ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

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Matematické modely proudění vazké nestlačitelné tekutiny na omezených oblastech

Mathematical models of the flow of viscous incompressible fluid on bounded domains

DISERTAČNÍ PRÁCE

Mgr. Petra Vacková

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Školitel: doc. RNDr. Petr Kučera, CSc.

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ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE
Fakulta stavební
Thákurova 7, 166 29 Praha 6

PROHLÁŠENÍ

Jméno doktoranda: Mgr. Petra Vacková

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Abstrakt

Disertační práce se zabývá některými kvalitativními vlastnostmi řešení dvoudimenzionálních Stokesových a Navier-Stokesových rovnic se smíšenými okrajovými podmínkami v omezené oblasti. Tato oblast odpovídá kanálu, který je naplněn nestlačitelnou tekutinou. Předepisujeme homogenní Dirichletovu nebo Navierovu okrajovou podmínku na části hranice, která odpovídá pevné stěně kanálu a "do-nothing"okrajovou podmínku na části hranice, která odpovídá vstupu a výstupu kanálu. V práci je dokázána regularita řešení Stokesových rovnic. V další části dokazujeme lokální existenci řešení Navierových-Stokesových rovnic v čase. V poslední části se zabýváme systémem Navierovy-Stokesovy soustavy a rovnicemi vedení tepla v kapalině. V této části opět dokazujeme lokální existenci řešení tohoto systému v čase.

Abstract

The PhD-thesis deals with some qualitative properties of the solution of the two-dimensional Stokes and Navier-Stokes equations with mixed boundary conditions in a bounded domain. This domain corresponds to a channel filled with an incompressible fluid. We prescribe homogeneous Dirichlet or Navier's boundary conditions on boundaries corresponding to the channel walls and do-nothing boundary conditions on the channel input and output. Regularity of solutions of the Stokes equations is proved in this thesis. In the next section we prove the local in time existence of solutions to the Navier-Stokes equations. In the last part we deal with the system of the Navier-Stokes equations and the heat equation and we prove the local in time existence of solutions of this system.

Obsah

Osnova disertační práce

Předložená disertační práce se zabývá matematickými modely proudění vazké nestlačitelné tekutiny. Předkládá vybrané výsledky o kvalitativních vlastnostech řešení těchto modelů. Práce obsahuje úvod a další tři části.

V úvodu jsou formulovány základní pojmy teorie Navierových-Stokesových rovnic, známé výsledky a otevřené problémy.

V první kapitole se zabýváme dvoudimenzionální stacionární soustavou Stokesových rovnic na omezené oblasti. Tato oblast odpovídá kanálu vyplněném proudící tekutinou. Na části hranice, která odpovídá pevné stěně, předepisujeme Navierovu okrajovou podmínku, na části hranice, která odpovídá vstupu a výstupu kanálu, předepisujeme okrajovou podmínku (7). V této kapitole studujeme regularitu řešení soustavy Stokesových rovnic. Hlavní část druhé kapitoly tvoří článek [4].

V druhé kapitole se zabýváme systémem dvoudimenzionálních Navierových-Stokesových rovnic se třemi typy okrajových podmínek. Předepisujeme Navierovu nebo homogenní Dirichletovu okrajovou podmínku na (navzájem disjunktních) částech pevné hranice, která odpovídá pevné stěně kanálu. Na vstupu a výstupu z kanálu předepisujeme okrajovou podmínku (7). V této kapitole je dokázána existence a jednoznačnost řešení na nějakém dostatečně malém časovém intervalu. Hlavní část třetí kapitoly tvoří článek [5].

Ve třetí kapitole se zabýváme systémem dvourozměrných Navierových-Stokesových rovnic se smíšenými okrajovými podmínkami v kanálu doplněném o rovnici vedení tepla v kapalinách. Zde předepisujeme Navierovu okrajovou podmínku pro rychlost tekutiny a Newtonovu okrajovou podmínku pro teplotu na pevné stěně kanálu. Na vstupu a výstupu z kanálu předepisujeme okrajovou podmínku (7) pro rychlost a tlak a Neumannovu okrajovou podmínku pro teplotu. Podobně jako v předchozí kapitole, i zde je dokázána existence a jednoznačnost řešení na nějakém dostatečně malém časovém intervalu. Hlavní část čtvrté kapitoly tvoří článek [6].

1 Úvod

1.1 Základní pojmy teorie Navierových-Stokesových rovnic

Existuje mnoho fyzikálních jevů, jejichž chování lze popsat obyčejnými nebo parciálními diferenciálními rovnicemi. Tyto rovnice jsou doplněny okrajovými podmínkami a pokud jde o nestacionární jevy, také počátečními podmínkami. V této disertační práci se zabýváme matematickými modely proudění nestlačitelné tekutiny. Tyto modely jsou odvozeny pomocí zákona zachování hmoty a zákona zachování hybnosti.

Předpokládejme, že $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. Nejpoužívanějším modelem proudění nestlačitelné vazké tekutiny je systém nestacionárních Navierových-Stokesových rovnic:

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \mathcal{P} = \boldsymbol{g} \text{ v } \Omega \times (0, T), \tag{1}$$

$$div\, \boldsymbol{u} = 0 \text{ v } \Omega \times (0, T), \tag{2}$$

$$\boldsymbol{u}(., 0) = \boldsymbol{u}_0 \text{ v } \Omega, \tag{3}$$

kde neznámé funkce jsou $\boldsymbol{u}$ a $\mathcal{P}$. $\boldsymbol{u} = (u_1, \ldots, u_n)$ představuje rychlost proudící tekutiny, komponenty $u_i = (\boldsymbol{x}, t)$ jsou funkcemi prostorových proměnných $\boldsymbol{x} = [x_1, \ldots, x_n]$ a času t. $\mathcal{P}$ značí tlak a je rovněž funkcí prostorových proměnných $\boldsymbol{x} = [x_1, \ldots, x_n]$ a t. Symboly $\boldsymbol{g}$ a $\boldsymbol{u}_0$ značíme po řadě objemovou sílu a počáteční rychlost. Vektorová rovnice (1) se nazývá Navierova-Stokesova rovnice. Je odvozena pomocí zákona zachování hybnosti. Rovnice (2) se nazývá rovnicí kontinuity a je odvozena ze zákona zachování hmoty. Symbol ν označuje viskozitu kapaliny a předpokládáme pro zjednodušení, že $\nu = 1$.

Pokud modelujeme proudění nestlačitelné tekutiny v ohraničené oblasti Ω, musíme k rovnicím (1)-(3) přidat okrajové podmínky. Jednou z nejčastěji používaných okrajových podmínek je homogenní Dirichletova okrajová podmínka:

$$\boldsymbol{u} = \boldsymbol{0} \text{ na } \partial\Omega. \tag{4}$$

Další okrajovou podmínkou je tzv. Navierova okrajová podmínka:

$$a)\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad b)\ [\mathbb{T}_d(\boldsymbol{u}) \cdot \boldsymbol{n}]_\tau + \gamma\, \boldsymbol{u} = \boldsymbol{0} \text{ na } \partial\Omega \times (0, T). \tag{5}$$

$\mathbb{T}_d(\boldsymbol{u})$ značí dynamický tenzor napětí přiřazený rychlostnímu poli $\boldsymbol{u}$ a dolní index τ označuje tečnou složku. Předpokládáme, že tekutina je newtonovská, tedy $\mathbb{T}_d(\boldsymbol{u}) = 2\nu(\nabla \boldsymbol{u})_s$, kde $(\nabla \boldsymbol{u})_s$ je symetrická část $\nabla \boldsymbol{u}$, γ je koeficient tření mezi kapalinou a hranicí. Předpokládáme $\gamma \geq 0$. Připomeňme si, že "celý"tenzor napětí je $\mathbb{T} = \mathbb{T}(\boldsymbol{u}, p) = -p\mathbb{I} + \mathbb{T}_d(\boldsymbol{u})$. Podmínka b) vyjadřuje, že tangenciální složka napětí, kterou tekutina na hranici působí, je úměrná rychlosti.

V některých pracech se rovněž předepisuje tzv. okrajová podmínka Navierova typu:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ na } \partial\Omega \times (0, T), \tag{6}$$

kde symbol $\boldsymbol{curl}$ označuje operátor rotace.

Tyto podmínky jsou předepsány na části hranice, kde předpokládáme kontakt mezi tekutinou a stěnou. V mnoha pracích je řešena úloha (1)–(3) a na celé hranici oblasti je předepsána jedna okrajová podmínka. Modelujeme-li proudění tekutiny v kanálu, je vhodné předepsat některou z okrajových podmínek (4)–(6) pouze na části hranice, která odpovídá pevné stěně kanálu, dále tuto část hranice budeme značit Γ_D . Část hranice, která odpovídá vstupu a výstupu z kanálu, budeme nadále značit Γ_N. Na této části hranice potom předepisujeme buď

$$\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - \mathcal{P}\boldsymbol{n} = \boldsymbol{\sigma} \text{ na } \Gamma_N \times (0,T) \tag{7}$$

nebo

$$\frac{\nu}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] \cdot \boldsymbol{n} - \mathcal{P}\boldsymbol{n} = \boldsymbol{\sigma} \text{ na } \Gamma_N \times (0,T). \tag{8}$$

Levá strana (8) odpovídá normálové složce tenzoru napětí.

Předchozí modely, jak již bylo zmíněno, popisují proudění vazké nestlačitelné newtonské tekutiny. Kromě zde popsaného modelu existuje řada dalších modelů, které popisují proudění tekutin, kupř. řada modelů popisujících stlačitelné proudění nebo proudění nenewtonských tekutin.

1.2 Prostory funkcí, slabá a silná řešení.

V této kapitole zavedeme některé prostory funkcí, které budeme potřebovat v dalším textu. Dále uvedeme některé definice. Poznamenejme, že prostory $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{W}^{1,2}(\Omega)$ a $\boldsymbol{W}_0^{1,2}(\Omega)$ jsou definovány obvyklým způsobem.

- $\mathcal{V} = \{\boldsymbol{v} \in \mathcal{C}^\infty(\Omega)^3;\ \overline{supp\, \boldsymbol{v}} \subset \Omega,\ div\, \boldsymbol{v} = 0 \text{ na } \Omega\}$
- $\mathcal{V}_T = \{\boldsymbol{v} \in \mathcal{C}^\infty(\Omega \times [0,T))^3;\ \overline{supp\, \boldsymbol{v}} \subset \Omega \times [0,T),\ div\, \boldsymbol{v} = 0 \text{ na } \Omega \times (0,T)\}$
- $\boldsymbol{L}^2_\sigma(\Omega)$ je uzávěr prostoru $\mathcal{V}$ v $\boldsymbol{L}^2(\Omega)$,
- $\boldsymbol{W}^{1,2}_\sigma(\Omega) = \boldsymbol{W}^{1,2}(\Omega) \cap \boldsymbol{L}^2_\sigma(\Omega)$
- $\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)$ je uzávěr prostoru $\mathcal{V}$ v $\boldsymbol{W}^{1,2}_0(\Omega)$,
- $(\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega))^*$ je duální k prostoru $\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)$

Protože dále uvedeme přehled nejdůležitějších výsledků teorie Navierových-Stokesových rovnic pro řešení úlohy (1)–(4) na ohraničených oblastech, uvádíme pro tyto úlohy definici slabého řešení. Slabá řešení Navierových-Stokesových rovnic na celém prostoru jsou definována stejným způsobem.

Definice 1 *Nechť* $\boldsymbol{g} \in \boldsymbol{L}^2(0,T;(\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega))^*)$ *a* $\boldsymbol{\gamma} \in \boldsymbol{L}^2_\sigma(\Omega)$. *Měřitelná funkce* $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)) \cap \boldsymbol{L}^\infty(0,T;(\boldsymbol{L}^2_\sigma(\Omega))$ *je slabým řešením úlohy (1)–(4), pokud*

$$\int_0^T \int_\Omega \left(-\boldsymbol{u} \cdot \frac{\partial \boldsymbol{v}}{\partial t} + \nu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v} \right) d(\Omega)\, dt = \int_\Omega \boldsymbol{\gamma} \cdot \boldsymbol{v}(.,0)\, d(\Omega) + \int_0^T \langle \boldsymbol{g}, \boldsymbol{v} \rangle\, dt.$$

pro všechna $\boldsymbol{v} \in \mathcal{V}_T$.

Poznamenejme, že slabá řešení na ohraničených oblastech, která splňují okrajové podmínky (5) nebo (6), jsou definována analogickým způsobem. Zde $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{W}^{1,2}_{\sigma}(\Omega)) \cap \boldsymbol{L}^\infty(0,T;(\boldsymbol{L}^2_\sigma(\Omega))$.

Slabé řešení (1) - (4) s pravou stranou $\boldsymbol{g} \in \boldsymbol{L}^2(0,T,\boldsymbol{L}^2_\sigma(\Omega))$ splňuje energetickou nerovnost, pokud

$$\int_{\Omega \times \{t\}} |\boldsymbol{u}|^2\, \mathrm{d}(\Omega) + 2\int_0^t \int_\Omega |\nabla \boldsymbol{u}|^2\, \mathrm{d}(\Omega)\mathrm{dt} \leq \int_\Omega |\boldsymbol{\gamma}|^2\, \mathrm{d}(\Omega) + 2\int_0^t \int_\Omega \boldsymbol{g} \cdot \boldsymbol{u}\, \mathrm{d}(\Omega)\mathrm{dt}$$

platí pro každé $t \in (0,T]$.

Slabé řešení (1) - (4) s pravou stranou $\boldsymbol{g} \in \boldsymbol{L}^2(0,T,\boldsymbol{L}^2_\sigma(\Omega))$ splňuje silnou energetickou nerovnost, pokud

$$\int_{\Omega \times \{t_2\}} |\boldsymbol{u}|^2\, \mathrm{d}(\Omega) + 2\int_{t_1}^{t_2} \int_\Omega |\nabla \boldsymbol{u}|^2\, \mathrm{d}(\Omega)\mathrm{dt} \leq \int_{\Omega \times \{t_1\}} |\boldsymbol{u}|^2\, \mathrm{d}(\Omega) + 2\int_{t_1}^{t_2} \int_\Omega \boldsymbol{g} \cdot \boldsymbol{u}\, \mathrm{d}(\Omega)\mathrm{dt}$$

platí pro každé $t_2 \in (0,T]$ a pro skoro všechna $t_1 \in (0,t_2)$ včetně $t_1 = 0$.

Za jakých podmínek slabé řešení splňuje energetickou rovnost, je stále otevřeným problémem. Energetickou rovností rozumíme:

$$\int_{\Omega \times \{t_2\}} |\boldsymbol{u}|^2\, \mathrm{d}(\Omega) + 2\int_{t_1}^{t_2} \int_\Omega |\nabla \boldsymbol{u}|^2\, \mathrm{d}(\Omega)\mathrm{dt} = \int_\Omega |\boldsymbol{\gamma}|^2\, \mathrm{d}(\Omega) + 2\int_{t_1}^{t_2} \int_\Omega \boldsymbol{g} \cdot \boldsymbol{u}\, \mathrm{d}(\Omega)\mathrm{dt}.$$

Na závěr této kapitoly budeme definovat regulární řešení.

Definice 2 *Nechť* $\boldsymbol{g} \in \boldsymbol{L}^2(0,T;(\boldsymbol{L}^2_\sigma(\Omega)))$, $\boldsymbol{u}_0 \in \boldsymbol{L}^3(\Omega)$ *a* $\boldsymbol{u}$ *je slabým řešení úlohy (1)–(4). Řekneme, že* $\boldsymbol{u}$ *je regulárním řešením, jestliže*

$$\nabla \boldsymbol{u} \in L^\infty_{loc}((0,T];\ \boldsymbol{L}^2(\Omega)).$$

1.3 Některé vybrané výsledky a otevřené problémy teorie Navierových-Stokesových rovnic

Matematická teorie Navierových-Stokesových rovnic se za posledních sto let zabývala zejména čtyřmi hlavními otázkami. První otázkou je, zda pro daná data existuje nějaké slabé řešení, kde daty úlohy rozumíme počáteční podmínku a pravou stranu rovnice, tj. objemovou sílu. Druhá otázka je, zda se slabé slabé řešení chová v souladu ze zákonem zachování energie. Třetí otázkou je, zda slabé řešení, pokud existuje, je jednoznačné. Čtvrtá a současně neznámější otázka se týká regularity řešení. Máme-li vhodná (hladká) data řešení, zda je řešení regulární, problém "regularita vs. blow-up". Tento problém byl zařazen mezi problémy tisíciletí Clayova institutu v USA.

Nyní uvedeme některé nejznámějších výsledky, které na tyto otázky odpovídají. Ne všechny z těchto otázek byly ovšem dosud zodpovězeny.

Existenci slabého řešení na celém prostoru za předpokladu, že $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega)$, dokázal J. Leray v roce 1934. Navíc ukázal, že toto řešení splňuje energetickou nerovnost.

V roce 1950 publikoval E. Hopf, viz [10] výsledek, ve kterém dokázal existenci slabého řešení na omezené oblasti. Toto řešení navíc splňuje silnou energetickou nerovnost.

Existenci slabého řešení na jakékoliv oblasti s dostatečně hladkou hranicí, s počáteční rychlostí $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega)$ a pravou stranou $\boldsymbol{g} \in \boldsymbol{L}^2(0, T, \boldsymbol{L}^2_\sigma(\Omega))$ dokázali J. Heywood [9], K. Masuda [22] a R. Temam. Toto řešení navíc splňuje silnou energetickou nerovnost.

Otázka, zda existuje slabé řešení, které splňuje energetickou rovnost, zůstává stále otevřenou. Uvedeme zde výsledek, který publikoval v roce 1960 J.L.Lions.

Věta 3 *a) Nechť* $\boldsymbol{u}$ *je slabé řešení úlohy (1)–(4) a* $\boldsymbol{u} \in L^4(0, T;\ \boldsymbol{L}^4(\Omega))$*. Potom* $\boldsymbol{u}$ *splňuje energetickou rovnost.*

b) Nechť $\boldsymbol{u}$ *je slabé řešení úlohy (1)–(3) na celém prostoru a* $\boldsymbol{u} \in L^4(0, T;\ \mathbb{R}^3)$*. Potom* $\boldsymbol{u}$ *splňuje energetickou rovnost.*

Tento výsledek rozšířil M. Shinbrot v [25]. Dokázal, že $\boldsymbol{u}$ splňuje energetickou rovnost, pokud $\boldsymbol{u} \in \boldsymbol{L}^p(0, T; \boldsymbol{L}^q(\Omega))$, kde $\frac{1}{p} + \frac{1}{q} = \frac{5}{4}$ a $q \geq 4$.

Nyní uvedeme některé dílčí výsledky, které se týkají jednoznačnosti řešení. G.Prodi [23] a J.L.Lions [21] dokázali následující větu.

Věta 4 *a) Existuje maximálně jedno regulární řešení úlohy (1)–(4).*

b) Existuje maximálně jedno regulární řešení úlohy (1)– (3) na celém prostoru.

J.Serrin dokázal následující výsledek v [24].

Věta 5 *Nechť* $\boldsymbol{u}$ *a* $\boldsymbol{w}$ *jsou slabá řešení úlohy (1) - (4) nebo (1) - (3) , řešíme-li úlohu na celém prostoru, se stejnými danými daty* $\boldsymbol{g}$ *a* $\boldsymbol{u}_0$*. Nechť dále* $\boldsymbol{u}$ *splňuje energetickou nerovnost a* $\boldsymbol{w}$ *je regulární řešení. Potom* $\boldsymbol{u} = \boldsymbol{w}$*.*

Tento výsledek rozšířili H.Kozono a H.Sohr v [12] Dokázali, že platí i tehdy, když $\boldsymbol{w} \in \boldsymbol{L}^\infty(0, T;\ \boldsymbol{L}^3(\Omega))$.

Jak jsme již zmínili, otázka regularity řešení je stále otevřená. Existuje sice mnoho dílčích výsledků, které ale poskytují pouze dílčí odpovědi. Jedním z dílčích výsledků je existence silného řešení na nějakém, libovolně malém, časovém intervalu. Tento výsledek dokázali A.A.Kiselev a O.A.Ladyženskaya (viz [11]).

Věta 6 *Nechť* $\partial\Omega \in \mathcal{C}^2, \boldsymbol{u}_0 \in \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)$ *a* $\boldsymbol{g} \in \boldsymbol{L}^\infty(0, T; H)$*. Pak existuje* $T^* > 0$ *a pouze jedno silné řešení (1) - (4) na* $(0, T^*)$*.*

Výsledek Kiseleva a Ladyženské rozšířil Y.Giga, který dokázal existenci regulárního řešení na nějakém, libovolně malém, časovém předpokladu za předpokladu, že $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega) \cap \boldsymbol{L}^3(\Omega)$.

V [9] a [18], J. Heywood a O.A. Ladyženskaya dokázali existenci regulárního řešení na pevně daném časovém intervalu s počáteční podmínkou, která je dostatečně malá v normě $\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)$.

Velmi důležitý výsledek je formulován v následující větě.

Věta 7 *Nechť $\boldsymbol{u}$ je slabé řešení s počáteční rychlostí $\boldsymbol{u}_0 \in \boldsymbol{L}^3(\Omega)$ na časovém intervalu $(0,T)$ tak, že*

$$\int_0^T \Big(\int_\Omega |\boldsymbol{u}|^q \Big)^{\frac{p}{q}} < \infty,$$

kde $q \in [3;\infty]$ a $2/p + 3/q = 1$. Pak $\boldsymbol{u}$ je regulární řešení na časovém intervalu $(0,T)$.

Toto kritérium nazýváme "Prodi-Serrinovými podmínkami". Bylo dokázáno v [23] a [24] pro $r \in (3;\infty]$ a v [7] pro $r = 3$. Na to navázala řada matematických prací. Existuje řada výsledků, kde je regularita řešení dokázána, pokud jedna nebo dvě složky rychlosti splňují toto kritérium nebo kritéria tohoto typu.

1.4 Řešení Navierových-Stokesových rovmic se smíšenovými okrajovými podmínkami

V hlavní části předkládané disertační práce se zabýváme systémem Navierových-Stokesových rovnic se smíšenými okrajovými podmínkami. V matematické literatuře se nejčastěji potkáváme s úlohami, kde je předepsána pouze jedna okrajová podmínka na celé hranici, nejčastěji jde o homogenní Dirichletovu podmínku. Tyto úlohy jsou poměrně hluboce rozpracovány. Pro takové úlohy je dokázána řada výsledků. Je dokázána globální existence slabých řešení. Jednoznačnost slabého řešení zůstává otevřeným problémem, ale je dokázána jednoznačnost regulárního řešení. Globální existence regulárního řešení zůstává stále otevřeným problémem, zde je dokázána globální existence silných řešení pro dostatečně malá počáteční data a existence regulárních řešení lokálně v čase..

Tyto úlohy neodpovídají vždy přesně fyzikální realitě. V poslední době autoři v mnoha článcích zabývajících se Navier-Stokesovou rovnicí řeší problém, kdy předepisují více typů okrajových podmínek na různých částech hranice. Tyto problémy lépe popisují fyzikální realitu. Modelujeme-li proudění vazké nestlačitelné tekutiny v kanále, je vhodné předepsat homogenní Dirichletovu okrajovou podmínku nebo Navierovu okrajovou podmínku na části hranice, která odpovídá pevné stěně. Na vstupu a výstupu z kanálu je přirozené předepsat okrajové podmínky (7) nebo (8).

Okrajové podmínky (7) a (8) však nemohou vyloučit existenci zpětných toků, které mohou přivést zpět do kanálu nekontrolované množství kinetické energie. Z tohoto důvodu neumíme pro řešení těchto úloh odvodit energetickou nerovnost nebo podobný odhad. Proto globální existence slabého řešení pro tyto úlohy a pro libovolná data zůstává otevřeným problémem.

Tato disertační práce byla motivována články, které tyto problémy řešily. Nyní některé z nich zmíníme.

Kračmar a Neustupa řešili tuto úlohu v [13] a v [14] pomocí vhodných variačních nerovností, kdy na hranici předepsali další okrajovou podmínku, která omezovala energii zpětného toku.

Kučera a Skalák v [17] řešili tuto úlohu pro třírozměrné proudění. Na pevné stěně předepsali homogenní Dirichletovu okrajovou podmínku, na vstupu nehomogenní Dirichletovu okrajovou podmínku a na výstupu okrajovou podmínku (7). Dokázali existenci a jednoznačnost řešení na nějakém malém časovém intervalu pro libovolná (velká) data problému. Na druhou stranu autoři požadovali velmi silné požadavky pro počáteční rychlost.

Podobný výsledek pro řešení Boussinesqových rovnic byl publikován v [26].

Kučera se v [15] zabýval řešením Navierovy-Stokesovy úlohy, kde je na pevné stěně kanálu předepsána okrajová podmínka (4) a na vstupu a výstupu je předepsána okrajová podmínka (7). Autor tuto úlohu přeformuloval do tvaru operátorů, které operují mezi Banachovým prostorem X (prostor řešení) a Banachovým prostorem Y (prostor dat). Pro daný operátor N, který odpovídal Navierově-Stokesově rovnici, ukázal, že $N(X)$ je neprázdná otevřená množina v prostoru Y.

Beneš a Kučera dále rozšířili některé z těchto výsledků na řešení Navierových-Stokesových rovnic, které jsou "regulárnější"v prostoru. Pro trojrozměrné problémy byl tento výsledek publikován v [2].

Kučera dále rozšířil výsledek publikovaný v [15] na řešení časově periodických Navier-Stokesových rovnic se stejnými okrajovými podmínkami. Tento výsledek byl publikován v [16].

Beneš a Kučera se zabývali v [3] dvourozměrným Navier-Stokesovým problémem se smíšenými okrajovými podmínkami kanálu. Na pevné stěně kanálu je předepsána okrajová podmínka (4) a na vstupu a výstupu kanálu je předepsána okrajová podmínka (7) . Zde je také dokázán velmi důležitý výsledek pro řešení ustáleného Stokesova problému se smíšenými okrajovými podmínkami a dostatečně hladkými daty.

V první a druhé kapitole této disertační práce navazujeme na výsledky uvedené v tomto odstavci. Ve druhé kapitole se zabýváme systémem dvourozměrného Stokesova problému se smíšenými okrajovými podmínkami. Předepisujeme Navierovu hranici na pevné stěně a okrajovou podmínku (7) na vstupu a výstupu kanálu. Dokazujeme hladkost řešení této úlohy v okolí bodu, ve kterém okrajové podmínky mění svůj typ. Ve druhé kapitole dokazujeme lokální existenci v čase řešení tohoto problému pro širokou třídu počátečních rychlostí.

1.5 Matematické modely proudění vazkých nestlačitelných tepelně vodivých tekutin se smíšenovými okrajovými podmínkami

Nechť $\Omega \subset \mathcal{R}^2$ je omezená oblast která představuje kanál vyplněný tepelně vodivou, nestlačitelnou, vazkou tekutinou. Připomeňme si, že $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, kde $\Gamma_D \cap \Gamma_N = \emptyset$. Γ_D představuje pevnou stěnu kanálu, kde předepisujeme Navierovu okrajovou podmínku pro rychlost a Newtonovu okrajovou podmínku pro teplotu. Γ_N představuje vstup a výstup kanálu, kde předepisujeme okrajovou podmínku (7) pro rychlost a tlak a Neumannovu okrajovou podmínku pro teplotu. Γ_D a Γ_N se skládají z úseček a svírají pravý úhel ve všech bodech, ve kterých okrajové podmínky mění svůj typ. Uvažujeme následující soustavu rovnic

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \mathcal{P} = (1-\theta)\boldsymbol{f} \qquad \text{v } \Omega \times (0,T), \tag{9}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{v } \Omega \times (0,T), \tag{10}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla\theta - \Delta\theta = \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{u}) + \theta \boldsymbol{f} \cdot \boldsymbol{u} + h \qquad \text{v } \Omega \times (0,T), \tag{11}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [2\boldsymbol{e}(\boldsymbol{u})\boldsymbol{n}] \cdot \boldsymbol{\tau} + \gamma \boldsymbol{u} \cdot \boldsymbol{\tau} = 0 \qquad \text{na } \Gamma_S \times (0,T), \tag{12}$$

$$-\frac{\partial \theta}{\partial \boldsymbol{n}} = \theta - \theta_\infty \qquad \text{na } \Gamma_S \times (0,T), \tag{13}$$

$$-\mathcal{P}\boldsymbol{n} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \boldsymbol{0} \qquad \text{na } \Gamma_S \times (0,T), \tag{14}$$

$$\frac{\partial \theta}{\partial \boldsymbol{n}} = 0 \qquad \text{na } \Gamma_N \times (0,T), \tag{15}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{v } \Omega, \tag{16}$$

$$\theta(0) = \theta_0 \qquad \text{v } \Omega. \tag{17}$$

Symboly $\boldsymbol{u}$, $\mathcal{P}$ a $\boldsymbol{u}_0$ mají stejný význam jako v předchozí části. Symboly θ, θ_0, θ_∞, h a $\boldsymbol{f}$ označují po řadě teplotu, počáteční teplotu, vnější teplotu, danou funkci a dvourozměrný vektor. Teplota ϑ je další neznámá funkce, $\boldsymbol{e}(u) = \big(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\big)/2$.

Odpovídající stacionární systém byl studován v [1]. Autoři zde předepsali homogenní Dirichletovu okrajovou podmínku pro rychlost a nehomogenní Dirichletovu okrajovou podmínku pro teplotu na pevné stěně. Na vstupu a výstupu předepsali okrajovou podmínku (8) pro rychlost a tlak a nehomogenní Neumannovu podmínku pro teplotu. Autoři dokázali, že pro dostupná data existuje řešení tohoto problému.

V této disertační práci dokazujeme lokální existenci slabého řešení problému (9)–(17).

2 Chapter 1

2.1 Regularita řešení Stokesých rovnic se smíšenými okrajovými podmínkami.

Převážná část této kapitoly je tvořena konferenčním příspěvkem [4]:

Michal Beneš, Petr Kučera, Petra Vacková: Existence and regularity of the Stokes system with the do-nothing and Navier's boundary conditions. Proceedings of the 19th Conference on Applied Mathematics APLIMAT 2020 (2020), pp. 47–56.

Autorský podíl M. Beneše a P. Kučery je 20% and P. Vackové 60%. Text má vlastní značení, definice, věty a literaturu. Na stránkách je uvedeno dvojí číslování, číslo stránky ve sborníku konference a číslo stránky v disertační práci.

V tomto článku se zabýváme dvoudimenzionální Stokesovou úlohou na omezené oblasti se smíšenými okrajovými podmínkami. Na části hranice předepisujeme Navierovu okrajovou podmínku a na části hranice okrajovou podmínku (7). V článku je studována lokální regularita řešení této úlohy na okolí hraničního bodu, kde se mění okrajové podmínky. Získané poznatky nám umožní dokázat regularitu řešení na celé oblasti.

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EXISTENCE AND REGULARITY OF THE STOKES SYSTEM WITH THE DO-NOTHING AND NAVIER'S BOUNDARY CONDITIONS

BENEŠ Michal (CZ), KUČERA Petr (CZ), VACKOVÁ Petra (CZ)

Abstract. In this paper we study qualitative properties of weak solutions of the Stokes equations with the mixed boundary conditions in bounded domains. We prescribe "do–nothing" boundary conditions and Navier's boundary conditions in two different parts of the boundary. Our aim is to study regularity of solutions in the neighbourhood of points in which boundary conditions change their type.

Keywords: Stokes equations, do–nothing boundary conditions, Navier's boundary conditions

Mathematics subject classification: Primary 35Q30, 35B35; Secondary 76D05, 76E09

1 Introduction

Mathematical models of viscous incompressible flow have been studied by many mathematicians in recent decades. Usually the flow is modeled as a solution of the system of the Navier-Stokes equations. If we solve this system in a bounded domain, we usually prescribe the Dirichlet boundary condition on the boundary. The theory of the Navier-Stokes equations with these boundary conditions is deeply elaborated. Unfortunately, these conditions are not natural in some situations. When we model fluid flow in a channel, we can prescribe either a boundary condition

$$\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - \mathcal{P}\boldsymbol{n} = \boldsymbol{\sigma}$$

or the condition

$$\frac{\nu}{2}\left[\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right] \cdot \boldsymbol{n} - \mathcal{P}\boldsymbol{n} = \boldsymbol{\sigma}$$

on the input and on the output of the channel (see e.g. [3], [16]), where $\boldsymbol{u} = (u_1, u_2)$ denotes the velocity field, $\mathcal{P}$ is the appropriate pressure, ν is the kinematic viscosity, $\boldsymbol{\sigma}$ is a prescribed vector function on the input and on the output of the channel and $\boldsymbol{n}$ is the outer unit normal vector. Unfortunately, these boundary conditions cannot exclude eventual backflows that can bring back to the channel an uncontrolled amount of kinetic energy to the channel. Consequently, these boundary conditions do not enable us to derive a priori estimate of a weak solution. Therefore, the question of the existence of the weak solution for arbitrary data remains open. Some qualitative properties of the solution of the Navier–Stokes equations with the boundary condition (1) on the input and on the output are studied in [8], [9], [10], [11] and [12].

Dirichlet boundary conditions are usually prescribed on the part of the boundary that corresponds to the fixed wall of the channel. These boundary conditions model fluid adherence to a solid wall. The behavior of a fluid in contact with a solid wall can also be modeled by the boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \;\; (\mathbb{T}_d \cdot \boldsymbol{n})_\tau + \gamma \boldsymbol{u} = 0.$$

These boundary conditions are called Navier's boundary conditions. They have been formulated in 1824 by H. Navier. The dynamic stress tensor associated with the velocity field $\boldsymbol{u}$ is denoted by $\mathbb{T}_d(\boldsymbol{u})$. $(\mathbb{T}_d(\boldsymbol{u}) \cdot \boldsymbol{n})_\tau$ means the tangential component of the vector $\mathbb{T}_d(\boldsymbol{u}) \cdot \boldsymbol{n}$. We suppose that the considered fluid is Newtonian, consequently $\mathbb{T}_d(\boldsymbol{u}) = \nu \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \right)$. Recall that the whole stress tensor is $\mathbb{T} = \mathbb{T}(\boldsymbol{u}, \mathcal{P}) = -\mathcal{P}\, \mathbb{I} + \mathbb{T}_d(\boldsymbol{u})$. Coefficient of friction between the fluid and the boundary is denoted by γ and we suppose $\gamma > 0$. Regularity of solutions of the Navier-Stokes equations with various types of boundary conditions for sufficiently smooth data is one of their important properties. It can be easily shown that the regularity of the solution of Navier-Stokes equations corresponds to the regularity of the solution of the Stokes equations.

In this paper we solve the steady Stokes problem with the mixed boundary condition. We prescribe two types of boundary conditions, Navier's boundary conditions and "do-nothing" boundary condition. We prove existence and uniqueness of the solution of this system and its regularity. We study the properties of the solution in the neighbourhood of points, where Navier's boundary conditions and "do-nothing" boundary condition are changed.

1.1 Description of the domain

In this subsection we describe the domain Ω which represents the channel filled up by a moving fluid.

- $\Omega \subset \mathbb{R}^2$ is a bounded domain.
- $\Gamma_1, \Gamma_2 \subset \partial\Omega$ such that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$.
- One–dimensional measure of Γ_1 is positive.
- All parts of Γ_1 are in parts smooth curves.
- All adjacent smooth parts of Γ_1 takes right angle.
- Quadratic form corresponding with symmetric matrix $\nabla \boldsymbol{n} + \nabla \boldsymbol{n}^T$ is seminegative in every point of Γ_1.
- All parts of Γ_2 are abscissas.
- $\partial\Omega - (\Gamma_1 \cup \Gamma_2) = \{A_1; \dots A_n\}$.
- Γ_1 and Γ_2 take right angle in all points $A_1, \dots A_n$.
- We prescribe the Navier boundary condition on Γ_1 and "do nothing" boundary condition on Γ_2.

1.2 Clasical formulation of the problem

The classical formulation of our problem is as follows.

$$\begin{aligned} -\nu\,\Delta \boldsymbol{u} + \nabla \mathcal{P} &= \boldsymbol{f} \quad \text{in } \Omega, &(1)\\ div\, \boldsymbol{u} &= 0 \quad \text{in } \Omega, &(2)\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \ [(\mathbb{T}_d \cdot \boldsymbol{n})]_\tau + \gamma \boldsymbol{u} &= 0 \quad \text{on } \Gamma_1, &(3)\\ -\mathcal{P}\boldsymbol{n} + \nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} &= 0 \quad \text{on } \Gamma_2, &(4) \end{aligned}$$

where $\boldsymbol{f} \in L^2(\Omega)^2$ is a body force. For simplicity we suppoose $\nu = 1$.

Remark 1 *The system (1)–(2) is called the steady Stokes system. Let $g \in W^{1,2}(\Omega)$. The system with equation*

$$div\, \boldsymbol{u} = g \quad \text{in } \Omega$$

instead of equation (2) is called the generalised Stokes system.

1.3 Notation definition of some function spaces

- We denote two-dimensional vector functions and spaces of these functions by bold letters.
- $\mathrm{B}_\tau(P)$ denotes the ball of radius τ centered at the point P
- Let $\mathcal{E}(\overline{\Omega}) := \{\boldsymbol{u} \in \mathcal{C}^\infty(\overline{\Omega})^2;\ div\, \boldsymbol{u} = 0,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_1\}$.
- Let $k \in \mathbb{N}$, $1 < p < \infty$. We define $\boldsymbol{W}^{k,p}_\kappa(\Omega)$ as the closure of $\mathcal{E}(\overline{\Omega})$ in the norm of $\boldsymbol{W}^{k,p}(\Omega)$.
- The dual space of $\boldsymbol{W}^{k,p}(\Omega)$ we denote by $\boldsymbol{W}^{k,p}(\Omega)^*$.
- For simplicity we use $\boldsymbol{L}^p_\kappa(\Omega)$ instead of $\boldsymbol{W}^{0,p}_\kappa(\Omega)$.
- The scalar product of the space $\boldsymbol{W}^{1,2}_\kappa(\Omega)$ we denote by $(\!(.,.)\!)_{1,2,\kappa}$. The norm of this space we denote by $\|.\|_{1,2,\kappa}$

2 Weak formulation of the problem

Definition 1 *Let $\boldsymbol{f} \in \boldsymbol{W}^{1,2}_\kappa(\Omega)^*$. We say that $\boldsymbol{u} \in \boldsymbol{W}^{1,2}_\kappa(\Omega)$ is a weak solution of the problem (1)–(4) iff*

$$\int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \int_{\Gamma_1} \gamma\, \boldsymbol{u}\cdot\boldsymbol{v} - \int_{\Gamma_1} \boldsymbol{u}\cdot\nabla\boldsymbol{n}\cdot\boldsymbol{v}^T = \langle \boldsymbol{f};\ \boldsymbol{v}\rangle \tag{5}$$

for every $\boldsymbol{v} \in \boldsymbol{W}^{1,2}_\kappa(\Omega)$.

Let $\mathcal{A} : \boldsymbol{W}^{1,2}_\kappa(\Omega) \times \boldsymbol{W}^{1,2}_\kappa(\Omega) \to \boldsymbol{W}^{1,2}_\kappa(\Omega)^*$ be a bilinear form such that

$$\mathcal{A}(\boldsymbol{u};\boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \int_{\Gamma_1} \gamma\, \boldsymbol{u}\cdot\boldsymbol{v} - \int_{\Gamma_1} \boldsymbol{u}\cdot\nabla\boldsymbol{n}\cdot\boldsymbol{v}^T.$$

It is easy to see that the bilinear form $\mathcal{A}$ is continuous. For $\boldsymbol{u} = \boldsymbol{v}$ we get

$$\mathcal{A}(\boldsymbol{u};\boldsymbol{u}) = \int_{\Omega} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \int_{\Gamma_1} \gamma\, \boldsymbol{u} \cdot \boldsymbol{u} - \frac{1}{2} \int_{\Gamma_1} \boldsymbol{u}^T \cdot \left(\nabla \boldsymbol{n} + \nabla \boldsymbol{n}^T \right) \cdot \boldsymbol{u}.$$

By assumption one–dimensional measure of Γ_1 is positive and $\left(\nabla \boldsymbol{n} + \nabla \boldsymbol{n}^T \right)$ is seminegative on Γ_1. These facts combined with the Friedrichs inequality give the following inequalities

$$c\, \|\boldsymbol{u}\|^2_{\boldsymbol{W}^{1,2}_{\kappa}(\Omega)} \leq \int_{\Omega} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \int_{\Gamma_1} \gamma\, \boldsymbol{u} \cdot \boldsymbol{u} \leq \mathcal{A}(\boldsymbol{u};\boldsymbol{u}).$$

Applying the Lax-Milgram theorem we obtain the following result.

Theorem 1 *Let $\boldsymbol{f} \in \boldsymbol{W}^{1,2}_{\kappa}(\Omega)^*$. There exists a unique weak solution of the problem (1)–(4).*

We introduce one example of the solution of the problem (1)–(4) now.

Example 1 *Suppose that $a,\ b,\ c,\ \varepsilon$ are positive numbers, such that $c/b^2 > \varepsilon$. Let*

$$\Omega = (-a;\ a) \times (-b;\ b)$$

with the boundary

$$\partial\Omega = \Gamma_1 \cup \Gamma_2,$$

where

$$\Gamma_1 = \{[x;\ -b];\ x \in [-a;\ a]\} \cup \{[x;\ b];\ x \in [-a;\ a]\}$$

and

$$\Gamma_2 = \{[-a;\ y];\ y \in [-b;\ b]\} \cup \{[a;\ y];\ y \in [-b;\ b]\}.$$

It can be easily verified that the pair $\{\boldsymbol{u};\ \mathcal{P}\}$ where the velocity field $\boldsymbol{u} = \boldsymbol{u}(x,y) = (c - \varepsilon\, y^2;\ 0)$ and the pressure $\mathcal{P} \equiv 0$ is the solution of the problem (1)–(4) with the right hand side $\boldsymbol{f} \equiv -2\,\varepsilon$ and the coefficient $\gamma = b\,\varepsilon/(c - \varepsilon b^2)$.

3 Regularity of weak solutions of the Stokes system

In this section we prove the theorem about regularity of the weak solution of the problem (1)–(4), which is the main result of the paper.

Theorem 2 *Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{W}^{1,2}_{\kappa}(\Omega)$, $1 < p < \infty$, be a weak solution of the problem (1)–(4). Then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and the corresponding pressure $\mathcal{P} \in \boldsymbol{W}^{1,p}(\Omega)$.*

In the following three remarks we suppose that all assumptions and notation of Theorem 2 hold.

Remark 2 *Let $G \subset \overline{G} \subset \Omega$. It is proved in [2] that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(G)$ and $\mathcal{P} \in \boldsymbol{W}^{1,p}(G)$.*

Remark 3 *We suppose that $B_1 \in \Gamma_1$ and $B_2 \in \Gamma_2$ are abitrary points. Then there exists open sets U_1 and U_2 such that $B_1 \in U_1$, $B_2 \in U_2$, $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega_1)$ and $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega_2)$, where $\Omega_1 = \Omega \cap U_1$ and $\Omega_2 = \Omega \cap U_2$. The corresponding property is proved in [1, Appendix A]. This result we can prove in a similar way with small technical difficulties.*

Remark 4 *Suppose that two smooth parts of Γ_1 form right angle at some point B_3. Then there exist open set U_3 that $B_3 \in U_3$ and $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega_3)$ and, where $\Omega_3 = \Omega \cap U_3$ (see e.g.[14]).*

Let $A \in \partial\Omega - (\Gamma_1 \cup \Gamma_2)$. This means $A \in \partial\Omega$ is the point in which boundary conditions change their type. By assuption Γ_1 and Γ_2 form right angle at the point A. To prove Theorem 2 we have to prove that $\boldsymbol{u}$ is smooth in some open neighbourhood of A.

We use the so called Kondrat'ev method. This method is developed in [4], [5], [6], [7], [13] and [15]. In [1, Appendix A] the authors used this method to prove regularity of the weak solution of the Stokes system with the homogeneous Dirichlet boundary conditions on the fixed wall of the channel and with the "do–nothing" boundary condition on the input and on the output of the channel.

We gradually define the so-called pencil operator $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}(\lambda)$ and investigate the structure of its eigenvalues. Using these results and applying [7, Theorems 1.4.3. and 1.4.4] and [1, Theorem A.4] we prove our result.

For simplicity we suppose that the origin of the coordinate system O is identical with the point A and that Γ_1 and Γ_2, respectively, coincide with positive parts of the axis x and y in $\Omega \cap \mathrm{B}_\epsilon(A)$ for sufficiently small ϵ.

Let $\eta(|\boldsymbol{x}|) \in C^\infty(\mathbb{R}^2)$, $0 \leq \eta(|\boldsymbol{x}|) \leq 1$,

$$\eta(|\boldsymbol{x}|) = \quad \left\{ \begin{array}{lll} 1 & \text{for} & |\boldsymbol{x}| < \epsilon/2, \\ 0 & \text{for} & |\boldsymbol{x}| > \epsilon. \end{array} \right\}$$

(η is a called cut-off function). Denote $\ddot{\boldsymbol{u}} = \eta\, \boldsymbol{u}$, $\ddot{\mathcal{P}} = \eta\, \mathcal{P}$ and $\mathcal{K}$ an infinite angle with the vertex O and size $\frac{\pi}{2}$. Then (1)–(3) yield

$$-\Delta(\ddot{\boldsymbol{u}}) + \nabla\ddot{\mathcal{P}} \;=\; \ddot{\boldsymbol{f}} \quad \text{in } \mathcal{K} \tag{6}$$

$$div\, \ddot{\boldsymbol{u}} \;=\; \ddot{h} \quad \text{in } \mathcal{K} \tag{7}$$

where $\ddot{\boldsymbol{f}} = \boldsymbol{f} - 2\nabla\boldsymbol{u}\, \nabla\eta - \boldsymbol{u}\, \Delta\eta + \mathcal{P}\, \nabla\eta \in \boldsymbol{L}^2(\mathcal{K})$, $\ddot{h} = \boldsymbol{u} \cdot \nabla\eta \in W^{1,2}(\mathcal{K})$.

Note that the behavior of $\boldsymbol{u}$ and $\mathcal{P}$ near O characterizes the regularity of $\boldsymbol{u}$ and $\mathcal{P}$ in a neighbourhood of A.

In the polar coordinates $(r,\ \omega)$, $r \in \langle 0;\ \infty)$, $\omega \in \langle 0;\ \frac{\pi}{2}\rangle$, the system (6)–(7) becomes

$$-\left(\frac{\partial^2 \bar{u}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{u}_1}{\partial \omega^2}\right) + \frac{\partial \bar{\mathcal{P}}}{\partial r}\cos\omega - \frac{1}{r}\frac{\partial \bar{\mathcal{P}}}{\partial \omega}\sin\omega \;=\; \bar{f}_1, \tag{8}$$

$$-\left(\frac{\partial^2 \bar{u}_2}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{u}_2}{\partial \omega^2}\right) + \frac{\partial \bar{\mathcal{P}}}{\partial r}\sin\omega + \frac{1}{r}\frac{\partial \bar{\mathcal{P}}}{\partial \omega}\cos\omega \;=\; \bar{f}_2, \tag{9}$$

$$\frac{\partial \bar{u}_1}{\partial r}\cos\omega - \frac{1}{r}\frac{\partial \bar{u}_1}{\partial \omega}\sin\omega + \frac{\partial \bar{u}_2}{\partial r}\sin\omega + \frac{1}{r}\frac{\partial \bar{u}_2}{\partial \omega}\cos\omega \;=\; \bar{h}. \tag{10}$$

We use the substitution $r = e^\tau$, $\tau \in \mathbb{R}$, and put $Q = \mathcal{P}e^\tau$. The system (8)–(10) yields

$$\frac{\partial^2 \tilde{u}_1}{\partial \tau^2} - \frac{\partial^2 \tilde{u}_1}{\partial \omega^2} + \cos\omega \left(\frac{\partial Q}{\partial \tau} - Q\right) - \sin\omega \frac{\partial Q}{\partial \omega} = \tilde{f}_1, \tag{11}$$

$$\frac{\partial^2 \tilde{u}_2}{\partial \tau^2} - \frac{\partial^2 \tilde{u}_2}{\partial \omega^2} + \sin\omega \left(\frac{\partial Q}{\partial \tau} - Q\right) + \cos\omega \frac{\partial Q}{\partial \omega} = \tilde{f}_2, \tag{12}$$

$$\cos\omega \frac{\partial \tilde{u}_1}{\partial \tau} - \sin\omega \frac{\partial \tilde{u}_1}{\partial \omega} + \frac{\partial \tilde{u}_2}{\partial \tau} \sin\omega + \frac{\partial \tilde{u}_2}{\partial \omega} \cos\omega = \tilde{h}. \tag{13}$$

This system holds in $\tilde{S} = \{(\tau,\ \omega);\ \tau \in \mathbb{R},\ 0 < \omega, \frac{\pi}{2}\}$.

We apply the Fourier transform with respect to τ for any $\lambda \in \mathbb{C}$ on the previous system $((\tau, \omega) \to (\lambda, \omega))$. We get following system:

$$-(i\lambda)^2 \hat{u}_1 - \frac{\partial^2 \hat{u}_1}{\partial \omega^2} + (i\lambda - 1)\cos\omega \hat{Q} - \sin\omega \frac{\partial \hat{Q}}{\partial \omega} = \hat{f}_1, \tag{14}$$

$$-(i\lambda)^2 \hat{u}_2 - \frac{\partial^2 \hat{u}_2}{\partial \omega^2} + (i\lambda - 1)\sin\omega \hat{Q} + \cos\omega \frac{\partial \hat{Q}}{\partial \omega} = \hat{f}_2, \tag{15}$$

$$i\lambda \cos\omega \hat{u}_1 - \sin\omega \frac{\partial \hat{u}_1}{\partial \omega} + i\lambda \sin\omega \hat{u}_2 + \cos\omega \frac{\partial \hat{u}_2}{\partial \omega} = \hat{h}. \tag{16}$$

Note that for arbitrary $\lambda \in \mathbb{C}$ we have $\hat{\boldsymbol{f}} \in \boldsymbol{L}^2((0;\ \frac{\pi}{2}))$ and $\hat{h} \in W^{1,2}((0;\ \frac{\pi}{2}))$.

Let $\mathcal{A}(\lambda) : \boldsymbol{W}^{2,2}((0;\pi/2))^2 \times W^{1,2}((0;\pi/2)) \to L^2((0;\pi/2))^2 \times W^{1,2}((0;\pi/2))$ be the matrix operator which corresponds to system (14)–(16) , i.e.

$$\mathcal{A}(\lambda) = \begin{pmatrix} -\frac{\partial}{\partial\omega^2} - (i\lambda)^2 & 0 & (i\lambda - 1)\cos\omega - \sin\omega\frac{\partial}{\partial\omega} \\ 0 & -\frac{\partial}{\partial\omega^2} - (i\lambda)^2 & (i\lambda - 1)\sin\omega + \cos\omega\frac{\partial}{\partial\omega} \\ (i\lambda)\cos\omega - \sin\omega\frac{\partial}{\partial\omega} & (i\lambda)\sin\omega + \cos\omega\frac{\partial}{\partial\omega} & 0 \end{pmatrix}. \tag{17}$$

We consider this operator for each parameter $\lambda \in \mathbb{C}$.

Now we did the same adjustment for the "do-nothing" boundary condition and Navier's condition. For the "do nothing" condition we suppose $\boldsymbol{n} = (0, -1)$ and $\omega = 0$. We get:

$$\frac{\partial \hat{u}_1}{\partial \omega} = 0, \tag{18}$$

$$\hat{Q} - \frac{\partial \hat{u}_2}{\partial \omega} = 0. \tag{19}$$

For the Navier's condition we suppose $\boldsymbol{n} = (-1, 0), \boldsymbol{t} = (0, 1)$ and $\omega = \frac{\pi}{2}$ and then we get:

$$\hat{u}_1 = 0 \tag{20}$$

$$\frac{\partial \hat{u}_2}{\partial \omega} = 0 \tag{21}$$

We define corresponding matrix operators $\mathcal{B}_1 = \mathcal{B}_1(\lambda)$ and $\mathcal{B}_2 = \mathcal{B}_2(\lambda)$ such that

$$\mathcal{B}_i : \boldsymbol{W}^{2,2}((0;\pi/2))^2 \times W^{1,2}((0;\pi/2)) \to \mathbb{C}^2$$

for $i=1,2$, respectively, which correspond to the boundary conditions (18)–(19) and (20)–(21).

$$\mathcal{B}_1(\lambda) = \begin{pmatrix} \frac{\partial}{\partial\omega}\big|_0 & 0 & 0 \\ 0 & \frac{\partial}{\partial\omega}\big|_0 & -1\big|_0 \end{pmatrix} \tag{22}$$

and

$$\mathcal{B}_2(\lambda) = \begin{pmatrix} 1\big|_{\frac{\pi}{2}} & 0 & 0 \\ 0 & \frac{\partial}{\partial\omega}\big|_{\frac{\pi}{2}} & 0 \end{pmatrix}. \tag{23}$$

The pencil operator $\widehat{\mathcal{L}}(\lambda) = \widehat{\mathcal{L}}(\lambda)(\lambda)$,

$$\widehat{\mathcal{L}}(\lambda) : W^{2,2}((0;\pi/2))^2 \times W^{1,2}((0;\pi/2)) \to L^2((0;\pi/2))^2 \times W^{1,2}((0;\pi/2)) \times \mathbb{C}^2 \times \mathbb{C}^2 \tag{24}$$

is the parameter dependent operator defined by

$$\widehat{\mathcal{L}}(\lambda) = [\mathcal{A}(\lambda);\, \mathcal{B}_1(\lambda);\, \mathcal{B}_2(\lambda)]\,. \tag{25}$$

$\widehat{\mathcal{L}}(\lambda)$ is considered for all $\lambda \in \mathbb{C}$ and it corresponds to the problem (14)–(16) with the boundary conditions (18)–(21).

Definition 2 *Suppose that $\lambda_0 \in \mathbb{C}$, $\widehat{\boldsymbol{F}}(.,\lambda_0) \in \mathcal{D}(\widehat{\mathcal{L}}(\lambda))$, $\widehat{\boldsymbol{F}}(.,\lambda_0) \neq \mathbf{0}$ and $\widehat{\boldsymbol{F}}(.,\lambda_0)$ is holomorphic at λ_0. We say that λ_0 is an eigenvalue of $\widehat{\mathcal{L}}(\lambda)$ and $\widehat{\boldsymbol{F}}(.,\lambda_0)$ the corresponding eigenfunction if*

$$\widehat{\mathcal{L}}(\lambda_0)\widehat{\boldsymbol{F}}(.,\lambda_0) = \mathbf{0}.$$

Definition 3 *Let λ_0 be an eigenvalue of $\widehat{\mathcal{L}}(\lambda)$. We say that it is a simple eigenvalue if*

$$\widehat{\mathcal{L}}'(\lambda_0)\widehat{\boldsymbol{F}}(.,\lambda_0) = \mathbf{0}$$

only for $\widehat{\boldsymbol{F}}(.,\lambda_0) = \mathbf{0}$.

The following theorem is the simplified version of Theorems 1.4.3 and 1.4.4 in [7].

Theorem 3 *Let $(\overline{\boldsymbol{\vartheta}}, \overline{q}) \in W^{1,2}(\Omega)^2 \times L^2(\Omega)$ be the weak solution of a generalized steady Stokes systems with a right hand side $\overline{\boldsymbol{\sigma}} = (\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3}) \in L^p(\Omega)^2 \times W^{1,p}(\Omega)$, $p > 1$, $A \in \partial\Omega$. Denote by $\widehat{\mathcal{B}} = \widehat{\mathcal{B}}(\lambda)$ its corresponding pencil operator. Then the following propositions hold:*

- *Assume that λ_0 is the only eigenvalue of $\widehat{\mathcal{B}}(\lambda)$ in the strip $Im\,\lambda \in (2/p-2, 0)$. Suppose additionally that this eigenvalue is simple. Assume that the lines $Im\,\lambda = 0$ and $Im\,\lambda = 2/p - 2$ are free of eigenvalues of the pencil operator $\widehat{\mathcal{B}}(\lambda)$. Then there exists a cut-off function $\eta = \eta(r)$ and $\delta > 0$ such that $(\overline{\boldsymbol{\vartheta}}, \overline{q}) = (\overline{\boldsymbol{\vartheta}}(r,\omega), \overline{q}(r,\omega))$ admits in a neighborhood $\mathcal{O}$ of the corner point A the asymptotic representation*

$$\eta(r)\begin{pmatrix} \overline{\boldsymbol{\vartheta}} \\ \overline{q} \end{pmatrix} = c\begin{pmatrix} \overline{\boldsymbol{\vartheta}}_{sing} \\ \overline{q}_{sing} \end{pmatrix} + \begin{pmatrix} \overline{\boldsymbol{\vartheta}}_{reg} \\ \overline{q}_{reg} \end{pmatrix}, \tag{26}$$

where $\left(\overline{\boldsymbol{\vartheta}}_{reg}, \overline{q}_{reg}\right) \in W^{2,p}(\Omega_\delta)^2 \times W^{1,p}(\Omega_\delta)$ *and* $\Omega_\delta = U_\delta(A) \cap \Omega$. *Constant* c *is called generalized intensity factor and the corresponding singular function is given by*

$$\begin{pmatrix} \overline{\boldsymbol{\vartheta}}_{sing} \\ \overline{q}_{sing} \end{pmatrix} = r^{i\lambda} \begin{pmatrix} \dot{\boldsymbol{\vartheta}} \\ r^{-1}\dot{q} \end{pmatrix},$$

where $(\dot{\boldsymbol{\vartheta}}, \dot{q}) = (\dot{\boldsymbol{\vartheta}}(\omega), \dot{q}(\omega))$ *is the corresponding eigenfunction of* $\widehat{\mathcal{B}}(\lambda_0)$.

- *Suppose that the line* $Im\,\lambda = 2/p - 2$ *does not contain eigenvalues of the pencil operator* $\widehat{\mathcal{L}}(\lambda)$ *and* $(\overline{\boldsymbol{\vartheta}}, \overline{q}) \in W^{2,p}(\Omega_\delta)^2 \times W^{1,p}(\Omega_\delta)$. *Then*

$$\|\overline{\boldsymbol{\vartheta}}\|_{W^{2,p}(\Omega_A)^2} + \|\overline{q}\|_{W^{1,p}(\Omega_A)} \leq c\,\|\overline{\boldsymbol{\sigma}}\|_{L^p(\Omega)^2}, \tag{27}$$

where $\Omega_A = U_\tau(A) \cap \Omega$ *for some* $\tau < \delta/2$ *and* $c = c(\Omega_\delta, \tau)$.

3.1 Calculation of the characteristic determinant

To be able to apply the previous theorem we have to find eigenvalues and corresponding eigenfunctions of our pencil operator. The general solution of the equation $\mathcal{A}(\lambda)(\hat{u}_1, \hat{u}_2, \hat{Q})^T = 0$ for $\lambda \neq 0$ is in the form $[\hat{u}_1, \hat{u}_2, \hat{Q}]^T = \Sigma_{i=1}^4 C_i \Phi_i(\lambda, \omega)$. The following is the corresponding fundamental system:

$$\Phi_1 = \begin{pmatrix} \cos(i\lambda\omega) \\ -\sin(i\lambda\omega) \\ 0 \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} \sin(i\lambda\omega) \\ \cos(i\lambda\omega) \\ 0 \end{pmatrix},$$

$$\Phi_3 = \begin{pmatrix} -\frac{i\lambda}{2}\cos[(i\lambda-2)\omega] \\ \sin(i\lambda\omega) + \frac{i\lambda}{2}\sin[(i\lambda-2)\omega] \\ -2i\lambda\cos[(i\lambda-1)\omega] \end{pmatrix}, \ \Phi_4 = \begin{pmatrix} \frac{i\lambda}{2}\sin[(i\lambda-2)\omega] \\ \cos(i\lambda\omega) + \frac{i\lambda}{2}\cos[(i\lambda-2)\omega] \\ 2i\lambda\sin[(i\lambda-1)\omega] \end{pmatrix}.$$

The general solution and the boundary conditions (18) – (21) goes to a homogeneous system of linear algebraic equations (which depends on λ). This system admits a nontrivial solution if and only if the corresponding determinant vanishes, i.e.

$$D(\lambda) = \begin{vmatrix} 0 & i\lambda & 0 & \frac{i\lambda}{2}(i\lambda-2) \\ -i\lambda & 0 & i\lambda(\frac{i\lambda}{2}+2) & 0 \\ \cos\frac{i\lambda\pi}{2} & \sin\frac{i\lambda\pi}{2} & \frac{i\lambda}{2}\cos\frac{i\lambda\pi}{2} & -\frac{i\lambda}{2}\sin\frac{i\lambda\pi}{2} \\ -i\lambda\cos\frac{i\lambda\pi}{2} & -i\lambda\sin\frac{i\lambda\pi}{2} & \frac{i\lambda}{2}(4-i\lambda)\cos\frac{i\lambda\pi}{2} & \frac{i\lambda}{2}(i\lambda-4)\sin\frac{i\lambda\pi}{2} \end{vmatrix} = 3i\lambda^3\sin i\lambda\pi = 0.$$

Consequently, $D(\lambda) = 0$ for $\lambda = ik$, $k \in \mathbb{Z}$, $k \neq 0$. For $\lambda = -i$ the corresponding eigenfunction is $(\cos\omega;\ -\sin\omega;\ -1)^T$. Consequently, corresponding singular function (see Theorem 3) is $(r\cos\omega;\ -r\sin\omega;\ -1)^T \approx (x;\ -y;\ 1)^T$ which belongs to $\boldsymbol{W}^{2,2}(\Omega)$. For $\lambda = 0$ we have the general solution in the form $[\hat{u}_1, \hat{u}_2, \hat{Q}]^T = \Sigma_{i=1}^4 C_i \Psi_i(0, \omega)$, where the following is the fundamental system:

$$\Psi_1 = \begin{pmatrix} \cos(2\omega) \\ -\sin(2\omega) - 2\omega \\ 4\cos\omega \end{pmatrix}, \Psi_2 = \begin{pmatrix} -\sin(2\omega) - 2\omega \\ \cos(2\omega) \\ 4\sin\omega \end{pmatrix}, \Psi_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \Psi_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The corresponding determinant has the form

$$D = \begin{vmatrix} 0 & -4 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ -1 & \pi & 1 & 0 \\ 0 & 2 & 0 & 0 \end{vmatrix}.$$

It follows that $D = 0$, the corresponding eigenfunction is $(0, 1, 0)^T$ and the corresponding singular function (see Theorem 3) is $(0;\ 1;\ 0)^T$ which also belongs to $\boldsymbol{W}^{2,2}(\Omega)$. This result combined with Theorem 3 gives Theorem 2.

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Current address

Beneš Michal, Ing., PhD.
Department of Mathematics
Faculty of Civil Engineering
Czech technical University in Prague
Thákurova 7, 166 29 Praha, Czech Republic
E-mail: michal.benes@cvut.cz

Kučera Petr, doc. RNDr., CSc.
Department of Mathematics
Faculty of Civil Engineering
Czech technical University in Prague
Thákurova 7, 166 29 Praha, Czech Republic
E-mail: petr.kucera@cvut.cz

Vacková Petra, Mgr.
Department of Mathematics
Faculty of Civil Engineering
Czech technical University in Prague
Thákurova 7, 166 29 Praha, Czech Republic
E-mail: petra.vackova@cvut.cz

3 Chapter 2

3.1 Existence řešení Navierových-Stokesových rovnic se smíšenými okrajovými podmínkami lokálně v čase.

Převážná část této kapitoly je tvořena článkem [5]:

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Autorský podíl M. Beneše a P. Kučery je 20% and P. Vackové 60%. Text má vlastní značení, definice, věty a literaturu. Na stránkách je uvedeno dvojí číslování, číslo stránky v časopise a číslo stránky v disertační práci.

V tomto článku je studován systém dvoudimenzionálních Navierových-Stokesových rovnic se smíšenými okrajovými podmínkami na omezené oblasti Ω. Tento systém modeluje dvoudimenzionální proudění tekutiny v kanálu, kdy na hranici, která odpovídá pevné stěně, je předepsána buď homogenní Dirichletova nebo Navierova okrajová podmínka a na hranici, která odpovídá vstupu a výstupu z kanálu, je předepsána "do-nothing" okrajová podmínka. Protože u této úlohy neumíme dokázat globální existenci slabého řešení, dokazujeme zde existenci řešení na nějakém, libovolně malém, časovém intervalu. Existence řešení je dokázána i za předpokladu, že počáteční rychlost náleží do prostoru, který je jen nepatrně silnější než $\boldsymbol{L}^2(\Omega)$.

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Local in time existence of solution of the Navier-Stokes equations with various types of boundary conditions

Michal Beneš[1] · **Petr Kučera**[1] · **Petra Vacková**[1]

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Abstract
In this paper we deal with the two-dimensional Navier-Stokes system with three types of boundary conditions, including the so called "do-nothing" boundary condition. We prove the local in time existence and uniqueness of a solution for the initial velocity, which can belong to a class of functions that can be at least a little stronger than $\boldsymbol{L}^2(\Omega)$.

Keywords Navier-Stokes equations · Mixed boundary conditions · Local in time existence of solutions · Navier's boundary conditions · Do-nothing boundary conditions

Mathematics Subject Classification Primary: 35Q30, 35 B35; secondary: 76D05, 76 E 09

1 Introduction

The Navier-Stokes equations are commonly solved on the whole space or on domains where we prescribe one type of boundary conditions. Mostly, no-slip boundary conditions or Navier's boundary conditions are prescribed on the boundary. Problems where we prescribe Dirichlet boundary conditions on the whole boundary are not always natural. When fluid flow in a finite channel is modelled, it is natural to prescribe either

$$\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - \mathcal{P}\boldsymbol{n} = \boldsymbol{g} \tag{1}$$

or

✉ Petr Kučera
petr.kucera@cvut.cz

[1] Department of Mathematics, Faculty of Civil Engineering, Czech Technical University in Prague, Prague, Czech Republic

$$\frac{\nu}{2}\Big((\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^{\top}\Big)\boldsymbol{n} - \mathcal{P}\boldsymbol{n} = \boldsymbol{g} \tag{2}$$

on the input and on the output of the channel. A Dirichlet boundary condition can be used on the fixed walls of the channel, but it cannot be prescribed on an assumed input and output because the velocity on the output depends on the flow in the whole channel and it is not known in advance. Unfortunately, boundary conditions (1) and (2) do not enable us to exclude backward flows that can bring an uncontrollable amount of kinetic energy into the channel. Therefore, we do not derive energy inequality when we solve this problem and the global existence of a weak solution to this problem remains an open problem. Consequently, we usually get two types of results. The first type is the existence of a solution on a sufficiently small time interval for arbitrary data. The second type of result concerns the existence of a solution on a fixed time interval. Specifically, suppose that a known solution of our problem is given. We can prove the existence of a solution for initial conditions with sufficiently small perturbations when the solution for the unperturbed initial conditions is known.

Some qualitative properties of the Navier-Stokes equations with the boundary condition (1) are studied for example in [2, 7, 8, 10]. In [7, 8], Kračmar and Neustupa prescribed an additional condition on the output (which bounds the kinetic energy of an eventual backward flow) and formulated steady and evolutionary Navier-Stokes problems by means of appropriate variational inequalities. In [10], Kučera and Skalák proved the local intime existence of a solution of the three-dimensional unsteady Navier-Stokes problem with boundary condition (1) on the part of the boundary. In [2], Beneš proved the local in time existence of the strong solution (in the sense that the solution possess second spatial derivatives) to the system (3)–(8). In [9], Kučera proved the global in time existence and uniqueness of a strong solution in a small neighbourhood of another known solution. Existence of solutions of problems with related boundary conditions for Navier-Stokes or Boussinesq equations are also studied for example in [1, 5, 12–14].

In this paper we deal with a two-dimensional problem, where we prescribe three types of boundary conditions on the boundary: homogeneous Dirichlet boundary conditions, Navier's boundary conditions and (1). We apply the results of the regularity of the solution to the neighbourhood of points where the boundary conditions change their type. We assume the parts of the boundary to be perpendicular at the points where the boundary conditions change their type. We use the known results of the regularity of a solution in the neighbourhood of points in which boundary conditions change their type. Then we obtain $W^{2,2}$-regularity of solutions for suitable data in the neighbourhood of these points. We prove the local in time existence of a solution for the initial velocity, which can belong to a class of functions that can be at least a little stronger than "$\boldsymbol{L}^2(\Omega)$".

Througout the paper, we denote by c a generic constant, i.e. a constant whose value may change from line to line. We admit that c may depend on Ω, but it never depends on a concrete function. On the other hand, numbered constant c_1, has fixed values throughout the whole paper.

Let $\boldsymbol{u} = (u_1, u_2)$ be a velocity field and $\boldsymbol{n} = (n_1, n_2)$ be a normal vector at some point of $\partial\Omega$. If we want to emphasize the matrix character of $\nabla\boldsymbol{u}$ and $\nabla\boldsymbol{n}$, we use symbols $(\nabla\boldsymbol{u})$ and $(\nabla\boldsymbol{n})$. Then $(\nabla\boldsymbol{u})_{i,j} = (\frac{\partial u_i}{\partial x_j})$ and $(\nabla\boldsymbol{n})_{i,j} = (\frac{\partial n_i}{\partial x_j})$. The transposed vectors to $\boldsymbol{u}$ and $\boldsymbol{n}$ and the transposed matrices to $(\nabla\boldsymbol{u})$ and $(\nabla\boldsymbol{n})$, respectively, are denoted by $\boldsymbol{u}^\top$, $\boldsymbol{n}^\top$, $(\nabla\boldsymbol{u})^\top$ and $(\nabla\boldsymbol{n})^\top$. Multiplication of the matrix $(\nabla\boldsymbol{u})$ and the transposed vector $\boldsymbol{n}^\top$ is denoted by $(\nabla\boldsymbol{u})\,\boldsymbol{n}^\top$. Scalar product of two vectors, $\boldsymbol{u}$ and $(\nabla\boldsymbol{u})\,\boldsymbol{n}^\top$ is denoted by $\boldsymbol{u}\cdot(\nabla\boldsymbol{u})\,\boldsymbol{n}^\top$.

We denote vector-valued functions and spaces of such functions by boldface letters with one exception. Bold marking is not used in subscripts of norms or scalar products. For example, the norm of the space $L^2(0,T;\,\boldsymbol{W}^{1,2}(\Omega))$ is denoted by $\|\cdot\|_{L^2(0,T;\;W^{1,2}(\Omega))}$.

1.1 Description of the domain

Let $\Omega \subset \mathrm{R}^2$ be a bounded domain which represents a channel filled up by a moving incompressible fluid. Suppose that

- $\Gamma_1, \Gamma_2, \Gamma_3 \subset \partial\Omega$, $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$.
- All parts of Γ_1 are smooth curves which belong to $\mathcal{C}^2$.
- Γ_1 is nonempty.
- All parts of Γ_2 and Γ_3 are line segments.
- $\partial\Omega - (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3) = \{\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n\}$, where $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n$ are points in which boundary conditions change their type.
- Γ_i and Γ_j, $i, j = 1, 2, 3$, are perpendicular in all points $\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n$.

1.2 Classical formulation of the problem

Let $T \in (0, \infty)$. We deal with the Navier-Stokes system

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\mathcal{P} = \boldsymbol{f} \quad \text{in } \Omega\times(0,T), \tag{3}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\times(0,T), \tag{4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1\times(0,T), \tag{5}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad [\mathbf{T}_d(\boldsymbol{u})\cdot\boldsymbol{n}]_\tau + \gamma\,\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_2\times(0,T), \tag{6}$$

$$-\mathcal{P}\boldsymbol{n} + \nu\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \boldsymbol{0} \quad \text{on } \Gamma_3\times(0,T), \tag{7}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{8}$$

Functions $\boldsymbol{u}$, $\mathcal{P}$, $\boldsymbol{f}$ and $\boldsymbol{u}_0$ are smooth enough, $\mathcal{P}$ represents pressure, $\boldsymbol{f}$ is body force and ν denotes the viscosity of the fluid. Recall that $\boldsymbol{u} = (u_1, u_2)$ is velocity field and $\boldsymbol{n} = (n_1, n_2)$ is a normal vector at some point of $\partial\Omega$. For simplicity we suppose that $\nu = 1$ throughout the paper.

Conditions (6) are the so called *Navier's boundary conditions*. The second condition claims that the tangential component of the stress is proportional to the velocity. The dynamic stress tensor associated with the velocity field $\boldsymbol{u}$ is denoted by $\mathbf{T}_d(\boldsymbol{u})$. The subscript τ denotes the tangential component. We suppose that the considered fluid is Newtonian, consequently $\mathbf{T}_d(\boldsymbol{u}) = 2\nu\,(\nabla\boldsymbol{u})_s$, where $(\nabla\boldsymbol{u})_s$ is the symmetric part of $\nabla\boldsymbol{u}$. Recall that the "whole" stress tensor is $\mathbf{T} = \mathbf{T}(\boldsymbol{u}, \mathcal{P}) = -\mathcal{P}\mathbf{I} + \mathbf{T}_d(\boldsymbol{u})$. Factor γ is the coefficient of friction between the fluid and the boundary. γ is supposed to be constant, $\gamma > 0$.

Note that the simplified system is the so-called non-stationary Stokes system, where Eq. (1) is replaced by

$$\frac{\partial\boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + \nabla\mathcal{P} = \boldsymbol{f} \quad \text{in } \Omega\times(0,T), \tag{9}$$

Eqs. (4)–(8) are the same as in the Navier-Stokes system.

1.3 Weak formulation of the problem

Let $\mathcal{E}(\overline{\Omega}) := \left\{\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega})^2; div\,\boldsymbol{v} = 0,\ supp\,\boldsymbol{v}\cap\overline{\Gamma_1} = \emptyset,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_2\right\}$.

The closure of $\mathcal{E}(\overline{\Omega})$ in the norm of $\boldsymbol{W}^{k,p}(\Omega)\ \big(= W^{k,p}(\Omega)^2\big)$ for $k \ge 0$ (k need not be an integer) and $1 \le p < \infty$ is denoted by $\boldsymbol{W}^{k,p}_\kappa$. Then $\boldsymbol{W}^{k,p}_\kappa$ is the Banach space. For simplicity, the space $\boldsymbol{W}^{0,2}_\kappa$ is denoted by $\boldsymbol{L}^2_\kappa$. Note that $\boldsymbol{L}^2_\kappa$ is the closed subspace of $\boldsymbol{L}^2(\Omega)$. The scalar product on $\boldsymbol{L}^2(\Omega)$ is denoted by $(\!(\cdot,\cdot)\!)_{L^2(\Omega)}$. The symbol $\langle\cdot,\cdot\rangle$ denotes duality between $\boldsymbol{W}^{1,2}_\kappa$ and $\big(\boldsymbol{W}^{1,2}_\kappa\big)^*$.

Definition 1 Let $T > 0$, $\boldsymbol{f} \in L^2\big(0,T;\big(\boldsymbol{W}^{1,2}_\kappa\big)^*\big)$ and $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$. We call a function $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}^{1,2}_\kappa)$ a *weak solution* of the problem (3)–(8) if

$$\int_0^T\int_\Omega\big[-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \nabla\boldsymbol{u}\nabla\boldsymbol{\phi} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}\big] + \gamma\int_0^T\int_{\Gamma_2}\boldsymbol{u}\cdot\boldsymbol{\phi}$$
$$-\int_{\Gamma_2}\boldsymbol{\phi}\cdot(\nabla\boldsymbol{n})^\top\boldsymbol{u}^\top - \int_\Omega\boldsymbol{u}_0\cdot\boldsymbol{\phi}(0) = \int_0^T\langle\boldsymbol{f},\boldsymbol{\phi}\rangle$$

for all $\boldsymbol{\phi} \in C^\infty([0,T];\boldsymbol{W}^{1,2}_\kappa)$ such that $\boldsymbol{\phi}(T) = \mathbf{0}$.

1.4 Auxiliary results

Let $\boldsymbol{u} \in L^2(0,T;\boldsymbol{W}^{2,2}_\kappa)$, $\mathcal{P} \in L^2(0,T;W^{1,2}(\Omega))$ be solution of (3)–(7) and $\boldsymbol{v} \in \boldsymbol{W}^{1,2}_\kappa$. If we multiply the term $(-\Delta\boldsymbol{u} + \nabla\mathcal{P}\big)$ by $\boldsymbol{v}$, we obtain the following identity for almost every $t \in (0,T)$. For simplicity we use $\boldsymbol{u}$ and $\mathcal{P}$, respectively, instead of $\boldsymbol{u}(t)$ and $\mathcal{P}(t)$.

$$\int_\Omega \left(-\Delta \boldsymbol{u} + \nabla \mathcal{P}\right) \cdot \boldsymbol{v} = \int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{v} - \int_{\partial\Omega} \boldsymbol{v} \cdot (\nabla \boldsymbol{u}) \boldsymbol{n}^\top + \int_{\Gamma_3} \mathcal{P} \boldsymbol{v} \cdot \boldsymbol{n}$$
$$= \int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{v} - \int_{\Gamma_2} \boldsymbol{v} \cdot (\nabla \boldsymbol{u}) \boldsymbol{n}^\top + \int_{\Gamma_3} \left(\mathcal{P} \boldsymbol{n} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\right) \cdot \boldsymbol{v}$$
$$= \int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{v} - \int_{\Gamma_2} \boldsymbol{v} \cdot \left((\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^\top\right) \boldsymbol{n}^\top + \int_{\Gamma_2} \boldsymbol{v} \cdot (\nabla \boldsymbol{u})^\top \boldsymbol{n}^\top$$
$$= \int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{v} + \gamma \int_{\Gamma_2} \boldsymbol{u} \cdot \boldsymbol{v} + \int_{\Gamma_2} \boldsymbol{v} \cdot (\nabla \boldsymbol{u})^\top \boldsymbol{n}^\top$$

Since $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_2 and all parts of Γ_2 are line segments we obtain $\int_{\Gamma_2} \boldsymbol{v} \cdot (\nabla \boldsymbol{u})^\top \boldsymbol{n}^\top = 0$. Consequently

$$\int_\Omega \left(-\Delta \boldsymbol{u} + \nabla \mathcal{P}\right) \cdot \boldsymbol{v} = \int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{v} + \gamma \int_{\Gamma_2} \boldsymbol{u} \cdot \boldsymbol{v}.$$

This identity motivates us to define the bilinear form $(\!(.,.)\!)_\kappa$ on $\boldsymbol{W}_\kappa^{1,2} \times \boldsymbol{W}_\kappa^{1,2}$,

$$(\!(\boldsymbol{\phi}, \boldsymbol{v})\!)_\kappa := \int_\Omega \nabla \boldsymbol{\phi} \nabla \boldsymbol{v} + \gamma \int_{\Gamma_2} \boldsymbol{\phi} \cdot \boldsymbol{v}. \tag{10}$$

It is easy to see that this bilinear form is symmetric and $\boldsymbol{W}_\kappa^{1,2}$-elliptic. Let

$$\boldsymbol{D} := \{\boldsymbol{\phi};\ \text{there exists}\ \boldsymbol{f} \in \boldsymbol{L}_\kappa^2\ \text{such that}\ (\!(\boldsymbol{\phi}, \boldsymbol{v})\!)_\kappa = (\!(\boldsymbol{f}, \boldsymbol{v})\!)_{L^2(\Omega)}\ \text{for every}\ \boldsymbol{v} \in \boldsymbol{W}_\kappa^{1,2}\}. \tag{11}$$

It can be shown as in [17, Chapter I., 2.6] that there exist functions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots \in \boldsymbol{W}_\kappa^{1,2} \subset \boldsymbol{L}_\kappa^2$ and real positive numbers $\lambda_1, \lambda_2, \ldots \nearrow \infty$ such that

$$(\!(\boldsymbol{\phi}_k, \boldsymbol{v})\!)_\kappa = \lambda_k (\!(\boldsymbol{\phi}_k, \boldsymbol{v})\!)_{L^2(\Omega)}$$

for every $\boldsymbol{v} \in \boldsymbol{W}_\kappa^{1,2}$. Functions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots$ form a system which is complete in both $\boldsymbol{L}_\kappa^2$ and $\boldsymbol{W}_\kappa^{1,2}$, orthonormal in $\boldsymbol{L}_\kappa^2$ and orthogonal in $\boldsymbol{W}_\kappa^{1,2}$. It is easy to see that this system is orthogonal and complete in $\boldsymbol{D}$. Moreover

$$\boldsymbol{L}_\kappa^2 = \left\{\boldsymbol{\phi}; \boldsymbol{\phi} = \sum_{k=1}^\infty a_k \boldsymbol{\phi}_k, a_k \in \mathbf{R} \text{ and } \sum_{k=1}^\infty a_k^2 < \infty\right\},$$
$$\boldsymbol{W}_\kappa^{1,2} = \left\{\boldsymbol{\phi}; \boldsymbol{\phi} = \sum_{k=1}^\infty a_k \boldsymbol{\phi}_k, a_k \in \mathbf{R} \text{ and } \sum_{k=1}^\infty \lambda_k a_k^2 < \infty\right\}$$

and

$$\boldsymbol{D} = \left\{\boldsymbol{\phi}; \boldsymbol{\phi} = \sum_{k=1}^\infty a_k \boldsymbol{\phi}_k, a_k \in \mathbf{R} \text{ and } \sum_{k=1}^\infty \lambda_k^2 a_k^2 < \infty\right\}.$$

Let $\alpha \in \mathbf{R}$. Define

$$V_\kappa^\alpha = \left\{ \phi; \phi = \sum_{k=1}^{\infty} a_k \phi_k, a_k \in \mathbf{R} \text{ and } \sum_{k=1}^{\infty} \lambda_k^\alpha a_k^2 < \infty \right\}.$$

Note that $V_\kappa^1 \equiv W_\kappa^{1,2}$, $V_\kappa^0 \equiv L_\kappa^2$ and $V_\kappa^2 \equiv D$. Let $\phi, \psi \in V_\kappa^\alpha$, $\varphi \in V_\kappa^\beta$, $\phi = \sum_{k=1}^\infty a_k \phi_k$, $\psi = \sum_{k=1}^\infty b_k \phi_k$, $\varphi = \sum_{k=1}^\infty c_k \phi_k$. V_κ^α are Hilbert spaces with the scalar product

$$(\!(\phi, \psi)\!)_{V_\kappa^\alpha} = \sum_{k=1}^{\infty} \lambda_k^\alpha a_k b_k.$$

By the symbol $\langle \cdot, \cdot \rangle_{\alpha,\beta}$ wedenote the bilinear form on $V_\kappa^\alpha \times V_\kappa^\beta$,

$$\langle \phi, \varphi \rangle_{\alpha,\beta} = \sum_{k=1}^{\infty} \lambda_k^{(\alpha+\beta)/2} a_k c_k.$$

It is obvious that

$$V_\kappa^{-\alpha} = (V_\kappa^\alpha)^*.$$

1.5 The main result

The main result of this paper is formulated in the following theorem.

Theorem 2 *Let $\alpha \in (0, 1]$, $u_0 \in V_\kappa^\alpha$ and $f \in L^2(0, T; V_\kappa^{\alpha-1})$. Then there exist $T^*, 0 < T^* \leq T$ and a unique weak solution of the problem* (3)–(8) *with the initial velocity u_0 and the right hand side f on the time interval* $(0, T^*)$.

2 Non-steady Stokes problem

Let X be an arbitrary Banach space, $\psi \in L^1(0, t; X)$. We say that there exists a time derivative of the function ψ, which is denoted by the symbol ψ', if $\psi' \in L^1(0, t; X)$ and the identity (1.15) in [17, Chapter III, Lemma 1.1] is satisfied.

Further we define function spaces

$$X_{\alpha,t} := \{w; w \in L^2(0, t; V_\kappa^{\alpha+1}),\ w' \in L^2(0, t; V_\kappa^{\alpha-1})\}$$

and

$$Y_{\alpha,t} := \{[g, v];\ g \in L^2(0, t; V_\kappa^{\alpha-1}),\ v \in V_\kappa^\alpha\}$$

for $t \in (0, T]$ respectively, with the norms

$$\|w\|_{X_{\alpha,t}} = \|w\|_{L^2(0,t;\ V_\kappa^{\alpha+1})} + \|w'\|_{L^2(0,t;\ V_\kappa^{\alpha-1})}$$

and

$$\|[\boldsymbol{g}, \boldsymbol{v}]\|_{Y_{\alpha,t}} = \|\boldsymbol{g}\|_{L^2(0,t;\ \boldsymbol{V}_\kappa^{\alpha-1})} + \|\boldsymbol{v}\|_{\boldsymbol{V}_\kappa^\alpha}.$$

Let $\alpha, \beta \in \mathbf{R}$, $\alpha > \beta$, $\boldsymbol{\phi} \in \boldsymbol{V}_\kappa^\alpha$, $\boldsymbol{\psi} \in \boldsymbol{V}_\kappa^{(\alpha+\beta)/2} \hookrightarrow \boldsymbol{V}_\kappa^\beta$. Then

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{\alpha,\ \beta} = (\!(\boldsymbol{\phi}, \boldsymbol{\psi})\!)_{\boldsymbol{V}_\kappa^{(\alpha+\beta)/2}}.$$

Using this fact and [17, Chapter III., Lemma 1.2] we obtain the following lemma.

Lemma 3 $X_{\alpha,t} \hookrightarrow \mathcal{C}([0,t]; \boldsymbol{V}_\kappa^\alpha) \subset L^\infty(0,t; \boldsymbol{V}_\kappa^\alpha)$.

Let $\boldsymbol{\phi} \in \boldsymbol{V}_\kappa^\alpha$, $\boldsymbol{\phi} = \sum_{k=1}^\infty a_k \boldsymbol{\phi}_k$ and $\boldsymbol{\Psi}$ be an operator defined by

$$\boldsymbol{\Psi}(\boldsymbol{\phi}) := \sum_{k=1}^\infty \lambda_k a_k \boldsymbol{\phi}_k.$$

It is easy to see that $\boldsymbol{\Psi}$ is the linear continuous operator from $\boldsymbol{V}_\kappa^\alpha$ to $\boldsymbol{V}_\kappa^{\alpha-2}$ for every $\alpha \in \mathbf{R}$.

Let $\boldsymbol{u} \in L^2(0,t; \boldsymbol{V}_\kappa^\alpha)$. It is obvious that

$$\boldsymbol{u}(s) = \sum_{k=1}^\infty g_k(s) \boldsymbol{\phi}_k,$$

where g_k are measurable functions on $(0, t)$ and

$$\sum_{k=1}^\infty \lambda_k^\alpha \int_0^t |g_k(s)|^2 \ \mathrm{ds} \ < \infty.$$

Let the operator $\mathcal{A}$ be defined by

$$\mathcal{A}(\boldsymbol{u}) := \sum_{k=1}^\infty \lambda_k g_k(s) \boldsymbol{\phi}_k.$$

It is easy to see that $\mathcal{A}$ is the linear continuous operator from $L^2(0,t; \boldsymbol{V}_\kappa^\alpha)$ to $L^2(0,t; \boldsymbol{V}_\kappa^{\alpha-2})$ for every $\alpha \in \mathbf{R}$.

Let $\mathcal{S}_{\alpha,t} : X_{\alpha,t} \to Y_{\alpha,t}$ be the operator defined by the following way: Let $\boldsymbol{u} \in X_{\alpha,t}$. Then

$$\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\boldsymbol{u}' + \mathcal{A}(\boldsymbol{u}),\ \boldsymbol{u}(0)].$$

It is easy to see that $\mathcal{S}_{\alpha,t}$ is the linear continuous operator from $X_{\alpha,t}$ to $Y_{\alpha,t}$ for every $\alpha \in \mathbf{R}$.

Let $\boldsymbol{u} \in X_{\alpha,t}$, $[\boldsymbol{g}, \boldsymbol{\eta}] \in Y_{\alpha,t}$ and $\alpha \in (0, 1]$. One can verify that $\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\boldsymbol{g}, \boldsymbol{\eta}]$ if and only if $\boldsymbol{u}$ is a weak solution of non-steady Stokes system (4)–(9) with the right hand side $\boldsymbol{g}$ and the initial velocity $\boldsymbol{\eta}$. Note also that the weak solution of the Stokes problem is defined analogously as the weak solution of the Navier-Stokes problem.

Lemma 4 *Let $\alpha \in (0,1]$ and $[\boldsymbol{g}, \boldsymbol{\eta}] \in Y_{\alpha,t}$. Then there exists unique $\boldsymbol{u} \in X_{\alpha,t}$ such that $\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\boldsymbol{g}, \boldsymbol{\eta}] \in Y_{\alpha,t}$.*

Proof Since $[\boldsymbol{g}, \boldsymbol{\eta}] \in Y_{\alpha,t}$ we have

$$\boldsymbol{g} = \sum_{k=1}^{\infty} \mu_k \boldsymbol{\phi}_k, \;\; \boldsymbol{\eta} = \sum_{k=1}^{\infty} a_k \boldsymbol{\phi}_k$$

with μ_k and a_k satisfying

$$\sum_{k=1}^{\infty} \lambda_k^{\alpha-1} \int_0^t \mu_k^2(s)ds + \sum_{k=1}^{\infty} \lambda_k^{\alpha} a_k^2 < \infty. \tag{12}$$

Let ϑ_k be a solution of the ordinary differential equation

$$\vartheta_k'(s) + \lambda_k \vartheta_k(s) = \mu_k(s) \tag{13}$$

(which holds for almost every $s \in (0,t)$) with the initial condition

$$\vartheta_k(0) = a_k, k = 1, 2, \ldots \tag{14}$$

Then

$$\vartheta_k(s) = \int_0^s e^{\lambda_k(\xi - s)} \mu_k(\xi) d\xi + a_k e^{-\lambda_k s}$$

for every $s \in (0,t)$. Hence $\vartheta_k \in \boldsymbol{W}^{1,2}((0,t))$. Let $s \in (0,t)$. Multiplying (13) by $2\,\lambda_k^{\alpha-1} \vartheta_k'$ and integrating over $(0,s)$ we get

$$2\lambda_k^{\alpha-1} \int_0^s {\vartheta_k'}^2(\xi) d\xi + \lambda_k^{\alpha} \vartheta_k^2(s) = \lambda_k^{\alpha} \vartheta_k^2(0) + 2\lambda_k^{\alpha-1} \int_0^s \mu_k(\xi) \vartheta_k'(\xi) d\xi.$$

Consequently, the inequality

$$\lambda_k^{\alpha-1} \int_0^s {\vartheta_k'}^2(\xi) d\xi + \lambda_k^{\alpha} \vartheta_k^2(s) \le \lambda_k^{\alpha} \vartheta_k^2(0) + \lambda_k^{\alpha-1} \int_0^s \mu_k^2(\xi) d\xi \tag{15}$$

holds for $k = 1, 2, \ldots$ and for every $s \in (0,t)$. Thus (15) yields

$$\sum_{k=1}^{\infty} \lambda_k^{\alpha-1} \int_0^s {\vartheta_k'}^2(\xi) d\xi + \sum_{k=1}^{\infty} \lambda_k^{\alpha} \vartheta_k^2(s) \le \sum_{k=1}^{\infty} \lambda_k^{\alpha} \vartheta_k^2(0) + \sum_{k=1}^{\infty} \lambda_k^{\alpha-1} \int_0^t \mu_k^2(\xi) d\xi \tag{16}$$

for every $s \in (0,t]$. Let

$$\boldsymbol{u} := \sum_{k=1}^{\infty} \vartheta_k \boldsymbol{\phi}_k.$$

The estimate (16) yields that

$$\boldsymbol{u} \in L^{\infty}(0, t; \boldsymbol{V}^{\alpha}_{\kappa}), \quad \boldsymbol{u}' \in L^{2}(0, t; \boldsymbol{V}^{\alpha-1}_{\kappa})$$

and $\boldsymbol{u}$ satisfies the inequality

$$\|\boldsymbol{u}\|^2_{L^{\infty}(0,t;\, V^{\alpha}_{\kappa})} + \|\boldsymbol{u}'\|^2_{L^2(0,t;\, V^{\alpha-1}_{\kappa})} \leq 2\, \|\boldsymbol{g}\|^2_{L^2(0,t;\, V^{\alpha-1}_{\kappa})} + 2\, \|\boldsymbol{\eta}\|^2_{V^{\alpha}_{\kappa}}. \tag{17}$$

Further, (13) yields also inequalities

$$\lambda_k^2 \vartheta_k^2(s) \leq 2\mu_k^2(s) + 2\vartheta'^2_k(s), k = 1, 2, \dots,$$

for every $s \in (0, t)$. Hence we get

$$\sum_{k=1}^{\infty} \lambda_k^{\alpha+1} \int_0^t \vartheta_k^2(s) ds \leq 2 \sum_{k=1}^{\infty} \int_0^t \lambda_k^{\alpha-1} \mu_k^2(s) ds + 2 \sum_{k=1}^{\infty} \lambda_k^{\alpha-1} \int_0^t \vartheta'^2_k(s) ds.$$

The last inequality and (16) yield

$$\sum_{k=1}^{\infty} \lambda_k^{\alpha+1} \int_0^t \vartheta_k^2(s) ds \leq 4 \sum_{k=1}^{\infty} \lambda_k^{\alpha-1} \int_0^t \mu_k^2(s) ds + 2 \sum_{k=1}^{\infty} \lambda_k^{\alpha} \vartheta_k^2(0) \tag{18}$$

for every $t \in (0, T)$. By (12), (14) and (18) we have $\boldsymbol{u} \in L^2(0, t; \boldsymbol{V}^{\alpha}_{\kappa})$ such that

$$\|\boldsymbol{u}\|_{L^2(0,t;\, V^{\alpha+1}_{\kappa})} \leq 4\, \|\boldsymbol{g}\|_{L^2(0,t;\, V^{\alpha-1}_{\kappa})} + 2\, \|\boldsymbol{\eta}\|_{V^{\alpha}_{\kappa}}.$$

Using the last inequality and (17) we obtain

$$\|\boldsymbol{u}\|_{L^2(0,t;\, V^{\alpha+1}_{\kappa})} + \|\boldsymbol{u}\|_{L^{\infty}(0,t;\, V^{\alpha}_{\kappa})} + \|\boldsymbol{u}'\|_{L^2(0,t;\, V^{\alpha-1}_{\kappa})} \leq\ 6\, \|\boldsymbol{g}\|_{L^2(0,t;\, V^{\alpha-1}_{\kappa})} + 4\, \|\boldsymbol{\eta}\|_{V^{\alpha}_{\kappa}}. \tag{19}$$

It is easy to see that $\boldsymbol{u} \in X_{\alpha,t}$ and $\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\boldsymbol{g}, \boldsymbol{\eta}]$.

Suppose that $\boldsymbol{u}_A, \boldsymbol{u}_B \in X_{\alpha,t}$ are solutions of this problem for given data $\boldsymbol{g}$ and $\boldsymbol{\eta}$. We prove that $\boldsymbol{u}_A = \boldsymbol{u}_B$.

Denote $\boldsymbol{w} = \boldsymbol{u}_A - \boldsymbol{u}_B$. It is easy to see that

$$\langle \mathcal{S}_{\alpha,t}(\boldsymbol{w}), \boldsymbol{h} \rangle_{\alpha-1,\alpha+1} = 0$$

for every $\boldsymbol{h} \in L^2(0, t; \boldsymbol{V}^{\alpha}_{\kappa})$ and almost everywhere on (0, t) and

$$\boldsymbol{w}(0) = \boldsymbol{0}.$$

Put $\boldsymbol{h} = \boldsymbol{w}$. One can verify that

$$\|\boldsymbol{w}(t)\|^2_{V^{\alpha}_{\kappa}} + 2\|\boldsymbol{w}(s)\|^2_{L^2(0,t;\, V^{\alpha+1}_{\kappa})} = \|\boldsymbol{w}(0)\|^2_{V^{\alpha}_{\kappa}} = 0.$$

Therefore we get that $\boldsymbol{u}_A = \boldsymbol{u}_B$. We have proved that $\boldsymbol{u} \in X_{\alpha,t}$ is unique solution of the equation $\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\boldsymbol{g}, \boldsymbol{\eta}]$. □

Remark 5 Using Lemma 4 we obtain that $\mathcal{S}_{\alpha,t}$ is the one-to-one operator from $X_{\alpha,t}$ to $Y_{\alpha,t}$ and onto $Y_{\alpha,t}$. Consequently, ${\mathcal{S}_{\alpha,t}}^{-1}$ is the linear continuous operator from $Y_{\alpha,t}$ onto $X_{\alpha,t}$. Using (19) we get

$$\|\boldsymbol{u}\|_{X_{\alpha,t}} = \|{\mathcal{S}_{\alpha,t}}^{-1}([\boldsymbol{g},\boldsymbol{\eta}])\|_{X_{\alpha,t}} \leq 6\ \|[\boldsymbol{g},\boldsymbol{\eta}]\|_{Y_{\alpha,t}}. \tag{20}$$

3 Properties of some function spaces

First we show some properties of function spaces. Let $\boldsymbol{\phi} \in \boldsymbol{D}$ and $\boldsymbol{A}_1 \subset \overline{\Gamma_1} \cap \overline{\Gamma_2}$. This means $\boldsymbol{A}_1$ is the point at which the boundary conditions change their type. Recall that Γ_1 and Γ_2 are perpendicular at the point $\boldsymbol{A}_1$. Orlt and Sändig [15] and (11) yield that there exist open sets U_1 and Ω_1, $\boldsymbol{A}_1 \in U_1$, $\Omega_1 = U_1 \cap \Omega$ such that $\boldsymbol{\phi} \in \boldsymbol{W}^{2,2}(\Omega_1)$.

Similarly, let $\boldsymbol{A}_2, \boldsymbol{A}_3 \subset \partial\Omega$, $\boldsymbol{A}_2 \subset \overline{\Gamma_1} \cap \overline{\Gamma_3}$ and $\boldsymbol{A}_3 \subset \overline{\Gamma_2} \cap \overline{\Gamma_3}$. Then there exist open sets U_2, U_3, Ω_2 and Ω_3, $\Omega_2 := U_2 \cap \Omega$, $\Omega_3 := U_3 \cap \Omega$, such that $\boldsymbol{A}_2 \in U_2$, $\boldsymbol{A}_3 \in U_3$, $\boldsymbol{\phi} \in \boldsymbol{W}^{2,2}(\Omega_2)$ (see [3, Apendix A]) and $\boldsymbol{\phi} \in \boldsymbol{W}^{2,2}(\Omega_3)$ (see [4, Theorem 2]).

Let $\Omega_4 \subset \overline{\Omega_4} \subset \Omega$. By [3, Apendix A], $\boldsymbol{\phi} \in \boldsymbol{W}^{2,2}(\Omega_4)$. Using these facts we deduce

$$\boldsymbol{V}^2_\kappa \hookrightarrow \boldsymbol{W}^{2,2}(\Omega).$$

Further, recall that $\boldsymbol{V}^0_\kappa \equiv \boldsymbol{L}^2_\kappa$. Applying [16, Chapter II., Lemma 3.2.3] we obtain that

$$\boldsymbol{V}^\beta_\kappa \hookrightarrow \boldsymbol{W}^{\beta,2}_\kappa(\Omega) \tag{21}$$

holds for every $\beta \in [0;2]$. Let $\beta_1 > \beta_2 > \beta_3$ and $\boldsymbol{\phi} \in \boldsymbol{V}^{\beta_1}_\kappa$. Then $\boldsymbol{\phi} \in \boldsymbol{V}^{\beta_2}_\kappa \subset \boldsymbol{V}^{\beta_3}_\kappa$ and

$$\|\boldsymbol{\phi}\|_{V^{\beta_2}_\kappa} \leq \|\boldsymbol{\phi}\|^{(\beta_2-\beta_3)/(\beta_1-\beta_3)}_{V^{\beta_1}_\kappa} \|\boldsymbol{\phi}\|^{(\beta_1-\beta_2)/(\beta_1-\beta_3)}_{V^{\beta_3}_\kappa}. \tag{22}$$

One can see that

$$\boldsymbol{V}^{\beta_1}_\kappa \hookrightarrow\hookrightarrow \boldsymbol{V}^{\beta_2}_\kappa \tag{23}$$

for $\beta_1, \beta_2 \in \mathbf{R}$, $\beta_1 > \beta_2$. Let α be given by Theorem 2. We have

$$\boldsymbol{V}^{\alpha+1}_\kappa \hookrightarrow\hookrightarrow \boldsymbol{V}^r_\kappa \hookrightarrow\hookrightarrow \boldsymbol{V}^{\alpha-1}_\kappa$$

for every r, $0 < r < 1$. Applying [17, Chapter III, Theorem 2.1.] and Lemma 3 we get that

$$X_{\alpha,t} \hookrightarrow\hookrightarrow L^2(0,t;\boldsymbol{V}^q_\kappa), \quad q < \alpha + 1, \tag{24}$$

and

$$X_{\alpha,t} \hookrightarrow L^\infty(0,t;\boldsymbol{V}^\alpha_\kappa). \tag{25}$$

Consequently

$$X_{\alpha,t} \hookrightarrow\hookrightarrow L^p(0,t;\, \boldsymbol{V}^{\alpha}_{\kappa}) \tag{26}$$

for every p, $2 \le p < \infty$. Using (21), (22) (for $\beta_1 = \frac{8+7\alpha}{8}$, $\beta_2 = \frac{2+\alpha}{2}$, $\beta_3 = \alpha$), (24) (for $q = \frac{8+7\alpha}{8}$) and (25) one obtains the embedding

$$X_{\alpha,t} \hookrightarrow\hookrightarrow L^{(8-\alpha)/(4-2\alpha)}(0,t;\, \boldsymbol{V}^{(2+\alpha)/2}_{\kappa}) \hookrightarrow L^{(8-\alpha)/(4-2\alpha)}(0,t;\, \boldsymbol{W}^{1,4/(2-\alpha)}(\Omega)). \tag{27}$$

Using (26) and [11, Part 8.3.3.(i)] we obtain the embedding

$$X_{\alpha,t} \hookrightarrow\hookrightarrow L^{(16-2\alpha)/\alpha}(0,t;\, \boldsymbol{V}^{\alpha}_{\kappa}) \hookrightarrow L^{(16-2\alpha)/\alpha}(0,t;\, \boldsymbol{L}^{16/(8-7\alpha)}(\Omega)). \tag{28}$$

Let $Z_{\alpha,t} := L^{(8-\alpha)/(4-2\alpha)}(0,t;\, \boldsymbol{V}^{(2+\alpha)/2}_{\kappa}) \cap L^{(16-2\alpha)/\alpha}(0,t;\, \boldsymbol{V}^{\alpha}_{\kappa})$ be the reflexive Banach space with the norm

$$\|.\|_{Z_{\alpha,t}} = \|.\|_{L^{(8-\alpha)/(4-2\alpha)}(0,t;\, V^{(2+\alpha)/2}_{\kappa})} + \|.\|_{L^{(16-2\alpha)/\alpha}(0,t;\, V^{\alpha}_{\kappa})}.$$

Equations (27) and (28) yield that

$$X_{\alpha,t} \hookrightarrow\hookrightarrow Z_{\alpha,t}. \tag{29}$$

4 Proof of Theorem 2

First, we will define the operator $\mathcal{B}_{\alpha,t} : Z_{\alpha,t} \times Z_{\alpha,t} \to L^2(0,t;\, \boldsymbol{V}^{\alpha-1}_{\kappa})$, which corresponds to the convective term in (3). Next we define the operator $\mathcal{T}_{\alpha,t} : Z_{\alpha,t} \to Z_{\alpha,t}$. We show that there exist $T^*, 0 < T^* \le T$, and a closed ball $K \subset Z_{\alpha,t}$ such that $\mathcal{T}_{\alpha,t}(K) \subset K$ provided $t \in (0, T^*]$. Next we show that the operator $\mathcal{T}_{\alpha,t}$ is compact. Consequently, there exists $\boldsymbol{u} \in Z_{\alpha,t}$ such that $\mathcal{T}_{\alpha,t}(\boldsymbol{u}) = \boldsymbol{u}$. Here, $\boldsymbol{u}$ corresponds to the weak solution of problem (3)–(8) on the time interval $(0, T^*)$.

Let $\boldsymbol{w} \in Z_{\alpha,t}$, $\boldsymbol{v} \in \boldsymbol{V}^{1-\alpha}_{\kappa} = (\boldsymbol{V}^{\alpha-1}_{\kappa})^*$. Then $\langle \mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w}), \boldsymbol{v}\rangle_{\alpha-1,1-\alpha} : (0,t) \to \mathbf{R}$ is a function such that

$$\langle \mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w}), \boldsymbol{v}\rangle_{\alpha-1,1-\alpha}(s) = \int_{\Omega} \big(\boldsymbol{w}(s)\cdot\nabla\big)\boldsymbol{w}(s)\cdot\boldsymbol{v}$$

for almost every on $s \in (0,t)$. Note that

$$\boldsymbol{V}^{1-\alpha}_{\kappa} \hookrightarrow \boldsymbol{L}^q(\Omega) \quad \text{for } 1 < q < \frac{2}{\alpha}. \tag{30}$$

Using (26) and (28) and (30) we obtain

$$\begin{aligned} |\langle \mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w}), \boldsymbol{v}\rangle_{\alpha-1,1-\alpha}| &\le c\, \|\boldsymbol{w}\|_{L^{16/(8-7\alpha)}} \|\boldsymbol{w}\|_{W^{1,4/(2-\alpha)}} \|\boldsymbol{v}\|_{L^{16/11\alpha}} \\ &\le c\, \|\boldsymbol{w}\|_{V^{\alpha}_{\kappa}} \|\boldsymbol{w}\|_{V^{(2+\alpha)/2}_{\kappa}} \|\boldsymbol{v}\|_{V^{1-\alpha}_{\kappa}} \end{aligned} \tag{31}$$

almost everywhere on $(0, t)$. Estimate (31) yields

$$\|\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})\|_{V^{\alpha-1}_{\kappa}} \le c\, \|\boldsymbol{w}\|_{V^{\alpha}_{\kappa}} \|\boldsymbol{w}\|_{V^{(2+\alpha)/2}_{\kappa}}.$$

Using (26) and (28) and (30) again we get

$$\begin{aligned}\|\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})\|^2_{L^2(0,t;\ V_\kappa^{\alpha-1})} &\le c\int_0^t \|\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})\|^2_{V_\kappa^{\alpha-1}}\\ &\le c\ t^{2\alpha/(8-\alpha)}\|\boldsymbol{w}\|^2_{L^{(16-2\alpha)/\alpha}(0,t;\ V_\kappa^{\alpha})}\|\boldsymbol{w}\|^2_{L^{(8-\alpha)/(4-2\alpha)}(0,t;\ V_\kappa^{(2+\alpha)/2})}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})\|_{L^2(0,t;\ V_\kappa^{\alpha-1})} &\le c\ t^{\alpha/(8-\alpha)}\|\boldsymbol{w}\|_{L^{(16-2\alpha)/\alpha}(0,t;\ V_\kappa^{\alpha})}\|\boldsymbol{w}\|_{L^{(8-\alpha)/(4-2\alpha)}(0,t;\ V_\kappa^{(2+\alpha)/2})}\\ &\le c\ t^{\alpha/(8-\alpha)}\|\boldsymbol{w}\|^2_{Z_{\alpha,t}}.\end{aligned} \tag{32}$$

We have verified that definition of $\mathcal{B}_{\alpha,t}$ is correct. It is obvious that $\mathcal{B}_{\alpha,t}$ is continuous operator.

Let $\boldsymbol{f}$ and $\boldsymbol{u}_0$ be given by Theorem 2. Then $[\boldsymbol{f},\boldsymbol{u}_0]\in Y_{\alpha,t}$. $\mathcal{T}_{\alpha,t}: Z_{\alpha,t}\to Z_{\alpha,t}$ be the operator defined by the identity

$$\mathcal{T}_{\alpha,t}(\boldsymbol{w}) = (\mathcal{S}_{\alpha,t})^{-1}\left(\left[-\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})+\boldsymbol{f},\boldsymbol{u}_0\right]\right).$$

Applying (20) we get

$$\begin{aligned}\|\mathcal{T}_{\alpha,t}(\boldsymbol{w})\|_{Z_{\alpha,t}} &\le c\ \left\|\mathcal{T}_{\alpha,t}(\boldsymbol{w})\right\|_{X_{\alpha,t}} = c\ \|\mathcal{S}_{\alpha,t}{}^{-1}([-\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})+\boldsymbol{f},\boldsymbol{u}_0])\|_{X_{\alpha,t}}\\ &\le c\ \left\|[-\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})+\boldsymbol{f},\boldsymbol{u}_0]\right\|_{Y_{\alpha,t}}\\ &\le c\ \left(\|\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})\|_{L^2(0,t;\ V_\kappa^{\alpha-1})} + \|\boldsymbol{f}\|_{L^2(0,t;\ V_\kappa^{\alpha-1})} + \|\boldsymbol{u}_0\|_{V_\kappa^{\alpha}}\right).\end{aligned} \tag{33}$$

Note that the constants c do not depend on t. Using (32) we obtain

$$\begin{aligned}\|\mathcal{T}_{\alpha,t}(\boldsymbol{w})\|_{Z_{\alpha,t}} &\le c_1\ \left(t^{\alpha/(8-\alpha)}\|\boldsymbol{w}\|^2_{Z_{\alpha,t}} + \|\boldsymbol{f}\|_{L^2(0,t;\ V_\kappa^{\alpha-1})} + \|\boldsymbol{u}_0\|_{V_\kappa^{\alpha}}\right)\\ &\le c_1\ \left(t^{\alpha/(8-\alpha)}\|\boldsymbol{w}\|^2_{Z_{\alpha,t}} + \|\boldsymbol{f}\|_{L^2(0,T;\ V_\kappa^{\alpha-1})} + \|\boldsymbol{u}_0\|_{V_\kappa^{\alpha}}\right).\end{aligned}$$

It is easy to see that there exist T_1, $0<T_1\le T$ and $K>0$ such that

$$c_1\left((T_1)^{\alpha/(8-\alpha)}K^2 + \|\boldsymbol{f}\|_{L^2(0,T;\ V_\kappa^{\alpha-1})} + \|\boldsymbol{u}_0\|_{V_\kappa^{\alpha}}\right) < K.$$

Denote $\boldsymbol{B}_K = \{\boldsymbol{w}\in Z_{\alpha,t};\|\boldsymbol{w}\|_{Z_{\alpha,t}}\le K\}$. Last inequality yields that $\mathcal{T}_{\alpha,t}(\boldsymbol{B}_K)\subset \boldsymbol{B}_K$ for $t\le T_1$. Next, (29) and (33) yield that operator $\mathcal{T}_{\alpha,t}$ is compact. Using well known Schauder's principle (see e.g. [6, Chapter V., §2.]) we prove that there exists $\boldsymbol{u}\in X_{\alpha,t}\subset Z_{\alpha,t}$ such that $\mathcal{T}_{\alpha,t}(\boldsymbol{u})=\boldsymbol{u}$. Put $T^*=\min(T_1,T)$. One can verify that $\boldsymbol{u}$ is a weak solution of (3)–(8) on the time interval $(0,T^*)$.

The proof of the uniqueness of the solution would be more or less straightforward copy of the procedures used in [17, Chapter III, Lemma 3.4] with some technical differences. In the cited work, the uniqueness of the two-dimensional solution of Navier-Sokes equations with no-slip boundary conditions is proved. Author uses the estimate proved in Lemma 3.4 in the same book. This estimate would be replaced by the estimate (28) and by the inequality given by the embedding $X_{\alpha,t} \hookrightarrow L^2(0, t; L^\infty(\Omega))$.

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Availability of data and materials This point is not applicable in our situation.

Code availability This point is not applicable in our situation.

Declarations

Conflict of interest The authors declare no conflict of interest.

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4 Kapitola 3

4.1 Existence řešení systému Navierových-Stokesových rovnic a rovnice vedení tepla v tekutinách s různými typy okrajových podmínek lokálně v čase.

Převážná část této kapitoly je tvořena zaslaným článkem [6]:

Michal Beneš, Petr Kučera, Petra Vacková: On buoyancy-driven viscous incompressible flows with various types of boundary conditions. Submitted to the Journal of Mathematical Analysis and Applications.

Autorský podíl M.Beneše je 34%, P.Kučery a P.Vackové 33%. Text má vlastní značení, definice, věty a literaturu.

V tomto článku se zabýváme systémem dvoudimenzionálních Navierových-Stokesových rovnic a rovnicí vedení tepla v tekutinách na omezené oblasti. Tento systém modeluje proudění nestlačitelné, tepelně vodivé tekutiny v kanálu. Na hranici kanálu předepisujeme smíšené okrajové podmínky. Na pevné stěně předepisuje Navierovu okrajovou podmínku pro rychlost a Newtonovu okrajovou podmínku pro teplotu, na vstupu a výstupu kanálu předepisujeme okrajovou podmínku (7) pro rychlost a tlak a Neumannovu okrajovou podmínku pro teplotu. Podobně jako v předchozí kapitole, tyto podmínky nám znemožňují dokázat globální existenci slabého řešení na pevně daném časovém intervalu. Dokazujeme proto existenci slabého řešení na nějakém, libovolně malém, časovém intervalu. Podobně jako v předchozí kapitole, existence řešení je dokázána i za předpokladu, že počáteční rychlost náleží do prostoru, který je jen nepatrně silnější než $\boldsymbol{L}^2(\Omega)$.

On buoyancy-driven viscous incompressible flows with various types of boundary conditions

Michal Beneš[a,*], Petr Kučera[a], Petra Vacková[a]

[a]*Department of Mathematics,*
Faculty of Civil Engineering,
Czech Technical University in Prague,
Thákurova 7, 166 29 Prague 6, Czech Republic

Abstract

In this paper we study the existence and uniqueness of solutions to the initial-boundary-value problem for time-dependent flows of heat-conducting viscous incompressible fluids through the two-dimensional channel. The boundary conditions are of two types: the so-called "do nothing" boundary condition on the outflow and the so called Navier boundary conditions on the solid walls of the channel. The considered mixed boundary conditions do not enable us to derive an energy-type estimate of the solution. We prove the existence and uniqueness of a solution on a (sufficiently short) time interval for arbitrarily large data.

Keywords:
Navier-Stokes equations, heat-conducting fluids, dissipative heat; adiabatic heat, local in time existence of solutions, Navier boundary conditions, do-nothing boundary conditions
2000 MSC: 35Q30, 35K05, 76D03

1. Introduction

1.1. The model

The full system of the Navier-Stokes equations represents the most commonly used mathematical model in thermodynamics of incompressible flu-

*Corresponding address: Department of Mathematics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7, 166 29 Prague 6, Czech Republic

Email address: michal.benes@cvut.cz (Michal Beneš)

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ids. Due to many applications in engineering and industry, the Navier-Stokes equations have been studied during the last decades from both theoretical as well as numerical point of view. We assume that the flow of a viscous incompressible heat-conducting fluid is governed by balance equations for linear momentum, mass and internal energy in the form

$$\varrho\,(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla\mathcal{P} = \varrho(1-\alpha_0\theta)\boldsymbol{f}, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{1.2}$$

$$c_p\varrho\,(\theta_t + \boldsymbol{u}\cdot\nabla\theta) - \kappa\Delta\theta = \nu\boldsymbol{e}(\boldsymbol{u}):\boldsymbol{e}(\boldsymbol{u}) + \varrho\alpha_0\theta\boldsymbol{f}\cdot\boldsymbol{u} + h. \tag{1.3}$$

Here $\boldsymbol{u}$ represents the unknown velocity, $\mathcal{P}$ is the unknown pressure and θ stands for the unknown temperature of the fluid. $\boldsymbol{f}$ represents the given external force (such as gravity) and h is a heat source term. Tensor $\boldsymbol{e}(\boldsymbol{u})$ denotes the symmetric part of the velocity gradient. Thermodynamic and transport properties represent the kinematic viscosity ν, density ϱ, heat conductivity κ, specific heat at constant pressure c_p and thermal expansion coefficient of the fluid α_0. The energy balance equation (1.3) takes into account the phenomena of the viscous energy dissipation and adiabatic heat effects. For rigorous derivation of the model like (1.1)–(1.3) we refer the readers to [11]. Brief discussion and overview of theoretical problems to this system and related references can be found in [7].

To complete the mathematical model, the evolution equations (1.1)–(1.3) have to be completed by the given initial conditions. Moreover, if the fluid does not occupy the whole space but e.g. a two-dimensional channel as shown in Figure 1, it is also necessary to apply suitable boundary conditions. The boundary conditions specify required behavior of the fluid on the boundary of the domain and depend on the particular problem at hand. The *no-slip boundary condition* (commonly used for laminar flows with small or medium velocity) or the *Navier slip boundary condition* for the velocity (reflecting rugosity effects of solid surfaces), respectively, are the most widely accepted on fixed walls

$$\boldsymbol{u} = \boldsymbol{0}, \tag{1.4}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \lambda[2\nu\boldsymbol{e}(\boldsymbol{u})\boldsymbol{n}]\cdot\boldsymbol{\tau} + \boldsymbol{u}\cdot\boldsymbol{\tau} = 0, \tag{1.5}$$

respectively. Here, $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the unit exterior normal and the unit tangent vector to the boundary, respectively. The parameter λ is the so-called *slip length.* The limit case $\lambda \to \infty$ leads to the *perfect slip condition* (the fluid may slip along the fixed wall without being stressed in the tangential direction). On the other hand, setting $\lambda = 0$, (1.5) reduces to the the *no-slip boundary condition* (1.4).

Mathematical modeling of flows in *physically large* domains (e.g. piping systems, exterior unbounded domains) is not practical from the computational point of view. Therefore, the unbounded physical regions are usually truncated to smaller bounded domains by assuming an artificial boundary. *The Neumann type boundary conditions* of the form

$$-\mathcal{P}\boldsymbol{n}+\nu\boldsymbol{e}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{0} \quad \text{or the outflow condition} \quad -\mathcal{P}\boldsymbol{n}+\nu\partial\boldsymbol{u}/\partial\boldsymbol{n} = \boldsymbol{0} \tag{1.6}$$

(used in many numerical simulations of *classical fluids*) are commonly applied on the artificial part of the boundary. As first observed by Heywood et al. in [10], boundary conditions (1.6) do not exclude the possibility of backward flows that could eventually bring an uncontrollable amount of kinetic energy back to the simulation domain. Consequently, we are not able to derive a priori estimate of a weak solution (*the energy inequality*), which is well known for the problem with *energy preserving* boundary conditions. In order to derive an a priori energy estimate, [12, 13, 14] prescribed an additional constraint on the output (which bounds the kinetic energy of the backward flow) and formulated steady and evolutionary Navier-Stokes problems by means of appropriate variational inequalities. However, this still leaves the question open whether one can prove global existence of solutions for the original problem.

The existence of a solution for the Boussinesq equations (neglecting dissipative and adiabatic heating) with the boundary conditions (1.4) and $(1.6)_2$ on *a sufficiently short time interval* has been proven in [15]. Moreover, taking in to account the dissipative and adiabatic heating, the quadratic source term on the right-hand side of the energy equation (1.3) causes major additional mathematical difficulties. Global existence result has been proven only in case of non-Newtonian fluids, see [6, 19]. Local existence and uniqueness of the strong solutions are shown in [2, 3].

In the present paper, we extend our previous result in [5] to the case of heat conducting fluids. Combining the Navier boundary conditions and the do-nothing boundary conditions for the velocity and the Neumann type boundary conditions for the temperature of the fluid, we prove the local existence and global uniqueness of the solution to the more general non-isothermal model taking into account dissipative and adiabatic heating.

1.2. Classical formulation of the problem

Let $\Omega \subset \mathrm{R}^2$ be a bounded rectangular domain, which represents a channel filled up by a moving fluid, see Figure 1. Γ_1 is a fixed wall and Γ_2 represents the outflow (or inflow) boundary of the channel.

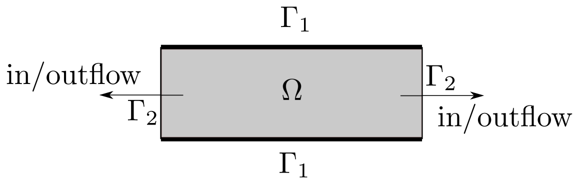

Figure 1: The domain Ω represents a straight channel filled up by a moving heat-conducting fluid.

Further, let $T \in (0, \infty)$, $\Omega_T = \Omega \times (0, T)$, $\Gamma_{1T} = \Gamma_1 \times (0, T)$ and $\Gamma_{2T} = \Gamma_2 \times (0, T)$. We consider the following system of equations

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \mathcal{P} = (1 - \theta)\boldsymbol{f} \qquad \text{in } \Omega_T, \tag{1.7}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega_T, \tag{1.8}$$

$$\theta_t + \boldsymbol{u} \cdot \nabla \theta - \Delta \theta = \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{u}) + \theta \boldsymbol{f} \cdot \boldsymbol{u} + h \qquad \text{in } \Omega_T, \tag{1.9}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [2\nu \boldsymbol{e}(\boldsymbol{u})\boldsymbol{n}] \cdot \boldsymbol{\tau} + \gamma \boldsymbol{u} \cdot \boldsymbol{\tau} = 0 \qquad \text{on } \Gamma_{1T}, \tag{1.10}$$

$$-\frac{\partial \theta}{\partial \boldsymbol{n}} = \theta - \theta_\infty \qquad \text{on } \Gamma_{1T}, \tag{1.11}$$

$$-\mathcal{P}\boldsymbol{n} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \boldsymbol{0} \qquad \text{on } \Gamma_{2T}, \tag{1.12}$$

$$\frac{\partial \theta}{\partial \boldsymbol{n}} = 0 \qquad \text{on } \Gamma_{2T}, \tag{1.13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega, \tag{1.14}$$

$$\theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{1.15}$$

For notational simplicity, we normalized material constants ϱ, ν, κ, α_0 and c_p to one. In (1.10), the factor γ is the coefficient of friction between the fluid and the boundary. γ is supposed to be constant, $\gamma > 0$.

1.3. Preliminaries

Throughout the paper, we denote by c, c_1, c_2, c_3 ... generic constants, i.e. constants whose values may change from line to line. We admit that these constants may depend on Ω, but they never depend on a concrete function.

Let $\boldsymbol{u} = (u_1, u_2)$ be a velocity field and $\boldsymbol{n} = (n_1, n_2)$ be a normal vector at some point of $\partial\Omega$. If we want to emphasize the matrix character of $\nabla\boldsymbol{u}$ and $\nabla\boldsymbol{n}$, we use symbols $(\nabla\boldsymbol{u})$ and $(\nabla\boldsymbol{n})$. Then $(\nabla\boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j}$ and $(\nabla\boldsymbol{n})_{ij} = \frac{\partial n_i}{\partial x_j}$. The transposed vectors to $\boldsymbol{u}$ and $\boldsymbol{n}$ and the transposed matrices to $(\nabla\boldsymbol{u})$ and $(\nabla\boldsymbol{n})$, respectively, are denoted by $\boldsymbol{u}^\top$, $\boldsymbol{n}^\top$, $(\nabla\boldsymbol{u})^\top$ and $(\nabla\boldsymbol{n})^\top$. Multiplication of the matrix $(\nabla\boldsymbol{u})$ and the transposed vector $\boldsymbol{n}^\top$ is denoted by $(\nabla\boldsymbol{u})\,\boldsymbol{n}^\top$. Scalar product of two vectors, $\boldsymbol{u}$ and $(\nabla\boldsymbol{u})\,\boldsymbol{n}^\top$ is denoted by $\boldsymbol{u}\cdot(\nabla\boldsymbol{u})\,\boldsymbol{n}^\top$.

We denote vector-valued functions and spaces of such functions by boldface letters with one exception. Bold marking is not used in subscripts of norms or scalar products. For example, the norm of the space $L^2(0,T;\boldsymbol{W}^{1,2}(\Omega))$ is denoted by $\|\cdot\|_{L^2(0,T;W^{1,2}(\Omega))}$.

Let $\boldsymbol{u} \in L^2(0,T;\boldsymbol{W}^{2,2}_\kappa)$ and $\mathcal{P} \in L^2(0,T;W^{1,2}(\Omega))$ satisfy the boundary conditions (1.10) and (1.12) and $\boldsymbol{v} \in \boldsymbol{W}^{1,2}_\kappa$. If we multiply the term $\Big(-\Delta\boldsymbol{u} + \nabla\mathcal{P}\Big)$ by $\boldsymbol{v}$, we obtain the following identity for almost every $t \in (0,T)$. For simplicity of notation, suppressing temporal variables, we write $\boldsymbol{u}$ and $\mathcal{P}$, respectively, instead of $\boldsymbol{u}(t)$ and $\mathcal{P}(t)$. We can write

$$\begin{aligned}
\int_\Omega \Big(-\Delta\boldsymbol{u} + \nabla\mathcal{P}\Big)\cdot\boldsymbol{v} &= \int_\Omega \nabla\boldsymbol{u}\nabla\boldsymbol{v} - \int_{\partial\Omega} \boldsymbol{v}\cdot(\nabla\boldsymbol{u})\boldsymbol{n}^\top + \int_{\Gamma_2} \mathcal{P}\boldsymbol{v}\cdot\boldsymbol{n} \\
&= \int_\Omega \nabla\boldsymbol{u}\nabla\boldsymbol{v} - \int_{\Gamma_1} \boldsymbol{v}\cdot(\nabla\boldsymbol{u})\boldsymbol{n}^\top + \int_{\Gamma_2} \Big(\mathcal{P}\boldsymbol{n} - \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\Big)\cdot\boldsymbol{v} \\
&= \int_\Omega \nabla\boldsymbol{u}\nabla\boldsymbol{v} - \int_{\Gamma_1} \boldsymbol{v}\cdot\Big((\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^\top\Big)\boldsymbol{n}^\top \\
&\quad + \int_{\Gamma_1} \boldsymbol{v}\cdot(\nabla\boldsymbol{u})^\top\boldsymbol{n}^\top \\
&= \int_\Omega \nabla\boldsymbol{u}\nabla\boldsymbol{v} + \gamma\int_{\Gamma_1} \boldsymbol{u}\cdot\boldsymbol{v} + \int_{\Gamma_1} \boldsymbol{v}\cdot(\nabla\boldsymbol{u})^\top\boldsymbol{n}^\top.
\end{aligned}$$

Since $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_1 and all parts of Γ_1 are line segments we obtain

$$\int_{\Gamma_1} \boldsymbol{v}\cdot(\nabla\boldsymbol{u})^\top\boldsymbol{n}^\top = -\int_{\Gamma_1} \boldsymbol{v}\cdot(\nabla\boldsymbol{n})^\top\boldsymbol{u}^\top = 0.$$

Consequently, we have

$$\int_\Omega \Big(-\Delta\boldsymbol{u} + \nabla\mathcal{P}\Big)\cdot\boldsymbol{v} = \int_\Omega \nabla\boldsymbol{u}\nabla\boldsymbol{v} + \gamma\int_{\Gamma_1} \boldsymbol{u}\cdot\boldsymbol{v}. \tag{1.16}$$

1.4. Weak formulation of the problem

Let

$$\mathcal{E}(\overline{\Omega}) := \left\{\boldsymbol{v} \in C^\infty(\overline{\Omega})^2;\ \operatorname{div}\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_1\right\}.$$

The closure of $\mathcal{E}(\overline{\Omega})$ in the norm of $\boldsymbol{W}^{k,p}(\Omega)$ $\big(= W^{k,p}(\Omega)^2\big)$ for $k \geq 0$ (k need not be an integer) and $1 \leq p < \infty$ is denoted by $\boldsymbol{W}^{k,p}_\kappa$. Then $\boldsymbol{W}^{k,p}_\kappa$ is the Banach space. For simplicity, the space $\boldsymbol{W}^{0,2}_\kappa$ is denoted by $\boldsymbol{L}^2_\kappa$. Note that $\boldsymbol{L}^2_\kappa$ is the closed subspace of $\boldsymbol{L}^2(\Omega)$. The scalar product on $\boldsymbol{L}^2(\Omega)$ is denoted by $(\!(\cdot,\cdot)\!)_{L^2(\Omega)}$. By $(\cdot,\cdot)$ we denote the scalar product on $L^2(\Omega)$.

Definition 1.1. *Let $T > 0$, $\boldsymbol{f} \in \mathbb{R}^2$, $h \in L^2(\Omega_T)$, $\theta_\infty \in L^2(\Gamma_{1T})$, $\theta_0 \in L^2(\Omega)$ and $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$. A weak solution of* (1.7)–(1.15) *on the time interval* $(0,T)$ *is a pair $[\boldsymbol{u},\theta]$ such that*

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{W}^{1,2}_\kappa),$$
$$\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$$

and

$$\int_{\Omega_T}\left[-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \nabla\boldsymbol{u}\nabla\boldsymbol{\phi} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}\right] + \gamma\int_{\Gamma_{1T}}\boldsymbol{u}\cdot\boldsymbol{\phi} = \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\phi}(0) + \int_{\Omega_T}(1-\theta)\boldsymbol{f}\cdot\boldsymbol{\phi}$$

for all $\boldsymbol{\phi} \in C^\infty([0,T];\ \mathcal{E}(\overline{\Omega}))$ such that $\boldsymbol{\phi}(T) = \boldsymbol{0}$ and

$$\int_{\Omega_T}[-\theta\partial_t\varphi + \nabla\theta\cdot\nabla\varphi] + \int_{\Gamma_{1T}}\theta\varphi + \int_{\Omega_T}\boldsymbol{u}\cdot\nabla\theta\,\varphi = \int_\Omega\theta_0\varphi(0) + \int_{\Gamma_{1T}}\theta_\infty\varphi + \int_{\Omega_T}\boldsymbol{e}(\boldsymbol{u}):\boldsymbol{e}(\boldsymbol{u})\varphi + \int_{\Omega_T}\theta\boldsymbol{f}\cdot\boldsymbol{u}\varphi + \int_{\Omega_T}h\varphi$$

for all $\varphi \in C^\infty([0,T];\ C^\infty(\overline{\Omega}))$ such that $\varphi(T) = 0$.

1.5. Auxiliary results

The weak formulation of the problem, namely the identity (1.16), motivates us to define the bilinear form $(\!(\cdot,\cdot)\!)_\kappa$ on $\boldsymbol{W}^{1,2}_\kappa \times \boldsymbol{W}^{1,2}_\kappa$,

$$(\!(\boldsymbol{\phi},\boldsymbol{v})\!)_\kappa := \int_\Omega \nabla\boldsymbol{\phi}\nabla\boldsymbol{v} + \gamma\int_{\Gamma_2}\boldsymbol{\phi}\cdot\boldsymbol{v}.$$

It is easy to see that this bilinear form is symmetric and $\boldsymbol{W}^{1,2}_{\kappa}$-elliptic. Let

$D_\kappa = \{\boldsymbol{\phi};$ there exists $\boldsymbol{\xi} \in \boldsymbol{L}^2_\kappa$ such that $(\!(\boldsymbol{\phi}, \boldsymbol{v})\!)_\kappa = (\!(\boldsymbol{\xi}, \boldsymbol{v})\!)_{L^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{W}^{1,2}_\kappa\}$.

By the same arguments as in [25, Chapter I., 2.6], there exists a sequence of eigenfunctions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots \in D_\kappa$, which form an orthonormal basis of $\boldsymbol{L}^2_\kappa$,

$$(\!(\boldsymbol{\phi}_k, \boldsymbol{v})\!)_\kappa = \lambda_k (\!(\boldsymbol{\phi}_k, \boldsymbol{v})\!)_{L^2(\Omega)}$$

for every $\boldsymbol{v} \in \boldsymbol{W}^{1,2}_\kappa$, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$, such that $\lambda_k \to +\infty$ for $k \to +\infty$. It is easy to see that the system of functions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots$ is orthogonal and complete in both $\boldsymbol{W}^{1,2}_\kappa$ and D_κ. Further, let $\alpha \in \mathbb{R}$. Define the space

$$\boldsymbol{V}^\alpha_\kappa = \Big\{\boldsymbol{\phi};\ \boldsymbol{\phi} = \sum_{k=1}^\infty a_k \boldsymbol{\phi}_k,\ a_k \in \mathbb{R} \text{ and } \sum_{k=1}^\infty \lambda_k^\alpha a_k^2 < \infty\Big\}.$$

Note that $\boldsymbol{V}^0_\kappa \equiv \boldsymbol{L}^2_\kappa$, $\boldsymbol{V}^1_\kappa \equiv \boldsymbol{W}^{1,2}_\kappa$ and $\boldsymbol{V}^2_\kappa \equiv D_\kappa$. Let $\boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{V}^\alpha_\kappa$, $\boldsymbol{\varphi} \in \boldsymbol{V}^\beta_\kappa$, $\boldsymbol{\phi} = \sum_{k=1}^\infty a_k \boldsymbol{\phi}_k$, $\boldsymbol{\psi} = \sum_{k=1}^\infty b_k \boldsymbol{\phi}_k$, $\boldsymbol{\varphi} = \sum_{k=1}^\infty c_k \boldsymbol{\phi}_k$. $\boldsymbol{V}^\alpha_\kappa$ are Hilbert spaces with the scalar product

$$(\!(\boldsymbol{\phi}, \boldsymbol{\psi})\!)_{V^\alpha_\kappa} = \sum_{k=1}^\infty \lambda_k^\alpha a_k b_k.$$

By the symbol $\langle \cdot, \cdot \rangle_{\alpha,\beta}$ we denote the bilinear form on $\boldsymbol{V}^\alpha_\kappa \times \boldsymbol{V}^\beta_\kappa$,

$$\langle \boldsymbol{\phi}, \boldsymbol{\varphi} \rangle_{\alpha,\beta} = \sum_{k=1}^\infty \lambda_k^{(\alpha+\beta)/2} a_k c_k.$$

It is clear that $\boldsymbol{V}^{-\alpha}_\kappa = (\boldsymbol{V}^\alpha_\kappa)^*$, where $(\boldsymbol{V}^\alpha_\kappa)^*$ represents the dual space corresponding to $\boldsymbol{V}^\alpha_\kappa$.

Remark 1.2. *It was shown in [4] that*

$$\boldsymbol{V}^2_\kappa \hookrightarrow \boldsymbol{W}^{2,2}(\Omega).$$

Further, recall that $\boldsymbol{V}^0_\kappa \equiv \boldsymbol{L}^2_\kappa$. Applying [24, Chapter II., Lemma 3.2.3] we obtain that

$$\boldsymbol{V}^\alpha_\kappa \hookrightarrow \boldsymbol{W}^{\alpha,2}_\kappa(\Omega) \tag{1.17}$$

holds for every $\alpha \in [0; 2]$.

Analogously, we shall introduce the function spaces for solving the energy equation. Define the bilinear form $((\cdot,\cdot))$ on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ as

$$((u,v)) := \int_\Omega \nabla u \cdot \nabla v + \int_{\Gamma_1} uv, \qquad u, v \in W^{1,2}(\Omega).$$

It is easy to see that this bilinear form is continuous, coercive and symmetric on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$. Thus, there exist eigenfunctions $\xi_1, \xi_2, \ldots \in W^{1,2}(\Omega) \subset L^2(\Omega)$ and real positive numbers $0 < \mu_1 \le \mu_2 \le \mu_3 \ldots$, $\mu_k \to \infty$ for $k \to \infty$, such that

$$((\xi_k, v)) = \mu_k(\xi_k, v)$$

for every $v \in W^{1,2}(\Omega)$. Functions $\xi_1, \xi_2, \ldots$ form a system which is complete in both $L^2(\Omega)$ and $W^{1,2}(\Omega)$, orthonormal in $L^2(\Omega)$ and orthogonal in $W^{1,2}(\Omega)$. Let

$$D = \{\phi; \text{ there exists } \chi \in L^2(\Omega) \text{ such that } ((\phi, v)) = (\chi, v) \quad \forall v \in W^{1,2}(\Omega)\}.$$

It is easy to see that the system $\xi_1, \xi_2, \ldots \in W^{1,2}(\Omega) \subset L^2(\Omega)$ is orthogonal and complete in D. Further, let $s \in \mathbb{R}$ and define the space

$$\mathcal{H}^s = \Big\{\phi;\ \phi = \sum_{k=1}^{\infty} a_k \xi_k,\ a_k \in \mathbb{R} \text{ and } \sum_{k=1}^{\infty} \mu_k^s a_k^2 < \infty\Big\}.$$

In particular, $\mathcal{H}^0 \equiv L^2(\Omega)$, $\mathcal{H}^1 \equiv W^{1,2}(\Omega)$ and $\mathcal{H}^2 \equiv D$. Let $\phi, \psi \in \mathcal{H}^s$, $\varphi \in \mathcal{H}^r$, $\phi = \sum_{k=1}^\infty a_k \xi_k$, $\psi = \sum_{k=1}^\infty b_k \xi_k$, $\varphi = \sum_{k=1}^\infty c_k \xi_k$. $\mathcal{H}^s$, $s \in \mathbb{R}$, are Hilbert spaces with the scalar product

$$(\phi, \psi)_{\mathcal{H}^s} = \sum_{k=1}^{\infty} \mu_k^s a_k b_k.$$

By the symbol $\langle \cdot, \cdot \rangle_{s,r}$ we denote the bilinear form on $\mathcal{H}^s \times \mathcal{H}^r$,

$$\langle \phi, \varphi \rangle_{s,r} = \sum_{k=1}^{\infty} \mu_k^{(r+s)/2} a_k c_k.$$

Finally, it is easily seen that $\mathcal{H}^{-s} = (\mathcal{H}^s)^*$.

Remark 1.3. *Using the regularity of solutions to a Neumann boundary value problem for the Laplace operator in polygonal domains, see [17], we obtain*

$$\mathcal{H}^2 \hookrightarrow W^{2,2}(\Omega).$$

Analogously to (1.17)*, we have*

$$\mathcal{H}^s \hookrightarrow W^{s,2}(\Omega) \tag{1.18}$$

for every $s \in [0;2]$.

Remark 1.4. *In the rest of the paper, we will denote by* ϵ *arbitrarily small positive real number. Throughout the paper, to shorten notation, we set*

$$s_1 = 2\alpha_1 - 1 - \epsilon, \tag{1.19}$$

where

$$\alpha_1 = \min\{\alpha, 3/4\} \tag{1.20}$$

for some given α.

2. The main result

The main result of this paper is formulated in the following theorem.

Theorem 2.1. *Let* $\boldsymbol{f} \in \mathbb{R}^2$, $h \in L^2(\Omega_T)$ *and* $\theta_\infty \in L^2(\Gamma_{1T})$. *Further, let* $\alpha \in (1/2, 1]$ *and suppose that*

$$\boldsymbol{u}_0 \in \boldsymbol{V}_\kappa^\alpha \quad \textit{and} \quad \theta_0 \in \mathcal{H}^{s_1}. \tag{2.1}$$

Then there exist $T^*, 0 < T^* \le T$, *and a weak solution of the problem* (1.7)–(1.15) *on the time interval* $(0, T^*)$.

If $\alpha = 1$ *and* $[\boldsymbol{u}, \theta]$ *is a solution on* $(0, T)$, *then the solution is unique.*

3. Decoupled problems

Before we proceed to prove the main result of this paper, we show the existence, uniqueness and regularity for appropriate linear problems. In particular, we establish the well-posedness of the Stokes problem with the mixed boundary conditions (1.10) and (1.12) and the linear heat equation with the boundary conditions (1.11) and (1.13).

3.1. The heat equation

Let $s \in \mathbb{R}$ and $t \in (0, T]$. We now define function spaces

$$X^\theta_{s,t} := \{\theta;\ \theta \in L^2(0, t; \mathcal{H}^{s+1}) \cap C([0, t]; \mathcal{H}^s),\ \theta' \in L^2(0, t; \mathcal{H}^{s-1})\}$$

and

$$Y^\theta_{s,t} := \{[f, \omega];\ f \in L^2(0, t; \mathcal{H}^{s-1}),\ \omega \in \mathcal{H}^s\}$$

endowed with the norms

$$\|\theta\|_{X^\theta_{s,t}} = \|\theta\|_{L^2(0,t;\mathcal{H}^{s+1})} + \|\theta\|_{C([0,t];\mathcal{H}^s)} + \|\theta'\|_{L^2(0,t;\mathcal{H}^{s-1})}$$

and

$$\|[f, \omega]\|_{Y^\theta_{s,t}} = \|f\|_{L^2(0,t;\mathcal{H}^{s-1})} + \|\omega\|_{\mathcal{H}^s}.$$

Remark 3.1. *By similar arguments as in [25, Chapter III] we may conclude that if a function θ belongs to $L^2(0, t; \mathcal{H}^{s+1})$ and its derivative θ' belongs to $L^2(0, t; \mathcal{H}^{s-1})$, then θ is almost everywhere equal to a function which is continuous from $[0, t]$ into $\mathcal{H}^s$. For technical reasons, the space $C([0, t]; \mathcal{H}^s)$ is directly included in the definition of the space $X^\theta_{s,t}$.*

Let $\phi \in \mathcal{H}^s$, $\phi = \sum_{k=1}^\infty a_k \xi_k$ and Ψ be an operator defined by

$$\Psi(\phi) := \sum_{k=1}^\infty \mu_k a_k \xi_k.$$

It is easy to see that Ψ is the linear continuous operator from $\mathcal{H}^s$ to $\mathcal{H}^{s-2}$ for every $s \in \mathbb{R}$.

The functions $\theta \in L^2(0, t; \mathcal{H}^s)$ are characterized by their series expansion

$$\theta(\tau) = \sum_{k=1}^\infty g_k(\tau)\xi_k,$$

where g_k are measurable functions on $(0, t)$ and

$$\|\theta\|_{L^2(0,t;\mathcal{H}^s)} := \sum_{k=1}^\infty \mu_k^s \int_0^t |g_k(\tau)|^2 \, \mathrm{d}\tau < \infty.$$

Now, let the operator $\mathcal{K}_{s,t}$ be defined as

$$\mathcal{K}_{s,t}(\theta) := \sum_{k=1}^{\infty} \mu_k g_k \xi_k.$$

It is easy to see that $\mathcal{K}_{s,t}$ is the linear continuous operator from $L^2(0,t;\mathcal{H}^s)$ to $L^2(0,t;\mathcal{H}^{s-2})$ for every $s \in \mathbb{R}$. Finally, let $\mathcal{L}_{s,t} : X^{\theta}_{s,t} \to Y^{\theta}_{s,t}$ be the operator defined by the following way: Let $\theta \in X^{\theta}_{s,t}$. Then

$$\mathcal{L}_{s,t}(\theta) = [\theta' + \mathcal{K}_{s,t}(\theta),\ \theta(0)].$$

Lemma 3.2. *Let $s \in (0,1]$ and $[f,\omega] \in Y^{\theta}_{s,t}$. Then there exists unique $\theta \in X^{\theta}_{s,t}$ such that $\mathcal{L}_{s,t}(\theta) = [f,\omega]$ and*

$$\|\theta\|_{X^{\theta}_{s,t}} = \|\mathcal{L}_{s,t}{}^{-1}([f,\omega])\|_{X^{\theta}_{s,t}} \le c\|[f,\omega]\|_{Y^{\theta}_{s,t}} \tag{3.1}$$

with c independent of t.

Proof. Since $[f,\omega] \in Y^{\theta}_{s,t}$ we have

$$f = \sum_{k=1}^{\infty} \nu_k \xi_k \quad \text{and} \quad \omega = \sum_{k=1}^{\infty} a_k \xi_k$$

with ν_k and a_k satisfying

$$\sum_{k=1}^{\infty} \mu_k^{s-1} \int_0^t \nu_k^2(\tau)\mathrm{d}\tau + \sum_{k=1}^{\infty} \mu_k^s a_k^2 < \infty. \tag{3.2}$$

Let ϑ_k be the solution of the ordinary differential equation

$$\vartheta_k'(\tau) + \mu_k \vartheta_k(\tau) = \nu_k(\tau) \tag{3.3}$$

(which holds for almost every $\tau \in (0,t)$) with the initial condition

$$\vartheta_k(0) = a_k, \quad k = 1, 2, \ldots \tag{3.4}$$

Then

$$\vartheta_k(\tau) = \int_0^{\tau} e^{\mu_k(\zeta-\tau)}\nu_k(\zeta)\mathrm{d}\zeta + a_k e^{-\mu_k\tau}$$

for every $\tau \in (0,t)$. Hence $\vartheta_k \in W^{1,2}((0,t))$. Let $\tau \in (0,t)$. Multiplying (3.3) by $2\mu_k^{s-1}\vartheta_k'$ and integrating over $(0,\tau)$ we get

$$2\mu_k^{s-1}\int_0^\tau {\vartheta_k'}^2(\zeta)\mathrm{d}\zeta + \mu_k^s\vartheta_k^2(\tau) = \mu_k^s\vartheta_k^2(0) + 2\mu_k^{s-1}\int_0^\tau \nu_k(\zeta)\vartheta_k'(\zeta)\mathrm{d}\zeta.$$

Consequently, the inequality

$$\mu_k^{s-1}\int_0^\tau {\vartheta_k'}^2(\zeta)\mathrm{d}\zeta + \mu_k^s\vartheta_k^2(\tau) \le \mu_k^s a_k^2 + \mu_k^{s-1}\int_0^\tau \nu_k^2(\zeta)\mathrm{d}\zeta \tag{3.5}$$

holds for $k = 1, 2, \ldots$ and for every $\tau \in (0,t)$. Thus (3.5) yields

$$\sum_{k=1}^\infty \mu_k^{s-1}\int_0^\tau {\vartheta_k'}^2(\zeta)\mathrm{d}\zeta + \sum_{k=1}^\infty \mu_k^s\vartheta_k^2(\tau) \le \sum_{k=1}^\infty \mu_k^s a_k^2 + \sum_{k=1}^\infty \mu_k^{s-1}\int_0^t \nu_k^2(\zeta)\mathrm{d}\zeta \tag{3.6}$$

for every $\tau \in (0,t]$. Let

$$\theta := \sum_{k=1}^\infty \vartheta_k\xi_k.$$

The estimate (3.6), together with (3.2), yields

$$\theta \in L^\infty(0,t;\mathcal{H}^s) \quad \text{ and } \quad \theta' \in L^2(0,t;\mathcal{H}^{s-1})$$

and θ satisfies the inequality

$$\|\theta\|^2_{L^\infty(0,t;\,\mathcal{H}^s)} + \|\theta'\|^2_{L^2(0,t;\mathcal{H}^{s-1})} \le 2\|f\|^2_{L^2(0,t;\mathcal{H}^{s-1})} + 2\|\omega\|^2_{\mathcal{H}^s}. \tag{3.7}$$

Further, (3.3) yields also inequalities

$$\mu_k^{s+1}\vartheta_k^2(\tau) \le 2\mu_k^{s-1}\nu_k^2(\tau) + 2\mu_k^{s-1}{\vartheta_k'}^2(\tau)$$

for every $k = 1, 2, \ldots$ and for almost every $\tau \in (0,t)$. Hence we get

$$\sum_{k=1}^\infty \mu_k^{s+1}\int_0^t \vartheta_k^2(\tau)\mathrm{d}\tau \le 2\sum_{k=1}^\infty\int_0^t \mu_k^{s-1}\nu_k^2(\tau)\mathrm{d}\tau + 2\sum_{k=1}^\infty \mu_k^{s-1}\int_0^t \vartheta_k'^2(\tau)\mathrm{d}\tau.$$

The last inequality and (3.6) yield

$$\sum_{k=1}^\infty \mu_k^{s+1}\int_0^t \vartheta_k^2(\tau)\mathrm{d}\tau \le 4\sum_{k=1}^\infty \mu_k^{s-1}\int_0^t \nu_k^2(\tau)\mathrm{d}\tau + 2\sum_{k=1}^\infty \mu_k^s a_k^2 \tag{3.8}$$

for every $t \in (0,T)$. By (3.2), (3.4) and (3.8) we have $\theta \in L^2(0,t;\mathcal{H}^{s+1})$ such that

$$\|\theta\|_{L^2(0,t;\mathcal{H}^{s+1})} \le 4\|f\|_{L^2(0,t;\mathcal{H}^{s-1})} + 2\|\omega\|_{\mathcal{H}^s}. \tag{3.9}$$

In view of Remark 3.1 and using (3.7) and (3.9) we have $\theta \in C([0,t];\mathcal{H}^s)$ such that

$$\|\theta\|_{L^2(0,t;\mathcal{H}^{s+1})} + \|\theta\|_{C([0,t];\mathcal{H}^s)} + \|\theta'\|_{L^2(0,t;\mathcal{H}^{s-1})} \le 6\|f\|_{L^2(0,t;\mathcal{H}^{s-1})} + 4\|\omega\|_{\mathcal{H}^s}.$$

It is easy to see that $\theta \in X^\theta_{s,t}$, $\mathcal{L}_{s,t}(\theta) = [f,\omega]$ and

$$\|\theta\|_{X^\theta_{s,t}} \le c\left(\|f\|_{L^2(0,t;\mathcal{H}^{s-1})} + \|\omega\|_{\mathcal{H}^s}\right)$$

(with c independent of t).

Finally, suppose that $\theta_1, \theta_2 \in X^\theta_{s,t}$ are solutions of this problem for given data f and ω. We prove that $\theta_1 = \theta_2$.

Denote $\theta_{12} = \theta_1 - \theta_2$. It is easy to see that

$$\langle \mathcal{L}_{s,t}(\theta_{12}), v\rangle_{s-1,s+1} = 0$$

for every $v \in L^2(0,t;\mathcal{H}^{s+1})$ and almost everywhere on $(0,t)$ and

$$\theta_{12}(0) = 0.$$

Put $v = \theta_{12}$. One can verify that

$$\|\theta_{12}(t)\|^2_{\mathcal{H}^s} + 2\|\theta_{12}(\tau)\|^2_{L^2(0,t;\mathcal{H}^{s+1})} = \|\theta_{12}(0)\|^2_{\mathcal{H}^s} = 0.$$

Therefore we get that $\theta_1 = \theta_2$. We have proved that $\theta \in X^\theta_{s,t}$ is the unique solution of the equation $\mathcal{L}_{s,t}(\theta) = [f,\omega]$.

Remark 3.3. *Note that $\mathcal{L}_{s,t}$ is the one-to-one operator from $X^\theta_{s,t}$ to $Y^\theta_{s,t}$ and onto $Y^\theta_{s,t}$. Consequently, ${\mathcal{L}_{s,t}}^{-1}$ is the linear continuous operator from $Y^\theta_{s,t}$ onto $X^\theta_{s,t}$. Further, $\mathcal{L}_{s,t}(\theta) = [f,\omega]$ with $\theta \in X^\theta_{s,t}$ and $[f,\omega] \in Y^\theta_{s,t}$, with some $s \in (0,1]$, iff θ is the solution of the linear heat equation with the right hand side f and the initial data ω in the sense that*

$$\langle \theta'(\tau), v\rangle_{s-1,1-s} + ((\theta(\tau), v)) = \langle f(\tau), v\rangle_{s-1,1-s}$$

for all $v \in W^{1,2}(\Omega)$ and for almost every $\tau \in (0,t)$ and $\theta(0) = \omega$.

3.2. The Stokes problem

We can here proceed analogously to the preceding section. However, let us mention that the results in this section have previously been proved in our work [5].

First, define the spaces

$$X_{\alpha,t} := \{\boldsymbol{w};\ \boldsymbol{w} \in L^2(0,t;\boldsymbol{V}_\kappa^{\alpha+1}) \cap C([0,t];\boldsymbol{V}_\kappa^{\alpha}),\ \boldsymbol{w}' \in L^2(0,t;\boldsymbol{V}_\kappa^{\alpha-1})\}$$

and

$$Y_{\alpha,t} := \{[\boldsymbol{g},\boldsymbol{v}];\ \ \boldsymbol{g} \in L^2(0,t;\ \boldsymbol{V}_\kappa^{\alpha-1}),\ \boldsymbol{v} \in \boldsymbol{V}_\kappa^{\alpha}\}$$

for $t \in (0,T]$ respectively, with the norms

$$\|\boldsymbol{w}\|_{X_{\alpha,t}} = \|\boldsymbol{w}\|_{L^2(0,t;\ V_\kappa^{\alpha+1})} + \|\boldsymbol{w}\|_{C([0,t];V_\kappa^{\alpha})} + \|\boldsymbol{w}'\|_{L^2(0,t;\ V_\kappa^{\alpha-1})}$$

and

$$\|[\boldsymbol{g},\boldsymbol{v}]\|_{Y_{\alpha,t}} = \|\boldsymbol{g}\|_{L^2(0,t;\ V_\kappa^{\alpha-1})} + \|\boldsymbol{v}\|_{V_\kappa^{\alpha}}.$$

Let $\boldsymbol{\phi} \in \boldsymbol{V}_\kappa^{\alpha}$, $\boldsymbol{\phi} = \sum_{k=1}^{\infty} a_k \boldsymbol{\phi}_k$ and $\boldsymbol{\Psi}$ be an operator defined by

$$\boldsymbol{\Psi}(\boldsymbol{\phi}) := \sum_{k=1}^{\infty} \lambda_k a_k \boldsymbol{\phi}_k.$$

It is easy to see that $\boldsymbol{\Psi}$ is the linear continuous operator from $\boldsymbol{V}_\kappa^{\alpha}$ to $\boldsymbol{V}_\kappa^{\alpha-2}$ for every $\alpha \in \mathbb{R}$.

Let $\boldsymbol{u} \in L^2(0,t;\boldsymbol{V}_\kappa^{\alpha})$. It is obvious that

$$\boldsymbol{u}(s) = \sum_{k=1}^{\infty} g_k(s)\boldsymbol{\phi}_k,$$

where g_k are measurable functions on $(0,t)$ and

$$\sum_{k=1}^{\infty} \lambda_k^{\alpha} \int_0^t |g_k(\tau)|^2\, d\tau < \infty.$$

Let the operator $\boldsymbol{\mathcal{A}}$ be defined by

$$\boldsymbol{\mathcal{A}}(\boldsymbol{u}) := \sum_{k=1}^{\infty} \lambda_k g_k \boldsymbol{\phi}_k.$$

It is easy to see that $\mathcal{A}$ is the linear continuous operator from $L^2(0,t;\ \boldsymbol{V}_\kappa^\alpha)$ to $L^2(0,t;\ \boldsymbol{V}_\kappa^{\alpha-2})$ for every $\alpha \in \mathbb{R}$.

Let $\mathcal{S}_{\alpha,t} : X_{\alpha,t} \to Y_{\alpha,t}$ be the operator defined by the following way: Let $\boldsymbol{u} \in X_{\alpha,t}$. Then

$$\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\boldsymbol{u}' + \mathcal{A}(\boldsymbol{u}),\ \boldsymbol{u}(0)].$$

It is easy to see that $\mathcal{S}_{\alpha,t}$ is the linear continuous operator from $X_{\alpha,t}$ to $Y_{\alpha,t}$ for every $\alpha \in \mathbb{R}$.

Lemma 3.4. *Let $\alpha \in (0,1]$ and $[\boldsymbol{g},\boldsymbol{\eta}] \in Y_{\alpha,t}$. Then there exists unique $\boldsymbol{u} \in X_{\alpha,t}$ such that $\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\boldsymbol{g},\boldsymbol{\eta}] \in Y_{\alpha,t}$.*

Proof. See [5].

Remark 3.5. *Using Lemma 3.4 we obtain that $\mathcal{S}_{\alpha,t}$ is the one-to-one operator from $X_{\alpha,t}$ to $Y_{\alpha,t}$ and onto $Y_{\alpha,t}$. Consequently, ${\mathcal{S}_{\alpha,t}}^{-1}$ is the linear continuous operator from $Y_{\alpha,t}$ onto $X_{\alpha,t}$. Using [5] we have*

$$\|\boldsymbol{u}\|_{X_{\alpha,t}} = \|{\mathcal{S}_{\alpha,t}}^{-1}([\boldsymbol{g},\boldsymbol{\eta}])\|_{X_{\alpha,t}} \le c\|[\boldsymbol{g},\boldsymbol{\eta}]\|_{Y_{\alpha,t}}, \tag{3.10}$$

where c is independent of t.

Remark 3.6. *Let $\boldsymbol{u} \in X_{\alpha,t}$, $[\boldsymbol{g},\boldsymbol{\eta}] \in Y_{\alpha,t}$ and $\alpha \in (0,1]$. Note that $\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\boldsymbol{g},\boldsymbol{\eta}]$ if and only if*

$$\langle \boldsymbol{u}'(\tau), \boldsymbol{v}\rangle_{\alpha-1,1-\alpha} + ((\boldsymbol{u}(\tau),\boldsymbol{v}))_\kappa = \langle \boldsymbol{g}(\tau), \boldsymbol{v}\rangle_{\alpha-1,1-\alpha}$$

for all $\boldsymbol{v} \in \boldsymbol{W}_\kappa^{1,2}$ and for almost every $\tau \in (0,t)$ and $\boldsymbol{u}(0) = \boldsymbol{\eta}$. Note that $\boldsymbol{u}$ is a solution of the so-called non-steady Stokes system with the right hand side $\boldsymbol{g}$ and the initial velocity $\boldsymbol{\eta}$.

4. Properties of some function spaces

First, let us present some embedding theorems which will be used throughout the paper. In particular, using (1.17), (1.18) and Theorem 8.3.3.(i) and Theorem 5.7.5 in [16], we have

$$\begin{cases} \boldsymbol{V}_\kappa^{1-\alpha} \hookrightarrow \boldsymbol{L}^q(\Omega), & 1 \le q < 2/\alpha, \quad \alpha \in (1/2,1], \\ \boldsymbol{V}_\kappa^{1} \hookrightarrow \boldsymbol{L}^q(\Omega), & 1 \le q < +\infty, \\ \boldsymbol{V}_\kappa^{0} \hookrightarrow \boldsymbol{L}^2(\Omega) & \end{cases} \tag{4.1}$$

and

$$\begin{cases} \mathcal{H}^1 \hookrightarrow L^q(\Omega), & 1 \le q < +\infty, \\ \mathcal{H}^s \hookrightarrow L^q(\Omega), & 1 \le q < 2/(1-s), \quad 0 < s < 1, \\ \mathcal{H}^s \hookrightarrow L^q(\Omega), & 1 \le q \le +\infty, \qquad\qquad s > 1. \end{cases} \tag{4.2}$$

For the convenience of the reader we also recall the well-known embeddings

$$W^{1,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & 1 \le q < +\infty, \qquad\qquad p = 2, \\ L^q(\Omega), & 1 \le q \le 2p/(2-p), \quad 1 \le p < 2, \\ L^\infty(\Omega), & p > 2 \end{cases} \tag{4.3}$$

and

$$W^{k,p}(\Omega) \hookrightarrow W^{r,q}(\Omega) \tag{4.4}$$

provided

$$1 < p < q < \infty, \quad k > 0, \quad r = k - 2\left(\frac{1}{p} - \frac{1}{q}\right) > 0.$$

Further, let $\beta_1 > \beta_2 > \beta_3$ and $\boldsymbol{\phi} \in \boldsymbol{V}_\kappa^{\beta_1}$. Then $\boldsymbol{\phi} \in \boldsymbol{V}_\kappa^{\beta_2} \subset \boldsymbol{V}_\kappa^{\beta_3}$ and

$$\|\boldsymbol{\phi}\|_{V_\kappa^{\beta_2}} \le \|\boldsymbol{\phi}\|_{V_\kappa^{\beta_1}}^{(\beta_2-\beta_3)/(\beta_1-\beta_3)} \|\boldsymbol{\phi}\|_{V_\kappa^{\beta_3}}^{(\beta_1-\beta_2)/(\beta_1-\beta_3)}. \tag{4.5}$$

One can see that

$$\boldsymbol{V}_\kappa^{\beta_1} \hookrightarrow\hookrightarrow \boldsymbol{V}_\kappa^{\beta_2}$$

for $\beta_1, \beta_2 \in \mathbb{R}$, $\beta_1 > \beta_2$. Let α be given by Theorem 2.1. We have

$$\boldsymbol{V}_\kappa^{\alpha+1} \hookrightarrow\hookrightarrow \boldsymbol{V}_\kappa^{r} \hookrightarrow\hookrightarrow \boldsymbol{V}_\kappa^{\alpha-1}$$

for every r, $0 < r < 1$. Applying [25, Chapter III, Theorem 2.1.] we get

$$X_{\alpha,t} \hookrightarrow\hookrightarrow L^2(0,t;\boldsymbol{V}_\kappa^{q}), \quad q < \alpha + 1, \tag{4.6}$$

and

$$X_{\alpha,t} \hookrightarrow\hookrightarrow L^p(0,t;\boldsymbol{V}_\kappa^{\alpha}) \tag{4.7}$$

for every p, $2 \le p < \infty$. Using (1.17), (4.5) (for $\beta_1 = \frac{8+7\alpha}{8}$, $\beta_2 = \frac{2+\alpha}{2}$, $\beta_3 = \alpha$) and (4.6) (for $q = \frac{8+7\alpha}{8}$) one obtains the embedding

$$X_{\alpha,t} \hookrightarrow\hookrightarrow L^{(8-\alpha)/(4-2\alpha)}(0,t;\boldsymbol{V}_\kappa^{(2+\alpha)/2}) \hookrightarrow L^{(8-\alpha)/(4-2\alpha)}(0,t;\boldsymbol{W}^{1,4/(2-\alpha)}(\Omega)). \tag{4.8}$$

Using (4.7) and [16, Part 8.3.3.(i)] we obtain the embedding

$$X_{\alpha,t} \hookrightarrow\hookrightarrow L^{(16-2\alpha)/\alpha}(0,t;\boldsymbol{V}_\kappa^\alpha) \hookrightarrow L^{(16-2\alpha)/\alpha}(0,t;\boldsymbol{L}^{16/(8-7\alpha)}(\Omega)). \tag{4.9}$$

Let

$$Z_{\alpha,t} := L^{(8-\alpha)/(4-2\alpha)}(0,t;\boldsymbol{V}_\kappa^{(2+\alpha)/2}) \cap L^{(16-2\alpha)/\alpha}(0,t;\boldsymbol{V}_\kappa^\alpha)$$

be the reflexive Banach space with the norm

$$\|.\|_{Z_{\alpha,t}} = \|.\|_{L^{(8-\alpha)/(4-2\alpha)}(0,t;V_\kappa^{(2+\alpha)/2})} + \|.\|_{L^{(16-2\alpha)/\alpha}(0,t;V_\kappa^\alpha)}.$$

(4.8) and (4.9) yield that

$$X_{\alpha,t} \hookrightarrow\hookrightarrow Z_{\alpha,t} \text{ and } \|\boldsymbol{u}\|_{Z_{\alpha,t}} \le c\|\boldsymbol{u}\|_{X_{\alpha,t}} \tag{4.10}$$

for all $\boldsymbol{u} \in X_{\alpha,t}$ (with c independent of t).

Remark 4.1. *To handle the dissipative term $\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{u})$ in the energy equation (1.9) we need the following estimates and embeddings. First, let $\boldsymbol{u} \in X_{\alpha,t}$ with some $\alpha \in (1/2, 1]$. Raising and integrating the interpolation inequality (see [18, Theorem 9.6])*

$$\|\boldsymbol{u}(\tau)\|_{V_\kappa^{\alpha+1/2}} \le c\|\boldsymbol{u}(\tau)\|_{V_\kappa^{\alpha+1}}^{1/2}\|\boldsymbol{u}(\tau)\|_{V_\kappa^\alpha}^{1/2}$$

from 0 to t, we obtain

$$\begin{aligned}\left(\int_0^t \|\boldsymbol{u}(\tau)\|_{V_\kappa^{\alpha+1/2}}^4 \mathrm{d}s\right)^{1/4} &\le c\left(\int_0^t \|\boldsymbol{u}(\tau)\|_{V_\kappa^{\alpha+1}}^2\|\boldsymbol{u}(\tau)\|_{V_\kappa^\alpha}^2 \mathrm{d}s\right)^{1/4}\\ &\le c\|\boldsymbol{u}\|_{L^2(0,t;V_\kappa^{\alpha+1})}^{1/2}\|\boldsymbol{u}\|_{L^\infty(0,t;V_\kappa^\alpha)}^{1/2}\\ &\le c\|\boldsymbol{u}\|_{X_{\alpha,t}},\end{aligned}$$

where c is independent of t. Hence, we have

$$X_{\alpha,t} \hookrightarrow L^4(0,t;\boldsymbol{V}_\kappa^{\alpha+1/2})$$

and, using the embedding

$$\boldsymbol{V}_\kappa^{\alpha+1/2} \hookrightarrow \boldsymbol{W}_\kappa^{1,4/(3-2\alpha)},$$

we also have

$$X_{\alpha,t} \hookrightarrow L^4(0,t;\boldsymbol{W}_\kappa^{1,4/(3-2\alpha)}), \qquad \|\boldsymbol{u}\|_{L^4(0,t;W_\kappa^{1,4/(3-2\alpha)})} \leq c\|\boldsymbol{u}\|_{X_{\alpha,t}} \tag{4.11}$$

for all $\boldsymbol{u} \in X_{\alpha,t}$ *(with* c *independent of* t*). Thus we have*

$$X_{\alpha,t} \hookrightarrow L^4(0,t;\boldsymbol{W}_\kappa^{1,2+\delta_0}), \qquad \|\boldsymbol{u}\|_{L^4(0,t;W_\kappa^{1,2+\delta_0})} \leq c\|\boldsymbol{u}\|_{X_{\alpha,t}}$$

for all $\boldsymbol{u} \in X_{\alpha,t}$*, with* $\delta_0 = (4\alpha-2)/(3-2\alpha) > 0$*.*

Define the space

$$Z^\theta_{\alpha,t} := L^{1/(1-\alpha+\epsilon)}(0,t;\mathcal{H}^{1+\epsilon/2}).$$

Using the interpolation inequality analogous to that in (4.5) we deduce

$$\|\theta\|_{L^{1/(1-\alpha+\epsilon)}(0,t;\mathcal{H}^{1+\epsilon})} \leq c\|\theta\|^{2-2\alpha+2\epsilon}_{L^2(0,t;\mathcal{H}^{2\alpha-\epsilon})}\|\theta\|^{2\alpha-1-2\epsilon}_{L^\infty(0,t;\mathcal{H}^{2\alpha-1-\epsilon})}$$

for all $\theta \in X^\theta_{2\alpha-1-\epsilon,t}$. Hence we have

$$X^\theta_{2\alpha-1-\epsilon,t} \hookrightarrow L^{1/(1-\alpha+\epsilon)}(0,t;\mathcal{H}^{1+\epsilon}).$$

Finally, by the compact embedding

$$\mathcal{H}^{1+\epsilon} \hookrightarrow\hookrightarrow \mathcal{H}^{1+\epsilon/2}$$

we also have

$$X^\theta_{2\alpha-1-\epsilon,t} \hookrightarrow\hookrightarrow Z^\theta_{\alpha,t}, \text{ and } \|\theta\|_{Z^\theta_{\alpha,t}} \leq c\|\theta\|_{X^\theta_{2\alpha-1-\epsilon,t}} \tag{4.12}$$

for all $\theta \in X^\theta_{2\alpha-1-\epsilon,t}$ (and with c independent of t).

5. Proof of Theorem 2.1

5.1. Existence of the solution

The existence of the variational solution is based on a fixed point argument. First we briefly outline the structure of the proof in the following four steps.

Step 1. For a given couple $\boldsymbol{w} \in Z_{\alpha,t}$ and $\vartheta \in Z^{\theta}_{\alpha_1,t}$, where $\alpha_1 = \min\{\alpha, 3/4\}$, we consider the problem

$$\mathcal{S}_{\alpha,t}(\boldsymbol{u}) = [\mathcal{F}_{\alpha,t}(\vartheta) - \mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w}), \boldsymbol{u}_0], \tag{5.1}$$

where $\mathcal{F}_{\alpha,t}$ is the operator corresponding to the right hand side of (1.7) and $\mathcal{B}_{\alpha,t}$ corresponds to the convective term in (1.7). Applying Lemma 3.4 we show the existence and uniqueness of $\boldsymbol{u} \in X_{\alpha,t}$ for this problem.

Step 2. Now, with $\boldsymbol{u} \in X_{\alpha,t}$ in hand, consider the problem

$$\mathcal{L}_{s,t}(\theta) = [\mu_{\alpha,t} - \mathcal{C}_{\alpha,t}(\boldsymbol{u}, \vartheta), \theta_0], \tag{5.2}$$

where the operator $\mathcal{C}_{\alpha,t}$ corresponds to the convective, dissipative and adiabatic terms in the energy equation (1.9) and $\mu_{\alpha,t}$ represents the data of the problem, namely θ_∞ and h, see (5.20) below. The existence and uniqueness of $\theta \in Z^{\theta}_{\alpha_1,t}$ is proven using Lemma 3.2.

Step 3. Recall that for given $\boldsymbol{w} \in Z_{\alpha,t}$ and $\vartheta \in Z^{\theta}_{\alpha_1,t}$ we have uniquely determined $\boldsymbol{u} \in X_{\alpha,t} \hookrightarrow Z_{\alpha,t}$ and $\theta \in Z^{\theta}_{\alpha_1,t}$. Now, consider the mapping $\mathcal{M}_{\alpha,t} : Z_{\alpha,t} \times Z^{\theta}_{\alpha_1,t} \to Z_{\alpha,t} \times Z^{\theta}_{\alpha_1,t}$ such that

$$[\boldsymbol{u}, \theta] = \mathcal{M}_{\alpha,t}(\boldsymbol{w}, \vartheta). \tag{5.3}$$

Using some a priori estimates we show that $\mathcal{M}_{\alpha,t}$ is completely continuous and there exists a closed ball $B \subset Z_{\alpha,t} \times Z^{\theta}_{\alpha_1,t}$ such that $\mathcal{M}_{\alpha,t}(B) \subset B$ for all $0 \le t \le T_1$, with some (sufficiently small) T_1.

Step 4. Finally, we apply Schauder fixed point theorem to conclude that there exists a couple $[\boldsymbol{u}, \theta] \in Z_{\alpha,t} \times Z^{\theta}_{\alpha_1,t}$ such that $[\boldsymbol{u}, \theta] = \mathcal{M}_{\alpha,t}(\boldsymbol{u}, \theta)$. Here, $[\boldsymbol{u}, \theta]$ corresponds to the weak solution of the problem (1.7)–(1.15) on the time interval $(0, T_1)$.

According to the particular steps introduced above, we now proceed with the detailed proof of the existence result.

First, we deal with the operator equation (5.1) and define the operators $\mathcal{F}_{\alpha,t}$ and $\mathcal{B}_{\alpha,t}$ and prove their continuity. To do this, take $\vartheta \in Z^{\theta}_{\alpha_1,t}$, $\boldsymbol{w}, \boldsymbol{z} \in Z_{\alpha,t}$ and $\boldsymbol{v} \in \boldsymbol{V}^{1-\alpha}_{\kappa} = (\boldsymbol{V}^{\alpha-1}_{\kappa})^*$. Then $\langle \mathcal{F}_{\alpha,t}(\vartheta), \boldsymbol{v} \rangle_{\alpha-1,1-\alpha} : (0,t) \to \mathbb{R}$ and $\langle \mathcal{B}_{\alpha,t}(\boldsymbol{w}, \boldsymbol{z}), \boldsymbol{v} \rangle_{\alpha-1,1-\alpha} : (0,t) \to \mathbb{R}$ are functions such that

$$\langle \mathcal{F}_{\alpha,t}(\vartheta), \boldsymbol{v} \rangle_{\alpha-1,1-\alpha}(\tau) = \int_\Omega (1 - \vartheta(\tau)) \boldsymbol{f} \cdot \boldsymbol{v}$$

and

$$\langle \mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{z}),\boldsymbol{v}\rangle_{\alpha-1,1-\alpha}(\tau) = \int_\Omega \big(\boldsymbol{w}(\tau)\cdot\nabla\big)\boldsymbol{z}(\tau)\cdot\boldsymbol{v}$$

for almost all $\tau \in (0,t)$.

Let $\vartheta_1,\vartheta_2 \in Z^\theta_{\alpha_1,t}$ and $\boldsymbol{w}_1,\boldsymbol{w}_2 \in Z_{\alpha,t}$. Then

$$\begin{aligned}|\langle \mathcal{F}_{\alpha,t}(\vartheta_1)-\mathcal{F}_{\alpha,t}(\vartheta_2),\boldsymbol{v}\rangle_{\alpha-1,1-\alpha}(\tau)| &= \left|\int_\Omega(\vartheta_2(\tau)-\vartheta_1(\tau))\boldsymbol{f}\cdot\boldsymbol{v}\right| \\ &\le c_1\|\vartheta_2(\tau)-\vartheta_1(\tau)\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)} \\ &\le c_2\|\vartheta_2(\tau)-\vartheta_1(\tau)\|_{W^{1+\epsilon/2,2}(\Omega)}\|\boldsymbol{v}\|_{V^0_\kappa} \\ &\le c_3\|\vartheta_2(\tau)-\vartheta_1(\tau)\|_{\mathcal{H}^{1+\epsilon/2}}\|\boldsymbol{v}\|_{V^{1-\alpha}_\kappa}\end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{V}^{1-\alpha}_\kappa$ and for almost every $\tau \in (0,t)$ and hence

$$\|\mathcal{F}_{\alpha,t}(\vartheta_1)-\mathcal{F}_{\alpha,t}(\vartheta_2)\|_{L^2(0,t;V^{\alpha-1}_\kappa)} \le c\,t^{(2\alpha_1-1-2\epsilon)/2}\|\vartheta_2-\vartheta_1\|_{Z^\theta_{\alpha_1,t}}. \tag{5.4}$$

Using (4.1), (4.7) and (4.9) we have

$$\begin{aligned}&|\langle \mathcal{B}_{\alpha,t}(\boldsymbol{w}_1,\boldsymbol{w}_1)-\mathcal{B}_{\alpha,t}(\boldsymbol{w}_2,\boldsymbol{w}_2),\boldsymbol{v}\rangle_{\alpha-1,1-\alpha}(\tau)| \\ &\quad= |\langle \mathcal{B}_{\alpha,t}(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{w}_1)+\mathcal{B}_{\alpha,t}(\boldsymbol{w}_2,\boldsymbol{w}_1-\boldsymbol{w}_2),\boldsymbol{v}\rangle_{\alpha-1,1-\alpha}(\tau)| \\ &\quad\le c_1\|\boldsymbol{w}_1(\tau)-\boldsymbol{w}_2(\tau)\|_{L^{16/(8-7\alpha)}(\Omega)}\|\boldsymbol{w}_1(\tau)\|_{W^{1,4/(2-\alpha)}(\Omega)}\|\boldsymbol{v}\|_{L^{16/11\alpha}(\Omega)} \\ &\qquad\quad+ c_2\|\boldsymbol{w}_2(\tau)\|_{L^{16/(8-7\alpha)}(\Omega)}\|\boldsymbol{w}_1(\tau)-\boldsymbol{w}_2(\tau)\|_{W^{1,4/(2-\alpha)}(\Omega)}\|\boldsymbol{v}\|_{L^{16/11\alpha}(\Omega)} \\ &\quad\le c_1\|\boldsymbol{w}_1(\tau)-\boldsymbol{w}_2(\tau)\|_{V^\alpha_\kappa}\|\boldsymbol{w}_1(\tau)\|_{V^{(2+\alpha)/2}_\kappa}\|\boldsymbol{v}\|_{V^{1-\alpha}_\kappa} \\ &\qquad\quad+ c_2\|\boldsymbol{w}_2(\tau)\|_{V^\alpha_\kappa}\|\boldsymbol{w}_1(\tau)-\boldsymbol{w}_2(\tau)\|_{V^{(2+\alpha)/2}_\kappa}\|\boldsymbol{v}\|_{V^{1-\alpha}_\kappa}\end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{V}^{1-\alpha}_\kappa$ and for almost every $\tau \in (0,t)$. Thus, we can write

$$\begin{aligned}&\|\mathcal{B}_{\alpha,t}(\boldsymbol{w}_1,\boldsymbol{w}_1)-\mathcal{B}_{\alpha,t}(\boldsymbol{w}_2,\boldsymbol{w}_2)\|_{L^2(0,t;V^{\alpha-1}_\kappa)} \\ &\quad\le c_1\,t^{\alpha/(8-\alpha)}\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^{(16-2\alpha)/\alpha}(0,t;\ V^\alpha_\kappa)}\|\boldsymbol{w}_1\|_{L^{(8-\alpha)/(4-2\alpha)}(0,t;\ V^{(2+\alpha)/2}_\kappa)} \\ &\qquad+ c_2\,t^{\alpha/(8-\alpha)}\|\boldsymbol{w}_2\|_{L^{(16-2\alpha)/\alpha}(0,t;\ V^\alpha_\kappa)}\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^{(8-\alpha)/(4-2\alpha)}(0,t;\ V^{(2+\alpha)/2}_\kappa)} \\ &\quad\le c_3\,t^{\alpha/(8-\alpha)}\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{Z_{\alpha,t}}\left(\|\boldsymbol{w}_1\|_{Z_{\alpha,t}}+\|\boldsymbol{w}_2\|_{Z_{\alpha,t}}\right).\end{aligned} \tag{5.5}$$

We have verified that the operators

$$\mathcal{F}_{\alpha,t} : Z^\theta_{\alpha_1,t} \to L^2(0,t;\boldsymbol{V}^{\alpha-1}_\kappa) \tag{5.6}$$

and

$$\mathcal{B}_{\alpha,t} : Z_{\alpha,t} \times Z_{\alpha,t} \to L^2(0,t;\boldsymbol{V}_\kappa^{\alpha-1}) \tag{5.7}$$

are well defined and continuous operators with respect to norm topologies indicated in (5.6) and (5.7). Let $\vartheta \in Z^\theta_{\alpha_1,t}$, $\boldsymbol{w} \in Z_{\alpha,t}$ and let $\boldsymbol{f}$ and $\boldsymbol{u}_0$ be given by Theorem 2.1. Then $[\mathcal{F}_{\alpha,t}(\vartheta) - \mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w}), \boldsymbol{u}_0] \in Y_{\alpha,t}$. Lemma 3.4 now gives the uniquely determined $\boldsymbol{u} \in X_{\alpha,t}$, the solution of the operator equation (5.1). Finally, let $\mathcal{T}_{\alpha,t} : Z_{\alpha,t} \times Z^\theta_{\alpha_1,t} \to X_{\alpha,t}$ be the operator defined by the identity

$$\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta) = (\mathcal{S}_{\alpha,t})^{-1}\big(\big[\mathcal{F}_{\alpha,t}(\vartheta) - \mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w}), \boldsymbol{u}_0\big]\big).$$

Clearly, the mapping

$$\mathcal{T}_{\alpha,t} : \quad [\boldsymbol{w},\vartheta] \to \boldsymbol{u} : \quad Z_{\alpha,t} \times Z^\theta_{\alpha_1,t} \to X_{\alpha,t} \tag{5.8}$$

is a continuous operator, which follows from Remark 3.5 and continuity of $\mathcal{F}_{\alpha,t}$ and $\mathcal{B}_{\alpha,t}$.

Now, let us examine the equation (5.2). To do this, define the operator $\mathcal{C}_{\alpha,t}$ corresponding to the convective, dissipative and adiabatic terms in the energy equation (1.9). Let $\boldsymbol{z} \in X_{\alpha,t}$, $\vartheta \in Z^\theta_{\alpha_1,t}$ and $\varphi \in W^{\epsilon+2(1-\alpha_1),2}(\Omega) \equiv W^{1-s_1,2}(\Omega)$ (according to notation (1.19)). Then $\langle \mathcal{C}_{\alpha,t}(\boldsymbol{z},\vartheta), \varphi\rangle_{s_1-1,1-s_1} : (0,t) \to \mathbb{R}$ is a function such that

$$\langle \mathcal{C}_{\alpha,t}(\boldsymbol{z},\vartheta), \varphi\rangle_{s_1-1,1-s_1}(\tau) = \int_\Omega \boldsymbol{e}(\boldsymbol{z}(\tau)) : \boldsymbol{e}(\boldsymbol{z}(\tau))\varphi + \int_\Omega \vartheta(\tau)\boldsymbol{f}\cdot\boldsymbol{z}(\tau)\varphi - \int_\Omega \boldsymbol{z}(\tau)\cdot\nabla\vartheta(\tau)\varphi$$

for almost all $\tau \in (0,t)$.

Let $\vartheta_1, \vartheta_2 \in Z^\theta_{\alpha_1,t}$ and $\boldsymbol{z}_1, \boldsymbol{z}_2 \in X_{\alpha,t}$. Note that in view of (1.20) we have $X_{\alpha,t} \hookrightarrow X_{\alpha_1,t}$ and

$$\|\boldsymbol{v}\|_{X_{\alpha_1,t}} \le c\|\boldsymbol{v}\|_{X_{\alpha,t}} \tag{5.9}$$

for all $\boldsymbol{v} \in X_{\alpha,t}$, where c is independent of t. Then

$$\begin{aligned}
&|\langle \mathcal{C}_{\alpha,t}(\boldsymbol{z}_1,\vartheta_1) - \mathcal{C}_{\alpha,t}(\boldsymbol{z}_2,\vartheta_2), \varphi\rangle_{s_1-1,1-s_1}(\tau)| \\
&\quad \leq \left| \int_\Omega (\boldsymbol{e}(\boldsymbol{z}_1(\tau)) - \boldsymbol{e}(\boldsymbol{z}_2(\tau))) : \boldsymbol{e}(\boldsymbol{z}_1(\tau))\varphi \right| \qquad (5.10) \\
&\qquad + \left| \int_\Omega \boldsymbol{e}(\boldsymbol{z}_2(\tau)) : (\boldsymbol{e}(\boldsymbol{z}_1(\tau)) - \boldsymbol{e}(\boldsymbol{z}_2(\tau)))\,\varphi \right| \\
&\qquad + \left| \int_\Omega (\vartheta_1(\tau) - \vartheta_2(\tau))\,\boldsymbol{f}\cdot \boldsymbol{z}_1(\tau)\varphi \right| + \left| \int_\Omega \vartheta_2(\tau)\boldsymbol{f}\cdot(\boldsymbol{z}_1(\tau) - \boldsymbol{z}_2(\tau))\,\varphi \right| \\
&\qquad + \left| \int_\Omega (\boldsymbol{z}_1(\tau) - \boldsymbol{z}_2(\tau))\cdot \nabla\vartheta_2(\tau)\varphi \right| + \left| \int_\Omega \boldsymbol{z}_1(\tau)\cdot\nabla\left(\vartheta_1(\tau) - \vartheta_2(\tau)\right)\varphi \right|. \qquad (5.11)
\end{aligned}$$

Now, we estimate successively all terms on the right hand side of (5.11). Using the Hölder inequality and the embeddings (4.1) and (4.2) we have

$$\begin{aligned}
&\left| \int_\Omega (\boldsymbol{e}(\boldsymbol{z}_1(\tau)) - \boldsymbol{e}(\boldsymbol{z}_2(\tau))) : \boldsymbol{e}(\boldsymbol{z}_1(\tau))\varphi \right| \\
&\quad \leq c_1 \|\boldsymbol{e}(\boldsymbol{z}_1(\tau)) - \boldsymbol{e}(\boldsymbol{z}_2(\tau))\|_{L^{4/(3-2\alpha_1)}(\Omega)} \|\boldsymbol{e}(\boldsymbol{z}_1(\tau))\|_{L^{4/(3-2\alpha_1)}(\Omega)} \|\varphi\|_{L^{2/(2\alpha_1-1)}(\Omega)} \\
&\quad \leq c_2 \|\boldsymbol{z}_1(\tau) - \boldsymbol{z}_2(\tau)\|_{W^{1,4/(3-2\alpha_1)}_\kappa} \|\boldsymbol{z}_1(\tau)\|_{W^{1,4/(3-2\alpha_1)}_\kappa} \|\varphi\|_{\mathcal{H}^{\epsilon+2(1-\alpha_1)}} \qquad (5.12)
\end{aligned}$$

and in the same manner we can see, using (4.3), that

$$\begin{aligned}
&\left| \int_\Omega \boldsymbol{e}(\boldsymbol{z}_2(\tau)) : (\boldsymbol{e}(\boldsymbol{z}_1(\tau)) - \boldsymbol{e}(\boldsymbol{z}_2(\tau)))\,\varphi \right| \\
&\quad \leq c_1 \|\boldsymbol{e}(\boldsymbol{z}_2(\tau))\|_{L^{4/(3-2\alpha_1)}(\Omega)} \|\boldsymbol{e}(\boldsymbol{z}_1(\tau)) - \boldsymbol{e}(\boldsymbol{z}_2(\tau))\|_{L^{4/(3-2\alpha_1)}(\Omega)} \|\varphi\|_{L^{2/(2\alpha_1-1)}(\Omega)} \\
&\quad \leq c_2 \|\boldsymbol{z}_2(\tau)\|_{W^{1,4/(3-2\alpha_1)}_\kappa} \|\boldsymbol{z}_1(\tau) - \boldsymbol{z}_2(\tau)\|_{W^{1,4/(3-2\alpha_1)}_\kappa} \|\varphi\|_{\mathcal{H}^{\epsilon+2(1-\alpha_1)}}\,. \qquad (5.13)
\end{aligned}$$

For the adiabatic terms, we can write (using the Hölder inequality and (4.2))

$$\begin{aligned}
&\left| \int_\Omega (\vartheta_1(\tau) - \vartheta_2(\tau))\,\boldsymbol{f}\cdot\boldsymbol{z}_1(\tau)\varphi \right| \\
&\quad \leq c_1 \|\boldsymbol{z}_1(\tau)\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)} \|\vartheta_1(\tau) - \vartheta_2(\tau)\|_{L^{4/(2-\epsilon)}(\Omega)} \|\varphi\|_{L^{2/(2\alpha_1-1)}(\Omega)} \\
&\quad \leq c_2 \|\boldsymbol{z}_1(\tau)\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)} \|\vartheta_1(\tau) - \vartheta_2(\tau)\|_{\mathcal{H}^{1+\epsilon/2}} \|\varphi\|_{\mathcal{H}^{\epsilon+2(1-\alpha_1)}} \qquad (5.14)
\end{aligned}$$

and, similarly,

$$\left|\int_\Omega \vartheta_2(\tau)\boldsymbol{f}\cdot(\boldsymbol{z}_1(\tau)-\boldsymbol{z}_2(\tau))\,\varphi\right|$$
$$\le c_1\left\|\boldsymbol{z}_1(\tau)-\boldsymbol{z}_2(\tau)\right\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)}\left\|\vartheta_2(\tau)\right\|_{L^{4/(2-\epsilon)}(\Omega)}\left\|\varphi\right\|_{L^{2/(2\alpha_1-1)}(\Omega)}$$
$$\le c_2\left\|\boldsymbol{z}_1(\tau)-\boldsymbol{z}_2(\tau)\right\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)}\left\|\vartheta_2(\tau)\right\|_{\mathcal{H}^{1+\epsilon/2}}\left\|\varphi\right\|_{\mathcal{H}^{\epsilon+2(1-\alpha_1)}}\,. \quad (5.15)$$

Finally, for the convective terms we have

$$\left|\int_\Omega (\boldsymbol{z}_1(\tau)-\boldsymbol{z}_2(\tau))\cdot\nabla\vartheta_2(\tau)\varphi\right|$$
$$\le c_1\left\|\boldsymbol{z}_1(\tau)-\boldsymbol{z}_2(\tau)\right\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)}\left\|\nabla\vartheta_2(\tau)\right\|_{L^{4/(2-\epsilon)}(\Omega)}\left\|\varphi\right\|_{L^{2/(2\alpha_1-1)}(\Omega)}$$
$$\le c_2\left\|\boldsymbol{z}_1(\tau)-\boldsymbol{z}_2(\tau)\right\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)}\left\|\vartheta_2(\tau)\right\|_{W^{1,4/(2-\epsilon)}(\Omega)}\left\|\varphi\right\|_{W^{\epsilon+2(1-\alpha_1),2}(\Omega)}$$
$$\le c_3\left\|\boldsymbol{z}_1(\tau)-\boldsymbol{z}_2(\tau)\right\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)}\left\|\vartheta_2(\tau)\right\|_{\mathcal{H}^{1+\epsilon/2}}\left\|\varphi\right\|_{\mathcal{H}^{\epsilon+2(1-\alpha_1)}}\,, \quad (5.16)$$

where we have used (4.4) in the form

$$\mathcal{H}^{1+\epsilon/2}\hookrightarrow W^{1+\epsilon/2,2}(\Omega)\hookrightarrow W^{1,4/(2-\epsilon)}(\Omega)$$

and similarly

$$\left|\int_\Omega \boldsymbol{z}_1(\tau)\cdot\nabla\left(\vartheta_1(\tau)-\vartheta_2(\tau)\right)\varphi\right|$$
$$\le c_1\left\|\boldsymbol{z}_1(\tau)\right\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)}\left\|\nabla\left(\vartheta_1(\tau)-\vartheta_2(\tau)\right)\right\|_{L^{4/(2-\epsilon)}(\Omega)}\left\|\varphi\right\|_{L^{2/(2\alpha_1-1)}(\Omega)}$$
$$\le c_2\left\|\boldsymbol{z}_1(\tau)\right\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)}\left\|\vartheta_1(\tau)-\vartheta_2(\tau)\right\|_{W^{1,4/(2-\epsilon)}(\Omega)}\left\|\varphi\right\|_{W^{\epsilon+2(1-\alpha_1),2}(\Omega)}$$
$$\le c_3\left\|\boldsymbol{z}_1(\tau)\right\|_{L^{1/(1-\alpha_1+\epsilon/4)}(\Omega)}\left\|\vartheta_1(\tau)-\vartheta_2(\tau)\right\|_{\mathcal{H}^{1+\epsilon/2}}\left\|\varphi\right\|_{\mathcal{H}^{\epsilon+2(1-\alpha_1)}}\,. \quad (5.17)$$

Hence, combining (5.12)–(5.17) together with (5.11) we deduce

$$\left\|\mathcal{C}_{\alpha,t}(\boldsymbol{z}_1,\vartheta_1)-\mathcal{C}_{\alpha,t}(\boldsymbol{z}_2,\vartheta_2)\right\|_{L^2(0,t;\mathcal{H}^{2(\alpha_1-1)-\epsilon})}$$
$$\le c_1\left\|\boldsymbol{z}_1-\boldsymbol{z}_2\right\|_{L^4(0,t;W_\kappa^{1,4/(3-2\alpha_1)})}\left\|\boldsymbol{z}_1\right\|_{L^4(0,t;W_\kappa^{1,4/(3-2\alpha_1)})}$$
$$+\,c_2\left\|\boldsymbol{z}_1-\boldsymbol{z}_2\right\|_{L^4(0,t;W_\kappa^{1,4/(3-2\alpha_1)})}\left\|\boldsymbol{z}_2\right\|_{L^4(0,t;W_\kappa^{1,4/(3-2\alpha_1)})}$$
$$+\,c_3\left\|\boldsymbol{z}_1\right\|_{L^\infty(0,t;L^{1/(1-\alpha_1+\epsilon/4)}(\Omega))}\left\|\vartheta_1-\vartheta_2\right\|_{L^2(0,t;\mathcal{H}^{1+\epsilon/2})}$$
$$+\,c_4\left\|\boldsymbol{z}_1-\boldsymbol{z}_2\right\|_{L^\infty(0,t;L^{1/(1-\alpha_1+\epsilon/4)}(\Omega))}\left\|\vartheta_2\right\|_{L^2(0,t;\mathcal{H}^{1+\epsilon/2})}\,. \quad (5.18)$$

Note that using the Hölder inequality we can write

$$\|\vartheta_1 - \vartheta_2\|_{L^2(0,t;\mathcal{H}^{1+\epsilon/2})} \leq t^{(2\alpha_1-1-2\epsilon)/2} \|\vartheta_1 - \vartheta_2\|_{Z^\theta_{\alpha_1,t}} ,$$
$$\|\vartheta_2\|_{L^2(0,t;\mathcal{H}^{1+\epsilon/2})} \leq t^{(2\alpha_1-1-2\epsilon)/2} \|\vartheta_2\|_{Z^\theta_{\alpha_1,t}} .$$

Further, in view of (4.1) we have

$$\|\boldsymbol{z}_1\|_{L^\infty(0,t;L^{1/(1-\alpha_1+\epsilon/4)}(\Omega))} \leq c \|\boldsymbol{z}_1\|_{L^\infty(0,t;V^{\alpha_1}_\kappa)} ,$$
$$\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{L^\infty(0,t;L^{1/(1-\alpha_1+\epsilon/4)}(\Omega))} \leq c \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{L^\infty(0,t;V^{\alpha_1}_\kappa)}$$

and also taking into account (4.11) and (5.9) we finally arrive at

$$\begin{aligned}
&\|\mathcal{C}_{\alpha,t}(\boldsymbol{z}_1, \vartheta_1) - \mathcal{C}_{\alpha,t}(\boldsymbol{z}_2, \vartheta_2)\|_{L^2(0,t;\mathcal{H}^{2(\alpha_1-1)-\epsilon})} \\
&\quad \leq c_1 \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{X_{\alpha_1,t}} \left(\|\boldsymbol{z}_1\|_{X_{\alpha_1,t}} + \|\boldsymbol{z}_2\|_{X_{\alpha_1,t}} + t^{(2\alpha_1-1-2\epsilon)/2} \|\vartheta_2\|_{Z^\theta_{\alpha_1,t}} \right) \\
&\qquad + c_2 \, t^{(2\alpha_1-1-2\epsilon)/2} \|\boldsymbol{z}_1\|_{X_{\alpha_1,t}} \|\vartheta_1 - \vartheta_2\|_{Z^\theta_{\alpha_1,t}} \\
&\quad \leq c_3 \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{X_{\alpha,t}} \left(\|\boldsymbol{z}_1\|_{X_{\alpha,t}} + \|\boldsymbol{z}_2\|_{X_{\alpha,t}} + t^{(2\alpha_1-1-2\epsilon)/2} \|\vartheta_2\|_{Z^\theta_{\alpha_1,t}} \right) \\
&\qquad + c_4 \, t^{(2\alpha_1-1-2\epsilon)/2} \|\boldsymbol{z}_1\|_{X_{\alpha,t}} \|\vartheta_1 - \vartheta_2\|_{Z^\theta_{\alpha_1,t}} .
\end{aligned} \tag{5.19}$$

We have verified that the definition of $\mathcal{C}_{\alpha,t}$ is correct and that

$$\mathcal{C}_{\alpha,t} : X_{\alpha,t} \times Z^\theta_{\alpha_1,t} \to L^2(0,t;\mathcal{H}^{2(\alpha_1-1)-\epsilon})$$

is a continuous operator.

We now define the functional $\mu_{\alpha,t} \in L^2(0,t;\mathcal{H}^{2(\alpha_1-1)-\epsilon})$ corresponding to the data of the problem, namely θ_∞ and h. Let $\varphi \in \mathcal{H}^{2(1-\alpha_1)+\epsilon} \equiv \mathcal{H}^{1-s_1}$. Define the function $\langle \mu_{\alpha,t}, \varphi \rangle_{s_1-1,1-s_1} : (0,t) \to \mathbb{R}$ such that

$$\langle \mu_{\alpha,t}, \varphi \rangle_{s_1-1,1-s_1}(\tau) \equiv \int_{\Gamma_1} \theta_\infty(\tau)\varphi + \int_\Omega h(\tau)\varphi \tag{5.20}$$

for almost all $\tau \in (0,t)$. Recall that $\alpha_1 = \min\{\alpha, 3/4\}$ and hence we have (see Theorem 1.5.1.2 in [9])

$$\mathcal{H}^{2(1-\alpha_1)+\epsilon} \hookrightarrow W^{2-2\alpha_1+\epsilon,2}(\Omega) \hookrightarrow W^{1/2+\epsilon,2}(\Omega) \hookrightarrow L^2(\Gamma_1)$$

and thus

$$L^2(0,t;\mathcal{H}^{2(1-\alpha_1)+\epsilon}) \hookrightarrow L^2(0,t;L^2(\Gamma_1)).$$

It is easy to see that $\mu_{\alpha,t}$ is well defined and continuous.

Let $\boldsymbol{u} \in X_{\alpha,t}$, $\vartheta \in Z^\theta_{\alpha_1,t}$ and let θ_0 be given by Theorem 2.1. Then $[\mu_{\alpha,t} - \mathcal{C}_{\alpha,t}(\boldsymbol{u},\vartheta),\theta_0] \in Y^\theta_{s_1,t}$ with $s_1 = 2\alpha_1 - 1 - \epsilon$. Lemma 3.2 now gives the uniquely determined $\theta \in X^\theta_{s_1,t}$, the solution of the operator equation (5.2). Define the operator

$$\mathcal{R}_{\alpha,t} : \quad [\boldsymbol{w},\vartheta] \to \theta : \quad Z_{\alpha,t} \times Z^\theta_{\alpha_1,t} \to X^\theta_{s_1,t} \tag{5.21}$$

by

$$\mathcal{R}_{\alpha,t}(\boldsymbol{w},\vartheta) = (\mathcal{L}_{\alpha,t})^{-1}\big([\mu_{\alpha,t} - \mathcal{C}_{\alpha,t}(\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta),\vartheta),\theta_0]\big), \text{ where } \mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta) \in X_{\alpha,t}.$$

From Lemma 3.3 and the continuity of $\mathcal{T}_{\alpha,t}$ ad $\mathcal{C}_{\alpha,t}$ it may be concluded that $\mathcal{R}_{\alpha,t}$ is a continuous operator with respect to norm topologies indicated in (5.21).

In the next step, we construct the operator $\mathcal{M}_{\alpha,t}$, see (5.3). We first recall that $X_{\alpha,t} \hookrightarrow\hookrightarrow Z_{\alpha,t}$ and $X^\theta_{s_1,t} \equiv X^\theta_{2\alpha_1-1-\epsilon,t} \hookrightarrow\hookrightarrow Z^\theta_{\alpha_1,t}$, see (4.10) and (4.12). We are now in a position to introduce the mapping

$$\mathcal{M}_{\alpha,t} : \quad [\boldsymbol{w},\vartheta] \to [\boldsymbol{u},\theta] : \quad Z_{\alpha,t} \times Z^\theta_{\alpha_1,t} \to Z_{\alpha,t} \times Z^\theta_{\alpha_1,t}$$

by

$$\mathcal{M}_{\alpha,t}(\boldsymbol{w},\vartheta) = [\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta),\mathcal{R}_{\alpha,t}(\boldsymbol{w},\vartheta)]\,.$$

From what has already been proved, it follows that $\mathcal{M}_{\alpha,t}$ is completely continuous (or *compact* since $Z_{\alpha,t}$ and $Z^\theta_{\alpha_1,t}$ are both reflexive Banach spaces) and using (3.1), (3.10), (4.10) and (4.12) we can write

$$\begin{aligned}
&\|\mathcal{M}_{\alpha,t}(\boldsymbol{w},\vartheta)\|_{Z_{\alpha,t}\times Z^\theta_{\alpha_1,t}} = \|\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta)\|_{Z_{\alpha,t}} + \|\mathcal{R}_{\alpha,t}(\boldsymbol{w},\vartheta)\|_{Z^\theta_{\alpha_1,t}} \\
&\quad \le c_1\left(\|\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta)\|_{X_{\alpha,t}} + \|\mathcal{R}_{\alpha,t}(\boldsymbol{w},\vartheta)\|_{X^\theta_{s_1,t}}\right) \\
&\quad \le c_2\left(\|[\mathcal{F}_{\alpha,t}(\vartheta) - \mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w}),\boldsymbol{u}_0]\|_{Y_{\alpha,t}} + \|\mu_{\alpha,t} - \mathcal{C}_{\alpha,t}(\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta),\vartheta),\theta_0\|_{Y^\theta_{s_1,t}}\right) \\
&\quad \le c_2\Big(\|\mathcal{F}_{\alpha,t}(\vartheta)\|_{L^2(0,t;V^{\alpha-1}_\kappa)} + \|\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})\|_{L^2(0,t;V^{\alpha-1}_\kappa)} + \|\boldsymbol{u}_0\|_{V^\alpha_\kappa} \\
&\qquad\qquad + \|\mathcal{C}_{\alpha,t}(\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta),\vartheta)\|_{L^2(0,t;\mathcal{H}^{s_1-1})} + \|\mu_{\alpha,t}\|_{L^2(0,t;\mathcal{H}^{s_1-1})} + \|\theta_0\|_{\mathcal{H}^{s_1}}\Big)\,,
\end{aligned} \tag{5.22}$$

where the constants c_1 and c_2 are independent of t. On account of the estimates (5.4), (5.5) and (5.19) we arrive at

$$\|\mathcal{F}_{\alpha,t}(\vartheta)\|_{L^2(0,t;V_\kappa^{\alpha-1})} \le c_1 t^{(2\alpha_1-1-2\epsilon)/2}\|\vartheta\|_{Z^\theta_{\alpha_1,t}} + c_2\, t^{1/2}, \tag{5.23}$$

$$\|\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})\|_{L^2(0,t;V_\kappa^{\alpha-1})} \le c\, t^{\alpha/(8-\alpha)}\|\boldsymbol{w}\|^2_{Z_{\alpha,t}} \tag{5.24}$$

and

$$\begin{aligned}
&\left\|\mathcal{C}_{\alpha,t}(\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta),\vartheta)\right\|_{L^2(0,t;\mathcal{H}^{s_1-1})} \\
&\quad\le c_1 \left\|\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta)\right\|^2_{X_{\alpha,t}} + c_2 t^{(2\alpha_1-1-2\epsilon)/2} \left\|\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta)\right\|_{X_{\alpha,t}} \|\vartheta\|_{Z^\theta_{\alpha_1,t}} \\
&\quad\le c_3 \left\|\mathcal{T}_{\alpha,t}(\boldsymbol{w},\vartheta)\right\|^2_{X_{\alpha,t}} + c_4 t^{(2\alpha_1-1-2\epsilon)} \|\vartheta\|^2_{Z^\theta_{\alpha_1,t}} \\
&\quad\le c_5 \left(\|\mathcal{F}_{\alpha,t}(\vartheta)\|^2_{L^2(0,t;V_\kappa^{\alpha-1})} + \|\mathcal{B}_{\alpha,t}(\boldsymbol{w},\boldsymbol{w})\|^2_{L^2(0,t;V_\kappa^{\alpha-1})} + \|\boldsymbol{u}_0\|^2_{V_\kappa^\alpha}\right) \\
&\qquad + c_4 t^{(2\alpha_1-1-2\epsilon)} \|\vartheta\|^2_{Z^\theta_{\alpha_1,t}}\,.
\end{aligned} \tag{5.25}$$

Now, set

$$B_K = \left\{[\boldsymbol{w},\vartheta] \in Z_{\alpha,t}\times Z^\theta_{\alpha_1,t};\quad \|[\boldsymbol{w},\vartheta]\|_{Z_{\alpha,t}\times Z^\theta_{\alpha_1,t}} \le K\right\}.$$

Substituting (5.25) into (5.22) and using (5.23) and (5.24) we arrive at

$$\begin{aligned}
\|\mathcal{M}_{\alpha,t}(\boldsymbol{w},\vartheta)\|_{Z_{\alpha,t}\times Z^\theta_{\alpha_1,t}} \le c_1\, t^{(2\alpha_1-1-2\epsilon)/2} K + c_2\, t^{1/2} + c_3\, t^{\alpha/(8-\alpha)} K^2 \\
c_4\, t^{2\alpha_1-1-2\epsilon} K^2 + c_5\, t + c_6\, t^{2\alpha/(8-\alpha)} K^4 + C,
\end{aligned}$$

where c_1, c_2, …, c_6 and C are independent of t, $0 \le t \le T$. It is easy to see that there exist T_1, $0 < T_1 \le T$, and $K > 0$ such that

$$\|\mathcal{M}_{\alpha,t}(\boldsymbol{w},\vartheta)\|_{Z_{\alpha,t}\times Z^\theta_{\alpha_1,t}} < K \text{ for arbitrary } \ t \in (0,T_1] \ \text{ and } \ [\boldsymbol{w},\vartheta] \in B_K$$

and hence $\mathcal{M}_{\alpha,t}(B_K) \subset B_K$ for $t \le T_1$.

It has already been shown that $\mathcal{M}_{\alpha,t}$ is compact so by the Schauder fixed point theorem (see [8, p. 279, Theorem 11.1 and p. 280, Corollary 11.2]) it has a fixed point $[\boldsymbol{u},\theta]$ in B_K. By the construction of the operator $\mathcal{M}_{\alpha,t}$ we see that the couple $[\boldsymbol{u},\theta]$ is a weak solution of (1.7)–(1.15) on the time interval $(0,T_1)$.

5.2. Uniqueness

Here we prove the uniqueness of the solution provided $\alpha = 1$. Regularity of the solution and interpolation-like inequalities reduce the proof to an application of Gronwall's inequality.

Remark 5.1. *If we neglect the dissipative term in the energy equation* (1.9) *then the uniqueness holds for arbitrary* $\alpha \in (1/2, 1]$.

Suppose that there are two solutions $[\boldsymbol{u}_1, \theta_1], [\boldsymbol{u}_2, \theta_2] \in X_{\alpha,t} \times X^{\theta}_{\alpha_1,t}$, $\alpha = 1$, $\alpha_1 = 3/4$, of the problem (1.7)–(1.15) on $(0, T)$. Set $\tilde{\boldsymbol{u}} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\tilde{\theta} = \theta_1 - \theta_2$. Then we have

$$\int_{\Omega_T} \partial_t \tilde{\boldsymbol{u}} \cdot \boldsymbol{\phi} + \int_{\Omega_T} \nabla \tilde{\boldsymbol{u}} \nabla \boldsymbol{\phi} + \gamma \int_{\Gamma_{1T}} \tilde{\boldsymbol{u}} \cdot \boldsymbol{\phi} + \int_{\Omega_T} (\tilde{\boldsymbol{u}} \cdot \nabla) \boldsymbol{u}_2 \cdot \boldsymbol{\phi} + \int_{\Omega_T} (\boldsymbol{u}_1 \cdot \nabla) \tilde{\boldsymbol{u}} \cdot \boldsymbol{\phi} + \int_{\Omega_T} \tilde{\theta} \boldsymbol{f} \cdot \boldsymbol{\phi} = 0 \quad (5.26)$$

and

$$\int_0^T \langle \partial_t \tilde{\theta}, \varphi \rangle + \int_{\Omega_T} \nabla \tilde{\theta} \cdot \nabla \varphi + \int_{\Gamma_T} \tilde{\theta} \varphi + \int_{\Omega_T} \tilde{\boldsymbol{u}} \cdot \nabla \theta_1 \, \varphi + \int_{\Omega_T} \boldsymbol{u}_2 \cdot \nabla \tilde{\theta} \, \varphi = \int_{\Omega_T} \boldsymbol{e}(\tilde{\boldsymbol{u}}) : \boldsymbol{e}(\boldsymbol{u}_1) \varphi + \int_{\Omega_T} \boldsymbol{e}(\boldsymbol{u}_2) : \boldsymbol{e}(\tilde{\boldsymbol{u}}) \varphi + \int_{\Omega_T} \tilde{\theta} \boldsymbol{f} \cdot \boldsymbol{u}_1 \varphi + \int_{\Omega_T} \theta_2 \boldsymbol{f} \cdot \tilde{\boldsymbol{u}} \varphi \quad (5.27)$$

for every $[\boldsymbol{\phi}, \varphi] \in X_{\alpha,t} \times X^{\theta}_{\alpha,t}$. Further, $\tilde{\boldsymbol{u}}(0) = \boldsymbol{0}$ and $\tilde{\theta}(0) = 0$.

Let $\tau \in [0, T]$ be arbitrary. For $\boldsymbol{\phi}(t)$ in (5.26) choose the function

$$\boldsymbol{\phi}(t) := \begin{cases} \tilde{\boldsymbol{u}}(t) & \text{for } 0 \le t \le \tau, \\ 0 & \text{for } \tau < t \le T. \end{cases}$$

Thus, we obtain

$$\frac{1}{2} \|\tilde{\boldsymbol{u}}(\tau)\|^2_{L^2_\kappa} + c_1 \int_0^\tau \|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}_\kappa} \le \left| \int_{\Omega_\tau} (\tilde{\boldsymbol{u}} \cdot \nabla) \boldsymbol{u}_2 \cdot \tilde{\boldsymbol{u}} \right| + \left| \int_{\Omega_\tau} (\boldsymbol{u}_1 \cdot \nabla) \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}} \right| + \left| \int_{\Omega_\tau} \tilde{\theta} \boldsymbol{f} \cdot \tilde{\boldsymbol{u}} \right|. \quad (5.28)$$

To estimate all terms on the right-hand side of (5.28) we use the Young inequality with a parameter δ and the interpolation inequalities (see [1, Theorem 5.8])

$$\|\tilde{\boldsymbol{u}}(t)\|_{L^4(\Omega)} \le c\,\|\tilde{\boldsymbol{u}}(t)\|^{1/2}_{W^{1,2}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|^{1/2}_{L^2(\Omega)}$$

and

$$\|\tilde{\theta}(t)\|_{L^4(\Omega)} \le c\,\|\tilde{\theta}(t)\|^{1/2}_{W^{1,2}(\Omega)}\|\tilde{\theta}(t)\|^{1/2}_{L^2(\Omega)},$$

which hold for a.e. $t \in (0,\tau)$. Accordingly, we have

$$\begin{aligned}\left|\int_{\Omega_\tau}(\boldsymbol{u}_1\cdot\nabla)\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right| &\le \int_0^\tau \|\boldsymbol{u}_1(t)\|_{L^4(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|_{W^{1,2}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|_{L^4(\Omega)}\\ &\le c\int_0^\tau \|\boldsymbol{u}_1(t)\|_{L^4(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|^{3/2}_{W^{1,2}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|^{1/2}_{L^2(\Omega)}\\ &\le \delta\int_0^\tau \|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}(\Omega)} + C(\delta)\int_0^\tau \|\boldsymbol{u}_1(t)\|^4_{L^4(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{5.29}$$

further,

$$\begin{aligned}\left|\int_{\Omega_\tau}(\tilde{\boldsymbol{u}}\cdot\nabla)\boldsymbol{u}_2\cdot\tilde{\boldsymbol{u}}\right| &\le \int_0^\tau \|\boldsymbol{u}_2(t)\|_{W^{1,2}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|^2_{L^4(\Omega)}\\ &\le c\int_0^\tau \|\boldsymbol{u}_2(t)\|_{W^{1,2}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|_{W^{1,2}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|_{L^2(\Omega)}\\ &\le \delta\int_0^\tau \|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}(\Omega)} + C(\delta)\int_0^\tau \|\boldsymbol{u}_2(t)\|^2_{W^{1,2}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|^2_{L^2(\Omega)}\end{aligned} \tag{5.30}$$

and

$$\left|\int_{\Omega_\tau}\tilde{\theta}\boldsymbol{f}\cdot\tilde{\boldsymbol{u}}\right| \le c\int_0^\tau\left(\|\tilde{\theta}(t)\|^2_{L^2(\Omega)} + \|\tilde{\boldsymbol{u}}(t)\|^2_{L^2(\Omega)}\right). \tag{5.31}$$

Substituting (5.29)–(5.31) into (5.28) we arrive at

$$\begin{aligned}\frac{1}{2}\|\tilde{\boldsymbol{u}}(\tau)\|^2_{L^2_\kappa} + c_1\int_0^\tau \|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}_\kappa} &\le \delta\int_0^\tau\left(\|\tilde{\theta}(t)\|^2_{W^{1,2}(\Omega)} + \|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}(\Omega)}\right)\\ +C(\delta)\int_0^\tau\left(1+\|\boldsymbol{u}_1(t)\|^4_{L^4(\Omega)} + \|\boldsymbol{u}_2(t)\|^2_{W^{1,2}(\Omega)}\right)&\|\tilde{\boldsymbol{u}}(t)\|^2_{L^2(\Omega)} + c\int_0^\tau \|\tilde{\theta}(t)\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.32}$$

Similarly, choosing

$$\varphi(t) := \begin{cases} \tilde{\theta}(t) & \text{for } 0 \le t \le \tau, \\ 0 & \text{for } \tau < t \le T, \end{cases}$$

as a test function in (5.27), we get

$$\frac{1}{2}\|\tilde{\theta}(\tau)\|^2_{L^2(\Omega)} + c_1 \int_0^\tau \|\tilde{\theta}(t)\|^2_{W^{1,2}(\Omega)} \le \left| \int_{\Omega_\tau} \tilde{\boldsymbol{u}} \cdot \nabla \theta_1 \, \tilde{\theta} \right| + \left| \int_{\Omega_\tau} \boldsymbol{u}_2 \cdot \nabla \tilde{\theta} \, \tilde{\theta} \right|$$
$$+ \left| \int_{\Omega_\tau} \boldsymbol{e}(\tilde{\boldsymbol{u}}) : \boldsymbol{e}(\boldsymbol{u}_1)\tilde{\theta} \right| + \left| \int_{\Omega_\tau} \boldsymbol{e}(\boldsymbol{u}_2) : \boldsymbol{e}(\tilde{\boldsymbol{u}})\tilde{\theta} \right| + \left| \int_{\Omega_\tau} \tilde{\theta} \boldsymbol{f} \cdot \boldsymbol{u}_1 \tilde{\theta} \right| + \left| \int_{\Omega_\tau} \theta_2 \boldsymbol{f} \cdot \tilde{\boldsymbol{u}} \tilde{\theta} \right|. \tag{5.33}$$

We can further estimate the right-hand side of (5.33) by Young's inequality and interpolation-like inequalities as

$$\begin{aligned} \left| \int_{\Omega_\tau} \tilde{\boldsymbol{u}} \cdot \nabla \theta_1 \, \tilde{\theta} \right| &\le \int_0^\tau \|\tilde{\boldsymbol{u}}(t)\|_{L^4(\Omega)} \|\nabla \theta_1(t)\|_{L^2(\Omega)} \|\tilde{\theta}(t)\|_{L^4(\Omega)} \\ &\le c \int_0^\tau \|\theta_1(t)\|_{W^{1,2}(\Omega)} \|\tilde{\theta}(t)\|^{1/2}_{L^2(\Omega)} \|\tilde{\theta}(t)\|^{1/2}_{W^{1,2}(\Omega)} \|\tilde{\boldsymbol{u}}(t)\|^{1/2}_{L^2(\Omega)} \|\tilde{\boldsymbol{u}}(t)\|^{1/2}_{W^{1,2}(\Omega)} \\ &\le \delta \int_0^\tau \left(\|\tilde{\theta}(t)\|^2_{W^{1,2}(\Omega)} + \|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}(\Omega)} \right) \\ &\quad + C(\delta) \int_0^\tau \|\theta_1(t)\|^2_{W^{1,2}(\Omega)} \left(\|\tilde{\theta}(t)\|^2_{L^2(\Omega)} + \|\tilde{\boldsymbol{u}}(t)\|^2_{L^2(\Omega)} \right) \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} \left| \int_{\Omega_\tau} \boldsymbol{u}_2 \cdot \nabla \tilde{\theta} \, \tilde{\theta} \right| &\le \int_0^\tau \|\boldsymbol{u}_2(t)\|_{L^4(\Omega)} \|\nabla \tilde{\theta}(t)\|_{L^2(\Omega)} \|\tilde{\theta}(t)\|_{L^4(\Omega)} \\ &\le c \int_0^\tau \|\boldsymbol{u}_2(t)\|_{L^4(\Omega)} \|\tilde{\theta}(t)\|^{3/2}_{W^{1,2}(\Omega)} \|\tilde{\theta}(t)\|^{1/2}_{L^2(\Omega)} \\ &\le \delta \int_0^\tau \|\tilde{\theta}(t)\|^2_{W^{1,2}(\Omega)} + C(\delta) \int_0^\tau \|\boldsymbol{u}_2(t)\|^4_{L^4(\Omega)} \|\tilde{\theta}(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.35}$$

For the dissipative terms in (5.33) we have

$$\left|\int_{\Omega_\tau} \boldsymbol{e}(\tilde{\boldsymbol{u}}) : \boldsymbol{e}(\boldsymbol{u}_1)\tilde{\theta}\right| \le c\int_0^\tau \|\tilde{\boldsymbol{u}}(t)\|_{W^{1,2}(\Omega)}\|\boldsymbol{u}_1(t)\|_{W^{1,4}(\Omega)}\|\tilde{\theta}(t)\|_{L^4(\Omega)}$$

$$\le c\int_0^\tau \|\tilde{\boldsymbol{u}}(t)\|_{W^{1,2}(\Omega)}\|\boldsymbol{u}_1(t)\|_{W^{1,4}}\|\tilde{\theta}(t)\|^{1/2}_{W^{1,2}(\Omega)}\|\tilde{\theta}(t)\|^{1/2}_{L^2(\Omega)}$$

$$\le \delta\int_0^\tau \left(\|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}(\Omega)} + \|\tilde{\theta}(t)\|^2_{W^{1,2}(\Omega)}\right) + C(\delta)\int_0^\tau \|\boldsymbol{u}_1(t)\|^4_{W^{1,4}(\Omega)}\|\tilde{\theta}(t)\|^2_{L^2(\Omega)} \tag{5.36}$$

and in the same way we arrive at the inequality

$$\left|\int_{\Omega_\tau} \boldsymbol{e}(\boldsymbol{u}_2) : \boldsymbol{e}(\tilde{\boldsymbol{u}})\tilde{\theta}\right| \le c\int_0^\tau \|\boldsymbol{u}_2(t)\|_{W^{1,4}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|_{W^{1,2}(\Omega)}\|\tilde{\theta}(t)\|_{L^4(\Omega)}$$

$$\le \delta\int_0^\tau \left(\|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}(\Omega)} + \|\tilde{\theta}(t)\|^2_{W^{1,2}(\Omega)}\right) + C(\delta)\int_0^\tau \|\boldsymbol{u}_2(t)\|^4_{W^{1,4}(\Omega)}\|\tilde{\theta}(t)\|^2_{L^2(\Omega)}. \tag{5.37}$$

The last two terms in (5.33) can be handled as

$$\left|\int_{\Omega_\tau} \tilde{\theta}\boldsymbol{f}\cdot\boldsymbol{u}_1\tilde{\theta}\right| \le c\int_0^\tau \|\boldsymbol{u}_1(t)\|_{L^\infty(\Omega)}\|\tilde{\theta}(t)\|^2_{L^2(\Omega)} \tag{5.38}$$

and

$$\left|\int_{\Omega_\tau} \theta_2\boldsymbol{f}\cdot\tilde{\boldsymbol{u}}\tilde{\theta}\right| \le c_1\int_0^\tau \|\theta_2(t)\|_{L^4(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|_{L^4(\Omega)}\|\tilde{\theta}(t)\|_{L^2(\Omega)}$$

$$\le c_2\int_0^\tau \|\theta_2(t)\|_{L^4(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|_{W^{1,2}(\Omega)}\|\tilde{\theta}(t)\|_{L^2(\Omega)}$$

$$\le \delta\int_0^\tau \|\tilde{\boldsymbol{u}}(t)\|_{W^{1,2}(\Omega)} + C(\delta)\int_0^\tau \|\theta_2(t)\|^2_{L^4(\Omega)}\|\tilde{\theta}(t)\|^2_{L^2(\Omega)}. \tag{5.39}$$

Substituting (5.34)–(5.39) into (5.33) gives

$$\frac{1}{2}\|\tilde{\theta}(\tau)\|^2_{L^2(\Omega)} + c_1\int_0^\tau \|\tilde{\theta}(t)\|^2_{W^{1,2}(\Omega)} \le \delta\int_0^\tau \left(\|\tilde{\boldsymbol{u}}(t)\|^2_{W^{1,2}(\Omega)} + \|\tilde{\theta}(t)\|^2_{W^{1,2}(\Omega)}\right)$$

$$+ C(\delta)\int_0^\tau \|\theta_1(t)\|^2_{W^{1,2}(\Omega)}\|\tilde{\boldsymbol{u}}(t)\|^2_{L^2(\Omega)}$$

$$+ C(\delta)\int_0^\tau \Big(\|\theta_1(t)\|^2_{W^{1,2}(\Omega)} + \|\boldsymbol{u}_2(t)\|^4_{L^4(\Omega)} + \|\boldsymbol{u}_1(t)\|^4_{W^{1,4}(\Omega)} + \|\boldsymbol{u}_2(t)\|^4_{W^{1,4}(\Omega)}$$

$$+\|\boldsymbol{u}_1(t)\|_{L^\infty(\Omega)} + \|\theta_2(t)\|^2_{L^4(\Omega)}\Big)\|\tilde{\theta}(t)\|^2_{L^2(\Omega)}. \tag{5.40}$$

Choosing δ sufficiently small and summing (5.32) and (5.40) we conclude that

$$\|\tilde{\boldsymbol{u}}(\tau)\|_{V_\kappa^0}^2 + \|\tilde{\theta}(\tau)\|_{L^2(\Omega)}^2 \leq \int_0^\tau \sigma(t) \left(\|\tilde{\boldsymbol{u}}(t)\|_{V_\kappa^0}^2 + \|\tilde{\theta}(t)\|_{L^2(\Omega)}^2 \right)$$

for all $\tau \in [0, T]$, where $\sigma \in L^1(0, T)$. Now, the uniqueness follows from the fact that $\tilde{\boldsymbol{u}}(0) = \boldsymbol{0}$ and $\tilde{\theta}(0) = 0$ using the Gronwall inequality, see [23, Section 1.6].

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