac{1}{2}}||^2 \leq ||X\sigma^{-\frac{1}{2}}||.
\end{aligned}
\tag{4.5}
$$

Putting together results (4.4) and (4.5), for $|\lambda| = 1$ and $X \in \mathsf{Ker}(\mathcal{T}_1 - \lambda\mathcal{I})$ the inequalities in (4.5) are saturated and consequently

$$\mathcal{V}\mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}}) = X\sigma^{-\frac{1}{2}}.$$

Using the definition (4.1) yields

$$\mathcal{V}\mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}}) = \mathcal{V}\left(\mathcal{T}(X)\sigma^{-\frac{1}{2}}\right) = \mathcal{T}_1^\dagger\left(\mathcal{T}_1(X)\sigma^{-1}\right)\sigma^{\frac{1}{2}}.$$

For $|\lambda| = 1$ and $X \in \mathsf{Ker}(\mathcal{T}_1 - \lambda\mathcal{I})$ one thus obtains

$$\mathcal{V}\mathcal{V}^\dagger(X\sigma^{-\frac{1}{2}}) = \lambda\mathcal{T}_1^\dagger(X\sigma^{-1})\sigma^{\frac{1}{2}} = X\sigma^{-\frac{1}{2}},$$

which results in

$$\mathcal{T}_1^\dagger(X\sigma^{-1}) = \bar{\lambda}X\sigma^{-1},$$
$$X \in \mathsf{Ker}(\mathcal{T}_1 - \lambda\mathcal{I}) \Leftrightarrow X\sigma^{-1} \in \mathsf{Ker}(\mathcal{T}_1^\dagger - \bar{\lambda}\mathcal{I}).$$

As $\mathcal{T}_1 = \mathcal{T}$ in the case of QMCH, the proof of the first relation in (4.3) is completed as the obtained equivalence implies

$$X \in \mathsf{Ker}(\mathcal{T} - \lambda\mathcal{I}) \Leftrightarrow X\sigma^{-1} \in \mathsf{Ker}\left(\mathcal{T}^\dagger - \bar{\lambda}\mathcal{I}\right)$$

for any $\lambda \in \sigma_{\mathrm{atr}}$.

For QMDS, one needs to be a little bit more careful, since the correspondence of the superoperators is $\mathcal{T}_1 = \exp[\mathcal{L}]$. For any $\lambda \in \sigma_{\mathrm{atr}}$ one has $e^\lambda \in \sigma(\mathcal{T}_1)$, $|e^\lambda| = 1$, which means that for certain eigenvalues λ, their associated eigenspaces $\mathsf{Ker}(\mathcal{L} - \lambda\mathcal{I})$ may correspond to a single eigenspace $\mathsf{Ker}(\mathcal{T}_1 - e^\lambda I)$. As an example, if $\sigma_{\mathrm{atr}} = \{0, -2i\pi, 2i\pi\}$, the quantum operation $\mathcal{T}_1$ has only a single eigenvalue modulo one, since $e^\lambda = 1$ for all $\lambda \in \sigma_{\mathrm{atr}}$. If such a case occurs, one simply chooses a different fixed time t for which the values $e^{\lambda t}$ do not coincide for any $\lambda \in \sigma_{\mathrm{atr}}$ and proceeds with the definition of the superoperator $\mathcal{V}$ for this particular value t. Consequently for any $\lambda \in \sigma_{\mathrm{atr}}$

$$X \in \mathsf{Ker}(\mathcal{L} - \lambda\mathcal{I}) \Leftrightarrow X\sigma^{-1} \in \mathsf{Ker}\left(\mathcal{L}^\dagger - \bar{\lambda}\mathcal{I}\right),$$

which concludes the proof of the first relation in (4.3) for any QMP $\mathcal{T}_\tau$ associated with a generator $\mathcal{G}$ and a $\mathcal{T}$-state σ.

The second relation in (4.3) can be proved analogously using a linear contraction $\mathcal{W}$ defined as

$$\mathcal{W}(X) = \sigma^{\frac{1}{2}} \mathcal{T}_1^\dagger (\sigma^{-\frac{1}{2}} X), \ X \in \mathsf{B}(\mathcal{H}).$$

The third and the fourth relations in (4.3) are a simple consequence of the first and the second relation in (4.3). Having $X \in \mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$ with $\lambda \in \sigma_{\mathrm{atr}}$, the first relation in (4.3) implies that for any two, not necessarily different $\mathcal{T}$-states σ_1 and σ_2 combining the first and the second relation in (4.3) yields

$$X \in \mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I}) \Leftrightarrow X\sigma_2^{-1} \in \mathsf{Ker}(\mathcal{G}^\dagger - \bar{\lambda}I) \Leftrightarrow \sigma_1 X\sigma_2^{-1} \in \mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I}),$$

which can be also inverted to obtain the fourth relation in (4.3), which concludes the proof.

$\square$

The consequences of this result are significant. First and foremost, it provides algebraic relations between the attractor spaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Atr}(\mathcal{G}^\dagger)$. Furthermore, it can be used to define a new inner product $(.,.)_\sigma$ as

$$(X, Y)_\sigma = \mathsf{Tr}\left[X^\dagger Y \sigma^{-1}\right]. \tag{4.6}$$

The new inner product (4.6) allows to define the concept of σ-orthogonality of two sets of operators $\mathcal{X}, \mathcal{Y} \subset \mathsf{B}(\mathcal{H})$:

Definition 4.1.2. *Consider a QMP $\mathcal{T}_\tau$ on a Hilbert space $\mathcal{H}$, associated with a faithful $\mathcal{T}$-state σ. Let $\mathcal{X}, \mathcal{Y} \subset \mathsf{B}(\mathcal{H})$ and $X \in \mathcal{X}, Y \in \mathcal{Y}$. Operators X and Y are called σ-orthogonal if $(X, Y)_\sigma = 0$. The sets $\mathcal{X}$ and $\mathcal{Y}$ called σ-orthogonal, i.e. $\mathcal{X} \perp_\sigma \mathcal{Y}$, if X and Y are σ-orthogonal for all $X \in \mathcal{X}, Y \in \mathcal{Y}$.*

Using the theorem 4.1.1 and the definition of σ-orthogonality, the following lemma can be formulated:

Lemma 4.1.3. *Consider a QMP $\mathcal{T}_\tau$ associated with a generator $\mathcal{G}$ and a faithful $\mathcal{T}$-state $\sigma > 0$, let $\lambda_1, \lambda_2 \in \sigma_{atr}$. Then the following holds:*

$$\mathsf{Ker}(\mathcal{G} - \lambda_1 I) \cap \mathsf{Ran}(\mathcal{G} - \lambda_1 I) = \{0\},$$
$$\mathsf{Ker}(\mathcal{G} - \lambda_1 I) \perp_\sigma \mathsf{Ran}(\mathcal{G} - \lambda_1 I),$$
$$\mathsf{Ker}(\mathcal{G} - \lambda_1 I) \perp_\sigma \mathsf{Ker}(\mathcal{G} - \lambda_2 I).$$

Proof. Consider $X \in \mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$ and $Y \in \mathsf{Ran}(\mathcal{G} - \lambda\mathcal{I})$ with $\lambda \in \sigma_{\mathrm{atr}}$. Thus, there must be $Z \in \mathsf{B}(\mathcal{H})$ such that $Y = \mathcal{G}(Z) - \lambda Z$. A simple calculation

$$
\begin{aligned}
(X, Y)_\sigma =& \mathsf{Tr}[X^\dagger Y \sigma^{-1}] = \mathsf{Tr}[X^\dagger (\mathcal{G}(Z) - \lambda Z)\sigma^{-1}] = \\
& \mathsf{Tr}[(\mathcal{G}(X\sigma^{-1})e)^\dagger Z] - \lambda \mathsf{Tr}[X^\dagger Z \sigma^{-1}] = \\
& \mathsf{Tr}[(\overline{\lambda} X \sigma^{-1})^\dagger Z] - \lambda \mathsf{Tr}[X^\dagger Z \sigma^{-1}] = 0,
\end{aligned}
$$

yields the first two relations.

Taking $X_1 \in \mathsf{Ker}(\mathcal{G} - \lambda_1)$, $X_2 \in \mathsf{Ker}(\mathcal{G} - \lambda_2 I)$ with $\lambda_{1,2} \in \sigma_{\mathrm{atr}}$, one obtains

$$
\begin{aligned}
(X_1, X_2)_\sigma =& \mathsf{Tr}[X_1^\dagger X_2 \sigma^{-1}] = \frac{1}{\overline{\lambda_1}} \mathsf{Tr}[\mathcal{G}(X_1^\dagger) X_2 \sigma^{-1}] = \\
& \frac{1}{\overline{\lambda_1}} \mathsf{Tr}[X_1^\dagger \mathcal{G}^\dagger (X_2 \sigma^{-1})] = \frac{\overline{\lambda_2}}{\overline{\lambda_1}} \mathsf{Tr}[X_1^\dagger X_2 \sigma^{-1}] = \frac{\overline{\lambda_2}}{\overline{\lambda_1}}, (X_1, X_2)_\sigma
\end{aligned}
$$

which implies that $(X_1, X_2)_\sigma = 0$ for $\lambda_1 \neq \lambda_2$.

$\square$

The space $\mathsf{B}(\mathcal{H})$ can be thus split into two parts, the attractor space $\mathsf{Atr}(\mathcal{G})$ and the rest $\mathsf{Van}(\mathcal{G})$ defined as

$$\mathsf{Van}(\mathcal{G}) = \bigcap_{\lambda \in \sigma_{\mathrm{atr}}} \mathsf{Ran}(\mathcal{G} - \lambda\mathcal{I}).$$

Lemma 4.1.3 then implies the relations

$$\mathsf{Atr}(\mathcal{G}) \oplus \mathsf{Van}(\mathcal{G}) = \mathsf{B}(\mathcal{H}),$$
$$\mathsf{Atr}(\mathcal{G}) \cap \mathsf{Van}(\mathcal{G}) = \{0\},$$
$$\mathsf{Atr}(\mathcal{G}) \perp_\sigma \mathsf{Van}(\mathcal{G}).$$

Since the subspaces $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$ and $\mathsf{Ran}(\mathcal{G} - \lambda\mathcal{I})$ are invariant under the application of the QMP $\mathcal{T}_\tau$, the same holds for the subspaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Van}(\mathcal{G})$. Having the Hilbert space $\mathsf{B}(\mathcal{H})$ divided into two invariant parts, with the first part $\mathsf{Atr}(\mathcal{G})$ consisting of operators fixed/rotating under the application of the QMP $\mathcal{T}_\tau$ and the second part vanishing in the asymptotics, with the additional property that these two parts are σ-orthogonal, allows one to write the analytic form of the asymptotic state $\rho_\infty(\tau)$ in terms of attractors for any initial state $\rho(0)$.

4.2 Asymptotic dynamics and attractor equations

As a next step in the derivation of the asymptotic state, one can choose any orthonormal basis of all subspaces $\mathsf{Ker}(\mathcal{G} - \lambda \mathcal{I})$ $\{X_{\lambda,i}|, i \in \{1, \ldots, \mathsf{d}(\lambda)\}\}$ with respect to the inner product (4.6). According to the previous section, elements $X_{\lambda,i}$ corresponding to different eigenvalues λ are σ-orthogonal and thus the set $\{X_{\lambda,i}|\lambda \in \sigma_{\mathrm{atr}}, i \in \{1, \ldots, \mathsf{d}(\lambda)\}\}$ forms an orthonormal basis of the subspace $\mathsf{Atr}(\mathcal{G})$ with respect to the inner product (4.6). Starting from an initial state $\rho(0)$, using the σ-orthogonality of the two invariant subspaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Van}(\mathcal{G})$ and the fact that in the asymptotic regime the blocks corresponding to the part $\mathsf{Van}(\mathcal{G})$ vanish, the asymptotic evolution takes the form

$$\rho_\infty(n) = \sum_{\lambda \in \sigma_{\mathrm{atr}}, i=1}^{\mathsf{d}(\lambda)} \lambda^n \mathsf{Tr}\left[(X^{\lambda,i})^\dagger \rho(0)\right] X_{\lambda,i} \tag{4.7}$$

for the case of QMCH, resp.

$$\rho_\infty(t) = \sum_{\lambda \in \sigma_{\mathrm{atr}}, i=1}^{\mathsf{d}(\lambda)} \exp[\lambda t] \, \mathsf{Tr}\left[(X^{\lambda,i})^\dagger \rho(0)\right] X_{\lambda,i} \tag{4.8}$$

for the case of QMDS, in the sense that

$$\lim_{\tau \to \infty} ||\rho(\tau) - \rho_\infty(\tau)|| = 0.$$

It is important to emphasize that, in general, the elements of the basis $X_{\lambda,i}$ do not represent a quantum states, i.e. generally $X_{\lambda,i} \notin \mathsf{S}(\mathcal{H})$. Linear combinations (4.7) and (4.8) nevertheless always represent a valid quantum state, i.e. $\rho_\infty(\tau) \in \mathsf{S}(\mathcal{H})$.

The issue of nondiagonalizability of the generator $\mathcal{G}$ can be therefore partially solved by restriction of the evolution to the asymptotic regime, since the part of the generator responsible for the asymptotic evolution is diagonalizable. Furthermore, the expressions for the asymptotic state ρ_∞ (4.7) and (4.8) contain an orthonormal basis of dual attractors $X^{\lambda,i}$, which can be, according to theorem (4.3), chosen as

$$X^{\lambda,i} = \frac{1}{\mathsf{Tr}[X_{\lambda,i}^\dagger X_{\lambda,i} \sigma^{-1}]} X_{\lambda,i} \sigma^{-1} \tag{4.9}$$

and the asymptotic state thus directly involves the faithful $\mathcal{T}$-state σ.

The asymptotic dynamics of any QMP $\mathcal{T}_\tau$ in Schrödinger picture is thus fully described in terms of attractors given by equations (4.7), (4.8) and (4.9). To be able to calculate the asymptotic dynamics, one has to have access to the attractor spaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Atr}(\mathcal{G}^\dagger)$. It is highly unpractical to derive these spaces using brute force, i.e by solving equations $\mathcal{G}(X) = \lambda X$ (resp. $\mathcal{G}^\dagger(X) = \overline{\lambda} X$). Instead, a set of algebraic equations, called the **attractor equations** can be derived. For QMCHs, these equations are specified in the following theorem:

Theorem 4.2.1. *Let $\mathcal{T}_n$ be a faithful QMCH associated with a generator $\mathcal{T}$ and a faithful $\mathcal{T}$-state $\sigma > 0$. Let K_j, $j \in \mathcal{J}$ be a Kraus operators defining the generator $\mathcal{T}$. Then each element $X \in \mathsf{Ker}(\mathcal{T} - \lambda\mathcal{I})$ with $\lambda \in \sigma_{atr}$ fulfils the following set of algebraic equations called the **attractor equations** for QMCH $\mathcal{T}_n$*

$$
\begin{aligned}
K_j X \sigma^{-1} &= \lambda X \sigma^{-1} K_j, \\
K_j^{\dagger} X \sigma^{-1} &= \bar{\lambda} X \sigma^{-1} K_j^{\dagger}, \\
K_j \sigma^{-1} X &= \lambda \sigma^{-1} X K_j, \\
K_j^{\dagger} \sigma^{-1} X &= \bar{\lambda} \sigma^{-1} X K_j^{\dagger},
\end{aligned}
\tag{4.10}
$$

for all $j \in \mathcal{J}$. In the Heisenberg picture, any $X \in \mathsf{Ker}\left(\mathcal{T}^{\dagger} - \bar{\lambda}\mathcal{I}\right)$ fulfils the set of equations

$$
\begin{aligned}
K_j X &= \lambda X K_j, \\
K_j^{\dagger} X &= \bar{\lambda} X K_j^{\dagger}.
\end{aligned}
\tag{4.11}
$$

for all $j \in \mathcal{J}$. If σ is an invariant state of the QMCH $\mathcal{T}_n$, or the QMCH $\mathcal{T}_n$ is TP, then the implication in the other direction is also true, i.e. any solution X of the equations (4.10) fulfils $X \in \mathsf{Ker}(\mathcal{T} - \lambda\mathcal{I})$.

Proof. For each index j define a linear superoperator $\mathcal{K}_j^{(1)} : \mathsf{Atr}(\mathcal{T}) \to \mathsf{B}(\mathcal{H})$ as

$$
\mathcal{K}_j^{(1)}(X) = \lambda X \sigma^{-1} K_j \sigma^{\frac{1}{2}} - K_j X \sigma^{-\frac{1}{2}}, \ X \in \mathsf{Ker}(\mathcal{T} - \lambda\mathcal{I}).
$$

To prove the validity of the first set of equations, consider $X \in \mathsf{Ker}(\mathcal{T} - \lambda\mathcal{I})$ to obtain the following estimate:

$$
\begin{aligned}
0 \leq \sum_j \|\mathcal{K}_j^{(1)}(X)\|^2 &= \sum_j \mathsf{Tr}\left[(\mathcal{K}_j^{(1)}(X))^{\dagger}\mathcal{K}_j^{(1)}(X)\right] = \\
\mathsf{Tr}\left[X\sigma^{-1}X^{\dagger}\mathcal{T}^{\dagger}(I)\right] &+ |\lambda|^2\mathsf{Tr}[X\sigma^{-1}\mathcal{T}(\sigma)\sigma^{-1}X^{\dagger}] - \\
\lambda\mathsf{Tr}[X\sigma^{-1}\mathcal{T}(X^{\dagger})] &- \bar{\lambda}\mathsf{Tr}[\mathcal{T}(X)\sigma^{-1}X^{\dagger}] \leq \\
(1 - |\lambda|^2)\mathsf{Tr}&[X\sigma^{-1}X^{\dagger}],
\end{aligned}
$$

where in the last inequality, assumptions $\mathcal{T}(\sigma) \leq \sigma$, $\mathcal{T}^{\dagger}(I) \leq I$, $\mathcal{T}(X) = \lambda X$ and $\mathcal{T}(X^{\dagger}) = \bar{\lambda}X^{\dagger}$ were simultaneously used. This implies $\mathcal{K}_j^{(1)} = 0$, which yields

$$
K_j X \sigma^{-\frac{1}{2}} = \lambda X \sigma^{-1} K_j \sigma^{\frac{1}{2}} \Leftrightarrow K_j X \sigma^{-1} = \lambda X \sigma^{-1} K_j,
$$

i.e. the first set of equations. The second, third and the fourth set of equations can be obtained analogously, using the following superoperators:

$$
\begin{aligned}
\mathcal{K}_j^{(2)} &= \bar{\lambda}\sigma^{\frac{1}{2}}X\sigma^{-1}K_j^{\dagger} - \sigma^{\frac{1}{2}}K_j^{\dagger}X\sigma^{-1}, \\
\mathcal{K}_j^{(3)} &= \lambda\sigma^{-1}XK_j\sigma^{\frac{1}{2}} - K_j\sigma^{-1}X\sigma^{\frac{1}{2}}, \\
\mathcal{K}_j^{(4)} &= \bar{\lambda}\sigma^{-\frac{1}{2}}XK_j^{\dagger} - \sigma^{\frac{1}{2}}K_j^{\dagger}\sigma^{-1}X
\end{aligned}
$$

and thus any element $X \in \mathsf{Ker}(\mathcal{T} - \lambda \mathcal{I})$ must fulfil equations (4.10).

If the QMCH $\mathcal{T}_n$ is TP, for any $X \in \mathsf{B}(\mathcal{H})$ fulfilling equations (4.10) one obtains

$$\mathcal{T}^\dagger(X\sigma^{-1}) = \sum_j K_j^\dagger X\sigma^{-1} K_j = \overline{\lambda} X\sigma^{-1} \sum_j K_j^\dagger K_j = \overline{\lambda} X\sigma^{-1},$$

or $X\sigma^{-1} \in \mathsf{Ker}\left(\mathcal{T}^\dagger - \overline{\lambda}\mathcal{I}\right)$. According to theorem 4.1.1 this implies $X \in \mathsf{Ker}(\mathcal{T} - \lambda\mathcal{I})$.

Similarly, if the corresponding σ is an invariant state of the QMCH $\mathcal{T}_n$, for any solution $X \in \mathsf{B}(\mathcal{H})$ of the equations (4.10) one obtains

$$\mathcal{T}(X) = \sum_j K_j X K_j^\dagger = \sum_j K_j X\sigma^{-1}\sigma K_j^\dagger = \lambda X\sigma^{-1} \sum_j K_j \sigma K_j^\dagger =$$
$$= \lambda X\sigma^{-1}\mathcal{T}(\sigma) = \lambda X,$$

i.e. $X \in \mathsf{Ker}(\mathcal{T} - \lambda\mathcal{I})$.

Attractor equations in the Heisenberg picture are obtained by using theorem 4.1.1. Taking $X \in \mathsf{Ker}\left(\mathcal{T}^\dagger - \overline{\lambda}\mathcal{I}\right)$, according to theorem 4.1.1 one has $X\sigma \in \mathsf{Ker}(\mathcal{T} - \lambda\mathcal{I})$ and according to the already found attractor equations (4.10):

$$K_j X = K_j (X\sigma)\sigma^{-1} = \lambda(X\sigma)\sigma^{-1} K_j = \lambda X K_j$$

and analogously for the other equations. $\qquad\square$

The attractor equations for a QMDS $\mathcal{T}_t$ can be derived in a similar way. For a Lindbladian $\mathcal{L}$ cast in the form (3.8) one can formulate the following theorem:

Theorem 4.2.2. *Let $\mathcal{T}_t$ be a QMDS associated with a Lindbladian generator $\mathcal{L}$ and a faithful $\mathcal{T}$-state $\sigma > 0$. Let $H \in \mathsf{A}(H)$ be the corresponding Hamiltonian, L_j, $j \in \mathcal{J}$ the noise operators and G the optical potential defining the generator $\mathcal{L}$. Then each element $X \in \mathsf{Ker}(\mathcal{L} - \lambda\mathcal{I})$ with $\lambda \in \sigma_{atr}$, $\lambda = ia$, $a \in \mathbb{R}$ fulfils the following set of algebraic equations, called the **attractor equations** for QMDS $\mathcal{T}_t$.*

$$[L_j, X\sigma^{-1}] = [L_j, \sigma^{-1}X] = [L_j^\dagger, X\sigma^{-1}] = [L_j^\dagger, \sigma^{-1}X] = 0,$$
$$[G, X\sigma^{-1}] = [G, \sigma^{-1}X] = 0,$$
$$[H, X\sigma^{-1}] = -aX\sigma^{-1}, \tag{4.12}$$
$$[H, \sigma^{-1}X] = -a\sigma^{-1}X,$$

for all $j \in \mathcal{J}$. In the Heisenberg picture, any $X \in \mathsf{Ker}\left(\mathcal{L}^\dagger - \overline{\lambda}\mathcal{I}\right)$ fulfils the set of equations

$$[L_j, X] = [L_j^\dagger, X] = 0,$$
$$[G, X] = 0, \tag{4.13}$$
$$[H, X] = -aX.$$

for all $j \in \mathcal{J}$. If σ is an invariant state of the QMDS $\mathcal{T}_t$, or the QMDS $\mathcal{T}_t$ is TP, then the implication in the other direction is also true, i.e. any solution X of the equations (4.12) fulfils $X \in \text{Ker}(\mathcal{L} - \lambda\mathcal{I})$.

Proof. Consider $X \in \text{Ker}(\mathcal{L} - \lambda\mathcal{I})$ with $\lambda = ia \in \sigma_{\text{atr}}$. According to theorem 4.1.1 the following equalities hold

$$\mathcal{L}^\dagger(\sigma^{-1}X) = -ia\sigma^{-1}X, \quad \mathcal{L}^\dagger(\sigma^{-1}X^\dagger) = ia\sigma^{-1}X^\dagger. \tag{4.14}$$

Using (3.7), these equalities can be rewritten as

$$\sum_j L_j^\dagger X\sigma^{-1}L_j = -iaX\sigma^{-1} + K^\dagger X\sigma^{-1} + X\sigma^{-1}K,$$

$$\sum_j L_j^\dagger \sigma^{-1}X^\dagger L_j = ia\sigma^{-1}X^\dagger + K^\dagger\sigma^{-1}X^\dagger + \sigma^{-1}XK,$$

$$\sum_j L_j^\dagger X^\dagger\sigma^{-1}L_j = iaX^\dagger\sigma^{-1} + K^\dagger X^\dagger\sigma^{-1} + X^\dagger\sigma^{-1}K, \tag{4.15}$$

$$\sum_j L_j^\dagger L_j \leq K + K^\dagger,$$

where the equations (4.14) are accompanied by the condition that the QMDS $\mathcal{T}_t$ generated by the Lindbladian generator $\mathcal{L}$ is TNI.

Furthermore, additional trivial equalities hold:

$$\mathcal{T}_t^\dagger(\sigma^{-1}X^\dagger) = e^{iat}\sigma^{-1}X^\dagger, \quad \mathcal{T}_t^\dagger(X\sigma^{-1}) = e^{-iat}X\sigma^{-1}. \tag{4.16}$$

Using the operator Schwarz inequality, equations (4.16) yield

$$\mathcal{T}_t^\dagger(\sigma^{-1}X^\dagger X\sigma^{-1}) \leq \mathcal{T}_t^\dagger(\sigma^{-1}X^\dagger)\mathcal{T}_t^\dagger(X\sigma^{-1}) = \sigma^{-1}X^\dagger X\sigma^{-1},$$

which holds for any $t \geq 0$ and one can thus conclude that

$$\mathcal{L}^\dagger(\sigma^{-1}X^\dagger X\sigma^{-1}) \leq 0,$$

which can be again rewritten using (3.7) as

$$\sum_j L_j^\dagger \sigma^{-1}X^\dagger X\sigma^{-1}L_j \leq K^\dagger\sigma^{-1}X^\dagger X\sigma^{-1} + \sigma^{-1}X^\dagger X\sigma^{-1}K. \tag{4.17}$$

Next, define superoperators $\mathcal{V}_j : \mathsf{B}(\mathcal{H}) \to \mathsf{B}(\mathcal{H})$ as

$$\mathcal{V}_j(X) = [X\sigma^{-1}, L_j].$$

Using the previous inequalities one obtains

$$0 \leq \sum_j ||\mathcal{V}_j(X)||^2 = \sum_j \mathsf{Tr}\left[(\mathcal{V}_j(X))^\dagger \mathcal{V}_j(X)\right] =$$

$$\mathsf{Tr}\left[\sum_j L_j^\dagger \sigma^{-1} X^\dagger X \sigma^{-1} L_j\right] + \mathsf{Tr}\left[\sigma^{-1} X^\dagger \left(\sum_j L_j^\dagger L_j\right) X \sigma^{-1}\right] -$$

$$\mathsf{Tr}\left[\left(\sum_j L_j^\dagger \sigma^{-1} X^\dagger L_j\right) X \sigma^{-1}\right] - \mathsf{Tr}\left[\sigma^{-1} X^\dagger \left(\sum_j L_j^\dagger X \sigma^{-1} L_j\right)\right] \leq$$

$$\mathsf{Tr}\left[K^\dagger \sigma^{-1} X^\dagger X \sigma^{-1} + \sigma^{-1} X^\dagger X \sigma^{-1} K\right] + \mathsf{Tr}[\sigma^{-1} X^\dagger (K + K^\dagger) X \sigma^{-1}] -$$
$$\mathsf{Tr}[(ia\sigma^{-1} X^\dagger + K^\dagger \sigma^{-1} X^\dagger + \sigma^{-1} X^\dagger K) X \sigma^{-1}] -$$
$$\mathsf{Tr}[\sigma^{-1} X^\dagger (-iaX\sigma^{-1} + K^\dagger X \sigma^{-1} + X \sigma^{-1} K)] = 0,$$

where in the last inequality relations (4.15) and (4.17) were used. One thus must have $\mathcal{V}_j = 0$, which yields the set of equations

$$[X\sigma^{-1}, L_j] = 0, \quad \forall j. \tag{4.18}$$

To obtain the remaining commutation relations, consider $X \in \mathsf{Ker}(\mathcal{L} - \lambda \mathcal{I})$. Using the first, already proved set of attractor equations in (4.12) and multiplying by operator $\sigma^{-1} X^\dagger$, this equation can be rewritten into the following form:

$$X\sigma^{-1} K X^\dagger - K X \sigma^{-1} X^\dagger - iaX\sigma^{-1} X^\dagger = -X\sigma^{-1} \mathcal{L}(\sigma) \sigma^{-1} X^\dagger. \tag{4.19}$$

For further considerations, denote the left side of the equation by $\mathcal{Z}_1$, i.e.

$$\mathcal{Z}_1 = X\sigma^{-1} K X^\dagger - K X \sigma^{-1} X^\dagger - iaX\sigma^{-1} X^\dagger. \tag{4.20}$$

As σ is a $\mathcal{T}$-state, the right-hand side of the equation (4.19) implies that $\mathcal{Z}_1 \geq 0$. On the other hand, by using $\mathcal{L}^\dagger(X\sigma^{-1}) = -iaX\sigma^{-1}$ one obtains

$$0 \leq \mathsf{Tr}[\mathcal{Z}_1] = \mathsf{Tr}[X\sigma^{-1} K X^\dagger - K X \sigma^{-1} X^\dagger - iaX\sigma^{-1} X^\dagger] =$$

$$\mathsf{Tr}\left[\left(\sum_j L_j^\dagger L_j - K - K^\dagger\right) X \sigma^{-1} X^\dagger\right] =$$

$$\mathsf{Tr}\left[\mathcal{L}^\dagger(I) X \sigma^{-\frac{1}{2}} \left(X\sigma^{-\frac{1}{2}}\right)^\dagger\right] \leq 0,$$

and thus $\mathcal{Z}_1 = 0$. The same procedure can be applied on the equation $\mathcal{L}(X\sigma^{-1}) = -iaX\sigma^{-1}$, which can be transformed analogously into the following form:

$$\sigma^{-1} X^\dagger K X \sigma^{-1} - \sigma^{-1} X^\dagger X \sigma^{-1} K + ia\sigma^{-1} X^\dagger X \sigma^{-1} = -\sigma^{-1} X^\dagger \mathcal{L}^\dagger X \sigma^{-1}. \tag{4.21}$$

Consider the left-hand side of equation (4.21), denoted as $\mathcal{Z}_2$, i.e.

$$\mathcal{Z}_2 = \sigma^{-1}X^\dagger K X \sigma^{-1} - \sigma^{-1}X^\dagger X \sigma^{-1}K + ia\sigma^{-1}X^\dagger X \sigma^{-1},$$

As the QMDS $\mathcal{T}_t$ is TNI, one must have $\mathcal{Z}_2 \geq 0$. On the other hand, using $\mathcal{L}^\dagger(\sigma^{-1}X^\dagger) = ia\sigma^{-1}X^\dagger$ implies

$$0 \leq \mathsf{Tr}[\mathcal{Z}_2] = \mathsf{Tr}[\sigma^{-1}X^\dagger K X \sigma^{-1} - \sigma^{-1}X^\dagger X \sigma^{-1}K + \mathcal{L}^\dagger(\sigma^{-1}X^\dagger)X\sigma^{-1}] =$$

$$\mathsf{Tr}\left[\left(\sum_j L_j^\dagger L_j - K - K^\dagger\right)\sigma^{-1}X^\dagger X \sigma^{-1}\right] =$$

$$\mathsf{Tr}\left[\mathcal{L}^\dagger(I)\left(X\sigma^{-\frac{1}{2}}\right)^\dagger X\sigma^{-\frac{1}{2}}\right] \leq 0$$

and consequently $\mathcal{Z}_2 = 0$.

Next, investigate the operator $\mathcal{W}$ defined as

$$\mathcal{W} = KX\sigma^{-\frac{1}{2}} - X\sigma^{-1}K\sigma^{\frac{1}{2}} + iaX\sigma^{-\frac{1}{2}}.$$

Its adjoint reads

$$\mathcal{W}^\dagger = \sigma^{-\frac{1}{2}}X^\dagger K^\dagger - \sigma^{\frac{1}{2}}K^\dagger\sigma^{-1}X^\dagger - ia\sigma^{\frac{1}{2}}X^\dagger.$$

The Hilbert-Schmidt norm of the operator W reads

$$||\mathcal{W}||_{\mathrm{HS}}^2 = \mathsf{Tr}[\mathcal{W}^\dagger\mathcal{W}] =$$
$$- \mathsf{Tr}[\sigma^{-\frac{1}{2}}K^\dagger\sigma^{-1}X^\dagger K X \sigma^{-\frac{1}{2}} + \sigma^{\frac{1}{2}}K^\dagger\sigma^{-1}X^\dagger X \sigma^{-1}K\sigma^{\frac{1}{2}} -$$
$$ia\sigma^{\frac{1}{2}}K^\dagger\sigma^{-1}X^\dagger X\sigma^{-\frac{1}{2}}]+$$
$$\mathsf{Tr}[ia\sigma^{-\frac{1}{2}}X^\dagger X\sigma^{-1}K\sigma^{\frac{1}{2}} - ia\sigma^{-\frac{1}{2}}X^\dagger K X\sigma^{-\frac{1}{2}} +$$
$$a^2\sigma^{-\frac{1}{2}}X^\dagger X\sigma^{-\frac{1}{2}}]-$$
$$\mathsf{Tr}[\sigma^{-\frac{1}{2}}X^\dagger K^\dagger X\sigma^{-1}K\sigma^{\frac{1}{2}} - \sigma^{-\frac{1}{2}}X^\dagger K^\dagger K X\sigma^{-\frac{1}{2}} -$$
$$ia\sigma^{-\frac{1}{2}}X^\dagger K^\dagger X\sigma^{-\frac{1}{2}}] =$$
$$\mathsf{Tr}[\mathcal{Z}_1(iaI - K^\dagger)] - \mathsf{Tr}[\mathcal{Z}_2\sigma K^\dagger] = 0$$

and consequently $\mathcal{W} = 0$, which can be rewritten into the equation

$$[K, X\sigma^{-1}] = -iaX\sigma^{-1}.$$

Using theorem 4.1.1 with $\sigma_1 = \sigma_2 = \sigma^{-1}$, one obtains

$$[K, \sigma^{-1}X] = -ia\sigma^{-1}X.$$

Analogously, procedure for $X^\dagger \in \mathsf{Ker}(\mathcal{L} - \bar{\lambda}I)$ results in

$$[K^\dagger, X\sigma^{-1}] = iaX\sigma^{-1}, \quad [K^\dagger, \sigma^{-1}X] = ia\sigma^{-1}X.$$

Now, using $K = iH + G = \frac{1}{2}\sum_j L_j^\dagger L_j$ one finds

$$[G, X\sigma^{-1}] = [G, \sigma^{-1}X] = 0,$$
$$[H, X\sigma^{-1}] = -aX\sigma^{-1},$$
$$[H, \sigma^{-1}X] = -a\sigma^{-1}X.$$

Consequently, any element of the subspace $\mathsf{Ker}(\mathcal{L} - \lambda\mathcal{I})$ with $\lambda = ia \in \sigma_{\text{atr}}$ must fulfil equations (4.12).

If the corresponding QMDS $\mathcal{T}_t$ is TP, σ is a $\mathcal{T}$-state and $X \in \mathsf{B}(\mathcal{H})$ fulfils equations (4.12), one has

$$\mathcal{L}^\dagger(X\sigma^{-1}) = \sum_j L_j^\dagger X\sigma^{-1}L_j - K^\dagger X\sigma^{-1} - X\sigma^{-1}K =$$

$$= X\sigma^{-1}\left(\sum_j L_j^\dagger L_j - K^\dagger - K\right) - iaX\sigma^{-1} =$$

$$= X\sigma^{-1}\mathcal{L}^\dagger(I) + \overline{\lambda}X\sigma^{-1} = \overline{\lambda}X\sigma^{-1}$$

and thus $X\sigma^{-1} \in \mathsf{Ker}\left(\mathcal{L}^\dagger - \overline{\lambda}\mathcal{I}\right)$. According to theorem 4.1.1 this implies $X \in \mathsf{Ker}(\mathcal{L} - \lambda\mathcal{I})$. Similarly, if the corresponding σ is an invariant state of the QMDS $\mathcal{T}$, for any solution $X \in \mathsf{B}(\mathcal{H})$ of the equations (4.12) one obtains

$$\mathcal{L}(X) = \sum_j L_j X L_j^\dagger - KX - XK^\dagger =$$

$$\sum_j L_j X\sigma^{-1}\sigma L_j^\dagger - KX\sigma^{-1}\sigma - X\sigma^{-1}\sigma K^\dagger =$$

$$X\sigma^{-1}\left(\sum_j L_j \sigma L_j^\dagger - K\sigma - \sigma K^\dagger\right) + iaX\sigma^{-1}\sigma =$$

$$X\sigma^{-1}\mathcal{L}(\sigma) + \lambda X = \lambda X$$

resulting in $X \in \mathsf{Ker}(\mathcal{L} - \lambda\mathcal{I})$.

For attractors in the Heisenberg picture, the theorem 4.1.1 can be utilized. Taking $X \in \mathsf{Ker}\left(\mathcal{L}^\dagger - \overline{\lambda}\mathcal{I}\right)$, one has $X\sigma \in \mathsf{Ker}(\mathcal{L} - \lambda\mathcal{I})$ and thus X must fulfil equations (4.12), yielding

$$[L_j, (X\sigma)\sigma^{-1}] = [L_j, X] = 0,$$
$$[L_j^\dagger, (X\sigma)\sigma^{-1}] = [L_j^\dagger, X] = 0,$$
$$[G, (X\sigma)\sigma^{-1}] = [G, X] = 0,$$
$$[H, (X\sigma)\sigma^{-1}] = -a(X\sigma)\sigma^{-1} = -aX.$$

$$\square$$

Generally, theorem 4.2.1, resp. theorem 4.2.2 gives a necessary condition for an operator to be an attractor of the QMP $\mathcal{T}_\tau$ and operators obtained by solution of

attractors equations are just candidates for attractors. These candidates must be therefore confirmed (or disproved) by application of the QMP $\mathcal{T}_\tau$. However, if the QMP $\mathcal{T}_\tau$ either possesses a faithful $\mathcal{T}$-state σ which is also an invariant state of the QMP $\mathcal{T}_\tau$, or the QMP $\mathcal{T}_\tau$ is TP, fulfilling equations (4.10), resp. equations (4.12) constitute also the sufficient condition for an operator to be an attractor of the QMP $\mathcal{T}_\tau$ and all solutions of the attractor equations are thus elements of the subspace $\mathsf{Atr}(\mathcal{G})$. Theorem 4.2.1, resp. theorem 4.2.2 provide a systematic way for finding attractor spaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Atr}(\mathcal{G}^\dagger)$.

Because of the simpler structure of equations (4.11), resp. (4.13), it is often more natural to first calculate the attractor space $\mathsf{Atr}(\mathcal{G}^\dagger)$. The attractor space $\mathsf{Atr}(\mathcal{G})$ is then simply obtained by using the theorem 4.1.1. Both the attractors in the Schrödinger and the Heisenberg picture exhibit interesting algebraic properties, which can considerably simplify the construction of the corresponding attractor space. For their simplicity, these are presented in a single theorem covering both cases of QMP:

Lemma 4.2.3. *Let $\mathcal{T}_n$ be a QMCH associated with a generator $\mathcal{T}$ and a faithful $\mathcal{T}$-state σ, let $\lambda_1, \lambda_2 \in \sigma_{atr}$. Let $X_1 \in \mathsf{Ker}(\mathcal{T} - \lambda_1 I)$, $X_2 \in \mathsf{Ker}(\mathcal{T} - \lambda_2 I)$, $Y_1 \in \mathsf{Ker}(\mathcal{T}^\dagger - \lambda_1 I)$, $Y_2 \in \mathsf{Ker}(\mathcal{T}^\dagger - \lambda_2 I)$. Then, if σ is invariant state, or the QMCH $\mathcal{T}_n$ is TP, one has*

$$X_1 X_2 \sigma^{-1} \in \mathsf{Ker}(\mathcal{T} - \lambda_1 \lambda_2 I),$$
$$Y_1 Y_2 \in \mathsf{Ker}(\mathcal{T}^\dagger - \lambda_1 \lambda_2 I).$$

Let $\mathcal{T}_t$ be a QMDS associated with a Lindbladian generator $\mathcal{L}$ and a faithful $\mathcal{T}$-state σ, let $\lambda_1, \lambda_2 \in \sigma_{atr}$. Let $X_1 \in \mathsf{Ker}(\mathcal{L} - \lambda_1 I)$, $X_2 \in \mathsf{Ker}(\mathcal{L} - \lambda_2 I)$, $Y_1 \in \mathsf{Ker}(\mathcal{L}^\dagger - \lambda_1 I)$, $Y_2 \in \mathsf{Ker}(\mathcal{L}^\dagger - \lambda_2 I)$. Then, if σ is invariant state, or the QMCH $\mathcal{T}_t$ is TP, one has

$$X_1 X_2 \sigma^{-1} \in \mathsf{Ker}(\mathcal{T} - (\lambda_1 + \lambda_2 I)),$$
$$Y_1 Y_2 \in \mathsf{Ker}(\mathcal{T}^\dagger - (\lambda_1 + \lambda_2 I)).$$

Proof. For the case of QMCH $\mathcal{T}_n$, consider $X_1 \in \mathsf{Ker}(\mathcal{T} - \lambda_1 I)$ and $X_2 \in \mathsf{Ker}(\mathcal{T} - \lambda_2 I)$. It follows that $\sigma X_2 \sigma^{-1} \in \mathsf{Ker}(\mathcal{T} - \lambda_2 I)$ and according to equations (4.10) one has

$$
\begin{aligned}
K_j (X_1 X_2 \sigma^{-1}) \sigma^{-1} =& K_j (X_1 \sigma^{-1})(\sigma X_2 \sigma^{-1} \sigma^{-1}) = \\
& \lambda_1 X_1 \sigma^{-1} K_j (\sigma X_2 \sigma^{-1} \sigma^{-1}) = \\
& \lambda_1 \lambda_2 (X_1 \sigma^{-1})(\sigma X_2 \sigma^{-1} \sigma^{-1}) K_j = \\
& \lambda_1 \lambda_2 X_1 X_2 \sigma^{-1} K_j
\end{aligned}
$$

and similarly for other equations. The statement for the attractors of QMCH in the Heisenberg picture is then a simple consequence of the theorem 4.1.1.

For the case of QMDS $\mathcal{T}_t$, consider $X_1 \in \mathsf{Ker}(\mathcal{L} - \lambda_1 I)$ and $X_2 \in \mathsf{Ker}(\mathcal{L} - \lambda_2 I)$, with $\lambda_1 = ia_1$ and $\lambda_2 = ia_2$. Again, one has $\sigma X_2 \sigma^{-1} \in \mathsf{Ker}(\mathcal{L} - \lambda_2 I)$ and due to equations (4.12)

$$[L_j, (X_1 X_2 \sigma^{-1}) \sigma^{-1}] = [L_j, (X_1 \sigma^{-1})(\sigma X_2 \sigma^{-1} \sigma^{-1})] =$$
$$[L_j, X_1 \sigma^{-1}] \sigma X_2 \sigma^{-2} + X_1 \sigma^{-1} [L_j, \sigma X_2 \sigma^{-1} \sigma^{-1}] = 0$$

and similarly for other equations (4.12). For the commutation relation with the Hamiltonian H one obtains

$$[H, (X_1 X_2 \sigma^{-1}) \sigma^{-1}] = [H, (X_1 \sigma^{-1})(\sigma X_2 \sigma^{-1} \sigma^{-1})] =$$
$$= [H, X_1 \sigma^{-1}] \sigma X_2 \sigma^{-2} + X_1 \sigma^{-1} [H, \sigma X_2 \sigma^{-1} \sigma^{-1}] =$$
$$= -a_1 X_1 \sigma^{-1} \sigma X_2 \sigma^{-2} + X_1 \sigma^{-1} (-a_2 \sigma X_2 \sigma^{-1} \sigma^{-1}) =$$
$$= -(a_1 + a_2)(X_1 X_2 \sigma^{-1}) \sigma^{-1}$$

and similarly for the other equation (4.12) concerning the Hamiltonian. The statement for the attractors of QMDS in the Heisenberg picture is then a simple consequence of the theorem 4.1.1.

$\square$

Relations obtained in lemma 4.2.3 often simplify the construction of attractor spaces in both Schrödinger and Heisenberg pictures, a pair of linearly independent attractors found by means of e.g. solution of attractor equations can be used to generate other linearly independent attractors. Furthermore, lemma 4.2.3 also states that the attractor space $\mathsf{Atr}(\mathcal{G}^\dagger)$ and the kernel $\mathsf{Ker}(\mathcal{G}^\dagger - \lambda I)$ with $\lambda = 1$ for QMCHs and $\lambda = 0$ for QMDSs form C^*-algebras [83]. This property is going to be important in the next chapter when deriving the Gibbs-like form of asymptotic states of QMPs.

4.3 Algebraic relation between the Schrödinger and the Heisenberg picture

Theorem 4.1.1 was so far a crucial part in developing the attractor formalism, describing the bijection between attractor spaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Atr}(\mathcal{G}^\dagger)$ for any faithful QMP $\mathcal{T}_\tau$ associated with a generator $\mathcal{G}$. This led to the analytic form of the asymptotic states (4.7) and (4.8) and to the set of equations fulfilled by any attractor in both Schrödinger and Heisenberg picture given by theorems 4.2.1 and 4.2.2. Relations (4.3) can be furthermore generalized by utilizing operator monotone functions [91, 92], resulting in a whole family of transformations between elements of spaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Atr}(\mathcal{G}^\dagger)$.

Definition 4.3.1. *A function $k : [0, \infty) \to [0, \infty)$ is called an operator monotone function, if for any two operators $A \geq B \geq 0$ we have*

$$k(A) \geq k(B).$$

It was shown [91] that any operator monotone function can be written in an integral representation as

$$k(y) = a + by + \int_0^\infty \frac{y(1+s)}{y+s} d\mu(s),$$

with $a = k(0) \geq 0$, $b = \lim\limits_{t \to \infty} \frac{k(t)}{t}$ and $\mu(s)$ being a finite measure. According to theorem 4.1.1, for any two faithful $\mathcal{T}$-states $\sigma_1, \sigma_2 > 0$, operator $\Delta_{\sigma_1,\sigma_2}$ is a bijection of the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$, $\lambda \in \sigma_{\mathrm{atr}}$ onto itself, which is strictly positive on the Hilbert space $\mathcal{H}$. Based on this fact, one can construct more general strictly positive bijections:

Lemma 4.3.2. *Let $\mathcal{T}_\tau$ be a faithful QMP associated with a generator $\mathcal{G}$ and two, not necessarily different $\mathcal{T}$-states $\sigma_1, \sigma_2 > 0$. Then for any $\lambda \in \sigma_{atr}$, any operator monotone function k defines strictly positive bijections $k(\Delta_{\sigma_1,\sigma_2})$ and $k(\Delta_{\sigma_1,\sigma_2}^{-1})$ of the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$ onto itself.*

Proof. According to the theorem 4.1.1, the superoperator $\Delta_{\sigma_1,\sigma_2}$ is a strictly positive bijection on the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$. This also applies to the superoperator $\Delta_{\sigma_1,\sigma_2} + sI$ with $s > 0$. Moreover, it is easy to see that the maps $\Delta_{\sigma_1,\sigma_2}$ and $(\Delta_{\sigma_1,\sigma_2} + sI)^{-1}$ commute. For any finite measure $\mu(s)$, the map

$$k(\Delta_{\sigma_1,\sigma_2}) = aI + b\Delta_{\sigma_1,\sigma_2} + \int_0^\infty \frac{\Delta_{\sigma_1,\sigma_2}(1+s)}{\Delta_{\sigma_1,\sigma_2} + sI} d\mu(s)$$

with $a, b \geq 0$ is a strictly positive bijection on the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$. Obviously, k is an operator monotone function and therefore any operator monotone function k defines a strictly positive bijection $k(\Delta_{\sigma_1,\sigma_2})$ on the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$. The superoperator $\Delta_{\sigma_1,\sigma_2}^{-1}$ is according to the theorem 4.1.1 also a strictly positive bijection on the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$ and one can thus follow exactly the same steps as for the superoperator $\Delta_{\sigma_1,\sigma_2}$ to complete the proof. $\qquad\square$

Using the lemma 4.3.2, one can generalize results of the theorem 4.1.1. By choosing an arbitrary operator monotone function k, one can define a map $R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})$ which provides a bijection between kernels $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$ and $\mathsf{Ker}(\mathcal{G}^\dagger - \overline{\lambda}\mathcal{I})$. This results in the following corollary.

Corollary 4.3.3. *Let $\mathcal{T}_\tau$ be a faithful QMP on the Hilbert space $\mathcal{H}$ associated with a generator $\mathcal{G}$, let $\sigma_1, \sigma_2 > 0$ be two, not necessarily different faithful $\mathcal{T}$-states. Then*

$$\begin{aligned}
X \in \mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I}) &\Leftrightarrow R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X) \in \mathsf{Ker}(\mathcal{G}^\dagger - \overline{\lambda}I), \\
X \in \mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I}) &\Leftrightarrow k(\Delta_{\sigma_1,\sigma_2})(X) \in \mathsf{Ker}(\mathcal{G} - \lambda I)
\end{aligned} \tag{4.22}$$

In principle it is possible to use any combination of superoperators R_σ^{-1}, L_σ^{-1} and $k(\Delta_{\sigma_1,\sigma_2})$, $k(\Delta_{\sigma_1,\sigma_2}^{-1})$ in the same order as in the relation (4.22) to obtain the desired transformations, the choice made in the corollary 4.3.3 however offers advantages as the superoperators $R_{\sigma_2}^{-1}$ and $k(\Delta_{\sigma_1,\sigma_2})$ commute with each other.

Equipped with generalized relations (4.22), one can define a whole family of inner products $(.,.)_k$ on the Hilbert space $\mathcal{H}$ which includes the previously used inner product $(.,.)_\sigma$ for any $X, Y \in \mathsf{B}(\mathcal{H})$ as

$$(X, Y)_k := (X, R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(Y))_{HS} = \mathsf{Tr}\left[X^\dagger R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(Y)\right].$$

Following the same reasoning as in the section 4.1, the concept of k-orthogonality can be defined for any sets $\mathcal{X}, \mathcal{Y} \subset \mathsf{B}(\mathcal{H})$ as

$$\mathcal{X} \perp_k \mathcal{Y} \Leftrightarrow (X, Y)_k = 0, \ \forall X \in \mathcal{X}, Y \in \mathcal{Y},$$

with implications analogous to the previously used concept of σ-orthogonality:

Corollary 4.3.4. *Let $\mathcal{T}_\tau$ be a faithful QMP associated with a generator $\mathcal{G}$ and two not necessarily different $\mathcal{T}$-states $\sigma_1, \ \sigma_2 > 0$, let $\lambda_1, \lambda_2 \in \sigma_{atr}$. Then the following holds:*

$$\mathsf{Ker}(\mathcal{G} - \lambda_1) \perp_k \mathsf{Ran}(\mathcal{G} - \lambda_1 I),$$
$$\mathsf{Ker}(\mathcal{G} - \lambda_1) \perp_k \mathsf{Ker}(\mathcal{G} - \lambda_2 I).$$

The dual vectors with respect to the inner product generated by the operator monotone function k $X^{\lambda,i}$ take the form

$$X^{\lambda,i} = \frac{1}{\mathsf{Tr}\left[X_{\lambda,i}^\dagger R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X_{\lambda,i})\right]} R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X_{\lambda,i}), \qquad (4.23)$$

There are two prominent examples of operator monotone functions which are of significant importance in connection with the presented theory. The first one is given by

$$k(y) = y^\alpha, \ \alpha \in (0, 1].$$

The integral representation of the function k reads

$$k(y) = \int_0^\infty \frac{ys^{\alpha-1}}{y+s} \frac{\sin(\alpha\pi)}{\pi} ds,$$

which means that k is an operator monotone function with

$$a = b = 0, \ d\mu(s) = \frac{\sin(\alpha\pi)}{\pi} \frac{s^{\alpha-1}}{1+s} ds.$$

The bijection of the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$, $\lambda \in \sigma_{\mathrm{atr}}$ onto itself is thus given by

$$k(\Delta_{\sigma_1,\sigma_2})(X) = \sigma_1^{-\alpha} X \sigma_2^\alpha.$$

Using the bijection $L_{\sigma_1}^{-1} k(\Delta_{\sigma_1,\sigma_2}^{-1})$ instead of (4.22), this result can be generalized for any $\alpha \in [-1, 1]$. The general transformation between subspaces $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$

and $\mathsf{Ker}(\mathcal{G}^\dagger - \bar{\lambda}I)$ using the studied operator monotone function k thus takes the form

$$X \in \mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I}) \Leftrightarrow \sigma_1^{-\alpha} X \sigma_2^{\alpha-1} \in \mathsf{Ker}(\mathcal{G}^\dagger - \bar{\lambda}I) \tag{4.24}$$

For each $\alpha \in [-1, 1]$ one can define an inner product $(.,.)_\alpha$ as

$$(X, Y)_\alpha = \mathsf{Tr}\left[X^\dagger \sigma_1^{-\alpha} Y \sigma_2^{\alpha-1}\right].$$

An especially important example is $\alpha = \frac{1}{2}$, $\sigma_1 = \sigma_2 \equiv \sigma$. In such case the bijection $R_\sigma^{-1} k(\Delta_{\sigma,\sigma})$ preserves hermicity of the input and it can be used to express the relation between observables and states. This feature will be utilized in the next chapter to reveal transformation between constants of motion and asymptotic states of given QMP $\mathcal{T}_\tau$.

The second example is provided by the operator monotone function

$$k(y) = \ln(1 + y),$$

with integral representation

$$k(y) = \int_1^\infty \frac{y}{y + s} ds.$$

It follows that it is possible to define a pair of bijections given by

$$k(\Delta_{\sigma_1,\sigma_2}) = \ln\left[I + \Delta_{\sigma_1,\sigma_2}\right],$$
$$k(\Delta_{\sigma_1,\sigma_2}^{-1}) = \ln\left[I + \Delta_{\sigma_1,\sigma_2}^{-1}\right],$$

which map the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$, $\lambda \in \sigma_{\mathrm{atr}}$ onto itself. As one can write

$$\ln\left[I + \Delta_{\sigma_1,\sigma_2}\right] = \ln\left[I + \Delta_{\sigma_1,\sigma_2}^{-1}\right] + \ln\left[\Delta_{\sigma_1,\sigma_2}\right],$$

the map $\ln\left[\Delta_{\sigma_1,\sigma_2}\right]$ is not necessarily a bijection, but an endomorphism of the subspace $\mathsf{Ker}(\mathcal{G} - \lambda\mathcal{I})$ onto itself. A straightforward calculation reveals that

$$\ln\left[\Delta_{\sigma_1,\sigma_2}\right] = L_{\ln[\sigma_1]} - R_{\ln[\sigma_2]}. \tag{4.25}$$

The map (4.25) will be important in the next chapter for the construction of the representation of the asymptotic states in terms of observable quantities and for the formulation of the Jaynes principle for QMPs.

4.4 Evolution within attractor spaces

The evolution generated by a QMP is generally contractive, i.e. after a sufficiently long time, the state of the system is contained in the attractor space. In this asymptotic regime, the state does not need to be invariant, as such behavior is represented only by certain eigenvalues ($\lambda = 1$ for QMCHs and $\lambda = 0$ for QMDSs) and oscillating time evolution is possible. The evolution in the

asymtpotic regime is thus generally not trivial and it is interesting to explore its properties.

It turns out that the evolution within attractor spaces shares some features with the evolution generated by unitary operators. It is reversible, however, the reverse direction is in a certain sense governed by a faithful $\mathcal{T}$-state σ. For QMCH, the evolution in the reversed time direction is generated by the superoperator $\mathcal{T}^{\ddagger}$, which is a generator adjoint to the generator $\mathcal{T}$, but with respect to a special kind of inner product, utilizing a faithful $\mathcal{T}$-state σ. For QMDS, it is shown that not the attractors X in the Schrödinger picture, but operators $X\sigma^{-1}$ (i.e. the attractors in the Heisenberg picture) evolve according to the von Neumann equation generated by the Hamiltonian $-H$.

Starting with a QMCHs, consider the inner product $(.,.)_{\frac{1}{2}}$ with $\sigma_1 = \sigma_2 \equiv \sigma > 0$ defined in the previous section. Taking an adjoint $\mathcal{T}^{\ddagger}$ to the generator $\mathcal{T}$ of the QMCH $\mathcal{T}_n$ with respect to the inner product $(.,.)_{\frac{1}{2}}$ yields

$$
\begin{aligned}
(X, \mathcal{T}(Y))_{\frac{1}{2}} =& \mathsf{Tr}\left[X^{\dagger}\sigma^{-\frac{1}{2}}\mathcal{T}(X)\sigma^{-\frac{1}{2}}\right] = \mathsf{Tr}\left[X^{\dagger}\sigma^{-\frac{1}{2}}\left(\sum_j K_j Y K_j^{\dagger}\right)\sigma^{-\frac{1}{2}}\right] = \\
& \mathsf{Tr}\left[\left(\sum_j K_j^{\dagger}\sigma^{-\frac{1}{2}}X^{\dagger}\sigma^{-\frac{1}{2}}K_j\right)Y\right] = \\
& \mathsf{Tr}\left[\left(\sum_j \sigma^{\frac{1}{2}}K_j^{\dagger}\sigma^{-\frac{1}{2}}X\sigma^{-\frac{1}{2}}K_j\sigma^{\frac{1}{2}}\right)^{\dagger}\sigma^{-\frac{1}{2}}Y\sigma^{-\frac{1}{2}}\right] = (\mathcal{T}^{\ddagger}(X), Y)_{\frac{1}{2}}
\end{aligned}
$$

resulting in

$$
\mathcal{T}^{\ddagger}(X) = \sum_j \sigma^{\frac{1}{2}}K_j^{\dagger}\sigma^{-\frac{1}{2}}X\sigma^{-\frac{1}{2}}K_j\sigma^{\frac{1}{2}}. \tag{4.26}
$$

A quick check reveals that $\mathcal{T}^{\ddagger}$ represents a Kraus representation of a quantum operation with Kraus operators $\tilde{K}_j$ defined as

$$
\tilde{K}_j = \sigma^{\frac{1}{2}}K_j^{\dagger}\sigma^{-\frac{1}{2}},
$$

it is thus CPTNI map and hence it represents a well defined quantum evolution. The following theorem proves that the quantum operation $\mathcal{T}^{\ddagger}$ is capable of reversing the dynamics of the QMCH $\mathcal{T}_n$ within the attractor space:

Theorem 4.4.1. *Let $\mathcal{T}_n$ be a QMCH associated with the generator $\mathcal{T}$ and a $\mathcal{T}$-state $\sigma > 0$. Then the quantum operation $\mathcal{T}^{\ddagger}$ defined by (4.26) reverses the dynamics in the attractor space, i.e for any $X \in \mathsf{Atr}(\mathcal{T})$ one has*

$$
\mathcal{T}^{\ddagger}\mathcal{T}(X) = \mathcal{T}\mathcal{T}^{\ddagger}(X) = X.
$$

Proof. An arbitrary element $X \in \mathsf{Atr}(\mathcal{T})$ can be written as

$$X = \sum_{\lambda \in \sigma_{\mathrm{atr}}, i=1}^{\mathsf{d}(\lambda)} c_{\lambda,i} X_{\lambda,i}, \ c_{\lambda,i} \in \mathbb{C}.$$

Explicit calculation reveals that

$$\mathcal{T}^{\ddagger} \mathcal{T}(X) = \mathcal{T}^{\ddagger} \left(\sum_j K_j X K_j^{\dagger} \right) = \mathcal{T}^{\ddagger} \left(\sum_j K_j X \sigma^{-1} \sigma K_j^{\dagger} \right) =$$

$$\mathcal{T}^{\ddagger} \left(\sum_j \sum_{\lambda \in \sigma_{atr}} \sum_{i=1}^{\mathsf{d}(\lambda)} c_{\lambda,i} K_j X_{\lambda,i} \sigma^{-1} \sigma K_j^{\dagger} \right) =$$

$$\mathcal{T}^{\ddagger} \left(\sum_{\lambda \in \sigma_{atr}} \sum_{i=1}^{\mathsf{d}(\lambda)} \lambda c_{\lambda,i} X_{\lambda,i} \sigma^{-1} \mathcal{T}(\sigma) \right) = \mathcal{T}^{\ddagger} \left(\sum_{\lambda \in \sigma_{atr}} \sum_{i=1}^{\mathsf{d}(\lambda)} \lambda c_{\lambda,i} X_{\lambda,i} \right) =$$

$$\sum_{\lambda \in \sigma_{\mathrm{atr}}} \sum_{i=1}^{\mathsf{d}(\lambda)} \lambda c_{\lambda,i} \sigma^{\frac{1}{2}} \left(\sum_j K_j^{\dagger} \sigma^{-\frac{1}{2}} X_{\lambda,i} \sigma^{-\frac{1}{2}} K_j \right) \sigma^{\frac{1}{2}} =$$

$$\sum_{\lambda \in \sigma_{atr}} \sum_{i=1}^{\mathsf{d}(\lambda)} \lambda \bar{\lambda} c_{\lambda,i} \sigma^{\frac{1}{2}} \left(\sigma^{-\frac{1}{2}} X_{\lambda,i} \sigma^{-\frac{1}{2}} \right) \mathcal{T}^{\dagger}(I) \sigma^{\frac{1}{2}} =$$

$$\sum_{\lambda \in \sigma_{atr}} \sum_{i=1}^{\mathsf{d}(\lambda)} |\lambda|^2 c_{\lambda,i} X_{\lambda,i} = X.$$

The relation $\mathcal{T}\mathcal{T}^{\ddagger}(X) = X$ can be obtained analogously for any $X \in \mathsf{Atr}(\mathcal{T})$. On the subspace $\mathsf{Atr}(\mathcal{T}) \subset \mathsf{B}(\mathcal{H})$, the generator $\mathcal{T}^{\ddagger}$ therefore fulfils $\mathcal{T}^{\ddagger}\mathcal{T} = \mathcal{T}\mathcal{T}^{\ddagger} = \mathcal{I}$. The generator $\mathcal{T}^{\ddagger}$ is thus capable of reversing the dynamics within the attractor space $\mathsf{Atr}(\mathcal{T})$ and in this sense, the dynamics within the attractor space $\mathsf{Atr}(\mathcal{T})$ is unitary. $\square$

In case of QMDS $\mathcal{T}_t$ associated with a Lindbladian generator $\mathcal{L}$ and the $\mathcal{T}$ state $\sigma > 0$, one can prove that the dynamics on the attractor space $\mathsf{Atr}(\mathcal{L}^{\dagger})$ is governed by the master equation of von Neumann type.

Theorem 4.4.2. *Let $\mathcal{T}_t$ be a QMDS associated with a Lindbladian generator $\mathcal{L}$ and a $\mathcal{T}$-state $\sigma > 0$. Then for any $X \in \mathsf{Atr}(\mathcal{L})$, one has*

$$\frac{d(X\sigma^{-1})}{dt} = i[H, X\sigma^{-1}],$$

i.e the dynamics of the operators $X\sigma^{-1} \in \mathsf{Atr}(\mathcal{L}^{\dagger})$ is of von Neumann type.

Proof.

$$\mathcal{L}(X) = -i[H,X] + \sum_j \left(L_j X \sigma^{-1} \sigma L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, X\sigma^{-1}\sigma\} \right) =$$
$$-i[H,X] + X\sigma^{-1}\left(\mathcal{L}(\sigma) - i[H,\sigma]\right) =$$
$$-i[H,X] - iX\sigma^{-1}H\sigma + iX\sigma^{-1}\sigma H = i(HX - X\sigma^{-1}H\sigma),$$

using $\mathcal{L}(\sigma) = 0$. The master equation takes the form

$$\frac{d(X\sigma^{-1})}{dt} = \frac{dX}{dt}\sigma^{-1} = \mathcal{L}(X)\sigma^{-1} = i(HX - X\sigma^{-1}H\sigma)\sigma^{-1} = i[H, X\sigma^{-1}].$$

It follows that instead of attractors $X \in \mathsf{Atr}(\mathcal{L})$, their multiplication by σ^{-1}, i.e. elements of the subspace $\mathsf{Atr}(\mathcal{L}^\dagger) \subset \mathsf{B}(\mathcal{L})$ undergo the unitary time evolution, with the reverse dynamics generated by the Hamiltonian $-H$. $\qquad\square$

4.5 General T-state

Having investigated the asymtotic evolution of a QMP $\mathcal{T}_\tau$ equipped with a faithful $\mathcal{T}$-state $\sigma > 0$, now it is time to turn attention to the case of QMP equipped with a $\mathcal{T}$-state σ such that $\mathsf{Rank}\,\sigma < \mathsf{Dim}\,\mathcal{H}$. The $\mathcal{T}$-projector P on a $\mathcal{T}$-state σ thus fulfils $P < I$. This brings nontrivial complications, as it is not possible take advantage of relations (4.3), since the inverse σ^{-1} on the space $\mathcal{H}$ does not exist. Nevertheless, the inverse σ_P^{-1} on the subspace $P\mathcal{H}$ exists, yielding

$$\sigma\sigma_P^{-1} = \sigma_P^{-1}\sigma = P.$$

To simplify the notation, the inverse σ_P^{-1} on the subspace $P\mathcal{H}$ will be denoted as σ^{-1}. From definition of the $\mathcal{T}$-projector, any invariant state ρ of the QMP $\mathcal{T}_\tau$ must fulfil $\rho \in PS(\mathcal{H})P$. However, this must be true for all asymptotic states of the QMP $\mathcal{T}_\tau$. As all asymptotic states are constructed via linear combinations of attractors in the Schrödinger picutre, one obtains $\mathsf{Atr}(\mathcal{G}) \subset PB(\mathcal{H})P$ and the attractors in the Schrödinger picture $X_{\lambda,i}$ are therefore supported by the subspace $P\mathcal{H}$. This is not true for the attractors in the Heisenberg picture $X^{\lambda,i}$, which are generally nontrivial on the whole Hilbert space $\mathcal{H}$ and their full form is needed for the description of the dependence of the asymptototic state $\rho_\infty(\tau)$ on the initial state $\rho(0)$.

To proceed, recall section 3.3 in which the $\mathcal{T}$-projector P and its complementary projector Q were defined. Denoting corresponding the projection on the subspace $P\mathcal{H}$ as $\mathcal{P}$ and the projection on the subspace $Q\mathcal{H}$ as $\mathcal{Q}$, i.e.

$$A_P \equiv \mathcal{P}(A) = PAP,$$
$$A_Q \equiv \mathcal{Q}(A) = QAQ,$$

any operator $X \in \mathsf{B}(\mathcal{H})$ can be divided into three parts

$$X = X_P + X_Q + X_{\text{coh}}, \tag{4.27}$$

with $X_{\text{coh}} = PXQ + QXP$. For any $X \in \mathsf{Atr}(\mathcal{G})$, one must have $X = X_P$, as any element of the attractor space in the Schrödinger picture must be nontrivial only on the subspace $P\mathcal{H}$. However, for an attractor Y in the Heisenberg picture this is generally no longer true. Following the steps of the theorem 4.1.1, an analogous result can be derived (and subsequently generalized for any operator monotone function k) for a $\mathcal{T}$-state associated with an arbitrary $\mathcal{T}$-projector P, providing analogous transformations on the subspace $P\mathcal{H}$:

Corollary 4.5.1. *Let $\mathcal{T}_\tau$ be a QMP associated with a generator $\mathcal{G}$ and two, not necessarily different $\mathcal{T}$-states σ_1, σ_2, with projection P onto $\mathcal{T}$-state. Let $\lambda \in \sigma_{atr}$. Then*

$$X \in \mathit{Ker}(\mathcal{G} - \lambda\mathcal{I}) \Leftrightarrow R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X) \in P\mathit{Ker}(\mathcal{G}^\dagger - \overline{\lambda}I)P,$$
$$X \in \mathit{Ker}(\mathcal{G} - \lambda\mathcal{I}) \Leftrightarrow k(\Delta_{\sigma_1,\sigma_2})(X) \in P\mathit{Ker}(\mathcal{G} - \lambda I)P \tag{4.28}$$

The left (resp. right) multiplication opeartor $L_{\sigma^{-1}}$ (resp. $R_{\sigma^{-1}}$) is thus a bijection between subspaces $\mathsf{Atr}(\mathcal{G})$ and $P\mathsf{Atr}(\mathcal{G}^\dagger)P$. Having an attractor $Y \in \mathsf{Atr}(\mathcal{G}^\dagger)$ in the Heisenberg picture, one is able to obtain the associated attractor in the Schrödinger picture, since the projection $Y_P \in P\mathsf{Atr}(\mathcal{G}^\dagger)P$ fulfils

$$L_\sigma(Y_P) \in \mathsf{Atr}(\mathcal{G}),$$
$$R_\sigma(Y_P) \in \mathsf{Atr}(\mathcal{G}).$$

However, for the transformation in the other direction, corollary 4.5.1 does not give the complete form of the attractor in Heisenberg picture. Starting with an attractor $X \in \mathsf{Atr}(\mathcal{G})$, relations (4.28) enable one to obtain an operator $Y_P \in P\mathsf{Atr}(\mathcal{G}^\dagger)P$, which is generally not a full attractor in the Heisenberg picture, as the corresponding attractor $Y \in \mathsf{Atr}(\mathcal{G}^\dagger)$ may be nontrivial on the subspace $Q\mathcal{H}$ and therefore $Y \neq Y_P$. Fortunately, it turns out that having knowledge of an orthonormal basis of the subspace $P\mathsf{Atr}(\mathcal{G}^\dagger)P$, one can obtain the full attractor space $\mathsf{Atr}(\mathcal{G}^\dagger)$ in the Heisenberg picture using results of Albert [69, 70]. For a QMP associated with a generator $\mathcal{G}$, Albert showed that for any $Y \in \mathsf{Ker}(\mathcal{G}^\dagger - \overline{\lambda}I)$, the corresponding projections Y_P and Y_Q satisfy equation

$$Q\mathcal{G}^\dagger(Y_P) = -\left(\mathcal{G}_Q^\dagger - \overline{\lambda}Q\right)(Y_Q), \tag{4.29}$$

or alternatively

$$Y_Q = -\left(\mathcal{G}_Q^\dagger - \overline{\lambda}Q\right)^{-1} Q\mathcal{G}^\dagger(Y_P). \tag{4.30}$$

Furthermore,

$$Y_{\text{coh}} = 0, \ \forall Y \in \mathsf{Atr}(\mathcal{G}^\dagger)$$

and the full attractor in the Heisenberg picture thus takes the form

$$Y = Y_P + Y_Q = Y_P - \left(\mathcal{G}_Q^\dagger - \bar{\lambda}\mathcal{Q}\right)^{-1} \mathcal{Q}\left(\mathcal{G}^\dagger(Y_P)\right). \tag{4.31}$$

The full form of attractors in the Heisenberg picture is obtained by first solving attractor equations in the Heisenberg picture on the subspace $P\mathcal{H}$ and subsequent solution of the equation (4.29), or application of formulas (4.30) or (4.31). Since the formulas (4.30) and (4.31) contains a superoperator inverse $\left(\mathcal{G}_Q^\dagger - \bar{\lambda}\mathcal{Q}\right)^{-1}$ which may be difficult to obtain, the equation (4.29) is usually the most convenient method of obtaining the full form of attractors in the Heisenberg picture in many practical applications.

4.6 Summary

In this chapter, the general algebraic analysis of the asymptotic evolution of finite-dimensional QMPs was presented. The asymptotic evolution of QMP in the Schrödinger and the Heisenberg picture is fully determined by its attractor spectrum and the corresponding right and left eigenvectors. These eigenvectors form so-called attractor spaces $\mathsf{Atr}(\mathcal{G})$ and $\mathsf{Atr}(\mathcal{G}^\dagger)$. Crucially, elements of these attractor spaces are linked through relations (4.3).

For faithful QMPs, linearly independent elements of the mentioned attractor spaces can be derived via attractor equations (4.10) and (4.12) in the Schrödinger picture, resp. (4.11) and (4.13). Furthermore, properties of attractor spaces, derived in lemma 4.2.3 may help with this derivation.

For QMPs without faithful $\mathcal{T}$-state, attractor equations determine the attractor space $\mathsf{Atr}(\mathcal{G})$ and the subspace $P\mathsf{Atr}(\mathcal{G}^\dagger)P$. Obtaining the attractor space $\mathsf{Atr}(\mathcal{G}^\dagger)$ requires subsequent solution of the equation (4.29), resp. (4.30).

The evolution within the attractor space is in certain sense unitary. For QMCH with the generator $\mathcal{T}$, the evolution within the reversed direction is given by the quantum operation $\mathcal{T}^\ddagger$, which is an adjoint superoperator to the generator $\mathcal{T}$ with respect to the inner product $(.,.)_{\frac{1}{2}}$. For QMDS $\mathcal{T}_t$, instead of attractors, the elements of the subspace $P\mathsf{Atr}(\mathcal{G}^\dagger)P$ undergo the reversible evolution, generated by the Hamiltonian $-H$.

Chapter 5

Gibbs-like states and the Jaynes principle

The description of the asymptotics of a QMP developed in the previous chapter gives a full answer to the problem of the dependence of the asymptotic state on the initial state. The asymptotic state is provided as a linear combination of attractors (4.7), (4.8), with the coefficients of this linear combination being dependent on the initial state. However, attractors themselves do not generally represent quantum states. One must consider a linear combination

$$\sum_{\lambda \in \sigma_{\mathrm{atr}}} \sum_{j=1}^{d_\lambda} c_{\lambda,j} X_{\lambda,j}, \tag{5.1}$$

with properly chosen coefficients $c_{\lambda,j}$ to obtain a valid quantum state. Starting from an initial state $\rho(0)$, in asymptotic regime coefficients $c_{\lambda,j}$ read

$$c_{\lambda,j} = e^{i\varphi_\lambda}|c_{\lambda,j}|, \ |c_{\lambda,j}| = \mathsf{Tr}\left[\left(X^{\lambda,j}\right)^\dagger \rho(0)\right], \tag{5.2}$$

with an additional phase factor φ_λ determined by the length of the time evolution evolution and the associated eigenvalue $\lambda \in \sigma_{\mathrm{atr}}$. If one is however interested in the form of the set of all asymptotic state, expression (5.1) with (5.2) cannot provide a simple answer since the possible range of coefficients (5.2) is generally unknown.

To resolve this issue and to get a deep, more physically motivated insight into asymptotics of QMPs, an alternative representation of asymptotic states can be introduced. Rather than expressing asymptotic states in terms of attractors in the Schrödinger picture, one can use advantageous properties of attractors in the Heisenberg picture and the exponential map to provide the representation of all asymptotic states, called Gibbs-like form of the asymptotic states. This form may be viewed as the broadening of the concept of generalized Gibbs states (2.6). Simultaneously, it is in agreement with the classical version of Gibbs states (2.19) and their generalization emerging from the maximum caliber principle (2.25).

Despite the fact that elements of subspaces $\mathsf{Atr}(\mathcal{G})$, resp. $\mathsf{Atr}(\mathcal{G}^\dagger)$ do not generally represent neither states nor observables, they contain important subspaces of invariant states $\mathsf{Inv}(\mathcal{G})$ and asymptotic states $\mathsf{As}(\mathcal{G})$, resp. integrals of motion $\mathsf{Inv}(\mathcal{G}^\dagger)$ and constants of motion $\mathsf{As}(\mathcal{G}^\dagger)$. The elements of subspaces $\mathsf{As}(\mathcal{G})$, resp. $\mathsf{As}(\mathcal{G}^\dagger)$ span attractor spaces $\mathsf{Atr}(\mathcal{G})$, resp. $\mathsf{Atr}(\mathcal{G}^\dagger)$, which allow to express the Gibbs-like form of asymptotic states in terms of real valued observable quantities. Furthermore, by a special choice of these observables as constants of motion associated with the particular QMP, one can guarantee additional elegant properties of resulting Gibbs-like form of asymptotic states.

Expressing asymptotic states in the Gibbs-like form allow the formulation of several extremal principles, which determine asymptotic evolution based on the knowledge of the dynamics and initial conditions, provided by the full, resp. partial knowledge of the initial state [98]. As such, these principles are a generalization, resp. extension of the Jaynes principle for any finite-dimensional QMP. Importantly, herein presented forms of Jaynes principle are derived with respect to the dynamics generated by the QMP and the resulting principles are therefore result of the dynamics of the system and not just a statistical inference.

5.1 Constants of motion

So far the asymptotics of QMP was described in terms of attractors in the Schrödinger and the Heisenberg picture. Any asymptotic state $\rho \in \mathsf{As}(\mathcal{G})$ is therefore given as a linear combination of attractors $X_{\lambda,i} \in \mathsf{Atr}(\mathcal{G})$. However, as the attractors in the Schrödinger picture do not generally represent quantum states, one obtains

$$\mathsf{As}(\mathcal{G}) \subset \mathsf{Atr}(\mathcal{G}).$$

As a consequence, only certain linear combinations of the attractors in the Schrödinger picture constitute valid quantum states. Furthermore, it is often desirable to describe the asymptotics in terms of observable quantities. To obtain such a description, it is necessary to identify the corresponding subspace of asymptotic observables $\mathsf{As}(\mathcal{G}^\dagger)$ and its relation with the attractor spaces $\mathsf{Atr}(\mathcal{G}^\dagger)$ and $\mathsf{Atr}(\mathcal{G})$.

Assume a QMCH associated with a generator $\mathcal{T}$ and consider an attractor $X^{\lambda,i} \in \mathsf{Ker}(\mathcal{T}^\dagger - \overline{\lambda}\mathcal{I})$. Then for any $n \in \mathbb{N}$ one obtains

$$\mathcal{T}^\dagger(\lambda^n X^{\lambda,i}) = \lambda^n \overline{\lambda} X^{\lambda,i} = \lambda^{n-1} X^{\lambda,i},$$
$$\mathcal{T}^\dagger\left(\overline{\lambda}^n \left(X^{\lambda,i}\right)^\dagger\right) = \overline{\lambda}^n \lambda \left(X^{\lambda,i}\right)^\dagger = \overline{\lambda}^{n-1} \left(X^{\lambda,i}\right)^\dagger. \tag{5.3}$$

Similarly, for QMDS $\mathcal{T}_t$ associated with a generator $\mathcal{L}$ and its corresponding attractor $X^{\lambda,i} \in \mathsf{Ker}(\mathcal{L}^\dagger - \overline{\lambda}\mathcal{I})$, for any $t_2 \in \mathbb{R}$ and $t_1 \geq 0$ one obtains

$$\mathcal{T}_{t_1}^\dagger(e^{\lambda t_2} X^{\lambda,i}) = e^{\lambda t_2} e^{\overline{\lambda} t_1} X^{\lambda,i} = e^{\lambda(t_2 - t_1)} X^{\lambda,i},$$
$$\mathcal{T}_{t_1}^\dagger\left(e^{-\lambda t_2} \left(X^{\lambda,i}\right)^\dagger\right) = e^{-\lambda t_2} e^{\lambda t_1} \left(X^{\lambda,i}\right)^\dagger = e^{-\lambda(t_2 - t_1)} \left(X^{\lambda,i}\right)^\dagger. \tag{5.4}$$

Relations (5.3) and (5.4) imply that the attractors $\lambda^n X^{\lambda,i}$, $\overline{\lambda}^n (X^{\lambda,i})^\dagger$, resp. $e^{\lambda t} X^{\lambda,i}$, $e^{-\lambda,i}(X^{\lambda,i})^\dagger$ evolve in reverse direction with time. This implies that the expectation values of these operators remain constant during evolution under a QMP as one has

$$
\begin{aligned}
\langle \lambda^n X^{\lambda,i} \rangle_{\rho(m)} &= \mathsf{Tr}\left[\lambda^n X^{\lambda,i} \mathcal{T}(\rho(m-1))\right] = \mathsf{Tr}\left[\mathcal{T}^\dagger(\lambda^n X^{\lambda,i})\rho(m-1)\right] = \\
&\quad \langle \lambda^{n-1} X^{\lambda,i} \rangle_{\rho(m-1)}, \\
\langle e^{\lambda t_2} X^{\lambda,i} \rangle_{\rho(t_3)} &= \mathsf{Tr}\left[e^{\lambda t_2} X^{\lambda,i} \mathcal{T}_{t_1}(\rho(t_3 - t_1))\right] = \mathsf{Tr}\left[\mathcal{T}_{t_1}^\dagger(e^{\lambda t_2} X^{\lambda,i})\rho(t_3 - t_1)\right] = \\
&\quad \langle e^{\lambda(t_2 - t_1)} X^{\lambda,i} \rangle_{\rho(t_3 - t_1)}
\end{aligned}
$$

and similarly for the operators $\overline{\lambda}^n (X^{\lambda,i})^\dagger$ and $e^{-\lambda t}(X^{\lambda,i})^\dagger$. This backward evolution holds also for any linear combination of these operators, e.g. for real linear combination

$$
C_{\lambda,j}^{(+)}(n) = \frac{1}{2}\left(\lambda^n X^{\lambda,j} + \overline{\lambda}^n (X^{\lambda,j})^\dagger\right)
$$

for the case of QMCH and analogously for the case of QMDS. Together with the complementary real linear combination

$$
C_{\lambda,j}^{(-)}(n) = \frac{1}{2i}\left(\lambda^n X^{\lambda,j} - \overline{\lambda}^n (X^{\lambda,j})^\dagger\right)
$$

operators $C_{\lambda,j}^{(\pm)}(n)$ are constants of motion corresponding to the given QMCH $\mathcal{T}_n$. The set $\left\{ C_{\lambda,j}^{(\pm)}(n) \right\}$ forms a Hermitian orthonormal basis of the attractor space $\mathsf{Atr}(\mathcal{T}^\dagger)$ (considering the fact that $C_{\overline{\lambda},j}^{(\pm)}(n) = \pm C_{\lambda,j}^{(\pm)}(n)$). Any real combination of the constants of motion $C_{\lambda,j}^{(\pm)}(n)$ is also a constant of motion. The subspace $\mathsf{As}(\mathcal{T}^\dagger)$ is therefore formed by all real combinations of operators $C_{\lambda,j}^{(\pm)}$. Making analogous considerations for the case of QMDS, this leads to the following corollary:

Corollary 5.1.1. *Consider a QMP $\mathcal{T}_\tau$ associated with a generator $\mathcal{G}$. Then an orthonormal basis of the subspace $\mathsf{Atr}(\mathcal{G}^\dagger)$ of all the constants of motion corresponding to the QMP $\mathcal{T}_\tau$ reads*

$$
\begin{aligned}
C_{\lambda,j}^{(+)}(n) &= \frac{1}{2}\left(\lambda^n X^{\lambda,j} + \overline{\lambda}^n (X^{\lambda,j})^\dagger\right), \\
C_{\lambda,j}^{(-)}(n) &= \frac{1}{2i}\left(\lambda^n X^{\lambda,j} - \overline{\lambda}^n (X^{\lambda,j})^\dagger\right).
\end{aligned}
\tag{5.5}
$$

in the case of QMCH with $\lambda \in \sigma_{atr}$. Similarly, for QMDS we obtain

$$
\begin{aligned}
C_{\lambda,j}^{(+)}(t) &= \frac{1}{2}\left(e^{\lambda t} X^{\lambda,j} + e^{-\lambda t}(X^{\lambda,j})^\dagger\right), \\
C_{\lambda,j}^{(-)}(t) &= \frac{1}{2i}\left(e^{\lambda t} X^{\lambda,j} - e^{-\lambda t}(X^{\lambda,j})^\dagger\right),
\end{aligned}
\tag{5.6}
$$

with $\lambda \in \sigma_{atr}$. All the presented observable quantities fulfil

$$\langle C_{\lambda,j}^{(\pm)}(\tau_2) \rangle_{\rho(\tau_3)} = \langle C_{\lambda,j}^{(\pm)}(\tau_2 + \tau_1) \rangle_{\rho(\tau_3+\tau_1)}, \quad \forall \tau_1, \tau_2, \tau_3 \geq 0.$$

Furthermore, all the presented constants of motion fulfil

$$\mathcal{T}_{\tau_1}^{\dagger}(C_{\lambda,j}^{(\pm)}(\tau_1 + \tau_2)) = C_{\lambda,j}^{(\pm)}(\tau_2)$$

and thus constants of motion are Hermitian trajectories of QMPs in the Heisenberg picture undergoing the time-reversed evolution within the attractor space $\mathsf{Atr}(\mathcal{G}^{\dagger})$. The set of all asymptotic observables $\mathsf{As}(\mathcal{G}^{\dagger}) \subset \mathsf{Atr}(\mathcal{G}^{\dagger})$ is formed by all real linear combinations of constants of motion $C_{\lambda,i}^{(\pm)}(\tau)$.

Constants of motion with $\lambda = 1$ for QMCHs, resp. with $\lambda = 0$ for QMDSs clearly do not undergo any evolution. These are called integrals of motion. The set of all integrals of motion forms a subspace $\mathsf{Inv}(\mathcal{G}^{\dagger})$ of the space of all constants of motion $\mathsf{As}(\mathcal{G}^{\dagger})$. More precisely, for a QMCH associated with a generator $\mathcal{T}$, the set of integral of motion fulfils

$$\mathsf{Inv}(\mathcal{T}^{\dagger}) \subset \mathsf{Ker}(\mathcal{T}^{\dagger} - I).$$

Similarly, for QMDS equipped with a Lindbladian generator $\mathcal{L}$, the set of integrals of motion fulfils

$$\mathsf{Inv}(\mathcal{L}^{\dagger}) \subset \mathsf{Ker}(\mathcal{L}^{\dagger}).$$

Integrals of motion can be linked with invariant states through relations (4.22) with proper choice of the operator monotone function k, e.g. $k(y) = y^{\frac{1}{2}}$. Similarly to the set of integrals of motion, the set of invariant states $\mathsf{Inv}(\mathcal{G})$, being a subset of the set of all asymptotic states $\mathsf{As}(\mathcal{G})$ can be also constructed from the corresponding kernels. For a QMCH associated with a generator $\mathcal{T}$, the set of invariant states fulfils

$$\mathsf{Inv}(\mathcal{T}) \subset \mathsf{Ker}(\mathcal{T} - I).$$

Analogously, for QMDS equipped with a Lindbladian generator $\mathcal{L}$, the set of invariant states fulfils

$$\mathsf{Inv}(\mathcal{L}) \subset \mathsf{Ker}(\mathcal{L}).$$

Having established constants of motion, the goal is to use them for the description of all asymptotic states and asymptotic trajectories. From that point of view, it is useful to look into the effect of the QMP $\mathcal{T}_{\tau}$ in the Schrödinger picture on the projection of constants of motion $C_j^{(\pm)}(\tau)$ on the subspace $P\mathcal{H}$. This formula is a special case of more general relations provided by the following lemma.

Lemma 5.1.2. *Let $\mathcal{T}_\tau$ be a QMP associated with a generator $\mathcal{G}$, $\mathcal{T}$-projector P and a $\mathcal{T}$-state σ, let $C(\tau) \in A(\mathcal{H})$ be a constant of motion and $X \in B(\mathcal{H})$. Then*

$$\begin{aligned}
\mathcal{T}_{\tau_2}(PC(\tau_1)PX) &= PC(\tau_1 + \tau_2)P\mathcal{T}_{\tau_2}(X), \\
\mathcal{T}_{\tau_2}(XPC(\tau_1)P) &= \mathcal{T}_{\tau_2}(X)PC(\tau_1 + \tau_2)P.
\end{aligned} \tag{5.7}$$

Proof. First, consider a QMCH $\mathcal{T}_n$ with a generator $\mathcal{T}$ associated with Kraus operators K_j. Using the notation introduced in section 4.5 and applying attractor equations (4.11), one obtains for any $Y \in \mathsf{Ker}\left(\mathcal{T}^\dagger - \overline{\lambda}\mathcal{I}\right)$ and $X \in \mathsf{B}(\mathcal{H})$

$$\mathcal{T}(XY_P) = \sum_j K_j XY_P K_j^\dagger = \sum_j K_j X K_j^\dagger \lambda Y_P = \mathcal{T}(X)\lambda Y_P,$$

$$\mathcal{T}(Y_P X) = \sum_j K_j Y_P X K_j^\dagger = \lambda Y_P \sum_j K_j X K_j^\dagger = \lambda Y_P \mathcal{T}(X).$$

By iteration, this can be easily generalized to obtain

$$\mathcal{T}_m(XY_P) = \mathcal{T}_m(X)\lambda^m Y_P, \ \ \mathcal{T}_m(Y_P X) = \lambda^m Y_P \mathcal{T}_m(X).$$

Consider a constant of motion $C(n) \sim \lambda^n Y \pm \overline{\lambda}^n Y^\dagger$ one obtains

$$\begin{aligned}
\mathcal{T}_m(XPC(n)P) &\sim \lambda^n \mathcal{T}_m(XY_P) \pm \overline{\lambda}^n \mathcal{T}_m(XY_P^\dagger) = \\
&\mathcal{T}_m(X)\lambda^{m+n}Y_P \pm \mathcal{T}_m(X)\overline{\lambda}^{m+n}Y_P^\dagger = \\
&\mathcal{T}_m(X)PC(m+n)P
\end{aligned}$$

and analogously for the second equality.

For a QMDS $\mathcal{T}_t$ associated with a Lindblad generator $\mathcal{L}$, first one obtains for any $Y \in \mathsf{Ker}(\mathcal{G}^\dagger - \overline{\lambda}I)$ and $X \in \mathsf{B}(\mathcal{H})$

$$\begin{aligned}
\mathcal{L}(XY_P) &= \sum_j L_j XY_P L_j^\dagger - KXY_P - XY_P K^\dagger = \\
&\sum_j L_j X L_j^\dagger Y_P - KXY_P - XK^\dagger Y_P + \lambda XY_P = (\mathcal{L} + \lambda)(X)Y_P, \\
\mathcal{L}(Y_P X) &= \sum_j L_j Y_P X L_j^\dagger - KY_P X - Y_P XK^\dagger = \\
&Y_P \sum_j L_j X L_j^\dagger - Y_P KX + \lambda Y_P X - Y_P K^\dagger X = Y_P(\mathcal{L} + \lambda)(X),
\end{aligned}$$

and consequently

$$\begin{aligned}
\frac{dXY_P}{dt} &= \mathcal{L}(XY_P) = (\mathcal{L} + \lambda)(X)Y_P, \\
\frac{dY_P X}{dt} &= \mathcal{L}(Y_P X) = Y_P(\mathcal{L} + \lambda)(X)
\end{aligned}$$

which results in

$$\mathcal{T}_t(XY_P) = \exp[(\mathcal{L} + \lambda)t](X)Y_P = \mathcal{T}_t(X)e^{\lambda t}Y_P,$$
$$\mathcal{T}_t(Y_P X) = Y_P \exp[(\mathcal{L} + \lambda)](X) = e^{\lambda t}Y_P\mathcal{T}_t(X).$$

Finally, by taking a constant of motion $C(t) \sim e^{\lambda t}Y \pm e^{-\lambda t}Y^\dagger$, one can calculate

$$\mathcal{T}_{t_2}(XPC(t_1)P) \sim e^{\lambda t_1}\mathcal{T}_{t_2}(XY_P) \pm e^{-\lambda t_1}\mathcal{T}_{t_2}(XY_P^\dagger) =$$
$$= \mathcal{T}_{t_2}(X)e^{\lambda(t_1+t_2)}Y_P \pm \mathcal{T}_{t_2}(X)e^{-\lambda(t_1+t_2)}Y_P =$$
$$= \mathcal{T}_{t_2}(X)PC(t_1 + t_2)P$$

and similarly for the other equality. $\qquad\square$

An important special case of lemma 5.1.2 occurs for $X = I$. For this case left sides of both equations (5.7) coincide and one therefore obtains the following corollary:

Corollary 5.1.3. *Let $\mathcal{T}_\tau$ be a QMP associated with a generator $\mathcal{G}$, $\mathcal{T}$-projector P and a $\mathcal{T}$-state σ, let $C(\tau)$ be a constant of motion. Then*

$$\mathcal{T}_{\tau_2}(PC(\tau_1)P) = \mathcal{T}_{\tau_2}(I)PC(\tau_1 + \tau_2)P = PC(\tau_1 + \tau_2)P\mathcal{T}_{\tau_2}(I). \qquad (5.8)$$

The above obtained properties of constants of motion will be important in the following chapter, as they allow to derive simple evolution properties of the alternative form of the asymptotic states, called the Gibbs-like form.

5.2 Gibbs-like states and asymptotic trajectories

According to lemma 4.2.3, subspaces $\mathsf{Atr}(\mathcal{G}^\dagger)$ and $\mathsf{Ker}(\mathcal{G}^\dagger - \bar{\lambda}\mathcal{I})$ with $\lambda = 1$ for QMCHs and $\lambda = 0$ for QMDSs possess an additional algebraic property in case of TP QMPs - they form a C^*-algebras. For such a QMP, one can exploit this property to construct a first exponential representation of asymptotic states using the Taylor expansion:

Theorem 5.2.1. *Let $\mathcal{T}_\tau$ be a TP QMP equipped with a generator $\mathcal{G}$, $\mathcal{T}$-projector P and a $\mathcal{T}$-state σ. Let $\{I, Z_j\}$ be any Hermitian base of the subspace $\mathsf{As}(\mathcal{G}^\dagger)$. Then any element $\rho \in \mathsf{As}(\mathcal{G})$ can be written as*

$$\rho = \frac{1}{Z}\sigma^{\frac{1}{2}} \exp\left[-\sum_j \beta_j Z_j\right] \sigma^{\frac{1}{2}}, \ \beta_j \in \mathbb{R},$$

$$Z = \mathsf{Tr}\left[\sigma^{\frac{1}{2}} \exp\left[-\sum_j \beta_j Z_j\right] \sigma^{\frac{1}{2}}\right]. \qquad (5.9)$$

If the state ρ is not strictly positive on the subspace $P\mathcal{H}$, some of the parameters β_j fulfil $\beta_j \in \{\pm\infty\}$. If $\{I, Z_j\}$ form a Hermitian basis of the subspace $\mathsf{Inv}(\mathcal{G}^\dagger)$ instead, any element $\rho \in \mathsf{Inv}(\mathcal{G})$ can be written as (5.9).

Proof. Taking any observable $A = A^\dagger \in \mathsf{As}(\mathcal{G}^\dagger)$ and using the exponential map, the outcome is again an observable from the same subspace:

$$\exp[A] = \sum_{n=0}^{\infty} \frac{A^n}{n!} \in \mathsf{As}(\mathcal{G}^\dagger). \tag{5.10}$$

Now, using theorem 4.3.3 with the operator monotone function $k(y) = \sqrt{y}$, one obtains

$$\frac{1}{\mathsf{Tr}\left[\sigma^{\frac{1}{2}} \exp[A]\sigma^{\frac{1}{2}}\right]} \sigma^{\frac{1}{2}} \exp[A]\sigma^{\frac{1}{2}} \in \mathsf{As}(\mathcal{G}), \tag{5.11}$$

i.e. any observable $A \in \mathsf{As}(\mathcal{G}^\dagger)$ can be used to define an asymptotic state strictly positive on the subspace $P\mathcal{H}$ by means of equation (5.11). To show that any $\rho \in \mathsf{As}(\mathcal{G})$ can be written by means of (5.11), first consider an element $\rho \in \mathsf{As}(\mathcal{G})$ strictly positive on the subspace $P\mathcal{H}$. The projection A_P of the attractor in the Heisenberg picture $A \in \mathsf{As}(\mathcal{G}^\dagger)$ on the subspace $P\mathcal{H}$ fulfils $\sigma^{\frac{1}{2}} A \sigma^{\frac{1}{2}} = \sigma^{\frac{1}{2}} A_P \sigma^{\frac{1}{2}} = \rho$. Using the logarithmic map

$$\ln[I + X] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n$$

one obtains

$$\ln[A] = \ln[I + (A - I)] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(A - I)^n \in \mathsf{As}(\mathcal{G}^\dagger).$$

Therefore, $\ln[A]$ can be written as a linear combination

$$\ln[A] = \alpha I - \sum_j \beta_j Z_j, \ \alpha, \beta_j \in \mathbb{R}.$$

Finally, using both exponential and logarithmic function as $\rho = \sigma^{\frac{1}{2}} \exp[\ln[A]]\sigma^{\frac{1}{2}}$ results in

$$\rho = \sigma^{\frac{1}{2}} \exp\left[\alpha I - \sum_j \beta_j Z_j\right] \sigma^{\frac{1}{2}}.$$

The observable I commutes with all other observables Z_j, one can use it to normalize the resulting state

$$\rho = \frac{1}{Z} \sigma^{\frac{1}{2}} \exp\left[-\sum_j \beta_j Z_j\right] \sigma^{\frac{1}{2}},$$

with Z given by (5.9).

Next, consider a general element $\rho \in \mathsf{As}(\mathcal{G})$. Using the $\mathcal{T}$-state σ, and defining a one-parameter family of states $\omega(s) \subset \mathsf{As}(\mathcal{G})$ which is strictly positive on the subspace $P\mathcal{H}$ at some vicinity of $s = 0$:

$$\omega(s) := (1 - s)\rho + s\sigma, \ s \in [0, \varepsilon)$$

and using the already derived result for states strictly positive on subspace $P\mathcal{H}$, one can write

$$\omega(s) = \frac{1}{Z}\sigma^{\frac{1}{2}} \exp\left[-\sum_j \beta_j(s) Z_j \right] \sigma^{\frac{1}{2}}.$$

The original state ρ is then obtained as the limit

$$\rho = \lim_{s \to 0_+} \frac{1}{Z}\sigma^{\frac{1}{2}} \exp\left[-\sum_j \beta_j(s) Z_j \right] \sigma^{\frac{1}{2}}.$$

As ρ is not strictly positive on $P\mathcal{H}$, some of the parameters $\beta_j(s)$ must fulfil $\lim_{s \to 0_+} \beta(s) = \pm\infty$.

The same reasoning leads to the result for $\rho \in \mathsf{Inv}(\mathcal{G})$.

$\square$

The second, and arguably more useful representation is provided by the relation (4.25), according to which any $\rho \in \mathsf{As}(\mathcal{G})$ and arbitrary two $\mathcal{T}$-states σ_1, σ_2 fulfil

$$\ln[\Delta_{\sigma_1,\sigma_2}](\rho) = \ln \sigma_1 \rho - \rho \ln \sigma_2 \in \mathsf{Atr}(\mathcal{G}). \tag{5.12}$$

Based on this relation, the following representation of asymptotic states can be defined.

Theorem 5.2.2. *Let $\mathcal{T}_\tau$ be a QMP associated with a generator $\mathcal{G}$, $\mathcal{T}$-projector P and a $\mathcal{T}$-state σ. Let $\{Z_j\}$ be a Hermitian basis of the subspace $\mathsf{As}(\mathcal{G}^\dagger)$. Then any element $\rho \in \mathsf{As}(\mathcal{G})$ can be written on the subspace $P\mathcal{H}$ as*

$$\rho = \exp\left[\ln \sigma - \sum_j \beta_j P Z_j P \right] \oplus 0_Q, \ \beta_j \in \mathbb{R} \tag{5.13}$$

with the logarithmic and exponential maps being applied only on the subspace $P\mathcal{H}$ and 0_Q being a null operator on the subspace $Q\mathcal{H}$. If the state ρ is not strictly positive on the subspace $P\mathcal{H}$, some of the parameters β_j fulfil $\beta_j \in \{\pm\infty\}$. If $\{Z_j\}$ form a Hermitian basis of the subspace $\mathsf{Inv}(\mathcal{G}^\dagger)$ instead, any element $\rho \in \mathsf{Inv}(\mathcal{G})$ can be written as (5.13)

Proof. Consider $\rho \in \mathsf{Inv}(\mathcal{G})$, strictly positive on the subspace $P\mathcal{H}$. The state ρ can be chosen as a $\mathcal{T}$-state. Taking $\sigma_1 = \rho$, $\sigma_2 = \sigma$, the relation (5.12) becomes

$$\rho(\ln \rho - \ln \sigma) \in \mathsf{Atr}(\mathcal{G}),$$

with logarithmic map being applied only on the subspace $P\mathcal{H}$. According to theorem 4.1.1, this implies

$$\ln \rho - \ln \sigma = - \sum_j \beta_j P Z_j P \in P\mathsf{Inv}(\mathcal{G}^\dagger)P, \tag{5.14}$$

with $\{Z_j\} \subset \mathsf{Inv}(\mathcal{G}^\dagger)$ and $\beta_j \in \mathbb{R}$, since the operator on the left is Hermitian. By application of the exponential map on the subspace $P\mathcal{H}$ one obtains ρ in the form

$$\rho = \exp\left[\ln \sigma - \sum_j \beta_j P Z_j P\right] \oplus 0_Q, \ \beta_j \in \mathbb{R}$$

To generalize the result for all asymptotic states, consider the lowest common multiple λ of the elements of σ_{atr}. Next, define a new QMP $\tilde{\mathcal{T}}_n$ as

$$\tilde{\mathcal{T}}_n = \mathcal{T}_{\lambda n}, \quad \tilde{\mathcal{T}}_n = \exp[\mathcal{L}\lambda n],$$

respectively for QMCHs and QMDSs. The obtained QMCH $\tilde{\mathcal{T}}_n$, equipped with the generator $\tilde{\mathcal{T}}$ is associated with the same attractor space as the QMP $\mathcal{T}_\tau$, however all attractors correspond to the eigenvalue $\lambda = 1$, i.e.

$$\mathsf{Inv}(\tilde{\mathcal{T}}) = \mathsf{As}(\mathcal{G}), \ \mathsf{Inv}(\tilde{\mathcal{T}}^\dagger) = \mathsf{As}(\mathcal{G}^\dagger).$$

Having a state $\rho \in \mathsf{As}(\mathcal{G})$ strictly positive on the subspace $P\mathcal{H}$, it is an invariant state of the QMCH $\tilde{\mathcal{T}}_n$ and one can use already proved result to write ρ in the form (5.13). Going back to the QMP $\mathcal{T}_\tau$, relation (5.14) becomes

$$\ln \rho - \ln \sigma = - \sum_j \beta_j P Z_j P \in P\mathsf{As}(\mathcal{G}^\dagger)P, \tag{5.15}$$

resulting in the form (5.13). Using the same reasoning as in theorem 5.2.1, one can generalize this result for any $\rho \in \mathsf{As}(\mathcal{G})$.

$\square$

For sake of clarity, the trivial part on the subspace $Q\mathcal{H}$ is omitted in the rest of the thesis, i.e.

$$\rho = \exp\left[\ln \sigma - \sum_j \beta_j P Z_j P\right], \ \beta_j \in \mathbb{R},$$

however, to avoid any possible confusion, it is stressed that the exponential operator acts only on the subspace $P\mathcal{H}$.

Similarly to theorem 5.2.1, if the QMP $\mathcal{T}_\tau$ is TP, the subspace $\mathsf{As}(\mathcal{G}^\dagger)$ is generated by observables $\{I, Z_j\}$ and the identity operator can be used to normalize the state, leading to

$$\rho = \frac{1}{Z} \exp\left[\ln\sigma - \sum_j \beta_j P Z_j P\right], \ \beta_j \in \mathbb{R},$$

$$Z = \mathsf{Tr}\left[\exp\left[\ln\sigma - \sum_j \beta_j P Z_j P\right]\right]. \tag{5.16}$$

Expressions (5.9) and (5.16) are generally different, i.e. using the same $\mathcal{T}$-state σ, and the same set of parameters β_j will result in a two different asymtptotic states, as the $\mathcal{T}$-state does not have to commute with the observables $\{Z_j\}$.

A natural choice for a Hermitian basis $\{Z_j\}$ is the set of all linearly independent constants of motion $\{C_j(\tau)\}$. Using such representation provides a simpler description of evolution of asymptotic states. To uncover them, it is helpful to show that there is a special kind of a $\mathcal{T}$-state σ_I:

Lemma 5.2.3. *Let $\mathcal{T}_\tau$ be a QMP equipped with a generator $\mathcal{G}$. The state $\sigma_I \in \mathsf{Inv}(\mathcal{G})$ defined as*

$$\sigma_I = \lim_{T \to \infty} \frac{1}{T} \sum_0^{T} d\tau \mathcal{T}_\tau(\rho_I),$$

with ρ_I being the maximally mixed state, is a $\mathcal{T}$-state which furthermore fulfils

$$[\sigma_I, \rho] = 0, \ \forall \rho \in \mathsf{As}(\mathcal{G}),$$

Proof. It is clear that $\sigma_I \in \mathsf{Inv}(\mathcal{G})$, as it is defined as a time-average of an evolution of the identity operator. Furthermore, for any $\mathcal{T}$-state σ, one has $\mathsf{Rank}\,\rho_I \geq \mathsf{Rank}\,\sigma$ and consequently $\mathsf{Rank}\,\sigma_I \geq \mathsf{Rank}\,\sigma$. The quantum state σ_I is therefore a $\mathcal{T}$-state. To proceed, choose a Hermitian basis $\{Z_j\}$ of the subspace $\mathsf{As}(\mathcal{G}^\dagger)$. According to theorem 5.2.2 any $\rho \in \mathsf{As}(\mathcal{G})$ can be written as

$$\rho = \exp\left[\ln\sigma_I - \sum_j \beta_j P Z_j P\right].$$

Operators Z_j can be constructed as linear combinations of constants of motion $\{C_j(\tau)\}$ evaluated at a particular reference time $\tau = \tau_R$ and thus, according to corollary 5.1.3, projections $P Z_j P$ commute with states $\mathcal{T}_\tau(\rho_I)$ for any $\tau \geq 0$ and consequently they commute with an arbitrary linear combination of these states such as the $\mathcal{T}$-state σ_I. As a result, one has

$$\sigma_I \rho = \sigma_I \exp\left[\ln \sigma_I - \sum_j \beta_j P Z_j P\right] =$$

$$= \exp\left[\ln \sigma_I - \sum_j \beta_j P Z_j P\right] \sigma_I = \rho \sigma_I.$$

$\square$

Using a general Hermitian basis $\{\, Z_j \,\}$ of of the subspace $\mathsf{As}(\mathcal{G}^\dagger)$, one can write down not only all individual asymptotic states, but also all asymptotic trajectories $\rho(\tau)$. Indeed, employing time-dependent coefficients $\beta_j(\tau)$ the trajectory $\rho(\tau)$ take the form

$$\rho(\tau) = \exp\left[\ln \sigma - \sum_j \beta_j(\tau) P Z_j P\right],$$

In general, the particular form of the time dependence of parameters $\beta_j(\tau)$ is unknown. However, by choosing a basis composed of constants of motion $\{C_j(\tau)\}$, the parameters $\beta_j(\tau)$ become time independent. Consequently, the corresponding asymptotic trajectory is fully described solely by the choice of a $\mathcal{T}$-state σ and the set of real parameters $\{\beta_j\}$.

Theorem 5.2.4. *Let $\mathcal{T}_\tau$ be a QMP associated with a generator $\mathcal{G}$, $\mathcal{T}$-projector P and a $\mathcal{T}$-state σ, let $\{C_j(\tau)\}$ be the set of constants of motion. Then the evolution of any asymptotic state $\rho(\tau) \in \mathsf{As}(\mathcal{G})$ on the subspace $P\mathcal{H}$ reads*

$$\rho(\tau) = \exp\left[\ln \sigma - \sum_j \beta_j P C_j(\tau) P\right], \tag{5.17}$$

i.e. the parameters β_j are time independent.

Proof. The evolution of the asymptotic state $\rho(\tau)$ takes the form

$$\rho(\tau) = \exp\left[\ln \sigma - \sum_j \beta_j(\tau) P C_j(\tau) P\right].$$

Utilizing the $\mathcal{T}$-state σ_I defined by theorem 5.2.3 and the relation

$$\ln \sigma - \ln \sigma_I \in \mathsf{Inv}(\mathcal{G}^\dagger) \tag{5.18}$$

yields

$$\rho(\tau) = \exp\left[\ln\sigma_I - \ln\sigma_I + \ln\sigma - \sum_j \beta_j(\tau)PC_j(\tau)P\right] =$$

$$= \exp\left[\ln\sigma_I - \sum_j \beta_j'(\tau)PC_j(\tau)P\right],$$

with $\beta_j'(\tau)$ being parameters $\beta_j(\tau)$ linearly shifted by some time independent constants according to (5.18). Using relations (5.7) with $X = \sigma_I$ and assuming an asymptotic trajectory $\{\rho(\tau) \mid \tau \geq 0\}$, one obtains

$$\mathcal{T}_{\tau_2}(\rho(\tau_1)) = \mathcal{T}_{\tau_2}\left(\exp\left[\ln\sigma_I - \sum_j \beta_j(\tau_1)PC_j(\tau_1)P\right]\right) =$$

$$\mathcal{T}_{\tau_2}\left(\exp\left[-\sum_j \beta_j(\tau_1)PC_j(\tau_1)P\right]\sigma_I\right) =$$

$$\mathcal{T}_{\tau_2}\left(\sum_{N=0}^{\infty}\frac{(-1)^N}{N!}\left(\sum_j \beta_j(\tau_1)PC_j(\tau_1)P\right)^N \sigma_I\right) =$$

$$\sum_{N=0}^{\infty}\frac{(-1)^N}{N!}\mathcal{T}_{\tau_2}\left(\left(\sum_j \beta_j(\tau_1)PC_j(\tau_1)P\right)^N \sigma_I\right) = \qquad (5.19)$$

$$\sum_{N=0}^{\infty}\frac{(-1)^N}{N!}\left(\sum_j \beta_j(\tau_1)PC_j(\tau_1+\tau_2)P\right)^N \mathcal{T}_{\tau_2}(\sigma_I) =$$

$$\sum_{N=0}^{\infty}\frac{(-1)^N}{N!}\left(\sum_j \beta_j(\tau_1)PC_j(\tau_1+\tau_2)P\right)^N \sigma_I =$$

$$\exp\left[\ln\sigma_I - \sum_j \beta_j(\tau_1)PC_j(\tau_1+\tau_2)P\right] = \rho(\tau_1+\tau_2)$$

and consequently $\beta_j(\tau_1 + \tau_2) = \beta_j(\tau_1)$. The parameters $\beta_j(\tau)$ are fixed by the initial value $\beta_j(0) \equiv \beta_j$ and the evolution takes the form (5.17). $\qquad\square$

Using projections of constants of motion on the subspace $P\mathcal{H}$ therefore results in an elegant form of representation of asymptotic trajectories $\rho(\tau)$, which are in a fixed basis $\{C_j(\tau)\}$ uniquely determined by the chosen $\mathcal{T}$-state σ and set of parameters $\{\beta_j\}$.

Furthermore, exploiting properties of the derived form of asymptotic states (5.17), allows a generalization of the formula (5.17) while preserving above de-

rived property of time independence of parameters β_j associated with constants of motion. Consider a maximal rank $\mathcal{T}$-trajectory $\{\omega(\tau)\}$ which reads

$$\omega(\tau) = \exp\left[\ln\sigma - \sum_j \beta_j PC_j(\tau)P\right],$$

with σ being an arbitrary $\mathcal{T}$-state and $\beta_j \in \mathbb{R}$. Then

$$\ln\omega(\tau) - \ln\sigma = -\sum_j \beta_j PC_j(\tau)P \in P\mathsf{As}(\mathcal{G}^\dagger)P$$

and thus for an arbitrary $\mathcal{T}$-trajectory $\{\rho(\tau)\}$ which reads

$$\rho(\tau) = \exp\left[\ln\sigma - \sum_j \gamma_j PC_j(\tau)P\right]$$

one obtains

$$\rho(\tau) = \exp\left[\ln\omega(\tau) - \sum_j (\gamma_j + \beta_j)PC_j(\tau)P\right]. \tag{5.20}$$

The concept of $\mathcal{T}$-state can be therefore generalized in the following way. Consider an asymptotic trajectory $\sigma(\tau)$ such that for each $\tau \geq 0$, the asymptotic state $\sigma(\tau)$ is a $\mathcal{T}$-state. Such an asymptotic trajectory is then called a $\mathcal{T}$-trajectory. For these trajectories, one can formulate the following corollary.

Corollary 5.2.5. *Consider a QMP $\mathcal{T}_\tau$ associated with the generator $\mathcal{G}$, $\mathcal{T}$-projector P and a $\mathcal{T}$-trajectory $\sigma(\tau)$. Let $\{C_j(\tau)\}$ be the set of all linearly independent constants of motion. Then any asymptotic trajectory $\rho(\tau)$ can be expressed on the subspace $P\mathcal{H}$ as*

$$\rho(\tau) = \exp\left[\ln\sigma(\tau) - \sum_j \beta_j PC_j(\tau)P\right], \ \beta_j \in \mathbb{R} \cup \{\pm\infty\}. \tag{5.21}$$

The expression (5.21) is the most general form of asymptotic trajectories. On one hand, by using $\mathcal{T}$-trajectory $\omega(\tau)$ instead of a $\mathcal{T}$-state, some properties of the representation (5.13) are lost, e.g. invariant states are no longer represented solely by integrals of motion - other constants of motion are needed as well. On the other hand, using $\mathcal{T}$-trajectories offers a more general approach, which can be helpful in some situations. An important example of the $\mathcal{T}$-trajectory is the $\mathcal{T}$-trajectory $\sigma_I(\tau)$ which is obtained as an asymptotic evolution of the identity operator, i.e.

$$\sigma_I(\tau \gg 1) = \mathcal{T}_\tau(I).$$

Analogously to the $\mathcal{T}$-state σ_I, asymptotic states from the trajectory $\sigma_I(\tau)$ commute with all constants of motion $C_j(\tau)$, which makes it advantageous in certain scenarios. Furthermore, as shown in the following section, the $\mathcal{T}$-trajectory $\sigma_I(\tau)$ is important for the formulation of the Jaynes principle for QMPs, describing the asymptotic evolution of QMPs, when complete information about the state of the system is unavailable.

5.3   Jaynes principle

The constants of motion $C(\tau)$ introduced in this chapter are natural quantities for the description of asymptotic states and asymptotic trajectories of QMPs. They contain all information about the initial state relevant for the asymptotics of the QMP and consequently one can use them for an elegant description of the asymptotic states and asymptotic trajectories in an exponential Gibbs-like form.

In this section, the evolution of TP QMPs is formulated in the form of an extremal principle analogous to Jaynes principle introduced in chapter 2 often used in classical physics. However, the exact form of this principle depends on the initial information available about the system, yielding several variations of this principle. It is shown that Gibbs-like states minimize the relative quantum entropy (2.26) with respect to a certain $\mathcal{T}$-state σ or a $\mathcal{T}$-trajectory $\sigma(\tau)$. Each introduced extremal principle deals with a different situation and they must be therefore treated separately. The first principle describes the situation when the initial state $\rho(0)$ is fully known:

Theorem 5.3.1. *Consider a TP QMP $\mathcal{T}_\tau$ equipped with a generator $\mathcal{G}$, $\mathcal{T}$-projector P and a $\mathcal{T}$-trajectory $\sigma(\tau)$ and let $\{I, C_1(\tau), \ldots, C_d(\tau)\}$ be a basis of the subspace $\mathsf{As}(\mathcal{G}^\dagger)$ comprised of constants of motion. Let $\rho(0)$ be the initial state with expectation values $c_j = \langle C_j(\tau) \rangle_{\rho(\tau)}$. Then individual states $\rho(\tau)$ from its asymptotic trajectory minimize, under the given expectation values, quantum relative entropy $S(\rho(\tau)|\sigma(\tau))$ for any sufficiently long time τ. The asymptotic evolution of the system takes the Gibbs-like form*

$$\rho(\tau) = \frac{1}{Z} \exp\left[\ln \sigma(\tau) - \sum_j \beta_j P C_j(\tau) P\right], \qquad (5.22)$$

where the partition function Z of the time independent Lagrange multipliers β_j ensures $\mathsf{Tr}[\rho(\tau)] = 1$.

Proof. Consider a $\mathcal{T}$-trajectory $\sigma(\tau)$. The support of the states from this trajectory is defined by the $\mathcal{T}$-projector P. From definition of the quantum relative entropy, (2.26), any state $\rho \in \mathsf{S}(\mathcal{H})$ such that the projector on its support P_ρ fulfils $P \leq P_\rho$ yields

$$S(\rho|\sigma(\tau)) = +\infty$$

and therefore, the individual states ρ from the trajectory $\rho(\tau)$ must have their support contained in the support of the states from the trajectory $\sigma(\tau)$. One therfore has

$$\rho(\tau) = P\rho(\tau)P.$$

The expectation values yield

$$\langle C_j(\tau)\rangle_{\rho(\tau)} = \mathrm{Tr}\left[C_j(\tau)P\rho(\tau)P\right] = \mathrm{Tr}\left[PC_j(\tau)P\rho(\tau)\right] = \langle PC_j(\tau)P\rangle_{\rho(\tau)}.$$

To find the minimum of the quantum relative entropy $S(\rho(\tau)|\sigma(\tau))$, consider a function Λ defined as

$$\Lambda = S(\rho(\tau)|\sigma(\tau)) - \alpha(\tau)\mathrm{Tr}[P\rho(\tau)P] + \sum_j \beta_j(\tau)\mathrm{Tr}\left[C_j(\tau)P\rho(\tau)P\right] =$$

$$\mathrm{Tr}\left[\rho(\tau)\left(\ln\rho(\tau) - \ln\sigma(\tau) - \alpha(\tau)P + \sum_j \beta_j(\tau)PC_j(\tau)P\right)\right].$$

Its variation yields

$$\delta\Lambda = \mathrm{Tr}\left[\delta\rho(\tau)\left(\ln\rho(\tau) - \ln\sigma(\tau) + (1+\alpha(\tau))P + \sum_j \beta_j(\tau)PC_j(\tau)P\right)\right].$$

To find the minimum of the relative quantum entropy, one must have $\delta\Lambda = 0$ for any Hermitian variation $\delta\rho(\tau)$ and thus

$$\rho(\tau) = \frac{1}{Z(\tau)}\exp\left[\ln\sigma(\tau) - \sum_j \beta_j(\tau)PC_j(\tau)P\right],$$

$$Z(\tau) = \exp[1+\alpha(\tau)] = \mathrm{Tr}\left[\exp\left[\ln\sigma(\tau) - \sum_j \beta_j(\tau)PC_j(\tau)P\right]\right].$$

The asymptotic trajectory therefore takes the form (5.21) with generally time-dependant parameters $\beta_j(\tau)$. However, according to (5.20), the transition from using a $\mathcal{T}$-state to using a $\mathcal{T}$-trajectory (and back) in a description of an arbitrary asymptotic trajectory manifests itself only by linear shift of the parameters β_j. Consequently, all considerations made in the proof of theorem 5.2.4 are also valid if one replaces a $\mathcal{T}$-state σ with a $\mathcal{T}$-trajectory $\sigma(\tau)$. As a result, parameters $\alpha(\tau)$ and $\beta_j(\tau)$ are in fact time-independent and the individual states $\rho(\tau)$ from the asymptotic trajectory take the form

$$\rho(\tau) = \frac{1}{Z} \exp\left[\ln \sigma(\tau) - \sum_j \beta_j PC_j(\tau)P\right],$$

$$Z = \exp[1 + \alpha] = \mathrm{Tr}\left[\exp\left[\ln \sigma(\tau) - \sum_j \beta_j PC_j(\tau)P\right]\right].$$

$\square$

Even though expectation values constitute time dependent constrains, the Lagrange multipliers β_j are time independent parameters. Their physical meaning is prescribed solely by a given QMP and can not be, as expected, revealed by the Jaynes principle itself. Furthermore, according to theorem 5.3.1 an arbitrary $\mathcal{T}$-trajectory can be used in the description of the asymptotic evolution. Often, the preferred choice may be to choose $\sigma_I(\tau)$ as a $\mathcal{T}$-state as it commutes with all projections of constants of motion on the subspace $P\mathcal{H}$ $PC_j(\tau)P$, however any choice is possible, as the change of the $\mathcal{T}$-trajectory manifests itself only by shifting values of parameters β_j.

Theorem 5.3.1 determines the asymptotic evolution based on knowledge of the initial state $\rho(0)$. This information may be however unavailable, as one is often equipped only with knowledge of expectation values of constants of motion $c_j = \langle C_j(\tau)\rangle_{\rho(\tau)}$ and not with the initial state $\rho(0)$ itself. Furthermore, not all expectation values may be available, but only expectation values c_j with $j \in \mathsf{K}$. Such situation often arises in statistical physics when one deals with macroscopic systems with huge number of constituents parts. As shown below, this situation is treatable, provided that one is equipped with additional information namely that the state of the system is supported on the subspace $P\mathcal{H}$. This may be interpreted in the following way: the system evolves towards the asymptotics. Once it is in the asymptotic regime, i.e. its state fulfils $\rho(\tau) \in \mathsf{As}(\mathcal{G})$, the constants of motion $C_j(\tau)$ $j \in \mathsf{K}$ are measured. Afterwards, the state of the system is located within the subspace $P\mathcal{H}$ and the following theorem may be applied.

Theorem 5.3.2. *Consider a TP QMP $\mathcal{T}_\tau$ equipped with a generator $\mathcal{G}$, $\mathcal{T}$-projector P and let $\sigma_I(\tau)$ be the $\mathcal{T}$-trajectory given by the asymptotic evolution of the maximally mixed state. Let the quantum system be in the asymptotic regime of the evolution. Assume that expectation values c_j of some linearly independent constants of motion $C_j(\tau)$ with $j \in \mathsf{K}$ is the only knowledge provided about its asymptotic trajectory. Then states $\rho(\tau)$ from the corresponding asymptotic trajectory minimize, under the given expectation values, quantum relative entropy $S(\rho(\tau)|\sigma_I(\tau))$ for any sufficiently long time τ. The asymptotic trajectory takes the Gibbs-like form*

$$\rho(\tau) = \frac{1}{Z} \exp\left(\ln \sigma_I(\tau) - \sum_{j \in \mathsf{K}} \beta_j PC_j(\tau)P\right), \tag{5.23}$$

where the partition function Z of time independent Lagrange multipliers β_j ensures $\mathsf{Tr}[\rho(\tau)] = 1$.

Proof. First, one must provide the initial state which best describes the knowledge of the system, i.e. all available information about the system must be incorporated in the initial state, while any additional information is ignored. According to assumptions, this state $\rho_R(\tau)$ exists on the subspace $P\mathcal{H}$ at some reference time $\tau \equiv 0$, it reads

$$\rho_R(0) = \frac{1}{Z} \exp\left[-\sum_{j \in \mathsf{K}} \beta_j P C_j(0) P\right],$$

as all available constants of motion are evaluated on the subspace $P\mathcal{H}$. According to lemma 5.1.2, the evolution of such initial state reads

$$\rho_R(\tau) = \frac{1}{Z} \mathcal{T}_\tau(I) \exp\left[-\sum_{j \in \mathsf{K}} \beta_j P C_j(\tau) P\right] =$$
$$\frac{1}{Z} \exp\left[\ln \mathcal{T}_\tau(I) - \sum_{j \in \mathsf{K}} \beta_j P C_j(\tau) P\right].$$

Switching to the asymptotic regime one obtains the asymptotic trajectory $\rho(\tau)$ given by $\rho(\tau) = \lim_{\tau \to \infty} \rho_R(\tau)$ as

$$\rho(\tau) = \frac{1}{Z} \exp\left[\ln \sigma_I(\tau) - \sum_{j \in \mathsf{K}} \beta_j P C_j(\tau) P\right].$$

According to theorem 5.3.1, such a quantum trajectory minimizes the relative quantum entropy $S(\rho(\tau)|\sigma_I(\tau))$. $\qquad\square$

Despite the similarity of theorems 5.3.1 and 5.3.2 they treat different situations. While theorem 5.3.1 deals with Jaynes principle determining asymptotic evolution of the given initial state, theorem 5.3.2 concerns with the Jaynes principle for for quantum systems which are known to be in the asymptotic regime and whose expectation values of some constants of motion are known. Consequently, the Jaynes principle for the latter case determines the asymptotic trajectory along the system is evolving, but it can not provide the actual position of the quantum system on this trajectory.

Unlike theorem 5.3.1, theorem 5.3.2 requires the specific $\mathcal{T}$-trajectory $\sigma_I(\tau)$. The evolution of the maximally mixed state plays therefore a special role among all $\mathcal{T}$-trajectories. This exact specification of the $\mathcal{T}$-trajectory in the theorem 5.3.2 results in quite a different treatment of parameters β_j. Within theorem 5.3.1, for a fixed choice of a $\mathcal{T}$-trajectory $\mathcal{T}$-trajectory $\sigma(\tau)$, parameters β_j are specified by the information about the initial state $\rho(0)$. By making a different choice of a $\mathcal{T}$-trajectory $\omega(\tau)$, these parameters are shifted accordingly. Even if

some of the parameters β_j vanish for a specific choice of the $\mathcal{T}$-trajectory, they are still a part of the description of the asymptotic trajectory $\rho(\tau)$. Contrary to that, the $\mathcal{T}$-trajectory in the theorem 5.3.2 is specified as $\sigma_I(\tau)$. Values of parameters β_j corresponding to constants of motion $C_j(\tau)$ with $j \in \mathsf{K}$ are therefore specified with respect to this $\mathcal{T}$-trajectory and unlike in theorem 5.3.1 they cannot be shifted. Most importantly, constants of motion $C_j(\tau)$ with $j \notin \mathsf{K}$ are not incorporated in the description of the asymptotic trajectory $\rho(\tau)$ and therefore, there are no parameters β_j assigned to such constants of motion.

One interesting case of the theorem 5.3.2 deserves a separate attention. Suppose the known information consists of knowledge of expectation values of integrals of motion. Interestingly, even though integrals of motion do not undergo any time evolution, the maximally mixed state can evolve nontrivially even in the asymptotics and thus stationarity of the asymptotic evolution in such case is not guaranteed. Only if the maximally mixed state fulfils

$$\mathcal{T}_{\tau \gg 1}(I) = \sigma_I(\tau) = \sigma_I,$$

then, under conditions of theorem 5.3.2 the asymptotic evolution is stationary.

However, stationarity can be regarded as an additional information about the system. If, beside the knowledge of the expectation values of integrals of motion one knows that the asymptotic evolution is stationary and the $\mathcal{T}$-trajectory $\sigma_I(\tau)$ is not a $\mathcal{T}$-state, one can incorporate this additional information by time-averaging of the trajectory $\sigma_I(\tau)$. These considerations lead to the following corollary:

Corollary 5.3.3. *Consider a TP QMP $\mathcal{T}_\tau$ equipped with a generator $\mathcal{G}$, $\mathcal{T}$-projector P and let $\sigma_I(\tau)$ be the evolution of the maximally mixed state. Let the quantum system be in the asymptotic regime of the evolution. Assume two types of information about the asymtptotic trajectory - knowledge of expectation values c_j of some linearly independent integrals of motion C_j with $j \in \mathsf{K}$ and the fact that the asymptotic evolution is stationary. Then the asymptotic invariant state ρ minimizes, under the given expectation values, quantum relative entropy $S(\rho|\sigma_I)$ for any sufficiently high time τ, with the $\mathcal{T}$-state σ_I being the time-averaged $\mathcal{T}$-trajectory $\sigma_I(\tau)$. The asymptotic invariant state ρ takes the Gibbs-like form*

$$\rho = \frac{1}{Z} \exp\left(\ln \sigma_I - \sum_{j \in \mathsf{K}} \beta_j P C_j P \right),$$

where the partition function Z of time independent Lagrange multipliers β_j ensures $\mathsf{Tr}[\rho] = 1$.

All the presented extremal principles may be regarded as a Jaynes principle for different situations, treating both asymptotic and invariant states of QMPs. For a general QMP $\mathcal{T}_\tau$, they are however different from the Jaynes principle 2.4.1 and the resulting invariant states do not coincide with generalized Gibbs states (2.22). Only if the QMP under investigation $\mathcal{T}_\tau$ is unital, the $\mathcal{T}$-trajectory can be always chosen as the maximally mixed state, resp. the evolution of the

maximally mixed state is trivial as it is an invariant state of $\mathcal{T}_\tau$ and the $\mathcal{T}$-projector becomes $P = I$. In such case, the resulting invariant states coincide with the generalized Gibbs states (2.22).

5.4   Summary

Algebraic properties of attractor spaces derived in the previous chapter (mainly the results discussed in lemma 4.2.3 and the results of section 4.3) allow the introduction of an alternative representation of the asymptotic states to the representation (4.7), resp. (4.8). The new representation strongly resembles the commonly used generalized Gibbs states (2.6), from whose it generally differs by the incorporation of a $\mathcal{T}$-state σ inside the exponential function. For this similarity, this representation of the asymptotic states is called Gibbs-like form of the asymptotic states, or Gibbs-like states. Due to special properties, Gibbs-like form allows an elegant description of the set of invariant states and the set of asymptotic states.

The most general description of Gibbs-like states utilizes a set of linearly independent Hermitian operators Z_j from the subspace $P\mathsf{Atr}(\mathcal{L}^\dagger)P$. Associating each operator Z_j with a parameter β_j with real or infinite value, the corresponding Gibbs-like state is fully determined by particular values of parameters β_j, taking generally different forms (5.9) and (5.13).

A natural choice for operators Z_j are constants of motion $C_j(\tau)$ associated with the system under investigation. Such a choice ensures neat properties of the time evolution of asymptotic states, i.e. asymptotic trajectories, as the parameters β_j become time-independent and the asymptotic trajectory takes the form (5.17). This expression can be furthermore generalized to utilize an arbitrary $\mathcal{T}$-trajectory $\sigma(\tau)$ instead of a fixed $\mathcal{T}$-state. A general asymptotic trajectory is then given by (5.21).

Importantly, it can be shown that Gibbs-like states (5.13) minimize the quantum relative entropy with respect to the chosen $\mathcal{T}$-state σ. This in turn allows the formulation of an extremal principle, which identifies the asymptotic state or the asymptotic trajectory based on the initial knowledge about the system. There are two prominent examples of such situations. In the first example, one assumes a complete knowledge of an initial state of the system. In the second example, one assumes knowledge of mean values of some (not necessarily all) conserved quantities of the QMP, which is in the asymptotic regime. These examples are treated in depth, resulting in two versions of Jaynes principle for TP QMPs, given by theorems 5.3.1 and 5.3.2. Furthermore, it is showed how an additional type of knowledge (namely the stationarity of the asymptotic evolution) can be incorporated in the Jaynes principle.

Crucially, all versions of Jaynes principle for TP QMPs are derived from the actual asymptotic dynamics of the QMP under investigation and giving a dynamic based derivation of the minimization of the relative quantum entropy. Due to this, the Jaynes principle is not a consequence of the theory of information, but the result of the actual dynamics of the system under investigation.

The initial information about the system determines the form of the Jaynes principle used, but the Jaynes principle iteslf stems from the algebraic properties of the attractor spaces associated with QMPs. The program realized here is similar to the one of Boltzmann [99, 100] and it yields the Gibbs-like states as the correct description of the asymptotics. The traditional Jaynes principle arises in the case, when the given QMP is unital. The $\mathcal{T}$-state can be then chosen as $\sigma \sim I$ and the minimization of the relative quantum entropy is equivalent to the maximization of the von Neumann entropy and the resulting asymptotic states are given by generalized Gibbs states (2.6).

Chapter 6

Excitation transfer in open networks

Previous two chapters developed a formalism of attractor spaces and asymptotic states of any finite-dimensional QMP. This chapter is devoted to application of obtained results on an actual quantum system, to show versitality and range of possible applications of the formalism.

In physics, chemistry and biology, particularly interesting processess are those of capable of transport of excitations. Especially in biology, a significant amount of research was devoted to study fundamental principles behind high efficiency of transportation of excitation during the process of photosynthesis [44, 65, 101, 102, 103, 104, 105, 106].

Two commonly used models of excitation transport during the process of photosynthesis are the Haaken-Strobl model [44, 105, 106] and the excitation energy transfer network (EET network) model, which will be the focus of this chapter.

The EET network model assumes a network of qubits, i.e. quantum systems equipped with a Hilbert space $\mathcal{H}^{(1)} \equiv \mathbb{C}^2$. Pairs of qubits can exchange excitation via the Hamiltonian $H \in \mathsf{A}(\mathcal{H})$, with $\mathcal{H} = (\mathcal{H}^{(1)})^{\otimes N}$. In a fixed orthonormal basis of qubits, denoted as $\{\,|\,0\rangle,\,|\,1\rangle\,\}$ and called the computational basis of qubits the Hamiltonian H reads

$$H = \sum_{j=1}^{N} \varepsilon_j L_{jj} + \sum_{j>k} V_{jk} \left(e^{i\phi_{jk}} L_{jk} + e^{-i\phi_{jk}} L_{kj} \right), \ \phi_{jk} \in [0, 2\pi), \qquad (6.1)$$

with

$$\begin{aligned} J^{(+)} &= |1\rangle \langle 0|, \ J^{(-)} = |0\rangle \langle 1|, \\ L_{jk} &= J_k^{(+)} J_j^{(-)}. \end{aligned} \qquad (6.2)$$

Furthermore, qubits are expose to irreversible effects of their environment. There are two processes usually incorporated into the description. The first process is the dissipation, described by the superoperator $\mathcal{L}_{\text{dis}}$ which reads

$$\mathcal{L}_{\text{dis}}(X) = \sum_{j=1}^{N} \mu_j \left(J_j^{(-)} X J_j^{(+)} - \frac{1}{2}\{L_{jj}, X\} \right). \tag{6.3}$$

The second process is the dephasing, provided by the superoperstor $\mathcal{L}_{\text{deph}}$ given by

$$\mathcal{L}_{\text{deph}}(X) = \sum_{j=1}^{N} \kappa_j \left(L_{jj} X L_{jj} - \frac{1}{2}\{L_{jj}, X\} \right), \tag{6.4}$$

To study the excitation transfer, an additional qubit, called the sink is liked to the network. It incoherently interacts with one or more of other qubits via the term $\mathcal{L}_{\text{s}}$ as

$$\mathcal{L}_{\text{s}}(X) = \sum_{j} \nu_j \left(L_{js} X L_{js}^{\dagger} - \frac{1}{2}\{L_{js}^{\dagger} L_{js}, X\} \right), \tag{6.5}$$

with s being the index of the sink. The evolution of the $N+1$ qubits is then given by the Lindbladian $\mathcal{L}$, which reads

$$\mathcal{L}(X) = -i[H, X] + \mathcal{L}_{\text{dis}}(X) + \mathcal{L}_{\text{deph}}(X) + \mathcal{L}_{\text{s}}(X). \tag{6.6}$$

The thorough study of EET network through numerical simulations resulted in number of important observations. It appears that under certain circumstances, irreversible effects imposed by the local dephasing represented by the superoperator (6.4) actually enhance the probability of the transport of the excitation towards the sink [65], contrary to the entanglement, which seems to be just a secondary product of the evolution and it does not assist with the transport of the excitation [102]. The efficiency of the transport can be further enhanced by optimization of phase factors ϕ_{jk} in (6.1) [103]. To introduce additional complexity, the relaxation process, described by the superoperator $\mathcal{L}_{\text{R}}$ defined as

$$\mathcal{L}_{\text{R}}(X) = \sum_{j \neq k} \gamma_{jk} \left(L_{jk} X L_{jk}^{\dagger} - \frac{1}{2}\{L_{jk}^{\dagger} L_{jk}, X\} \right) \tag{6.7}$$

was added in [104] and studied for the case $N = 2$, i.e. a pair of qubits with the second qubit being linked with the sink. The relaxation term (6.7) is interesting, as its form resembles the second term in Hamiltonian (6.1), describing the transfer of excitations within the network. However, there are two main differences between these two terms. First, unlike in the term (6.1), the transfer of excitations defined by relaxation term (6.7) is incoherent. Second, parameters γ_{jk} found in the relaxation term (6.7) can take any real values, resulting in possibly asymmetric interaction, whereas the coefficients V_{jk} are identical

for both operators L_{jk} and L_{kj} and hence the interaction is symmetric. As a result, the incoherent excitation transfer represented by the term (6.7) offers a richer structure of possible links between individual qubits via the interaction operators L_{jk}.

Following sections are devoted to thorough investigation of the asymptotics of a QMDS $\mathcal{T}_t$, related to the above introduced EET network, which may be called an incoherent EET network [107]. Instead of qubits, a network of qudits is assumed, each qudit equipped with the Hilbert space $\mathcal{H}^{(1)} \equiv \mathbb{C}^d$ and the computational basis $\{ \, |0\rangle, \ldots, |d-1\rangle \, \}$. To study the incoherent excitation transfer, the coherent transfer is dropped, resulting in the Hamiltonian H which reads

$$H = \sum_{j=1}^{N} H_j^{(1)},$$

$$H^{(1)} = \varepsilon \sum_{k=0}^{d-1} k \, |k\rangle \langle k| .$$

(6.8)

Generalization of operators $J^{(\pm)}$ for qudits reads

$$J^{(-)} = \sum_{k=0}^{d-2} |k\rangle \langle k+1| ,$$

$$J^{(+)} = \sum_{k=0}^{d-2} |k+1\rangle \langle k| ,$$

$$L_{jk} = J_j^{(-)} J_k^{(+)}.$$

(6.9)

Contrary to the Hamiltonian (6.1), it is therefore assumed $V_{jk} = 0$, $,\forall j, k$. However, some of the obtained results can be modified to hold for a direct generalization of the Hamiltonian (6.1), which fulfils the condition $V_{jk} = 0$ for j, k such that $\gamma_{jk} = 0$.

The QMDS $\mathcal{T}_t$ under investigation is given by the Lindbladian $\mathcal{L}$ which reads

$$\mathcal{L}(X) = -i[H, X] + \mathcal{L}_{\mathrm{R}}(X).$$

(6.10)

The dephasing term (6.4) is dropped as the term (6.7) provides dephasing on its own. Furthermore, the dissipation term (6.3) and the sink term (6.5) are also dropped, as they can be incorporated in the network by a suitable choice of coefficients γ_{jk}. The distribution of nonzero interaction strengths $\gamma_{jk} > 0$, the so-called interaction topology plays a crucial role in determination of attractor spaces $\mathsf{Atr}(\mathcal{L})$ and $\mathsf{Atr}(\mathcal{L}^\dagger)$. On their own, coefficients γ_{jk} do not fulfil any detailed balance condition [13]. For this reason, the QMDS $\mathcal{T}_t$ is generally not unital, which makes it a suitable choice for showcasing the attractor formalism, as through different choices of interaction topologies, it is possible to achieve a number of different regimes, e.g. maximally mixed state ρ_I not being a $\mathcal{T}$-state or the $\mathcal{T}$-projector P fulfiling $P < I$.

There are three main goals of this chapter. First goal is to determine, which properties of interaction topology determine the algebraic structure of attractor spaces $\mathsf{Atr}(\mathcal{L})$ and $\mathsf{Atr}(\mathcal{L}^\dagger)$. Second goal is to determine, how the individual attractors depend on the details of the dynamics, i.e. on particular values of the interaction strengths γ_{jk}. The third goal is to analyze a network of N qubits linked to two sinks and show that the attractor formalism is able to reconstruct and even extend the results obtained in [104]. First, however, some additional notation is needed.

The computational basis of qudits defines the computational basis of the whole network. This basis consists of kets $|\mathbf{z}\rangle$, with $\mathbf{z} \in \{0, \ldots, d-1\}^N$. Each element of the computational basis therefore reads

$$|\mathbf{z}\rangle = |k_1 \ldots k_N\rangle, \ k_j \in \{0, \ldots, d-1\}.$$

As the opearators L_{jk} do not change the number of excitation during the interaction of qudits, the EET network associated with the Lindbladian (6.10) preserves the total number of excitations inserted into network. As a result, it is convenient to split the Hilbert space $\mathcal{H}$ into subspaces $\mathcal{H}_n$, with $n \in \{0, \ldots, (d-1)N\}$. The orthonormal basis of each subspace $\mathcal{H}_n$ is given by kets $|\mathbf{z}_n\rangle$ representing states with n excitations, i.e.

$$|\mathbf{z}_n\rangle = |k_1 \ldots k_N\rangle, \ k_j \in \{0, \ldots, d-1\}, \ \sum_{j=1}^{N} k_j = n.$$

Additionally, each subspace $\mathcal{H}_n$ is associated with its own identity operator I_n, which reads

$$I_n = \sum_{\mathbf{z}_n} |\mathbf{z}_n\rangle \langle \mathbf{z}_n|.$$

6.1 Graph representations

Dynamical properties of the QMDS $\mathcal{T}_t$ are determined by the interaction topology of the network. A EET network in which the qudits can be divided into two sets such that $\gamma_{jk} = 0$ for j and k being indices of qudits from different sets will have clearly asymptotic properties much different than a EET network such that $\gamma_{jk} > 0, \forall j \neq k$. The interaction topology of EET networks can be described by certain graph structures, each characterizing the EET network on a different level. The most intuitive graph structure describing the interaction topology of the EET network is the **interaction graph** – a weighted directed graph $G(V, E, \Gamma)$. Each vertex $j \in V$ represents particular qubit j, an edge $e = (j, k) \in E$ represents interaction between qudits j and k with positive interaction strengths $0 < \gamma_{jk} \in \Gamma$ being the weight of the edge $e = (j, k)$. Interaction graph serves an elegant visual representation of the network as well as a base from which the other graph representation are constructed.

The algebraic structure of attractor spaces is determined by the graph of components $\mathsf{C}(G)$ corresponding to the interaction graph $G(V, E, \Gamma)$. Two EET networks with the identical graph of components have the same algebraic structure of attractor spaces. The simplest graph of components $\mathsf{C}(G)$, given by a single vertex, occurs for strongly connected interaction graph G and yields the most basic attractor structure. The attractor space for the more complex graphs of components can be constructed using the attractor space corresponding to strongly connected interaction graph.

The dynamics of the EET network takes place on the level of individual kets $|\mathbf{z}\rangle$, which are connected through the Lindblad operators L_{jk}. Since each Lindblad operator L_{jk} is associated with a different interaction strength γ_{jk}, each excitation configuration $\mathbf{z} \in \{0, \ldots, d-1\}^N$ occurs with a specific probability, hidden within the form of the corresponding attractor. For description of the relationship of kets $|\mathbf{z}\rangle$, another graph structure, describing the connection between different excitation configurations $\mathbf{z} \in \{0, \ldots, d-1\}^N$ is needed – the **index graph**, $\mathcal{G}(G)$. Similarly to the interaction graph, the index graph is a weighted directed graph $\mathcal{G}(G) = (\mathbf{Z}, E_\mathcal{G}, \Gamma)$. Vertices $\mathbf{z} \in \mathbf{Z}$ of the index graph correspond to the elements of the computational basis $|\mathbf{z}\rangle$. Two vertices $\mathbf{z}_1$ and $\mathbf{z}_2$ are connected with an edge $(\mathbf{z}_1, \mathbf{z}_2) \in E_\mathcal{G}$ with weight $\gamma_{jk} \in \Gamma$ if and only if $L_{jk}|\mathbf{z}_1\rangle = |\mathbf{z}_2\rangle$ and $\gamma_{jk} > 0$. The index graph thus contains information about connection of individual elements of the computational basis through the operators L_{jk}.

Due to nature of the Lindbladian (6.10) discussed in the previous section the index graph $\mathcal{G}(G)$ can be divided into $(d-1)N+1$ disconnected subgraphs $\mathcal{G}_n(G)$, called the n-index graphs. Each n-index graph $\mathcal{G}_n(G)$ contains vertices $\mathbf{z}_n$ corresponding to elements of computational basis $|\mathbf{z}_n\rangle \in \mathcal{H}_n$. Furthermore, if the interaction graph G is strongly connected, then each n-index graph $\mathcal{G}_n(G)$ forms a disconnected component of the index graph $\mathcal{G}(G)$. As demonstrated in example depicted on figure 6.1, 0-index graph $\mathcal{G}_0(G)$ and $(d-1)N$-index graph $\mathcal{G}_{(d-1)N}$ each contain only a single vertex and for a number of applications they do not need to be taken into the consideration.

6.2 T-state of EET network

Generally, apart from case of unital QMPs, obtaining a $\mathcal{T}$-state may represent a difficult task. In case of a EET network, due to incoherent transfer of excitations provided by Lindblad operators L_{jk}, the Lindbladian $\mathcal{L}$ with $V_{jk} = 0$, $\forall j, k$ tends to destroy any coherence present in a general quantum state $\rho \in \mathsf{S}(\mathcal{H})$. It is therefore possible to assume the existence of a diagonal $\mathcal{T}$-state σ, i.e.

$$\sigma = \sum_{\mathbf{z}} \sigma_{\mathbf{z}} |\mathbf{z}\rangle \langle \mathbf{z}|, \ \sigma_{\mathbf{z}} \geq 0, \ \forall \mathbf{z}. \tag{6.11}$$

To justify this assumption, consider an example of network of $N = 4$ qubits ($d = 2$) associated with an interaction graph depicted on the figure 6.1. Since the Lindblad operators L_{jk} do not create nor destroy excitations, the equation
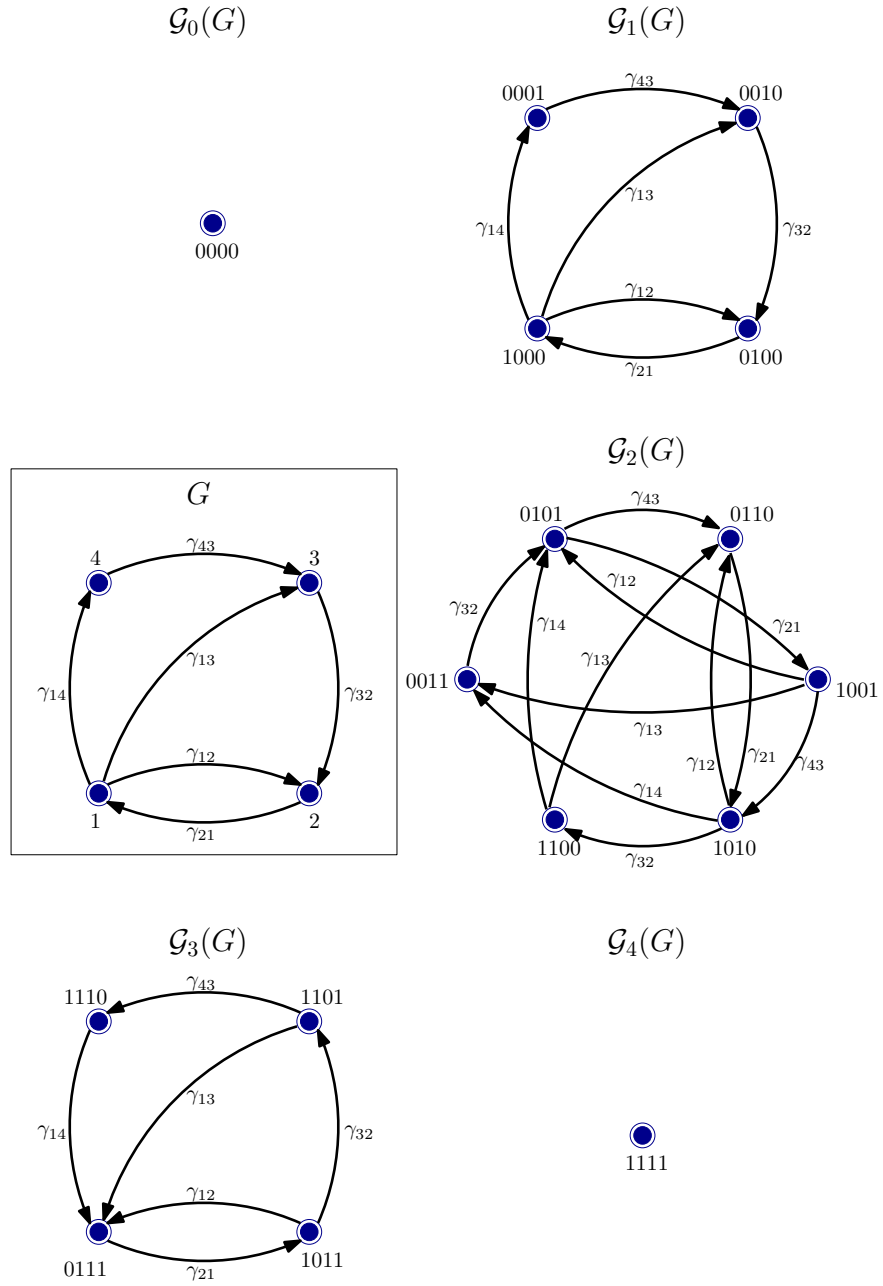
Figure 6.1: An example of an interaction graph $G(V, E, \Gamma)$ (in rectangle) and the corresponding index graph $\mathcal{G}(G)$ divided into n-index graphs $\mathcal{G}_n(G)$ for case of $N = 4$ qubits (i.e. d=2). The n-index graphs $\mathcal{G}_0(G)$ and $\mathcal{G}_{(d-1)N}(G) \equiv \mathcal{G}_4(G)$ are trivial. All n-index graphs $\mathcal{G}_n(G)$ are not connected with each other.

$\mathcal{L}(\sigma) = 0$ with σ given by (6.11) can be decomposed into set of $(d-1)N+1$ equations $\mathcal{L}(\sigma^{(n)}) = 0$, with

$$\sigma^{(n)} = \sum_{\mathbf{z}_n} \sigma_{\mathbf{z}_n} |\mathbf{z}_n\rangle \langle \mathbf{z}_n| . \tag{6.12}$$

Consider the case $n = 3$. The equation $\mathcal{L}(\sigma^{(3)}) = 0$ can be written in matrix form as

$$\begin{pmatrix} \Gamma_1 & -\gamma_{12} & -\gamma_{13} & -\gamma_{14} \\ -\gamma_{21} & \Gamma_2 & 0 & 0 \\ 0 & -\gamma_{32} & \Gamma_3 & 0 \\ 0 & 0 & -\gamma_{43} & \Gamma_4 \end{pmatrix} \begin{pmatrix} \sigma_{0111} \\ \sigma_{1011} \\ \sigma_{1101} \\ \sigma_{1110} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6.13}$$

with

$$\Gamma_j = \sum_{k \neq j} \gamma_{kj}.$$

It is straightforward to check that the matrix in the left-hand side of equation (6.13) has rank equal to three and consequently, the solution of this equation forms a one-dimensional subspace, hinting the existence of the diagonal quantum state σ fulfiling the equation $\mathcal{L}(\sigma) = 0$. It needs to be stressed, that these considerations are valid only for the case $V_{jk} = 0$, $\forall j, k$. If this is not the case, an existence of a diagonal $\mathcal{T}$-state is not guaranteed.

To determine a diagonal $\mathcal{T}$-state σ, it is appropriate to introduce an alternative notation. Although this notation is unpractical for solving particular examples, it is convenient for writing the equations (6.13) for a general case. Elements $|\mathbf{z}\rangle$ represent the number of excitations and their distribution through present in the network. Alternatively, same information can be provided by listing indices of all qudits with particular number of excitations, represented by kets $|n\rangle$. Grouping all indices of qudits with fixed number n of excitations into sets $\mathbf{j}_n$, sets $\mathbf{j}_n$ are then arranged to define kets $|\mathbf{j}\rangle = |\mathbf{j}_0, \ldots, \mathbf{j}_{d-1}\rangle$. As an example, consider $N = 3$, $d = 3$ and $|\mathbf{z}\rangle = |010\rangle$. Then $\mathbf{j}_0 = \{1, 3\}$, $\mathbf{j}_1 = \{2\}$, $\mathbf{j}_2 = \emptyset$ and consequently $|\mathbf{z}\rangle \equiv |\mathbf{j}\rangle = |\{1, 3\}, \{2\}, \emptyset\rangle$.

In this notation the quantum state σ (6.11) takes the form

$$\sigma = \sum_{\mathbf{j}} \sigma_{\mathbf{j}} |\mathbf{j}\rangle \langle \mathbf{j}| .$$

Furthermore, it is convenient to additionally incorporate the character of interactions into this new notation. Consider an element $|\mathbf{j}\rangle$ and a Lindblad operator L_{ab}. With respect to the ket $|\mathbf{j}\rangle$, the index a is a k-th element of the set $\mathbf{j}_m$ and it can be thus denoted as $j_m^{(k)}$. Similarly, the index b, being the l-th element of the set $\mathbf{j}_n$ can be denoted as $j_n^{(l)}$ and thus $L_{ab} = L_{j_m^{(k)} j_n^{(l)}}$. To factor the character of interactions into the new notation, the coefficient $\sigma_{\mathbf{j}'}$ corresponding to the element $|\mathbf{j}'\rangle$ such that

$$|\mathbf{j}'\rangle = L_{j_m^{(k)} j_n^{(l)}} |\mathbf{j}\rangle$$

will be denoted as $\sigma_{\mathbf{j},(j_m^{(k)}, j_m^{(k)})}$. This coefficient differs from the coefficient $\sigma_{\mathbf{j}}$ by exchange of indices $j_m^{(k)}$ and $j_n^{(l)}$ due to the application of the corresponding Lindblad operator.

In the context of graph representations, all possible coefficients $\sigma_{\mathbf{j},(j_m^{(k)}, j_m^{(k)})}$ are associated with vertices, which can be connected to the vertex $\mathbf{z}$, corresponding to the element $|\mathbf{j}\rangle$ by a single edge.

The equation $\mathcal{L}(\sigma) = 0$ can be written in the new notation as a set of algebraic equations $\Lambda(G)$ for coefficients $\sigma_{\mathbf{j}}$ as

$$
\begin{aligned}
\Gamma_{\mathbf{j}} \sigma_{\mathbf{j}} &= \sum_{m+1,n=0}^{d-2} \sum_{k,l} \gamma_{j_n^{(l)} j_m^{(k)}} \sigma_{\mathbf{j},(j_m^{(k)}, j_n^{(l)})}, \quad \forall \mathbf{j}, \\
\Gamma_{\mathbf{j}} &= \sum_{m+1,n=0}^{d-2} \sum_{k,l} \gamma_{j_m^{(k)} j_n^{(l)}}
\end{aligned}
\tag{6.14}
$$

Such a set of equations was already studied in context of classical Markov processes in [13].

A quick look reveals a neat graphical representation of each of equations (6.14) in the context of n-index graphs $\mathcal{G}_n(G)$. The coefficient $\Gamma_{\mathbf{j}}$ is the sum of weights of all edges such that the vertex $\mathbf{z}$, corresponding to the coefficient $\sigma_{\mathbf{j}}$ is their tail. The coefficients $\gamma_{j_n^{(l)} j_m^{(k)}}$ represent weights of edges such that the vertex $\mathbf{z}$, corresponding to the coefficient $\sigma_{\mathbf{j}}$ is their head. Consequently, the left side of the equation (6.14) can be interpreted as the flow of excitation from the vertex $\mathbf{z}$ (the outward flow) and the right side is the flow of excitation in the vertex $\mathbf{z}$ (the inward flow). An example of this interpretation is depicted on figure 6.2. As there are no edges between different n-index graphs $\mathcal{G}_n(G)$, the set of equations (6.14) can be divided into $(d-1)N+1$ sets of equations $\Lambda_n(G)$, each corresponding to the fixed number of excitations $n \in \{0, \ldots, (d-1)N\}$ in the network. Furthermore, the n-index graphs $\mathcal{G}_0(G)$ and $\mathcal{G}_{(d-1)N}(G)$ contain a single isolated vertex. Consequently, equations $\Lambda_n(G)$ for $n \in \{0, (d-1)N\}$ are trivial and the corresponding elements $\sigma_{\mathbf{z}}$ can be chosen arbitrarily.

To solve equations (6.14), first, consider a strongly connected interaction graph G. This implies that each n-index graph $\mathcal{G}_n(G)$ is also a strongly connected graph and consequently, each set of equations $\Lambda_n(G)$ has a unique solution, up to a normalization. This solution can be obtained by standard triangular method. As an example, consider previously derived set of equations $\Lambda_3(G)$ (6.13) for $N = 4$ qubits. Denoting $\Gamma_j^{(0)} \equiv \Gamma_j$ and $\gamma_{kj}^{(0)} \equiv \gamma_{kj}$, the matrix equation (6.13) can be converted into upper triangular form by means of following iteration process

$$
\begin{aligned}
\Gamma_j^{(n+1)} &= \Gamma_j^{(n)} \Gamma_{N-n}^{(n)} - \gamma_{N-nj}^{(n)} \gamma_{jN-n}^{(n)}, \\
\gamma_{kj}^{(n+1)} &= \gamma_{kj}^{(n)} \Gamma_{N-n}^{(n)} + \gamma_{N-nj}^{(n)} \gamma_{kN-n}^{(n)},
\end{aligned}
\tag{6.15}
$$

$$\gamma_{21}\sigma_{0111} - \gamma_{12}\sigma_{1011} - \gamma_{13}\sigma_{1101} - \gamma_{14}\sigma_{1110} = 0$$
$$(\gamma_{12} + \gamma_{32})\sigma_{1101} - \gamma_{21}\sigma_{0111} = 0$$
$$(\gamma_{13} + \gamma_{43})\sigma_{1101} - \gamma_{32}\sigma_{1011} = 0$$
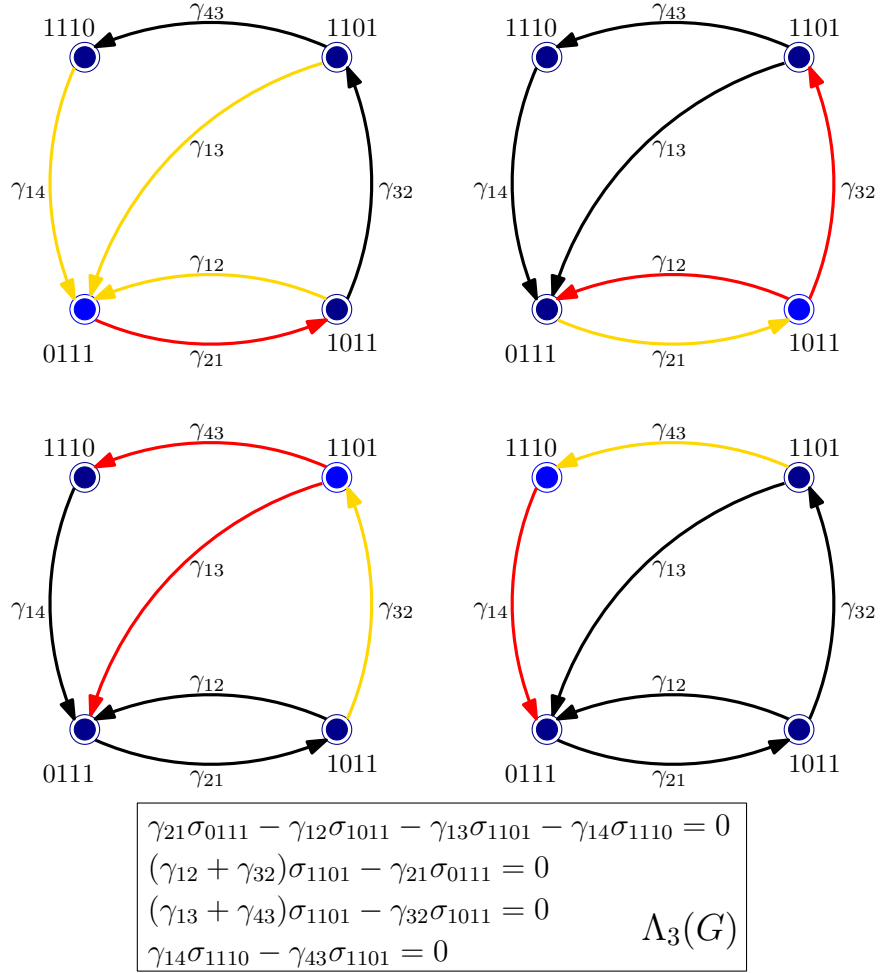$$\gamma_{14}\sigma_{1110} - \gamma_{43}\sigma_{1101} = 0$$

$$\Lambda_3(G)$$

Figure 6.2: Construction of the set of equations $\Lambda_3(G)$ for the interaction graph $G(V, E, \Gamma)$ from figure 6.1, based on the in and out flow of excitaitons from the particular vertices of the 3-index graph $\mathcal{G}_3(G)$. Each graph has a highlighted vertex (bright blue) which can be associated with a single equation from the set $\Lambda_3(G)$. Yellow edges contribute to the inward flow, red edges contribute to the outward flow.

resulting in

$$\sigma_{0111} = \Gamma_2^{(2)} = \Gamma_2\Gamma_3\Gamma_4 = (\gamma_{12} + \gamma_{32})(\gamma_{13} + \gamma_{43})\gamma_{14} =$$
$$\gamma_{12}\gamma_{13}\gamma_{14} + \gamma_{12}\gamma_{43}\gamma_{14} + \gamma_{32}\gamma_{13}\gamma_{14}\gamma_{32}\gamma_{43}\gamma_{14}. \tag{6.16}$$

Each addend of the solution (6.16) can be interpreted as a weight $\Omega(g)$ of a certain subgraph $g \subset \mathcal{G}_3(G)$, more specifically a directed maximal tree corresponding to the vertex $\mathbf{z}_3 = 0111$. An analogous result can be obtained for remaining coefficients $\sigma_{\mathbf{z}_3}$ - each element $\sigma_{\mathbf{z}_3}$ is thus proportional to the sum of weights of all directed maximal trees corresponding to the vertex $\mathbf{z}_3$.

This process can be then straightforwardly applied to a more general case of a network of N d-dimensional qubits associated with a strongly connected interaction graph G, with the following result: For any vertex $\mathbf{z}_n \in \mathcal{G}_n(G)$, one obtains

$$\sigma_{\mathbf{z}_n} = \sum_{T_{\mathbf{z}_n}(\mathcal{G}_n(G))} \Omega(T_{\mathbf{z}_n}(\mathcal{G}_n(G))), \tag{6.17}$$

i.e. the coefficient $\sigma_{\mathbf{z}_n}$ is proportional to the value of all directed maximal trees corresponding to the vertex $\mathbf{z}_n$.

Results obtained for strongly connected interaction graphs allow to study more general interaction topologies. In doing so, some of the less interesting interaction topologies will be ignored - particularly such that the interaction graph contains some disconnected components. If that is the case, the asymptotics can be studied separately on each component.

As a first extension of obtained results, assume an interaction graph with two components, connected with edges in one direction. Therefore, the interaction graph G contains one terminal component G_T of the size $N_T < N$. For $n \leq (d-1)N_T$, all n-index graphs $\mathcal{G}_n(G)$ also contain only one terminal component $\mathcal{G}_n^T(G)$, although the total number of components of the graph $\mathcal{G}_n(G)$ can be larger than two. For $n > (d-1)N_T$, any n-index graph $\mathcal{G}_n(G)$ is strongly connected. The latter case is analogous to the case of a strongly connected interaction graph, the focus will be therefore put on the case $n \leq (d-1)N_T$.

Consider a vertex $\mathbf{z}_n \in \mathcal{G}_n(G) \backslash \mathcal{G}_n^T(G)$. It is easy to deduce that there are no maximal trees corresponding to the vertex $\mathbf{z}_n$, as there are no edges (v, w) with $v \in \mathcal{G}_n^T(G)$ and $w \in \mathcal{G}_n(G)$. It is thus impossible to have a subgraph $g \subset \mathcal{G}_n(G)$ such that each of its vertices v from the terminal component $\mathcal{G}_n^T(G)$ fulfils $\mathsf{D}(v) = 1$ and that is simultaneously acyclic. As a result, one conclude that

$$\sigma_{\mathbf{z}_n} = 0, \mathbf{z}_n \in \mathcal{G}_n(G) \backslash \mathcal{G}_n^T(G).$$

Next, consider $\mathbf{z}_n \in \mathcal{G}_n^T(G)$. In such case, the result (6.17) still holds. However, one can notice that considering any two different vertices $\mathbf{z}_n^{(1)}, \mathbf{z}_n^{(2)} \in \mathcal{G}_n^T(G)$, each maximal tree $T_{\mathbf{z}_n^{(1)}}(\mathcal{G}_n(G))$ can be associated with a maximal tree $T_{\mathbf{z}_n^{(2)}}(\mathcal{G}_n(G))$ such that both maximal trees $T_{\mathbf{z}_n^{(1)}}(\mathcal{G}_n(G))$ and $T_{\mathbf{z}_n^{(2)}}(\mathcal{G}_n(G))$ have the same edges except those in the terminal component $\mathcal{G}_n^T(G)$. The value of edges outside the
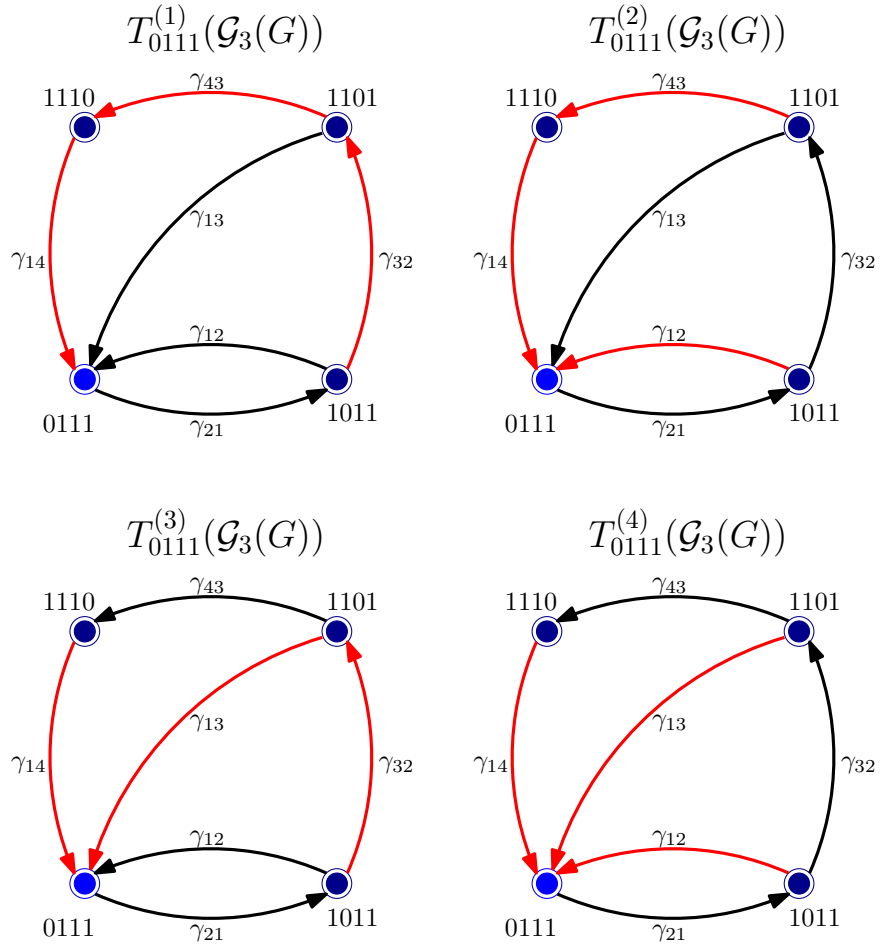
$$\sigma_{0111} = \gamma_{32}\gamma_{43}\gamma_{14} + \gamma_{43}\gamma_{14}\gamma_{12} + \gamma_{32}\gamma_{13}\gamma_{14} + \gamma_{12}\gamma_{13}\gamma_{14}$$

Figure 6.3: All $\mathbf{z}_n$-trees $T_{0111}^{(j)}(\mathcal{G}_3(G))$ for the vertex $\mathbf{z}_n = 0111$ (bright blue) associated with the interaction graph from figure 6.1. The edges belonging to the trees are red. The resulting element σ_{0111} is according to (6.17) calculated as a sum of weights of all corresponding trees $T_{0111}^{(j)}(\mathcal{G}_3(G))$ (below).

terminal component $\mathcal{G}_n^T(G)$ can be thus canceled out in the formula (6.17), yielding

$$\sigma_{\mathbf{z}_n} = \sum_{T_{\mathbf{z}_n}(\mathcal{G}_n^T(G))} \Omega(T_{\mathbf{z}_n}(\mathcal{G}_n^T(G))). \tag{6.18}$$

As the most general case, consider an interaction graph G with several terminal components G_{T_k}. Then n-index graph $\mathcal{G}_n(G)$ is either strongly connected, or it has one or several terminal components. Focusing on the case of several terminal components $\mathcal{G}_n^{T_k}(G)$, each set of equations $\Lambda_n(G)$ has a number of linearly independent solutions equal to the number of terminal components. These solutions read

$$\sigma_{\mathbf{z}_n} = \sum_{T_{\mathbf{z}_n}(\mathcal{G}_n^{T_k}(G))} \Omega(T_{\mathbf{z}_n}(\mathcal{G}_n^{T_k}(G))), \qquad \mathbf{z}_n \in \mathcal{G}_n^{T_k}(G),$$
$$\sigma_{\mathbf{z}_n} = 0, \qquad \mathbf{z}_n \in \mathcal{G}_n(G) \backslash \mathcal{G}_n^{T_k}(G). \tag{6.19}$$

Solutions (6.19) on particular terminal components $\mathcal{G}_n^{T_k}(G)$ can be then used to construct a solution $\sigma \in \mathsf{S}(\mathcal{H})$ with maximal possible rank. Apart normalization, coefficients of a $\mathcal{T}$-state σ in form (6.12) therefore read

$$\sigma_{\mathbf{z}_n} = \sum_{T_{\mathbf{z}_n}(\mathcal{G}_n^{T_k}(G))} \Omega(T_{\mathbf{z}_n}(\mathcal{G}_n^{T_k}(G))), \qquad \mathbf{z}_n \in \mathcal{G}_n^{T_k}(G)$$
$$\sigma_{\mathbf{z}_n} = 0, \qquad \mathbf{z}_n \in \mathcal{G}_n(G) \backslash \left(\bigcup_k \mathcal{G}_n^{T_k}(G) \right). \tag{6.20}$$

To summarize obtained results, for any strongly connected interaction graph $G(V, E, \Gamma)$, resulting $\mathcal{T}$-state σ takes up to a normalization the form (6.17) and consequently the $\mathcal{T}$-projector P fulfils $P = I$. On the contrary, if the interaction graph $G(V, E, \Gamma)$ contains one or several terminal components G_{T_k}, then a $\mathcal{T}$-state σ is up to a normalization given by (6.20), which implies $P < I$. In such case, the derivation of the attractor space $\mathsf{Atr}(\mathcal{L}^\dagger)$ is significantly more complicated, as attractor equations in the Heisenberg picture do not determine full form of corrsponding attractors, which need to be obtained by means of equation (4.29).

There are several examples, of interaction strengths γ_{jk} which result in a simple form of a $\mathcal{T}$-state σ. First, it is beneficial to analyze, in which cases the QMDS $\mathcal{T}_t$ is unital. Such case can be obtained only for strongly connected interaction graphs $G(V, E)$, as the resulting $\mathcal{T}$-state σ must fulfil $\sigma \sim I > 0$. Entering $\sigma_{\mathbf{j}} = 1$ in (6.14), one obtains

$$\sum_{m+1,n=0}^{d-2} \sum_{k,l} \left(\gamma_{j_m^{(k)} j_n^{(l)}} - \gamma_{j_n^{(l)} j_m^{(k)}} \right) = 0, \quad \forall \mathbf{j}. \tag{6.21}$$

Columns of matrix defined by equations (6.14) always sum up to zero. In order to obtain an unital QMDS $\mathcal{T}_t$, rows of this matrix must according to equations (6.21) also sum up to zero. Generally, this is fulfilled only for the symmetric interaction, i.e. $\gamma_{jk} = \gamma_{kj}$, $\forall j \neq k$. However, in case of qubit network ($d = 2$), condition (6.21) can be further relaxed. Consider the set of equations $\Lambda_1(G)$, for which the unitality condition reads

$$\sum_{k \neq j} (\gamma_{jk} - \gamma_{kj}) = 0, \quad \forall j \in \{\, 1, \ldots, N \,\}. \tag{6.22}$$

It can be shown that the equation (6.21) for any $\mathbf{j}$ can be in case of qubit network written as a linear combination of equations (6.22). More specifically, consider $\mathbf{j} = \{\, \mathbf{j}_0, \mathbf{j}_1 \,\}$. The equation (6.21) for such $\mathbf{j}$ can be obtained by summing all equations (6.22) for all $j \in \mathbf{j}_1$ and thus the unitality of the network of qubits is determined by equations (6.22), i.e. solely by the interaction graph $G(V, E, \Gamma)$.

Subunital QMPs $\mathcal{T}_t$ can be treated in a similar way. A EET network of qudits can be subunital only for weakly connected interaction graphs $G(V, E, \Gamma)$ with strongly connected components $G^{(n)}(V^{(n)}, E^{(n)}, \Gamma^{(n)})$. It is straightforward to show that the qudit network associated with such an interaction graph is subunital, if the interaction within each of the components $G^{(n)}$ is symmetric, i.e. if $\gamma_{jk} = \gamma_{kj}$, $\forall j, k \in V^{(n)}, j \neq k$. The subunitality is therefore not affected by interaction strenghts γ_{jk} with j and k belonging to different components. As for the unitality case, the condition of symmetry can be relaxed in case of qubits, resulting in

$$\sum_{k \in V^{(n)}, k \neq j} (\gamma_{jk} - \gamma_{kj}) = 0, \quad \forall j \in V^{(n)}, \ \forall n. \tag{6.23}$$

Another interesting case with a surprisingly elegant solution is the "light bulb" model. Assume that each interaction strength γ_{jk} fulfils $\gamma_{jk} = \alpha_j a_{jk}$, with $a_{jk} = a_{kj}$. The symmetric term a_{jk} may be interpreted as a distance between qudits j and k, while the term α_j plays the role of interaction strength (or "brightness") of qudit j. Such model of interaction results in a completely connected network, with $\gamma_{jk} > 0$ for all possible pairs $j \neq k$. By insertion into equations (6.14), one obtains

$$\sum_{m+1,n=0}^{d-2} \sum_{k,l} a_{j_m^{(k)} j_n^{(l)}} \left(\alpha_{j_m^{(k)}} \sigma_{\mathbf{j}} - \alpha_{j_n^{(l)}} \sigma_{\mathbf{j},(j_m^{(k)}, j_n^{(l)})} \right) = 0, \quad \forall \mathbf{j}, \tag{6.24}$$

which can be simplified to

$$\alpha_{j_m^{(k)}} \sigma_{\mathbf{j}} = \alpha_{j_n^{(l)}} \sigma_{\mathbf{j},(j_m^{(k)}, j_n^{(l)})}, \quad \forall \mathbf{j}. \tag{6.25}$$

Resulting coefficients $\sigma_{\mathbf{j}}$ therefore do not depend on distances a_{jk}, but only on the "brightness" α_j. In the standard notation $\sigma_{\mathbf{z}}$, $\mathbf{z} = \{\, z_1, \ldots, z_N \,\}$, $z_n \in \{\, 0, \ldots, d-1 \,\}$ they read

$$\sigma_{\mathbf{z}} = \prod_{n=1}^{N} \alpha_n^{d-z_n}. \tag{6.26}$$

Due to independence of the coefficients $\sigma_{\mathbf{z}}$ on the distance a_{jk}, all directed maximal trees corresponding to a fixed vertex $\mathbf{z}$ have the same value, simplifying the coefficient $\sigma_{\mathbf{z}}$ to a single term. Similar results can be intuitively derived for a several qudit networks with bulb model, connected with edges only in a single direction. Resulting coefficients $\sigma_{\mathbf{z}}$ then combine results (6.19) and (6.26).

6.3 Attractor spaces of EET network

A $\mathcal{T}$-state σ derived in the previous section enables determination of attractor spaces $\mathsf{Atr}(\mathcal{L})$ and $\mathsf{Atr}(\mathcal{L}^{\dagger})$. In case of EET networks associated with a strongly connected interaction graph $G(V, E, \Gamma)$, the $\mathcal{T}$-projector P fulfils $P = I$ and according to the chapter 4 both these attractor spaces are fully determined by attractor equations (4.12) and (4.13). If the interaction graph $G(V, E, \Gamma)$ is only weakly connected, the corresponding $\mathcal{T}$-projector P fulfils $P < I$ and attractor equations in the Heisenberg picture (4.13) determine only the subspace $P\mathsf{Atr}(\mathcal{L}^{\dagger})P$. Obtaining the full form of attractors in the Heisenberg picture in this case requires additional solution of the equation (4.29).

It is convenient to start with the case $P = I$, i.e. a network associated with a strongly connected interaction graph $G(V, E, \Gamma)$. For such case, even without direct solution of attractor equations (4.13), it is possible to determine a number of linearly independent attractors in the Heisenberg picture. due to character of Lindblad operators L_{jk}. As the number of excitations remains unchanged under the application of operators L_{jk}, identity operators I_n on subspaces $\mathcal{H}_n$ consisting of the kets corresponding to n excitations, defined as

$$I_n = \sum_{\mathbf{z}_n} |\mathbf{z}_n\rangle \langle \mathbf{z}_n|, \quad n \in \{0, \dots, (d-1)N\} \tag{6.27}$$

commute both with the Lindblad operators L_{jk}, $L_{jk}^{\dagger}$ and the Hamiltonian H. All these operators therefore correspond to the eigenvalue $\lambda = 0$, i.e. $I_n \in \mathsf{Ker}(\mathcal{L})$. Furthermore, the operators I_0 and $I_{(d-1)N}$ read

$$I_0 = |\mathbf{0}_N\rangle \langle \mathbf{0}_N|, \; I_{(d-1)N} = |\mathbf{d\text{-}1}_N\rangle \langle \mathbf{d\text{-}1}_N|, \tag{6.28}$$

with $|\mathbf{n}_N\rangle \equiv |n \dots n\rangle \in \mathcal{H}$. In fact, kets $|\mathbf{0}_N\rangle$ and $|\mathbf{d\text{-}1}_N\rangle$ are a common eigenvectors of all Lindblad operators L_{jk}, $L_{jk}^{\dagger}$ and the Hamiltonian H. One can therefore construct additional attractors $I_{\pm}$ as

$$I_+ = |\mathbf{0}_N\rangle \langle \mathbf{d\text{-}1}_N|, \quad I_- = |\mathbf{d\text{-}1}_N\rangle \langle \mathbf{0}_N|, \tag{6.29}$$

corresponding to eigenvalues $\lambda_{\mp} = \mp i\varepsilon(d-1)N$ respectively, i.e. $I_{\pm} \in \mathsf{Ker}(\mathcal{L}^{\dagger} - \lambda_{\mp}I)$. Due to $|\mathbf{0}_N\rangle$ and $|\mathbf{d\text{-}1}_N\rangle$ being common eigenvectors of all Lindblad operators and the Hamiltonian, the set of operators $\{I_0, I_{\pm}, I_{(d-1)N}\}$ forms a

decoherence free subspace (DFS) [31, 32] of the EET network associated with a strongly connected interaction graph G.

To resolve whether obtained $(d-1)N + 3$ operators generate the whole attractor space $\mathsf{Atr}(\mathcal{L}^\dagger)$, attractor equations (4.13) must be examined. Due to the binary nature of Lindblad operators L_{jk}, fixing the indices j and k, it suffices to study attractor equations only on subspace corresponding to the pair of qudits directly involved in the action of the Lindblad operator L_{jk}. Assume therefore that a solution of the attractor equations on this subspace takes the form

$$X = \sum_{\mathbf{z},\mathbf{z}'} a_{\mathbf{z}'}^{\mathbf{z}} \, |\mathbf{z}\rangle \, \langle \mathbf{z}'| \,,$$

with $\mathbf{z}$ and $\mathbf{z}'$ being index corresponding to two qudits. Inserting the operator X into attractor equation $[X, L_{jk}] = 0$, one obtains

$$
\begin{aligned}
[X, L_{jk}] = & \\
\sum_{z_j, z_j'=1}^{d-1} \sum_{z_k, z_k'=0}^{d-2} & \left(a_{z_j'-1z_k'+1}^{z_j-1z_k+1} - a_{z_j'z_k'}^{z_jz_k} \right) |z_j - 1z_k + 1\rangle \langle \mathbf{z}'| + \\
\sum_{z_j=1}^{d-1} \sum_{z_k=0}^{d-2} \sum_{z=0}^{d-1} & a_{z\,d-1}^{z_jz_k} |z_j - 1z_k + 1\rangle \langle z\, d-1| + \\
\sum_{z_j=1}^{d-1} \sum_{z_k=0}^{d-2} \sum_{z=0}^{d-1} & a_{0\,z}^{z_jz_k} |z_j - 1z_k + 1\rangle \langle 0\, z| - \\
\sum_{z_j'=0}^{d-2} \sum_{z_k'=1}^{d-1} \sum_{z=0}^{d-1} & a_{z_j'z_k'}^{d-1z} |d - 1\, z\rangle \langle z_j' + 1z_k' - 1| - \\
\sum_{z_j'=0}^{d-2} \sum_{z_k'=1}^{d-1} \sum_{z=0}^{d-1} & a_{z_j'z_k'}^{z\,0} |z\, 0\rangle \langle z_j' + 1z_k' - 1| = 0.
\end{aligned}
\tag{6.30}
$$

Together with analogous equations $[X, L_{jk}^\dagger] = 0$, $[X, L_{kj}] = 0$ and $[X, L_{kj}^\dagger] = 0$, these form a set of equations responsible for selection of attractors in the Heisenberg picture. The additional commutator equation for Hamiltonian is responsible for determination of the corresponding eigenvalue. Since the Hamiltonian (6.8) contains the operators L_{jk} in a linear fashion, the only part of the Hamiltonian which must be checked is the free evolution given by the operators $H_j^{(1)}$.

According to commutation relations with operators L_{jk}, $L_{jk}^\dagger$, L_{kj} and $L_{kj}^\dagger$, coefficients $a_{\mathbf{z}'}^{\mathbf{z}}$ such that $\mathbf{z}$ or $\mathbf{z}'$ contains at least one index $z \in \{0, d-1\}$ and simultaneously $\mathbf{z} \neq \mathbf{z}'$ must read $a_{\mathbf{z}'}^{\mathbf{z}} = 0$. The only exception for nondiagonal solution is if

$$\mathbf{z}, \mathbf{z'} \in \{\, \mathbf{0}_2, \mathbf{d\text{-}1}_2 \,\} \wedge \mathbf{z} \neq \mathbf{z}', \tag{6.31}$$

as such terms are not present in the equation (6.30). The first term of equation (6.30) then connects coefficients associated with shifted configuration given by values z_j, z_k, z_j' and z_k'. As a result, a nondiagonal operator X can be an attractor in the Heisenberg picture only if it fulfils condition (6.31) on each pair of qudits j and k, resulting in attractors $I_\pm$.

Except for attractors $I_\pm$, all attractors in the Heisenberg picture corresponding to the strongly connected qudit network must be diagonal. All such solutions can be generated by operators I_n and consequently, the set of $(d-1)N+3$ operators given by (6.27) and (6.29) represents all linearly independent solutions of attractor equations (4.13). The attractor space $\mathsf{Atr}(\mathcal{L}^\dagger)$ is consequently determined by the set of operators $\mathcal{V}(\mathcal{L}^\dagger)$ which reads

$$\mathcal{V}(\mathcal{L}^\dagger) = \{\, I_n, I_\pm \mid n \in \{\, 0, \ldots, (d-1)N \,\} \,\} \tag{6.32}$$

Having obtained the whole attractor space $\mathsf{Atr}(\mathcal{L}^\dagger)$, there is no need to solve the attractor equations (4.12) as one can exploit the relation (4.3) to obtain attractors in the Schrödinger picture. Since attractors I_n are identity operators on subspaces $\mathcal{H}_n$ corresponding to kets $|\mathbf{z}_n\rangle$ with fixed number of excitations, diagonal attractors $\sigma_n \in \mathsf{Atr}(\mathcal{L})$, corresponding to the eigenvalue $\lambda = 0$, are given by (6.12) with coefficients (6.17). Nondiagonal attractors $I_\pm \in \mathsf{Atr}(\mathcal{L}^\dagger)$ map to nondiagonal attractors $\sigma_\pm \in \mathsf{Atr}(\mathcal{L})$, which take exactly the same form, i.e. $\sigma_\pm = I_\pm$, however due to relations (4.3) these operators fulfil $\sigma_\pm \in \mathsf{Ker}(\mathcal{L} - \lambda_\pm I)$. The basis $\mathcal{V}(\mathcal{L})$ of the attractor space $\mathsf{Atr}(\mathcal{L})$ therefore reads

$$\mathcal{V}(\mathcal{L}) = \{\, \sigma_n, \sigma_\pm \mid n \in \{\, 0, \ldots, (d-1)N \,\} \,\} \tag{6.33}$$

For weakly connected interaction graph $G(V, E, \Gamma)$, the situation is significantly more complicated. Before discussing the structure of attractor spaces, the $\mathcal{T}$-projector P must be determined, as it fulfils $P < I$. This can be done in two ways. The first possibility is to derive the $\mathcal{T}$-projector using the graph of components $\mathsf{C}(G)$, while respecting the nature of interactions , i.e. if there is a directed edge (G_1, G_2) between two components $G_1, G_2 \in \mathsf{C}(G)$, then the component G_2 must be completely filled with excitations before any excitations stay in the component C_1, as in an example depicted on figure 6.4. The other possibility is to obtain the $\mathcal{T}$-projector P through the index graph $\mathcal{G}(C)$ as

$$P = \sum_{n=0}^{(d-1)N} \sum_{\mathbf{z}_n \in \mathcal{G}_n^T(G)} |\mathbf{z}_n\rangle \langle \mathbf{z}_n| , \tag{6.34}$$

Attractor equations (4.12) and (4.13) are valid only on the subspace $P\mathcal{H}$ and consequently, these equations determine the subspace $P\mathsf{Atr}(\mathcal{L}^\dagger)P$ and the attractor space $\mathsf{Atr}(\mathcal{L})$. Attractors in the Schrödinger picture can be obtained using the results of the previous section, as any solution of equations $\Lambda_n(G)$ which is nonzero only on a terminal component $\mathcal{G}_n^{T_k}(G)$, given by (6.19) results in a linearly independent attractor in the Schrödinger picture. However, for large networks, the index graph $\mathcal{G}(G)$ becomes increasingly more complex and therefore it is useful to have an insight, how to determine the attractor space
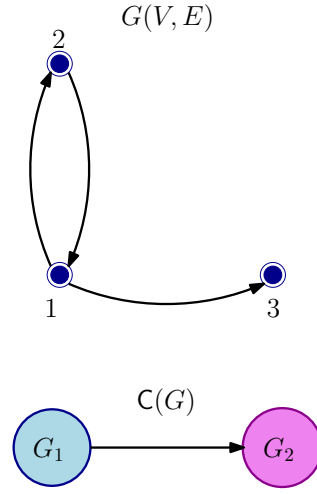
$$P = |000\rangle\langle000| + |001\rangle\langle001| + |011\rangle\langle011| + |101\rangle\langle101| + |111\rangle\langle111|$$

Figure 6.4: An example of a weakly connected interaction graph $G(V, E)$ of 3 qubits ($d = 2$), its corresponding graph of components $\mathsf{C}(G)$ and the resulting $\mathcal{T}$-projector P.

$\mathsf{Atr}(\mathcal{L})$ from the structure of the interaction graph G, resp. its corresponding graph of components $\mathsf{C}(G)$ alone. It turns out that generally, a large part of the attractor space $\mathsf{Atr}(\mathcal{L})$ associated with a graph of components $\mathsf{C}(\mathcal{G})$ can be determined using graphs of components $\mathsf{C}(G')$ corresponding to simpler interaction graphs G'. This procedure is quite similar to intuitive derivation of the $\mathcal{T}$-projector P.

To showcase this procedure, consider a several examples of interaction topologies, with corresponding graphs of components depicted on the figure 6.5. each with corresponding Lindbladians $\mathcal{L}_j$. Using the graph of components $\mathsf{C}(G^{(j)})$, the $\mathcal{T}$-projector P_j corresponding to each of these interaction topologies reads
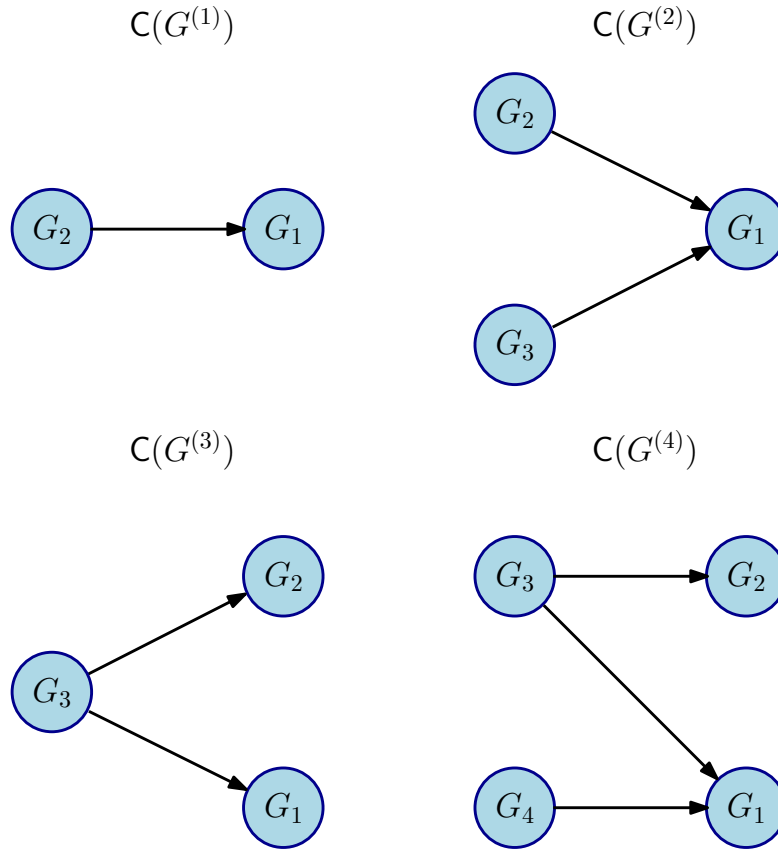
Figure 6.5: Several examples of graph of components $\mathsf{C}(G^{(j)})$ associated with weakly connected interaction graphs. The size of each component G_k is N_k. Each of this graph of components results in an unique $\mathcal{T}$-projector P_j and the structure of the attractor space, given by the set of operators $\mathcal{V}(\mathcal{L}_j)$.

$$P_1 = I_0^{(2)} \otimes I^{(1)} + \left(I^{(2)} - I_0^{(2)} \right) \otimes I_N^{(1)},$$

$$P_2 = I_0^{(3)} \otimes I_0^{(2)} \otimes I^{(1)} + \left(I^{(3)} \otimes I^{(2)} - I_0^{(3)} \otimes I_0^{(2)} \right) \otimes I_N^{(1)} =$$
$$I_0^{(3)} \otimes P_1^{(1,2)} + \left(I^{(3)} - I_0^{(3)} \right) \otimes I^{(2)} \otimes I_N^{(1)},$$

$$P_3 = I_0^{(3)} \otimes I^{(2)} \otimes I^{(1)} + \left(I^{(3)} - I_0^{(3)} \right) \otimes I_N^{(2)} \otimes I_N^{(1)},$$

$$P_4 = I_0^{(4)} \otimes I_0^{(3)} \otimes I^{(2)} \otimes I^{(1)} +$$
$$\left(I_0^{(4)} \otimes \left(I^{(3)} - I_0^{(3)} \right) + \left(I^{(4)} - I_0^{(4)} \right) \otimes I_0^{(3)} \right) \otimes I_N^{(2)} \otimes I_N^{(1)} +$$
$$\left(I^{(4)} - I_0^{(4)} \right) \otimes I_0^{(3)} \otimes \left(I^{(2)} - I_0^{(2)} \right) \otimes I_N^{(1)} +$$
$$\left(I^{(4)} - I_0^{(4)} \right) \otimes \left(I^{(3)} - I_0^{(3)} \right) \otimes I_N^{(2)} \otimes I_N^{(1)} =$$
$$I_0^{(4)} \otimes P_3^{(1,2,3)} + \left(I^{(4)} - I_0^{(4)} \right) \otimes P_1^{(2,3)} \otimes I_N^{(1)},$$

(6.35)

with $I_n^{(k)}$ representing the attractor I_n on the component G_k and $I_N^{(k)} \equiv I_{(d-1)N_k}^{(k)}$ for brevity. Furthermore, projectors P_j demonstrate a certain level of inner structure, as they can be written using more basic building blocks, in case of projectors P_1 and P_3 its the projector I corresponding to components G_k, the projector P_2 utilizes projector I and P_1 and the projector P_4 can be decomposed in terms of projectors I, P_1 and P_3.

Decomposition of the projectors P_j to projectors corresponding to simpler structures may be also utilized for description of sets $\mathcal{V}(\mathcal{L}_j)$ generating the attractor space $\mathsf{Atr}(\mathcal{L})$. Denoting $\mathsf{Atr}(\mathcal{L}_C)^{(k)}$ the attractor space of the isolated component G_k, the sets $\mathcal{V}(\mathcal{L}_j)$ read

$$\mathcal{V}(\mathcal{L}_1) = \{ \sigma_0^{(2)} \otimes \mathcal{V}(\mathcal{L}_C)^{(1)}, \left(\mathcal{V}(\mathcal{L}_C)^{(2)} \setminus \{ \sigma_0^{(2)} \} \right) \otimes \sigma_N^{(1)}, \sigma_{\pm\pm} \},$$

$$\mathcal{V}(\mathcal{L}_2) = \{ \sigma_0^{(3)} \otimes \mathcal{V}(\mathcal{L}_1)^{(1,2)}, \left(\mathcal{V}(\mathcal{L}_C)^{(3)} \setminus \sigma_0^{(3)} \right) \otimes \mathcal{V}(\mathcal{L}_C)^{(2)} \otimes \sigma_N^{(1)},$$
$$\sigma_{\pm\pm\pm}, \sigma_{\pm 0 \pm}, \sigma_{\pm\pm 0} \},$$

$$\mathcal{V}(\mathcal{L}_3) = \{ \sigma_0^{(3)} \otimes \mathcal{V}(\mathcal{L}_C)^{(2)} \otimes \mathcal{V}(\mathcal{L}_C)^{(1)}, \left(\mathcal{V}(\mathcal{L}_C)^{(3)} \setminus \sigma_0^{(3)} \right) \otimes \sigma_N^{(2)} \otimes \sigma_N^{(1)},$$ (6.36)
$$\sigma_{\pm\pm\pm}, \sigma_{N\pm\pm}, \sigma_{\pm N \pm} \},$$

$$\mathcal{V}(\mathcal{L}_4) = \{ \sigma_0^{(4)} \otimes \mathcal{V}(\mathcal{L}_3)^{(1,2,3)}, \left(\mathcal{V}(\mathcal{L}_C)^{(4)} \setminus \sigma_0^{(4)} \right) \otimes \mathcal{V}(\mathcal{L}_1)^{(2,3)} \otimes \sigma_N^{(1)},$$
$$\sigma_{\pm} \otimes \mathcal{V}(\mathcal{L}_C)^{(2)} \otimes \sigma_{0\pm}, \sigma_{\pm\pm\pm\pm} \},$$

with $\sigma_{\mu\nu} = \sigma_\mu^{(1)} \otimes \sigma_\nu^{(2)}$ and $\mu, \nu \in \{ 0, N, \pm \}$, $\sigma_N^{(j)} \equiv \sigma_{(d-1)N_j}^{(j)}$. It is clear that there is a connection between parts of sets $\mathcal{V}(\mathcal{L}_j)$ and certain terms of projectors P_j, e.g. the first term of projector P_4, which reads $I_0^{(4)} \otimes P_3^{(1,2,3)}$ corresponds

to the set of attractors $\{\,\sigma_0^{(4)} \otimes \mathcal{V}(\mathcal{L}_3)^{(1,2,3)}\,\} \subset \mathcal{V}(\mathcal{L}_4)$, however, some of attractors cannot be deduced from the form of the corresponding projector P_j, e.g. attractors $\sigma_{\pm\cdots\pm}$.

Analogous results hold for sets $\mathcal{V}(\mathcal{L}_j^\dagger)$, generating subspaces $P\mathsf{Atr}(\mathcal{L}_j^\dagger)P$. To obtain attractor spaces $\mathsf{Atr}(\mathcal{L}_j^\dagger)$ requires an additional solution of equation (4.29) for each operator $X \in \mathcal{V}(\mathcal{L}_j)$. Such a derivation goes well beyond the scope of this thesis, nevertheless, an important special case of the interaction topology represented by graph of components $G^{(3)}$ depicted on figure 6.5 will be more closely investigated in the following section.

Interestingly, all here made considerations are valid even for the case $V_{jk} \neq 0$ and hence, the algebraic structure of the attractor spaces is unaffected by presence of the coherent excitation transfer, provided that the coefficients V_{jk} vanish for j and k such that $\gamma_{jk} = 0$. The nonvanishing coefficients V_{jk} do not affect the P part of attractors in the Heisenberg picture, however, due to different structure of $\mathcal{T}$-state σ, they have effect on the detailed form of the attractors in the Schrödinger picture.

6.4   Qubit network with two sinks

Consider a EET network consisting of $N + 2$, $N \in \mathbb{N}$ qubits $(d = 2)$. N of these qubits, labeled $j \in \{\,1,\ldots,N\,\}$ form a strongly connected component G_C of the interaction graph with interaction strengths γ_{jk}. Each of two remaining qubits forms a terminal component S_k, $k \in \{\,1,2\,\}$, weakly connected with the component C_G with interaction strengths ν_{jk}, $j \in \{\,1,\ldots,N\,\}$, $k \in \{\,1,2\,\}$. These qubits are called sinks.

The primary goal of this section is complete specification of attractor spaces $\mathsf{Atr}(\mathcal{L})$ and $\mathsf{Atr}(\mathcal{L}^\dagger)$. Following this derivation, it is interesting to study the asympototic behavior of qubit network with two sinks with a single excitation inserted, to determine what properties determine the probabilities of the transport of this excitation in the sinks S_1 and S_2.

The projector P can be according to formula (6.35) written as

$$P = |\mathbf{0}_N\rangle \langle \mathbf{0}_N| \otimes I_S + (I_N - |\mathbf{0}_N\rangle \langle \mathbf{0}_N|) \otimes |\mathbf{1}_S\rangle \langle \mathbf{1}_S| \qquad (6.37)$$

with I_N and I_S being identity operators on the subspace corresponding to the component G_C and combined components S_1 and S_2 respectively. Consequently, the complementary projector $Q = I - P$ reads

$$Q = (I_N - |\mathbf{0}_N\rangle \langle \mathbf{0}_N|) \otimes (I_S - |\mathbf{1}_S\rangle \langle \mathbf{1}_S|). \qquad (6.38)$$

The attractor space $\mathsf{Atr}(\mathcal{L})$, derived using relations (6.36) can be divided into two parts. First is the DFS, given by operators
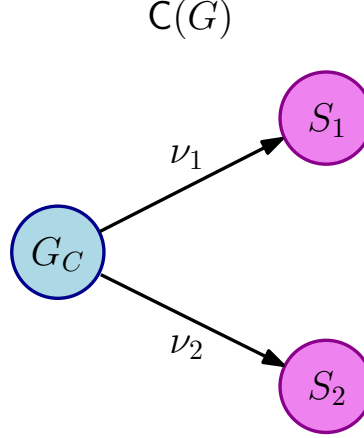
Figure 6.6: The graph of components corresponding to the studied system. The component G_C is the strongly connected network of N qubits. Components S_1 and S_2 are both comprised of a single qubit.

$$\sigma_{\mathbf{zz'}}^{00} = |\mathbf{0}_N\rangle\langle\mathbf{0}_N| \otimes |\mathbf{z}_S\rangle\langle\mathbf{z}'_S|,$$
$$\sigma_{\mathbf{z}}^{01} = |\mathbf{0}_N\rangle\langle\mathbf{1}_N| \otimes |\mathbf{z}_S\rangle\langle\mathbf{1}_S|,$$
$$\sigma_{\mathbf{z}}^{10} = |\mathbf{1}_N\rangle\langle\mathbf{0}_N| \otimes |\mathbf{1}_S\rangle\langle\mathbf{z}_S|,$$
$$\sigma^{11} = |\mathbf{1}_S\rangle\langle\mathbf{1}_N| \otimes |\mathbf{1}_S\rangle\langle\mathbf{1}_S|,$$

with $\mathbf{z}_S \in \{0,1\}^2$. Diagonal attractors $\sigma_{\mathbf{zz}}^{00}$ and σ^{11} correspond to the eigenvalue $\lambda = 0$. For $\mathbf{z} \neq \mathbf{z'}$, the eigenvalues of the attractors $\sigma_{\mathbf{zz'}}^{00}$ are $\lambda \in \{0, \pm i\varepsilon, \pm 2i\varepsilon\}$ and for attractors $\sigma_{\mathbf{z}}^{01}$ and $\sigma_{\mathbf{z}}^{10}$, the eigenvalues shift by additional value of $iN\varepsilon$, i.e. $\lambda \in \{\pm Ni\varepsilon, \pm(N+1)i\varepsilon, \pm(N+2)i\varepsilon\}$. Furthermore, there are additional $N-1$ attractors σ_n, $n \in \{1, \ldots, N-1\}$ given by

$$\sigma_n = \sigma_n^{(C)} \otimes |\mathbf{1}_S\rangle\langle\mathbf{1}_S|, \ \ \lambda = 0,$$

with $\sigma_n^{(C)}$ being the solution of the equations $\Lambda_n(G_C)$. These attractors represent solutions with $n+2$ excitations in the network, $n \in \{1, \ldots, N-1\}$.

Based upon previous sections, one can expect that obtaining the attrator space $\mathsf{Atr}(\mathcal{L}^\dagger)$ will require an additional work. The P part of the DFS is the same as in the Schrödinger picture, i.e.

$$(X_{\mathbf{zz'}}^{00})_P = \sigma_{\mathbf{zz'}}^{00},$$
$$(X_{\mathbf{z}}^{01})_P = \sigma_{\mathbf{z}}^{01},$$
$$(X_{\mathbf{z}}^{10})_P = \sigma_{\mathbf{z}}^{10},$$
$$(X^{11})_P = \sigma^{11},$$

with the eigenvalues given by $\sigma \in \mathsf{Ker}(\mathcal{L} - \lambda\mathcal{I}) \Leftrightarrow X_P \in P\mathsf{Ker}\left(\mathcal{L}^\dagger - \overline{\lambda}\mathcal{I}\right)P$. The P-part of the attractor space in the Heisenberg picture is then completed by operators X_n, which read

$$(X_n)_P = I_n^{(C)} \otimes |\mathbf{1}_S\rangle \langle\mathbf{1}_S| .$$

In order to obtain the attractor space in the Heisenberg picture $\mathsf{Atr}(\mathcal{L}^\dagger)$, equations (4.29) must be solved. This is easily done for the attractors X_n, resulting in

$$X_n = I_{n+2}.$$

Furthermore, it is straightforward to show that

$$\mathcal{L}^\dagger((X_{\mathbf{z}}^{01})_P) = \mathcal{L}^\dagger((X_{\mathbf{z}}^{10})_P) = \mathcal{L}^\dagger((X^{11})_P) = 0,$$

and that

$$\mathcal{L}^\dagger((X_{\mathbf{zz}'}^{00})_P) = 0,$$

except for cases

$$\begin{aligned}
&\mathbf{z} = \mathbf{z}' \in \{\, 01, 10 \,\}, \\
&\mathbf{z} = 11 \wedge \mathbf{z}' \in \{\, 01, 10 \,\}, \\
&\mathbf{z}' = 11 \wedge \mathbf{z} \in \{\, 01, 10 \,\}.
\end{aligned} \tag{6.39}$$

First, consider $X_P \in P\mathsf{Atr}(\mathcal{L}^\dagger)P$ such that $\mathcal{L}^\dagger(X_P) = 0$. Equation (4.29) then yields $X_Q = 0$ and thus $X = X_P \in \mathsf{Atr}(\mathcal{L}^\dagger)$. Consequently, only 2 diagonal and 4 nondiagonal attractors in the Heisenberg picture have nontrivial extensions to the whole Hilbert space H.

Starting with the diagonal attractors $X_{\mathbf{zz}}^{00}$, without loss of generality assume that the excitaiton is in the second sink, i.e. $\mathbf{z} = 01$. The P-part of the attractor reads

$$(X_{\mathbf{zz}}^{00})_P = |\mathbf{0}_N\rangle \langle\mathbf{0}_N| \otimes |0_{S_1}\rangle \langle 0_{S_1}| \otimes |1_{S_2}\rangle \langle 1_{S_2}| . \tag{6.40}$$

Assuming that $(X_{\mathbf{zz}}^{00})_Q$ takes a diagonal form and that it does contain only kets corresponding to a single excitation, one has

$$(X_{\mathbf{zz}}^{00})_Q = \sum_{n=1}^{N} x_j |\mathbf{0}_{N-1}\rangle \langle\mathbf{0}_{N-1}| \otimes |1_j\rangle \langle 1_j| \otimes |\mathbf{0}_S\rangle \langle\mathbf{0}_S| . \tag{6.41}$$

Inserting (6.40) and (6.41) into equation (4.29), one obtains

$$\begin{aligned}
&\Gamma_j x_j - \sum_{j\neq k=1}^{N} \gamma_{jk} x_k = \nu_{j2}, \\
&\Gamma_j = \sum_{k\neq j} \gamma_{jk} + \nu_{j1} + \nu_{j2}.
\end{aligned} \tag{6.42}$$

Equations (6.42) strongly resemble equations (6.14) for the set $\Lambda_1(G)$ with two differences. First, rows of the matrix associated with equations (6.42) sum up to zero, whereas for the matrix associated with equations $\Lambda_1(G)$ columns sum up to zero and second, the right hand side of equations (6.42) is nonzero as it contains interaction strengths between qubits from the component G_C and the sink S_2.

Solution of equations (6.42) is therefore analogous to solution of equations (6.14) and can be summed up by a following formula:

$$
x_j = \frac{\displaystyle\sum_{\substack{T_{S_1 S_2}(G), \\ \mathsf{P}(j, S_2) \subset T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G))}{\displaystyle\sum_{T_{S_1 S_2}(G)} \Omega(T_{S_1 S_2}(G))}.
\tag{6.43}
$$

Coefficients x_j of the Q-part (6.41) of the attractor $X_{\mathbf{zz}}^{00}$ for $\mathbf{z} = 01$ is given by values of all directed maximal trees of the interaction graph $G(V, E)$ corresponding to vertices $\{S_1, S_2\}$ containing a path $\mathsf{P}(j, S_2)$ from the vertex j to the sink vertex S_2, normalized by weight of all directed maximal trees of the interaction graph $G(V, E)$ corresponding to vertices $\{S_1, S_2\}$.

For nondiagonal attractors, consider e.g. the attractor $Y_{\mathbf{zz}'}^{00}$ with $\mathbf{z} = 01$ and $\mathbf{z} = 11$. The P-part of the attractor reads

$$
(X_{\mathbf{zz}'}^{00})_P = |\mathbf{0}_N\rangle \langle \mathbf{0}_N| \otimes |0_{S_1}\rangle \langle 1_{S_1}| \otimes |1_{S_2}\rangle \langle 1_{S_2}|.
\tag{6.44}
$$

Analogously to the previous case of diagonal attractors, the Q-part of this attractor can be assumed in the form

$$
(X_{\mathbf{zz}'}^{00})_Q = \sum_{n=1}^{N} x_j |\mathbf{0}_{N-1}\rangle \langle \mathbf{0}_{N-1}| \otimes |1_j\rangle \langle 1_j| \otimes |\mathbf{0}_{S_1}\rangle \langle \mathbf{1}_{S_1}| \otimes |\mathbf{0}_{S_2}\rangle \langle \mathbf{0}_{S_2}|.
\tag{6.45}
$$

Equations (6.14) then read

$$
\left(\Gamma_j - \frac{\nu_{j1}}{2}\right) x_j - \sum_{k=1}^{N} \gamma_{jk} x_k = \nu_{j2}.
\tag{6.46}
$$

Interestingly, equations (6.46) have the same form as equations (6.42) with a small modification that the coefficient Γ_j is weakened by half of the interaction strength between qubit j and the sink S_1. The coefficients y_j of attractors $X_{\mathbf{zz}'}^{00}$, with $\mathbf{z} \neq \mathbf{z}'$ therefore take the same form as the coefficients corresponding to attractors $X_{\mathbf{zz}}^{00}$, with the weight of an edge, representing interaction between the strongly connected part G_C and one of the sinks S_k reduced by the factor of $\frac{1}{2}$. Although contrary to the strongly connected EET network, attractors in the Schrödinger picture corresponding to the sinks do not depend on particular values of interaction strengths γ_{jk}, resp. ν_{jk}, associated attractors in the Heisenberg picture do depend on these interaction strengths, as a result of equation (4.29).

To see the correspondence between coefficients associated with diagonal attractors and nondiagonal attractors, coefficients x_j corresponding to diagonal attractors $X_{\mathbf{zz}}^{00}$, given by (6.43) may be rewritten as

$$
x_j = \frac{1}{\mathsf{N}} \left(\sum_{\substack{T_{S_1 S_2}(G), \\ \mathsf{P}(j,S_2) \subset T_{S_1 S_2}(G), \\ (k,S_1) \notin T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G)) + \sum_{\substack{T_{S_1 S_2}(G), \\ \mathsf{P}(j,S_2) \subset T_{S_1 S_2}(G), \\ (k,S_1) \in T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G)) \right),
$$

$$
\mathsf{N} = \left(\sum_{\substack{T_{S_1 S_2}(G), \\ (k,S_1) \notin T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G)) + \sum_{\substack{T_{S_1 S_2}(G), \\ (k,S_1) \in T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G)) \right).
$$
(6.47)

The coefficients x_j of nondiagonal attractors $X_{\mathbf{zz}'}^{00}$ may be then written as

$$
x_j = \frac{1}{\mathsf{N}} \left(\sum_{\substack{T_{S_1 S_2}(G), \\ \mathsf{P}(j,S_2) \subset T_{S_1 S_2}(G), \\ (k,S_1) \notin T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G)) + \frac{1}{2} \sum_{\substack{T_{S_1 S_2}(G), \\ \mathsf{P}(j,S_2) \subset T_{S_1 S_2}(G), \\ (k,S_1) \in T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G)) \right),
$$

$$
\mathsf{N} = \left(\sum_{\substack{T_{S_1 S_2}(G), \\ (k,S_1) \notin T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G)) + \frac{1}{2} \sum_{\substack{T_{S_1 S_2}(G), \\ (k,S_1) \in T_{S_1 S_2}(G)}} \Omega(T_{S_1 S_2}(G)) \right).
$$
(6.48)

Nondiagonal attractors with nontrivial Q-part do matter in certain special scenarios, e.g. if initially, one sink is filled with excitation and the strongly connected component is in a superposition of states with zero excitations and a single excitation. Consider e.g. $N = 2$, $d = 2$ and an initial state $\rho(0)$ which reads

$$
\rho(0) = |\psi\rangle \langle\psi| , \ |\psi\rangle = (a |00\rangle + |10\rangle) \otimes |01\rangle , \ |a|^2 + |b|^2 = 1. \tag{6.49}
$$

Such an initial state describes a single excitation in the sink S_2 and superposition between zero exciations and a single excitation in the first qubit of the strongly connected component. Due to superposition of kets $|00\rangle$ and $|10\rangle$, the initial state contains correlations, which are not completely destroyed in the asymptotic evolution, due to attractors $X_{\mathbf{zz}'}^{00}$ with $\mathbf{z}$ and $\mathbf{z}'$ being the second or the third case of (6.39).
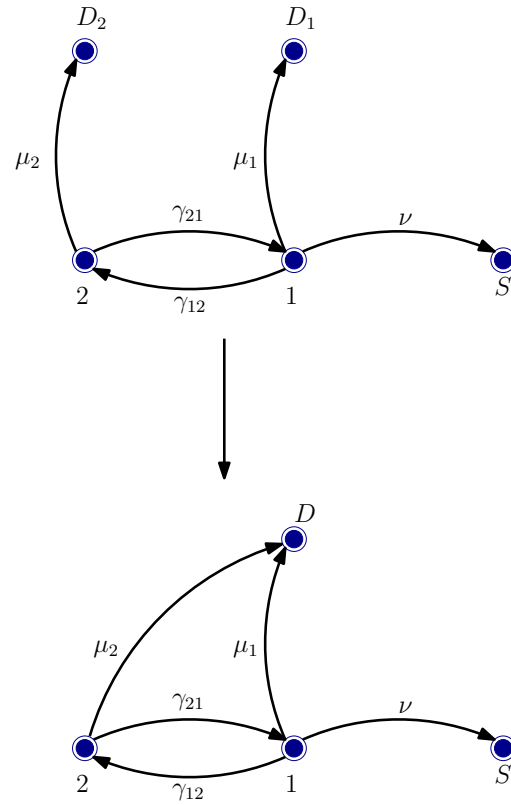
Figure 6.7: A network of two qubits and three qubit sinks (upper picture). Each network qubit has an associated sink D_j simulating dissipation, with the dissipation rate μ_j. The goal is to obtain the excitation transfer rate to the sink S. For a single excitation inserted into the network, the same system can be represented by a analogous network with a single dissipation sink D (lower picture).

Obtained results allow calculation of some of the results presented in [104], namely the efficiency P of the transfer of the excitation to the sink, for the case $V_{jk} = 0$, i.e. without the coherent excitation transfer. The analogy of the system studied in [104] is depicted on the figure 6.7. Inserting the excitation in the first qubit, the efficiency P is determined by the coefficient x_1 in (6.43). The numerator of this coefficient is given by sum of weights of all maximal trees $T_{DS}(G)$ containing path from the vertex 1 to the vertex S. There is only one such tree and its weight reads $\Omega(T_{DS}(G)) = \gamma_{21}\nu$. The denominator is then given by sum of weights of all maximal trees $T_{DS}(G)$. The resulting efficiency reads

$$P = \frac{\gamma_{21}\nu}{\gamma_{21}\nu + \gamma_{21}\mu_1 + \gamma_{12}\mu_2 + \mu_1\mu_2 + \mu_2\nu}, \tag{6.50}$$

confirming the result of [104]. Furthermore, this result can be straightforwardly extended for a longer chain of qubits using the obtained coefficients (6.43):

$$
\begin{aligned}
P =& \frac{\nu \prod_{j=1}^{N-1} \gamma_j^{j+1}}{Q_N}, \\
Q_N =& (\gamma_{N-1}^N + \mu_N)Q_{N-1} + \gamma_{N-1}^N \mu_N (Q_{N-2} + \gamma_{N-2}^{N-1}(Q_{N-3} + \\
& \gamma_{N-3}^{N-2}(Q_{N-3} + \ldots \gamma_2^3(Q_1 + \gamma_1^2) + \gamma_1^2\mu_2)\ldots))), \\
Q_1 =& \mu_1 + \nu,
\end{aligned}
$$

where γ_{jk} is denoted as γ_k^j for clarity. This formula shows an application of results obtained in this section.

6.5 Summary

To illustrate the attractor method of description of the asymptotic evolution on a physically relevant example, the so-called incoherent EET network was studied in depth. The studied system is related to the EET network, which is used mainly to study excitation transfer during the process of photosynthesis. Incoherent EET network faciliates a number of asymptotic regimes, allowing to study cases of non-unital QMP, non-faithful QMP etc.

The main factor affecting the resulting attractor spaces is the interaction topology of the network. This can be encoded in a weighted directed graph, called the interaction graph. The interaction graph allows to define representations, each describing the given network on different level – the graph of components and the index graph.

The complex derivation of a $\mathcal{T}$-state σ resulted in a set of balance equations (6.14), which determine the dependence of the $\mathcal{T}$-state (and consequently attractors in the Schrödinger picture) on the details of the dynamics, i.e. values of interaction strengths.

After deriving the attractor spaces for the case of the strongly connected interaction graph, the dependence of algebraic structure of the general attractor

spaces on the interaction topology was studied. The resulting attractor structures can be fully described by the topological properties of the index graph. As the index graph becomes overly complicated for large networks, however, it is possible to obtain relevant insight into connection between attractor spaces with the interaction graph, whose structure is significantly simpler than that of the index graph. It turns out that using the concept of the graph of components, attractor spaces of an interaction graph which is not strongly connected can be obtained by appropriate tensor product of elements from attractor spaces corresponding to strongly connected components. This is shown on a few examples of interesting interaction topologies.

Lastly, an in depth study of a strongly connected network of qubits attached to two qubit sinks is made. This requires calculation of full form of attractors in the Heisenberg picture by means of equation (4.29). The same process which was used in the derivation of $\mathcal{T}$-state results in obtaining a set of algebraic equations (6.42). This set of equations is similar to the set of equations (6.14) for a single excitation and thus it can be interpreted in a similar way through special subgraphs of the 1-index graph, i.e. of the interaction graph. This investigation allowed to determine the probabilities of absorbing a single excitation injected into the network at particular sinks, hidden within (6.43). These results are in agreement with [104] and can be directly generalized for more complex networks.

Chapter 7

Summary and outlook

The class of finite-dimensional homogenous quantum Markov processes (QMPs) is essential for the description of many naturally occurring quantum processes. QMPs represent a nontrivial step from reversible processes, being able to describe a large number of open system phenomena such as dissipation, dephasing, decoherence etc. Their usefulness stems from the fact that all mathematical properties of QMPs are determined by a single time independent superoperator $\mathcal{G}$, called the generator of QMP.

Despite the simplicity of QMPs compared to a general irreversible quantum process, the generator $\mathcal{G}$ is usually not diagonalizable and consequently, analytic solution of the dynamics of QMP is often not available. This however is not true for the asymptotic part of the evolution, i.e. the limit $\tau \to \infty$, with τ representing discrete or continuous time. The asymptotic part of the generator $\mathcal{G}$ of QMP is always diagonalizable, which makes the analytical solution of the dynamics possible.

Many QMPs have a complex structure of the asymptotic space, containing more than one invariant state with the additional possibility of a nontrivial evolution within the asymptotic space. Mathematical obstacles accompanying the investigation of the asymptotic evolution of QMPs can be resolved by introducing a procedure, which can uncover the analytical form of the asymptotic space, allowing detailed investigation of the properties of a given QMP.

The attractor method, described in chapter 4 of this thesis offers such a procedure. Generally, this procedure shares a lot of common features with the standard process of diagonalization of the Hamiltonian used for the description of the time evolution of closed quantum systems. There are however several key differences which make the solution of the asymtptotics of QMPs a much more complex task.

Because the generator $\mathcal{G}$ is generally not normal, there is an ambiguous relation between attractor spaces in the Schrödinger and Heisenberg picture. In order to uncover the dependence of the asymptotic state on the initial state, both these attractor spaces need to be determined. For a large class of QMPs, called faithful QMPs, attractors in both pictures can be obtained by solving

the so-called attractor equations. Since the form of the generator $\mathcal{G}$ is distinct for both studied classes of QMPs, the form of attractor equations for QMCHs (4.10) and (4.11) differs from those corresponding to the QMDSs (4.12) and (4.13). Furthermore, if the QMP under investigation is not faithful, attractor equations determine the form of attractors only on a subspace $P\mathcal{H}$ determined by a certain projector P.

These complications are closely connected with the exact form of the so-called $\mathcal{T}$-state. The $\mathcal{T}$-state is an arbitrary subinvariant state σ of the studied QMP $\mathcal{T}$ which has maximal possible rank. The $\mathcal{T}$-state σ can be used to uncover a large family of transformations between attractor spaces in the Schrödinger and the Heisenberg picture (4.22), it directly enters the attractor equations in the Schrödinger picture (4.10) and (4.12), and it determines the projector P, the $\mathcal{T}$-projector, on which the attractor equations hold. The knowledge of an arbitrary $\mathcal{T}$-state σ is therefore essential for the determination of the asymptotic evolution of QMPs.

On its own, the attractor method allows to determine the dependence of the asymptotic state on the initial state. However, individual attractors do not generally represent physically relevant objects, i.e. neither observables nor quantum states, as they are not generally Hermitian. A valid asymptotic quantum state is constructed only by making a linear combinations (4.7) resp. (4.8). By making a specific linear combinations of attractors in the Heisenberg picture (5.5) resp. (5.6), one may identify an important set of operators playing the role of constants of motion – observables, whose expectation values do not change during the evolution.

By exploiting algebraic properties of the attractor space in the Heisenberg picture and the algebraic relations (4.22), it is possible to cast all asymptotic states in a new form, called Gibbs-like states. There are two distinct forms of Gibbs-like states, the form (5.9), valid for any TP QMP, and the form (5.13) valid for any QMP. Unlike the general expressions (4.7) and (4.8), the parameters β_j, which determine the particular Gibbs-like state can take any real value independently on the QMP under investigation, which considerably simplifies the description of the set of all asymptotic states corresponding to the studied QMP. Furthermore, Gibbs-like states allow one to study asymptotic trajectories of QMPs. Expressed in terms of constants of motion, any asymptotic trajectory can be written as (5.21).

The revealed properties of asymptotic dynamics of TP QMPs can be employed to formulate and prove the generalized Jaynes principle [98], as is done in chapter 5. There are several distinct cases of the Jaynes principle for QMPs, depending on the amount of initial information about the system available. While the first Jaynes principle 5.3.1 requires full knowledge of the initial state, the second Jaynes principle 5.3.2 assumes knowledge of expectation values of some constants of motion. Knowledge of mean values of constants of motion is not the only kind of information which can be incorporated into the resulting asymptotic trajectory. This is shown on a special case of the second Jaynes principle 5.3.3, demonstrating an incorporation of an additional information about the asymptotic dynamics, namely its stationarity. Crucially, all the presented ver-

sions of the Jaynes principle are derived as a result of the asymptotic dynamics of the given QMP and therefore, the Jaynes principle should not be viewed as a principle emerging from the information theory. The information theory therefore only dictates which version of Jaynes principle should be used. In this sense, the procedure used here, which resulted in the derivation of the Jaynes principle is similar to the one of Boltzmann [99, 100].

Lastly, a detailed investigation of asymptotics of a QMP called the incoherent EET network [107] is made in chapter 6. This QMP was chosen as its $\mathcal{T}$-state is apart from special cases different from the identity operator I. Furthermore, its $\mathcal{T}$-projector P fulfils for a large spectrum of interaction topologies $P < I$. These properties allows to discuss the construction of the attractor spaces in both Schrödinger and Heisenberg picture, when both these spaces do not coincide. Furthermore, for a specially chosen interaction topology of studied QMPs, i.e. a strongly connected qubit network interacting with a pair of qubit sinks, the construction of the full form of attractors in the Heisenberg picture is done. This full form then allows the analytical calculation of important properties of such network, namely the probabilities of absorption of a single excitation by the first and the second sink, revealing the dependence of these probabilities on the details of the dynamics and on the initial state of the network.

The class of quantum networks, such as the EET network are an example of a broad collection of quantum systems whose asymptotics can be successfully treated by the attractor method, particularly quantum networks equipped with binary interactions (or other interactions of low order), as in such case, the attractor equations can be treated locally -(as in the case of the EET network) and their solution is significantly simplified. Usually, such quantum networks can be represented by a certain graph structure. The topology of the corresponding graph structure has in many cases a crucial impact on the form of the resulting attractor spaces. However, in some complex scenarios with different kinds of interaction, some of the underlying graph structures can however become irrelevant, as shown in [96] on case of simultaneous two-qubit and three-qubit interaction – if the interaction graph corresponding to the two-qubit interaction is strongly connected, then the attractor spaces are unaffected by the topology of the interaction graphs corresponding to three-qubit interactions and similar situation can occur vice versa. The problem of relevance of the graph structures associated with quantum networks therefore requires more attention.

QMPs are often employed in a study of thermodynamic properties of quantum systems, e.g. exchange of heat, work, various kinds of currents, etc. due to their simplicity (compared with a general open quantum systems). The attractor method offers a coherent way to formulate the asymptotic quantum thermodynamic properties of any QMP. If the given QMP is TP, the identity operator I is a constant of motion and can be used to define the partition sum Z. Any asymptotic state (and more generally any asymptotic trajectory) can be then written in a compact way as a Gibbs-like state, with well defined partition sum Z. The partition sum corresponding to a TP QMPs seemingly possesses the properties analogous to the partition sum associated with the classical Gibbs ensembles (2.20). The extent of this similarity as well as the investigation of

trace-nonincreasing QMPs is a subject of future research.

Lastly, the asymptotics of QMPs is largly influenced by the form of its corresponding $\mathcal{T}$-states. At the moment, there is no easy way to obtain the analytical form of an arbitrary $\mathcal{T}$-state. The QMP therefore needs to be constructed in such a way that at least one $\mathcal{T}$-state is known, else the brute force must be used to obtain an arbitrary $\mathcal{T}$-state. Furthermore, some properties of $\mathcal{T}$-states remain ambiguous, e.g. what properties of QMP can ensure that the asymptotic evolution of the identity operator $\sigma_I(\tau)$ is not a fixed $\mathcal{T}$-state, but rather a $\mathcal{T}$-trajectory with nontrivial time evolution. This problem is relevant for the Jaynes principle 5.3.2 and such QMP could draw a significant attention, as it could have a nontrivial asymptotic evolution even if the only information about the initial state would be given by expectation values of integrals of motion corresponding to the given QMP.

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