Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Department of Physics
Field of study: Mathematical Engineering

Physical aspects of financial markets

BACHELOR'S THESIS

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Year: 2022

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David Dobáš

Acknowledgement

I would like to thank Petr Jizba, Ph.D. for all the inspiration and insights he gave me during the work on this thesis. I am greatful to be introduced to the fascinating topic of econophysics and I am looking forward to exploring it more.

David Dobáš

Název práce:
Fyzikální aspekty finančních trhů

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Studijní program: Aplikace přírodních věd
Obor: Matematické inženýrství
Druh práce: Bakalářská práce

Vedoucí práce: Ing. Petr Jizba, Ph.D.
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Abstrakt: Finanční trhy jsou komplexní systémy s mnoha analogiemi ve fyzice. V této práci se v první řadě věnujeme některým principům finančních trhů a jejich srovnáním se zákony termodynamiky. Následně se detailněji zabýváme standardní teorií oceňování opcí a jejími nedostatky. Zjistíme, že pro popis trhů je třeba rozumět systémům se silnými korelacemi mezi jejich součástmi. Tím dospějeme k teorii kritických jevů ve fyzice, jejíž základy vyložíme a jejíž metody použijeme k popisu finančních trhů během normálních fází i během krizí.
Klíčová slova: Ekonofyzika, finanční trhy, opce, Black-Scholesova rovnice, kritické jevy, krize

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Abstract: Financial markets are complex systems with many analogies in physics. In this thesis, we take a look at some principles of financial markets and their comparison with the laws of thermodynamics. Afterwards, we focus on the standard option pricing theory and its shortcomings. We find out, that to be able to describe markets, we need to understand sytems with large correlations between its constituents. That leads us to the theory of critical phenomena in physics. We explain the basics of it and then we use it to describe markets during normal phases and also during crises.
Key words: Econophysics, financial markets, options, Black-Scholes equation, critical phenomena, crises

Contents

Introduction

Why should the markets be of interest for a physicist?

Financial market is a huge system composed of many entities, which interact in various ways. This leads to very complex behavior with a significant effect on the whole society. What makes the markets interesting for scientists is the fact that there is an enormous amount of data, which can be examined and used to test hypotheses. Studying them can also give an insight to other complex systems, where the data are more difficult to obtain.

There are several established theories, which are used in quantitative finance to model markets. These, however, show up not to be based on solid empirical evidence, yet they are widely used in practice. That can eventually lead to huge losses for traders.

Physicists, on the other hand, are very sensitive to overlooking empirical data. Their aim is to find models which could persist observations and tests. Financial markets offer a great challenge for them.

In this thesis, we show several approaches physicists made. In the first chapter, we introduce basic notions and principles of financial markets, and we try to find analogies in phenomenological thermodynamics. Second chapter deals with the cornerstone of quantitative finance, option pricing theory, and it also shows its shortcomings. We will see, that one of the key phenomena occurring in markets is the collective behavior, which leads to several important consequences. That is the topic of the third chapter, where we find analogies with the physics of criticality.

Chapter 1

Physics and markets

In this chapter, we will make a brief introduction to quantitative finance, its principles and assumptions, and we will try to find some connections with physics. Next chapters will dive deeper in some concepts.

One of the key concepts in physics, which gave rise to the quantum mechanics, is the fact that a measurement of some observable variable can change the state, which is measured. Measurement can even affect previous measurements.

Financial markets are also influenced by measurement. As a matter of fact, they are measured very intensively in some aspects, for example price changes can be observed in microseconds. On the other hand, some other factors are not easily measurable, as the system is composed of millions of actors, each with different motivation and actions. Anyway, the market is accommodated to measurement. Continual availability of prices and accessibility of information is fundamental for the functioning of markets.

What is the level of complexity we need to deal with, when describing markets? Let us make a comparison with quantum states. We can not say how complex some quantum state is, we can describe it just in terms of possible measurements. An electron could have a free will or communicate with other electrons telepathically. We just do not have anything to say about it, as long as we can not measure it.

In these terms, we can look at financial markets. Critics could say that any description neglecting free will, inscrutability, irrationality and psychology of market participants is useless. But we can consider these as hardly measurable (or impossible to measure) and try to find characteristics, which are observable and tractable without precise knowledge of every detail.

In physics, this is the aim of statistical mechanics. It deals with large systems, which are practically indescribable in detail (we can not compute movement of each individual particle), but are possible to describe in statistical means. In case of independent non-interacting particles, big deviations vanish and macroscopically observable quantities occur. That leads to a model of ideal gas. When interactions take place, collective behaviour can occur, which in specific situations leads to critical phenomena. These are the reasons, why many of the analogies between physics and markets can be found in thermodynamics, statistical physics and physics of criticality.

Markets are composed of many participants, which are connected by various forms of interaction. These connections change over time, strengthen or loosen, new forms of them appear frequently. This makes markets very complex and interesting, but also very hard to deal with mathematically and difficult to model. Some approaches are being developed in the theory of complex systems, namely the network theory. We will see some basic models in the next chapters.

Next sections will be devoted to basic principles of our description of markets. We will try to show analogies in physics if possible. However, we will also try to figure out some specifics of markets, which do not have an analogy in physics.

1.1 Basics of markets, terminology

In this section we will follow [1], although the notions mentioned here are well established. Financial market is basically a field where trades can occur. One of the basic assets traded are **stocks**. A stock is basically a share in the company. Stocks can be bought or sold, supply and demand determine the price of them. Profits of a company can be reinvested or distributed to stakeholders in the form of **dividend**. Financial markets are not composed only of stock markets, one can trade **commodities** (raw products such as metals, food products, oil etc.) for example. Of great importance is also the foreign **exchange**, which accounts for currencies trades. One can buy a **bond**, which entitle the owner to get paid a certain sum after some time. Bonds are often issued by states and can also be traded.

The next level of complexity is added with **derivatives**. As the name suggests, their value is derived from the value of an underlying asset. Derivatives allow traders to avoid risks as well undergo some. We briefly introduce some of the main derivatives:

- **Forwards and futures**: They are agreements between two parties, where one of them promises to buy an asset at some specific time for a specific price and another party promises to sell. Forward contract can happen between arbitrary parties and is tailored, futures contracts are traded on an exchange.
- **Options**: Call option is a right to buy a particular asset for an agreed amount at a specified time in the future. Put option is similar, but it is the right to sell.
- **Swaps**: Swap is an agreement between two parties to exchange, or swap, future cashflows. For example, one side agrees to pay the other a fixed interest rate and the other pays a floating rate.

We will go in deeper detail with options in the next chapter, as there is a nice mathematical background and a huge playground for an econophysicist.

A market is said to be *liquid*, if an investor can easily buy or sell an asset at any time without a significant change in the price of that asset. A typical example of liquid market is the foreign exchange market, on the other hand one can encounter low liquidity when trying to sell a car or a house. *Market friction* amounts for all kinds of trading costs, including provisions, taxes etc. Market friction is low, if these costs are negligible compared with the volume traded.

1.2 No arbitrage principle

Many models in finance, including the Black-Sholes theory, assume the absence of arbitrage possibilities. An arbitrage is a situation, in which one can make unbounded profits without accompanying risks. To illustrate this, suppose there are three exchange offices in Prague. In one of them, we could buy 1 EUR for 25 CZK. In the second one, we could sell 1 EURO for 1.1 USD, and in the third one we could sell 1.1 USD for 26 CZK. This would give us a profit of 1 CZK for every 25 CZK invested. If we invest 1 million CZK in this trade, we get a profit of 40 thousand CZK without any risk. This is named *free lunch* in financial jargon.

No arbitrage principle, as used in many models, says, the market is free of any arbitrage possibilities. The argument is that if there were any, market participants would instantaneously use them to make profits, leading the opportunity to vanish. In our example with exchange offices, this would happen due to the increased demand for euros in the first office, leading the price to rise adequately.

We see that this principle would not work without the traders searching for the arbitrage opportunities. If all traders believed that no arbitrage opportunities exist, nobody would search for them thus leading them to occur again. Is this contradictory? Just due to the strict formulation, that all arbitrage opportunities vanish instantaneously.

It would be the same as saying, that every thermodynamic system, when deviated from equilibrium, instantaneously reaches equilibrium again. This is not true and physicist therefore choose much more careful statements. One of the postulates of thermodynamics says:

Isolated system reaches ***over time*** *one of the possible thermodynamic equilibrium, in which it remains until it is forced to change this state by external forces.*

If the system is deviated out of equilibrium, we can just state it will reach new equilibrium state over time.

This can lead us to make a more precise formulation of no arbitrage principle

Financial markets make every arbitrage opportunity vanish over time.

This formulation is much more realistic, on the other hand it is harder to use in models. As in thermodynamics, examination of states in equilibrium is much easier than those out of equilibrium, it is easier to assume no arbitrage. Thermodynamic systems also fluctuate around the equilibrium state, however as far as the fluctuations are small, the predictions of equilibrium models work. The same is believed with no arbitrage principle.

Let us remind a first law of thermodynamics (or law of energy conservation) in terms of perpetuum mobile of the first kind:

It is impossible to construct an engine which would work in a cycle and produce continuous work, or kinetic energy, from nothing.

Paul and Bachnagel [2] offer a formulation of no arbitrage principle reminding the first law of thermodynamics:

There is no periodically working financial process which generates a risk-free profit from nothing.

Let us examine this formulation a little. It does not say that it is not possible to make money without risk. If we put our money on a bank account with an interest rate, after some time we get more back. However, the amount of money earned due to the interest rate depends on the time we keep our deposit in the bank. There is no way we could speed this process up and use it periodically. However, if the bank gives us a specific amount of money each time we put a deposit there, we could repeat the depositing and earn risk-less profits.

Although it is appealing to treat no arbitrage principle as an analogy to energy conservation in physics, we suggest being more careful. There have been a massive growth in efficiency during the past centuries, making the vast majority of people richer. Can we find any energy which compensates this growth? Maybe some dark energy?

Therefore, we suggest seeing no arbitrage principle just as a mechanism leading different markets (i.e. exchange offices) to price equilibrium.

1.3 Efficient market hypothesis

Efficient market hypothesis (EMH) is one of the cornerstones of the modern financial theory. However, we will see that the formulations can differ, as well as the mathematical representations and their consequences.

One formulation, given by Paul and Baschnagel [2] states, that "all necessary information for the future price evolution is contained in the present prices." Another one, given by [3], says that EMH is "a theory that the price of a security reflects all currently available information about its economic value." One of the main econophysics books by Mantegna and Stanley [4] says, that "a market is highly efficient in the determination of the most rational price of the traded asset." Last but not least, a book on econophysics modelling by Slanina [5]states, that "all information you might try to use to make a profit from price movements has already been incorporated into the price."

Although all of these formulations might seem similar, there are differences leading many authors to come to different conclusions. What does it mean to have "all necessary information for the future of price"? Necessary for what? To make future estimates? Or to make profit? How do we define economic value of an asset? In which sense do mean "rational price"?

Every approach concludes with a statement, that analysis of price histories is useless, as all the information is already absorbed in the price. To base this on solid ground, we should try to find a mathematical representation. There are two main approaches.

1.3.1 Markov property

Paul and Bachnagel deduce, that EMH can be represented as a Markov property. Briefly stated, Markov property of a random process means, that it has no memory.

Definition 1.3.1 (Markov property)**.** A random process $X_t, t \geq 0$ is said to have a Markov property, if

$$P[X_t = j | X_{t_n} = i_n, X_{t_{n-1}} = i_{n-1}, ..., X_{t_1} = i_1] = P[X_t = j | X_{t_n} = i_n] \tag{1.1}$$

for all $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$ and $i_1, ..., i_n, j$ such that conditional probability is well-defined.

Gamers account has often a Markov property. If he bets on colors in roulette, the same amount all the time, the next state of his account is determined just by the present state and the roulette. History of his wins and losses is irrelevant.

If asset prices had a Markov property, the probability distribution of future price changes knowing the whole history of the price would be the same, as if we knew just the present price. We shall point out, that this is a strong property. It is concerning not just an expectation or other moments, but the whole distribution.

We state that the Markov property is too restrictive. Let us suggest a simple model. The prices could behave the way, that if there have been many successive rises or drops in a row, the probability of extreme events can rise. The probabilities could be centered so that the expected value of future price change would be zero, but the probability distribution would allow for higher extremes. Here, the prices would lose the Markov property, because the probability distribution would depend on the number of successive rises or drops.

Does this model violate EMH? That depends on the formulation of EMH. The information about the distribution could not be used to make profits, because the expected value of a price change would be zero. That leads us to the second option to mathematically formulate EMH.

1.3.2 Martingale property

What is important for a price is that there needs to be a buyer for a seller. The buyer expects the price to rather rise, if he did not, he would have no reason to buy the asset. Te seller expects the price to rather fall, if not, he could keep the asset and sell later with profit. They have opposite expectations.

We suggest, that the average expectation of the traders about the price change should tend to zero. If there are more trades based on the expectation of a price drop, there will more sell orders pushing the price down. On the other hand, if there are more trades expecting price rise, the price will go up adequately. The price always reacts to the expectations and beliefs of traders.

We will compare this with a martingale property.

Definition 1.3.2 (Fair process)**.** A discrete-time random process $\{X_1, X_2, ...\}$ is called fair, if

1. $\mathbb{E}[|X_n|] < \infty, \forall n \in \mathbb{N}$
2. $\mathbb{E}[X_{n+1}|X_n, X_{n-1}, ..., X_1] = 0, \acute{\text{I}}, \forall n \in \mathbb{N}$

Having a random process $X_1, X_2, ...$, let us define a new process of partial sums $Y_1, Y_2, ...$. It is clear, that this process suffices equation $\mathbb{E}[Y_{n+1}|Y_n, Y_{n-1}, ..., Y_1] = Y_n$

Definition 1.3.3 (Martingale)**.** A discrete-time random process $\{Y_1, Y_2, ...\}$, is called a martingale, if

1. $\mathbb{E}[|Y_n|] < \infty, \forall n \in \mathbb{N}$
2. $\mathbb{E}[Y_{n+1}|Y_n, Y_{n-1}, ..., Y_1] = Y_n, \acute{\text{I}}, \forall n \in \mathbb{N}$

This definition is sufficient for illustration, one more general is given in appendix. If prices are martingales, it means that the best estimate (in the probabilistic sense) for the future price, given all the history of the price, is the present value of it. The differences of prices should then be a fair process.

Is this a good mathematical representation of the motivation given? We stated that the average expectation of traders about the future price change should tend to zero. One group always expects the prices to drop, the other to rise. Both groups can access the same information. The argument given by [6] is that there is no reason one group should consistently be better in predicting the future, than the other. Therefore the average estimate of traders should correspond to the expected value in probabilistic sense. The best-known derivation of martingale property regarding prices was given by Samuelson in 1965 [7].

We shall note that many models of prices (such as geometric Brownian motion with drift) are not martingales. This is because we neglected interest rates, inflation, risk aversion etc. Trader do not consider just drops or rises of the asset prices. They compare the investment with other ones. For example, if we expect the price to rise a little, but we would earn more putting the money in the bank and getting interest, we are still motivated to sell. Or due to inflation, trader expects all the prices to rise on average. Then he compares his expectation with this average. If all these factors could be modelled by an exponential growth with parameter μ, we could modify the required property as follows:

$$\mathbb{E}[Y_s|Y_t, Y_{t_n}, ..., Y_{t_1}] = \exp[\mu(t-s)]Y_t \quad \forall n \in \mathbb{N}, 0 \leq t_1 \leq ... \leq t_n \leq t \leq s \tag{1.2}$$

1.3.3 Information point of view

We can also analyse EMH from the information point of view. In martingales, we took into consideration only the past of the process. In markets, however, there are many other relevant information, such as annual reports, forecasts, analyses and others. One measure of effectivity, as proposed by Eugene Fama, is how fast the prices react to new information.

Some formulations of EMH state, that the new information is immediately incorporated into the price, making any analyses useless. However, in practice there are many funds making money on fast data mining (such as sentiment analysis) followed by algorithmic trading, making them faster in usage of information than others. These new technologies then lead to even more efficient markets, faster to respond to new information.

On the other hand, obtaining and analysing an information is not free, and the faster we want to do it, the more costly it gets. For some speed, the cost can be so large, that it would not pay off for anybody. This limitation is discussed by the theory of marginally efficient markets. Slanina [5] finds here an analogy to the third law of thermodynamics.

Carnot's theorem *No engine operating between two reservoirs is more efficient than a Carnot engine operating between them.*

The efficiency of Carnot engine is given by $\eta = 1 - \frac{T_C}{T_H}$, where T_H and T_C are the absolute temperature of the hot and the cold reservoir respectively. This means that by approaching lower temperatures of the cold reservoir we get higher efficiencies. Third law of thermodynamics limits this.

Third law of thermodynamics *It is impossible to cool any system to absolute zero temperature in a finite number of steps.*

The third law therefore implies the impossibility of 100 percent efficient engine. Slanina proposes, that the situation with markets is the same. The 100 percent efficiency can not be reached due to diverging costs of gathering and analysing information in short time.

1.4 Nonexistence of independent description

In the first paragraphs of this chapter, we mentioned an analogy between markets and quantum physics. We stated that, similarly as the quantum state is affected by measurement, the markets are also influenced by it. Accessibility of information is a key element of markets, and they are accommodated to it. However, there is one thing which is very special to markets. They are influenced not just by measurement, but also by our way of describing them.

Physical systems are independent on our description (as far as we know). We can just observe them and try to develop deeper insight into the principles of the nature. However, if we figure out some description of the financial systems and we use it for trading, we influence the system by it. The effect could be negligible, but when we make the description available for the rest of traders, it can change the behavior of the system significantly.

How could we prevent this to happen? One option is to keep our description just for ourselves and not use it in the market. This would have no impact not only to markets, but also to science and the effect would be the same as inventing nothing. Secondly, our description could be useless for behavior on the market, leading nobody to use it. Or it could be “invariant under publicity”, holding true both if traders know it or not.

We are parts of the system we try to describe, therefore every statement we make is a measurement in some sense. This makes our aim to describe markets even more difficult.

In the next chapters, we will show models assuming, that prices of assets follow geometric Brownian motion, which is based on Gaussian distribution. This decription leads to analytically solvable solutions and is followed by many textbooks, yet it hugely underestimates extreme situations. What we will also show is that this assumption is invalidated by empirical data. N. N. Taleb is one of the main critics of usage of Gaussian distribution inappropriately, showing how much traders underestimate their risks using it [8]. Not only they have unexpected losses during financial crashes, but also make the whole crisis worse. Therefore, their usage of Gaussian models further invalidates itself. We can call this **amplification of incorrect description**.

This leads us to another question - is it possible to make predictions in financial markets? This problem was proposed by Lucas in his paper [9], now known as Lucas critique, in which he dealt with models predicting impact of government regulations of market. These models have many parameters, which can be found using historical data. But can we use them to predict future effects of regulations?

Suppose central bank finds a strong anticorrelation between inflation and unemployment in the past. In order to reduce unemployment, the bank imposes measures to increase inflation. This then leads companies to change their inflation predictions and therefore change their employment policies, which can lead to opposite effect than predicted. The model does not take into account, how the behavior of market participants and the market as whole change due to the regulation proposed, it is calibrated only to the past data.

The predictions need not be connected with regulations though. Each time some prediction change behavior of market, it is vulnerable not to hold. If a respected authority predicts a crisis, it is probable that investors will try to eliminate risks, therefore change their portfolios and weaken the possible effects. If market participants figure out the problems, which could cause the crisis, they will try to eliminate them.

Eventually, due to this effect most of the crises are and will be unpredicted. On the other hand, most of the positive changes are unpredicted too. Consider the invention of internet, for example. If anyone could predict it 10 years before it coming, it would be in place much sooner. The phenomenon of black swans, the major but unpredicted events, is thoroughly discussed in Talebs work [8].

Chapter 2

Option pricing

In this chapter, we will introduce stochastic modelling of prices as well as standard option pricing theory. We will also compare theoretical assumptions and results with empirical evidence and show several shortcomings of these models. In the next two sections we will follow mainly [10]

2.1 Modelling of prices

First thing we need is a mathematical model of prices. As we have shown earlier, the prices are influenced by many factors, such as new information coming to the market or beliefs of the investors. These are hard to predict themselves, moreover the market mechanisms such as the EMH act against any predictability. On the other hand, we have shown, that due to interests on bank accounts and inflation, we can expect a growing trend, an exponential growth on average.

X_t being the price at the time t, our model of prices can be as follows:

$$\frac{dX_t}{X_t} = \mu dt + \text{“randomness”} \tag{2.1}$$

Why do we compare the increment of the price to its total value? It is because of the behavior of the market participants. The important quantity for traders is not the absolute value of a price, but the relative changes. They count the returns in percentages.

The question arising here is how to choose the randomness part adequately. The first hypothesis could be, that the randomness should be of the gaussian type, meaning that the price difference between two times should be a normally distributed random variable. The rationale behind this is, that if there are many random inputs, the average influence should be normally distributed due to the Central limit theorem (CLT). However, the CLT has some crucial assumptions, which has to be fulfilled for it to hold. We will discuss these later.

For now, let us build a model based on normal distribution. Related to prices, it was proposed by Louis Bachelier in 1900, concerning motion of particles it was elaborated by Albert Einstein in 1905 and the mathematically formalised by Norbert Wiener in 1923. We are talking about Brownian motion, or Wiener process in mathematical context.

Definition 2.1.1 (Wiener process)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probabilistic space. A random process $\{W_t\}_{t\geq 0}$ is called Wiener process, if the following properties are satisfied:

- $W_0 = 0$ a.s.
- It has independent increments, that is $\forall t_1, t_2, t_3, 0 \leq t_1 \leq t_2 \leq t_3$, $W_{t_3} - W_{t_2}$ and $W_{t_2} - W_{t_1}$ are independent random variables
- It has normally distributed increments, $\forall s, t, 0 \leq s \leq t$, $W_t - W_s \sim \mathcal{N}(0, t-s)$
- It has continuous sample paths, that is $W_t(\omega)$ is continuous $\forall \omega \in \Omega$ as a function of t

Although the definition looks appealing, it deserves showing such a process exists. We refer an interested reader to [11], as this step is not crucial to our topic.

Wiener process has several important properties.

Theorem 2.1.2 (Properties of Wiener process)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probabilistic space, $\{W_t\}_{t\geq 0}$ a Wiener process. Then $\{W_t\}_{t\geq 0}$ has the following properties:

- It is nowhere differentiable
- It is 1/2 self-similar, i.e. $(W_{Tt_1}, W_{Tt_2}, ..., W_{Tt_n}) \overset{d}{=} (T^{1/2}W_{t_1}, T^{1/2}W_{t_2}, ..., T^{1/2}W_{t_n})$ for any $t_1, t_2, ...t_n \geq 0$, where $\overset{d}{=}$ means equality in distribution.
- It is a Markov process, i.e. $P[W_t = j | W_{t_n} = i_n, W_{t_{n-1}} = i_{n-1}, ..., W_{t_1} = i_1] = P[W_t = j | W_{t_n} = i_n]$, $\forall 0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$.
- It is a martingale, details in appendix

We can find these properties very satisfactory related to finance. The first two properties say, that we can expect similar behavior on different timescales, something we see on every price graph. If someone shows us a graph of prices with hidden axes, we can hardly say if it is an evolution in one day or one year. The second two properties remind us of the discussion about EMH. We discussed, that the Markov property may be too strong, however, the model still might be useful.

The Wiener process is the cornerstone of the theory of stochastic differential equations. Thus, we can write our model of price evolution in the following form:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t \tag{2.2}$$

We interpret this equation in terms of integrals:

$$X_t - X_0 = \int_0^t \mu X_s \, ds + \int_0^t \sigma X_s \, dW_s \tag{2.3}$$

What remains is to explain the meaning of $\int_0^t \cdot \, dW_s$. This is the Itô integral. The construction of Itô integral is well-described in many books, we mention one by L.C. Evans [11]. What is important for us is the linearity of the Itô integral and the following rules.

Theorem 2.1.3 (Simple form of the Itô lemma)**.** Let f be a twice differentiable function. Then for $0 \leq s < t$ the following equation holds:

$$f(W_t) - f(W_s) = \int_s^t f'(W_x) \, dW_x + \frac{1}{2} \int_s^t f''(W_x) \, dW_x \tag{2.4}$$

In the next theorem, we use the notation f_1, f_2 as the partial derivative of f with respect to first or second variable respectively.

Theorem 2.1.4 (Extended form of the Itô lemma)**.** Let $f(t,x)$ be a function with continuous second order partial derivatives. Then for $0 \leq s < t$ the following equation holds:

$$f(t, W_t) - f(s, W_s) = \int_s^t f_1(x, W_x) + \frac{1}{2} f_{22}(x, W_x)\, dx + \int_s^t f_2(x, W_x)\, dW_x \tag{2.5}$$

Let us consider a random process in the following form:

$$X_t = \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] \tag{2.6}$$

Intending to use the extended Itô lemma 2.1.4, we find that

$$\begin{aligned} f(t,x) = \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma x\right] \quad & f_1 = \left(\mu - \frac{1}{2}\sigma^2\right) f(t,x) \\ f_2 = \sigma f(t,x) \quad & f_{22} = \sigma^2 f(t,x) \end{aligned}$$

Substituting into (2.5) and using linearity, we get that X_t satisfies the equation:

$$X_t - X_0 = \mu \int_0^t X_s\, ds + \sigma \int_0^t X_s\, dW_s \tag{2.7}$$

or equivalently

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{2.8}$$

which is exactly the equation for which we want to find a solution. Therefore, the stochastic process (2.6) is the solution of our stochastic differential equation, and it is our model of prices. We call processes of this form Geometric Brownian Motion.

To prepare for the next section, we will need even more general version of Itô lemma.

Definition 2.1.5 (Itô process)**.** A process of the form

$$X_t = X_0 + \int_0^t A_s^{(1)}\, ds + \int_0^t A_s^{(2)}\, dW_s \tag{2.9}$$

where $A^{(1)}, A^{(2)}$ are random processes adapted to Brownian motion (see definition in appendix), is called Itô process.

Theorem 2.1.6 (Uniqueness of Itô process coefficients)**.** If a stochastic process $\{X_t\}_{t\geq 0}$ has the form (2.9), then the processes $A^{(1)}, A^{(2)}$ are determined uniquely.

Theorem 2.1.7 (General Itô lemma)**.** Let X be an Itô process and $f(t,x)$ a function with continuous second order partial derivatives. Then for $0 \leq s < t$

$$\begin{aligned} f(t, X_t) - f(s, X_s) = \int_s^t \left[f_1(y, X_y) + A_y^{(1)} f_2(y, X_y) + \frac{1}{2}[A_y^{(2)}]^2 f_{22}(y, X_y)\right] dy \\ + \int_s^t A_y^{(2)} f_2(y, X_y)\, dB_y \end{aligned} \tag{2.10}$$

2.2 Black-Scholes option pricing

Now, as we have the apparatus of stochastic calculus at hand, we would like to make use of it. In 1973, Black, Scholes and Merton derived an equation usable for option pricing. We will derive it here and then discuss it.

As we stated in the section introducing financial derivatives, an option is a right to buy (call option) or sell (put option) a stock at a specific time (time of maturity) for a specific price (strike price). If one can exercise the option only at the time of maturity, it is called European option. If one can exercise it before or at the time of maturity, we call the option American.

The important feature of options is that they are just a right, not an obligation. If it is not profitable for the owner of the option to exercise it, he can let it expire. As the option is just a right, we expect there will be some price or fee, for which the issuer would be willing to sell it. Our task is to find a rational price for an option.

At the time of maturity T, the purchaser of a European call option can be in two situations, depending on the price of the underlying stock S_T and the strike price K:

1. $S_T > K$, then he exercises the stock and have a profit of $S_t - K$
2. $S_T \leq K$, then he lets the option expire, because he can purchase the stock for a better price than the strike price

This final state can be compactly written as $(S_T - K)^+ = \max(0, S_T - K)$. To find a rational price for an option, we will build an associated portfolio.

2.2.1 Self-financing portfolio

To find a rational price for an option, we want to find a strategy for an issuer to omit risk. It will be as follows:

- Sell the option for price V_0
- Invest V_0 in a portfolio consisting of the underlying stock and an account with risk-less interest rate, which is self-financing, i.e. which will not need any other financial investments during time
- Manage the portfolio such a way that its value at time T will be $(S_T - K)^+$, which means that issuer will not be at a loss.

We suppose that the price of an underlying asset follows our pricing model from previous chapter, i.e.

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] \tag{2.11}$$

Then we suppose there is a non-risky asset available, such as a bank account, where we suppose continuous interest. This is called *bond*. The value of our investment (β_0 being initial deposit) is therefore

$$\beta_t = \beta_0 e^{rt} \tag{2.12}$$

We want to build a portfolio, where we will be able to continuously change the amount invested in the stock and in the bond. Therefore, there will be two processes, a_t and b_t denoting the investment in the stock and bond respectively in the time t. The value of our portfolio the time t is

$$V_t = a_t S_t + b_t \beta_t \tag{2.13}$$

We allow a_t and b_t to be both positive or negative, meaning that the stock or money is borrowed (both is possible on the market). We will neglect all transaction costs for simplicity.

Now what does it mean for a portfolio to be self-financing? It means that it gets along just with the initial investment, the purchase of stock must be balanced by corresponding reduction of investment in the bond. This also means that the change of the wealth V_t can result only from changes of the prices of S_t and β_t. In terms of differentials, this translates as

$$dV_t = a_t dX_t + b_t d\beta_t \tag{2.14}$$

which we interpret as

$$V_t - V_0 = \int_0^t a_s \, dS_s + \int_0^t b_s \, d\beta_s \tag{2.15}$$

2.2.2 Black-Scholes equation

Firstly, the issuer sells the option for V_0. Then, he invests to a self-financing portfolio described above. But how exactly should he manage the portfolio to achieve the desired value $(K - S_T)^+$ at the maturity time T?

We suppose, that there is a deterministic function $u(t, x)$, such that

$$V_t = a_t dS_t + b_t d\beta_t = u(T - t, S_t), \quad t \in [0, T] \tag{2.16}$$

and we impose the terminal condition

$$V_T = u(0, S_T) = (S_T - K)^+ \tag{2.17}$$

We know, that the process $\{S_t\}_{t\geq 0}$ follows (2.9) with $A_t^{(1)} = \mu S_t, A_t^{(1)} = \sigma S_t$ and $V_t = u(T - t, S_t)$. We intend to use the general Itô lemma 2.1.7, therefore we write $u(T - t, x) = f(t, x)$, and we see, that

$$f_1(t, x) = -u_1(T - t, x), \quad f_2(t, x) = u_2(T - t, x), \quad f_{22}(t, x) = u_{22}(T - t, x)$$

Now, using 2.1.7 we get

$$\begin{aligned} V_t - V_0 = f(t, S_t) - f(0, S_0) = \\ \int_0^t \left[-u_1(T - s, S_s) + \mu S_s u_2(t - s, S_s) + \frac{1}{2}\sigma^2 S_s^2 u_{22}(T - s, S_s) \right] ds \\ + \int_0^t \sigma S_s u_2(T - s, S_s) \, dW_s \end{aligned} \tag{2.18}$$

Equation (2.15) also holds. We will rewrite it a little

$$\begin{aligned} V_t - V_0 = \int_0^t a_s \, dS_s + \int_0^t \frac{V_s - a_s S_s}{\beta_s} r \beta_s \, ds \\ = \int_0^t [(\mu - r) a_s S_s + r V_s] \, ds + \int_0^t \sigma a_s S_s dW_s \end{aligned} \tag{2.19}$$

where firstly we used the equation $V_t = a_t S_t + b_t \beta_t$ to obtain $\beta_t = \frac{V_t - a_t S_t}{\beta_t}$, and we also substituted $d\beta_t = r\beta_t dt$. Then we used the relation (2.8) for dS_t and the linearity of Itô integral.

We know, that coefficients of an Itô process are determined uniquely 2.1.6, therefore the following equations hold:

$$a_t \overset{!}{=} u_2(T-t, X_t) \tag{2.20}$$

$$\begin{aligned}(\mu - r)a_t S_t + rV_t &= (\mu - r)X_t u_2(T-t, S_t) + ru(T-t, S_t) \\ &\overset{!}{=} -u_1(T-t, S_t) + \mu X_t u_2(T-t, S_t) \\ &\quad + \frac{1}{2}\sigma^2 X_s^2 u_{22}(T-t, S_t)\end{aligned} \tag{2.21}$$

S_s may assume any positive value, therefore we can write

$$-u_1(t,x) = \frac{1}{2}\sigma^2 x^2 u_{22}(t,x) + rxu_2(t,x) - ru(t,x) \tag{2.22}$$

We rewrite the equation in a more common way

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.23}$$

where V denotes the value of the portfolio and S the price of the stock. This is the celebrated **Black-Scholes equation**

2.2.3 Solution of the Black-Scholes equation

We will show in short, how to find a solution to the Black-Scholes equation. Firstly, we will remind us of the boundary conditions:

- $t = T : V(T,S) = (S_T - K)^+$
- $S = 0 : V(t, 0) = 0$
- $S \to \infty : V(t,S) \sim S$

The first condition was discussed earlier. The second basically says that whenever price S reaches zero, then due to (2.8) the price will remain zero and the call option is worthless. The third condition says, that for sufficiently high price S, it is definitely bigger than the strike price K, which is getting negligible for $S \gg K$. The value of the option therefore approaches the price S.

We make two substitutions and ansatz:

$$V = Kf(x,\tau), \qquad S = Ke^x, \qquad t = T - \frac{\tau}{(\sigma^2/2)} \tag{2.24}$$

which transform our equation and boundary condition to

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial x^2} + (\kappa - 1)\frac{\partial f}{\partial x} - \kappa f \tag{2.25}$$

$$\tau = 0 : f(x, 0) = \max(e^x - 1) \tag{2.26}$$

where $\kappa = \frac{2r}{\sigma^2}$. Then we make another ansatz:

$$f(x,\tau) = e^{ax+b\tau} g(x,\tau) \tag{2.27}$$

for a, b real but undetermined. This leaves us with

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} + (2a+\kappa-1)\frac{\partial g}{\partial x} + [a^2 - b + (\kappa-1)a - \kappa]f \tag{2.28}$$

Here, a and b can be chosen to make the expressions in brackets zero, which gives the heat equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} \tag{2.29}$$

with the boundary condition

$$g(x,0) = (e^{(\kappa+1)x/2} - e^{(\kappa-1)x/2})^+ \tag{2.30}$$

The equation has a standard solution in the form

$$\frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} g(y,0) \exp\left(-\frac{(x-y)^2}{4\tau}\right) dy \tag{2.31}$$

which after integration and restoration of the original variables leads to the solution

$$V(S,t) = S\Phi(d_1(S,t)) - Ke^{-r(T-t)}\Phi(d_2(S,t)) \tag{2.32}$$

$$d_1(S,t) = \frac{ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{2.33}$$

$$d_2(S,t) = d_1(S,t) - \sigma\sqrt{T-t} \tag{2.34}$$

where $\Phi(x)$ is the value of the normal cumulative distribution function at x.

What is remarkable about this solution? We shall remind that we started with a random process, and we ended with a deterministic solution. What does this solution tell us? We can derive a rational price for an option, and also we got a way to manage the accompanying self-financing portfolio.

The rational price for an option, given the price S_0 at the time $t = 0$ is given by

$$V(S_0,0) = S_0\Phi(d_1(S_0,0)) - Ke^{-rT}\Phi(d_2(S_0,0)) \tag{2.35}$$

The strategy to maintain risk-less portfolio is then given by the obtained solution (2.32), (2.13) and (2.20):

$$a_t = \frac{\partial V}{\partial S}(S,t) \qquad b_t = \frac{V(S,t) - a_t S}{\beta_t} \tag{2.36}$$

The strategy is risk-less in the sense that if we sell a call option for a price given by (2.35), use the money to establish a portfolio and then manage the portfolio using (2.36), we are guaranteed to end with a portfolio value of $(S_T = K)^+$, which exactly compensates the loss made by the execution of the option at the maturity time. On the other hand, if we manage to sell the option for a higher price than the rational one, we make a risk-less profit which is an arbitrage. Therefore, the option prices should tend to be the rational ones.

In finance, there is a slightly different notation. The value a_t is usually denoted as Δ and the strategy $\Delta = \frac{\partial V}{\partial S}$ is named delta-hedging.

2.3 Assumptions of the Black-Scholes model

The derivation of Black-Scholes formula is so appealing, that the assumptions are often forgotten. However, they are crucial and even worse, they mostly do not correspond to the empirical evidence.

1. **There is no credit risk**: This means that there is no risk connected with the issuer of the option or the buyer, for example that the issuer will not be able to fulfill his obligation to sell, when a call option is exercised. If we had to include this type of risk, the value of the option would be different. This assumption is fragile during crises (such as one in 2008).

2. **The market is perfectly liquid**: There are no transaction costs, there are no barriers to buying or selling a stock or to deposit money in bank. This depends on the chosen market.

3. **Continuous trading and divisible underlying**: It is possible to trade continuously and the amount of stocks in portfolio need not be an integer. The continuous trading is more problematic because of transaction costs.

4. **The time evolution of the asset prices follow a geometric Brownian motion**: Therefore this model of prices does not violate the EMH. On the other hand, we will show that this assumption does not correspond to empirical data. The extremes just happen too often. This can lead to a huge underestimation of risks.

5. **The risk-free rate r and the volatility σ are constant**: This can be relaxed to an assumption, that r and σ are known functions of time. However, interest-rate is not known in advance and the volatility is hard to estimate.

6. **The underlying pays no dividends**: However, the Black-Scholes equation can be modified to allow for dividends.

7. **There are no arbitrage opportunities**: This assumption is manifested by the usage of bank account interest rate r as the best risk-less rate one could get. No arbitrage principle says that there could not be a better way to invest in a risk-less manner.

In the next sections, we will shortly comment the assumption of constant volatility, then we will elaborate the assumption of geometric Brownian motion.

2.3.1 Volatility smile

As we mentioned, the assumption of constant volatility and interest rate can be weakened. It can be shown, that if we replace r and σ with their time averages $\frac{1}{T-t}\int_t^T r(s)\,ds$, $\frac{1}{T-t}\int_t^T \sigma(s)\,ds$, the Black-Scholes formula remains valid. There are also generalized models, which assume interest rate and volatility to be stochastic processes. [12]

However, we can also invert the Black-Scholes formula to derive volatility. Let us assume that market finds correct price for a European call option, denoted by $V^{(m)}$. The Black-Scholes value of an option depends on the volatility of the underlying,

$V^{(B-S)} = V^{(B-S)}(S_t, T-t; K, r, \sigma)$. Therefore, knowing the interest rate r, the price of the underlying S_t, the strike price K and the time to maturity $T-t$, we can obtain the so-called implied volatility from the relation $V^{(m)} \stackrel{!}{=} V^{(B-S)}(S_t, T-t; K, r, \sigma_{imp})$.

We argued that if the Black-Scholes model assumptions were correct, then it provides rational prices for options. Now we assume that the market also finds rational prices. Therefore, the implied volatility for one chosen option should be independent on its strike price K. However, empirical data show the dependence. In the following figure, we show the dependence of implied volatility on strike price for S&P 500 options.

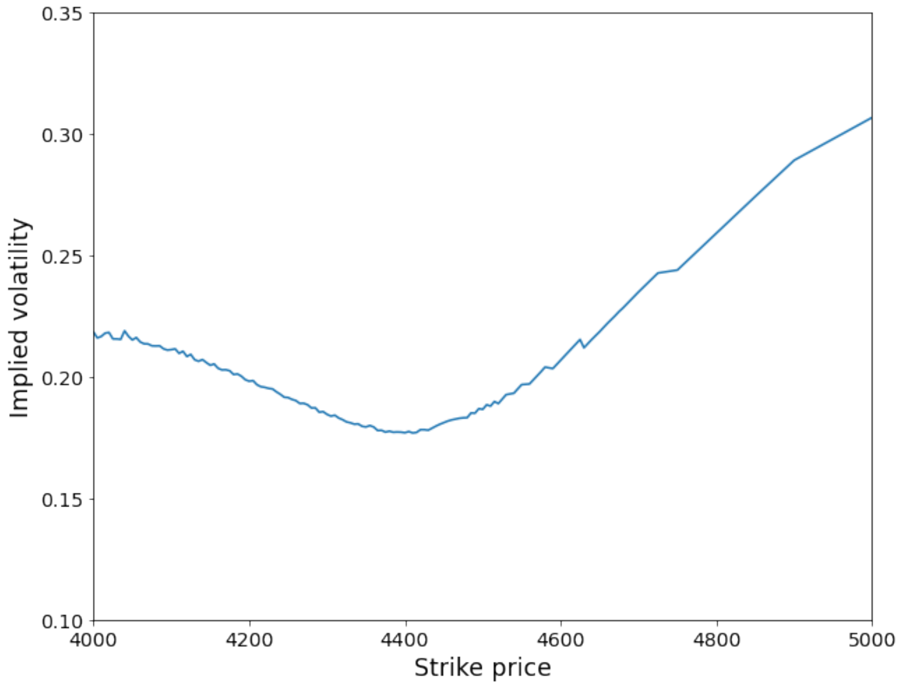

Figure 2.1: Volatility smile for S&P 500 options

This specific shape is called volatility smile in financial literature. Interesting fact to point out is, that before the market crash in 1987, there was no volatility smile observed for stock options [12]. The reason for the smile to appear since may be the so-called "crashophobia", the fear of traders that the crash similar to 1987 may happen again. The geometric Brownian motion used in the Black-Scholes model underestimates these extreme events, which leads traders to use higher volatility for extreme strike prices.

Whatever the reason may be, the volatility smile shows that the Black-Scholes model is not used by the market as is. That means that the model can not recognize the best prices, because if it could and the market did not use it, there would be arbitrage opportunities.

2.3.2 Beyond the geometric Brownian motion

The model of prices used in the Black-Scholes model is the geometric Brownian motion, the price evolution is thus of the form:

$$S_t = \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right] \tag{2.37}$$

The important quantity in finance is the logarithm of price. We will elaborate the differences between price logarithms in two different times:

$$\ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \tag{2.38}$$

$$\Delta \ln S_t = \ln S_{t+\Delta t} - \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + \sigma(W_{t+\Delta t} - W_t) \tag{2.39}$$

We know that $(W_{t+\Delta t} - W_t) \sim \mathcal{N}(0, \Delta t)$ and also that the increments of the Wiener process in non-overlaping times are independent. Therefore, $\Delta \ln S_t \sim \mathcal{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right) \Delta t, \sigma \Delta t\right)$ and if we choose a constant time step, these differences will be i.i.d. If we make a histogram of these differences, we should observe a normal distribution, independently on the time difference (scale) we choose. This is an empirically verifiable statement.

To verify this, we use prices of 504 stocks issued by 500 companies included in the S&P 500 index. We count the logarithm of prices and then the differences using a constant interval. The intervals used are 1 minute during the period of 7 days (631 162 data points in total), then 1 hour during 730 days (2 128 683 data points) and finally 1 day during 20 years (1 795 472 data points). We must take in account that the trading stops during nights, weekends etc., therefore we must carefully filter the data so that only the appropriate differences will be used.

The resulting histograms (in blue) are shown on the three following pictures. We use the logarithmic scale on the y axis to make the difference between normal distribution and heavy-tails more visible. We count the sample standard deviation, scale the x axis according to it and then compare the data to the normal distribution with the sample mean and sample standard deviation as parameters (in red).

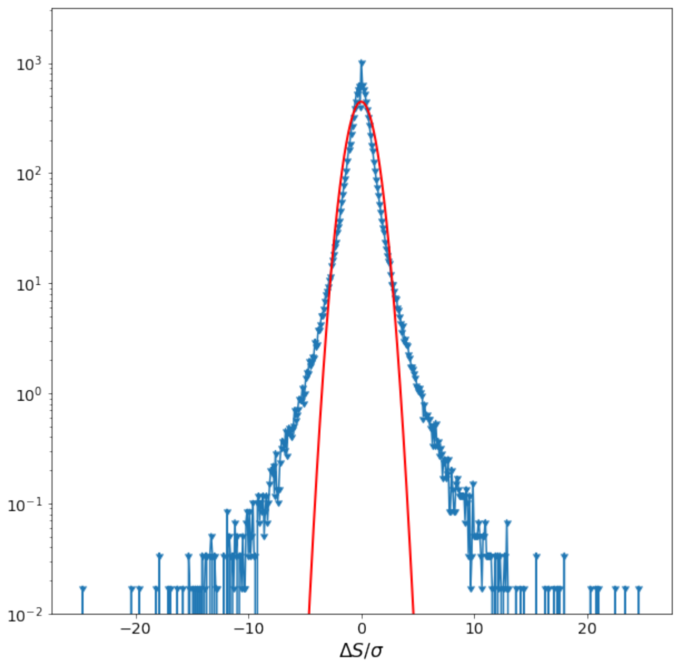

Figure 2.2: 7 days, 1 minute interval

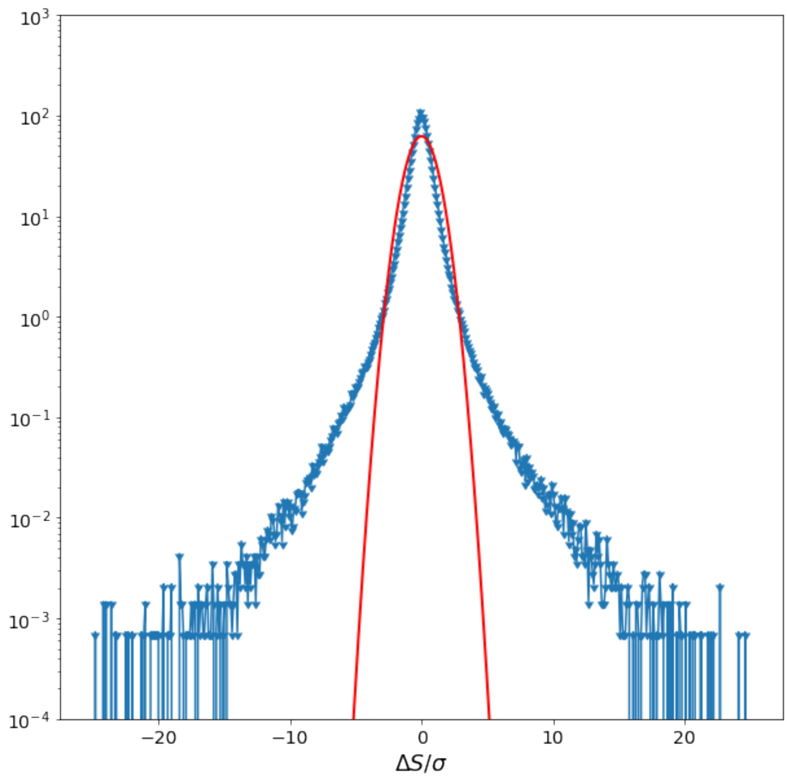

Figure 2.3: 730 days, 1 hour interval

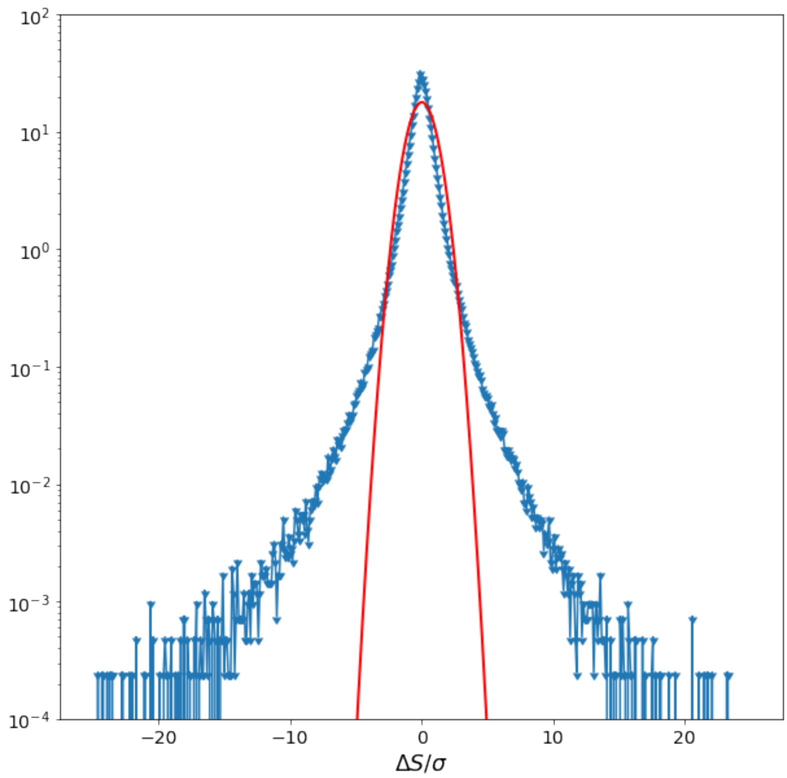

Figure 2.4: 20 years, 1 day interval

These histograms are similar to those obtained by Mantegna and Stanley [4]. We can clearly see that the normal distribution does not fit the data on any of the chosen timescales. The histograms show much heavier tails than normal distribution of any parameters can provide. This means that the normal distribution significantly underestimates extreme events.

What was the rationale behind the usage of geometric Brownian motion? Certainly the Central limit theorem (CLT). The price is determined by many random factors, which in sum should form a normal distribution. Let us see the CLT with all its assumptions.

Theorem 2.3.1 (Central Limit Theorem)**.** Let $(X_i)_{n=1}^{\infty}$ be a sequence of **independent**, identically distributed random variables with finite mean $\mathbb{E}X_i = \mu$ and **finite variance** $VarX_i = \sigma^2$. Then

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \xrightarrow{\mathcal{D}} X \sim \mathcal{N}(0,1) \tag{2.40}$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in the distribution (see appendix for definition).

The central limit theorem appears in many situations, in finance however, we do not observe normal distributions often. Some of the assumptions is therefore violated. Firstly, we will discuss the assumption of finite variance. Then, we will take a closer look on the independence of random inputs.

Stable distributions and generalized central limit theorem

We would like to find a generalization of the central limit theorem for random variables with infinite first or second moment. Having $(X_i)_{n=1}^{\infty}$ sequence of independent, identically distributed random variables, the question is, if it is possible to find constants a_n, b_n and some limiting distribution L so that the distribution of the sum

$$S_n = \frac{\sum_{i=1}^{n} X_i - a_n}{b_n}$$

converges to the limiting distribution L. Then we would say that the ditribution of X_i is in the domain of attraction of the distribution L. Lévy and Khintchine showed that any limiting distribution must be stable.

Definition 2.3.2 (Stable random variable)**.** A random variable X is called stable, if for every sequence $X_1, \dots, X_n$ of independent copies of the random variable X, there exist real-valued c_n, d_n such that

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} c_n X + d_n \tag{2.41}$$

Alternatively, we may define stability in terms of the probability density function.

Definition 2.3.3 (Stable probability density)**.** A probability density p is called stable, if it is invariant under convolution, i.e. for all $a_1, a_2 > 0, b_1, b_2 \in \mathbb{R}$, there exist $a > 0, b \in \mathbb{R}$ such that

$$p(a_1x + b_1) * p(a_2x + b_2) = \int_{-\infty}^{\infty} p(a_1(x-y) + b_1)p(a_2y + b_2)\, dy = p(ax + b) \tag{2.42}$$

It is a well known fact, that normal distribution satisfies this relation. Are there any other stable distributions? This was answered by Lévy and Khintchine in terms of the characteristic function.

Definition 2.3.4 (Characteristic function)**.** Let X be a random variable with a probability density p. Then the characteristic function φ of this random variable is defined as a Fourier transform of its probability density, i.e.

$$\varphi(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\, dx \tag{2.43}$$

Theorem 2.3.5 (Complete characterisation of stable densities)**.** A probability density $p_{\alpha,\beta}$ is stable $\Leftrightarrow$ its characteristic function satisfies

$$\ln \varphi(k) = \begin{cases} i\mu k - \sigma^\alpha |k|^\alpha \left[1 - i\beta \frac{k}{|k|} \tan(\frac{\pi}{2}\alpha)\right], & \text{if } \alpha \neq 1, 2 \\ i\mu k - \sigma|k| \left[1 + i\beta \frac{k}{|k|} \frac{2}{\pi} \ln|k|\right], & \text{if } \alpha = 1 \\ i\mu k - \frac{1}{2}\sigma^2 k^2, & \text{if } \alpha = 2 \end{cases} \tag{2.44}$$

for some $\mu \in \mathbb{R}, \sigma > 0, 0 < \alpha \leq 2, -1 \leq \beta \leq 1$

The constants μ, γ are scale factors, on the other hand, α and β determine the shape and properties of the probability density. The parameter α is of great importance, because it determines the asymptotic behavior of the probability density in the case of $0 < \alpha < 2$:

$$p_{\alpha,\beta}(x) \sim \frac{1}{|x|^{1+\alpha}} \qquad \text{for } |x| \to +\infty \tag{2.45}$$

From this behavior, it is clear that there are finite moments $\mathbb{E}(|X|^\delta)$ of such distribution just for $\delta < \alpha$. For example, if $\alpha \leq 1$, both mean value and variance do not exist, for $1 < \alpha \leq 2$ the variance does not exist.

The parameter β determines the asymmetry of $p_{\alpha,\beta}(x)$, for example, for $\beta = 0$, the $p_{\alpha,\beta}(x)$ is an even function of x.

There are only three cases, in which we can find the probability density in a closed form:

- $\alpha = 2$: the normal distribution $\mathcal{N}(\mu, \sigma^2)$
- $\alpha = 1, \beta = 0$: the Cauchy distribution, $p_{1,0}(x) = \frac{\sigma}{\pi[(x-\mu)^2+\sigma^2]}$
- $\alpha = \frac{1}{2}, \beta = 1$: the Lévy distribution,

$$p_{1/2,1}(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\left[-\frac{\sigma}{2(x-\mu)}\right] \quad \text{for } x > \mu$$

Now we are ready to formulate the generalized CLT. We denote $L_{\alpha,\beta}$ as the cumulative distribution function for the stable density $p_{\alpha,\beta}$.

Theorem 2.3.6 (Generalized central limit theorem)**.** Let $(X_i)_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables, whose probability density has an asymptotic behavior

$$p_{\alpha,\beta}(x) \sim \frac{\alpha a^\alpha C_\pm}{|x|^{1+\alpha}} \qquad \text{for } x \to +\pm\infty,\ 0 < \alpha < 2$$

for some constants $C_+, C_- \geq 0$ and $a > 0$. Then

$$\frac{\sum_{i=1}^n X_i - a_n}{an^{1/\alpha}} \xrightarrow{\mathcal{D}} X \sim L_{\alpha,\beta}$$

where

$$a_n = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ nE(X_1) & \text{if } 1 < \alpha < 2 \end{cases}$$

$$\beta = \begin{cases} \frac{C_- - C_+}{C_+ + C_-} & \text{if } \alpha \neq 1 \\ \frac{C_+ - C_-}{C_+ + C_-} & \text{if } \alpha = 1 \end{cases}$$

Now we know how the sums of i.i.d. variables with infinite variance or mean behave. They converge to some stable distribution. The stable distributions, except for normal distribution, are asymptotic power laws. This may be the reason why we see heavy tails in the empirical financial data. There is a discussion among scientists, if the prices can be modelled using stable laws. Akgiray and Booth [13], for example, argue that the empirical tail shapes differ from those of stable distributions and that the estimates of parameters may not be robust to these differences. Therefore, they suggest to use heavy-tailed distributions, which are in the domain of attraction of the normal distribution.

We have seen on the figures, that the heavy tails appear for 1 minute interval as well as for 1 day. This may signify that the underlying distribution is stable, however, it can also be just a very slow convergence to the normal distribution.

We may show a model with such a property, which was proposed by Mantegna and Stanley [4]. They suggest a random variable could behave as a stable distribution, but just within some allowed range.

$$p(x) = \begin{cases} Np_{\alpha,0}(x) & \text{for } -x_{\text{cut}} \leq x \leq x_{\text{cut}} \\ 0 & \text{otherwise} \end{cases} \tag{2.46}$$

where N is a normalization constant and x_{cut} is the cutoff parameter. By truncating the tails, the variance is made finite, therefore such a variable is in the domain of attraction of the normal distribution. However, for large values of x_{cut}, the convergence is very slow.

We can also truncate the tails smoothly by the exponential function [2].

$$p(x) \sim \begin{cases} c_- e^{-\lambda|x|}|x|^{-(1+\alpha)} & \text{for} \quad x \ll 0 \\ c_+ e^{-\lambda x} x^{-(1+\alpha)} & \text{for} \quad x \gg 0 \end{cases} \tag{2.47}$$

For λ small, the Lévy character is pronounced and the convergence to normal distribution is small.

Independence of variables

The second crucial assumption for CLT to hold is the independence of the random variables. There are many generalizations of the CLT, which relax this condition, however, all of them rely on at least low correlations between the variables. There is no limit theorem which would cover strong correlations. On the other hand, in the times of financial crashes, the correlations between market participants are very frequent, mass behavior takes place. The same happens in physical systems in the vicinity of critical points. Therefore, the next chapter is dedicated to critical phenomena in physics and their analogies in financial systems.

Chapter 3

Critical phenomena, crashes and crises

In this chapter, we will introduce critical phenomena in physics, their connections to financial markets, and then we will show some models of critical behavior in markets. For the critical phenomena in physics, our main sources are [14] and [15].

3.1 Critical phenomena in physics

Critical phenomena are those which happen near critical points. What a critical point is? For example, water has its critical point at $T_c = 647\text{K}$ and $p_c = 22.064\text{MPa}$. Under T_c, there is a sharp difference between the liquid and gas phases, in terms of different densities (there is nonzero difference between liquid density ρ_L and gas density ρ_G). At the critical temperature, the difference vanishes and no phase transition happens. And for even higher temperatures, $T \gg T_c$, the behavior gets closer and closer to the ideal gas.

The second typical example of a critical phenomenon is observed in ferromagnets. Ferromagnet is a material with macroscopic magnetization. It is caused by the presence of magnetic domains, within which the spins are aligned to the same direction. However, by rising the temperature, the magnetization M gets lower and at the critical temperature T_c, it vanishes completely. In fact, M approaches zero with an infinite slope.

What is so interesting about critical points? There are certain quantities, which diverge at the critical point. For the fluid systems, it is the isothermal compressibility $K_T = \rho^{-1}(\partial\rho/\partial P)_T$ and the specific heat at constant volume V, $C_V = -T(\partial^2 F/\partial T^2)_V$, where F is the Helmholtz free energy, analogously for a magnetic system it is the isothermal susceptibility $\chi_T = (\partial M/\partial H)_T$ and the specific heat at constant magnetic field $C_H = -T(\partial^2 G/\partial T^2)_H$, where G is the Gibbs free energy. These divergences are not precisely predicted by standard models such as the mean-field model.

Not only that these quantities diverge in the limit $T \to T_c$, they diverge as a power law of the form $(-\varepsilon)^{-\xi}$, where ξ is the characteristic exponent and $\varepsilon = (T - T_c)/T_c$ is the reduced temperature. Definition of some of these critical-point exponents can be found in Table 3.1.

The interesting thing here is that different systems can behave very similarly near the critical point, having critical exponents the same. Such an example is a

Type	α'	β	γ'
Fluid	$C_{V=V_C} \sim (-\varepsilon)^{-\alpha'}$	$\rho_L - \rho_G \sim (-\varepsilon)^{\beta}$	$K_T \sim (-\varepsilon)^{-\gamma'}$
Magnet	$C_{H=0} \sim (-\varepsilon)^{-\alpha'}$	$M_{H=0} \sim (-\varepsilon)^{\beta}$	$\chi_T \sim (-\varepsilon)^{-\gamma'}$

Table 3.1: Table of critical exponents.

single-axis ferromagnet and a simple fluid. We can then find so-called "universality classes", in which the systems have the same behavior near critical points.

3.1.1 Scaling

We define the reduced temperature as

$$\varepsilon \equiv \frac{T - T_C}{T_C} \tag{3.1}$$

We then examine the behavior of a function $f(\varepsilon)$ when ε reaches 0. Supposing that $f(\varepsilon)$ is continuous in the neighborhood of $\varepsilon = 0$, we define the critical-point exponent as

$$\lambda \equiv \lim_{\varepsilon \to 0} \frac{\ln f(\varepsilon)}{\ln \varepsilon} \tag{3.2}$$

Then we use a notation $f(\varepsilon) \sim \varepsilon^{\lambda}$. The function f can be written using correction terms:

$$f(\varepsilon) = A\varepsilon^{\lambda}(1 + B\varepsilon^{y} + \dots), \qquad y > 0 \tag{3.3}$$

The focus on the critical-point exponent is rationalized by the experimental fact, that near the critical point, the leading term dominates. On the log-log plot of experimental data, we should see a straight line signifying power-law behavior, from which we can determine the critical-point exponent. Determining the whole function might not be possible.

There are some relations between the critical-point exponents. We will introduce the Rushbrooke's inequality concerning α', β, γ' and then we will show that under additional assumptions, we can deduce equality.

Rushbrooke inequality

The only relations between the critical-point exponents we can find rigorously are inequalities. Here, we will examine the magnetic system for intensity $H = 0$ and $T \to T_C^-$

Firstly, we define

$$C_M \equiv T\left(\frac{\partial S}{\partial T}\right)_M = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_M, \quad C_H \equiv T\left(\frac{\partial S}{\partial T}\right)_H = -T\left(\frac{\partial^2 G}{\partial T^2}\right)_H \tag{3.4}$$

where we used Maxwell relations. We use a thermodynamic relation shown in [14]

$$\chi_T(C_H - C_M) = T\left(\frac{\partial M}{\partial T}\right)_H^2 \tag{3.5}$$

It can be shown that for stable systems, the capacity C_M must be positive. Therefore

$$C_H \geq \frac{1}{\chi_T} T\left(\frac{\partial M}{\partial T}\right)_H^2 \tag{3.6}$$

Using the definition of exponents in table Table 3.1

$$C_H \sim (-\varepsilon)^{-\alpha'}, \quad \chi_T \sim (-\varepsilon)^{-\gamma'}, \, (\partial M/\partial T)_H \sim (-\varepsilon)^{\beta-1}$$

we see can write

$$(-\varepsilon)^{-\alpha'} \geq \text{const} \cdot (-\varepsilon)^{-\gamma'}(-\varepsilon)^{2(\beta-1)} \tag{3.7}$$

from which we get the Rushbrooke's inequality

$$\alpha' + 2\beta + \gamma' \geq 2 \tag{3.8}$$

For some materials, the experimental values of critical-point exponents fail to add up to two unless errors are taken into account. Therefore, it can be suggested that the exponents should satisfy an equality, which is a prediction of the *scaling law hypothesis*

Scaling law hypothesis

The scaling law hypothesis is based on the assumption, that the Gibbs potential $G(\varepsilon, H)$ is a generalized homogenous function.

Definition 3.1.1 (Generalized homogenous function)**.** A function $f(x, y)$ is generalized homogenous function (GMF), if there exist some constants a, b such that

$$f(\lambda^a x, \lambda^b y) = \lambda f(x, y), \quad \forall \lambda > 0 \tag{3.9}$$

No rigorous justification for this hypothesis has been found yet, however, it gives interesting predictions. Let us suppose, that Gibbs potential is GMF, meaning

$$G(\lambda^{a_\varepsilon}\varepsilon, \lambda^{a_H}H) = \lambda G(\varepsilon, H) \tag{3.10}$$

We may count derivatives of this equation:

$$\begin{aligned}
\lambda^{a_H}\left(\frac{\partial G}{\partial H}\right)_T(\lambda^{a_\varepsilon}\varepsilon, \lambda^{a_H}H) &= \lambda\left(\frac{\partial G}{\partial H}\right)_T(\varepsilon, H) \\
\lambda^{2a_H}\left(\frac{\partial^2 G}{\partial H^2}\right)_T(\lambda^{a_\varepsilon}\varepsilon, \lambda^{a_H}H) &= \lambda\left(\frac{\partial^2 G}{\partial H^2}\right)_T(\varepsilon, H) \\
\lambda^{2a_\varepsilon}\frac{1}{T_c}\left(\frac{\partial^2 G}{\partial T^2}\right)_T(\lambda^{a_\varepsilon}\varepsilon, \lambda^{a_H}H) &= \lambda\frac{1}{T_c}\left(\frac{\partial^2 G}{\partial T^2}\right)_T(\varepsilon, H)
\end{aligned}$$

Using definitions and Maxwell relations

$$M = -\left(\frac{\partial G}{\partial H}\right)_T, \, \chi_T = -\left(\frac{\partial^2 G}{\partial H^2}\right)_T, \, C_H = -T\left(\frac{\partial^2 G}{\partial T^2}\right)_T$$

we obtain

$$\begin{aligned}
M(\varepsilon, H) &= \lambda^{a_H-1}M(\lambda^{a_\varepsilon}\varepsilon, \lambda^{a_H}H) \\
\chi_T(\varepsilon, H) &= \lambda^{2a_H-1}\chi_T(\lambda^{a_\varepsilon}\varepsilon, \lambda^{a_H}H) \\
C_H(\varepsilon, H) &= \lambda^{2a_\varepsilon-1}C_H(\lambda^{a_\varepsilon}\varepsilon, \lambda^{a_H}H)
\end{aligned}$$

Letting $H = 0$ and setting $\lambda = (-\varepsilon)^{-1/a_\varepsilon}$ in all three equations, we obtain

$$M(\varepsilon, 0) = (-\varepsilon)^{(1-a_H)/a_\varepsilon} M(-1, 0) \tag{3.11}$$

$$\chi_T(\varepsilon, 0) = (-\varepsilon)^{-(2a_H-1)/a_\varepsilon} \chi_T(-1, 0) \tag{3.12}$$

$$C_H(\varepsilon, 0) = (-\varepsilon)^{-(2a_\varepsilon-1)/a_\varepsilon} C_H(-1, 0) \tag{3.13}$$

Then by letting $\varepsilon \to 0^-$ and by the definition of the critical-point exponents, we obtain

$$\beta = \frac{1 - a_H}{a_\varepsilon} \tag{3.14}$$

$$\gamma' = \frac{2a_H - 1}{a_\varepsilon} \tag{3.15}$$

$$\alpha' = \frac{2a_\varepsilon - 1}{a_\varepsilon} \tag{3.16}$$

which leads by elimination of a_H and a_ε to the final equality

$$\alpha' + 2\beta + \gamma' = 2 \tag{3.17}$$

There are also other critical-point exponents, which one could define, for example δ defined for $\varepsilon = 0$ as $H \sim |M|^\delta sgn(M)$. But now we see that we can obtain them using only a_ε and a_H. Therefore, we can also obtain other equalities, such as *Widom equality* or *Griffiths equality*.

The second important consequence of the scaling hypothesis is that we can find some restrictions for the equation of state, as the quantities in it are derived from termodynamic potentials. We can take the equation

$$M(\lambda^{a_\varepsilon}\varepsilon, \lambda^{a_H} H) = \lambda^{1-a_H} M(\varepsilon, H) \tag{3.18}$$

Now we can choose $\lambda = H^{-1/a_H}$, which leads to

$$M(1, H^{-a_\varepsilon/a_H}\varepsilon) = H^{-(1-a_H)/a_H} M(H, \varepsilon) \tag{3.19}$$

Defining the scaled magnetization $M_H \equiv \frac{M}{H^{(1-a_H)/a_H}}$ and scaled temperature $\varepsilon_H = \frac{\varepsilon}{H^{a_\varepsilon/a_H}}$, we get

$$M_H = M(1, \varepsilon_H) = F(\varepsilon_H) \tag{3.20}$$

Therefore, we get a function of one variable, $F(X) = M(1, x)$, which we can compare for different materials. As shown in [15], there are several materials, such as $CrBr_3$, EuO, Ni or Pd_3Fe, whose scaled magnetization against the scaled temperature collapse on the same curve, which is predicted by the Heisenberg model mentioned below.

3.1.2 Universality

"Empirically, one finds that all systems in nature belong to one of a comparatively small number of universality classes." [15]. It seems that there are just two Hamiltonians, which can describe these classes.

The first is the Q-state Potts model, which assumes Q discrete spin orientations $\zeta_i \in 1, ...Q$ and has a Hamiltonian

$$\mathcal{H}_1(d, Q) = -J \sum_{\langle i,j \rangle} \delta(\zeta_i, \zeta_j) \tag{3.21}$$

Summation is made over neighboring spins, if they have the same orientation, they contribute $-J$ to the overall energy, otherwise they contribute nothing.

The second model, so-called n-vector model, allows for continuum of states, using spin vectors. The Hamiltonian is

$$\mathcal{H}_2(d,n) = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \tag{3.22}$$

where $\vec{S}_i = (S_{i1}, S_{i2}, ..., S_{in})$ is an n-dimensional unit vector and d is the dimensionality of the system.

Both models are generalizations of the Ising model, which corresponds to 2-state Potts model $\mathcal{H}_1(d,2)$ or to the 1-vector model $\mathcal{H}_2(d,1)$. The Ising model is important for interpreting the liquid-gas critical point or uniaxial ferromagnets. The $\mathcal{H}_2(d,2)$ is useful when interpreting λ-transition in ^{4}He , the $\mathcal{H}_2(d,3)$ is the standard Heisenberg model, which is used to interpret isotropic magnetic materials near critical points.

These models can encompass huge amount of different systems. However, finding exact solutions in closed form may not be possible. The Ising model has been solved in dimensions 1, 2. The solution in two dimensions was given by Lars Onsager in 1944 and it was the first exactly solved model exhibiting a phase transition. However, it has been shown that finding a partition function for the Ising model in higher dimensions is an NP-complete problem [16]. On the other hand, numerical simulations are possible.

3.1.3 Renormalization

Renormalization is a method, which, unlike scaling arguments leading only to relations between critical-point exponents, can make it possible to find exact values of the exponents. It is beyond the scope of this work to introduce the whole theory of renormalization and renormalization group, therefore, we will just show some basics on the percolation problem following [15].

The percolation problem assumes a d-dimensional lattice with each site having probability p of being occupied. With p small, the occupied sites will be surrounded by many empty sites, however with p increasing, clusters of occupied sites will show up. We may define the characteristic dimension $\xi(p)$ as the average size of clusters. At some point, the characteristic dimension will diverge as a power-law, $\xi(p) \sim |p - p_c|^{-\nu}$, and an infinite cluster will appear. The critical probability p_c, is referred to as the *connectivity threshold*, because for $p > p_c$ there is an infinite cluster going through the whole lattice.

We will show the method of renormalization in one dimension on the percolation problem. On Figure 3.1, we make the Kadanoff-cell transformation.

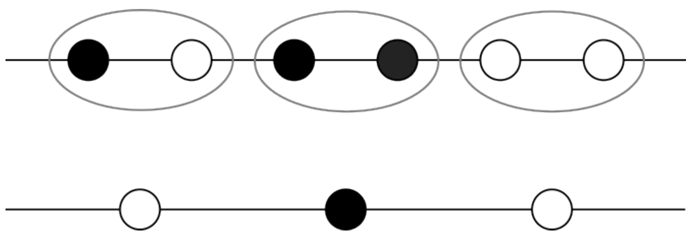

Figure 3.1: Kadanoff cells

First line is the original lattice. Then we make cells of size $b = 2$. Only if all the sites in the cell are occupied, we mark the cell as occupied, otherwise not. This transforms the original probability p of being occupied to the new probability $p' = R_b(p)$, where $R_b(p)$ denotes renormalization transformation. For general value of b, the probability of all b cells being occupied is p^b, therefore $p' = R_b(p) = p^b$. This is basically looking at the system at different scales, which is the main principle of renormalization.

Next, we would like to know, how is the characteristic dimension or correlation length $\xi(p)$ transformed. It is clear that the new correlation length will become

$$\xi(p') = b^{-1}\xi(p) \tag{3.23}$$

Successive Kadanoff-cell transformations take the system away from its critical point. If for example the initial probability $p = 0.9$, then after one $b = 2$ transformation we get $p' = 0.81$ and each other transformation lowers both the probability and the correlation length. The transformation drives the system out of the critical point.

However, if the system is initially at the critical point, than the correlation length ξ is infinite and it is such on every scale. Necessary condition for this to happen is $p' = p$. Therefore, we want to find a fixed point p^* of the transformation $R_b(p)$, meaning that $R(p^*) = p^*$.

For our particular choice $R_b(p) = p^b$, the fixed points are only $p^* = 0$ and $p^* = 1$. If the system begins near the fixed point $p^* = 1$, by transformations, it is carried away from the fixed point. We say that the fixed point is unstable under the scaling. On the other hand, the $p^* = 0$ is stable.

Now, we can expand the transformation near the fixed point:

$$R_b(p) = R_b(p^*) + \lambda_T(b)(p - p^*) + \mathcal{O}(p - p^*)^2 \tag{3.24}$$

where $\lambda_T(b)$ denotes the first derivative of the renormalization function evaluated at p^* and for our particular transformation and $p^* = 1$ we have $\lambda_T(b) = b$.

Knowing $R_b(p) = p'$, $R_b(p^*) = p^*$ and neglecting higher-order terms, we get

$$p' - p^* = \lambda_T(b)(p - p^*) \tag{3.25}$$

$$|p' - p^*|^{-\nu} = |\lambda_T(b)|^{-\nu}|p - p^*|^{-\nu} \tag{3.26}$$

For the choice $b = 1$, we get $\lambda_T(b) = 1$, therefore $p' = p$ and $\xi'(p) = \xi(p)$, which means, that ξ' and ξ are the same functions. Therefore

$$\xi(p) \sim |p - p_c|^{-\nu} \implies \xi'(p') \sim |p' - p_c|^{-\nu} \tag{3.27}$$

$$\xi'(p') = b^{-1}\xi(p) \implies |p' - p_c|^{-\nu} \sim b^{-1}|p - p_c|^{-\nu} \tag{3.28}$$

As the critical point and the fixed point are the same, we may use $p^* = p_c$ in (3.26) and then by comparing (3.26) and (3.28) we finally obtain

$$\nu = \frac{\ln b}{\ln \lambda_T(b)}$$

For our choice of transformation $\lambda_T(b) = b$, and therefore we obtained the critical-point exponent $\nu = 1$. However, this technique can be used for a general transformation.

3.1.4 Bigger picture

We showed some basics of the physics of critical phenomena. We saw that near critical points, we can observe power-law behavior with characteristic exponents, for which we can find useful relations or we can even determine them using renormalization. But what is the relation to other systems? Interesting interpretation is given in [15] and concerns correlations between spins.

Suppose a lattice of spins. There are many of them in the lattice, but they can interact only on finite distance, the correlations between them decay exponentially with distance. This characteristic length is determined by the temperature of the system. On the other hand, the number of paths using which the spins can interact grows exponentially with the characteristic length of interaction. Therefore, these two exponentials are in competition, where the winner is determined by the temperature. The temperature by which the exponentials balance is the critical temperature T_C. There, lower order terms play significant role, and we observe power-law behavior.

This interpretation can be also used in finance. The market is still between the exponential, gaussian behavior of equilibrium and the power-law behavior of some critical point. Near crash, the correlations between market participants grow, and at some point, they can be so high to cause a crash. The vicinity of crash could be determined by scale-invariant behavior of the market.

3.2 Observations and models in markets

3.2.1 Scale-free behavior of bubbles

First interesting result we will mention was obtained by Preis and Stanley [17]. They studied microtrends in price evololution and how volatility, transaction volumes and inter-trade times are connected to these microtrends. The question was, if there is any connection between small "crises", i.e. microtrends or bubbles occurring on small time scales, and the large crises such as that in the 1930's.

How one can proceed? Firstly, we need to recognize the microtrends in price time series. We define the price $P(t)$ at the time t to be a local minimum of order Δt, if there is no lower price in the interval $[t - \Delta t, t + \Delta t]$. The local maximum is defined analogously. The order Δt determines on which scale we want to determine the microtrends. Time positions of the successive extrema can be used to define a renormalized time scale. Let t_{min} and t_{max} be times of successive pair of local minimum and maximum. Then, the renormalised time scale for a positive microtrend is given by

$$\varepsilon(t) \equiv \frac{t - t_{\mathrm{min}}}{t_{\mathrm{max}} - t_{\mathrm{min}}} \tag{3.29}$$

Preis and Stanley analyzed the range $0 \leq \varepsilon \leq 2$ to see the effects of the trend switch around $\varepsilon = 1$ on volatility, transaction volume and inter-trade times. They used data from European Exchange, namely the German DAX Future contract. The dataset included all the transactions of three disjoint three-months period. The inter-trade times were down to 10ms, which allowed to study a wide range of time scales. For large scales, they used data from S&P 500.

At first, one can study volatility. Here, we can define local volatility as a squared price difference between successive trades, $\sigma^2(t) = (P(t) - P(t-1))^2$, where t is the time ordering of transactions, $t = 1, ..., T$. Then, one can average the local volatility over all positive or negative microtrends of order Δt to obtain mean volatility $\langle \sigma^2_{\text{pos}} \rangle(\varepsilon, \Delta t)$ for positive microtrends and $\langle \sigma^2_{\text{neg}} \rangle(\varepsilon, \Delta t)$ for negative ones. To normalize these volatilities, one can use average volatilities $\bar{\sigma}_{\text{pos}}$ and $\bar{\sigma}_{\text{neg}}$ which are averages over all Δt up to some Δt_{max} and over some discrete number of bins in the ε variable.

What Preis and Stanley observed was that the volatility profile was nearly identical for different values of Δt greater than some Δt_{cut}. This means that the volatility behaves similarly for a wide range of scales. Then they counted volatility aggregation for positive microtrends

$$\sigma^{2*}_{\text{pos}}(\varepsilon) = \frac{1}{\Delta t_{\text{max}} - \Delta t_{\text{cut}}} \sum_{\Delta t = \Delta t_{\text{cut}}}^{\Delta t_{\text{max}}} \frac{\langle \sigma^2_{\text{pos}} \rangle(\varepsilon, \Delta t)}{\bar{\sigma}_{\text{pos}}} \tag{3.30}$$

and $\sigma^{2*}_{\text{neg}}(\varepsilon)$ in an analogous way. $\Delta t_{\text{cut}} = 50$ and $\Delta t_{\text{max}} = 1000$ were used. When they then plotted log-log graphs of this volatility aggregation near the critical point $\varepsilon = 1$, they found that the data form a straight line, signalizing a power law dependence. Therefore we can write

$$\sigma^{2*}(|\varepsilon - 1|) \sim |\varepsilon - 1|^{\beta_{\sigma^2}} \tag{3.31}$$

where the empirically found critical exponents β_{σ^2} can be found in Table 3.2

What was also analyzed was the dependence of the trade volume, i.e. the number of contracts traded in each individual transaction, on the renormalized time. Using the same method as with volatility, i.e. counting mean volume for all microtrends of order Δt and normalizing by average volume, they obtained volume aggregation $v^*(\varepsilon)$, which also followed a power law near the critical point $\varepsilon = 1$

$$v^*(|\varepsilon - 1|) \sim |\varepsilon - 1|^{\beta_v} \tag{3.32}$$

with the exponents (for $\Delta t_{\text{cut}} = 50$ and $\Delta t_{\text{max}} = 1000$) in Table 3.2. Similarly, the inter-trade times were analyzed. Even here, the power-law behavior appeared.

	Positive microtrend		Negative microtrend	
	$\varepsilon < 1$	$\varepsilon > 1$	$\varepsilon < 1$	$\varepsilon > 1$
Volatility	0.01	-0.30	0.04	-0.54
Volume	-0.14	-0.20	-0.17	0
Inter-trade times	0.10	0.12	0.09	0.15

Table 3.2: Critical exponents for German DAX Future contract

Similar scaling exponents were found for the daily closing prices of the S&P500 index, therefore they are valid also for macrotrends. This means that the behavior of the volatility, volume and inter-trade times has no characteristic scale. Thus, one can not consider the big market crashes just as outliers, but as a result of the market scale-free behavior.

3.2.2 The Cont-Bouchaud model

The aim of the Cont-Bouchaud model is to show, what could be the nature of the market behavior in the normal phases or near the critical points [18], [2]. Let us suppose N_t traders, each trading with the same (average) amount of some given asset. At each time, i-th trader can be in one of the following states: he can buy ($\phi_i = 1$), sell ($\phi_i = -1$) or wait ($\phi_i = 0$). Let us suppose that the change in the asset price is proportional to the difference between supply and demand, which is determined by the sum of the states of individual traders

$$\Delta S(t_n) \propto \sum_{i=1}^{N} \phi_i(t_n) \tag{3.33}$$

Now let us suppose that the traders communicate, either directly or by having same trading strategies depending on the available information. This communication can be represented by a bond between these traders, which leads them to trade in the same way, to be in the same trading state. Let us define the probability of the bond creation as

$$p_b = \frac{b}{N_t} \tag{3.34}$$

where b is the average number of bonds of each trader. We do not have detailed information about how the bonds are created between individual traders, therefore, to model the market, we choose the bonds randomly. By doing this, we obtain groups of traders, clusters with the same trading strategy. Such a structure is called a random graph. Denoting the state of c-th cluster at the time t_n $\phi_c(t_n)$, size of the c-th s_c and let the total number of clusters be N_c. Then

$$\Delta S(t_n) \propto \sum_{c=1}^{N_c} s_c \phi_c(t_n) \tag{3.35}$$

The distribution of the price differences $\Delta S(t_n)$ is thus determined by the distribution of the sizes of clusters. However, the distribution of cluster-sizes for this model can be found [2] and is given by

$$p_c(s) \sim \frac{1}{s^{1+3/2}} \exp\left[-(1-b)^2 s\right] \qquad \text{for } 1 \ll s \ll N_t \tag{3.36}$$

for b approaching 1 from left and for $N_t \to \infty$. We see that this is the same form as of the exponentially truncated Lévy distribution. For b close to 1, the Lévy character is pronounced and the convergence to Gaussian under convolution is slow. As the price difference distribution $\Delta S(t_n)$ is proportional to the sum of independent variables with the distribution $p_c(s)$, this model predicts that we should observe the characteristic exponent close to $\alpha \simeq 3/2$ in the price changes distribution. This corresponds to several analyses of empirical data. For example, Mantegna and Stanley [4] finds that Lévy distribution with $\alpha = 1.4$ is the best fit for the price distribution data they had.

However, this prediction depends on the assumption that the parameter b is close to 1. Cont and Bouchaud suggest this to be a property of market, that it is driven towards $b \approx 1$. However, they do not offer the reason for this.

3.2.3 The Sornette-Johansen model

Sornette et al. proposed a model to describe crashes inspired by critical phenomena in physics. We examined the phase transition between ferromagnetic and paramagnetic phase in section 3.1. There we can use temperature T as a control parameter to approach the critical point at T_C. Could there be such a parameter in the market?

It could, however it can not be known by the market. If it had been known, the market would change its behavior to avoid risk, which would also lead to elimination of the parameter.

We will not go much into detail, we will only show some results from [2]. Sornette et al. introduce a *hazard rate* $h(t)$, which denotes the probability that a crash will occur in the interval $[t, t + \mathrm{d}t]$. When the price of some asset increases, traders focus their attention on it. Some of them speculates on further increase, however some of them fear of the trend reversal. This is a normal situation. However, if the attention of the traders is increased too much, if too many traders start to believe that the price will rise, a bubble emerges. This, however, also raises the fear of the losses. The market is then very sensitive to any fluctuation and a small downtrend can cause a big crash. The hazard rate $h(t)$ therefore means the stress, tension of the market.

The proposed form of a hazard rate is as follows

$$h(t) = \frac{1}{(t_c - t)^\gamma} \left[B_0 + B_1 \cos(\omega \ln(t_c - t) + \Psi)\right] \qquad \text{for } t \lesssim t_c \tag{3.37}$$

that means it follows a power-law with log-periodic correction. γ is a critical exponent, which should be universal, however, $B_0, B_1, \omega, t_c, \Psi$ are parameters which need to be found individually.

Assuming, that the deterministic part of price evolution follows a differential equation

$$\mathrm{d}S_{det} = \mu(t) S_{det}(t) \mathrm{d}t \tag{3.38}$$

we would like to find a connection between the drift $\mu(t)$ and the hazard rate $h(t)$. The drift reflects the reward traders demand when investing in the risky asset. As the hazard rate is proportional to the risk, one can assume $\mu(t) = \kappa h(t)$, introducing yet another constant κ. From this, one can find a relation for the deterministic part of the price S_{det}, however, the ability of predictions of this model is debatable due to the large number of constants to fit.

Here we can also regard to section 1.4, where we discussed the nonexistence of independent description. The question, which immediately arises, if there can be an independent prediction of crisis. If a single trader predicts crisis, he can reallocate his portfolio to avoid risk, by doing so, however, he influences the market. Furthermore, if there was a widely believed prediction of crash in the future, could the prediction persist? The market would definitely change its behavior to avoid the crash. This can lead to the elimination of the danger, however, it can also speed up the process of the market fall. In any case, the prediction dramatically influences the market, which will very probably behave in a different way than originally predicted.

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Appendix

A Remarks from probability theory

Definition A.1 (Random variable). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ a real function which is $\mathcal{A}$-measurable, i.e.

$$\forall c \in \mathbb{R}, X^{-1}(-\infty, c) \in \mathcal{A} \tag{39}$$

Then we call X a random variable, abbreviated r.v.

Here, we just give definition of convergence in distribution used in the main text.

Definition A.2 (Convergence in distribution). The sequence $(X_n)_{n=1}^{\infty}$ of random variables converges in distribution to the random variable X, denoted $X_n \xrightarrow{\mathcal{D}} X$, if

$$\lim_{n \to \infty} F_n(x) = F(x) \tag{40}$$

for every $x \in \mathbb{R}$ at which $F(x)$ is continuous, where F_n are cumulative distribution functions of X_n and F is the cumulative distribution function of X.

B Random processes

Definition B.1 (Random process). Let T be a nonempty set. Then we call a system of random variables $X_t, t \in T$ a random process on T.

Usually, we use $T = \mathbb{N}$ for discrete random processes or $T = \mathbb{R}$ or some interval for continuous-time processes. For $\omega \in \Omega$ given, $X(\omega, t)$ is a function of time, which we call a *trajectory of a random process*, or sample path. It is one concrete realisation of the random process, for example a price evolution of one given asset.

C Martingales

We motivated the usage of martingales for the Efficient Market Hypothesis in subsection 1.3.2. Here we want to show the general definition of martingale for a continuous-time random processes.

Definition C.1 (σ-algebra generated by a random variable). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, Y a random variable. Then we define $\sigma(Y) = \{Y^{-1}(A) \mid A \in \mathcal{B}(\mathbb{R}^n)\}$ to be the sigma-algebra generated by the random variable Y.

Definition C.2 (Conditional expectation with respect to a random variable)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, X, Y random variables, $\mathbb{E}(|X|), \mathbb{E}(|Y|) < +\infty$. Conditional expectation of r.v. X with respect to a r.v. Y is a $\sigma(Y)$-measurable function, denoted as $\mathbb{E}(X|Y)$, satisfying

$$\int_A \mathbb{E}(X|Y)\,\mathrm{d}\mathbb{P} = \int_A X\,\mathrm{d}\mathbb{P} \quad \forall A \in \sigma(Y)$$

It is straightforward to generalize this definition for any σ-algebra $\mathcal{A}^* \subset \mathcal{A}$.

Definition C.3 (Conditional expectation with respect to σ algebra)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, X random variable, $\mathbb{E}(|X|) < +\infty$, $\mathcal{A}^* \subset \mathcal{A}$ σ-algebra. The conditional expectation of the r.v. X with respect to $\mathcal{A}^*$ is a $\mathcal{A}^*$-measurable function, denoted as $\mathbb{E}(X|\mathcal{A}^*)$, satisfying

$$\int_A \mathbb{E}(X|\mathcal{A}^*)\,\mathrm{d}\mathbb{P} = \int_A X\,\mathrm{d}\mathbb{P} \quad \forall A \in \mathcal{A}^*$$

Let us mention several important properties of the conditional expectation:

1. If X is independent of $\mathcal{A}^*$, then $\mathbb{E}(X|\mathcal{A}^*) = \mathbb{E}(X)$
2. $\mathbb{E}(\mathbb{E}(X|\mathcal{A}^*)) = \mathbb{E}(X)$
3. Linearity: $\mathbb{E}(aX + Y|\mathcal{A}^*) = a\mathbb{E}(X|\mathcal{A}^*) + \mathbb{E}(Y|\mathcal{A}^*)$
4. If $\sigma(X) \subset \mathcal{A}^*$, then $\mathbb{E}(X|\mathcal{A}^*) = X$
5. $X, Y \in L^2(\Omega, \mathcal{A})$, $\sigma(X) \subset \mathcal{A}^*$. Then $\mathbb{E}(XY|\mathcal{A}^*) = X\mathbb{E}(Y|\mathcal{A}^*)$
6. Tower property: Let $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{A}$ be two σ-algebras. Then:

$$\mathbb{E}(X|\mathcal{F}) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}')|\mathcal{F})$$
$$\mathbb{E}(X|\mathcal{F}') = \mathbb{E}(\mathbb{E}(X|\mathcal{F})|\mathcal{F}')$$

7. Jensen inequality: Let φ be a convex function and $\mathbb{E}(|\varphi(X)|) \leq \infty$. Then $\varphi(\mathbb{E}(X|\mathcal{A}^*)) \leq \mathbb{E}(\varphi(X)|\mathcal{A}^*)$

If we interpret $\mathcal{A}^*$ as the information available to us, we can regard $\mathbb{E}(X|\mathcal{A}^*)$ as the best estimate of the r.v. X, given this information.

A random process is a set of random variables ordered by the parameter t. With t increasing, the information available about the process also increases. We describe this using filtration.

Definition C.4 (Filtration)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Then we call the set of σ-algebras $(\mathcal{A}_t)_{t\in T}$, for which $\forall s, t \in T, 0 \leq s \leq t : \mathcal{A}_s \subset \mathcal{A}_t \subset \mathcal{A}$, a filtration.

Definition C.5 (Random process adapted to filtration)**.** A random process $(X_t)_{t\in T}$ is adapted to a filtration $(\mathcal{A}_t)_{t\in T}$, if $\forall t \geq 0, \sigma(X_t) \subset \mathcal{A}_t$

It is clear, that a random process is always adapted to its natural filtration $\mathcal{A}_t^X = \sigma(X_s, 0 \leq s \leq t)$

Now we have everything ready to define the general martingale property for a continuous-time process.

Definition C.6 (Continuous-time martingale)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $(X_t)_{t\geq 0}$ a random variable adapted to a filtration $(\mathcal{A}_t)_{t\in T}$, for which $\mathbb{E}(|X_t|) < \infty, \forall t \geq 0$. Then we call $(X_t)_{t\geq 0}$ a martingale, if the so-called martingale-property is satisfied:

$$\mathbb{E}(X_t|\mathcal{A}_s) = X_s \; a.s. \; \forall t \geq s \geq 0$$

If, instead of equality, just the inequalities $\leq$, $\geq$ are satisfied, we call $(X_t)_{t\geq 0}$ a supermartingale or submartingale respectively.

Now, we would like to show that the Wiener process W_t (Brownian motion) is a martingale with resect to its natural filtration $(\mathcal{F}_t)_{t\in T}$.

$$\begin{aligned}\mathbb{E}(W_t|\mathcal{F}_s) &= \mathbb{E}(W_t - W_s + W_s|\mathcal{F}_s)\\ &= \mathbb{E}(W_t - W_s|\mathcal{F}_s) + \mathbb{E}(W_s|\mathcal{F}_s)\\ &= \mathbb{E}(W_t - W_s) + W_s = W_s\end{aligned}$$

where in second equation we used linearity, in the third that $W_t - W_s$ is independent of $\mathcal{F}_s$ for $t > s$ and in the last, we used that $\mathbb{E}(W_t - W_s) = 0$ for the Wiener process and $t > s$. Therefore, the Wiener process is a martingale.

Secondly, we would like to find some analogy for the geometric Brownian motion. It is defined as

$$X_t = X_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right] \tag{41}$$

We will find its conditional expectation with respect to the natural filtration of the underlying Wiener process.

$$\begin{aligned}\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}(X_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t}|\mathcal{F}_s)\\ &= X_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t}\mathbb{E}(e^{\sigma(W_t-W_s)+W_s}|\mathcal{F}_s)\\ &= X_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t}e^{\sigma W_s}\mathbb{E}(e^{\sigma(W_t-W_s)}|\mathcal{F}_s)\\ &= X_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t}e^{\sigma W_s}\mathbb{E}(e^{\sigma(W_t-W_s)})\\ &= X_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t}e^{\sigma W_s}e^{\frac{1}{2}\sigma^2(t-s)}\\ &= e^{\mu(t-s)}X_s\end{aligned}$$

This is exactly the form we motivated in (1.2). In third equation, we used the property 5 of the conditional expectation, in the fifth we used the mean of log-normal distribution.