# Counterexamples to a conjecture of Harris on Hall ratio 

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#### Abstract

The Hall ratio of a graph $G$ is the maximum value of $v(H) / \alpha(H)$ taken over all non-null subgraphs $H \subseteq G$. For any graph, the Hall ratio is a lower-bound on its fractional chromatic number. In this note, we present various constructions of graphs whose fractional chromatic number grows much faster than their Hall ratio. This refutes a conjecture of Harris.


## 1 Introduction

A graph $G$ is $k$-colorable if its vertices can be colored with $k$ colors so that adjacent vertices receive different colors. The minimum integer $k$ such that $G$ is $k$-colorable is called the chromatic number of $G$, and it is denoted by $\chi(G)$.

Various refinements and relaxations of the chromatic number have been considered in the literature. One of the classical and most studied ones is the

[^0]fractional chromatic number, which we denote by $\chi_{f}(G)$; see Section 2.1 for its definition.

A basic averaging argument reveals that $\chi_{f}(G) \geq v(G) / \alpha(G)$, where $v(G)$ and $\alpha(G)$ are the number of vertices and the size of a largest independent set in $G$, respectively. Moreover, since $\chi_{f}(G) \geq \chi_{f}(H)$ for a subgraph $H \subseteq G$, it holds that

$$
\chi_{f}(G) \geq \frac{v(H)}{\alpha(H)} \quad \text { for every non-null } H \subseteq G
$$

We define $\rho(G)$ - the Hall ratio of a graph $G$ - to be the best lower-bound obtained in this way, i.e.,

$$
\rho(G):=\max _{\emptyset \neq H \subseteq G} \frac{v(H)}{\alpha(H)}
$$

How tight is $\rho(G)$ as a lower bound for $\chi_{f}(G)$ ? In 2009, Johnson [11] suggested that there are graphs $G$ where the value of $\chi_{f}(G) / \rho(G)$ is unbounded. In earlier versions of [7] (see [8, Conjecture 6.2]), Harris explicitly conjectured the opposite.

Conjecture 1. There exists $C$ such that $\chi_{f}(G) \leq C \cdot \rho(G)$ for every graph $G$.
In 2016, Barnett [2] constructed graphs showing that if such a constant $C$ exists, then $C \geq 343 / 282 \sim 1.216$ improving an earlier bound 1.2 [3]. Our first result refutes Conjecture 1 .

Theorem 2. There exists $P_{0}$ such that for every $P \geq P_{0}$, there is a graph $G$ with $\rho(G) \leq P$ and $\chi_{f}(G)>P^{2} / 33$.

The proof of Theorem 22 is very short and simple, modulo some standard results about random graphs. The following two theorems strengthen Theorem 2 at the expense of somewhat more technical proofs.

Theorem 3. There exists $P_{0}$ such that for every $P \geq P_{0}$ there is a $K_{5}$-free graph $G$ with $\rho(G) \leq P$ and $\chi_{f}(G)>P^{2} / 82$.

Theorem 4. There exists $P_{0}$ such that for all $P \geq P_{0}$ there is a graph $G$ with $\rho(G) \leq P$ and $\chi_{f}(G) \geq e^{\ln ^{2}(P) / 5}$.

This note is organized as follows. In Section 2, we recall definitions and properties of the fractional chromatic number, and Erdős-Rényi random graphs. Proofs of our results are in Section 3. We conclude the note by Section 4 with related open problems.

## 2 Definitions and preliminaries

The join of two graphs $G_{1}$ and $G_{2}$, which we denote by $G_{1} \wedge G_{2}$, is obtained by taking vertex-disjoint copies of $G_{1}$ and $G_{2}$, and adding all the edges between
$V\left(G_{1}\right)$ and $V\left(G_{2}\right)$. More generally, for graphs $G_{1}, G_{2}, \ldots G_{\ell}$, we write $\bigwedge_{i=1}^{\ell} G_{i}$ to denote $\left(\bigwedge_{i=1}^{\ell-1} G_{i}\right) \wedge G_{\ell}$.

For a graph $H$ on the vertex-set $\{1, \ldots, \ell\}$ and a collection of $\ell$ vertex-disjoint graphs $G_{1}, \ldots, G_{\ell}$, we define $H\left\{G_{1}, \ldots, G_{\ell}\right\}$ to be the graph obtained by taking a union $G_{1}, \ldots, G_{\ell}$, and, for every edge $i j \in E(H)$, adding all the edges between $V\left(G_{i}\right)$ and $V\left(G_{j}\right)$. Note that if $G_{1} \cong \ldots \cong G_{\ell}$, then $H\left\{G_{1}, \ldots, G_{\ell}\right\}$ corresponds to the composition (also known as the lexicographic product) of $G$ and $H$. Also, observe that

$$
K_{\ell}\left\{G_{1}, \ldots, G_{\ell}\right\}=\bigwedge_{i=1}^{\ell} G_{i}
$$

### 2.1 Fractional chromatic number

We present a definition of the fractional chromatic number based on a linear programming relaxation of an integer program computing the ordinary chromatic number. For a graph $G$, let $\mathcal{I}(G)$ be the set of all independent sets. Let FRACC be the following linear program.

$$
\text { FRACC }\left\{\begin{array}{ll}
\text { Minimize } & \sum_{\substack{I \in \mathcal{I}(G)}} x_{I} \\
\text { subject to } \sum_{\substack{I \in \mathcal{I}(G) \\
v \in I}} x_{I} \geq 1 & \text { for } v \in V(G) \\
& x_{I} \geq 0
\end{array} \quad \text { for } I \in \mathcal{I}(G)\right.
$$

Furthermore, let FRACD be the following program, which is the dual of FRACC.

$$
\text { FRACD }\left\{\begin{array}{ll}
\text { Maximize } \sum_{v \in V(G)} y(v) & \\
\text { subject to } \sum_{v \in I} y(v) \leq 1 & \text { for } I \in \mathcal{I}(G) ; \\
& y(v) \geq 0
\end{array} \text { for } v \in V(G) .\right.
$$

Since these two linear programs are dual of each other, the LP-duality theorem ensures that they have the same value, which we denote by $\chi_{f}(G)$.

Let us now mention a different way to introduce the fractional chromatic number. As we have already mentioned, $\alpha(G) \geq v(G) / \chi_{f}(G)$. Moreover, the lower-bound stays valid even in the setting where the vertices have weights, and we measure the size of an independent set by the proportion of the weight it occupies rather than its cardinality.

More precisely, let $G=(V, E)$ be a graph and $w: V \rightarrow \mathbb{R}_{+}$a weight function. Let $\alpha(G, w)$ be the maximum sum of the weights of the vertices that form an independent set, i.e.,

$$
\alpha(G, w):=\max _{I \in \mathcal{I}} \sum_{v \in I} w(v) .
$$

If we rescale an optimal solution of FRACC by a factor $1 / \chi_{f}(G)$ and interpret it as a probability distribution on $\mathcal{I}$, the linearity of expectation yields that

$$
\alpha(G, w) \geq \mathbb{E}_{I}\left[\sum_{v \in I} w(v)\right]=\sum_{v \in V} w(v) \cdot \sum_{\substack{I \in \mathcal{I}(G) \\ v \in I}} \frac{x_{I}}{\chi_{f}(G)} \geq \frac{\sum_{v \in V} w(v)}{\chi_{f}(G)}
$$

On the other hand, any optimal solution of FRACD yields a weight function $w_{0}$ for which the bound is tight, i.e., $\alpha\left(G, w_{0}\right)=\sum_{v \in V} w_{0}(v) / \chi_{f}(G)$. Therefore,

$$
\chi_{f}(G)=\sup _{w: V \rightarrow[0,1]} \frac{\sum_{v \in V} w(v)}{\alpha(G, w)}
$$

Note that the Hall ratio can be viewed as an integral version of the above, since

$$
\rho(G)=\max _{w: V \rightarrow\{0,1\}} \frac{\sum_{v \in V} w(v)}{\alpha(G, w)}
$$

For other possible definitions of the fractional chromatic number, see [15].
We finish this section with a straightforward generalization of the fact that the fractional chromatic number of the composition of two graphs is equal to the product of their fractional chromatic numbers.

Proposition 5. Let $H$ be a graph with the vertex-set $\{1, \ldots, \ell\}$ and let $G_{1}, \ldots, G_{\ell}$ be graphs. It holds that $\chi_{f}(H) \cdot \min _{i \in[\ell]} \chi_{f}\left(G_{i}\right) \leq \chi_{f}\left(H\left\{G_{1}, \ldots, G_{\ell}\right\}\right)$. In particular, $\chi_{f}\left(\bigwedge_{i=1}^{\ell} G_{i}\right) \geq \ell \cdot \min _{i \in[\ell]} \chi_{f}\left(G_{i}\right)$.
Proof. Without loss of generality, we may assume that $V\left(G_{i}\right)=\left\{1, \ldots, v\left(G_{i}\right)\right\}$. Let $w_{1}^{H}, \ldots, w_{\ell}^{H}$ be any optimal solution of the dual program FRACD for $H$, and, for every $i \in[\ell]$, let $w_{1}^{i}, \ldots, w_{v\left(G_{i}\right)}^{i}$, be any optimal solution of FRACD for $G_{i}$.

Let $G:=H\left\{G_{1}, \ldots, G_{\ell}\right\}$. For a vertex $(i, j) \in V(G)$, where $i \in[\ell]$ and $j \in\left[v\left(G_{i}\right)\right]$, we set $y_{i, j}:=w_{i}^{H} \cdot w_{j}^{i}$. It holds that

$$
\sum_{(i, j) \in V(G)} y_{i, j}=\sum_{i \in[\ell]} w_{i}^{H} \cdot \sum_{j \in V\left(G_{i}\right)} w_{j}^{i}=\sum_{i \in[\ell]} w_{i}^{H} \cdot \chi_{f}\left(G_{i}\right) \geq \chi_{f}(H) \cdot \min _{i \in[\ell]} \chi_{f}\left(G_{i}\right)
$$

We claim that $\left(y_{i, j}\right)$, where $(i, j) \in V(G)$, is a feasible solution of FRACD for $G$.
Indeed, fix any $I \in \mathcal{I}(G)$. For $i \in[\ell]$, let $I_{i}:=\left\{j \in\left[v\left(G_{i}\right)\right]:(i, j) \in I\right\}$. Since $I_{i} \in \mathcal{I}\left(G_{i}\right)$, it holds that

$$
\sum_{j \in I_{i}} w_{i}^{H} \cdot w_{j}^{i}=w_{i}^{H} \cdot \sum_{j \in I_{i}} w_{j}^{i} \leq w_{i}^{H}
$$

On the other hand, the set $I_{H}:=\{i \in[\ell]: \exists(i, j) \in I\}$ is independent in $H$. Therefore,

$$
\sum_{(i, j) \in I} y_{i, j}=\sum_{i \in I_{H}} \sum_{j \in I_{i}} y_{i, j}=\sum_{i \in I_{H}} w_{i}^{H} \cdot \sum_{j \in I_{i}} w_{j}^{i} \leq \sum_{i \in I_{H}} w_{i}^{H} \leq 1 .
$$

We note that a similar composition of optimal solutions of FRACC yields $\chi_{f}\left(H\left\{G_{1}, \ldots, G_{\ell}\right\}\right) \leq \chi_{f}(H) \cdot \max _{i \in[\ell]} \chi_{f}\left(G_{i}\right)$, but we will never need this bound. However, we will use the following analogue of this bound for proper colorings.

Proposition 6. Let $H$ be a graph with the vertex-set $\{1, \ldots, \ell\}$ and let $G_{1}, \ldots, G_{\ell}$ be graphs. It holds that $\chi(H) \cdot \max _{i \in[\ell]} \chi\left(G_{i}\right) \geq \chi\left(H\left\{G_{1}, \ldots, G_{\ell}\right\}\right)$.
Proof. Let $k:=\max _{i \in[\ell]} \chi\left(G_{i}\right)$, and $d$ be a proper $\chi(H)$-coloring of $H$. Next, for every $i \in[\ell]$, let $c_{i}: V\left(G_{i}\right) \rightarrow[k]$ be a proper $k$-coloring of $G_{i}$. It is straightforward to verify that assigning each vertex $v \in V_{i}$ a color $\left(d(i), c_{i}(v)\right)$ yields a proper coloring of $\chi\left(H\left\{G_{1}, \ldots, G_{\ell}\right\}\right)$ using $\chi(H) \cdot k$ colors.

Finally, the following observation is going to be useful in the next section.
Observation 7. If every $H \subseteq G$ has at most $|V(H)|$ edges, then $\chi(G) \leq 3$.
Proof. Without loss of generality, we may assume $G$ is connected. Since $|E(G)| \leq$ $|V(G)|$, the graph $G$ contains at most one cycle and hence it is 3-colorable.

### 2.2 Sparse Erdős-Rényi random graphs

Let $G_{n, p}$ be a random graph on $\{1,2, \ldots, n\}$ where each pair of vertices forms an edge independently with probability $p$. We now recall some well-known properties of $G_{n, \frac{D}{n}}$ we are going to use.
Proposition 8. There exists $C_{0}$ such that for every $C \geq C_{0}$ the following is true: There exists $n_{0}=n_{0}(C) \in \mathbb{N}$ such that for every $n \geq n_{0}$ there is an $n$ vertex triangle-free graph $G=G^{1}(n, C)$ with the following properties:
(A) $1.001 \cdot C>\chi(G) \geq \chi_{f}(G) \geq \frac{n}{\alpha(G)}>C$, and
(B) for all $k \leq \sqrt{\ln n}$, every $k$-vertex subgraph of $G$ is 3 -colorable.

Proof. Suppose that $C$ and $n$ are sufficiently large, and let $D>1$ be such that $C=\frac{D}{2 \cdot \ln D}$. By [6] and [13], a random graph $G_{n, \frac{D}{n}}$ satisfies with high probability $\alpha\left(G_{n, \frac{D}{n}}\right)<n / C$ and $\chi\left(G_{n, \frac{D}{n}}\right)<1.001 \cdot C$, respectively.

Next, the expected number of subgraphs $H$ in $G_{n, \frac{D}{n}}$ with $v(H) \leq \sqrt{\ln n}$ and more than $v(H)$ edges is at most

$$
\sum_{k=3}^{\sqrt{\ln n}} 2^{k^{2}} \cdot n^{k} \cdot\left(\frac{D}{n}\right)^{k+1} \leq \sqrt{\ln n} \cdot \frac{D^{\sqrt{\ln n}+1}}{n^{1-\ln 2}}=O\left(n^{-0.3}\right)
$$

By Markov's inequality, $G_{n, \frac{D}{n}}$ has no such $H$ with high probability, hence the property (B) follows from Observation 7

Finally, Schürger [16] showed that the number of triangles in $G_{n, \frac{D}{n}}$ converges to the Poisson distribution with mean $\Theta\left(D^{3}\right)$. Therefore, $G_{n, \frac{D}{n}}$ is triangle-free with probability $e^{-\Theta\left(D^{3}\right)}>0$. Note that a similar estimate can also be deduced using the FKG inequality.

## 3 Counter-examples to Conjecture 1

We start with a simple construction of a sequence of graphs for which $\chi_{f}(G) \gg$ $\rho(G)$. Each graph $G$ is the join of the graphs $G^{1}\left(n_{i}, C\right)$ of very different orders.

Proof of Theorem [2. Let $C_{0}$ be the constant from Proposition 8, and $P_{0}:=8 C_{0}$.
Given $P \geq P_{0}$, let $\ell:=\lfloor P / 4\rfloor, C:=P / 8$, and $n_{1}:=n_{0}(C)$ from Proposition 8. For all $j \in[\ell-1]$, let $n_{j+1}:=\left\lceil e^{2 \cdot n_{j}^{2}}\right\rceil$, and, for all $i \in[\ell]$, let $G_{i}:=G^{1}\left(n_{i}, C\right)$. We set $G:=\bigwedge_{i=1}^{\ell} G_{i}$.

By Proposition 5, $\chi_{f}(G)>\ell \cdot C>P^{2} / 33$. It only remains to prove that $\rho(G) \leq P$, i.e., that $\alpha(G[X]) \geq v(G[X]) / P$ for every $X \subseteq V(G)$.

Fix $X \subseteq V(G)$, and let $X_{i}:=V\left(G_{i}\right) \cap X$ for $i \in[\ell]$. We split the indices into two categories, small and big, based on $\left|X_{i}\right|$ with respect to $v\left(G_{i}\right)=n_{i}$. Specifically, let

$$
\mathcal{S}:=\left\{i \in[\ell]:\left|X_{i}\right|<\sqrt{\ln n_{i}}\right\}, \quad \text { and } \quad \mathcal{B}:=[\ell] \backslash \mathcal{S} .
$$

Next, let $H_{S}$ and $H_{B}$ be the subgraphs of $G$ induced by $\bigcup_{i \in \mathcal{S}} X_{i}$ and $\bigcup_{i \in \mathcal{B}} X_{i}$, respectively, and $v_{s}$ and $v_{b}$ their respective orders. In both of these subgraphs, we can find quite large independent sets.

Claim 9. $H_{S}$ has an independent set of size at least $4 v_{s} / 3 P$.
Proof. Fix $i \in \mathcal{S}$ such that $\left|X_{i}\right|$ is maximized. Note that $\left|X_{i}\right| \geq v_{s} /|\mathcal{S}|$. The property ( (B) of $G_{i}$ established in Proposition 8 yields that $G\left[X_{i}\right]$ is 3-colorable, and hence its largest color class has size at least

$$
\frac{v_{s}}{3|\mathcal{S}|} \geq \frac{v_{s}}{3 \ell} \geq \frac{4 v_{s}}{3 P}
$$

which finishes the proof.
Claim 10. $H_{B}$ has an independent set of size at least $4 v_{b} / P$.
Proof. Let $m$ be the largest element of $\mathcal{B}$. Since $G_{m}$ is ( $0.51 \ell$ )-colorable, $G\left[X_{m}\right]$ contains an independent set of size at least 1.9•| $X_{m} \mid / \ell$. If $m=1$, then $\left|X_{m}\right|=v_{b}$. On the other hand, if $m \geq 2$, then

$$
1.9 \cdot\left|X_{m}\right| \geq\left|X_{m}\right|+0.9 \cdot \sqrt{\ln n_{m}}>\left|X_{m}\right|+1.2 \cdot n_{m-1}>\left|X_{m}\right|+\sum_{i=1}^{m-1} n_{i} \geq v_{b}
$$

We conclude that $H_{B}$ has an independent set of size at least $v_{b} / \ell \geq 4 v_{b} / P$.
If $v_{s} \geq 3|X| / 4$, then we find an independent set of size at least $|X| / P$ in $H_{S}$ by Claim 9. Otherwise, $v_{b} \geq|X| / 4$, and Claim 10 guarantees an independent set in $H_{B}$ of size at least $|X| / P$.

## $3.1 \quad K_{5}$-free and iterated constructions

As we have already noted in Section 2, the graph $G=\bigwedge_{i=1}^{\ell} G_{i}$ constructed in Theorem 2] can be equivalently viewed as $K_{\ell}\left\{G_{1}, G_{2}, \ldots, G_{\ell}\right\}$. An adaptation of the proof of Theorem 2 will show that replacing $K_{\ell}$ by a graph from Proposition 8 yields another graph $G^{2}$ with $\chi_{f}\left(G^{2}\right) \sim\left(\rho\left(G^{2}\right)\right)^{2}$. However, as all the graphs involved in the composition are now triangle-free, $G^{2}$ will be $K_{5}$-free.

But we do not need to stop here. Since we have now much better control on the chromatic numbers of small subgraphs in $G^{2}$ than in the original graph $G$, replacing the graphs $G_{i}=G^{1}\left(n_{i}, C\right)$ in the composition by $n_{i}$-vertex variants of $G^{2}$ yields a graph $G^{3}$ with $\chi_{f}\left(G^{3}\right) \sim\left(\rho\left(G^{3}\right)\right)^{3}$. Repeating this procedure $k$ times leads to a construction of a graph $G^{k+1}$ with $\chi_{f}\left(G^{k+1}\right) \sim\left(\rho\left(G^{k+1}\right)\right)^{k+1}$.

In order to present our proofs of Theorems 3 and 4 we need to introduce some additional notation. Let us start with recalling the Knuth's up-arrow notation

$$
a \uparrow^{(k)} b= \begin{cases}a^{b} & \text { if } k=1, \\ 1 & \text { if } k \geq 1 \text { and } b=0, \\ a \uparrow^{(k-1)}\left(a \uparrow^{(k)}(b-1)\right) & \text { otherwise }\end{cases}
$$

where $a, b, k \in \mathbb{N}$, and its inverse $a \downarrow^{(k)} n$, which is the largest integer $b$ such that $n \geq a \uparrow^{(k)} b$. Using this, we define the following Ackermann-type function $F_{k}(b)$ and its inverse $f_{k}(b)$ :

$$
F_{k}(b):=2 \uparrow^{(k)} b \quad \text { and } \quad f_{k}(b):=2 \downarrow^{(k)} b .
$$

Note that $F_{1}(b)=2^{b}$ and $f_{1}(b)=\left\lfloor\log _{2}(b)\right\rfloor$, and for every $k \in \mathbb{N}$ it holds that $F_{k}(1)=2$ and $F_{k}(2)=4$. The functions also satisfy the following properties:

Fact 11. For every $k \in \mathbb{N}$, the following holds:

1. $f_{k}\left(f_{k}\left(F_{k+1}(n+2)\right)\right)=F_{k+1}(n)$ for every $n \in \mathbb{N}$,
2. $f_{k+1}(4 M)<f_{k}\left(f_{k}(M)\right)$ for every $M \geq F_{k}\left(F_{k}(7)\right)$, and
3. $\sum_{b=0}^{n} F_{k}(b)<F_{k}(n+1)$ for every $n \in \mathbb{N}$.

For a proof, see Appendix A. We are now ready to present the main lemma.
Lemma 12. Let $C_{0}$ be the constant from Proposition 8 . For every $k \in \mathbb{N}$ and $C \geq C_{0}$ there is $n_{0}:=n_{0}(k, C)$ such that for all $n \geq n_{0}$ there is an $n$-vertex $K_{2^{k}+1}-$ free graph $G:=G^{k}(n, C)$ with the following properties:

- $\chi_{f}(G) \geq C^{k}$,
- $\rho(G) \leq 1.001 \cdot 3^{k} \cdot C$, and
- $G[W]$ is $3^{k}$-colorable for every $W \subseteq V(G)$ such that $|W| \leq f_{k}\left(f_{k}(n)\right)$.

Proof. For any fixed $C \geq C_{0}$, we proceed by induction on $k$. As the case $k=1$ follows by letting $G:=G^{1}(n, C)$ from Proposition 8 we may assume $k \geq 2$.

Let $M$ be the smallest positive integer such that $f_{k}(4 M) \leq f_{k-1}\left(f_{k-1}(M)\right)$. Note that $M \leq F_{k-1}\left(F_{k-1}(7)\right)$ by the second property of Fact 11. We set $n_{0}(k, C):=\max \left\{M, F_{k}\left(4 \cdot n_{0}(k-1, C)\right)\right\}$. Given $n \geq n_{0}(k, C)$, we define $m$ to be the largest integer such that

$$
m+\sum_{i=2}^{m} F_{k}(m+3 i-6) \leq n
$$

Note that $F_{k}(4 m-1)>n$, as otherwise the third property of Fact 11 yields

$$
(m+1)+\sum_{i=2}^{m+1} F_{k}(m+3 i-6) \leq F_{k}(4 m-1) \leq n
$$

contradicting the maximality of $m$. Therefore,

$$
F_{k}(4 m)>F_{k}(4 m-1)>n_{0}(k, C) \geq F_{k}\left(4 n_{0}(k-1, C)\right)
$$

and hence $m>n_{0}(k-1, C)$. We set $b_{1}:=m$, and $b_{i}:=F_{k}(m+3 i-6)$ for every $i=2,3, \ldots, m-1$. Finally, we set $b_{m}:=n-\sum_{i<m} b_{i}$.

Let $H:=G^{1}(m, C)$, and $G_{i}:=G^{k-1}\left(b_{i}, C\right)$ for all $i \in[m]$. We define $G:=H\left\{G_{1}, G_{2}, \ldots, G_{m}\right\}$. Clearly, the graph $G$ contains no $K_{2^{k}+1}$. In the following three claims, we show that $G$ has the desired three properties:
Claim 13. $\chi_{f}(G) \geq C^{k}$.
Proof. By the induction hypothesis, $\chi_{f}(H) \geq C$ and $\chi_{f}\left(G_{i}\right) \geq C^{k-1}$ for all $i \in[m]$. Therefore, Proposition 5yields the desired lower-bound on $\chi_{f}(G)$.

Claim 14. $\rho(G) \leq 1.001 \cdot 3^{k} \cdot C$.
Proof. Fix an $X \subseteq V(G)$. Our aim is to show that $\alpha(G[X]) \geq|X| /\left(1.001 \cdot 3^{k} \cdot C\right)$. For $i \in[m]$, let $X_{i}$ be $X \cap V\left(G_{i}\right)$. As in the proof of Theorem 2 let

$$
\mathcal{S}:=\left\{i \in[m]:\left|X_{i}\right| \leq f_{k-1}\left(f_{k-1}\left(b_{i}\right)\right)\right\} \quad \text { and } \quad \mathcal{B}:=[m] \backslash \mathcal{S}
$$

First, suppose the case $\left|\bigcup_{i \in \mathcal{S}} X_{i}\right| \geq|X| / 3$. By the definition of $\mathcal{S}$ and the properties of $G_{i}$, every subgraph $G\left[X_{i}\right]$, where $i \in \mathcal{S}$, has an independent set of size at least $\left|X_{i}\right| / 3^{k-1}$. On the other hand, $\chi(H)<1.001 \cdot C$, so the projection of at least one of the color classes of the optimal coloring of $H$ on $\bigcup_{i \in \mathcal{S}} X_{i}$ contains an independent set of size at least

$$
\sum_{i \in \mathcal{S}} \frac{\left|X_{i}\right|}{3^{k-1}} \cdot \frac{1}{1.001 \cdot C} \geq \frac{|X|}{1.001 \cdot 3^{k} \cdot C}
$$

Now suppose $\left|\bigcup_{i \in \mathcal{B}} X_{i}\right| \geq 2|X| / 3$, and let $z$ be the maximum index in $\mathcal{B}$. If $z=1$, then $\left|X_{1}\right| \geq 2|X| / 3$. On the other hand, if $z \geq 2$, then

$$
f_{k-1}\left(f_{k-1}\left(b_{z}\right)\right) \geq f_{k-1}\left(f_{k-1}\left(F_{k}(m+3 z-6)\right)\right)=F_{k}(m+3 z-8) \geq \sum_{i<z} b_{i}
$$

where the equality and the last inequality follow from the first and the third property of Fact 11, respectively. Therefore, $\left|X_{z}\right| \geq|X| / 3$ and $G_{z}\left[X_{z}\right]$ contains an independent set of the sought size by $\rho\left(G_{z}\right) \leq 1.001 \cdot 3^{k-1} \cdot C$.

Claim 15. $G[W]$ is $3^{k}$-colorable for every $W \subseteq V$ with $|W| \leq f_{k}\left(f_{k}(n)\right)$.
Fix a set $W \subseteq V$ of size at most $f_{k}\left(f_{k}(n)\right)$. Firstly, let $Z:=\{i: W \cap$ $\left.V\left(G_{i}\right) \neq \emptyset\right\}$. Clearly, $|Z| \leq|W| \leq f_{k}\left(f_{k}(n)\right)$. Since $f_{k}(n) \leq 4 m$ and $f_{k}(x) \ll$ $\log _{2} \log _{2}(x / 4)$, we conclude that $|Z| \leq \log _{2} \log _{2}(v(H))$. Therefore, there exists a proper 3-coloring of the induced subgraph $H[Z]$.

By the second property of Fact 11] for every $i \in[m]$ it holds that

$$
\left|V\left(G_{i}\right) \cap W\right| \leq|W| \leq f_{k}(4 m) \leq f_{k-1}\left(f_{k-1}(m)\right) \leq f_{k-1}\left(f_{k-1}\left(b_{i}\right)\right)
$$

Therefore, the induction hypothesis yields that each $V\left(G_{i}\right) \cap W$ induces a $3^{k-1}$ colorable subgraph of $G$, and hence $\chi(G[W]) \leq 3^{k}$ by Proposition 6,

Theorem 3 is a direct consequence of Lemma 12 applied with $k=2$. It remains to establish Theorem 4.

Proof of Theorem 4. Let $P_{0}:=\left(2 C_{0}\right)^{2}$. Given $P \geq P_{0}$, let $C:=\sqrt{P / 1.001}$ and $k:=\left\lfloor\log _{3} C\right\rfloor$. Applying Lemma 12 with $k$ and $C$ yields an $n_{0}(k, C)$-vertex graph $G$ with $\rho(G) \leq P$ and $\chi_{f}(G) \geq C^{\left\lfloor\log _{3} C\right\rfloor}>e^{0.9 \cdot \ln ^{2}(C)}>e^{\ln ^{2}(P) / 5}$.

## 4 Concluding remarks

We presented various constructions of graphs where the fractional chromatic number grows much faster than the Hall ratio, which refuted Conjecture 1. It is natural to ask whether the conclusion in Conjecture 1 can be relaxed and the fractional chromatic number of a graph is always upper-bounded by some function of its Hall ratio.

Question 16. Is there a function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $\chi_{f}(G) \leq g(\rho(G))$ for every graph $G$ ?

Theorem4 shows that if such a function $g$ exists, then $g(x) \geq e^{\ln ^{2}(x) / 5}$. While preparing our manuscript, we have learned that Dvořák, Ossona de Mendez and $\mathrm{Wu}[4$ constructed graphs with Hall ratio at most 18 and arbitrary large fractional chromatic number. Therefore, the answer to Question 16 is no.

Conjecture 1 was partially motivated by another conjecture of Harris concerned with fractional colorings of triangle-free graphs, which was inspired by a famous result of Johansson [10] (for a recent short proof, see [14]) stating that $\chi(G)=O(\Delta / \ln \Delta)$ for every triangle-free graph $G$ with maximum degree $\Delta$.

Conjecture 17 ([7] Conjecture 6.2]). There is $C$ such that $\chi_{f}(G) \leq C \cdot d / \ln d$ for every triangle-free d-degenerate graph $G$.

A classical result of Ajtai, Komlós, and Szemerédi [1] together with an averaging argument yield that $\rho(G)=O(d / \ln d)$ for $G$ and $d$ as above. Therefore, if Conjecture 1 could be recovered in the triangle-free setting, it would immediately yield the sought bound on $\chi_{f}$ in Conjecture 17

Question 18. Is there $C$ such that $\chi_{f}(G) \leq C \cdot \rho(G)$ for every triangle-free graph G?

In [11, it has been mentioned that the sequence of Mycelski graphs might provide a negative answer to Question 18, but we still do not know. For $K_{5}$-free graphs, Theorem 3 shows that the answer is definitely negative. As a possibly simpler question, does the answer stay negative in case of $K_{4}$-free graphs?

Question 19. Is there $C$ such that $\chi_{f}(G) \leq C \cdot \rho(G)$ for every $K_{4}$-free graph $G$ ?
Let us conclude with an additional motivation for studying Conjecture 17 Esperet, Kang and Thomassé [5] conjectured that dense triangle-free graphs must contain dense induced bipartite subgraphs.

Conjecture 20 ([5, Conjecture 1.5]). There exists $C>0$ such that any trianglefree graph with minimum degree at least d contains an induced bipartite subgraph of minimum degree at least $C \cdot \ln d$.

Erdôs-Rényi random graphs of the appropriate density show that the bound would be, up to the constant $C$, best possible. A relation between the fractional chromatic number and induced bipartite subgraphs proven in [5, Theorem 3.1] shows that if Conjecture 17 holds, then Conjecture 20 holds as well. Very recently, Kwan, Letzter, Sudakov and Tran [12] proved a slightly weaker version of Conjecture 20 where the bound $C \cdot \ln n$ is replaced by $C \cdot \ln n / \ln \ln n$.

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## A Proof of Fact 11

The definitions of $f_{k}$ and $F_{k+1}$ readily yield that $f_{k}\left(F_{k+1}(n+1)\right)=F_{k+1}(n)$. Therefore, $f_{k}\left(f_{k}\left(F_{k+1}(n+2)\right)\right)=F_{k+1}(n)$ proving the first property.

For every $k, n \in \mathbb{N}$, a straightforward induction yields that $F_{k}(n) \geq n+1$. This in turn implies that $F_{k+1}(n)=F_{k}\left(F_{k+1}(n-1)\right) \geq F_{k}(n) \geq 2^{n}$. Similarly, for all $k \in \mathbb{N}$, the functions $F_{k}(\cdot)$ and $f_{k}(\cdot)$ are monotone non-decreasing. Therefore, for all $k \in \mathbb{N}$ and $n \geq 7$, it holds that
$F_{k+1}(n)=F_{k}\left(F_{k}\left(F_{k}\left(F_{k}\left(F_{k+1}(n-4)\right)\right)\right)\right) \geq 2^{F_{k}\left(F_{k}\left(2^{n-4}\right)+1\right)} \geq 4 \cdot F_{k}\left(F_{k}(n+1)+1\right)$.
Since $F_{k}\left(f_{k}(M)+1\right)>M \geq F_{k}\left(f_{k}(M)\right)$, we assert that $f_{k+1}(4 M)<f_{k}\left(f_{k}(M)\right)$ for all $M \geq F_{k}\left(F_{k}(7)\right)$. Indeed, as otherwise

$$
4 M \geq F_{k+1}\left(f_{k+1}(4 M)\right) \geq F_{k+1}\left(f_{k}\left(f_{k}(M)\right)\right) \geq 4 \cdot F_{k}\left(F_{k}\left(f_{k}\left(f_{k}(M)\right)+1\right)+1\right)>4 M
$$

a contradiction. This concludes the proof of the second property.
The last property is proven by induction on $k$. Indeed, the case $k=1$ is the sum of a geometric progression. If $k \geq 2$, then by induction hypothesis

$$
\sum_{b=0}^{n} F_{k+1}(b)=\sum_{b=0}^{n} F_{k}\left(F_{k+1}(b-1)\right) \leq \sum_{i=0}^{F_{k+1}(n-1)} F_{k}(i)<F_{k}\left(F_{k+1}(n-1)+1\right)
$$

However, the right-hand side is at most $F_{k}\left(F_{k}\left(F_{k+1}(n-1)\right)\right)=F_{k+1}(n+1)$.


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