Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Application of the theory of orthogonal polynomials to solution of Heun's differential equation

Aplikace teorie ortogonálních polynomů v řešení Heunovy diferenciální rovnice

Master's thesis

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ZADÁNÍ DIPLOMOVÉ PRÁCE

Student:	Bc. Patrik Šnauko
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Pokyny pro vypracování:

1) Proveďte shrnutí základů teorie ortogonálních polynomů.
2) Proveďte shrnutí základních výsledků týkajících se Heunovy diferenciální rovnice.
3) Seznamte se se základy regulární poruchové teorie.
4) Aplikujte teorii ortogonálních polynomů pro zavedení a studium speciálních řešení Heunovy diferenciální rovnice.
5) Aplikujte poruchovou teorii na výpočet nejnižší vlastní hodnoty Jacobiho matice, která odpovídá ortogonálním polynomům z bodu č. 4 zadání.

Doporučená literatura:

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Boyd, Edinburgh, 1965.

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Author's declaration:
I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, April 29, 2022 Patrik Šnauko

Název práce:

Aplikace teorie ortogonálních polynomů v řešení Heunovy diferenciální rovnice

Autor: Patrik Šnauko

Program: Aplikovaná analýza a algebra

Druh práce: Diplomová práce

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Abstrakt: Tato práce z oblasti teorie ortogonálních polynomů se zaměřuje na využití zmíněné teorie při řešení Heunovy diferenciální rovnice. Zafixováním jednoho z parametrů v Heunově rovnici v závislosti na jiném jsme schopni podle článků [11] a [10] naleznout řešení ve tvaru mocninné řady a určit koeficienty této řady. Zmíněný postup lze aplikovat za předpokladu, že parametry této rovnice jsou kladné. V práci bude výsledek zobecněn na komplexní rovinu s výjimkou záporné poloosy. Právě v nalezeném řešení figurují ortogonální polynomy příslušející jisté Jacobiho matici. Dále bude zkoumána první vlastní hodnota (základní stav) této matice pomocí dvou přístupů k poruchové teorii, klasického dle knížek [6] a [8] a pomocí přístupu odvozeného v článku [5]. Bude ukázáno, že výsledky obou teorií se shodují.

Klíčová slova: Heunova diferenciální rovnice, implicitní funkce, Jacobiho matice, ortogonální polynomy, poruchové teorie

Title:

Application of the theory of orthogonal polynomials to solution of Heun's differential equation

Author: Patrik Šnauko

Abstract: This thesis from the field of the theory of orthogonal polynomials focuses on the application of this theory to solution of Heun's differential equation. Fixing one of the parametres of the equation in dependence on another, we will be able to find a solution in the form of power series and we will be able to determine coefficients of these series according to [11] and [10]. This approach holds under the assumption that all parametres of Heun's equation are positive. The result will be extended to the complex plane except for negative real numbers. In the found solution, orthogonal polynomials corresponding to a certain Jacobi matrix occur. Next part of the thesis focuses on finding the approximation for the ground state by two methods. Firstly, we will find perturbation series according to [6] and [8]. Next, we compare obtained results with those obtained by second method due to [5]. We will show that the results coincide.

Key words: Heun's differential equation, implicit function, Jacobi matrix, orthogonal polynomials, perturbation theory

Contents

Introduction

This master's thesis focuses on the theory of orthogonal polynomials and in particular on the application of this theory to the solution of Heun's differential equation.

Heun's equation is an ordinary linear differential equation of the second order in the complex plane with, in general, four singular points. Some sources, for instance [12], reffered to this equation as natural generalization of the Gauss differential equation. Similarly as the Gauss equation, Heun's equation occurs after separation of variables in a certain PDE. Namely, according to [12], this happens when adding three spins in quantum mechanics.

In the first chapter, we will summarize essential results from the theory of orthogonal polynomials. Two approaches will be described. The first one, via the so-called moment functional. This approach allows us to state that orthogonal polynomials corresponding to a given moment functional obey certain three-terms recurrence. Conversely, Favard's theorem claims that if some system of polynomials obeys a certain three-terms recurrence, then there exists a moment functional for which they are orthogonal. This will be the second approach – via semi-infinite Jacobi tridiagonal matrices. Also, we will define Hamburger moment problem. We distinguish two types of this problem – determinate and indeterminate. Some criterions for determinacy of the Hamburger moment problem will be given too. The case of the determinate Hamburger moment problem will be epsecially important for us. Results from the first chapter are mainly taken from [4] and [1].

The second chapter is focused on the theory of the Fuchsian differential equations. Heun's equation is introduced as a special case of this type of differential equations. Main refference for this part is [9]. Next, we introduce some notations and results from paper [11] which are connecting Heun's local function with some orthogonal polynomials.

The third chapter brings some original results for a given Jacobi matrix. Heun's local function will be found for Heun's equation with just one fixed parameter. Firstly, some restriction will be required on parametres to apply the theory derived in paper [10]. These results are also summarized in this chapter. Next, we extend the result to a significantly larger range of parametres.

The final, fourth, chapter focuses on the perturbation theory from two points of view. The first one is the classical perturbation theory due to Kato. Main referrences for this section are [6] and [8]. The second approach, via the implicit function theorem, is taken from [5]. Both methods will be applied to the Jacobi matrix from chapter three, in order to find it's ground state. The obtained results from the two methods are then compared.

Chapter 1

Theory of orthogonal polynomials

The following chapter is based mainly on the books [1] and [4]. First of all, we are going to introduce orthogonal polynomials as a certain sequence that obeys some relations given by the so-called moment functional. The vector space of all complex polynomials in a real variable x is denoted by $\mathbb{C}[x]$ in the following chapter. However, if convenient, the domain of these can be extended to the complex plane. Similarly, $\mathbb{R}[x]$ will denote vector space of all real polynomials of a real variable.

1.1 The moment functional and the orthogonal polynomials

Definition 1.1.0.0.1. Let $\{\mu_n\}_{n=0}^{\infty}$ be an arbitrary sequence of complex numbers. Let the linear functional $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$ be given by the condition

$$\mathcal{L}[x^n] = \mu_n$$

for all $n \geq 0$. In this case, linear functional $\mathcal{L}$ is called the *moment functional* corresponding to the sequence of *moments* $\{\mu_n\}_{n=0}^{\infty}$. For all $n \geq 0$ number μ_n is called the *moment of the n-th order*.

It is clear that for a polynomial $\pi(x) = \sum_{k=0}^{n} c_k x^k$ we have $\mathcal{L}[\pi(x)] = \sum_{k=0}^{n} c_k \mu_k$ due to linearity of the moment funcitonal. Now we are about to introduce orthogonal polynomials correspoding to a given moment functional.

Definition 1.1.0.0.2. Let $\mathcal{L}$ be a moment functional. A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying the following conditions for all nonnegative integers m and n

1. $P_n(x)$ is a polynomial of degree n,
2. $\mathcal{L}[P_m(x)P_n(x)] = 0$ for $m \neq n$,
3. $\mathcal{L}[P_n^2(x)] \neq 0$

is called *orthogonal polynomial sequence corresponding to the moment functional* $\mathcal{L}$.

In this text, „orthogonal polynomial sequence“ will be abbreviated „OPS“ due to [4]. Particularly important for us will be the case where the condition 3 in definition 1.1.0.0.2 is specified as $\mathcal{L}[P_n^2(x)] = 1$. In this case, we are talking about *orthonormal polynomial sequence.*

From conditions 1 and 3 in definition 1.1.0.0.2 it is obvious that $P_0(x) = a \in \mathbb{C} \setminus \{0\}$ and $\mathcal{L}[1] = \mu_0 \neq 0$.

Next proposition will give us some statements equivalent with definition 1.1.0.0.2.

Proposition 1.1.0.0.3. Let $\mathcal{L}$ be a moment functional and let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials. Then the following are equivalent:

a. $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS with respect to $\mathcal{L}$,

b. $\mathcal{L}[\pi(x)P_n(x)] = 0$ for every polynomial $\pi(x)$ of degree $m \leq n$ while $\mathcal{L}[\pi(x)P_n(x)] \neq 0$ for every polynomial $\pi(x)$ of degree $m = n$,

c. $\mathcal{L}[x^m P_n(x)] = K_n \delta_{m,n}$ where $K_n \neq 0$, $m = 0, 1, \ldots n$.

Remark 1.1.0.0.4. Let us consider OPS $\{P_n(x)\}_{n=0}^{\infty}$ corresponding to a moment functional $\mathcal{L}$. We already know that any polynomial of degree n can be represented as a linear combination

$$\pi(x) = \sum_{k=0}^{n} c_k P_k(x), \qquad c_n \neq 0.$$

We will take $P_m(x)$ for $m = 0, 1, \ldots, n$ and compute

$$\mathcal{L}[\pi(x)P_m(x)] = \sum_{k=0}^{n} c_k \mathcal{L}[P_k(x)P_m(x)] = c_m \mathcal{L}[P_m^2(x)].$$

Thus we have identity for coefficients c_k in the form

$$c_k = \frac{\mathcal{L}[\pi(x)P_k(x)]}{\mathcal{L}[P_k^2(x)]},$$

which strongly reminds identity for Fourier.

As a consequence of preceeding remark, we can state that OPS is determined uniquely up to an arbitrary nonzero factor. Indeed, if $\{P_n(x)\}_{n=0}^{\infty}$ is OPS corresponding to a moment functional $\mathcal{L}$, sequence $\{c_n P_n(x)\}_{n=0}^{\infty}$ is also OPS corresponding to the moment functional $\mathcal{L}$. Conversely, we can state

Consequence 1.1.0.0.5. Let $\{P_n(x)\}_{n=0}^{\infty}$ be OPS corresponding to a moment functional $\mathcal{L}$ and let $\{R_n(x)\}_{n=0}^{\infty}$ be another OPS corresponding to the moment functional $\mathcal{L}$. Then there exist constants $c_n \neq 0$ such that

$$R_n(x) = c_n P_n(x).$$

1.2 Existence of the OPS

Since we are aware of the fact that OPS corresponding to the given moment functional $\mathcal{L}$ is given unambiguously up to a factor, next to discuss will be an existence of the OPS to a given moment functional $\mathcal{L}$. For these purposes, we will use the following notation.

Notation 1.2.0.0.1. For a moment functional $\mathcal{L}$ with a sequence of the moments $\{\mu_n\}_{n=0}^{\infty}$ we will denote

$$\Delta_n := \det(\mu_{i+j})_{i,j=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}.$$

Theorem 1.2.0.0.2. Let $\mathcal{L}$ be a moment functional with moment sequence $\{\mu_n\}$. A necessary and sufficient condition for the existence of an OPS correspoding to the moment functional $\mathcal{L}$ is

$$\Delta_n \neq 0, \qquad n = 0, 1, \ldots.$$

Next definition will be very important in our approach to the so-called moment problem.

Definition 1.2.0.0.3. A moment functional $\mathcal{L}$ is called *positive-definite* (*PD*) if and only if $\mathcal{L}[\pi(x)] > 0$ for every polynomial $\pi(x)$ which is non-negative for every $x \in \mathbb{R}$ and is not identically zero.

In case of PD moment functional $\mathcal{L}$ one convenient property occurs.

Theorem 1.2.0.0.4. Let $\mathcal{L}$ be positive-definite. Then $\mathcal{L}$ has real moments and a corresponding OPS consisting of real polynomials exists.

Finaly, we state a criteiron for positive-definiteness for a moment functional $\mathcal{L}$.

Theorem 1.2.0.0.5. A moment functional $\mathcal{L}$ is PD if and only if its moments are all real and $\Delta_n > 0$ for each $n \geq 0$.

1.3 Three-terms recurrence, Favard's theorem and Jacobi matrices

In this section, we will show a one-to-one correspondence between moment functional and semi-infinite matrix. We will be able to state this result for a certain type of moment functionals.

Definition 1.3.0.0.1. A moment functional $\mathcal{L}$ is called *quasi-definite (QD)* in case that for every $n \geq 0$

$$\Delta_n \neq 0$$

holds true.

From this section on, we are going to assume that sequence $\{P_n(x)\}_{n=0}^{\infty}$ corresponding to at least quasi-definite moment functional $\mathcal{L}$ is orthonormal. In this case, we will use a symbol $\left\{\widehat{P}_n(x)\right\}_{n=0}^{\infty}$. Obviously, every polynomial can be expressed as a linear combination of orthonormal polynomials, thus

$$x\widehat{P}_n(x) = a_{n,n+1}\widehat{P}_{n+1}(x) + a_{n,n}\widehat{P}_n(x) + a_{n,n-1}\widehat{P}_{n-1}(x) + \ldots \tag{1.1}$$

Multiplying both sides of equation 1.1 by polynomials $\widehat{P}_i(x)$ for $i = 0, 1, \ldots n$ and applying the moment functional $\mathcal{L}$ we will get, according to the proposition 1.1.0.0.3,

$$\begin{aligned} a_{n,i} &= 0, \ \text{ pro } i = 0, 1, \ldots, n-2, \\ a_{n,n-1} &= \mathcal{L}\left[x\widehat{P}_n(x)\widehat{P}_{n-1}(x)\right], \\ a_{n,n} &= \mathcal{L}\left[x\widehat{P}_n(x)\widehat{P}_n(x)\right]. \end{aligned}$$

We also have that $a_{n,n+1} = \frac{\Delta_{n+1}}{\Delta_n} \neq 0, n \geq 0$. Now let as assume the following representation

$$x\widehat{P}_{n-1}(x) = a_{n-1,n}\widehat{P}_n(x) + R_{n-1}(x),$$

here $R_{n-1}(x)$ is a polynomial of degree not greater than $n-1$. Let us multiply both sides by $\widehat{P}_n(x)$ and apply the moment functional $\mathcal{L}$. Hence we have

$$a_{n,n-1} = a_{n-1,n}.$$

Due to this symmetry we see that expression (1.1) is reduced to

$$x\widehat{P}_n(x) = \alpha_{n-1}\widehat{P}_{n-1}(x) + \beta_n\widehat{P}_n(x) + \alpha_n\widehat{P}_{n+1}(x), \tag{1.2}$$

where $\alpha_n := a_{n,n+1}, \beta_n := a_{n,n}$. By the preceding it is obvious that terms of the sequence of orthonormal polynomials $\left\{\widehat{P}_n(x)\right\}$ corresponding to a certain at least quasi-definite moment functional $\mathcal{L}$ obeys three terms recurrence formula (1.2) with initial data

$$\alpha_0\widehat{P}_1(x) + (\beta_0 - x)\widehat{P}_0(x) = 0. \tag{1.3}$$

It means that if $\widehat{P}_0(x)$ is known, one is able to compute another $\widehat{P}_n(x)$ by the recursion. As $\widehat{P}_0(x)$ has to be polynomial of degree 0, natural choice is $\widehat{P}_0(x) = 1$.

Remark 1.3.0.0.2. Moreover, if the moment functional $\mathcal{L}$ is PD, $\beta_n \in \mathbb{R}$ and $\alpha_n > 0$ for every $n \geq 0$.

Now we are about to state a very useful conversion of the preceding thoughts, which is known as Favard Theorem.

Theorem 1.3.0.0.3. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be arbitrary sequences of complex numbers. Let $\left\{\widehat{P}_n(x)\right\}$ be a sequence of polynomials defined by three terms recurrence (1.2) with an initial data (1.3). Then there exists unambigiously given QD moment functional $\mathcal{L}$ such that $\left\{\widehat{P}_n(x)\right\}$ is its corresponding sequence of orthonormal polynomials. Moreover, if $\beta_n \in \mathbb{R}$ and $\alpha_n > 0$ for each $n \geq 0$, the moment functional $\mathcal{L}$ is PD.

Now let us conclude the preceding thoughts. First of all, let us denote

$$\mathcal{J} = \begin{pmatrix} \beta_0 & \alpha_0 & 0 & 0 & 0 & \dots \\ \alpha_0 & \beta_1 & \alpha_1 & 0 & 0 & \dots \\ 0 & \alpha_1 & \beta_2 & \alpha_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.4}$$

Definition 1.3.0.0.4. A matrix $\mathcal{J}$ given by (1.4) is called *Jacobi (tridiagonal) matrix.*

A symbol $\mathbb{C}^\infty$ will stand for a vector space of column vectors of complex numbers. Jacobi matrix $\mathcal{J}$ can be considered as an operator on this vector space. Moreover, if we will denote

$$\mathbf{P}(x) := \left(\widehat{P}_0(x), \widehat{P}_1(x), \dots, \widehat{P}_n(x), \dots\right)^T,$$

then obviously for every $x \in \mathbb{R}$ is $\mathbf{P}(x) \in \mathbb{C}^\infty$ and one can consider equation for formal eigenvalues

$$\mathcal{J}\mathbf{P}(x) = x\mathbf{P}(x). \tag{1.5}$$

With $\widehat{P}_0(x) = 1$ according to Favard theorem, equation (1.5) gives us unique QD moment functional $\mathcal{L}$ for which $\left\{\widehat{P}_n(x)\right\}$ is the sequence of orthonormal polynomials. Conversely, if we have a QD moment functional $\mathcal{L}$ we can construct Jacobi matrix as it was shown in the beggining of the section.

Remark 1.3.0.0.5. Again, we assume recurrence

$$xQ_n(x) = \alpha_{n-1}Q_{n-1}(x) + \beta_n Q_n(x) + \alpha_n Q_{n+1}(x).$$

However, this time we will replace initial condition with

$$Q_0(x) = 0,\ Q_1(x) = \frac{1}{\alpha_0}.$$

Thus we have another solution $\{Q_n(x)\}_{n=0}^{\infty}$ of the considered recurrence.

Definition 1.3.0.0.6. Polynomials introduced in the remark 1.3.0.0.5 are called the *polynomials of the second kind.*

1.4 Zeros of the orthogonal polynomials

There are also some interesting properties of orthogonal polynomials in the brunch of their roots. The following section will be focused on these. Firstly, let us generalize the definition of positive-definiteness.

Definition 1.4.0.0.1. Let E be subset of $\mathbb{R}$. A moment functional $\mathcal{L}$ is said to be *positive-definite on E* if

$$\mathcal{L}[\pi(x)] > 0$$

for every polynomial $\pi(x)$ which is non-negative and non-zero on E. Set E is called the *supporting set* of the moment functional $\mathcal{L}$.

Theorem 1.4.0.0.2. Let $I \subset \mathbb{R}$ be an interval which is supporting set for some PD moment functional $\mathcal{L}$. Zeros of orthogonal polynomials corresponding to $\mathcal{L}$ are all real, simple, and located in the interior of the interval I.

Due to the fact that there are exactly n different zeros of the polynomial $P_n(x)$ in an interval $I \subset \mathbb{R}$, we can order them increasingly

$$x_{n,1} < x_{n,2} < \cdots < x_{n,n}.$$

Without loss of generality let us assume that $P_n(x)$ is monic. Thus for every $x > x_{n,n}$ is

$$P_n(x) > 0.$$

Conversely, for $x < x_{n,1}$ one has

$$\text{sgn } P_n(x) = (-1)^n.$$

Since $P_n(x_{n,k}) = 0 = P_n(x_{n,k+1})$ for every $k = 1, \ldots, n-1$ and $P_n(x)$ is differentiable, in each interval $(x_{n,k}, x_{n,k+1})$ there is a zero of a polynomial $P_n'(x)$. Since $P_n'(x)$ the polynomial of degree $n-1$, one has all of the zeros.

The following theorem will be an important tool for us and will be often recalled as *the separation theorem for the zeros.*

Theorem 1.4.0.0.3. Zeros of polynomials $P_n(x)$ and $P_{n+1}(x)$ are mutually separated, i. e.

$$x_{n+1,i} < x_{n,i} < x_{n+1,i+1}$$

for each $i = 1, \ldots, n$.

Remark 1.4.0.0.4. By the separation theorem for the zeros, we have that $\{x_{n,k}\}_{n=k}^{\infty}$ is decreasing sequence and $\{x_{n,n-k+1}\}_{n=k}^{\infty}$ is increasing sequence. Thus there exist numbers ξ_i and η_i in $\overline{\mathbb{R}}$ such that

$$\xi_i = \lim_{n\to\infty} x_{n,i}$$
$$\eta_j = \lim_{n\to\infty} x_{n,n-j+1}$$

Definition 1.4.0.0.5. An interval $[\xi_i, \eta_i]$ is called *the true interval of orthogonality* of the OPS corresponding to a moment functional $\mathcal{L}$.

1.5 Representation theorem

An important question is under which conditions can be a moment functional $\mathcal{L}$ represented as a convergent Stieltjes integral over the real line, this means under which condition the expression

$$\mathcal{L}[x^n] = \int_{\mathbb{R}} x^n \mathrm{d}\psi(x) \in \mathbb{R} \tag{1.6}$$

holds for some bounded, non-decreasing function ψ continuous from the right.

Definition 1.5.0.0.1. A bounded, non-decreasing function ψ continuous from the right for which

$$\mu_n := \int_{\mathbb{R}} x^n \mathrm{d}\psi(x) \in \mathbb{R}; \qquad n = 0, 1, \ldots, \tag{1.7}$$

holds, is called *distribution function solving a problem* (1.6).

Definition 1.5.0.0.2. Under same assumptions as in the definition above, a set

$$\sigma(\psi) := \{x \in \mathbb{R}; (\forall\delta > 0)\,(\psi(x+\delta) - \psi(x-\delta) > 0)\}$$

is called *spectrum* of the function ψ. A point $x \in \sigma(\psi)$ is called *spectral point* of the function ψ.

Remark 1.5.0.0.3. Conversely, a point $x \in \mathbb{R}$ does not belong to the spectrum of the function ψ, if there is $\delta > 0$ such that $\psi(y) = \textit{constant}$ for every $y \in [x-\delta, x+\delta]$. Therefore, $\sigma(\psi)$ is a closed set.

Problem (1.6) is called *Hamburger moment problem.*

Definition 1.5.0.0.4. Two distribution functions ψ_1, ψ_2 are said to be *substantially equal* if and only if there is a constant C such that $\psi_1(x) = \psi_2(x) + C$ at all common points of continuity.

Definition 1.5.0.0.5. Hamburger moment problem is called *determinate* if there is unique distribution function ψ which obeys (1.6) up to substantially equal functions. Else, Hamburger moment problem is said to be *indeterminate.*

Connection between a spectrum defined in 1.5.0.0.2 and the spectrum of a certain operator will be discussed later.

1.5.1 Gauss quadrature

Remark 1.5.1.0.1. Let us take $n \in \mathbb{N}$ and a set $\{t_1, t_2, \dots, t_n\} \subset \mathbb{R}$. We set

$$F(x) := \prod_{i=1}^{n}(x - t_i).$$

Obviously, $\deg\left(\frac{F(x)}{x-t_i}\right) = n - 1$. Also,

$$F'(t_k) \neq 0 \qquad k = 1, 2, \dots, n$$

holds true. For $k = 1, 2, \dots, n$ one can define polynomials

$$l_k(x) = \frac{F(x)}{(x - t_k)F'(t_k)}.$$

Again, $\deg(l_k(x)) = n - 1$ and moreover

$$l_k(t_j) = \delta_{j,k}.$$

Then for every set of numbers $\{y_1, y_2, \dots, y_n\}$ degree of the polynomial

$$L_n(x) := \sum_{k=1}^{n} y_k l_k(x) \tag{1.8}$$

does not exceed $n - 1$. Moreover, it obeys property

$$L_n(t_j) = \sum_{k=1}^{n} y_k l_k(t_j) = \sum_{k=1}^{n} y_k \delta_{j,k} = y_j \qquad j = 1, 2, \dots, n.$$

Note that the polynomial (1.8) is the only solution of the task of searching polynomial which degree does not exceed $n - 1$ and its graph intersects points (t_i, y_i).

Definition 1.5.1.0.2. Polynomials constructed in the preceding remark and defined by the equation (1.8) are called *Lagrange interpolation polynomials* corresponding to the *nodes* t_i and *coordinates* y_i.

Let us remind that $x_{n,k}$ denotes $k-$th root of n-th orthogonal polynomial. The following theorem is known as the Gauss quadrature formula.

Theorem 1.5.1.0.3. Let $\mathcal{L}$ be a PD moment functional. Then for every $n \geq 0$, there exist numbers $A_{n,1}, \dots A_{n,n}$ such that for every polynomail $\pi(x) \in \mathbb{C}[x]$ of degree not exceeding $2n - 1$

$$\mathcal{L}[\pi(x)] = \sum_{k=1}^{n} A_{n,k}\pi(x_{n,k}). \tag{1.9}$$

holds true. Numbers $A_{n,1}, \dots A_{n,n}$ are positive and obey the condition

$$\sum_{k=1}^{n} A_{n,k} = \mu_0. \tag{1.10}$$

Due to this theorem we can prove the following proposition which is in the book [4] left as an excercise for the reader. First of all, we denote a set of roots of all orthogonal polynomials corresponding to a moment functional $\mathcal{L}$ by the symbol $\mathcal{N}(\mathcal{L})$, i.e.

$$\mathcal{N}(\mathcal{L}) = \bigcup_{n=0}^{\infty} \{x_{n,k}\}_{k=1}^{n}.$$

Proposition 1.5.1.0.4. Let $\mathcal{L}$ be a positive-definite moment functional. Then $\mathcal{N}(\mathcal{L})$ is a supporting set for $\mathcal{L}$.

Proof. Let us take polynomial $\pi(x)$ of degree n such that $\pi(x) > 0$ on $\mathcal{N}(\mathcal{L})$. Since $\mathcal{L}$ is PD and $\pi(x)$ of a degree less than $2n-1$, we have according to the theorem 1.9 numbers $A_{n,1}, \ldots, A_{n,n} > 0$ obeying the relation

$$\mathcal{L}[\pi(x)] = \sum_{k=1}^{n} A_{n,k} \pi(x_{n,k}) \geq 0.$$

The preceeding expression must be positive. Indeed, there must be any $j \in \{1, \ldots, n\}$ such that $\pi(x_{n,j}) \neq 0$. If there was not such number, the following expression would be hold

$$\pi(x) = c \prod_{k=1}^{n} (x - x_{n,k}) = cP_n(x)$$

for some $c \in \mathbb{R}$. However, it would contradict the positivity of the polynomial $\pi(x)$ on the set $\mathcal{N}(\mathcal{L})$. Indeed, according to the separation theorem for the zeros, polynomial $P_n(x)$ changes sign in the roots of the polynomial $P_{n+1}(x)$, which are eleements of the set $\mathcal{N}(\mathcal{L})$. □

Consequence 1.5.1.0.5. Under the same assumptions as in the proposition above, it is obvious that $\mathcal{N}(\mathcal{L}) \subset [\xi_1, \eta_1]$ and $\mathcal{N}(\mathcal{L})$ is a supporting set for the positive-definite moment functional $\mathcal{L}$. According to the theorem 1.4.0.0.2 the true interval of orthogonality $[\xi_1, \eta_1]$ is the smallest closed interval which is supporting for $\mathcal{L}$.

1.5.2 Representation theorem

Let $\mathcal{L}$ be a positive-definite moment functional with moments $\{\mu_n\}_{n=0}^{\infty}$. From theorem 1.9 we have that for every $n \in \mathbb{N}_0$ exist numbers $A_{n,1}, \ldots, A_{n,n}$ such that

$$\mathcal{L}[x^k] = \mu_k = \sum_{i=1}^{n} A_{n,i} x_{n,i}^k, \qquad k = 0, 1, \ldots, 2n-1.$$

We define a sequence $\{\psi_n\}_{n=0}^{\infty}$

$$\psi_n(x) := \begin{cases} 0 & \text{for} \quad x < x_{n,1} \\ \sum_{i=1}^{p} A_{n,i} & \text{for} \quad x_{n,p} \leq x < x_{n,p+1}, \text{ where } 1 \leq p < n \; . \\ \mu_0 & \text{for} \quad x \geq x_{n,n} \end{cases} \tag{1.11}$$

It is readily seen that ψ_n is

- bounded,

- continuous from the right,
- non-decreasing.
- and $\sigma(\psi_n) = \{x_{n,1}, \ldots, x_{n,n}\}$ and a size of the jump at the point $x_{n,i}$ is $A_{n,i}$.

Due to the last point of the preceding properties, one has

$$\int_{\mathbb{R}} x^k \mathrm{d}\psi_n(x) = \sum_{i=1}^{n} A_{n,i} x_{n,i}^k = \mu_k \qquad k = 0, 1, \ldots, 2n-1. \tag{1.12}$$

For our next step we will need the following theorems which are known as *the Helly selection principle* and *Helly's second theorem*

Theorem 1.5.2.0.1. Let $\{\phi_n\}_{n=0}^{\infty}$ be a uniformly bounded sequence of non-decreasing functions defined on real axis. Then there is a subsequence $\{\tilde{\phi}_n\}_{n=0}^{\infty}$ of the sequence $\{\phi_n\}_{n=0}^{\infty}$, such that $\{\tilde{\phi}_n\}_{n=0}^{\infty}$ converges pointwise in $\mathbb{R}$ to a bounded, non-decreasing function.

Theorem 1.5.2.0.2. Let $\{\phi_n\}_{n=0}^{\infty}$ be a uniformly bounded sequence of non-decreasing functions defined on compact interval $[a, b]$ and let $\phi_n \overset{[a,b]}{\longrightarrow} \phi$ pointwise, where ϕ is a bounded, non-decreasing function. Then for any real function f continuous on the interval $[a, b]$

$$\int_a^b f \mathrm{d}\phi_n \overset{n\to\infty}{\longrightarrow} \int_a^b f \mathrm{d}\phi$$

holds true.

Sequence $\{\psi_n\}_{n=0}^{\infty}$ obeys assumptions of the theorem 1.5.2.0.1, thus there is some subsequence $\{\tilde{\psi}_n\}_{n=0}^{\infty}$ which converges to a bounded, non-decreasing function ψ pointwise in real axis. Now consider true interval of orthogonality $[\xi_1, \eta_1]$. There are two possibilities. Firstly, $[\xi_1, \eta_1]$ is bounded, then due to the theorem 1.5.2.0.2 and (1.12) we have

$$\int_{\mathbb{R}} x^k \mathrm{d}\psi(x) = \int_{\xi_1}^{\eta_1} x^k \mathrm{d}\psi = \mu_k = \mathcal{L}[x^k] \qquad k = 0, 1, \ldots. \tag{1.13}$$

First equation holds, because for $x \leq \xi_1$ is $\psi(x) = 0$ and for $x \geq \eta_1$ is $\psi(x) = \mu_0$. Therefore we can write integration over smaller interval $[\xi_1, \eta_1]$. The second possibility is that interval $[\xi_1, \eta_1]$ is non-bounded. In this case, one does not get answer from Helly's second theorem and it is necessery to prove (1.13) directly. Answer for this question is given by the following theorem which proof can be found in [4].

Theorem 1.5.2.0.3. Let $\mathcal{L}$ be a positive-definite moment functional and the sequence $\{\psi_n\}_{n=0}^{\infty}$ defined by (1.11). Then there is a subsequence $\{\tilde{\psi}_n\}_{n=0}^{\infty}$ which converges on entire real axis to a distribution function ψ for which $\sigma(\psi)$ is an infinite set and ψ is a solution of the Hamburger moment problem (1.6).

Definition 1.5.2.0.4. A distribution function ϕ which is a solution of the Hamburger moment problem (1.6) is said to be a *representation* of the positive-definite moment functional $\mathcal{L}$. If $\phi = \psi$ (i. e. it is the distribution function from the theorem 1.5.2.0.3) we are talking about *natural representation* of $\mathcal{L}$.

1.5.3 About zeros of OG polynomials and the spectrum of the distribution function

Theorem 1.5.3.0.1. Let $\mathcal{L}$ be a positive-definite moment functional. Then there is a representation ϕ such that $\sigma(\phi) \subset [\xi_1, \eta_1]$. Moreover, true interval of orthogonality $[\xi_1, \eta_1]$ is subset of every closed interval which contains spectrum of any representation of $\mathcal{L}$.

Theorem 1.5.3.0.2. Let ϕ be a representation of a moment functional $\mathcal{L}$. Then

$$\sigma(\phi) \cap (x_{n,i}, x_{n,i+1}) \neq \emptyset$$

for $n = 2, 3, \dots$ and $1 \leq i \leq n-1$.

Particularly important for us will be the following remark.

Remark 1.5.3.0.3. Recall the notation

$$\xi_i = \lim_{n\to\infty} x_{n,i},$$
$$\eta_i = \lim_{n\to\infty} x_{n,n-i+1}.$$

Regarding those limits as elements of the extended real axis, one obviously has $\xi_{i-1} \leq \xi_i < \eta_j \leq \eta_{j+1}$. Thus, we can define

$$\xi = \begin{cases} -\infty & \text{if } (\forall i \in \mathbb{N})(\xi_i = -\infty) \\ \lim_{i\to\infty} \xi_i & \text{if } (\exists p \in \mathbb{N})(\xi_p > -\infty) \end{cases},$$

and

$$\eta = \begin{cases} \infty & \text{if } (\forall j \in \mathbb{N})(\eta_j = \infty) \\ \lim_{j\to\infty} \eta_j & \text{if } (\exists q \in \mathbb{N})(\eta_q < \infty) \end{cases}.$$

Adding

$$\xi_0 = -\infty, \qquad \eta_0 = \infty,$$

we have non-decreasing sequence

$$-\infty = \xi_0 \leq \xi_1 \leq \xi_2 \leq \cdots \leq \xi \leq \eta \leq \cdots \leq \eta_2 \leq \eta_1 \leq \eta_0 = \infty.$$

Theorem 1.5.3.0.4. Let ϕ be a representation of a moment functional $\mathcal{L}$.

1. If for some $k \in \mathbb{N}$ inequality $\xi_k < \xi_{k+1}$ holds, then
$$\sigma(\phi) \cap (\xi_k, \xi_{k+1}] \neq \emptyset. \tag{1.14}$$
2. If for some $k \in \mathbb{N}$ equality $\xi_k = \xi_{k+1}$ holds. then $\xi_k \in \sigma(\phi)$,
3. $\xi \in \sigma(\phi)$.

Theorem 1.5.3.0.5. Let ψ be the natural representation of a moment functional $\mathcal{L}$. If $\xi_1 > -\infty$, then for every $i \in \mathbb{N}, \xi_i$ is an element of $\sigma(\phi)$. Moreover, $\sigma(\phi)$ does not contain any smaller point than ξ except eventually ξ_i.

Similar result holds for η.

Proof of the following proposition is left as an excercise for reader in [4], we include proof. Note that according to [4], for coefficients of the Gauss quadrature the following equation holds

$$A_{n,k} = \frac{1}{\sum_{k=0}^{n} \widehat{P}_k^2(x_{n,k})}. \tag{1.15}$$

Proposition 1.5.3.0.6. Let $-\infty < \xi_1 < \xi_2 \dots$. Then the following inequalities

$$0 < \psi(\xi_i) - \psi(\xi_i -) \le \frac{1}{\sum_{k=0}^{\infty} \widehat{P}_k^2(\xi_i)}$$

hold.

Proof. The first inequality is a consequence of theorem 1.5.3.0.5. There exists a subsequence $\{\tilde{\psi}_n\}$ of the sequence $\{\psi_n\}$ such that it's pointwise limit is natural representation ψ. According to definition (1.11) and relation (1.15) we have

$$\tilde{\psi}_n(x_{n,k}) - \tilde{\psi}_n(x_{n,k-1}) = A_{n,k} = \frac{1}{\sum_{k=0}^{n} \widehat{P}_k^2(x_{n,k})}.$$

Due to the assumption of the proposition, for any $k \in \mathbb{N}$ inequality $\xi_{k-1} < \xi_k$ holds. Hence, there is $N \in \mathbb{N}$ such that for any $n > N$ inequality $x_{n,k-1} < \xi_k$ holds. Moreover, sequence $\{x_{n,k}\}$ is decreasing according to the separation theorem for zeros. Thus, for every $n \in \mathbb{N}$ inequality $\xi_k < x_{n,k}$ holds. Since that, we can find numbers a, b such as

$$x_{n,k-1} < a < \xi_k < x_{n,k} < b < \xi_{k+1} < x_{n,k+1} \tag{1.16}$$

from a certain $N \in \mathbb{N}$ on. By the choice of these numbers and by the definition (1.11) it follows that

$$\tilde{\psi}_n(b) - \tilde{\psi}_n(a) = \tilde{\psi}_n(x_{n,k}) - \tilde{\psi}_n(x_{n,k-1}) = \frac{1}{\sum_{k=0}^{n} \widehat{P}_k^2(x_{n,k})} \le \frac{1}{\sum_{k=0}^{N} \widehat{P}_k^2(x_{n,k})}.$$

Combining of (1.16), monotony of the function ψ and theorem 1.5.3.0.5 one has

$$\psi(\xi_k) - \psi(\xi_k -) = \psi(b) - \psi(a) \le \frac{1}{\sum_{k=0}^{N} \widehat{P}_k^2(x_{n,k})}.$$

Taking limit $N \to \infty$ in the last inequality, we have proved the coveted inequality from the statement. □

Theorem 1.5.3.0.7. Let $\mathcal{L}$ be a positive-definite moment functional and let $[\xi_1, \eta_1]$ be bounded subset of $\mathbb{R}$. Then $\mathcal{L}$ is determinate.

1.6 The operator of multiplication by the coordinate

Let us assume operator Q on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}\psi)$ given by

$$\begin{aligned} \mathrm{Dom}(Q) &:= \{f \in \mathcal{H}; x \cdot f(x) \in \mathcal{H}\}, \\ (Qf)(x) &:= x \cdot f(x). \end{aligned}$$

The spectrum of this operator will be denoted by $\mathrm{spec}(Q)$ and it's resolvent set by $\varrho(Q)$. Our goal is to show that $\sigma(\psi) = \mathrm{spec}(Q)$. For later purposes, let us recall well-known Weyl's criterion for self-adjoint operators.

Theorem 1.6.0.0.1. Let $\mathcal{H}$ be a Hilbert space and T an self-adjoint operator on $\mathcal{H}$. Then

1. $\lambda \in \varrho(T)$ only if there exists $m > 0$ such that $(\forall \varphi \in \mathrm{Dom}(T))\,(\|(T-\lambda)\varphi\| \geq m\|\varphi\|)$
2. $\lambda \in \mathrm{spec}(T)$ only if there exists sequence $(\varphi_n)_{n=1}^{\infty} \subset \mathrm{Dom}(T)$ such that $(\forall n \in \mathbb{N})(\|\varphi_n\| = 1)$ and $\|(T-\lambda)\varphi_n\| \to 0$.

As can be seen from the theorem 1.6.0.0.1, we need to show that Q is self-adjoint, if we wish to apply Weyl's criterion.

Proposition 1.6.0.0.2. Q is self-adjoint operator.

Proof. We need to find domain of Q^* and how does it work on it's domain. We know that

$$\mathrm{Dom}(Q^*) = \{g \in \mathcal{H}; (\exists h \in \mathcal{H})(\forall f \in \mathrm{Dom}(Q))(\langle g, Qf\rangle = \langle h, f\rangle)\}.$$

One has

$$\langle g, Qf\rangle = \int_{\mathbb{R}} \overline{g(x)} x f(x) \mathrm{d}\psi(x) = \int_{\mathbb{R}} \overline{x g(x)} f(x) \mathrm{d}\psi(x).$$

If h exists, almost for every $x \in \mathbb{R}$ equality $h(x) = x \cdot g(x)$ must hold. Thus we have

$$\mathrm{Dom}(Q^*) = \mathrm{Dom}(Q),$$

and

$$(Q^* g)(x) := x \cdot g(x).$$

Thus $Q^* = Q$. □

Remark 1.6.0.0.3. Note that due to the fact, that Q is self-adjoint, $\mathrm{spec}(Q) \subset \mathbb{R}$. At the same time, $\sigma(\psi) \subset \mathbb{R}$ from the definition.

Theorem 1.6.0.0.4. Under the same assumptions about ψ and Q, equality

$$\sigma(\psi) = \mathrm{spec}(Q)$$

holds.

Proof. We need to show two inclusions. Firstly, consider that $\lambda \notin \sigma(\psi)$. By the definition of $\sigma(\psi)$, there exists $\varepsilon > 0$ such that function ψ is constant on $(\lambda - \varepsilon, \lambda + \varepsilon)$. Thus, Borel measure generated by the function ψ of this interval is zero. Let us denote

$$(B_\lambda f)(x) = b_\lambda(x) f(x) = \frac{1}{x - \lambda} f(x).$$

Function b_λ is bounded almost everywhere on $\mathbb{R}$ (exceptional set is $(\lambda - \varepsilon, \lambda + \varepsilon)$ which is of the measure zero). Thus $b_\lambda \in L^\infty(\mathbb{R}, \mathrm{d}\psi)$ and B_λ is bounded operator on $\mathcal{H}$. Bounding constant can be taken, for instance, as $\|b_\lambda\|_{L^\infty}$. We will show that B_λ is an inverse operator for $Q - \lambda$. We compute

$$\|(Q-\lambda)B_\lambda f - f\|^2 = \int_{\mathbb{R}} |(Q - \lambda 1)B_\lambda f(x) - f(x)|^2 \,\mathrm{d}\psi(x) = \int_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon)} |f(x) - f(x)|^2 \mathrm{d}\psi(x).$$

The second equality holds, since deleting set of the zero measure does not affect result of the integral. We have $B_\lambda = (Q-\lambda)^{-1} \in \mathcal{B}(\mathcal{H})$, thus $\lambda \in \varrho(Q)$. Therefore we have inclusion

$$\operatorname{spec}(Q) \subset \sigma(\psi).$$

Conversely, assume that $\lambda \in \sigma(\psi)$. Again by the definition of $\sigma(\psi)$ we have for every $\delta > 0$ that

$$\mu_B((\lambda-\delta,\lambda+\delta)) = \psi(\lambda+\delta) - \psi(\lambda-\delta) > 0.$$

Here μ_B stands for Borel measure generated by the function ψ. Thus characteristic function of the interval $(\lambda-\delta,\lambda+\delta]$ is measurable and

$$\int_{\mathbb{R}} \chi_{(\lambda-\delta,\lambda+\delta]}(x)\mathrm{d}\psi(x) = \mu_B\left((\lambda-\delta,\lambda+\delta]\right).$$

Let us denote $\delta_n = \frac{1}{n}$ and $I_n = \left(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}\right)$. Obviously,

$$\|\chi_{I_n}\|^2 = \int_{\mathbb{R}} \chi_{I_n}(x)\mathrm{d}\psi(x) = \mu_B(I_n) > 0.$$

With further notation

$$f_n := \frac{1}{\|\chi_{I_n}\|}\,\chi_{I_n},$$

one has $\|f_n\| = 1$. Let us compute

$$\begin{aligned}\|(Q-\lambda)f_n\|^2 &= \int_{\mathbb{R}} |x-\lambda|^2 \frac{1}{\|\chi_{I_n}\|^2}\chi_{I_n}\mathrm{d}\psi(x) = \frac{1}{\|\chi_{I_n}\|^2}\int_{\lambda-\frac{1}{n}}^{\lambda+\frac{1}{n}} |x-\lambda|^2\mathrm{d}\psi(x)\\ &\leq \frac{1}{\|\chi_{I_n}\|^2}\frac{1}{n^2}\int_{\lambda-\frac{1}{n}}^{\lambda+\frac{1}{n}} \mathrm{d}\psi(x) = \frac{\mu_B\left(\lambda-\frac{1}{n},\lambda+\frac{1}{n}\right)}{\|\chi_{I_n}\|^2}\frac{1}{n^2} \overset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

thus by Weyl's criterion we have $\lambda \in \operatorname{spec}(Q)$. Finally,

$$\sigma(\psi) \subset \operatorname{spec}(Q).$$

□

Remark 1.6.0.0.5. Finally, we will discuss a relationship between orthonormal polynomials and operator Q. A sequence of orthonormal polynomials $\left\{\widehat{P}_n\right\}_{n=0}^{\infty}$ can be given by the three-terms recurrence

$$x\widehat{P}_n(x) = \alpha_{n-1}\widehat{P}_{n-1}(x) + \beta_n\widehat{P}_n(x) + \alpha_n\widehat{P}_{n+1}(x),$$

with initial data

$$\begin{aligned}\widehat{P}_0(x) &= 1,\\ \widehat{P}_{-1}(x) &= 0.\end{aligned}$$

Let us compute matrix elements for the operator Q. Those are given by

$$\begin{aligned}\mathcal{Q}_{m,n} := \left\langle \widehat{P}_m, Q\widehat{P}_n\right\rangle &= \int_{\mathbb{R}} x\widehat{P}_m(x)\widehat{P}_n(x)\mathrm{d}\psi(x)\\ &= \int_{\mathbb{R}} \left(\alpha_{m-1}\widehat{P}_{m-1}(x) + \beta_m\widehat{P}_m(x) + \alpha_m\widehat{P}_{m+1}(x)\right)\widehat{P}_n(x)\mathrm{d}\psi(x).\end{aligned}$$

In our case

$$\mathcal{Q}_{m,n} = \alpha_{n-1}\delta_{m,n-1} + \beta_n\delta_{m,n} + \alpha_n\delta_{m,n+1}.$$

This result can be represented by semi-infinite matrix

$$\mathcal{Q} = \begin{pmatrix} \beta_0 & \alpha_0 & 0 & 0 & 0 & \dots \\ \alpha_0 & \beta_1 & \alpha_1 & 0 & 0 & \dots \\ 0 & \alpha_1 & \beta_2 & \alpha_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \mathcal{J}.$$

This is exactly already discussed Jacobi matrix.

1.7 Density of the set of polynomials in $L^2(\mathbb{R}, \mathrm{d}\psi)$ and unitary operator between $L^2(\mathbb{R}, \mathrm{d}\psi)$ and ℓ^2

In the following section, we will discuss isometric operator between $L^2(\mathbb{R}, \mathrm{d}\psi)$ and ℓ^2 generated by the solution ψ of the moment problem (1.7). Results are taken from book [1]. Assume system of orthonormal polynomials $\left\{\widehat{P}_n\right\}_{n=0}^{\infty}$ and a system of the polynomials of the second kind $\{Q_n\}_{n=0}^{\infty}$ for a given Jacobi matrix $\mathcal{J}$. For any $n \in \mathbb{N}_0$ we define a function

$$w_n(z,t) := -\frac{Q_n(z) - tQ_{n-1}(z)}{\widehat{P}_n(z) - t\widehat{P}_{n-1}(z)}. \tag{1.17}$$

It is the function of complex variable z and real parameter t. Obviously,

$$w_n(z,\infty) = w_{n-1}(z,0).$$

The following theorem will be important in the sequel text.

Theorem 1.7.0.0.1. Let $z \in \mathbb{C}$ be fixed such that $\mathrm{Im}(z) > 0$. Let the parameter t vary along the whole extended real axis. Then the function

$$w = w_n(z,t)$$

describes a circular contour $K_n(z)$ in the half-plane $\mathrm{Im}(z) > 0$. The center of this circle is at the point

$$-\frac{Q_n(z)\overline{\widehat{P}_{n-1}(z)} - Q_{n-1}(z)\overline{\widehat{P}_n(z)}}{\widehat{P}_n(z)\overline{\widehat{P}_{n-1}(z)} - \widehat{P}_{n-1}(z)\overline{\widehat{P}_n(z)}},$$

for its radius one has

$$\frac{1}{|z-\overline{z}|}\frac{1}{\sum_{k=0}^{n-1}|P_k(z)|^2}.$$

The equation of the circle $K_n(z)$ may be written in the form

$$\frac{w-\overline{w}}{z-\overline{z}} - \sum_{k=0}^{n-1}\left|w\widehat{P}_k(z) + Q_k(z)\right|^2 = 0.$$

Remark 1.7.0.0.2. Points w lying outside the circle $K_n(z)$ are described by inequality

$$\frac{w-\overline{w}}{z-\overline{z}} - \sum_{k=0}^{n-1} \left| w\widehat{P}_k(z) + Q_k(z) \right|^2 < 0$$

Conversely, points w lying inside the circle $K_n(z)$ are given by

$$\frac{w-\overline{w}}{z-\overline{z}} - \sum_{k=0}^{n-1} \left| w\widehat{P}_k(z) + Q_k(z) \right|^2 > 0.$$

Proposition 1.7.0.0.3. Under the same assumptions as in the theorem above, for every $z \in \mathbb{C}\backslash\mathbb{R}$ an inclusion

$$K_n(z) \subset K_{n-1}(z), \quad n \in \mathbb{N}_0$$

holds.

We conclude, that there is a decreasing sequence $\{K_n(z)\}$ for fixed point $z \in \mathbb{C}\setminus\mathbb{R}$. It means that there exists some limiting set $K_\infty(z)$. This can be either a circle or a point. Let us take $w \in K_\infty(z)$ arbitrary. Obviously for every $n \in \mathbb{N}_0$ we have

$$\sum_{k=0}^{n-1} \left| w\widehat{P}_k(z) + Q_k(z) \right|^2 < \frac{w-\overline{w}}{z-\overline{z}}.$$

Taking limit $n \to \infty$ on both sides of the previous expression, one gets

$$\sum_{k=0}^{\infty} \left| w\widehat{P}_k(z) + Q_k(z) \right|^2 < \infty.$$

Hence, we have the solution $\left\{ w\widehat{P}_k(z) + Q_k(z) \right\}_{k=0}^{\infty}$ of the recurrence

$$\alpha_{n-1}y_{n-1} + (\beta_n - z)y_n + \alpha_n y_{n+1} = 0,$$

belonging to ℓ^2 independent on the type of the limiting set.

Theorem 1.7.0.0.4. The solution of the recurrence

$$\alpha_{n-1}y_{n-1} + (\beta_n - z)y_n + \alpha_n y_{n+1} = 0,$$

belongs to ℓ^2 only if $K_\infty(z)$ is a circle.

Theorem 1.7.0.0.5. If $K_\infty(z)$ is a circle for some $z \in \mathbb{C}\setminus\mathbb{R}$, then $K_\infty(z)$ is a circle for any $z \in \mathbb{C}\setminus\mathbb{R}$.

Theorem 1.7.0.0.5 allows us to define the following.

Definition 1.7.0.0.6. Jacobi matrix $\mathcal{J}$ is called of a *type C* if $K_\infty(z)$ is a circle for some (and thus for any) $z \in \mathbb{C}\setminus\mathbb{R}$. Conversely, Jacobi matrix $\mathcal{J}$ is called of a *type D* if $K_\infty(z)$ is a point for some (and thus for any) $z \in \mathbb{C}\setminus\mathbb{R}$.

Remark 1.7.0.0.7. For Jacobi matrix $\mathcal{J}$ of a type D there exists unambigiously given function $w = w(z)$ such that

$$\sum_{k=0}^{\infty} \left| w\widehat{P}_k(z) + Q_k(z) \right|^2 < \infty, \quad (\forall z \in \mathbb{C} \setminus \mathbb{R}),$$

while $K_\infty(z) = \{w(z)\}$. This function is called *Weyl's function* and will be defined more specificaly later.

Theorem 1.7.0.0.8. Let $\{\mu_n\}_{n=0}^{\infty}$ be a positive sequence of moments. If the corresponding Jacobi matrix is of type C, the Hamburger moment problem (1.7) is indeterminate.

Let us consider a positive sequence of the moments $\{\mu_n\}_{n=0}^{\infty}$ and a function ψ solving Hamburger moment problem

$$\mu_k = \int_{\mathbb{R}} u^k \mathrm{d}\psi(u), \quad k \in \mathbb{N}_0.$$

Symbol L^2_ψ will denote space of all quadratic-integrable functions in a space with Borel measure given by function ψ, that says

$$f \in L^2_\psi \Leftrightarrow \int_{\mathbb{R}} |f(u)|^2 \mathrm{d}\psi(u) < \infty.$$

L^2_ψ is a Hilbert space. Scalar product is given by

$$\left(\forall f, g \in L^2_\psi\right) \left(\langle f, g \rangle_\psi := \int_{\mathbb{R}} f(u)\overline{g(u)} \mathrm{d}\psi(u) \right).$$

The system of orthonormal polynomials $\left\{\widehat{P}_n\right\}_{n=0}^{\infty}$ is obviously orthonormal in L^2_ψ. An element $x \equiv \{x_n\}_{n=0}^{\infty} \in \ell^2$ is said to be *finite* if $x_n \neq 0$ just for a finite number of indices $n \in \mathbb{N}_0$.

Let us construct a certain operator $U : \ell^2 \to L^2_\psi$. The constructions will be devided in several steps. Firstly, let us take a finite element $x \in \ell^2$. We define

$$f(u) \equiv (Ux)(u) := x_0\widehat{P}_0(u) + x_1\widehat{P}_1(u) + \cdots + x_n\widehat{P}_n(u) \ldots$$

Obviously, $f(u) \in L^2_\psi$. Moreover

$$\|f\|^2_\psi = \|Ux\|^2_\psi = \int_{\mathbb{R}} |f(u)|^2 \mathrm{d}\psi(u) = \sum_{i,j=0}^{\infty} x_i\overline{x_j} \int_{\mathbb{R}} P_i(u)P_j(u)\mathrm{d}\psi(u) = \sum_{i=0}^{\infty} |x_i|^2 = \|x\|^2,$$

here symbol $\|\cdot\|$ stands for the norm in ℓ^2. Similarly,

$$\langle Ux, Uy \rangle_\psi = \langle x, y \rangle.$$

In the second step, let us take $x \in \ell^2$ arbitrary. We define

$$f_n(u) := \sum_{k=0}^{n} x_k\widehat{P}_k(u).$$

Clearly,

$$\int_{\mathbb{R}} |f_n(u) - f_m(u)|^2 \mathrm{d}\psi(u) = \sum_{k=m+1}^{n} |x_k|^2.$$

Taking limits $m, n \to \infty$ on the both sides in the expression above, one gets

$$\lim_{n,m\to\infty} \int_{\mathbb{R}} |f_n(u) - f_m(u)|^2 \mathrm{d}\psi(u) = 0.$$

Hence $\{f_n\}_{n=0}^{\infty}$ is a Cauchy sequence in L^2_ψ, thus there exists a limit f in L^2_ψ. We define

$$(Ux)(u) := f(u).$$

We will show that operator U is an isometry. Indeed, let us take $x, y \in \ell^2$ arbitrary. Let us denote $f := Ux, g := Uy$. Then

$$\langle x, y\rangle = \sum_{k=0}^{\infty} x_k \overline{y_k} = \lim_{n\to\infty} \sum_{k=0}^{n} x_k \overline{y_k} = \lim_{n\to\infty} \langle f_n, g_n\rangle_\psi = \langle f, g\rangle_\psi.$$

Next, we justify the last equality. Indeed,

$$\langle f_n, g_n\rangle_\psi = \langle f, g\rangle_\psi + \langle f_n - f, g\rangle_\psi + \langle f, g_n - g\rangle_\psi + \langle f_n - f, g_n - g\rangle_\psi.$$

Thus

$$|\langle f_n, g_n\rangle_\psi - \langle f, g\rangle_\psi| \le \|f_n - f\|_\psi \|g\|_\psi + \|f\|_\psi \|g_n - g\|_\psi + \|f_n - f\|_\psi \|g_n - g\|_\psi.$$

Taking limit $n \to \infty$ in the expression above, we have proven the equality. The preceding process can be summarized in a theorem.

Theorem 1.7.0.0.9. Any solution of the Hamburger moment problem (1.7) ψ generates linear map $U : \ell^2 \to L^2_\psi$ given by

$$Ux := L^2_\psi - \lim_{n\to\infty} \sum_{k=0}^{n} x_k \widehat{P}_k(u).$$

Furthermore, $\mathrm{Dom}(U) = \ell^2$ and $\mathrm{Ran}(U) := \Delta_U$ is a subspace of L^2_ψ. The map U is isometry and for an inverse map

$$x_k = \int_{\mathbb{R}} f(u) \widehat{P}_k(u) \mathrm{d}\psi(u). \tag{1.18}$$

obeys.

(1.18) is an equation for Fourier coefficients. These can be introduced for any function $F \in L^2_\psi$, we do not have to be restricted for those belonging to Δ_U. With a function F we can associate Fourier series

$$\sum_{k=0}^{\infty} x_k \widehat{P}_k,$$

with x_k given by (1.18). We adopt the following notation

$$F(u) \sim \sum_{k=0}^{\infty} x_k \widehat{P}_k(u).$$

The partial sum

$$\sum_{k=0}^{n} x_k \widehat{P}_k$$

is the best approximation of the function F in L^2_ψ. It means

$$I_n := \int_{\mathbb{R}} \left| F(u) - \sum_{k=0}^{n} x_k \widehat{P}_k(u) \right|^2 \mathrm{d}\psi(u) = \min_{R_n} \int_{\mathbb{R}} |F(u) - R_n|^2 \, \mathrm{d}\psi(u),$$

where R_n is a polynomial with degree not exceeding n. We have

$$I_n = \int_{\mathbb{R}} |F(u)|^2 \mathrm{d}\psi(u) - \sum_{k=0}^{n} |x_k|^2.$$

Taking $n \to \infty$ one gets *Bessel inequality*

$$\sum_{k=0}^{\infty} |x_k|^2 \leq \int_{\mathbb{R}} |F(u)|^2 \mathrm{d}\psi(u).$$

In the case of equality, we are talking about *Parseval equality.*

Natural question arise, whether Δ_U coincides with L^2_ψ. The following theorem answers this question.

Theorem 1.7.0.0.10. Subspace Δ_U coincides with L^2_ψ only if the set of all polynomials is dense in L^2_ψ.

Consider a set of all solution ψ of the moment problem (1.7). Now, we define function

$$w_\psi(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\psi(u)}{u - z}. \tag{1.19}$$

Definition 1.7.0.0.11. A function defined by (1.19) is called *Weyl's function.*

Theorem 1.7.0.0.12. The set of all values of Weyl's functions $w_\psi(z)$ consider in a point $z \in \mathbb{C} \setminus \mathbb{R}$ coincides with a closed disc bounded by the circle $K_\infty(z)$.

For later purposes, we will prove a part of the statement above.

Proof. Let us take $z \in \mathbb{C} \setminus \mathbb{R}$ arbitrary. Symbol w will stand for the value $w_\psi(z)$. Let us consider a function

$$f(u) = \frac{1}{u - z} \in L^2_\psi$$

and let us find it's Fourier coefficients

$$x_k = \int_{\mathbb{R}} \frac{1}{u-z}\widehat{P}_k(u)\mathrm{d}\psi(u) = \int_{\mathbb{R}} \frac{\widehat{P}_k(u)-\widehat{P}_k(z)}{u-z}\mathrm{d}\psi(u) + \widehat{P}_k(z)\int_{\mathbb{R}} \frac{\mathrm{d}\psi(u)}{u-z} = Q_k(z) + w\widehat{P}_k(z).$$

Hence we have

$$f(u) \sim \sum_{k=0}^{\infty} \left(w\widehat{P}_k(z) + Q_k(z)\right)\widehat{P}_k(u).$$

According to Bessel inequality

$$\sum_{k=0}^{\infty} \left|w\widehat{P}_k(z) + Q_k(z)\right|^2 \leq \int_{\mathbb{R}} \frac{1}{|u-z|^2}\mathrm{d}\psi(u) = \int_{\mathbb{R}} \frac{1}{z-\overline{z}}\left(\frac{1}{u-z} - \frac{1}{u-\overline{z}}\right)\mathrm{d}\psi(u) = \frac{w-\overline{w}}{z-\overline{z}} \tag{1.20}$$

Thus, inequality (1.20) gives us that w is an element of closed disc bounded by $K_\infty(z)$. □

Theorem 1.7.0.0.12 is a prelimianary for a proof of the following important theorem.

Theorem 1.7.0.0.13. If a Jacobi matrix corresponding to the given positive sequence of moments $\{\mu_n\}_{n=0}^{\infty}$ is of the type D, then the moment problem (1.7) is determinate.

By the combination of theorem 1.7.0.0.8 and **??** one has

Theorem 1.7.0.0.14. The moment problem (1.7) is determinate only if the corresponding Jacobi matrix is of the type D.

Finally, let us concentrate on the question of density of the set of polynomials in L^2_ψ. Let us take a solution of the moment problem (1.7) ψ such that

$$\left|\int_{\mathbb{R}} u^k \mathrm{d}\psi(u)\right| < \infty, \quad \forall k \in \mathbb{N}_0.$$

Then obviously set of all polynomials (with restricted operations) is a subspace of L^2_ψ. Let us choose $z \in \mathbb{C}\setminus\mathbb{R}$ and find a value

$$w = \int_{\mathbb{R}} \frac{\mathrm{d}\psi(u)}{u-z}.$$

According to (1.20) we have

$$\sum_{k=0}^{\infty} \left|w\widehat{P}_k(z) + Q_k(z)\right|^2 \leq \frac{w-\overline{w}}{z-\overline{z}}. \tag{1.21}$$

Definition 1.7.0.0.15. Under the same assumptions as above, the solution ψ is said to be *N-extremal in* $z \in \mathbb{C}\setminus\mathbb{R}$ if the equality holds in (1.21)

Theorem 1.7.0.0.16. Under the same assumptions as above, if the solution ψ is N-extremal in some $z \in \mathbb{C}\setminus\mathbb{R}$, then ψ is N-extremal in any $z \in \mathbb{C}\setminus\mathbb{R}$.

Theorem 1.7.0.0.17. Under the same assumptions as above,

1. if the set of all polynomials is dense in L^2_ψ, then ψ is N-extremal in any $z \in \mathbb{C} \setminus \mathbb{R}$,
2. if the solution ψ of the moment problem (1.7) is N-extremal for some $z \in \mathbb{C} \setminus \mathbb{R}$, then the set of all polynomials is dense in L^2_ψ.

By the following definition, the previous theorem can be formulated in the more compact form.

Definition 1.7.0.0.18. Under the same assumptions as above, the solution ψ of the moment problem (1.7) is said to be *N-extremal* if one of the following conditions obey

1. ψ is a unique solution,
2. ψ is not unique solution, but a point

$$w = \int_{\mathbb{R}} \frac{\mathrm{d}\psi(u)}{u - z}$$

lies on a circle $K_\infty(z)$ for some (and thus for any) $z \in \mathbb{C} \setminus \mathbb{R}$.

Remark 1.7.0.0.19. Under the same assumptions as above, the solution ψ is N-extremal in some (and thus for any) $z \in \mathbb{C} \setminus \mathbb{R}$ only if ψ is N-extremal.

Finally, we can modify theorem 1.7.0.0.17.

Theorem 1.7.0.0.20. Under the same assumptions as above, the set of all polynomials is dense in L^2_ψ only if the solution ψ is N-extremal.

Note that in the case of determinate Hamburger problem (thus existence of the unique solution ψ), the set of all polynomials is dense in L^2_ψ.

1.8 Operators on ℓ^2 generated by Jacobi matrix

Let us take Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} \beta_0 & \alpha_0 & 0 & 0 & 0 & \dots \\ \alpha_0 & \beta_1 & \alpha_1 & 0 & 0 & \dots \\ 0 & \alpha_1 & \beta_2 & \alpha_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

such that $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$. Let us denote

$$\mathbf{P}^T(x) := \left(\widehat{P}_0(x), \widehat{P}_1(x), \dots, \widehat{P}_n(x), \dots\right).$$

For a chosen $x \in \mathbb{C}$, we regard $\mathbf{P}(x)$ as an element of the set $\mathbb{C}^\infty$. By the preceding it follows that forn any $x \in \mathbb{C}$ we can write

$$\mathcal{J}\mathbf{P}(x) = x\mathbf{P}(x). \tag{1.22}$$

This equation can be seen as an equation for formal eigenvalues.

Definition 1.8.0.0.1. Nonzero solution $\mathbf{f} \in \mathbb{C}^\infty$ of the equation

$$\mathcal{J}\mathbf{f} = \lambda \mathbf{f}$$

is called *formal eigenvector* of the matrix $\mathcal{J}$ associated with a eigenvalue λ. Matrix $\mathcal{J}$ is regarded as an operator on $\mathbb{C}^\infty$.

Remark 1.8.0.0.2. For any formal eigenvalue $\lambda \in \mathbb{C}$, the corresponding eigenspace is one-dimensional with basis formed by $\mathbf{P}(\lambda)$.

Let us focus on properties of some operators on the Hilbert space ℓ^2 defined by Jacobi matrix $\mathcal{J}$. First, we introduce operator $\dot{J}$ with a domain

$$\mathrm{Dom}\left(\dot{J}\right) = \mathrm{span}\{\mathbf{e}_n\}_{n=0}^\infty$$

by

$$(\forall \mathbf{f} \in \mathrm{Dom}(\dot{J}))(\dot{J}\mathbf{f} = \mathcal{J}\mathbf{f}).$$

Clearly, $\dot{J}$ is an operator on ℓ^2. Let us investigate it's adjoint operator. Since $\{\mathbf{e}_n\}_{n=0}^\infty$ is ON basis in ℓ^2, for any $\mathbf{g} \in \ell^2$ one can write

$$\mathbf{g} = \sum_{n=0}^\infty g_n \mathbf{e}_n.$$

Vector $\mathbf{g}$ is an element of $\mathrm{Dom}\left(\dot{J}^*\right)$ if there is a vector $\mathbf{h} = \sum_{n=0}^\infty h_n \mathbf{e}_n$ such as for every $\mathbf{f} \in \mathrm{Dom}\left(\dot{J}\right)$

$$\left\langle \mathbf{g}, \dot{J}\mathbf{f} \right\rangle = \langle \mathbf{h}, \mathbf{f} \rangle$$

holds. Obviously, it suffices to be held for $\mathbf{f} = \mathbf{e}_k$, where $k \in \mathbb{N}_0$ is chosen arbitrarily. We have

$$h_k = \alpha_{k-1} g_{k-1} + \beta_k g_k + \alpha_k g_{k+1}, \quad \forall k \in \mathbb{N}.$$

Thus we require

$$\mathbf{h} = \sum_{k=0}^\infty (\alpha_{k-1} g_{k-1} + \beta_k g_k + \alpha_k g_{k+1}) \mathbf{e}_k = \mathcal{J}\mathbf{g} \in \ell^2.$$

Adjoint operator $\dot{J}^*$ is given by

$$\mathrm{Dom}(\dot{J}^*) = \{\mathbf{f} \in \ell^2; \mathcal{J}\mathbf{f} \in \ell^2\} \tag{1.23}$$

and

$$\left(\forall \mathbf{f} \in \mathrm{Dom}(\dot{J}^*)\right) (\dot{J}^*\mathbf{f} = \mathcal{J}\mathbf{f}). \tag{1.24}$$

Therefore we have $\dot{J} \subset \dot{J}^*$, equivalently $\dot{J}$ is a symmetric operator. Hence, there exists closed extension J of the operator $\dot{J}$ and we know that

$$\mathrm{Dom}(\dot{J}^*) = \mathrm{Dom}(J) \dot{+} \mathrm{Ker}(\dot{J}^* - i) \dot{+} \mathrm{Ker}(\dot{J}^* + i).$$

To determine deficiency indices we have to find non-trivial solution $\mathbf{f} \in \ell^2$ of the equation

$$\dot{J}^*\mathbf{f} - i\mathbf{f} = 0.$$

According to (1.23) and (1.24) it means to solve equation

$$\mathcal{J}\mathbf{f} = i\mathbf{f}.$$

Solution of this equation is $\mathbf{P}(i)$. Thus we have a non-trivial solution only if

$$\sum_{n=0}^{\infty} \left|\widehat{P}(i)\right|^2 < \infty,$$

thus it has to be

$$\sum_{n=0}^{\infty} \left|\widehat{P}(\lambda)\right|^2 < \infty,$$

for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since coefficients of the polynomials $\widehat{P}_n(x)$ are real, $\dim \operatorname{Ker}(\dot{J}^* - i) = \dim \operatorname{Ker}(\dot{J}^* + i)$. Thus there are just two possibilities for the deficiency indices either $(1,1)$ or $(0,0)$. In the second case, one has

$$\dot{J}^* = J.$$

Thus $\dot{J}$ is essentialy self-adjoint operator on ℓ^2. We conclude that $\dot{J}$ is ESA (or equivalently has unique self-adjoint extension) only if the corresponding Hambuerger moment problem is determinate. We have a new criterion for the determinacy of the Hamburger moment problem.

Theorem 1.8.0.0.3. Hamburger moment problem (1.7) is determinate only if

$$\sum_{n=0}^{\infty} \left|\widehat{P}_n(z)\right|^2 = \infty.$$

holds for some (and thus for any) $z \in \mathbb{C} \setminus \mathbb{R}$.

Chapter 2

Fuchsian differential equations

In the following chapter we will introduce the Heun's equation as a special case of the Fuchsian differential equation. The main source for this chapter is book [9].

Definition 2.0.0.0.1. Function f is said to be *meromorphic on an open set* $\Omega \subset \mathbb{C}$ if there exists subset $A \subset \Omega$ such that

1. A has no limiting point in Ω,
2. f is holomorphic on $\Omega \setminus A$,
3. f has a pole in every point of the subset A.

Definition 2.0.0.0.2. Let $\Omega \subset \mathbb{C}$ be an open set. Let F be a complex function defined on Ω. Then

$$\left(\ell_F(F)\right)(z) = F^{(n)}(z) + \sum_{k=0}^{n-1} p_k(z) F^{(k)}(z) = 0 \tag{2.1}$$

is called *the Fuchsian differential equation* if coefficients p_k are meromorphic for $k \in \{0, 1, 2, \ldots, n-1\}$.

Remark 2.0.0.0.3. In particular, if we set $p = p_1$ and $q = p_0$ in the definition above, we get

$$\frac{\mathrm{d}^2 F(z)}{\mathrm{d}z^2} + p(z)\frac{\mathrm{d}F(z)}{\mathrm{d}z} + q(z)F(z) = 0. \tag{2.2}$$

This case is especially important for us, as the Heun equation is the differential equation of the second order.

Definition 2.0.0.0.4. Point $z_0 \in \mathbb{C}$ is said to be *an ordinary point* of the equation (2.1) if p_k is holomorphic in z_0 for $k = 0, 1, \ldots, n-1$. Else, point z_0 is said to be *singular point* of the equation (2.1). Singular point z_0 is said to be *regular singularity* of the equation (2.1) if the function p_k has pole of the order not exceeding $n - k$ in z_0 for $k = 0, 1, \ldots, n-1$.

Remark 2.0.0.0.5. Solution of the equation (2.1) will be found by the *Frobenius method.* Let us consider that

$$F(z) = z^{\rho}\varphi(z), \tag{2.3}$$

with $\rho \in \mathbb{C}$ and φ being holomorphic in some neighbourhood $\mathcal{U}$ of the point 0. Assume that 0 is an isolated singularity. Since φ is analytic in 0, it can be expressed as a power series. Due to linearity, one can assume that $\varphi(0) = 1$. Number ρ can be found by plugging expression (2.3) into equation (2.1) and multiplying this equation by z^n. Setting $a_k(z) = z^{n-k}$, the final condition for ρ reads

$$\sum_{k=0}^{n} a_k(0)\rho(\rho-1)\dots(\rho-k+1) = 0. \tag{2.4}$$

Following the analogous approach, one would get the same result for $0 \mapsto z_0 \in \mathbb{C}$.

Remark 2.0.0.0.6. Returning to remark 2.0.0.0.3 and considering z_0 as ordinary point or regular singularity, one gets existence of the limits

$$\lim_{z\to z_0}(z-z_0)p(z) =: A,$$
$$\lim_{z\to z_0}(z-z_0)^2 q(z) =: B.$$

Hence, the so-called *characteristic equation* (2.4), has the form

$$\rho^2 + (A-1)\rho + B = 0. \tag{2.5}$$

Definition 2.0.0.0.7. Solutions ρ_1, ρ_2 of the equation (2.5) are called *characteristic exponents* at the point z_0.

Next theorem is adopoted from [3].

Theorem 2.0.0.0.8. Let us assume differential equation (2.2) with characteristic exponents ρ_1, ρ_2 at a regular singular point z_0 such that $\rho_1 \neq \rho_2$ and $\rho_1 - \rho_2 \notin \mathbb{Z}$. Then

$$F_1(z) = (z-z_0)^{\rho_1}\sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad F_2(z) = (z-z_0)^{\rho_2}\sum_{n=0}^{\infty} b_n(z-z_0)^n$$

are linearly independent solutions of the equation (2.2) with the coefficients a_n, b_n given by a *certain recursive relation.* Both solutions F_1, F_2 converges on a disc given by

$$0 < |z-z_0| < R,$$

with R being no bigger than the radius of convergence of either $(z-z_0)p$ and $(z-z_0)^2 q$.

Definition 2.0.0.0.9. Consider a transformation

$$z = \frac{1}{\zeta}, \tag{2.6}$$
$$F(z) = \eta(\zeta).$$

If equation (2.1) takes the form of another Fuchsian differential equation, we are talking about the *Fuchsian differential equation in the point* ∞.

Remark 2.0.0.0.10. Note that after substitution (2.6), equation (2.2) takes the form

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\zeta^2} + \left(\frac{2}{\zeta} - \frac{1}{\zeta^2}p\left(\frac{1}{\zeta}\right)\right)\frac{\mathrm{d}\eta}{\mathrm{d}\zeta} + \frac{1}{\zeta^4}q\left(\frac{1}{\zeta}\right)\eta.$$

For us, particularly interesting example is the so-called *Heun's equation*

$$\frac{\mathrm{d}^2F(z)}{\mathrm{d}z^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)\frac{\mathrm{d}F(z)}{\mathrm{d}z}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}F(z)=0, \tag{2.7}$$

with $\alpha,\beta,\gamma,\delta,\epsilon,a,q$ being complex parameters such as $a\neq 0,1$. Equation (2.7) is clearly Fuchsian differential equation with singularities $(0,1,a,\infty$ with respectively ordered exponents $0,1-\gamma),(0,1-\delta),(0,1-\epsilon)$ and (α,β). Parameters are chosen such as they obey characteristic equation in the singular point ∞. This assumption leads to a requirement on parametres

$$\gamma+\delta+\epsilon=\alpha+\beta+1.$$

From now on, let us assume that $|a|>1$. If $\gamma\neq 0,-1,-2,\dots$, then there is a unique Frobenius solution in $z=0$, with the characteristic exponent zero, up to multiplicative constant. We have

$$F(z)=\sum_{r=0}^{\infty}c_rz^r,\quad c_0\neq 0. \tag{2.8}$$

Since $|a|>1$, series (2.8) converge for $|z|<1$. Coefficients c_r are given by

$$\begin{aligned}-qc_0+a\gamma c_1&=0\\ A_rc_{r-1}-(B_r+q)c_r+C_rc_{r+1}&=0,\quad \text{pro } r\geq 1,\end{aligned}$$

together with

$$\begin{aligned}A_r&=(r-1+\alpha)(r-1+\beta)\\ B_r&=r\left((r-1+\gamma)(1+a)+a\delta+\epsilon\right)\\ C_r&=(r+1)(r+\gamma)a.\end{aligned}$$

Setting $c_0=1$ or equivalently $F(0)=1$, the multiplicative constant which causes ambiguousness of the solution, is fixed. Solution (2.8) with the coefficients obeying conditions above is called *Heun's local function* and is denoted by

$$H\ell(a,q;\alpha,\beta,\gamma,\delta;z).$$

We will adopt notation from paper [11]. Let us denote

$$a=\frac{1}{k^2},\qquad q=-\frac{s}{k^2},$$

with $k\in(0,1)$. Heun's equation (2.7) then takes the form

$$\frac{\mathrm{d}^2F(z)}{\mathrm{d}z}+\left(\frac{\gamma}{z}-\frac{\delta}{1-z}-\frac{\epsilon k^2}{1-k^2z}\right)\frac{\mathrm{d}F(z)}{\mathrm{d}z}+\frac{s+\alpha\beta k^2z}{z(1-z)(1-k^2z)}F(z)=0.$$

We adopt the notation for the Heun's local function as well from [11]

$$\mathrm{Hn}(k^2,s;\alpha,\beta,\gamma,\delta;z).$$

We know that the orthogonal polynomails $\tilde{P}_n(x)$ obey three-terms recurrence [1]

$$b_{n-1}\tilde{P}_{n-1}(x)+(a_n-x)\tilde{P}_n(x)+b_n\tilde{P}_{n+1}(x)=0,\qquad n\in\mathbb{N}_0,$$

[1] In this section, deviation of the notation for OPS will be used according to [11]. The reason will be clear in the next chapter.

with initial conditions $\tilde{P}_{-1}(x) = 0$ and $\tilde{P}_0(x) = 1$. We will set

$$b_n = \sqrt{\lambda_n \nu_{n+1}}, \qquad a_n = \lambda_n + \nu_n + \gamma_n,$$

with

$$\begin{aligned} \lambda_n &= k^2(n+\alpha)(n+\beta), \\ \nu_n &= n(n+\gamma-1), \\ \gamma_n &= (1-k^2)\delta n. \end{aligned}$$

For $\alpha, \beta, \gamma > 0$, $\delta \in \mathbb{R}$ and $k \in (0,1)$, polynomials $\tilde{P}_n(x)$ are orthonormal with respect to a unique probability measure. It means, that the associated Hamburger problem is determinate. In particular we have

$$\mathrm{Hn}(k^2, s; \alpha, \beta, \gamma, \delta; z) = F(x,z),$$

where

$$F(x,z) := \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\nu_1 \nu_2 \dots \nu_n}} P_n(s + \alpha\beta k^2) z^n.$$

Chapter 3

Application of the theory of orthogonal polynomials to a solution of Heun's equation

Let us suppose that the Hamburger problem is determinate. Since that, there is a unique self-adjoint extension J of the operator $\dot{J}$ given by Jacobi matrix $\mathcal{J}$. Resolvent set of the operator J will be denoted by $\varrho(J)$ as usual. We have

$$J = \int \lambda \mathrm{d}E_\lambda,$$

with E_λ being a projection-valued measure. Therefore for probability measure one has

$$\mu(\cdot) = \langle \mathbf{e}_0, E(\cdot)\mathbf{e}_0 \rangle. \tag{3.1}$$

Moreover, probabilty measure from (3.1) is the only solution to the Hamburger moment problem. One has

$$\mu_k = \langle \mathbf{e}_0, J^k \mathbf{e}_0 \rangle = \int \lambda^k \mathrm{d}\mu(\lambda), \ \ k \geq 0.$$

Suppose that J is bounded below a certain positive constant γ. It says that

$$(\forall \mathbf{f} \in \mathrm{Dom}(J))(\langle \mathbf{f}, \mathcal{J}\mathbf{f} \rangle \geq \gamma \|\mathbf{f}\|).$$

In that case, J^{-1} existst and is bounded. Inequality

$$0 \leq J^{-1} \leq \frac{1}{\gamma}$$

holds true. Finally, let us assume that J^{-1} is a trace-class operator. In that case, all spectral points except for, eventually, zero are eigenvalues. Furthermore, the spectrum of J^{-1} is countable and 0 is it's limiting point. Let us denote

$$\mathrm{spec}\left(J^{-1}\right) = \left\{ \frac{1}{\lambda_n}; n \geq 1 \right\} \cup \{0\}.$$

Thus

$$\mathrm{spec}(J) = \{\lambda_n; n \geq 0\}.$$

Note that 0 cannot be an element of spectrum of J, because J is bounded below. It is also clear that $\operatorname{spec}(J) = \operatorname{spec}_{\mathrm{p}}(J)$. Moreover, eigenvalues can be ordered increasingly

$$0 < \gamma \le \lambda_1 \le \lambda_2 \le \dots.$$

Hence $\lim_{n\to\infty} \lambda_n = \infty$. We also suppose that for all $n \ge 0, \alpha_n > 0$. It follows that all eigenvalues are simple.

3.1 Useful identities

In the following section, we will summarize some useful identities which will be used later. This section is taken from paper [10].

Equation (1.22) can be rewritten as the well-known three-terms recurrence for orthonormal polynomials.

$$\begin{aligned} \widehat{P}_0(x) &= 1, \\ \alpha_0 \widehat{P}_1(x) + (\beta_0 - x)\widehat{P}_0(x) &= 0, \\ \alpha_{n-1}\widehat{P}_{n-1}(x) + (\beta_n - x)\widehat{P}_n(x) + \alpha_n \widehat{P}_{n+1}(x) &= 0, \ \text{ pro } n \ge 1. \end{aligned} \tag{3.2}$$

Besides the sequence $\left\{\widehat{P}_n(x)\right\}$, we also recall sequence $\{Q_n(x)\}$, given by

$$\begin{aligned} Q_0(x) &= 0, \\ Q_1(x) &= \frac{1}{\alpha_0}, \\ \alpha_{n-1}Q_{n-1}(x) + (\beta_n - x)Q_n(x) + \alpha_n Q_{n+1}(x) &= 0, \ \ n \ge 1. \end{aligned} \tag{3.3}$$

With matrix $\mathcal{J}$, equations (3.3) can be rewritten as

$$(\mathcal{J} - z)\mathbf{Q}(z) = \mathbf{e}_0, \tag{3.4}$$

with $\mathbf{Q}^T(z) = (Q_0(z), Q_1(z), \dots)$. Orthonormal polynomials and polynomials of the second kind are related, as the following proposition says.

Proposition 3.1.0.0.1. For all $n \in \mathbb{N}_0$

$$Q_n(z) = \left(\sum_{j=0}^{n-1} \frac{1}{\alpha_j \widehat{P}_j(z)\widehat{P}_{j+1}(z)}\right) \widehat{P}_n(z). \tag{3.5}$$

holds true.

Proof. Both sequences $\left\{\widehat{P}_n(x)\right\}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ obey the same recurrence (up to initial data). Multiplying the last equation of the recurrences (3.2) and (3.3) by $Q_n(x)$ and $\widehat{P}_n(x)$ respectively and substracting one from another one gets

$$\alpha_{n-1}\left(\widehat{P}_n(x)Q_{n-1}(x) - Q_n(x)\widehat{P}_n(x)\right) - \alpha_n\left(\widehat{P}_{n+1}(x)Q_n(x) - Q_{n+1}\widehat{P}_n(x)\right) = 0.$$

It follows

$$\alpha_k\left(\widehat{P}_{k+1}(x)Q_k(x) - Q_{k+1}\widehat{P}_k(x)\right) = A, \tag{3.6}$$

for any $k \in \mathbb{N}_0$, with A being a constant. Under an assumption $\widehat{P}_n(x) \neq 0$, we can use substitution $Q_n(x) = \widehat{P}_n(x)H_n(x)$. After this, equation (3.6) takes the form

$$H_{k+1}(x) - H_k(x) = \frac{A}{\alpha_k \widehat{P}_k(x)\widehat{P}_{k+1}(x)}. \tag{3.7}$$

Note that $H_0(x) = 0$. Taking sum in expression (3.7) for $k = 0, \dots, n-1$ one has

$$H_n(x) = \sum_{k=0}^{n-1} \frac{A}{\alpha_k \widehat{P}_k(x)\widehat{P}_{k+1}(x)}.$$

Thus

$$Q_n(z) = \left(\sum_{j=0}^{n-1} \frac{A}{\alpha_j \widehat{P}_j(z)\widehat{P}_{j+1}(z)} \right) \widehat{P}_n(z).$$

It remains to show that $A = 1$. Indeed, $H_1(z) = \frac{A}{\alpha_0 \widehat{P}_0(z)\widehat{P}_1(z)} = \frac{Q_1(z)}{\widehat{P}_1(z)} = \frac{1}{\alpha_0 \widehat{P}_1(z)}$. □

Since matrix $\mathcal{J}$ is tridiagonal, power $\mathcal{J}^k$ makes good sense for any $k \in \mathbb{N}_0$. Indeed, components in the matrix multiplication are given by convergent series, since these are reduced to finite sums. Applying (3.2) on matrix $\mathcal{J}$, we have

$$P_n(\mathcal{J})\mathbf{e}_0 = \mathbf{e}_n. \tag{3.8}$$

Proposition 3.1.0.0.2. OPS $\left\{\widehat{P}_n(x)\right\}_{n=0}^{\infty}$ forms orthonormal basis in the space L^2_μ, with measure μ being given by (3.1).

Proof. The set of all polynomials is dense in L^2_μ, as we are assuming determinate Hamburger moment problem. It remains to show that they are orthonormal. According to equation (3.1) we have for any polynomial $R(\lambda)$

$$\langle \mathbf{e}_0, R(\mathcal{J})\mathbf{e}_0 \rangle = \int R(\lambda) \mathrm{d}\mu(\lambda).$$

Thus

$$\delta_{m,n} = \langle \mathbf{e}_m, \mathbf{e}_n \rangle = \langle \widehat{P}_m(\mathcal{J})\mathbf{e}_0, \widehat{P}_n(\mathcal{J})\mathbf{e}_0 \rangle = \langle \mathbf{e}_0, \widehat{P}_m(\mathcal{J})\widehat{P}_n(\mathcal{J})\mathbf{e}_0 \rangle = \int \widehat{P}_m(\lambda)\widehat{P}_n(\lambda) \mathrm{d}\mu(\lambda).$$

In the third equality, (3.8) was used. □

Definition 3.1.0.0.3. For any $z \in \varrho(J)$ we define vector-valued function $\mathbf{w}$ as

$$\mathbf{w}(z) := (J - z)^{-1}\mathbf{e}_0 = \int \frac{\mathbf{P}(z)}{\lambda - z} \mathrm{d}\mu(\lambda).$$

Components w_k of the function above are called *function of the second kind* and obviously for any $z \in \varrho(J)$

$$w_k(z) = \langle \mathbf{e}_k, (J - z)^{-1}\mathbf{e}_0 \rangle = \int \frac{\widehat{P}_k(z)}{\lambda - z} \mathrm{d}\mu(\lambda).$$

holds.

Note that for $k = 0$ we have Weyl's function.

Proposition 3.1.0.0.4. For any $n \in \mathbb{N}_0$ and for any $z \in \varrho(J)$

$$w_n(z)\widehat{P}_n(z) = \langle \mathbf{e}_n, (J - z)^{-1}\mathbf{e}_n \rangle. \tag{3.9}$$

holds true.

Proof. We have

$$\int \widehat{P}_n(\lambda)\frac{\widehat{P}_n(\lambda) - \widehat{P}_n(z)}{\lambda - z}\mathrm{d}\mu(\lambda) = 0.$$

Moreover degree of the polynomial $\frac{\widehat{P}_n(\lambda)-\widehat{P}_n(z)}{\lambda-z}$ is not exceeding $n - 1$. Thus

$$\begin{aligned}
w_n(z)\widehat{P}_n(z) &= \widehat{P}_n(z)\int \frac{\widehat{P}_n(\lambda)}{\lambda - z}\mathrm{d}\mu(\lambda) + \int \widehat{P}_n(\lambda)\frac{\widehat{P}_n(\lambda) - \widehat{P}_n(z)}{\lambda - z}\mathrm{d}\mu(\lambda) \\
&= \int \frac{\widehat{P}_n(\lambda)^2}{\lambda - z}\mathrm{d}\mu(\lambda) = \langle \mathbf{e}_0, \widehat{P}_n(z)^2(J - z)^{-1}\mathbf{e}_0 \rangle \\
&= \langle \mathbf{e}_n, (J - z)^{-1}\mathbf{e}_n \rangle.
\end{aligned}$$

□

J^{-1} is trace-class only if for some orthonormal basis $\{\mathbf{x}_n\}_{n=0}^{\infty}$ in ℓ^2 (and thus for any)

$$\mathrm{tr}J^{-1} = \sum_{n=0}^{\infty}\langle \mathbf{x}_n, J^{-1}\mathbf{x}_n \rangle < \infty$$

holds. We can choose standard basis $\{\mathbf{e}_n\}_{n=0}^{\infty}$. Thus J^{-1} is trace-class operator only if

$$\mathrm{tr}J^{-1} = \sum_{n=0}^{\infty}\langle \mathbf{e}_n, J^{-1}\mathbf{e}_n \rangle = \sum_{n=0}^{\infty} w_n(0)\widehat{P}_n(0) < \infty. \tag{3.10}$$

In the second equality, (3.9) was used. In view of (1.22) and (3.4) we have

$$(\mathcal{J} - z)(w(z)\mathbf{P}(z) + \mathbf{Q}(z)) = \mathbf{e}_0.$$

It was claimed in the preceding chapter that $w(z)\mathbf{P}(z) + \mathbf{Q}(z) \in \ell^2$ for any $z \in \varrho(J)$. Thus

$$w(z)\mathbf{P}(z) + \mathbf{Q}(z) = (J - z)^{-1}\mathbf{e}_0 = \mathbf{w}(z) \in \ell^2.$$

It follows that

$$\langle \mathbf{e}_0, w(z)\mathbf{P}(z) + \mathbf{Q}(z) \rangle = w(z).$$

Another useful will be following *Markov's theorem.*

Theorem 3.1.0.0.5. Let $z \in \varrho(J)$. Then a limit $\lim_{n\to\infty} \frac{Q_n(z)}{\widehat{P}_n(z)}$ exists and

$$\lim_{n\to\infty} \frac{Q_n(z)}{\widehat{P}_n(z)} = -w(z) = -\langle \mathbf{e}_0, (J - z)^{-1}\mathbf{e}_0 \rangle = -\int \frac{\mathrm{d}\mu(\lambda)}{z - \lambda} = -\sum_{j=0}^{\infty} \frac{1}{\alpha_j \widehat{P}_j(z)\widehat{P}_{j+1}(z)}. \tag{3.11}$$

holds true.

Definition 3.1.0.0.6. Let $\mathcal{J}$ be Jacobi matrix. For any $k \in \mathbb{N}_0$, let the symbol $\mathcal{J}^{(k)}$ denote matrix obtained from matrix $\mathcal{J}$ by delating first k columns and first k rows. Matrix $\mathcal{J}^{(k)}$ is called the *k-th associated matrix* corresponding to the matrix $\mathcal{J}$. In particular, $\mathcal{J}^{(0)} = \mathcal{J}$.

In [10], one can find proof of the following proposition.

Proposition 3.1.0.0.7. Let $\mathcal{J}$ be Jacobi matrix corresponding to a determinate Hamburger moment problem. Then for all $k \in N_0$, matrix $\mathcal{J}^{(k)}$ corresponds to a determinate Hamburger moment problem.

For arbitrary $k \in \mathbb{N}_0$, matrix $\mathcal{J}^{(k)}$ determine a system of orthonormal polynomials $\left\{\widehat{P}_n^{(k)}(x); n \in \mathbb{N}_0\right\}$ unambiguously by

$$\begin{aligned} \widehat{P}_0^{(k)}(x) &= 1, \\ \mathcal{J}^{(k)}\mathbf{P}^{(k)}(x) &= x\mathbf{P}^{(k)}(x). \end{aligned} \tag{3.12}$$

These polynomials are called *k-th associated orthonormal polynomials*. Similarly we can define polynomials of the second kind $\left\{Q_n^{(k)}(x); n \in \mathbb{N}_0\right\}$, unambiguously as well, by

$$\begin{aligned} Q_j^{(k)}(x) &= 0, \quad j = 0, 1, \ldots, k \\ (\mathcal{J} - x)\mathbf{Q}^{(k)}(x) &= \mathbf{e}_k. \end{aligned} \tag{3.13}$$

Relation between k-th associated polynomials and their polynomials of the second kind reads

$$\widehat{P}_n^{(k)}(x) = \frac{Q_{n+k}^{(k-1)}(x)}{Q_{n+k}^{(k-1)}(0)}, \quad \text{for } n \geq 0, k \geq 0. \tag{3.14}$$

Similarly, $J^{(k)}$ denotes a unique self-adjoint operator on ℓ^2 corresponding to the k-th associated matrix $\mathcal{J}^{(k)}$. Since J is bounded below, $J^{(k)}$ is bounded below as well. For any $z \in \varrho(J)$ and for any $k \in \mathbb{N}_0$ we can define

$$\mathbf{w}^{(k)}(z) := (J - z)^{-1}\mathbf{e}_k \in \ell^2. \tag{3.15}$$

Column vector $\mathbf{w}^{(k)}(z)$ is a solution of the equation

$$(\mathcal{J} - z)\mathbf{w}^{(k)}(z) = \mathbf{e}_k \text{ v } \mathbb{C}^\infty, \quad \langle \mathbf{e}_0, \mathbf{w}^{(k)}(z)\rangle = w_k(z).$$

Furthermore,

$$(\forall z \in \varrho(J))(\mathbf{w}^{(k)}(z) = w_k(z)\mathbf{P}(z) + \mathbf{Q}^{(k)}(z) \in \ell^2). \tag{3.16}$$

According to paper [2], generalization of the Markov's theorem holds true.

Theorem 3.1.0.0.8. Under the same assumptions as in the theorem 3.1.0.0.5, the limit $\lim_{n\to\infty} \frac{Q_n^{(k)}(z)}{\widehat{P}_n(z)}$ exists and

$$\lim_{n\to\infty} \frac{Q_n^{(k)}(z)}{\widehat{P}_n(z)} = -w_k(z), \quad z \in \mathbb{C} \setminus [\gamma, \infty). \tag{3.17}$$

holds.

Due to (3.15) and (3.16), for $m \le n$ and for $z \in \varrho(J)$, we have

$$\langle \mathbf{e}_m, (J-z)^{-1}\mathbf{e}_n\rangle = \langle \mathbf{e}_m, w_n(z)\mathbf{P}(z) + \mathbf{Q}^{(n)}(z)\rangle = w_n(z)\widehat{P}_m(z).$$

The last equality holds because of (3.13). Conversely, for $m > n$, it is obvious that

$$\langle \mathbf{e}_m, (J-z)^{-1}\mathbf{e}_n\rangle = w_n(z)\widehat{P}_m(z) + Q_m^{(n)}(z).$$

At the same time

$$\langle \mathbf{e}_m, (J-z)^{-1}\mathbf{e}_n\rangle = \overline{\langle \mathbf{e}_n, (J-\overline{z})\mathbf{e}_m\rangle} = w_m(z)\widehat{P}_n(z).$$

Thus

$$(\forall m > n)\left(Q_m^{(n)}(z) = w_m(z)\widehat{P}_n(z) - w_n(z)\widehat{P}_m(z)\right). \tag{3.18}$$

In particular,

$$(\forall m \ge 1)\left(Q_m(z) = w_m(z) - w(z)\widehat{P}_m(z)\right). \tag{3.19}$$

Combining (3.18) and (3.19) one gets

$$(\forall m > n)(\forall z \in \mathbb{C}\setminus[\gamma,\infty))\left(Q_m^{(n)}(z) = Q_m(z)\widehat{P}_n(z) - Q_n(z)\widehat{P}_m(z)\right). \tag{3.20}$$

Finally, substituting (3.5) into (3.20) we have

$$(\forall m > n)(\forall z \in \mathbb{C}\setminus[\gamma,\infty))\left(Q_m^{(n)}(z) = \left(\sum_{j=n}^{m-1}\frac{1}{\alpha_j\widehat{P}_j(z)\widehat{P}_{j+1}(z)}\right)P_n(z)P_m(z)\right). \tag{3.21}$$

According to the last equality in (3.11) and (3.21) we have

$$w_n(z) = w(z)\widehat{P}_n(z) + Q_n(z) = -\left(\sum_{j=n}^{\infty}\frac{1}{\alpha_j\widehat{P}_j(z)\widehat{P}_{j+1}(z)}\right)P_n(z). \tag{3.22}$$

Plugging (3.22) into (3.10), expression for the trace reads

$$\operatorname{tr} J^{-1} = -\sum_{n=0}^{\infty}P_n(0)^2\sum_{j=n}^{\infty}\frac{1}{\alpha_j\widehat{P}_j(0)\widehat{P}_{j+1}(0)}.$$

Next, we define matrix $\mathcal{G}$ component-wise

$$\mathcal{G}_{m,n} := Q_m^{(n)}(0), \quad m,n \ge 0.$$

This matrix is obviously strictly lower-triangular. Due to (3.13) we have

$$\mathcal{J}\mathcal{G} = I. \tag{3.23}$$

Matrix $\mathcal{G}$ is obviously the only strictly lower-triangular obeying (3.23). Hence, this matrix can be interpreted as the *Green function* of the Jacobi matrix $\mathcal{J}$. Due to (3.22), we have

$$\mathcal{G}_{m,n} = \widehat{P}_n(0)\widehat{P}_m(0)\sum_{j=n}^{m-1}\frac{1}{\alpha_j\widehat{P}_j(0)\widehat{P}_{j+1}(0)}.$$

Proof of the following theorem can be found in [10].

Theorem 3.1.0.0.9. Let $\mathcal{J}$ be a Jacobi matrix and $\{\widehat{P}_n(x)\}$ be corresponding system of orthonormal polynomials. Let $\mathcal{G}$ be the Green function of the matrix $\mathcal{J}$. Then

$$\mathbf{P}(x) = (I - x\mathcal{G})\mathbf{P}(0). \tag{3.24}$$

Conversely, equation (3.24) determine a strictly lower-triangular matrix unambiguously.

$(I - x\mathcal{G})$ can be expanded to the series in the powers of $\mathcal{G}$, it says

$$(I - x\mathcal{G}) = I + \sum_{l=1}^{\infty} x^l \mathcal{G}^l.$$

This series is convergent as for arbitrary fixed matrix element the series terminate after finite terms. Thus

$$(\forall n \in \mathbb{N}_0) \left(\widehat{P}_n(x) = \widehat{P}_n(0) + \sum_{l=1}^{n} x^l \sum_{0 \le k_1 < k_2 < \cdots < k_l < n} \mathcal{G}_{n,k_l} \mathcal{G}_{k_l,k_{l-1}} \ldots \mathcal{G}_{k_2,k_1} \widehat{P}_{k_1}(0) \right). \tag{3.25}$$

3.2 Worked example

For complex number a, Pochhammer symbol is defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1). \tag{3.26}$$

Let us assume $\alpha, \beta, \gamma > 0$. Next, let us set

$$\begin{aligned} \alpha_n &:= k\sqrt{(n+1)(n+\alpha)(n+\beta)(n+\gamma)}, \\ \beta_n &:= k^2 n(n+\alpha-1) + (n+\beta)(n+\gamma), \end{aligned} \tag{3.27}$$

where $k \in (0,1)$.

Let us denote $u_n := (n+\beta)(n+\gamma)$ and $v_n := k^2 n(n+\alpha-1)$. With this notation we can write

$$\begin{aligned} \alpha_n &= \sqrt{u_n v_{n+1}}, \\ \beta_n &= u_n + v_n. \end{aligned} \tag{3.28}$$

It is easy to see that the relation

$$\frac{\alpha_n}{v_{n+1}} = \frac{u_n}{\alpha_n}$$

holds true. One also has

$$\frac{\alpha_n}{v_{n+1}} = \frac{1}{k}\sqrt{\frac{(n+\beta)(n+\gamma)}{(n+1)(n+\alpha)}}.$$

We already know that OPS $\left\{\widehat{P}_n(x)\right\}_{n=0}^{\infty}$ obeys the following three-term reccurence

$$\alpha_n \widehat{P}_{n+1}(x) + (\beta_n - x)\widehat{P}_n(x) + \alpha_{n-1}\widehat{P}_{n-1}(x) = 0.$$

Setting $x = 0$ and taking advantage of notatiton (3.28) this equation takes the form

$$\sqrt{u_n v_{n+1}}\widehat{P}_{n+1}(0) + (u_n + v_n)\widehat{P}_n(0) + \sqrt{u_{n-1}v_n}\widehat{P}_{n-1}(0) = 0,$$

which can be rewritten as

$$\sqrt{u_n v_{n+1}}\left(\widehat{P}_{n+1}(0) + \sqrt{\frac{u_n}{v_{n+1}}}\widehat{P}_n(0)\right) + v_n\left(\widehat{P}_n(0) + \sqrt{\frac{u_{n-1}}{v_n}}\widehat{P}_{n-1}(0)\right) = 0.$$

After substitution $x_n = \widehat{P}_{n+1}(0) + \sqrt{\frac{u_n}{v_{n+1}}}\widehat{P}_n(0)$ we obtain a two-term reccurence for the sequence $\{x_n\}$ with the initial data $x_0 = \widehat{P}_1(0) + \sqrt{\frac{u_0}{v_1}}\widehat{P}_0(0) = \frac{1}{\alpha_0}\left(\alpha_0\widehat{P}_1(0) + \beta_0\widehat{P}_0(0)\right) = 0$. It's solution is $x_n = 0$, hence

$$\widehat{P}_{n+1}(0) + \sqrt{\frac{u_n}{v_{n+1}}}\widehat{P}_n(0) = 0, \quad n \geq 0.$$

Thus we can express $\widehat{P}_n(0)$ in the form

$$\widehat{P}_n(0) = (-1)^n k^{-n}\sqrt{\frac{(\beta)_n(\gamma)_n}{n!(\alpha)_n}}.$$

With this result one can evaluate the Green function. Let us recall relation (3.21). According to that, we have

$$\mathcal{G}_{m,n} = (-1)^{n+m+1}k^{-n-m}\sqrt{\frac{(\beta)_n(\beta)_m(\gamma)_n(\gamma)_m}{n!m!(\alpha)_n(\alpha)_m}}\sum_{j=n}^{m-1}\frac{k^{2j}j!(\alpha)_j}{(\beta)_{j+1}(\gamma)_{j+1}}. \tag{3.29}$$

For coefficients given by equation (3.27) the corresponding Jacobi matrix is Hamburger determinate. Indeed,

$$\lim_{n\to\infty}\left|\frac{\widehat{P}_{n+1}(0)}{\widehat{P}_n(0)}\right| = \lim_{n\to\infty}\frac{1}{k}\sqrt{\frac{(n+\beta)(n+\gamma)}{(n+1)(n+\alpha)}} = \frac{1}{k} \in (1, \infty).$$

Thus $\lim_{n\to\infty}\left|\widehat{P}_n(0)\right| = \infty$ and therefore the series

$$\sum_{n=0}^{\infty}\left|\widehat{P}_n(z)\right|^2$$

diverges for $z = 0$ and hence for any $z \in \mathbb{C} \setminus \mathbb{R}$. It follows that the corresponding Jacobi matrix is Hamburger determinate.

For application of the formulas derived in [10], we need to prove, that J^{-1} is a trace-class operator, as shown in [10] and summarize in the preceding section. This happens if and only if

$$\mathrm{tr}J^{-1} = -\sum_{n=0}^{\infty}\widehat{P}_n(0)^2\sum_{j=n}^{\infty}\frac{1}{\alpha_j\widehat{P}_j(0)\widehat{P}_{j+1}(0)} < \infty. \tag{3.30}$$

In our case, equation (3.30) takes the form

$$\mathrm{tr} J^{-1} = \sum_{n=0}^{\infty} k^{-2n} \frac{(\beta)_n(\gamma)_n}{n!(\alpha)_n} \sum_{j=n}^{\infty} \frac{k^{2j}}{\sqrt{(j+1)(j+\alpha)(j+\beta)(j+\gamma)}} \sqrt{\frac{j!(\alpha)_j}{(\beta)_j(\gamma)_j}} \sqrt{\frac{(j+1)!(\alpha)_{j+1}}{(\beta)_{j+1}(\gamma)_{j+1}}}. \tag{3.31}$$

Recall that Pochhammer symbol obeys relation (3.26). In order to prove the convergence of (3.31), we will use the following asymptotic behavior of the Pochhammer symbol

$$\frac{(a)_n}{n!} = \frac{n^{-1+a}}{\Gamma(a)} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Thus there are constants $C_1, C_2 > 0$ such that

$$C_1 n^{\frac{1}{2}(-1-\alpha+\beta+\gamma)} \leq \sqrt{\frac{(\beta)_n(\gamma)_n}{n!(\alpha)_n}} \leq C_2 n^{\frac{1}{2}(-1-\alpha+\beta+\gamma)}, \quad n \geq 1. \tag{3.32}$$

Obviously, omitting the term with $n = 0$ in (3.31) does not influence the convergence. From (3.32) it is readily seen that the rest of (3.31) can be estimated from above by the expression

$$C \sum_{n=1}^{\infty} k^{-2n} n^{-1-\alpha+\beta+\gamma} \sum_{j=n}^{\infty} \frac{k^{2j}}{(j+1)^{1-\alpha+\beta+\gamma}},$$

where $C > 0$ is a constant. Solving the convergence problem, the constant C can be omitted. After changing the order of summation one has

$$\sum_{n=1}^{\infty} k^{-2n} n^{-1-\alpha+\beta+\gamma} \sum_{j=n}^{\infty} \frac{k^{2j}}{(j+1)^{1-\alpha+\beta+\gamma}} = \sum_{j=0}^{\infty} k^{2j} \sum_{n=1}^{\infty} \frac{n^{-1-\alpha+\beta+\gamma}}{(j+n+1)^{1-\alpha+\beta+\gamma}} = \sum_{j=0}^{\infty} k^{2j} \sum_{n=1}^{\infty} \frac{n^{\omega}}{(j+n+1)^{2+\omega}},$$

where $\omega := -1 - \alpha + \beta + \gamma$. Hence we have to prove the convergence of the sum

$$\sum_{j=0}^{\infty} k^{2j} \sum_{n=1}^{\infty} \frac{n^{\omega}}{(n+j+1)^{2+\omega}}.$$

Our goal is to estimate the second sum with integral. For this sake, let us define for $j \geq 0$

$$g_n(j) := \frac{n^{\omega}}{(n+j+1)^{2+\omega}}$$

and

$$f(j) := \sum_{n=1}^{\infty} g_n(j).$$

Considering $\omega > 0$, it is easy to check, that $g_n(j)$ is decreasing for $n \geq \left[\frac{\omega}{2}(j+1)\right]$ and increasing for $1 \leq n < \left[\frac{\omega}{2}(j+1)\right]$. In the case $\omega \leq 0$, $g_n(j)$ is decreasing for every $n \geq 1$.

First, let us compute the following indeterminate integral

$$I(j) := \int \frac{x^{\omega}}{(x+j+1)^{2+\omega}} \mathrm{d}x.$$

Using substitution $y = x + j + 1$ we get

$$I(j) = \int \frac{(y-j-1)^{\omega}}{y^{2+\omega}}\mathrm{d}y = \int \frac{1}{y^2}\left(1 - \frac{j+1}{y}\right)^{\omega}\mathrm{d}y.$$

Next, we use the substitution $t = -\frac{j+1}{y}$. Hence

$$I(j) = \frac{1}{j+1}\int (1+t)^{\omega}\mathrm{d}t = \frac{(1+t)^{1+\omega}}{(\omega+1)(j+1)},\ \ \omega \neq -1.$$

Thus

$$I(j) = \frac{x^{1+\omega}}{(\omega+1)(j+1)(x+j+1)^{1+\omega}},\ \ \omega \neq -1.$$

For $\omega = -1$ one has

$$I(j) = \frac{1}{j+1}\ln\left(\frac{x}{x+j+1}\right).$$

First of all, let us concentrate on the case $\omega > 0$. Then we have

$$\begin{aligned} f(j) &= \sum_{n=1}^{\left[\frac{\omega}{2}(j+1)\right]-1} g_n(j) + \sum_{\left[\frac{\omega}{2}(j+1)\right]}^{\infty} g_n(j) \qquad (3.33)\\ &\leq \int_1^{\left[\frac{\omega}{2}(j+1)\right]-1} \frac{x^{\omega}}{(x+j+1)^{2+\omega}}\mathrm{d}x + g_{\left[\frac{\omega}{2}(j+1)\right]}(j) + \int_{\left[\frac{\omega}{2}(j+1)\right]}^{\infty} \frac{x^{\omega}}{(x+j+1)^{2+\omega}}\mathrm{d}x \\ &\leq \int_0^{\infty} \frac{x^{\omega}}{(x+j+1)^{2+\omega}}\mathrm{d}x + \frac{\left(\frac{\omega}{2}(j+1)\right)^{\omega}}{\left(\left(\frac{\omega}{2}+1\right)(j+1)\right)^{\omega+2}} = \frac{1}{(\omega+1)(j+1)} + \frac{\left(\frac{\omega}{2}(j+1)\right)^{\omega}}{\left(\left(\frac{\omega}{2}+1\right)(j+1)\right)^{\omega+2}} \\ &= \frac{1}{(\omega+1)(j+1)}\left(1 + O\left(\frac{1}{j}\right)\right) \quad \text{as } j \to \infty. \end{aligned}$$

Next, we will solve the case $-1 < \omega < 0$. Then

$$\begin{aligned} f(j) &\leq \int_1^{\infty} \frac{x^{\omega}}{(x+j+1)^{2+\omega}}\mathrm{d}x \leq \frac{1}{(\omega+1)(j+1)} - \frac{1}{(\omega+1)(j+1)^2} \qquad (3.34)\\ &= \frac{1}{(\omega+1)(j+1)}\left(1 + O\left(\frac{1}{j}\right)\right) \quad \text{as } j \to \infty. \end{aligned}$$

For $\omega < -1$ one can estimate

$$f(j) \leq \int_1^{\infty} \frac{x^{\omega}}{(x+j+1)^{2+\omega}}\mathrm{d}x = \lim_{x\to\infty} \frac{x^{1+\omega}}{(\omega+1)(j+1)(x+j+1)^{1+\omega}} - \frac{1}{(\omega+1)(j+1)(j+2)^{1+\omega}} \qquad (3.35)$$

$$= \frac{1}{-(\omega+1)j^{2+\omega}} \cdot (1 + o(1)) \quad \text{as } j \to \infty.$$

For the remaining case, $\omega = -1$ one has

$$\begin{aligned} f(j) &\leq \int_1^{\infty} \frac{1}{x(x+j+1)}\mathrm{d}x = \lim_{x\to\infty} \frac{1}{j+1}\ln\left(\frac{x}{x+j+1}\right) - \frac{1}{j+1}\ln\left(\frac{1}{j+1}\right) \qquad (3.36)\\ &= \frac{1}{j+1}\ln(j+1) \end{aligned}$$

In view of the estimates (3.33), (3.34), (3.35) and (3.36), the following holds true

$$\sum_{j=0}^{\infty} k^{2j} \sum_{n=1}^{\infty} \frac{n^{\omega}}{(n+j+1)^{2+\omega}} \leq \sum_{j=0}^{\infty} f(j) k^{2j} < \infty.$$

We have proven that J^{-1} is a trace-class operator, thus we can use formulas derived in [10].

Each polynomials $\widehat{P}_n(x)$ can be written as

$$\widehat{P}_n(x) = \sum_{m=0}^{n} p(n,m) x^m.$$

Note that $p(0,n) = \widehat{P}_n(0)$. By (3.25) we have, for $n \geq 1$

$$\begin{aligned}
p(n,m) &= \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \widehat{P}_n(x) \Big|_{x=0} = \sum_{0 \leq l_1 < l_2 < \cdots < l_m < n} \mathcal{G}_{n,l_m} \mathcal{G}_{l_m,l_{m-1}} \ldots \mathcal{G}_{l_2,l_1} \widehat{P}_{l_1}(0) \\
&= (-1)^{n+m} k^{-n} \sqrt{\frac{(\beta)_n (\gamma)_n}{n!(\alpha)_n}} \sum_{0 \leq l_1 < l_2 < \cdots < l_m < n} k^{-2l_m - 2l_{m-1} - \ldots - 2l_1} \frac{(\beta)_{l_m} (\gamma)_{l_m}}{l_m! (\alpha)_{l_m}} \times \\
&\times \frac{(\beta)_{l_{m-1}} (\gamma)_{l_{m-1}}}{l_{m-1}! (\alpha)_{l_{m-1}}} \cdots \frac{(\beta)_{l_1} (\gamma)_{l_1}}{l_1! (\alpha)_{l_1}} \sum_{j_m = l_m}^{n-1} \frac{k^{2j_m} j_m! (\alpha)_{j_m}}{(\beta)_{j_m+1} (\gamma)_{j_m+1}} \cdots \times \\
&\times \sum_{j_1 = l_1}^{l_2 - 1} \frac{k^{2j_1} j_1! (\alpha)_{j_1}}{(\beta)_{j_1+1} (\gamma)_{j_1+1}}
\end{aligned}$$

By the virtue of [11] let us set

$$G(x,z) := \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n}} \widehat{P}_n(x) (kz)^n. \tag{3.37}$$

We will find Heun's equation for which the function $G(x,z) \equiv G(z)$, as defined above in (3.37), is a local Heun function. We have to be carefull because our coefficients α_n, β_n are not exactly the same as a_n, b_n in [11]. The relation between them is described in the following equations

$$\alpha_n = b_n, \qquad \beta_n = a_n + \beta\gamma - k^2 \alpha\beta.$$

The orthonormal polynomails given by these coefficients will be again denoted by $\tilde{P}_n(x)$. We already know that function $F(x,z) \equiv F(z)$ given by

$$F(x,z) := \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n}} \tilde{P}_n(x) (kz)^n$$

is the Heun function for the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} F(z) + \left(\frac{\gamma}{z} - \frac{\beta+1}{1-z} - \frac{\epsilon k^2}{1-k^2 z} \right) \frac{\mathrm{d}}{\mathrm{d}z} F(z) + \frac{x - \alpha\beta k^2 + \alpha\beta k^2 z}{z(z-1)(1-k^2 z)} F(z) = 0.$$

It is easy to check, that polynomials $\widehat{P}_n(x)$ and polynomials $\tilde{P}_n(x)$ are connected by the formula

$$\widehat{P}_n(x) = \tilde{P}_n(x + k^2 \alpha\beta - \beta\gamma),$$

thus the generating function (3.37) is a solution of the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} G(z) + \left(\frac{\gamma}{z} - \frac{\beta+1}{1-z} - \frac{\epsilon k^2}{1-k^2 z} \right) \frac{\mathrm{d}}{\mathrm{d}z} G(z) + \frac{x - \beta\gamma + \alpha\beta k^2 z}{z(1-z)(1-k^2 z)} G(z) = 0.$$

3.2.1 Special cases

In some special cases, particular simplification of the epxression for $p(n,m)$ is possible. First, let us assume

$$\alpha = \beta = \gamma = 1.$$

Under this assumption one has a solution of the equation

$$\frac{\mathrm{d}^2 G(z)}{\mathrm{d}z^2} + \frac{1-3z}{z(1-z)}\frac{\mathrm{d}G(z)}{\mathrm{d}z} + \frac{x-1+k^2 z}{z(1-z)(1-k^2 z)}G(z) = 0.$$

For coeffients $p(n,m)$ one has

$$\begin{aligned} p(m,n) &= \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\widehat{P}_n(x)\Big|_{x=0} = \sum_{0\le l_1<l_2<\dots<l_m<n} \mathcal{G}_{n,l_m}\mathcal{G}_{l_m,l_{m-1}}\dots\mathcal{G}_{l_2,l_1}\widehat{P}_{l_1}(0) \\ &= (-1)^{n+m}k^{-n} \sum_{0\le l_1<l_2<\dots<l_m<n} k^{-2l_m-2l_{m-1}-\dots-2l_1}\times \\ &\times \sum_{j_m=l_m}^{n-1}\frac{k^{2j_m}}{(j_m+1)^2}\sum_{j_{m-1}=l_{m-1}}^{l_m-1}\frac{k^{2j_{m-1}}}{(j_{m-1}+1)^2}\dots\sum_{j_1=l_1}^{l_2-1}\frac{k^{2j_1}}{(j_1+1)^2}, \end{aligned}$$

Note that the indices in this formula satisfy

$$0\le l_1\le j_1<l_2\le j_2<\dots<l_{m-1}\le j_{m-1}<l_m\le j_m<n.$$

Changing the order of summation in the above expression, we have

$$\begin{aligned} p(m,n) &= (-1)^{n+m}k^{-n}\sum_{0\le j_1<j_2\dots<j_m<n}\frac{k^{j_m}k^{2j_{m-1}}\dots k^{2j_1}}{(j_m+1)^2(j_{m-1}+1)^2\dots(j_1+1)^2}\times \\ &\times \sum_{l_m=j_{m-1}+1}^{j_m}k^{-2l_m}\sum_{l_{m-1}=j_{m-2}+1}^{j_{m-1}}k^{-2l_{m-1}}\dots\sum_{l_1=0}^{j_1}k^{-2l_1}. \end{aligned}$$

Hence,

$$p(m,n) = \frac{(-1)^{n+m}k^{-n}}{(1-k^2)^m}\sum_{0\le j_1<j_2\dots<j_m<n}\frac{(1-k^{2(j_1+1)})(1-k^{2(j_2-j_1)})\dots(1-k^{2(j_m-j_{m-1})})}{(j_1+1)^2(j_2+1)^2\dots(j_m+1)^2}.$$

In this case, the generating function (3.37) takes the form

$$G(x,z) = \sum_{n=0}^{\infty}(-kz)^n\widehat{P}_n(x),$$

which can also be simplified

$$\begin{aligned}
G(x,z) &= \sum_{n=0}^{\infty}(-kz)^n \sum_{m=0}^{n} p(m,n)x^m \\
&= \sum_{m=0}^{\infty} x^m \sum_{n=m}^{\infty} p(m,n)(-kz)^n \\
&= \sum_{m=0}^{\infty} \frac{(-x)^m}{(1-k^2)^m} \sum_{n=m}^{\infty} z^n \sum_{0\le j_1<j_2\dots<j_m<n} \frac{(1-k^{2(j_1+1)})(1-k^{2(j_2-j_1)})\dots(1-k^{2(j_m-j_{m-1})})}{(j_1+1)^2(j_2+1)^2\dots(j_m+1)^2} \\
&= \sum_{m=0}^{\infty} \frac{(-x)^m}{(1-k^2)^m} \sum_{0\le j_1<j_2<\dots<j_m<\infty} \frac{(1-k^{2(j_1+1)})(1-k^{2(j_2-j_1)})\dots(1-k^{2(j_m-j_{m-1})})}{(j_1+1)^2(j_2+1)^2\dots(j_m+1)^2} \sum_{n=j_m+1}^{\infty} z^n.
\end{aligned}$$

Thus we have

$$\begin{aligned}
G(x,z) = \frac{1}{1-z}\Bigg(1 + \sum_{m=0}^{\infty} \frac{(-x)^m}{(1-k^2)^m} \times \\
\times \sum_{0\le j_1<j_2<\dots<j_m<\infty} \frac{(1-k^{2(j_1+1)})(1-k^{2(j_2-j_1)})\dots(1-k^{2(j_m-j_{m-1})})}{(j_1+1)^2(j_2+1)^2\dots(j_m+1)^2} z^{j_m+1}\Bigg).
\end{aligned}$$

Second, let us assume

$$\alpha = \beta > 0, \gamma = 1.$$

Thus we are looking for a solution of the equation

$$\frac{\mathrm{d}^2 G(z)}{\mathrm{d}z^2} + \left(\frac{1}{z} - \frac{\alpha+1}{1-z} - \frac{(\alpha-1)k^2}{1-k^2 z}\right)\frac{\mathrm{d}G(z)}{\mathrm{d}z} + \frac{x-\alpha+\alpha^2k^2z}{z(1-z)(1-k^2z)}G(z) = 0.$$

Similarly as above, we can simplify expression for $p(n,m)$ as

$$\begin{aligned}
p(m,n) &= \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\widehat{P}_n(x)\Big|_{x=0} = \sum_{0\le l_1<l_2<\dots<l_m<n} \mathcal{G}_{n,l_m}\mathcal{G}_{l_m,l_{m-1}}\dots\mathcal{G}_{l_2,l_1}\widehat{P}_{l_1}(0) \\
&= (-1)^{n+m}k^{-n} \sum_{0\le l_1<l_2<\dots<l_m<n} k^{-2l_m-2l_{m-1}-\dots-2l_1} \times \\
&\times \sum_{j_m=l_m}^{n-1} \frac{k^{2j_m}}{(j_m+1)(j_m+\alpha)} \sum_{j_{m-1}=l_{m-1}}^{l_m-1} \frac{k^{2j_{m-1}}}{(j_{m-1}+1)(j_{m-1}+\alpha)} \times \\
&\times \dots \times \sum_{j_1=l_1}^{l_2-1} \frac{k^{2j_1}}{(j_1+1)(j_1+\alpha)}.
\end{aligned}$$

Note that

$$0 \le l_1 \le j_1 < l_2 \le j_2 < \dots < l_{m-1} \le j_{m-1} < l_m \le j_m < n.$$

Changing the order of summation one has

$$p(m,n) = \frac{(-1)^{n+m}k^{-n}}{(1-k^2)^m} \sum_{0\le j_1<j_2<\dots<j_m<n} \frac{(1-k^{2(j_1+1)})(1-k^{2(j_2-j_1)})\dots(1-k^{2(j_m-j_{m-1})})}{(j_1+1)(j_1+\alpha)(j_2+1)(j_2+\alpha)\dots(j_m+1)(j_m+\alpha)}.$$

The generating function is of the form

$$G(x,z) := \sum_{n=0}^{\infty} \widehat{P}_n(x) \frac{(\alpha)_n}{n!} (-kz)^n$$

and can be expressed as

$$\begin{aligned} G(x,z) = (1-z)^{-\alpha} + \sum_{m=1}^{\infty} \frac{(-x)^m}{(1-k^2)^m} \\ \times \sum_{0 \le j_1 < \dots < j_m < \infty} \frac{(1-k^{2(j_1+1)}) \dots (1-k^{2(j_m - j_{m-1})})}{(j_1+1)(j_1+\alpha)\dots(j_m+1)(j_m+\alpha)} \times \\ \times \frac{(\alpha)_{j_m+1}}{(j_m+1)!} \, {}_2F_1(1, j_m+\alpha+1; j_m+2; z) z^{j_m+1}. \end{aligned}$$

Finally, assume that

$$\alpha = \gamma > 0, \beta = 1.$$

Then a discussed equation is of the form

$$\frac{\mathrm{d}^2 G(z)}{\mathrm{d}z^2} + \frac{\alpha - (2+\alpha)z}{z(1-z)} \frac{\mathrm{d}G(z)}{\mathrm{d}z} + \frac{x - \alpha + \alpha k^2 z}{z(1-z)(1-k^2 z)} G(z) = 0.$$

The generating function

$$G(x,z) = \sum_{n=0}^{\infty} \widehat{P}_n(x)(-kz)^n$$

can be expressed as

$$\begin{aligned} G(x,z) = \frac{1}{1-z} \Bigg(1 + \sum_{m=1}^{\infty} \frac{(-x)^m}{(1-k^2)^m} \times \\ \times \sum_{0 \le j_1 < \dots < j_m < \infty} \frac{(1-k^{2(j_1+1)}) \dots (1-k^{2(j_m - j_{m-1})})}{(j_1+1)(j_1+\alpha)\dots(j_m+1)(j_m+\alpha)} \Bigg). \end{aligned}$$

Note that the orthonormal systems in the second and the third case are the same.

3.2.2 Generalization

Recall that our aim is to solve the Heun's differential equation of the special form

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} G(z) + \left(\frac{\gamma}{z} - \frac{\beta+1}{1-z} - \frac{\epsilon k^2}{1-k^2 z} \right) \frac{\mathrm{d}}{\mathrm{d}z} G(z) + \frac{x - \beta\gamma + \alpha\beta k^2 z}{z(1-z)(1-k^2 z)} G(z) = 0. \tag{3.38}$$

with an additional condition $\epsilon = \alpha - \gamma$. Assumption of positivity of α, β, γ in the approach above, allows us to show that an inverse operator for Jacobi matrix exists and is trace class. Hence, according to [10], the expression

$$G(x, z) := \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n}} \widehat{P}_n(x) (kz)^n \tag{3.39}$$

with

$$\begin{aligned} p(m, n) = (-1)^{n+m} k^{-n} \sqrt{\frac{(\beta)_n (\gamma)_n}{n! (\alpha)_n}} \sum_{0 \le l_1 < l_2 < \dots < l_m < n} & k^{-2l_m - 2l_{m-1} - \dots - 2l_1} \frac{(\beta)_{l_m} (\gamma)_{l_m}}{l_m! (\alpha)_{l_m}} \times \\ \times \frac{(\beta)_{l_{m-1}} (\gamma)_{l_{m-1}}}{l_{m-1}! (\alpha)_{l_{m-1}}} \cdots \frac{(\beta)_{l_1} (\gamma)_{l_1}}{l_1! (\alpha)_{l_1}} \sum_{j_m = l_m}^{n-1} \frac{k^{2j_m} j_m! (\alpha)_{j_m}}{(\beta)_{j_m+1} (\gamma)_{j_m+1}} \cdots \times \\ \times \sum_{j_1 = l_1}^{l_2 - 1} \frac{k^{2j_1} j_1! (\alpha)_{j_1}}{(\beta)_{j_1+1} (\gamma)_{j_1+1}} \end{aligned} \tag{3.40}$$

is possible expression for the Heun local function of the equation mentioned above.

In this section, our tactics will be slightly different. We will assume that (3.39) together with (3.40) is a candidate for the solution of the equation (3.38) whenever the RHS of (3.39) makes good sense. This approach allows us to extend the range of parametres α, β, γ if compared to the case when the existence of a trace-class inverse is guaranted. The result is summarize in the theorem below.

Theorem 3.2.2.0.1. Let $\alpha, \beta, \gamma \in \{z \in \mathbb{C} : \Re(z) > -1 \ \vee \ \Im(z) \neq 0\}$ and $k \in (0, 1)$. Then the function $G = G(z)$ defined by (3.39) and (3.40) is the local Heun function for the Heun differential equation (3.38).

Proof. The first goal is to show that the infinite matrix $\mathcal{G}$ defined by elements (3.29) is the Green function for the Jacobi matrix $\mathcal{J}$ defined by coefficients (3.27), this is to say that

$$\mathcal{J} \cdot \mathcal{G} = 1, \tag{3.41}$$

here the symbol $\cdot$ means matrix multiplication. This operation is meaningful since $\mathcal{J}$ is tridiagonal. In the sequel, the symbol $\cdot$ will be omitted. Let $n \ge 0$ be arbitrary. Let us compute

$$(\mathcal{J}\mathcal{G})_{n,l} = \alpha_{n-1} \mathcal{G}_{n-1,l} + \beta_n \mathcal{G}_{n,l} + \alpha_n \mathcal{G}_{n+1,l}. \tag{3.42}$$

As $\mathcal{G}$ is strictly lower-triangular, $(\mathcal{J}\mathcal{G})_{n,l}$ is not equal to 0 only if $l = 0, \dots, n$. For $l = n$ one has

$$(\mathcal{J}\mathcal{G})_{n,n} = \alpha_n \mathcal{G}_{n+1,n} = \alpha_n \frac{1}{k\sqrt{(n+1)(n+\alpha)(n+\beta)(n+\gamma)}} = 1.$$

For $l = 0, \dots, n-1$ we firstly give an auxiliary result

$$\mathcal{G}_{n,l} = -k\sqrt{\frac{(n-1+\beta)(n-1+\gamma)}{n(n-1+\alpha)}}\mathcal{G}_{n-1,l} + (-1)^{n+l+1}k^{-n-l}\sqrt{\frac{(\beta)_l(\gamma)_l}{l!(\alpha)_l}}\sqrt{\frac{(\beta)_n(\gamma)_n}{n!(\alpha)_n}} \times \frac{k^{2n-2}(n-1)!(\alpha)_{n-1}}{(\beta)_n(\gamma)_n}$$

$$\mathcal{G}_{n+1,l} = k^2\sqrt{\frac{(n+\beta)(n-1+\beta)(n+\gamma)(n-1+\gamma)}{n(n+1)(n+\gamma)(n-1+\gamma)}}\mathcal{G}_{n-1,l}+ + \sqrt{\frac{(\beta)_l(\gamma)_l}{l!(\alpha)_l}}\sqrt{\frac{(\beta)_{n+1}(\gamma)_{n+1}}{(n+1)!(\alpha)_{n+1}}}\left(\frac{k^{2n-2}(n-1)!(\alpha)_{n-1}}{(\beta)_n(\gamma)_n} + \frac{k^{2n}n!(\alpha)_n}{(\beta_{n+1}(\gamma)_{n+1}}\right)$$

Plugging this result into the equation (3.42) one gets

$$(\mathcal{J}\mathcal{G})_{n,l} = 0.$$

Hence, (3.41) holds true. Thus, according to theorem 3.1.0.0.9 one has that polynomials $\widehat{P}_n(x)$ obey equation (3.24). In our case this equation is equivalent to the equation (3.40). Polynomials $\widehat{P}_n(x)$ obey the reccurence

$$\alpha_0\widehat{P}_1(x) + (\beta_0 - x)\widehat{P}_0(x) = 0 \tag{3.43}$$
$$\alpha_n\widehat{P}_{n+1}(x) + (\beta_n - x)\widehat{P}_n(x) + \alpha_{n-1}\widehat{P}_{n-1}(x) = 0, \quad n \geq 1$$

with α_n and β_n being defined by (3.27). They represent the OPS corresponding to the Jacobi matrix $\mathcal{J}$ with coefficients (3.27). We will multiply both sides of equation (3.38) by $z(1-z)(1-k^2z)$ and plug expression (3.39) for $G(z)$ into (3.38). After routine manipulation with indices in the series, the LHS of the equation (3.38) reads

$$-\alpha_0\widehat{P}_1(x) - \beta_0\widehat{P}_0(x) + k\sqrt{\frac{\alpha\beta}{\gamma}}\left(\alpha_1\widehat{P}_2(x) + (\beta_1 - x)\widehat{P}_1(x) + \alpha_0\widehat{P}_0(x)\right)z$$
$$-k^2\sqrt{\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}}\left(\alpha_2\widehat{P}_3(x) + (\beta_2 - x)\widehat{P}_2(x) - \alpha_1\widehat{P}_1(x)\right)z^2$$
$$+\sum_{n=3}^{\infty}(-1)^{n+1}k^n\sqrt{\frac{(\alpha)_{n-1}(\beta)_{n-1}}{n-1)!(\gamma)_{n-1}}}\left(\alpha_n\widehat{P}_{n+1}(x) + (\beta_n - x)\widehat{P}_n(x) + \alpha_{n-1}\widehat{P}_{n-1}(x)\right)z^n$$

being equal to zero due to the reccurence (3.43). This shows that expressions (3.39) and (3.40) describe, indeed, the Heun local solution of the Heun equation (3.38). □

Chapter 4

Ground state of the Jacobi matrix

4.1 Perturbation series

The following section is adopted mainly from books [6] and [8].

4.1.1 Preliminaries

Let $\mathcal{X}$ be a Banach space with $\dim \mathcal{X} < \infty$. Let T be an operator on $\mathcal{X}$. Let us denote the eigenvalues of T by $\mathrm{spec}(T) = \{\lambda_h\}_{h=1}^{s}$. Every eigenvalue is a singular point for the resolvent $R(z)$ of the operator T. Without loss of generelity, let us take $\lambda_h = 0$ and expand $R(z)$ in the Laurent series in a neighbourhood of $\lambda_h = 0$. We have

$$R(z) = \sum_{n=-\infty}^{\infty} z^n A_n, \tag{4.1}$$

where

$$A_n = \frac{1}{2\pi i} \int_\Gamma z^{-n-1} R(z) \mathrm{d}z, \tag{4.2}$$

with Γ being a circle enclosing $\lambda_h = 0$ but no other eigenvalue. In (4.2), one can integrate over slightly smaller circle Γ'without changing the result. Thus

$$A_n A_m = \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \int_{\Gamma'} z^{-n-1} w^{-m-1} R(z) R(w) \mathrm{d}z \mathrm{d}w = (\eta_n + \eta_m - 1) A_{n+m+1}, \tag{4.3}$$

where

$$\eta_n = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} .$$

Since $A_{-1}^2 = -A_{-1}, -A_{-1}$ is a projection. Let us denote

$$P = -A_{-1} = -\frac{1}{2\pi i} \int_\Gamma R(z) \mathrm{d}z.$$

Letting

$$D := A_{-2}, \qquad S := A_0,$$

one gets from equation (4.3)

$$A_{-k} = -D^{k-1}, \text{ for } k \geq 2, \qquad A_k = S^{k+1}, \text{ for } k \geq 0.$$

Thus the expression (4.1) can be rewritten as

$$R(z) = -\frac{1}{z}P - \sum_{n=1}^{\infty} z^{-n-1}D^n + \sum_{n=0}^{\infty} z^n S^{n+1}.$$

For a general λ_h, not necessarily equal to 0, we have

$$R(z) = -\frac{1}{z-\lambda_h}P_h - \sum_{n=1}^{\infty}(z-\lambda_h)^{-n-1}D_h^n + \sum_{n=0}^{\infty}(z-\lambda_h)^n S_h^{n+1}. \tag{4.4}$$

From (4.3) one can conclude that

$$P_hD_h = D_hP_h = D_h, \quad P_hS_h = S_hP_h = 0.$$

Expression (4.4) can be seen as a decomposition of the resolvent $R(z)$ according to decomposition of the Banach space $\mathcal{X} = M_h \dotplus M_h'$, where

$$M_h = P_h\mathcal{X}, \quad M_h' = (1-P_h)\mathcal{X}.$$

Proposition 4.1.1.0.1. Operator T on $\mathcal{X}$ is nilpotent if and only if it's spectral radius is zero.

Since $R(z)$ converges everywhere in M_h and for all z except for $z = \lambda_h$, spectral radius of D_h must be zero and, regarding the proposition 4.1.1.0.1, D_h is nilpotent.

For $h, k \in \{1, \dots, s\}$ s. t. $h \neq k$ the following holds true

$$P_hP_k = \delta_{h,k}P_h, \quad \sum_{h=1}^{s} P_h = 1, \quad P_hT = TP_h.$$

Multiplying the integral in the equation (4.2) from the left or from the right by operator T it is readily seen that

$$A_nT = TA_n = \delta_{n,0} + A_{n-1}.$$

In particular, one has

$$(T-\lambda_h)P_h = P_h(T-\lambda_h) = D_h, \quad (T-\lambda_h)S_h = S_h(T-\lambda_h) = 1 - P_h. \tag{4.5}$$

Proposition 4.1.1.0.2. For $k, h \in \{1, \dots, s\}$

$$P_hD_h = D_hP_h = \delta_{h,k}D_h, \quad D_hD_k = 0, \text{ for } h \neq k.$$

Let us denote, for $h = 1, \dots, s$, $M_h = \mathrm{Ran}P_h$. One has $\mathcal{X} = M_1 \dotplus M_2 \cdots \dotplus M_s$. M_h are invariant subspaces for T.

Definition 4.1.1.0.3. For any $h \in \{1, \dots s\}$, M_h is called *algebraic subspace* of the operator T corresponding to the eigenvalue λ_h. P_h is called *eigenprojection* corresponding to the eigenvalue λ_h. Any non-zero vector $u \in M_h$ is called *generalized eigenvector* corresponding to the eigenvalue λ_h.

From (4.5) one has

$$T_{M_h} = TP_h^2 = P_hTP_h = \lambda_h P_h + D_h. \tag{4.6}$$

Now, taking sum from 1 to s in the expression (4.6), we have

$$T = S + D, \tag{4.7}$$

with

$$S = \sum_{h=1}^{s} \lambda_h P_h, \qquad D = \sum_{h=1}^{s} D_h. \tag{4.8}$$

According to the proposition 4.1.1.0.2 one has

$$D^n = \sum_{h=0}^{s} D_h^n = 0, \text{ for } n \geq \max_{1 \leq h \leq s} m_h.$$

The preceding shows that any operator T on $\mathcal{X}$ can be expressed as a sum of diagonalizable operator and nilpotent. This expression is unique in the following sense.

Theorem 4.1.1.0.4. Let T be expressed as a sum of diagonalizable operator S and nilpotent D which commutes with S. Then S and D must obey relations from the equation (4.8).

Definition 4.1.1.0.5. Expressions (4.7) and (4.8) are called *the spectral representation* of the operator T. An eigenvalue λ_h is said to be *semisiple* if $D_h = 0$ and is said to be *simple* if $m_h = 1$.

Let us note that $m_h = 1$ implies $D_h = 0$.

Let us consider operator on a Banach space $\mathcal{X}$ in the form

$$T(k) = T + kT^{(1)} + k^2T^{(2)} + \dots \tag{4.9}$$

T is called *unperturbed operator* and $kT^{(1)} + k^2T^{(2)} + \dots$ is called *perturbation*. According to [6], the number of eigenvalues of $T(k)$ is a constant s independent of k up to some special values of k. There are only a finite number of such *exceptional points* k in each compact subset of D_0, where D_0 is a set of all possible values of k. Recall that resolvent of $T(k)$ is defined as

$$R(k,z) = (T(k) - z)^{-1},$$

where z lies in the resolvent set of $T(k)$. Let us denote

$$A(k) = kT^{(1)} + k^2T^{(2)} + \dots,$$

It is convenient to write $R(k,z)$ as a power series in k with coefficients depending on z. This reads

$$\begin{aligned} R(k,z) &= R(z)\left(1 + A(k)R(z)\right)^{-1} \qquad (4.10)\\ &= R(z)\sum_{p=0}^{\infty} \left(-A(k)R(z)\right)^p \\ &= R(z) + \sum_{n=1}^{\infty} k^n R^{(n)}(z), \end{aligned}$$

where

$$R^{(n)}(z) = \sum_{\substack{n_1+\cdots+n_p=n \\ n_j\geq 1}} (-1)^p R(z)T^{(n_1)}R(z)T^{(n_2)}\ldots T^{(n_p)}R(z).$$

According to [6], series above is convergent for sufficiently small k and $z \in \Gamma$ if Γ is a compact subset of the resolvent set of $\rho(T)$ with $T = T(0)$. Let λ be one of the eigenvalues of $T = T(0)$ with algebraic multiplicity m. Let Γ be a positively-oriented circle in the resolvent set $\rho(T)$ enclosing λ and no other eigenvalue of T. The operator

$$P(k) = -\frac{1}{2\pi i}\int_\Gamma R(k,z)\mathrm{d}z$$

is a projection and is equal to the sum of eigenprojections for all the eigenvalues of $T(k)$ lying inside Γ ([6]). Integrating (4.10) term by term, one has

$$P(k) = P + \sum_{n=1}^{\infty} k^n P^{(n)} \tag{4.11}$$

with

$$P^{(n)} = -\frac{1}{2\pi i}\int_\Gamma R^{(n)}(z)\mathrm{d}z. \tag{4.12}$$

Lemma 4.1.1.0.6. Let $P(t)$ be a projection depending continuously on a parameter t varying in a connected region of complex numbers. Then the ranges $P(t)\mathcal{X}$ for different t are isomorphic to one another. In particular, $\dim P(t)$ is constant.

The series (4.11) converges for small $|k|$. It follows form the lemma 4.1.1.0.6 that the range $M(k) := P(k)\mathcal{X}$ is isomorphic with the algebraic eigenspace $M = M(0) = P\mathcal{X}$ of T for the eigenvalue λ. In particular,

$$\dim P(k) = \dim P = m.$$

Symbol $\{\lambda_h(k)\}_{h=1}^s$ again denotes the set of all eigenvalues of $T(k)$. With additional notation $M_h(k) := P_h(k)\mathcal{X}$, we have

$$\mathcal{X} = M_1(k) \dotplus M_2(k) \dotplus \cdots \dotplus M_s(k), \quad \dim M_h(k) = m_h, \quad \sum_{h=1}^{s} m_h = \dim \mathcal{X}.$$

From (4.5), the eigennilpotent for the eigenvalue $\lambda_h(k)$ is given by

$$D_h(k) = (T(k) - \lambda_h(k))P_h(k). \tag{4.13}$$

4.1.2 Perturbation series

Our starting point will be the power series for $T(k)$ given by (4.9). Let λ be one of the eigenvalues of the unperturbed operator $T = T(0)$ with algebraic multiplicity m and let P and D be the associated eigenprojection and eigennilpotent. The eigenvalue λ is in general split into several eigenvalues of $T(k)$, the total projection $P(k)$ for these is holomorphic at $k = 0$ and is given by

$$P(k) = \sum_{n=0}^{\infty} k^n P^{(n)}, \quad P^{(0)} = P, \tag{4.14}$$

with $P^{(n)}$ given by (4.12). From now on, suppose there is no splitting of the eigenvalue λ. In particular it is always true if $m = 1$. In order to determine the eigenvalue of $T(k)$ associated with λ, it is enough to solve the eigenvalue problem in the subspace $M(k) = P(k)\mathcal{X}$. This is equivalent to the eigenvalue problem for the operator

$$T_r(k) = T(k)P(k) = P(k)T(k) = P(k)T(k)P(k).$$

Thus

$$\lambda(k) = \frac{1}{m}\mathrm{tr}\left(T(k)P(k)\right) = \lambda + \frac{1}{m}\mathrm{tr}\left((T(k) - \lambda)P(k)\right). \tag{4.15}$$

Equations (4.14) for the eigenprojection, (4.15) for the eigenvalue and (4.13) fully describe the eigenvalue problem for $T(k)$. Now we will give an explicit form for those series. The coefficients of the series (4.14) are given by

$$P^{(n)} = -\frac{1}{2\pi i} \sum_{\substack{n_1+\dots+n_p=n \\ n_j \geq 1}} (-1)^p \int_\Gamma R(z)T^{(n_1)} \dots T^{(n_p)} R(z)\mathrm{d}z, \tag{4.16}$$

where Γ is a small, positively-oriented circle around λ. To evaluate the integral above, instead of $R(z)$ we will substitute it's Laurent series (4.4) at $z = \lambda$, for convenience it will be written in the form

$$R(z) = \sum_{n=-m}^{\infty} (z - \lambda)^n S^{(n+1)}, \tag{4.17}$$

with

$$S^{(0)} = -P, \quad S^{(n)} = S^n, \quad S^{(-n)} = -D^n, \quad n \geq 1.$$

Substituting (4.17) into the integrand of (4.16) one has

$$P^{(n)} = -\frac{1}{2\pi i} \sum_{p=1}^{n} (-1)^p \sum_{\substack{n_1+\dots n_p=n \\ k_1+\dots+k_{p+1}=p \\ n_j \geq 1, k_j \geq -m+1}} S^{(k_1)}T^{(n_1)}S^{(k_2)} \dots S^{(k_p)}T^{(n_p)}S^{(k_{p+1})}. \tag{4.18}$$

The just described results were derived without knowing that the series in question has a nonzero radius of convergence. The following Kato-Rellich Theorem taken from [8] will justify all the above results.

Definition 4.1.2.0.1. An operator-valued function $T(k)$ on a complex domain D is called an *analytic family* if and only if

1. for each $k \in D$, $T(k)$ is closed and has nonempty resolvent set,

2. for every k_0, there is a $\lambda_0 \in \rho(T(k_0))$ so that $\lambda_0 \in \rho(T(k))$ for k near k_0 and $(T(k) - \lambda_0)^{-1}$ is analytic operator-valued function of k near k_0.

Theorem 4.1.2.0.2. Let $T(k)$ be an analytic family. Let λ_0 be a nondegenerate discrete eigenvalue of $T(k_0)$. Then, for k near k_0, there is exactly one point $\lambda(k)$ of spec$(T(k))$ near λ_0 and this point is isolated and nondegenerate. $\lambda(k)$ is an analytic function of k for k near k_0.

Note that just derived results hold also for operators on an infinite-dimensional Banach space if λ_h is an isolated point of the sprectrum and the projection is finite-dimensional.

4.2 Implicit function approach to the perturbation series

In the following section some preliminaries will be needed. We begin with recalling several well-known propostions.

Proposition 4.2.0.0.1. Let $\mathcal{X}$ be a Banach space and let $A \in \mathcal{B}(\mathcal{X})$ such that $\|A\| < 1$. Then $(I - A)^{-1} \in \mathcal{B}(\mathcal{X})$ and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k,$$

with the series being convergent in $\mathcal{B}(\mathcal{X})$. Moreover,

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}$$

Proposition 4.2.0.0.2. Let $\mathcal{X}$ be a Banach space, let A be an operator on $\mathcal{X}$, let λ be an element of the resolvent set of A which will be denoted by $\varrho(A)$. Let us denote by R_λ the resolvent of the operator A evaluated at λ. Then

1. $B\left(\lambda, \frac{1}{\|R_\lambda\|}\right) \subset \varrho(A)$,

2. for each $\mu \in B\left(\lambda, \frac{1}{\|R_\lambda\|}\right)$ the operator-valued function R_μ can be expressed in the form of a power series

$$R_\mu = \sum_{n=0}^{\infty} (\mu - \lambda)^n R_\lambda^{n+1}.$$

This means that $R.$ is an analytic function in the neighbourhood of the point λ.

The implicit function theorem for analytic functions taken from [7] will be useful for our purposes.

Theorem 4.2.0.0.3. Let $F(z, w)$ be a function of two complex variables which is analytic in a neighbourhood of the point (z_0, w_0) and suppose that

$$F(z_0, w_0) = 0, \quad \partial_w F(z_0, w_0) \neq 0.$$

Then there are neighbourhoods $N(z_0)$ and $N(w_0)$ such that equation

$$F(z, w) = 0$$

has a unique root $w = w(z)$ in $N(w_0)$ for any given $z \in N(z_0)$. Moreover, the function $w = w(z)$ is single-valued and analyitic in $N(z_0)$ and satisfies the condition

$$w(z_0) = w_0.$$

Now we are ready to summarize some results from [5]. Let $\mathcal{H}$ be a Hilbert space. Let K be a self-adjoint operator on $\mathcal{H}$ and let $F \in \mathbb{R}$ be an isolated simple eigenvalue of K, i.e.

$$\operatorname{dist}(F, \operatorname{spec}(K) \setminus \{F\}) =: d > 0.$$

Throughout this section, f will denote a normalized eigenvector of K corresponding to the eigenvalue F. Also, let P be an orthogonal projection on $\mathbb{C}f$ and $Q := I - P$. Let V_1, V_2 be symmetric operators on $\mathcal{H}$. We are going to discuss operator $K + kV_1 + k^2V_2$, where k is a real parameter. In particular, we focus on the question whether an eigenvalue $F(k)$ of $K+kV_1+k^2V_2$ is in some sense inherited from the eigenvalue F of K.

For any operator A on $\mathcal{H}$ we will denote

$$\tilde{A} := QAQ : \operatorname{Dom}(A) \cap \operatorname{Ran}(Q) \to \operatorname{Ran}(Q).$$

Clearly, $\left(\tilde{K} - F\right)^{-1}$ is self-adjoint. It follows from the fact that for any operator A on $\mathcal{H}$ the relation

$$(A^*)^{-1} = \left(A^{-1}\right)^*$$

holds true if A^{-1} exists and is densily defined. Next we need $\tilde{V}_1$ and $\tilde{V}_2$ to be *relatively bounded bounded with respect* to the operator $\tilde{K}$. It means:

Definition 4.2.0.0.4. Under the same assumption as in the preceding text, an operator $\tilde{V}$ on Hilbert space $\mathcal{H}$ is called relatively bounded with respect to the operator K if the following two conditions are obeyed at the same time

1. $\operatorname{Dom}\tilde{K} \subset \operatorname{Dom}\tilde{V}$,
2. $\tilde{V}\left(\tilde{K} - F\right)^{-1}$ is bounded.

By regular perturbation theory, it is possible to express $F(k)$ and $f(k)$ as power series in the parameter k, i.e.

$$F(k) = F + k\lambda_1 + k^2\lambda_2 + \dots$$
$$f(k) = f + kg_1 + k^2g_2 + \dots,$$

with $\lambda_i \in \mathbb{R}, g_i \in \mathcal{H}$. $F(k)$ is the only eigenvalue near F for k near 0. Adding normalizing condition

$$\langle f, f(k)\rangle = 1,$$

equivalently $f(k) - f \in \operatorname{Ran}(Q)$, thus coefficients g_i are necessarily elements of $\operatorname{Ran}(Q)$.

Let us consider an eigenvalue for $K + kV_1 + k^2V_2$ in the form $F + \lambda$, where $\lambda \in \mathbb{R}$. Similarly for and eigenvector $f + g$ corresponding to the eigenvalue $F + \lambda$, where $g \in \operatorname{Ran}(Q)$ due to the normalization condition. Hence the equation

$$(K + kV_1 + k^2V_2)(f + g) = (F + \lambda)(f + g) \tag{4.19}$$

should be fullfilled. It is clear that for any vectors $u, v \in \mathcal{H}$ equation $u = v$ holds if and only if $Pu = Pv$ and $Qu = Qv$. Applying projection P on the equation (4.19) one obtains

$$\lambda f = P(kV_1 + k^2V_2)f + P(kV_1 + k^2V_2)g.$$

Note that $KP = PK$. Next we will apply the scalar product with f. Since V is symmetric and $P = P^*$, one gets

$$\lambda = \langle (kV_1 + k^2V_2)f, f\rangle + \langle (kV_1 + k^2V_2)f, g\rangle$$

Analogously, we will apply the projection Q in equation (4.19). Since $g = Qg$ we obtain

$$\left(\tilde{K} + k\tilde{V}_1 + k^2\tilde{V}_2 - F - \lambda\right) g = -Q(kV_1 + k^2V_2)f.$$

In summary, equation (4.19) is equivalent to the equations

$$\lambda = \langle (kV_1 + k^2V_2)f, f\rangle + \langle (kV_1 + k^2V_2)f, g\rangle \tag{4.20}$$

and

$$\left(\tilde{K} + k\tilde{V}_1 + k^2\tilde{V}_2 - F - \lambda\right) g = -Q(kV_1 + k^2V_2)f. \tag{4.21}$$

Regarding λ as another auxiliary parameter, our goal is to express $g = g(k, \lambda)$ and by plugging this expression into equation (4.20) to obtain an implicit equation $\lambda = G(k, \lambda)$. Let us introduce some additional notation. Set

$$\begin{aligned}\Gamma_0 &:= \left(\tilde{K} - F\right)^{-1},\\ \Gamma_\lambda &:= \left(\tilde{K} - F - \lambda\right)^{-1} = (I - \lambda\Gamma_0)^{-1}\,\Gamma_0.\end{aligned}$$

Thus Γ_0 is a self-adjoint operator acting in $\mathrm{Ran}(Q)$, so is Γ_λ if $\lambda \notin \mathrm{spec}\left(\tilde{K} - F\right)$. Moreover, Γ_0 has the following property.

Proposition 4.2.0.0.5. Under the same assumptions and notation as above,

$$\|\Gamma_0\| = \frac{1}{d}.$$

Proof. According to the proposition 4.2.0.0.2, $B\left(F, \frac{1}{\|\Gamma_0\|}\right) \subset \varrho(\tilde{K})$. Thus $\frac{1}{d} \leq \|\Gamma_0\|$. Conversely, $\mathrm{spec}(\tilde{K} - F) \cap B(0, d) = \emptyset$, thus we have

$$\mathrm{spec}(\Gamma_0) \subset \left\{\frac{1}{\lambda}; \lambda \in \mathrm{spec}\left(\tilde{K} - F\right)\right\} \cup \{0\} \subset \overline{B\left(0, \frac{1}{d}\right)}.$$

Γ_0 is self-adjoint and therefore

$$\|\Gamma_0\| = r_\sigma(\Gamma_0) \leq \frac{1}{d}.$$

Here $r_\sigma(\Gamma_0)$ stands for the spectral radius of the operator Γ_0. We conclude that $\|\Gamma_0\| = \frac{1}{d}$. □

In a view of equation (4.21), one needs to invert the operator

$$\tilde{K} + k\tilde{V}_1 + k^2\tilde{V}_2 - F - \lambda = \left(I + (k\tilde{V}_1 + k^2\tilde{V}_2)\Gamma_\lambda\right)\left(\tilde{K} - F - \lambda\right).$$

First bracket is obviously invertible since our assumption is that $|\lambda| < d$. Thus it is enough to show that the second bracket is invertible as well. Again, according to the proposition 4.2.0.0.1, it suffices that $\|\Gamma_\lambda(k\tilde{V} + k^2\tilde{V}_2)\| < 1$. We have

$$(k\tilde{V}_1 + k^2\tilde{V}_2)\Gamma_\lambda = (k\tilde{V}_1 + k^2\tilde{V}_2)\Gamma_0(I - \lambda\Gamma_0)^{-1}$$

By corollary after proposition 4.2.0.0.1 one has

$$\|(k\tilde{V}_1 + k^2\tilde{V}_2)\Gamma_\lambda\| \leq \left(|k|\|\tilde{V}_1\Gamma_0\| + |k|^2\|\tilde{V}_2\Gamma_0\|\right)\frac{1}{1 - \frac{|\lambda|}{d}}$$

which we want to be less than 1. Thus λ and k are supposed to obey the relation

$$d|k|\|\tilde{V}_1\Gamma_0\| + d|k|^2\|\tilde{V}_2\Gamma_0\| + |\lambda| < d \tag{4.22}$$

Consequently, there exists a unique solution to (4.21), given by

$$g(k,\lambda) = -\Gamma_\lambda\left(1 + (k\tilde{V}_1 + k^2\tilde{V}_2)\Gamma_\lambda\right)^{-1} Q(kV_1 + k^2V_2)f.$$

Function $g(k,\lambda)$ is obviously analytic in the domain (4.22). Plugging this result in to the equation (4.20), we obtain implicit equation

$$\lambda = G(k,\lambda),$$

where

$$G(k,\lambda) = \langle (kV_1 + k^2V_2)f, f\rangle - \left\langle Q(kV_1 + k^2V_2)f, \Gamma_\lambda\left(1 + (k\tilde{V}_1 + k^2\tilde{V}_2)\Gamma_\lambda\right)^{-1} Q(kV_1 + k^2V_2)f\right\rangle.$$

Since $\lambda - G(k,\lambda)$ is analytic and

$$(\lambda - G(k,\lambda))|_{(k,\lambda)=(0,0)} = 0, \qquad \partial_\lambda(\lambda - G(k,\lambda))|_{(k,\lambda)=(0,0)} = 1,$$

we have by theorem 4.2.0.0.3 that there exists a unique analytic function $\lambda = \lambda(k)$ defined on a neighbourhood of the origin such that $\lambda(0) = 0$ and $\lambda(k) = G(k,\lambda(k))$. Thus we have both eigen-value and eigen-vector given as uniquely determined analytic functions.

4.3 Ground state for the Jacobi matrix

Let us again consider Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} \beta_0 & \alpha_0 & 0 & 0 & 0 & \dots \\ \alpha_0 & \beta_1 & \alpha_1 & 0 & 0 & \dots \\ 0 & \alpha_1 & \beta_2 & \alpha_2 & 0 & \dots \\ 0 & 0 & \alpha_2 & \beta_3 & \alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.23}$$

with

$$\alpha_n := k\sqrt{(n+1)(n+\alpha)(n+\beta)(n+\gamma)},$$
$$\beta_n := k^2 n(n+\alpha-1) + (n+\beta)(n+\gamma),$$

where we take $k \in (0,1)$ and $\alpha, \beta, \gamma > 0$. As was described in the preceeding text, one can define operator $\dot{J}$ on the hull of cannonical basis $\{\mathbf{e}_n\}_{n=0}^{\infty}$ of the space ℓ^2. This operator is essentially self-adjoint. Since Jacobi matrix from equation (4.23) can be associated with determinate Hamburger moment problem, there exisist unique closed, self-adjoint extension J of the operator $\dot{J}$. Moreover, J^{-1} is a trace-class operator, as was shown above. Spectrum of the closure satisfies

$$\operatorname{spec}(J) = \operatorname{spec}_p(J) = \{\lambda_j; j \in \mathbb{N}_0\}.$$

We would like to apply the theory derived in [5] on the operator J which can be seen the following sum

$$J = J_0 + kJ_1 + k^2 J_2 \tag{4.24}$$

with the corresponding matrices

$$\begin{aligned}\mathcal{J}_0 &= \operatorname{diag}\{(n+\beta)(n+\gamma)\}_{n=0}^{\infty},\\ \mathcal{J}_2 &= \operatorname{diag}\{n(n+\alpha-1)\}_{n=0}^{\infty},\end{aligned}$$

and

$$\mathcal{J}_1 = \frac{1}{k}\begin{pmatrix} 0 & \alpha_0 & 0 & 0 & 0 & \dots \\ \alpha_0 & 0 & \alpha_1 & 0 & 0 & \dots \\ 0 & \alpha_1 & 0 & \alpha_2 & 0 & \dots \\ 0 & 0 & \alpha_2 & 0 & \alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For brevity, from now on we will identify operators J_* with the corresponding matrix $\mathcal{J}_*$ whenever convenient. It is clear that the ground state of the operator J_0 is $\lambda_0 = \beta\gamma$ with the multiplicity 1 and with normalized eigenvector $\mathbf{e}_0$. Let us denote a projection on the space $\mathbb{C}\mathbf{e}_0$ by P. Next, $Q := I - P$. We need to show that operators $\tilde{J}_j = QJ_jQ$ are relatively bounded with respect to the operator J_0 for $j = 1, 2$. This means to show that $\left\|\tilde{J}_j\Gamma_0\right\| < \infty$ with

$$\Gamma_0 = \left(\tilde{J}_0 - \lambda_0\right)^{-1}.$$

In our case

$$\Gamma_0 = \operatorname{diag}\left\{0, \frac{1}{1+\beta+\gamma}, \dots, \frac{1}{n^2+(\beta+\gamma)n}, \dots\right\}$$

Thus

$$\tilde{J}_2\Gamma_0 = \operatorname{diag}\left\{\frac{n(n+\alpha-1)}{n^2+(\beta+\gamma)n}\right\}_{n=0}^{\infty}.$$

We have

$$\|\tilde{J}_2\Gamma_0\|^2 = \sup_{\substack{\mathbf{f}\in\ell^2\\ \|\mathbf{f}\|=1}} \|\tilde{J}_2\Gamma_0 f\|^2 = \sup_{\substack{\mathbf{f}\in\ell^2\\ \|\mathbf{f}\|=1}} \sum_{n=1}^{\infty}\left|\frac{n(n+\alpha-1)}{n^2+(\beta+\gamma)n} f_n\right|^2 \leq M^2 \sup_{\substack{\mathbf{f}\in\ell^2\\ \|\mathbf{f}\|=1}} \sum_{n=0}^{\infty}|f_n|^2 = M^2 < \infty,$$

where $M = \sup_{n\in\mathbb{N}} \left|\frac{n(n+\alpha-1)}{(n+\beta)(n+\gamma)}\right| < \infty$ as the sequence $\left\{\frac{n(n+\alpha-1)}{(n+\beta)(n+\gamma)}\right\}$ is bounded.

For the second operator one has

$$\Gamma_0\tilde{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_1 & 0 & \dots \\ 0 & b_1 & 0 & a_2 & \dots \\ 0 & 0 & b_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $a_n = \frac{\sqrt{(n+1)(n+\alpha)(n+\beta)(n+\gamma)}}{(n+1)^2+(\beta+\gamma)(n+1)}$ and $b_n = \frac{\sqrt{(n+1)(n+\alpha)(n+\beta)(n+\gamma)}}{n^2+(\beta+\gamma)n}$. Note that both sequences $\{a_n\}$ and $\{b_n\}$ are positive and bounded. Let us denote their suprema respectively $a, b < \infty$. Let us take $\mathbf{f} \in \ell^2$ and compute

$$\begin{aligned}\left\|\tilde{J}_1\Gamma_0\mathbf{f}\right\|^2 &= |a_1 f_2|^2 + \sum_{n=1}^{\infty} |b_n f_n + a_{n+1} f_{n+2}|^2 \\ &\leq a^2\|\mathbf{f}\|^2 + \sum_{n=1}^{\infty} b_n |f_n|^2 + \sum_{n=1}^{\infty} a_{n+1}|f_{n+2}|^2 + 2\sum_{n=1}^{\infty} a_{n+1} b_n |f_n||f_{n+2}| \\ &\leq a^2\|\mathbf{f}\|^2 + b^2\sum_{n=1}^{\infty}|f_n|^2 + a^2\sum_{n=1}^{\infty}|f_{n+2}|^2 + 2ab\left(\sum_{n=1}^{\infty}|f_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}|f_{n+2}|^2\right)^{\frac{1}{2}} \\ &\leq \left(2a^2 + 2ab + b^2\right)\|\mathbf{f}\|.\end{aligned}$$

Our goal is to find the ground state λ_0 for which we have

$$\lambda_0 = \beta\gamma + \eta$$

with $\eta = \eta(k)$ being perturbation depending on the perturbation parameter k. According to *DuStoVit*, η is a unique solution of the implicit equation

$$\eta = G(k, \eta), \qquad \eta(0) = 0,$$

where

$$\begin{aligned}G(k,\eta) &= \left\langle \left(kJ_1 + k^2 J_2\right)\mathbf{e}_0, \mathbf{e}_0\right\rangle \\ &- \left\langle Q\left(kJ_1 + k^2J^2\right)\mathbf{e}_0, \Gamma(\eta)\left(I + Q\left(kJ_1 + k^2J_2\right)Q\Gamma(\eta)\right)^{-1} Q\left(kJ_1 + k^2J_2\right)\mathbf{e}_0\right\rangle\end{aligned} \tag{4.25}$$

and

$$\Gamma(\eta) = \left(\tilde{J}_0 - \beta\gamma - \eta\right)^{-1}.$$

It is easy to find out that

$$Q\Gamma(\eta) = \operatorname{diag}\left\{0, \frac{1}{1+\beta+\gamma}, \dots \frac{1}{n^2+(\beta+\gamma)n-\eta}, \dots\right\}. \tag{4.26}$$

We have

$$(kJ_1 + k^2J_2)\mathbf{e}_0 = kJ_1\mathbf{e}_0 = k\sqrt{\alpha\beta\gamma}\mathbf{e}_1,$$

hence

$$\left\langle \left(kJ_1 + k^2 J_2\right) \mathbf{e}_0, \mathbf{e}_0 \right\rangle = k \left\langle J_1 \mathbf{e}_0, \mathbf{e}_0 \right\rangle = k\sqrt{\alpha\beta\gamma} \left\langle \mathbf{e}_1, \mathbf{e}_0 \right\rangle = 0.$$

Moreover, the operator $Q(kJ_1 + k^2 J_2)Q$ can be regarded as an operator

$$\mathrm{Ran} Q = \mathcal{H}_1$$

with $\mathcal{H}_1$ being spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \dots\}$. Let us denote the restriction $Q(kJ_1 + k^2 J_2)Q$ to the subspace $\mathcal{H}_1$ by $W_1(k)$. Using basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots\}$ in $\mathcal{H}_1$ we can regard $W_1(k)$ as the tridiagonal matrix

$$\begin{pmatrix} k^2\alpha & k\sqrt{2(\alpha+1)(\beta+1)(\gamma+1)} & 0 & \dots \\ k\sqrt{2(\alpha+1)(\beta+1)(\gamma+1)} & 2k^2(\alpha+1) & k\sqrt{3(\alpha+2)(\beta+2)(\gamma+2)} & \dots \\ 0 & k\sqrt{3(\alpha+2)(\beta+2)(\gamma+2)} & 3k^2(\alpha+2) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Next, let us denote the restriction of $\Gamma(\eta)$ to the subspace $\mathcal{H}_1$ by $\Gamma_1(\eta)$. Explicitly,

$$\Gamma_1(\eta) = \mathrm{diag}\left\{ \frac{1}{n^2 + (\beta+\gamma)n - \eta} \right\}_{n=1}^{\infty}.$$

Furthemore, $\mathbf{f}_1 \in \mathcal{H}_1$ is a column vector

$$\mathbf{f}_1^T = (1, 0, 0, \dots).$$

With these notations we can rewrite equation (4.25) as

$$G(k,\eta) = -\frac{\alpha\beta\gamma k^2}{1+\beta+\gamma-\eta} \left\langle \mathbf{f}_1, \left(I + W_1(k)\Gamma_1(\eta)\right)^{-1} \mathbf{f}_1 \right\rangle. \tag{4.27}$$

Let k and η be chosen so that condition (4.22) holds with $\eta = \lambda$. Since that, we can express equation (4.27) in terms of the Neumann series of the operator $(I + \Gamma_1(\eta)W_1(k))^{-1}$. It says

$$\begin{aligned} G(k,\eta) &= -\frac{\alpha\beta\gamma k^2}{1+\beta+\gamma-\eta} \left\langle \mathbf{f}_1, I + \sum_{i=1}^{\infty} (-1)^i \left(W_1(k)\Gamma_1(\eta)\right)^i \mathbf{f}_1 \right\rangle \\ &= -\frac{\alpha\beta\gamma k^2}{1+\beta+\gamma-\eta} \left(1 + \sum_{i=1}^{\infty} (-1)^i \left\langle \mathbf{f}_1, \left(W_1(k)\Gamma_1(\eta)\right)^i \mathbf{f}_1 \right\rangle \right). \end{aligned} \tag{4.28}$$

Let us compare the implicit function approach for the ground state with Kato's theory. Again, our aim is to find the ground state for the operator (4.24). A particular simplification is possible for coefficients (4.18) in the series (4.17), since the ground state is a simple eigenvalue. We will find the perturbation series and compare it with results of the previous method up to the order k^2. Thus we have for our operator

$$\begin{aligned} P^{(1)} &= -PJ_1S - SJ_1P \\ P^{(2)} &= -PJ_2S - SJ_2P + PJ_1SJ_1S + SJ_1PJ_1S \\ &\quad + SJ_1SJ_1P - PJ_1PJ_1S^2 - PJ_1S^2J_1P - S^2J_1PJ_1P \end{aligned} \tag{4.29}$$

with P being the projection on the eigenspace for the ground state $\lambda_0 = \beta\gamma$ and $S = S(0)$ where $S(z)$ is the so-called reduced resolvent. In particular, we have that projection P can be represented by a matrix

$$P = \text{diag}\{1, 0, 0, \dots\}.$$

Operator S can be found by expanding the resolvent $(J_0 - \beta\gamma - z)^{-1}$ in the Laurent series for $z \in \varrho(J_0)$. Since $\beta\gamma$ is a simple eigenvalue, one has

$$(J_0 - \beta\gamma - z)^{-1} = -\frac{1}{z}P + \sum_{n=0}^{\infty} z^n S^{n+1} \tag{4.30}$$

On the other hand, it is straighforward to check that

$$(J_0 - \beta\gamma - z)^{-1} = \text{diag}\left\{-\frac{1}{z}, \frac{1}{1+\beta+\gamma-z}, \dots, \frac{1}{n^2+(\beta+\gamma)n-z}, \dots\right\}. \tag{4.31}$$

Comparing (4.30) with (4.31) and $D = 0$ gives

$$S(z) = \sum_{n=0}^{\infty} z^n S^{n+1} = \text{diag}\left\{0, \frac{1}{1+\beta+\gamma-z}, \dots, \frac{1}{n^2+(\beta+\gamma)n-z}, \dots\right\}.$$

Thus

$$S = S(0) = \text{diag}\left\{0, \frac{1}{1+\beta+\gamma}, \dots, \frac{1}{n^2+(\beta+\gamma)n}, \dots\right\}.$$

Note that $Q\Gamma(z)$ given by (4.26) is exactly the reduced resolvent. Determining P and S, one can compute coefficients (4.29)

$$P^{(1)} = -\frac{\sqrt{\alpha\beta\gamma}}{1+\beta+\gamma}\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$P^{(2)} = \frac{\sqrt{\alpha(\alpha+1)\beta(\beta+1)\gamma(\gamma+1)}}{(1+\beta+\gamma)(4+2\beta+2\gamma}\begin{pmatrix} 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$+ \frac{\alpha\beta\gamma}{1+\beta+\gamma}\begin{pmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

With these coefficients, we plug the expression

$$P(k) = P + kP^{(1)} + k^2P^{(2)} + O\left(k^3\right), \quad \text{for } k \to 0$$

into equation (4.15). Perturbed eigenvalue thus reads

$$\lambda(k) = \beta\gamma - \frac{\alpha\beta\gamma}{1+\beta+\gamma}k^2 + O\left(k^4\right), \quad \text{for } k \to 0. \tag{4.32}$$

Now, having computed series for $\lambda(k)$, let us turn back to the solution of equation (4.28). This will be provided by iterations

$$\eta_0 = 0, \qquad \eta_j = G(k, \eta_{j-1}), \quad \text{for } j \geq 1.$$

We thus have

$$\begin{aligned}
\eta_1 = G(k,0) &= -\frac{\alpha\beta\gamma k^2}{1+\beta+\gamma}\left(1 + \sum_{i=1}^{\infty}(-1)^i \left\langle \mathbf{f}_1, \left(W_1(k)\Gamma_1(0)\right)^i \mathbf{f}_1 \right\rangle\right) \\
&= -\frac{\alpha\beta\gamma k^2}{1+\beta+\gamma}\left(1 - \left(W_1(k)\Gamma_1(0)\right)_{1,1} + \left(\left(W_1(k)\Gamma_1(0)\right)^2\right)_{1,1} + O\left(k^4\right)\right) \\
&= -\frac{\alpha\beta\gamma k^2}{1+\beta+\gamma}\left(1 - \frac{\alpha}{1+\beta+\gamma}k^2 + \frac{2(\alpha+1)(\beta+1)(\gamma+1)}{(1+\beta+\gamma)(4+2\beta+2\gamma)}k^2 + O\left(k^4\right)\right) \\
&= -\frac{\alpha\beta\gamma}{1+\beta+\gamma}k^2 + O\left(k^4\right), \quad \text{for } k \to 0.
\end{aligned}$$

It follows that the first iteration of the perturbed ground state $\lambda_0^{(1)} = \beta\gamma + \eta_1(k)$ reads

$$\lambda_0^{(1)} = \beta\gamma - \frac{\alpha\beta\gamma}{1+\beta+\gamma}k^2 + O\left(k^4\right), \quad \text{for } k \to 0.$$

This expression coincides with the perturbation series in (4.32).

Conclusion

Contribution of this thesis can be devided into two groups

1. solutions of Heun's equation in case o a special, yet quite general form,
2. an approximation of the ground state of a certain Jacobi matrix.

We have found the solution to the Heun's equation of the form,

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}G(z) + \left(\frac{\gamma}{z} - \frac{\beta+1}{1-z} - \frac{\epsilon k^2}{1-k^2 z}\right)\frac{\mathrm{d}}{\mathrm{d}z}G(z) + \frac{x - \beta\gamma + \alpha\beta k^2 z}{z(1-z)(1-k^2 z)}G(z) = 0,$$

which differs from the general Heun's equation just by setting $\delta = \beta + 1$. Firstly, we need to suppose that α, β, γ are positive to apply results from [10]. Via this approach, we have found solution given by (3.39) together with (3.40). Omitting requirement of positivity, range of parameters were extended to $\alpha, \beta, \gamma \in \{z \in \mathbb{C} : \Re(z) > -1 \ \vee \ \Im(z) \neq 0\}$ as shown in theorem 3.2.2.0.1.

Then we apply the classical perturbation theory to the ground state of the Jacobi matrix corresponding to the orthogonal polynomials solving the equation above. This approach was compared with another approach using an implicit function. An implicit function for the ground state was found and by iteration, it turns out that it coincides with the classical perturbation series up to the second order of the perturbation parameter.

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