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Review of the master thesis by Šimon VEDL
"The Role of Shape Operator in Gauge Theories"

It is a great pleasure to write this report on the diploma thesis by Šimon Vedl. His work gives an original link between fundamental physics of gauge theories and the mathematical work of universal connections by Narasimhan and Ramanan. This allows to generalize the shape operator from the differential geometry of embedded manifolds to general principal bundles using the language of Clifford algebras. This singles out a preferred gauge in which the equations of motion simplify.

In addition, Šimon Vedl proves better dimension bounds for the universal connection and gives numerous examples of application, notably in electromagnetism, Yang–Mills theories and general relativity via the Palatini formalism.

The physical motivations for the work come from the idea of gauge theories in physics. On a manifold, representing space-time, physical quantities are represented via sections and connections in bundles. These quantities obey differential equations, describing their evolution. Typical examples are the Yang–Mills equations.

Although the basic principles from general relativity impose the manifold and the bundles over it to be defined intrinsically, it is often convenient to work in an ambient space. Most of the intrinsic operations have a simpler form for a well chosen embedding. To give an example, the Levi–Civita connection on a Riemannian manifold M is nothing but the usual derivative followed by an orthogonal projection to the tangent bundle if M is isometrically embedded into some $\mathbb{R}^N$.

In this example of Riemannian manifold, it turns out that every compact Riemannian manifold can be isometrically embedded into some Euclidean space. This is the famous Nash embedding theorem. In Chapter 1 of the thesis, Šimon Vedl describes embedded Riemannian manifolds in detail. In particular, he discusses the shape operator, which nicely encodes the second fundamental form, and an efficient treatment using Clifford algebras.

The natural question arise whether a similar "embedding" exists for any connection in a bundle over a compact manifold. To be more precise, one asks for a universal connection ∇ in a G-bundle B such that any connection in a G-bundle $P \to M$ can be obtained as pull-back of ∇ along a map $P \to B$ (G denotes a compact Lie group).

The Narasimhan–Ramanan theorem asserts the existence of such universal connections if the dimension of M is bounded by some integer d. For $G = U(n)$ the unitary group, the universal bundle is a complex Stiefel bundle over the Grassmannian $\mathrm{Gr}(n, N)$, where $N = (d+1)(2d+1)n^3$. The Stiefel manifold is the space of all unitary n-frames of $\mathbb{C}^N$. The associated n-plane of the frame gives the map to the Grassmannian. This material is exposed in a clear way in Chapter 2 and 3 of the thesis.

In Chapter 4, the central part of the thesis, Šimon Vedl generalizes the shape operator and the rotating blade from the embedded manifold setting to the general setting of universal connections. The idea is to consider the direct sum of two Stiefel bundles, the one for unitary n-frames and the orthogonal one for unitary $(N-n)$-frames. This mimics the decomposition into tangent and normal bundle of an embedded manifold. The associated vector bundle of the direct sum is a trivial $\mathbb{C}^N$-bundle over the Grassmannian.

This decomposition singles out a canonical gauge, called shape gauge. This gauge, as well as the shape operator and the rotating blade, can be transported to any G-bundle via the map to the universal bundle. The advantage is that the differential equations simplify in the shape gauge. For the Yang–Mills functional, seen as a function on the rotating blade, Šimon Vedl derives the Euler-Lagrange equations, which only involves first derivatives.

In the final Chapter 5, Šimon Vedl gives numerous examples, from electromagnetism, Yang–Mills theories to general relativity. He computes the shape operators and rotating blades. In these examples he improves the dimensional bound given by the Narasimhan–Ramanan theorem.

The reading of the manuscript triggered some questions to me:

- Is the universal connection in the Stiefel bundle flat? If yes, how can curvature arise by pulling back a flat connection?
- Is the shape gauge unique?
- Are there ideas on how to quantize the rotating blade?

The diploma thesis of Šimon Vedl represents a substantial step towards a better understanding of universal connections for physical theories. It opens research directions towards new geometric approaches to quantum mechanics.

Based on the quality and depth of the diploma thesis of Šimon Vedl, **I assess this work as excellent (A)**.

Sincerely,

Dr. Alexander Thomas