# Review on “Dynamics of percolated quantum walks on graphs” by Jan Mareš

Reviewer: Etsuo Segawa (Yokohama National University, Japan)

This dissertation considers a discrete-time quantum walk on graphs with dynamical percolation. Let $G = (V, E)$ be the original graph. The total state space of the discrete-time quantum walk is expressed by $\ell^2(E^{(d)})$, where $E^{(d)}$ is the set of the “half edges” induced by $E$, $E^{(d)}$; each half edge is cut in the middle of the original edge, and has the origin vertex losing the terminal vertex. Time evolution operator of this discrete-time quantum walk on $\ell^2(E^{(d)})$ is randomly given at each time step by $U_n = C'S_n$. Here $C' = \oplus_{u\in V} C'_u$ under the decomposition of $\ell^2(E^{(d)}) = \sum_{u\in V} \oplus\mathrm{span}\ \{|i\rangle$ : the origin vertex is u}, where $C'_u$ is a unitary operator on span$\{|i\rangle$ : the origin vertex is u}, and $R_K = \oplus_{e\in E}\tau_e$ under the decomposition of $\ell^2(E^{(d)}) = \sum_{e\in E} \oplus\mathrm{span}\{|i\rangle$ : i is a half edge of e}, where

$$\tau_e = \begin{cases} \sigma_1 & : \text{with prob. } p \\ \sigma_0 & : \text{with prob. } 1-p. \end{cases}$$

There are several choices of the shift operator, but the author shows every time evolution operator of discrete-time quantum walk can be described by the reflecting shift operator acting appropriate permutations on the local unitary coins from the right. This dissertation mainly treats the three-dimensional Grover matrix as the original local coin and graphs so that identity or clockwise or counter-clockwise permutations can be applied as the local permutations. Then the original graphs considered in this dissertation are mainly planar 3-regular graphs.

To see the average of the distributions with respect to the configurations at each time step, the author considers the following time iteration of $\rho_t \in \mathcal{B}(\ell^2(E^{(d)}))$ $(t = 0, 1, 2\dots)$

$$\rho_{t+1} = \mathcal{S}(\rho_t) := \sum_{K\in 2^E} \pi_K U_K \rho_t U_K^\dagger,$$

where $\pi_K$ is the realization probability of the configuration $K$. According to the spectral analysis on random unitary operations by [Novotny et al., Central Eur. J. Phys., 8 1001-1014 (2010)], the $t$-th iteration $\rho_t$ assymptotically desctibed by

$$\rho_t \sim \sum_{i,\lambda\in \mathrm{Spec}(\mathcal{S})\ \mathrm{with}|\lambda|=1} \lambda^t \mathrm{Tr}(\rho_0 X_{\lambda,i}^\dagger) X_{\lambda,i}$$

where $\{X_{\lambda,i}\}_i$ is CONS of $\ker(\lambda - \mathcal{S})$, and every attractor $X \in \mathcal{B}(\ell^2(E^{(d)}))$ satisfies

$$U_K X = \lambda X U_k \text{ for any } K \in 2^E.$$

The author focuses on the attractors $\{X_{\lambda,i}\}$ for the discrete-time quantum walk case, especially the special attractors named p-attractors which derive from $|\psi\rangle \in \ell^2(E^{(d)})$ such that

$$U_K|\psi\rangle = \lambda|\psi\rangle \text{ for any } K \in 2^E$$

and gives graph theoretical characterizations concretely. It is shown that the condition for the p-attractor is equivalent to satisfying $C|\psi\rangle = \lambda|\psi\rangle$ (the coin condition); and $R_K|\psi\rangle = R_L|\psi\rangle$ for any $K, L \in 2^E$ (the shift condition) simultaneously. This is a key towards such a characterization.

Then the following are main results of the author that the p-attractors for reflecting Grover walk and cyclic Grover walk cases are characterized by “common eigenvectors” using graph notions.

1. Reflecting shift case (the original Grover coin case): The p-attractor space depends on that the graph has whether following structures; (i) $\nexists$ odd-edges faces and $\nexists$ paired self loops; (ii) $\exists$ odd-edges faces and $\nexists$ paired self loops; (iii) $\nexists$ odd-edges faces and $\exists$ paired self loops; (iv) $\exists$ odd-edges faces and $\exists$ paired self loops. The vectors of p-attractor are represented by (A): a weighted even-cycle; (B): weighted subgraph constructed by a pair of odd cycles and a path connecting them; (C): weighted path between a pair of selfloops; (D): wighted subgraph constructed by a pair of self loop and odd cycle and a path connecting them; so that a weight is assigned $\{\pm 2, \pm 1\}$ to each edge satisfying with the Kirchhoff condition at every vertex. The case for (i) only needs (A)-type vectors. In the cases for (ii), (iii), (iv), $\{$(A)-type, (B)-type$\}$, $\{$(A)-type, (C)-type$\}$ and $\{$(A)-type, (B)-type, (D)-type$\}$ vectors construct the common eigenspace, respectively.

2. Cyclic shift case (Grover coin with cyclic permutation case): The local eigenvector of the Grover coin with the cyclic permutation associated with two eigenvalues $\{-\omega : \omega^3 = 1, \omega \neq 1\}$ is of the form $[1, \omega, \omega^2]^\top$. ($\omega = 1$ case is trivially extended to a whole space). If the common eigenvector exists for $-\omega$, then the shift condition automatically induces an edge coloring of the original graph with the colors $1 \leftrightarrow$ "red", $\omega \leftrightarrow$ "green" and $\omega^2 \leftrightarrow$ "blue". Conversely, if the original graph is three edge colored, then the coloring way gives a way of the construction of the cyclic Grover walk the cyclic exhibiting the common eigenvector of $-\omega$.

As an application of the results, the author considers the Grover walk with sinks on several graphs and compute the asymptotic transport probability (ATP) to the sink $v_* \in V$ The quantum walks are considered both non-percolated and percolated versions. The graphs are ladder graph; carbon nanotube; cayley trees. The ATP is obtained by the overlap between the initial state and trapped eigenspace. In the trapped eigenspace, there exist additional eigenvectors which do not described by the common eigenvector discussed in the above. The author clarifies this additional eigenspace case by case and also show numerical results.

The results of this dissertation are novel from the view point of percolated dynamical system and correct and also nice quality. In particular, the introduction of edge coloring notion toward a spectral analysis on the cyclic Grover walk gives an expansion to a new research area, for example, combinatorial and topological graph theories for the future's works. It is expected to find a graph structure induced by a quantum effect as a starting point of this manuscript.

The following are few comments and questions for the author.

1. page 40 line 2: "$\rho_{t\to\infty}(t)$" in RHS is not easy to understand.

2. page 40: It would be nice for a readability to add a remark that the coin and shift conditions are not only necessary condition but also sufficient condition for the attractor condition.

3. page 52 line 6: I expect that the cycle rotations (clockwise or counter clockwise) have a conflict in some face if the graph is embeded on a surface of the Mobius strip without any crossing of edges. Are the cycle rotations determined if the graph can be embeded on an orientable closed surface (e.g., sphare, torus) without any crossing edges ?

4. page 54 line 15: the use of "line graph" may lead to a misunderstanding because this is used in another notion in graph theory.

5. page 63 line 1: How do we have to consider the situation that two faces have common edges or vertices? Can such situation be exclusive? Maybe the reference [Higuchi et al., Linear Algebra and Its Applications, 583 (2019) pp. 257-281 especially the proof of Proposition 2 for case 2] is helpful.

6. page 64 Fig 3.7: it might be better to use triangle rather than hexagon in (b') because maybe the message of this figure is that even for the odd faces the additional eigenstate can be constructed.

7. Grover walk on Cayley tree: a similar argument can be seen for example in [Higuchi and Segawa, Journal of Physics A: Mathematics and Theoretical 51 (2018) 075303]

In conclusion, I recommend the thesis for defense.

In Yokohama, 9. 2. 2021

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