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Report on the PhD manuscript of M. David Fridrich from the Czech Technical University in Prague, Czech Republic.

To whom it may concern,

The manuscript of M. David Fridrich is constituted of 137 pages — 116 pages plus 11 pages of appendices and 125 bibliographic references. It is organized into seven relatively independent chapters as M. David Fridrich wisely adds a small introduction, a numerical validation and a conclusion section to each of them. This is greatly appreciated and allows for a very pleasant reading.

After a first introductory chapter, M. David Fridrich makes some preliminary presentation of classical Lagrangian and Eulerian frameworks for fluid dynamics (chapter 2). Then the third section deals with Lagrangian schemes in 1D where the notions of Lax-Wendroff (LW) + Harten-Lax-van Leer (HLL) based schemes are introduced along with their associated numerical schemes and results. The candidate presents and illustrates the main idea of the PhD of re-using LW schemes but in an hybrid context.

Next M. David Fridrich extends this family of schemes to the 2D Cartesian geometry and conduct a large numerical validation campaign. The 2D cylindrical extension of the approach is proposed in section 5, again with a complete numerical validation test-suite. The next step is the application of these numerical method to the important situation of elasto-plasticity models. After briefly presenting the so-called Wilkins model of hypo-elasticity in section 6, M. David Fridrich presents his Lagrangian numerical scheme in 1D and 2D Cartesian geometry derived from the ones previously designed. Each dimension being numerically validated by the classical benchmark test cases for this field of research.

This way of structuring the manuscript naturally leads to a concise final conclusive section followed by large appendices and complete bibliography. Again the large and exhaustive appendices are genuinely appreciated because they deal with important technical issues –namely boundary condition treatments and symmetry proofs– which are of utmost importance but would have blurred the message if within the core of the manuscript. Finally the bibliography is perfectly representative of the old and venerable seminal papers but also the most up-to-date works, and, more important, wisely neglecting the thousands and thousands of useless recent and ancient works on this vast subject.

This manuscript is a very nice and interesting piece of research work where new and important results are compiled. There are revolving around the development of new numerical

methods, based on Lax-Wendroff-HLL approach, their analysis, implementation and validation in the context of CFD.
I would like to mention right away that the manuscript is **extremely well written**, packed with useful and analysis for this new family of numerical methods which are further validated and tested.
The subject of the PhD is the improvement of LW numerical methods in a Lagrangian framework dedicated to become the engine of Lagrangian or ALE simulation codes of compressible fluid and elasto-plastic flows modeled by evolutionary partial differential equations (PDEs).
Classical numerical methods have been used since the origin of the CFD in the 50s of last century but they, maybe, have reached their limits. Therefore new ideas, or, old, but revisited ones, like LW scheme –which dates back from the 60s!– are of tremendous importance and the work of M. David Fridrich is very much in this line. Therefore I can ensure that **the subject is up to date** and of much interest for academics and researchers from national laboratories.
The **scientific value** of the work of M. David Fridrich is unquestionable as it has already been recognized by the community by the acceptation of five journal papers in top-ranked international journals like *Journal of Computation Physics*, *Computers & Fluids*, *Springer Proceedings in Mathematics & Statistics* and *Acta Polytechnica*. Needless to write that this is an impressive performance of M. David Fridrich to having produced enough new materials and results to open the doors so such demanding journals.

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Let me give more details of the novelties in the manuscript. First of all revisiting LW schemes by supplementing it with HLL type of dissipation is an appealing choice as both approaches have proven practical interest in the past. In a second-order accurate approach the choice of the predictor is crucial and several solutions are proposed by M. David Fridrich such as Lax-Friedrichs or less known Wendroff-White solutions.
Another key point lays in the artificial dissipation procedure and its edge-based limiting strategy. The dissipation can be added specifically to the variable of interest; mass, velocity or energy, leading to difference schemes sharing or not some behaviors. The amount of dissipation can also be piloted by ad-hoc coefficients. The dissipation and limiting is of paramount importance as it ensures that the numerical method produces essentially non oscillatory results (robustness). The a posteriori checking of 'no issue in the mesh' and the possible recomputation of the whole time step with a reduce timestep is problems are encountered is a nice idea. I would even pursue in this direction by best fitting the dissipation coefficients as well for the recomputation.
M. David Fridrich did invest some effort in performing the numerical analysis of his schemes; GCL preservation proofs, symmetry preservation, conservation...
Numerical results presented for Noh problem in particular are really good ones, the cylindrical symmetry is almost perfectly reproduced, as well for Sedov test case. This is truly impressive. Being 'only' Lagrangian, the scheme only performs as one could expect for the triple point. However when it will be supplemented with rezone and remap procedures, it will be appealing! As already mentioned this amount of work is done in 1D, then in 2D, then in 2D axisymetry! Then the hyper-elasto-plastic system of PDEs is considered to validate the numerical approach

on a more realistic physical model. 1D and 2D schemes are extended with the associated set of numerical tests. Again the results are clean and very promising for a future extension into an ALE code for high impact velocity problems.
I did not mention yet this point, but the choice of mathematical notation is clean and clear and consistent across the PhD. This drastically eases the reading and avoid confusion and headache. It makes the 2D mathematical descriptions and proofs easy to follow and agree with.

Novelties are scattered into the PhD and the impact of the work of M. David Fridrich has already been observed by practitioners in the field. As already mentioned several publications extracted from this PhD are already accepted in top international journals. These papers are compacted versions of the main chapters of the manuscript, hence this work has somewhat already been reviewed and validated by top-ranked journal reviewers before me. Quantitatively about 10 reviewers have given their expert agreement on M. David Fridrich 's work. More importantly, M. David Fridrich 's work has opened some paths to be explored even further by the researchers from the applied mathematical community in the upcoming years.

This PhD has been written with **great care and in a pedagogical way**. The difficult mathematical parts are well handled to facilitate the "work" of the reader. Though they are not hidden and the necessary details are present. This is truly appreciated and demonstrates if needed that M. David Fridrich has a **deep and broad understanding of his subject**. According to me this proves (if needed) that M. David Fridrich has attained and even **exceeded the maturity which one may expect from a junior researcher** at the end of his PhD. I wish I could read and review more PhD manuscripts like this one!

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In regards of the previous comments I consider that the research work in this manuscript not only attains but exceeds the standard required for a PhD work. The contribution in the field of scientific computation, applied mathematics and numerical analysis is clear and illustrated by the quality of his publications in top-journals and by the overall manuscript in hands. Let us also note that this work has already been presented in numerous conferences or workshops.

Therefore I strongly recommend without any doubt that the PhD be presented and defended in due time in its current form for the degree of doctor in Mathematics from the University of the Czech Technical University in Prague.

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