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ON GENERALIZED HEUN EQUATION WITH SOME MATHEMATICAL PROPERTIES

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Abstract. We study the analytic solutions of the generalized Heun equation, $\big(\alpha_0 + \alpha_1\, r + \alpha_2\, r^2 + \alpha_3\, r^3\big)\, y'' + \big(\beta_0 + \beta_1\, r + \beta_2\, r^2\big)\, y' + \big(\varepsilon_0 + \varepsilon_1\, r\big)\, y = 0$, where $|\alpha_3| + |\beta_2| \neq 0$, and $\{\alpha_i\}_{i=0}^3$, $\{\beta_i\}_{i=0}^2$, $\{\varepsilon_i\}_{i=0}^1$ are real parameters. The existence conditions for the polynomial solutions are given. A simple procedure based on a recurrence relation is introduced to evaluate these polynomial solutions explicitly. For $\alpha_0 = 0$, $\alpha_1 \neq 0$, we prove that the polynomial solutions of the corresponding differential equation are sources of finite sequences of orthogonal polynomials. Several mathematical properties, such as the recurrence relation, Christoffel-Darboux formulas and the norms of these polynomials, are discussed. We shall also show that they exhibit a factorization property that permits the construction of other infinite sequences of orthogonal polynomials.

Keywords: Heun equation, confluent forms of Heun's equation, polynomial solutions, sequences of orthogonal polynomials.

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1. Introduction

It seems as a simple question to ask: *Under what conditions does the differential equation*

$$\pi_3(r)\, y'' + \pi_2(r)\, y' + \pi_1(r)\, y = (\lambda_n + \mu_n \pi_0(r))\, y,$$

where λ_n and μ_n are constants and $\pi_j(r), j = 0, 1, 2, 3$ are polynomials of unknown degree to be found, has n-degree monic polynomial solutions $y_n = \sum_{k=0}^{n} c_k\, r^k$, $c_0 \neq 0$, $c_k = 1$?

A simple approach to deduce the possible degrees of $\pi_j, j = 0, 1, 2, 3$, is to examine the degrees for the (possible) polynomial solutions y_n:

For $n = 0$, $y_0(r) = 1$, we must have $\pi_1(r) = \lambda_0 + \mu_0 \pi_0(r)$ and the degree of the polynomial $\pi_1(r)$ must have the same degree as that of $\pi_0(r)$, so we may combine the same degree polynomial coefficients of y and write the equation as

$$\pi_3(r) y'' + \pi_2(r) y' + \pi_1(r) y = 0.$$

Next, for a polynomial solution of degree one, say $y_1(r) = r + \alpha$, the differential equation reduces to

$$\pi_2(r) + \pi_1(r)(r + \alpha) = 0$$

and the degree of π_2 should be the degree of $\pi_1(r)$ plus one.

Similarly, for a second-order polynomial solution, say $y(r) = r^2 + \alpha\, r + \beta$, it follows by substitution that

$$\pi_3(r) + \pi_2(r)(2r + \alpha) + \pi_1(r)(r^2 + \alpha r + \beta) = 0$$

which indicates that the degree of $\pi_3(r)$ should be the degree of π_2 plus one, which, in turn, is a polynomial of π_1 degree plus one.

This simple argument shows that for the polynomial solutions of the linear second-order differential equation with polynomial coefficients, the degree of the polynomial coefficients $\pi_j(r), j = 3, 2, 1$ must be of degree n, $n - 1$

and $n-2$, respectively. So, without the loss of generality, we may direct out attention to the following: *Under what conditions on the equation parameters α_k, β_k, and ε_k, for $k = 0, 1, \ldots, n$, does the differential equation*

$$\left(\sum_{k=0}^{n}\alpha_k\, r^k\right) y''(r) + \left(\sum_{k=0}^{n-1}\beta_k\, r^k\right) y'(r) + \left(\sum_{k=0}^{n-2}\varepsilon_k\, r^k\right) y(r) = 0,\; n\geq 2 \tag{1}$$

has polynomial solutions $y = \sum_{k=0}^{m}\mathcal{C}_j r^j$?

A logical approach is to examine the differential equation using the series solution

$$\begin{cases} y = \sum\limits_{j=0}^{\infty}\mathcal{C}_j\, r^j, \\ y' = \sum\limits_{j=0}^{\infty} j\,\mathcal{C}_j\, r^{j-1}, \\ y'' = \sum\limits_{j=0}^{\infty} j\,(j-1)\mathcal{C}_j\, r^{j-2} \end{cases}$$

in (1) and enforce the coefficients $\mathcal{C}_j = 0$ for all $j \geq m+1,\; m = 0,1,2,\cdots$ to find the condition so that $\mathcal{C}_m \neq 0$. This approach leads to a conclusion that for equation (1) to have m degree polynomial solution, it is necessary that

$$\varepsilon_{n-2} = -m\,(m-1)\,\alpha_n - m\,\beta_{n-1},\; n = 2,3,\cdots. \tag{2}$$

Did this answer the question? Indeed, no. Consider, for example, this simple equation

$$r^3 y'' + 2\,r^2 y' + (-2\,r + 5) y = 0.$$

Clearly, the necessary condition (2) is satisfied for $m = 1$ and one expects the existence of a first degree polynomial solution, say $y = r + b$, for an arbitrary value of $b \in \mathbb{R}$, however, $2\,r^2 + (-2r+5)(r+b) \neq 0$ for any real value of b.

Therefore, for $n \geq 3$, the condition (2) is *necessary* but not *sufficient* for the existence of polynomial solutions of the differential equation (1).

Note, for $n = 2$, equation (1) is the classical hypergeometric-type differential equation [1–4]

$$(\alpha_2\, r^2 + \alpha_1\, r + \alpha_0)\, y'' + (\beta_1\, r + \beta_0)\, y' + \varepsilon_0\, y = 0 \tag{3}$$

with the necessary and sufficient condition [2] for the polynomial solutions

$$\varepsilon_0 = -m\,(m-1)\,\alpha_2 - m\,\beta_1,\; m = 0,1,2,\cdots.$$

For $n = 3$, the differential equation (1) assumes the form

$$\begin{gathered} p_3(r)\, y'' + p_2(r)\, y' + p_1(r)\, y = 0, \\ \begin{cases} p_3(r) = \sum\limits_{j=0}^{3}\alpha_j r^j, \\ p_2(r) = \sum\limits_{j=0}^{2}\beta_j\, r^j, \\ p_1(r) = \sum\limits_{j=0}^{1}\varepsilon_j r^j, \qquad \alpha_j, \beta_j, \varepsilon_j \in \mathbb{R}, \end{cases} \end{gathered} \tag{4}$$

which includes as a special case or with elementary substitutions, the classical Heun differential equation [5, 6]

$$y'' + \left(\frac{\gamma}{r} + \frac{\delta}{r-1} + \frac{\varepsilon}{r-a}\right) y' + \frac{\alpha\,\beta\, r - q}{r(r-1)(r-a)}\, y = 0, \tag{5}$$

subject to the regularity (at infinity) condition

$$\alpha + \beta + 1 = \gamma + \delta + \varepsilon,$$

and its four confluent forms (Confluent, Doubly-Confluent, Biconfluent and Triconfluent Heun Equations). These equations are indispensable from the point of view of a mathematical analysis [5–11] and for its valuable applications in many areas of theoretical physics [5, 6, 12–20].

In the present work

- From equation (4), we will extract the possible differential equations that can be solved using two-term recurrence formulas.
- From equation (4), we will extract all the differential equations whose series solutions can be evaluated with a three-term recurrence formula.
- For $a_0 \neq 0$, we shall devise a procedure based on the Asymptotic Iteration Method [21] to find the series and polynomial solutions of the differential equation (4).
- In the neighbordhoud of a singular point $r = 0$, i.e., with $a_0 = 0$, we will prove that the series solution can be written as

$$y(r) = \sum_{k=0}^{\infty} (-1)^k \frac{P_{k;s}(\varepsilon_0)}{\alpha_1^k \left(\frac{\beta_0}{\alpha_1} + s\right)_k (1+s)_k} r^{k+s},$$

  where s is a root of the indicial equation. Also, we show that $\{P_{k;s}(\varepsilon_0)\}_{k=0}^{\infty}$ is an infinite sequence of orthogonal polynomials with several interesting properties.
- By imposing the termination conditions, we study the mathematical properties of the finite sequences of the orthogonal polynomials $\{P_{k;s}(\varepsilon_0)\}_{k=0}^{n}$ and explore the factorization property associated with these polynomials.

2. Elementary observations

The classical approach to study the analytical solutions of equation (4) relies on the nature of the singular points of the leading polynomial coefficients

$$\mathfrak{L} \equiv \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3$$

in addition to the point $r = \infty$ in the extended plane. For real coefficients and $\alpha_0 \neq 0$, the odd-degree polynomial $\mathfrak{L}$ is factored into either a product of a linear polynomial and an irreducible quadratic polynomial or a product of three linear factors.

In the first case, the polynomial $\mathfrak{L}$ can be written as

$$\mathfrak{L} = \alpha_3 (r - \xi)(r^2 + br + c)$$

where $r^2 + br + c$ is an irreducible polynomial. In this case, ξ is regular, real, singular point and ∞ is irregular for otherwise, the differential equation can be solved in terms of elementary functions according to the classical theory of ordinary differential equations. In this case, the differential equation can be written as

$$\frac{d^2y}{dr^2} + \left(\frac{\mu_1}{r-\xi} + \frac{\mu_2}{r^2+br+c}\right)\frac{dy}{dr} + \frac{\varepsilon_1 r + \varepsilon_0}{\alpha_3(r-\xi)(r^2+b\,r+c)} y = 0. \tag{6}$$

The second case, the polynomial $\mathfrak{L}$ can be written as

$$\mathfrak{L} = \alpha_3 (r - \xi_1)(r - \xi_2)(r - \xi_3)$$

where $\xi_j, j = 1, 2, 3$ and ∞ are all regular singular points, i.e., the differential equation of Fushsian type,

$$\frac{d^2y}{dr^2} + \left(\sum_{j=1}^{3} \frac{\mu_j}{r-\xi_j}\right)\frac{dy}{dr} + \frac{\varepsilon_1 r + \varepsilon_0}{\alpha_3(r-\xi_1)(r-\xi_2)(r-\xi_3)} y = 0. \tag{7}$$

where μ_j are constants depending on the differential equation parameters. One can then study the series solutions of equations (6) and (7) using the classical Frobenius method.

Another approach, recently adopted, to study (4), depends on the possible combination of the parameters $\alpha_j, j = 0, 1, 2, 3$ such that the polynomial $\mathfrak{L}$ does not vanish identically. There are fifteen possible combinations in total. These fifteen combinations can be classified into two main classes: the first class is characterized by $\alpha_0 \neq 0$, which has eight equations in total, the second class characterized by $\alpha_0 = 0$ includes the remaining seven equations. Each of these two classes will be studied in the next sections. First, we consider some elementary observations regarding the differential equation (4).

We assume no common factor among the polynomial coefficients $p_j(r),\ j = 1, 2, 3$, we start our study of equation (4) by asking the following simple question: *Under what conditions the series solutions of the differential equation (4) can be evaluated using a two-term recurrence relation [22]?* For, in this case, the two linearly independent series solutions can be found explicitly.

Theorem 2.1. *The necessary and sufficient conditions for the linear differential equation*

$$p_2(r)\,u''(r) + p_1(r)\,u'(r) + p_0(r)\,u(r) = 0, \tag{8}$$

to have a two-term recurrence relationship that relates the successive coefficients in its series solution is that in the neighbourhood of the singular regular point r_0 (where $p_2(r_0) = 0$), the equation (8) can be written as:

$$\underbrace{\left[q_{2,0} + q_{2,h}\,(r - r_0)^h\right]}_{q_2(r)}(r - r_0)^{2-m}\,u''(r) + \underbrace{\left[q_{1,0} + q_{1,h}\,(r - r_0)^h\right]}_{q_1(r)}\,r^{1-m}\,u'(r)$$
$$+ \underbrace{\left[q_{0,0} + q_{0,h}\,(r - r_0)^h\right]}_{q_0(r)}(r - r_0)^{-m}\,u(r) = 0, \tag{9}$$

where, for $m \in \mathbb{Z}$, $h \in \mathbb{Z}^+,\ j = 0, 1, 2,$

$$q_j(r) \equiv \sum_{k=0}^{\infty} q_{j,k}(r - r_0)^k = p_j(r)\,(r - r_0)^{m-j}, \tag{10}$$

when at least one of $q_{j,0}, j = 0, 1, 2$ and $q_{j,h}, j = 0, 1, 2,$ is different from zero. In this case, the two-term recurrence formula is given by

$$\frac{c_k}{c_{k-h}} = -\frac{(k + \lambda - h)[q_{2,h}\,(k + \lambda - h - 1) + q_{1,h}\,] + q_{0,h}}{(k + \lambda)[q_{2,0}\,(k + \lambda - 1) + q_{1,0}] + q_{0,0}}, \tag{11}$$

where $c_0 \neq 0$, and $\lambda = \lambda_1, \lambda_2$ are the roots of the indicial equation

$$q_{2,0}\,\lambda\,(\lambda - 1) + q_{1,0}\,\lambda + q_{0,0} = 0. \tag{12}$$

The closed form of the series solution generated by (11) can be written in terms of the generalized hypergeometric function as

$$\begin{aligned} u(r;\lambda) =& z^{\lambda}\sum_{k=0}^{\infty} c_{hk} r^{hk} = r^{\lambda}{}_3F_2\Big(1, \tfrac{2\lambda-1}{2h} + \tfrac{q_{1,h}}{2\,h\,q_{2,h}} - \tfrac{\sqrt{(q_{1,h}-q_{2,h})^2-4q_{0,h}q_{2,h}}}{2\,h\,q_{2,h}},\\ &\tfrac{2\lambda-1}{2h} + \tfrac{q_{1,h}}{2\,h\,q_{2,h}} + \tfrac{\sqrt{(q_{1,h}-q_{2,h})^2-4q_{0,h}q_{2,h}}}{2\,h\,q_{2,h}}; 1 + \tfrac{2\lambda-1}{2\,h} + \tfrac{q_{1,0}}{2\,h\,q_{2,0}} - \tfrac{\sqrt{(q_{1,0}-q_{2,h})^2-4q_{0,0}q_{2,0}}}{2\,h\,q_{2,0}},\\ &1 + \tfrac{2\lambda-1}{2h} + \tfrac{q_{1,0}}{2\,h\,q_{2,0}} + \tfrac{\sqrt{(q_{1,0}-q_{2,h})^2-4q_{0,0}q_{2,0}}}{2\,h\,q_{2,0}}; -\tfrac{q_{2,h}}{q_{2,0}}\,r^h\Big). \end{aligned} \tag{13}$$

Applying this theorem, equation (4) generates the following solvable equations:

- Differential equation:

$$r^2\,(\alpha_2 + \alpha_3\,r)\,u''(r) + r\,(\beta_1\, + \beta_2\,r)\,u'(r) + (\varepsilon_0 + \varepsilon_1\,r)\,u(r) = 0, \quad \varepsilon_0 \neq 0, \tag{14}$$

Recurrence relation: For $k = 1, 2, \cdots$, and $c_0 = 1$,

$$\frac{c_k}{c_{k-1}} = -\frac{(k + \lambda - 1)[\alpha_3\,(k + \lambda - 2) + \beta_2] + \varepsilon_1}{(k + \lambda)[\alpha_2\,(k + \lambda - 1) + \beta_1\,] + \varepsilon_0}, \tag{15}$$

where $\lambda = \lambda_+, \lambda_-$ are the roots of the indical equation

$$\alpha_2\,\lambda\,(\lambda - 1) + \beta_1\,\lambda + \varepsilon_0 = 0,$$

namely

$$\lambda_\pm = \frac{\alpha_2 - \beta_1 \pm \sqrt{(\alpha_2 - \beta_1)^2 - 4\alpha_2\varepsilon_0}}{2\alpha_2}.$$

The two linearly independent solutions generated by (15), in terms of the Gauss hypergeometric functions, are:

$$u_\pm = r^{\frac{\alpha_2-\beta_1\pm\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}} {}_2F_1\Bigg(\frac{\beta_2}{2\alpha_3}-\frac{\beta_1}{2\alpha_2}\pm\frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}-\frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3},$$
$$\frac{\beta_2}{2\alpha_3}-\frac{\beta_1}{2\alpha_2}\pm\frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}+\frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3 c_1}}{2\alpha_3};\frac{\alpha_2\pm\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{\alpha_2};-\frac{\alpha_3}{\alpha_2}r\Bigg). \tag{16}$$

- Differential equation:

$$(\alpha_1 r+\alpha_3 r^3)\,u''(r)+(\beta_0+\beta_2 r^2)\,u'(r)+\varepsilon_1 r\,u(r)=0,\qquad(\beta_0\neq 0), \tag{17}$$

Recurrence relation:

$$\frac{c_k}{c_{k-2}}=-\frac{(k+\lambda-2)[\alpha_3(k+\lambda-3)+\beta_2]+\varepsilon_1}{(k+\lambda)[\alpha_1(k+\lambda-1)+\beta_0]},\qquad(c_0,c_1\neq 0,\ k=2,3,\cdots), \tag{18}$$

where $\lambda=\lambda_+,\lambda_-$ are the roots of the indicial equation

$$\alpha_1\lambda(\lambda-1)+\beta_0\lambda=0,$$

i.e $\lambda_+=0,\lambda_-=1-\beta_0/\alpha_1$.

The two linearly independent series solutions generated by (18) are:

$$u_+(r)={}_2F_1\Bigg(\frac{\beta_2}{4\alpha_3}-\frac{1}{4}-\frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{4\alpha_3},$$
$$\frac{\beta_2}{4\alpha_3}-\frac{1}{4}+\frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{4\alpha_3};\frac{1}{2}+\frac{\beta_0}{2\alpha_1};-\frac{\alpha_3}{\alpha_1}r^2\Bigg), \tag{19}$$

and

$$u_-(r)=r^{1-\frac{\beta_0}{\alpha_1}}\times{}_2F_1\Bigg(\frac{1}{4}+\frac{\beta_2}{4\alpha_3}-\frac{\beta_0}{2\alpha_1}-\frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{4\alpha_3},$$
$$\frac{1}{4}+\frac{\beta_2}{4\alpha_3}-\frac{\beta_0}{2\alpha_1}+\frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{4\alpha_3};\frac{3}{2}-\frac{\beta_0}{2\alpha_1};-\frac{\alpha_3}{\alpha_1}r^2\Bigg). \tag{20}$$

- Differential equation

$$(\alpha_0+\alpha_3 r^3)\,u''(r)+\beta_2 r^2\,u'(r)+\varepsilon_1 r\,u(r)=0,\ \alpha_0\neq 0, \tag{21}$$

Recurrence relation:

$$\frac{c_k}{c_{k-3}}=-\frac{(k+\lambda-3)[\alpha_3(k+\lambda-4)+\beta_2]+\varepsilon_1}{\alpha_0(k+\lambda)(k+\lambda-1)},\qquad(c_0\neq 0), \tag{22}$$

where $\lambda=\lambda_1,\lambda_2$ are the roots of the indicial equation $\alpha_0\lambda(\lambda-1)=0$, namely, $\lambda_1=0$, $\lambda_2=1$.

The two linearly independent series solutions are:

$$u_1(r)={}_2F_1\Bigg(-\frac{\alpha_3-\beta_2+\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{6\alpha_3},\ \frac{-\alpha_3+\beta_2+\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{6\alpha_3};\frac{2}{3};-\frac{\alpha_3}{\alpha_0}r^3\Bigg), \tag{23}$$

and

$$u_2(r)=r\,{}_2F_1\Bigg(\frac{\alpha_3+\beta_2-\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{6\alpha_3},\ \frac{\alpha_3+\beta_2+\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{6\alpha_3};\frac{4}{3};-\frac{\alpha_3}{\alpha_0}r^3\Bigg). \tag{24}$$

Out of the three generic equations (14), (17) and (21), five exactly solvable differential equations (Cases 1, 4, 5, 8, and 10) of the type (4) follows and other five (Cases 2, 3, 6, 7, 9) that can be derived directly from them by taking the limits of the equation parameters. For direct use, the ten equations are listed in Table 1.

	DEs and their linearly independent solutions
1	$\alpha_2\, r^2 u'' + (\beta_1 r + \beta_2 r^2)\, u' + (\varepsilon_0 + \varepsilon_1 r)\, u = 0$ $u = r^{\frac{1}{2} - \frac{\beta_1}{2\alpha_2} + \frac{1}{2\alpha_2}\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}\, {}_1F_1\left(\frac{1}{2} - \frac{\beta_1}{2\alpha_2} + \frac{\varepsilon_1}{\beta_2} + \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}; 1 + \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{\alpha_2}; -\frac{\beta_2}{\alpha_2} r\right),$ $u = r^{-\frac{1}{2} + \frac{\beta_1}{2\alpha_2} - \frac{1}{2\alpha_2}\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}\, {}_1F_1\left(\frac{1}{2} - \frac{\beta_1}{2\alpha_2} + \frac{\varepsilon_1}{\beta_2} - \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}; 1 - \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{\alpha_2}; -\frac{\beta_2}{\alpha_2} r\right).$
2	$\alpha_2\, r^2 u'' + \beta_1 r\, u' + (\varepsilon_0 + \varepsilon_1 r)\, u = 0$ $u = r^{\frac{1}{2} - \frac{\beta_1}{2\alpha_2} + \frac{1}{2\alpha_2}\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}\, {}_0F_1\left(-; 1 + \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{\alpha_2}; -\frac{\varepsilon_1}{\alpha_2}\, r\right),$ $u = r^{-\frac{1}{2} + \frac{\beta_1}{2\alpha_2} - \frac{1}{2\alpha_2}\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}\, {}_0F_1\left(-; 1 - \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{\alpha_2}; -\frac{\beta_2}{\alpha_2}\, r\right).$
3	$\alpha_2\, r^2 u'' + \beta_2 r^2 u' + (\varepsilon_0 + \varepsilon_1 r)\, u = 0$ $u = r^{\frac{1}{2} - \frac{\sqrt{\alpha_2-4\varepsilon_0}}{2\sqrt{\alpha_2}}}\, {}_1F_1\left(\frac{1}{2} + \frac{\varepsilon_1}{\beta_2} - \frac{\sqrt{\alpha_2-4\varepsilon_0}}{2\sqrt{\alpha_2}}; 1 - \frac{\sqrt{\alpha_2-4\varepsilon_0}}{\sqrt{\alpha_2}}; -\frac{\beta_2}{\alpha_2}\, r\right),$ $u = r^{\frac{1}{2} + \frac{\sqrt{\alpha_2-4\varepsilon_0}}{2\sqrt{\alpha_2}}}\, {}_1F_1\left(\frac{1}{2} + \frac{\varepsilon_1}{\beta_2} + \frac{\sqrt{\alpha_2-4\varepsilon_0}}{2\sqrt{\alpha_2}}; 1 + \frac{\sqrt{\alpha_2-4\varepsilon_0}}{\sqrt{\alpha_2}}; -\frac{\beta_2}{\alpha_2}\, r\right).$
4	$(\alpha_2\, r^2 + \alpha_3\, r^3) u'' + \beta_1\, r\, u' + (\varepsilon_0 + \varepsilon_1 r)\, u = 0$ $u = r^{\frac{1}{2} - \frac{\beta_1}{2\alpha_2} + \frac{1}{2\alpha_2}\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}\, {}_2F_1\left(\frac{1}{2}\sqrt{\frac{\alpha_3-4\varepsilon_1}{\alpha_3}} - \frac{\beta_1}{2\alpha_2} + \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}, -\frac{1}{2}\sqrt{\frac{\alpha_3-4\varepsilon_1}{\alpha_3}} - \frac{\beta_1}{2\alpha_2} + \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}; 1 + \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{\alpha_2}; -\frac{\alpha_3}{\alpha_2}\, r\right),$ $u = r^{\frac{1}{2} - \frac{\beta_1}{2\alpha_2} - \frac{1}{2\alpha_2}\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}\, {}_2F_1\left(\frac{1}{2}\sqrt{\frac{\alpha_3-4\varepsilon_1}{\alpha_3}} - \frac{\beta_1}{2\alpha_2} - \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}, -\frac{1}{2}\sqrt{\frac{\alpha_3-4\varepsilon_1}{\alpha_3}} - \frac{\beta_1}{2\alpha_2} - \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2}; 1 - \frac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{\alpha_2}; -\frac{\alpha_3}{\alpha_2}\, r\right).$
5	$(\alpha_2\, r^2 + \alpha_3\, r^3) u'' + \beta_2 r^2\, u' + (\varepsilon_0 + \varepsilon_1 r)\, u = 0$ $u = r^{\frac{1}{2} - \frac{1}{2\sqrt{\alpha_2}}\sqrt{\alpha_2-4\varepsilon_0}}\, {}_2F_1\left(\frac{\beta_2}{2\alpha_3} - \frac{1}{2}\sqrt{\frac{\alpha_2-4\varepsilon_0}{\alpha_2}} + \frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}, \frac{\beta_2}{2\alpha_3} - \frac{1}{2}\sqrt{\frac{\alpha_2-4\varepsilon_0}{\alpha_2}} - \frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}; 1 - \frac{\sqrt{\alpha_2-4\varepsilon_0}}{\sqrt{\alpha_2}}; -\frac{\alpha_3}{\alpha_2}\, r\right),$ $u = r^{\frac{1}{2} + \frac{1}{2\sqrt{\alpha_2}}\sqrt{\alpha_2-4\varepsilon_0}}\, {}_2F_1\left(\frac{\beta_2}{2\alpha_3} + \frac{1}{2}\sqrt{\frac{\alpha_2-4\varepsilon_0}{\alpha_2}} - \frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}, \frac{\beta_2}{2\alpha_3} + \frac{1}{2}\sqrt{\frac{\alpha_2-4\varepsilon_0}{\alpha_2}} + \frac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}; 1 + \frac{\sqrt{\alpha_2-4\varepsilon_0}}{\sqrt{\alpha_2}}; -\frac{\alpha_3}{\alpha_2}\, r\right).$
6	$(\alpha_2\, r^2 + \alpha_3\, r^3) u'' + (\varepsilon_0 + \varepsilon_1\, r)\, u = 0$ $u = r^{\frac{1}{2} - \frac{1}{2\sqrt{\alpha_2}}\sqrt{\alpha_2-4\varepsilon_0}}\, {}_2F_1\left(-\frac{1}{2}\sqrt{\frac{\alpha_2-4\varepsilon_0}{\alpha_2}} - \frac{\sqrt{\alpha_3-4\varepsilon_1}}{2\sqrt{\alpha_3}}, -\frac{1}{2}\sqrt{\frac{\alpha_2-4\varepsilon_0}{\alpha_2}} - \frac{\sqrt{\alpha_3-4\varepsilon_1}}{2\sqrt{\alpha_3}}; 1 - \frac{\sqrt{\alpha_2-4\varepsilon_0}}{\sqrt{\alpha_2}}; -\frac{\alpha_3}{\alpha_2}\, r\right),$ $u = r^{\frac{1}{2} + \frac{1}{2\sqrt{\alpha_2}}\sqrt{\alpha_2-4\varepsilon_0}}\, {}_2F_1\left(\frac{1}{2}\sqrt{\frac{\alpha_2-4\varepsilon_0}{\alpha_2}} - \frac{\sqrt{\alpha_3-4\varepsilon_1}}{2\sqrt{\alpha_3}}, \frac{1}{2}\sqrt{\frac{\alpha_2-4\varepsilon_0}{\alpha_2}} + \frac{\sqrt{\alpha_3-4\varepsilon_1}}{2\sqrt{\alpha_3}}; 1 + \frac{\sqrt{\alpha_2-4\varepsilon_0}}{\sqrt{\alpha_2}}; -\frac{\alpha_3}{\alpha_2}\, r\right).$
7	$\alpha_2\, r^2\, u'' + (\varepsilon_0 + \varepsilon_1\, r)\, u = 0$ $u = r^{\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4\varepsilon_0}{\alpha_2}}}\, {}_0F_1\left(; 1 + \sqrt{1 + \frac{4\varepsilon_0}{\alpha_2}}; -\frac{\varepsilon_1}{\alpha_2}\, r\right), \qquad u = r^{\frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{4\varepsilon_0}{\alpha_2}}}\, {}_0F_1\left(; 1 - \sqrt{1 - \frac{4\varepsilon_0}{\alpha_2}}; -\frac{\varepsilon_1}{\alpha_2}\, r\right)$
8	$\alpha_1\, r\, u'' + (\beta_0 + \beta_2 r^2)\, u' + \varepsilon_1\, r\, u = 0,$ $u = {}_1F_1\left(\frac{\varepsilon_1}{2\beta_2}; \frac{1}{2} + \frac{\beta_0}{2\alpha_1}; -\frac{\beta_2}{2\alpha_1}\, r^2\right), \qquad u = r^{1-\frac{\beta_0}{\alpha_1}}\, {}_1F_1\left(\frac{1}{2} - \frac{\beta_0}{2\alpha_1} + \frac{\varepsilon_1}{2\beta_2}; \frac{3}{2} - \frac{\beta_0}{2\alpha_1}; -\frac{\beta_2}{2\alpha_1}\, r^2\right).$
9	$\alpha_1\, r\, u'' + \beta_0\, u' + \varepsilon_1\, r\, u = 0,$ $u = {}_0F_1\left(-; \frac{1}{2} + \frac{\beta_0}{2\alpha_1}; -\frac{\varepsilon_1}{4\alpha_1}\, z^2\right), \qquad u = r^{1-\frac{\beta_0}{\alpha_1}}\, {}_0F_1\left(-; \frac{3}{2} - \frac{\beta_0}{2\alpha_1}; -\frac{\varepsilon_1}{4\alpha_1}\, r^2\right).$
10	$\alpha_0\, u'' + \beta_2\, r^2 u' + \varepsilon_1\, r\, u = 0,$ $u = {}_1F_1\left(\frac{\varepsilon_1}{3\beta_2}; \frac{2}{3}; -\frac{\beta_2}{3\alpha_0}\, r^3\right), \qquad u = r\; {}_1F_1\left(\frac{1}{3} + \frac{\varepsilon_1}{3\beta_2}; \frac{4}{3}; -\frac{\beta_2}{3\alpha_0}\, r^3\right).$

Table 1. Ten solvable equations of the type (4) that follows from the generic equations (14), (17), and (21).

3. The solutions in the neighbourhood of an ordinary point

3.1. Series solutions

In the case of $\alpha_0 \neq 0$, $r = 0$, there is an ordinary point for the differential equations (4). The classical theory of differential equation ensure that the (4) has two linearly independent power series solutions in the neighbourhood of $r = 0$ and valid to the nearest real singular point of the leading polynomial coefficient $\mathfrak{L} \equiv \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 = 0$. Indeed, the polynomial $\mathfrak{L} = 0$ has the discriminate [23]:

$$\Delta = 18\,\alpha_3\,\alpha_2\,\alpha_1\,\alpha_0 - 4\,\alpha_2^3\,\alpha_0 + \alpha_2^2\,\alpha_1^2 - 4\,\alpha_3\,\alpha_1^3 - 27\,\alpha_3^2\,\alpha_0^2. \tag{25}$$

The nature of the $\mathfrak{L}$ roots as given by (25) along with the corresponding eight differential equations are summarized in Table 2.

For these differential equations, the following theorem, that can be easily proved using Frobenius method, holds.

Theorem 3.1. *(Formal series solutions) In the neighbourhood of the ordinary point $r = 0$, the coefficients of the series solution $y(r) = \sum_{k=0}^{\infty} \mathcal{C}_k\, r^k$ to the differential equation (4) satisfy the four-term recurrence relation*

$$\begin{aligned}((k-1)\,((k-2)\,\alpha_3 + \beta_2) + \varepsilon_1)\,\mathcal{C}_{k-1} + (k((k-1)\,\alpha_2 + \beta_1) + \varepsilon_0)\,\mathcal{C}_k \\ + (k+1)(k\alpha_1 + \beta_0)\,\mathcal{C}_{k+1} + (k+2)\,(k+1)\,\alpha_0\,\mathcal{C}_{k+2} = 0,\end{aligned} \tag{26}$$

where $k = 0, 1, 2, \cdots$, with $\mathcal{C}_{-1} = 0$ and arbitrary nonzero constants $\mathcal{C}_0$ and $\mathcal{C}_1$. The radius of convergence of these series solutions is extended from $r = 0$ to the nearest singular point of the leading polynomial coefficient $\mathfrak{L} = 0$.

The first few terms of the series solution are given explicitly by

$$\begin{aligned}\mathcal{C}_2 &= -\tfrac{\varepsilon_0}{2\alpha_0}\,\mathcal{C}_0 - \tfrac{\beta_0}{2\alpha_0}\,\mathcal{C}_1, \\ \mathcal{C}_3 &= \tfrac{(\alpha_1+\beta_0)\,\varepsilon_0 - \alpha_0\,\varepsilon_1}{6\alpha_0^2}\,\mathcal{C}_0 + \tfrac{\beta_0(\alpha_1+\beta_0) - \alpha_0(\beta_1+\varepsilon_0)}{6\,\alpha_0^2}\,\mathcal{C}_1, \\ &\cdots .\end{aligned}$$

For $\alpha_0 \neq 0$, using (26), we can extract the following differential equations with series solution from (4) using a three-term recurrence relation:

- Differential equation:

$$\left(\alpha_0 + \alpha_1\, r + \alpha_3\, r^3\right) u''(r) + \left(\beta_0 + \beta_2\, r^2\right) u'(r) + \varepsilon_1\, r\, u(r) = 0. \tag{27}$$

  Recurrence formula:

$$C_{k+2} = -\frac{(k+1)(k\,\alpha_1 + \beta_0)}{(k+1)\,(k+2)\,\alpha_0} C_{k+1} - \frac{(k-1)((k-2)\,\alpha_3 + \beta_2) + \varepsilon_1}{(k+1)\,(k+2)\,\alpha_0} C_{k-1}. \tag{28}$$

- Differential equation:

$$\left(\alpha_0 + \alpha_2 r^2 + \alpha_3 r^3\right)\, u''(r) + (\beta_1\, r + \beta_2\, r^2) u'(r) + \varepsilon_1\, r\, u(r) = 0. \tag{29}$$

  Recurrence formula:

$$C_{k+2} = -\frac{k(k-1)\alpha_2 + k\,\beta_1}{(k+1)\,(k+2)\,\alpha_0} C_k - \frac{(k-1)(k-2)\alpha_3 + (k-1)\beta_2 + \varepsilon_1}{(k+1)(k+2)\,\alpha_0} C_{k-1}. \tag{30}$$

- Differential equation:

$$\left(\alpha_0 + \alpha_2\, r^2\right)\, u''(r) + (\beta_1\, r + \beta_2\, r^2)\, u'(r) + \varepsilon_1\, r\, u(r) = 0. \tag{31}$$

  Recurrence formula:

$$C_{k+2} = -\frac{k\,(k-1)\alpha_2 + k\,\beta_1}{(k+1)\,(k+2)\,\alpha_0} C_k - \frac{(k-1)\beta_2 + \varepsilon_1}{(k+1)\,(k+2)\,\alpha_0} C_{k-1}. \tag{32}$$

DE	α_3	α_2	α_1	α_0	Discriminant	Roots of $\mathfrak{L}$	Domain definition
I	α_3	α_2	α_1	α_0	$\Delta_3 > 0$	$\xi_1 \neq \xi_2 \neq \xi_3$	$\lvert r\rvert < \min_{i=1,2,3} \xi_i$
					$\Delta_3 = 0$	$\xi_1 = \xi_2 = \xi_3 = \xi$	$\lvert r\rvert < \xi$
					$\Delta_3 < 0$	$\xi \in \mathbb{R}$	$\lvert r\rvert < \xi$
	Differential equation:				$(\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3) y'' + (\beta_0 + \beta_1 r + \beta_2 r^2) y' + (\varepsilon_0 + \varepsilon_1 r) y = 0$		
	Discriminant:				$\Delta_3 = 18 \alpha_3 \alpha_2 \alpha_1 \alpha_0 - 4 \alpha_2{}^3 \alpha_0 + \alpha_2{}^2 \alpha_1{}^2 - 4 \alpha_3 \alpha_1{}^3 - 27 \alpha_3{}^2 \alpha_0{}^2$		
II	0	α_2	α_1	α_0	$\Delta_3 > 0$	$\xi_1 \neq \xi_2$	$\lvert r\rvert < \min_{i=1,2} \xi_i$
					$\Delta_3 = 0$	$\xi_1 = \xi_2 = \xi$	$\lvert r\rvert < \xi$
					$\Delta_3 < 0$	None	$\lvert r\rvert < \infty$
	Differential Equation:				$(\alpha_0 + \alpha_1 r + \alpha_2 r^2) y'' + (\beta_0 + \beta_1 r + \beta_2 r^2) y' + (\varepsilon_0 + \varepsilon_1 r) y = 0$		
	Discriminant:				$\Delta_3 = \alpha_2^2(-4 \alpha_2 \alpha_0 + \alpha_1{}^2)$		
III	0	0	α_1	α_0	$\alpha_1\alpha_0 > 0$	$r = -\alpha_0/\alpha_1$	$-\infty < r < -\alpha_0/\alpha_1$
					$\alpha_1\alpha_0 < 0$	$r = -\alpha_0/\alpha_1$	$-\alpha_0/\alpha_1 < r < \infty$
	Differential Equation:				$(\alpha_0 + \alpha_1 r) y'' + (\beta_0 + \beta_1 r + \beta_2 r^2) y' + (\varepsilon_0 + \varepsilon_1 r) y = 0$		
	Discriminant:				$\Delta_3 = 0$		
IV	0	0	0	α_0	None	None	$-\infty < r < \infty$
	Differential Equation:				$\alpha_0 y'' + (\beta_0 + \beta_1 r + \beta_2 r^2) y' + (\varepsilon_0 + \varepsilon_1 r) y = 0$		
	Discriminant:				$\Delta_3 = 0$		
V	α_3	0	α_1	α_0	$\Delta_3 > 0$	$\xi_1 \neq \xi_2 \neq \xi_3$	$\lvert r\rvert < \min_{i=1,2,3} \xi_i$
					$\Delta_3 = 0$	$\xi_1 = \xi_2 = \xi_3 = \xi$	$\lvert r\rvert < \xi$
					$\Delta_3 < 0$	$\xi \in \mathbb{R}$	$\lvert r\rvert < \xi$
	Differential Equation:				$(\alpha_0 + \alpha_1 r + \alpha_3 r^3) y'' + (\beta_0 + \beta_1 r + \beta_2 r^2) y' + (\varepsilon_0 + \varepsilon_1 r) y = 0$		
	Discriminant:				$\Delta_3 = -4 \alpha_3 \alpha_1{}^3 - 27 \alpha_3{}^2 \alpha_0{}^2$		
VI	α_3	α_2	0	α_0	$\Delta_3 > 0$	$\xi_1 \neq \xi_2 \neq \xi_3$	$\lvert r\rvert < \min_{i=1,2,3} \xi_i$
					$\Delta_3 = 0$	$\xi_1 = \xi_2 = \xi_3 = \xi$	$\lvert r\rvert < \xi$
					$\Delta_3 < 0$	$\xi \in \mathbb{R}$	$\lvert r\rvert < \xi$
	Differential Equation:				$(\alpha_0 + \alpha_2 r^2 + \alpha_3 r^3) y'' + (\beta_0 + \beta_1 r + \beta_2 r^2) y' + (\varepsilon_0 + \varepsilon_1 r) y = 0$		
	Discriminant:				$\Delta_3 = -4 \alpha_2{}^3 \alpha_0 - 27 \alpha_3{}^2 \alpha_0{}^2$		
VII	α_3	0	0	α_0	$\alpha_0 \alpha_3 < 0$ or $\alpha_0 \alpha_3 > 0$	$\xi = \sqrt[3]{-\alpha_0/\alpha_3}$	$\lvert r\rvert < \xi$
	Differential Equation:				$(\alpha_0 + \alpha_3 r^3) y'' + (\beta_0 + \beta_1 r + \beta_2 r^2) y' + (\varepsilon_0 + \varepsilon_1 r) y = 0$		
	Discriminant:				$\Delta_3 = -27 \alpha_3{}^2 \alpha_0{}^2$		
VIII	0	α_2	0	α_0	$\alpha_2\alpha_0 < 0$	$r = \pm\sqrt{-\frac{\alpha_0}{\alpha_2}}$	$-\sqrt{-\frac{\alpha_0}{\alpha_2}} < r < \sqrt{-\frac{\alpha_0}{\alpha_2}}$
					$\alpha_2\alpha_0 > 0$	None	$-\infty < r < \infty$
	Differential Equation:				$(\alpha_0 + \alpha_2 r^2) y'' + (\beta_0 + \beta_1 r + \beta_2 r^2) y' + (\varepsilon_0 + \varepsilon_1 r) y = 0$		
	Discriminant:				$\Delta_3 = -4 \alpha_2{}^3 \alpha_0$		

Table 2. Tabulating the eight different types of differential equations, which apply to Theorem 3.1.

- Differential equation:

$$\left(\alpha_0 + \alpha_3 r^3\right) u''(r) + (\beta_1 r + \beta_2 r^2) u'(r) + r \varepsilon_1 u(r) = 0. \tag{33}$$

Recurrence formula:

$$C_{k+2} = -\frac{k \beta_1}{(k+1)(k+2)\alpha_0} C_k - \frac{(k-1)(k-2)\alpha_3 + (k-1)\beta_2 + \varepsilon_1}{(k+1)(k+2)\alpha_0} C_{k-1}. \tag{34}$$

- Differential equation:

$$\alpha_0 u''(r) + (\beta_1 r + \beta_2 r^2) u'(r) + \varepsilon_1 r u(r) = 0. \tag{35}$$

Recurrence formula:

$$C_{k+2} = -\frac{k \beta_1}{(k+1)(k+2)\alpha_0} C_k - \frac{(k-1)\beta_2 + \varepsilon_1}{(k+1)(k+2)\alpha_0} C_{k-1}. \tag{36}$$

- Differential equation:

$$\left(\alpha_0 + \alpha_2 r^2 + \alpha_3 r^3\right) u''(r) + \beta_1 r u'(r) + \varepsilon_1 r u(r) = 0. \tag{37}$$

Recurrence formula:

$$C_{k+2} = -\frac{k(k-1)\alpha_2 + k \beta_1}{(k+1)(k+2)\alpha_0} C_k - \frac{(k-2)(k-1)\alpha_3 + \varepsilon_1}{(k+1)(k+2)\alpha_0} C_{k-1}. \tag{38}$$

- Differential equation:

$$\left(\alpha_0 + \alpha_3 r^3\right) u''(r) + \beta_1 r u'(r) + \varepsilon_1 r u(r) = 0. \tag{39}$$

Recurrence formula:

$$C_{k+2} = -\frac{k \beta_1}{(k+1)(k+2)\alpha_0} C_k - \frac{(k-2)(k-1)\alpha_3 + \varepsilon_1}{(k+1)(k+2)\alpha_0} C_{k-1}. \tag{40}$$

- Differential equation:

$$\left(\alpha_0 + \alpha_2 r^2\right) u''(r) + \beta_1 r u'(r) + \varepsilon_1 r u(r) = 0. \tag{41}$$

Recurrence formula:

$$C_{k+2} = -\frac{k(k-1)\alpha_2 + k \beta_1}{(k+1)(k+2)\alpha_0} C_k - \frac{\varepsilon_1}{(k+1)(k+2)\alpha_0} C_{k-1}. \tag{42}$$

- Differential equation:

$$\left(\alpha_0 + \alpha_2 r^2\right) u''(r) + (\beta_1 + \beta_2 r^2) u'(r) + \varepsilon_1 r u(r) = 0. \tag{43}$$

Recurrence formula:

$$C_{k+2} = -\frac{k(k-1)\alpha_2 + \beta_1 k)}{(k+1)(k+2)\alpha_0} C_k - \frac{(k-1)\beta_2 + \varepsilon_1}{(k+1)(k+2)\alpha_0} C_{k-1}. \tag{44}$$

- Differential equation:

$$u''(r) + \beta_1 r u'(r) + \varepsilon_1 r u(r) = 0, \tag{45}$$

Recurrence formula:

$$C_{k+2} = -\frac{k \beta_1}{(k+1)(k+2)} C_k - \frac{\varepsilon_1}{(k+1)(k+2)} C_{k-1}. \tag{46}$$

3.2. Polynomial solutions

The series solution $y(r) = \sum_{k=0}^{\infty} \mathcal{C}_k r^k$ terminates to an n^{th}-degree polynomial if $\mathcal{C}_n \neq 0$ and $\mathcal{C}_j = 0$ for all $j \geq n+1$. It is not difficult to show by direct substitution that for polynomial solutions of $\mathbf{P}_n(r) = \sum_{k=0}^{n} \mathcal{C}_k r^k$, it is necessary that

$$\varepsilon_1 = -n(n-1)\alpha_3 - n\beta_2, \qquad n = 0, 1, 2, \cdots. \tag{47}$$

Furthermore, the polynomial solution coefficients $\{\mathcal{C}_k\}_{k=0}^n$ satisfy a four-term recurrence relation, see (26),

$$\begin{aligned}\Big((k-1)\big((k-2)\alpha_3+\beta_2\big)+\varepsilon_{1;n}\Big)\,\mathcal{C}_{k-1}+\Big(k\big((k-1)\alpha_2+\beta_1\big)+\varepsilon_{0;n}\Big)\,\mathcal{C}_k\\ +(k+1)(k\alpha_1+\beta_0)\,\mathcal{C}_{k+1}+(k+1)(k+2)\alpha_0\,\mathcal{C}_{k+2}=0\,,\quad k=0,\,1,\,\ldots,\,n+1\,,\end{aligned}\tag{48}$$

that generates a system of $(n+2)$ linear equations in $\{\mathcal{C}_k\}_{k=0}^n$: $\underbrace{\overbrace{\cdots\cdots\cdots\cdots}^{n-equations}\quad\cdot\quad}_{(n+2)-equations}$.

The first n equations are

$$\begin{cases} k=0, & \to \varepsilon_0\mathcal{C}_0+\beta_0\mathcal{C}_1+2\,\alpha_0\,\mathcal{C}_2=0\\ k=1, & \to \varepsilon_1\mathcal{C}_0+(\beta_1+\varepsilon_0)\mathcal{C}_1+2(\alpha_1+\beta_0)\mathcal{C}_2+6\,\alpha_0\,\mathcal{C}_3=0\\ k=2, & \to (\beta_2+\varepsilon_1)\mathcal{C}_1+(2\,\alpha_2+2\,\beta_1+\varepsilon_0)\mathcal{C}_2+3(2\,\alpha_1+\beta_0)\mathcal{C}_3+12\,\alpha_0\,\mathcal{C}_4=0\\ k=3, & \to (2\alpha_3+2\,\beta_2+\varepsilon_1)\mathcal{C}_2+(6\,\alpha_2+3\,\beta_1+\varepsilon_0)\mathcal{C}_3+4(3\,\alpha_1+\beta_0)\mathcal{C}_4+20\,\alpha_0\,\mathcal{C}_5=0\\ & \qquad\vdots\\ k=n-1, & \to \Big((n-2)\,\big((n-3)\,\alpha_3+\beta_2\big)+\varepsilon_{1;n}\Big)\mathcal{C}_{n-2}+\Big((n-1)\big((n-2)\,\alpha_2+\beta_1\big)+\varepsilon_{0;n}\Big)\mathcal{C}_{n-1}\\ & \qquad +n\big((n-1)\,\alpha_1+\beta_0\big)\mathcal{C}_n=0. \end{cases}\tag{49}$$

These equations permit the evaluation, using say Cramer's rule, of the coefficients $\{\mathcal{C}_k\}_{k=1}^n$ of the polynomial solution in terms of the non-zero constant $\mathcal{C}_0$.

The $(n+1)^{th}$ equation

$$\Big((n-1)\,(n-2)\,\alpha_3+(n-1)\,\beta_2+\varepsilon_1\Big)\,\mathcal{C}_{n-1}+\Big(n(n-1)\,\alpha_2+n\,\beta_1+\varepsilon_0\Big)\,\mathcal{C}_n=0\,,\tag{50}$$

gives our *sufficient condition* that relates $\varepsilon_0\equiv\varepsilon_{0;n}$ to the remaining parameters of the differential equation.

Finally, the $(n+2)^{th}$ equation

$$\varepsilon_{1;n}=-n\,(n-1)\,\alpha_3-n\,\beta_2\,,\quad n=0,\,1,\,\cdots\,,\tag{51}$$

re-establishes *the necessary condition* ($\varepsilon_1\equiv\varepsilon_{1;n}$) for the existence of the n-degree polynomial solution, see (47).

For a non-zero solution, the $n+1$ linear equations generated by the recurrence relation (48) require the vanishing of the $(n+1)\times(n+1)$-determinant (with four main diagonals and all other entries being zeros)

$$\Delta_{n+1}=\begin{vmatrix} S_0 & T_1 & \eta_1 & & & & \\ \gamma_1 & S_1 & T_2 & \eta_2 & & & \\ & \gamma_2 & S_2 & T_3 & \eta_3 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \gamma_{n-2} & S_{n-2} & T_{n-1} & \eta_{n-1}\\ & & & & \gamma_{n-1} & S_{n-1} & T_n\\ & & & & & \gamma_n & S_n \end{vmatrix},$$

where

$$\begin{aligned} S_k&=\varepsilon_{0;n}+k\big((k-1)\alpha_2+\beta_1\big)\,,\\ T_k&=k\big((k-1)\alpha_1+\beta_0\big)\,,\\ \gamma_k&=\varepsilon_{1;n}+(k-1)\big((k-2)\alpha_3+\beta_2\big)\,,\\ \eta_k&=k(k+1)\alpha_0\,,\end{aligned}$$

and for fixed n,

$$\varepsilon_{1;n}=-n\,(n-1)\,\alpha_3-n\,\beta_2\,.\tag{52}$$

A simple relation to evaluate this determinant in terms of lower-degree determinants is given by

$$\Delta_{k+1} = S_k\,\Delta_k - \gamma_k\,T_k\,\Delta_{k-1} + \gamma_k\,\gamma_{k-1}\,\eta_{k-1}\,\Delta_{k-2}\,, \quad (\Delta_{-2} = \Delta_{-1} = 0,\ \Delta_0 = 1,\ k = 0, 1, \ldots, n). \tag{53}$$

Although there is a classical theorem [24] that guarantees the simple distinct real roots of the three diagonal matrix, to the best of our knowledge, there is no such theorem available for the matrix-type (52). However, we shall assume, in the following example, that the matrix entries allow for the distinct real roots of the resulting polynomial of $\varepsilon_{0;n}$.

Illustrative example:

- For the zero-degree polynomial solution $\mathbf{P}_0\,(r) = 1$, i.e., $n = 0$, the coefficients $\mathcal{C}_j = 0$ for all $j \geq 1$ and the recurrence relation (28) for $k = 0, 1$ gives, respectively, the necessary and sufficient conditions

$$\varepsilon_{1;0} = 0\,, \qquad \varepsilon_{0;0} = 0. \tag{54}$$

- For a first-degree polynomial solution, $n = 1$, the coefficients $\mathcal{C}_j = 0$ for all $j \geq 2$ where $k = 0, 1, 2$ give the following three equations

$$\begin{cases} \varepsilon_{0;1}\,\mathcal{C}_0 + \beta_0\,\mathcal{C}_1 = 0, \\ \varepsilon_{1;1}\,\mathcal{C}_0 + (\beta_1 + \varepsilon_{0;1})\,\mathcal{C}_1 = 0, \\ (\beta_2 + \varepsilon_{1;1})\,\mathcal{C}_1 = 0. \end{cases} \tag{55}$$

So, for $\mathcal{C}_0 = 1$, it is necessary that $\varepsilon_{1;1} = -\beta_2$ and therefore, $\mathcal{C}_1 = -\varepsilon_{0;1}/\beta_0$ where $\varepsilon_{0;1}$ are now the roots of the quadratic equation

$$\beta_0\,\beta_2 + \beta_1\varepsilon_{0;1} + \varepsilon_{0;1}^2 = 0.$$

Let $\varepsilon^{\ell}_{0;1}, \ell = 1, 2$, denote, if any, the two distinct real roots $\varepsilon^{0}_{0;1} \neq \varepsilon^{1}_{0;1}$ of this quadratic equation. Then, for the two (distinct) differential equations

$$\begin{aligned} &\big(\alpha_0 + \alpha_1\,r + \alpha_2\,r^2 + \alpha_3\,r^3\big)\mathbf{P}''_{1;\ell}\,(r) + \big(\beta_0 + \beta_1\,r + \beta_2\,r^2\big)\,\mathbf{P}'_{1;\ell}\,(r) \\ &\quad + \big(\varepsilon^{\ell}_{0;1} - \beta_2\,r\big)\,\mathbf{P}_{1;\ell}\,(r) = 0\,, \quad \ell = 1,\,2\,, \end{aligned} \tag{56}$$

the first-order polynomial solutions are

$$\begin{cases} \mathbf{P}_{1;\ell}\,(r) = 1 - \dfrac{\varepsilon^{\ell}_{0;1}}{\beta_0}\,r, \\ \beta_0\,\beta_2 + \beta_1\,\varepsilon^{\ell}_{0;1} + (\varepsilon^{\ell}_{0;1})^2 = 0, \qquad \ell = 1,\,2\,. \end{cases} \tag{57}$$

- For a second-degree polynomial solution, $n = 2$, the coefficients $\mathcal{C}_j = 0$ for all $j \geq 3$ where $k = 0,\,1,\,2,\,3$ give the four linear equations

$$\begin{cases} \varepsilon_{0;2}\,\mathcal{C}_0 + \beta_0\,\mathcal{C}_1 + 2\,\alpha_0\,\mathcal{C}_2 = 0, \\ \varepsilon_{1;2}\,\mathcal{C}_0 + (\beta_1 + \varepsilon_{0;2})\,\mathcal{C}_1 + 2(\alpha_1 + \beta_0)\,\mathcal{C}_2 = 0, \\ (\beta_2 + \varepsilon_{1;2})\,\mathcal{C}_1 + (2\,\alpha_2 + 2\,\beta_1 + \varepsilon_{0;2})\,\mathcal{C}_2 = 0, \\ (2\alpha_3 + 2\,\beta_2 + \varepsilon_{1;2})\mathcal{C}_2 = 0. \end{cases} \tag{58}$$

The very last equation of (58), correspondent to $k = 3$, gives the necessary condition

$$\varepsilon_{1;2} = -2\,\alpha_3 - 2\,\beta_2\,, \tag{59}$$

and for $k = 0, 1$, the coefficients of the polynomial solution $y(r) = 1 + \mathcal{C}_1\,r + \mathcal{C}_2\,r^2$ read

$$\begin{cases} \mathcal{C}_1 = \dfrac{\begin{vmatrix} -\varepsilon_{0;2} & 2\alpha_0 \\ 2\,\alpha_3 + 2\,\beta_2 & 2\alpha_1 + 2\beta_0, \end{vmatrix}}{\begin{vmatrix} \beta_0 & 2\alpha_0 \\ \beta_1 + \varepsilon_{0;2} & 2\alpha_1 + 2\beta_0 \end{vmatrix}}, \\[4ex] \mathcal{C}_2 = \dfrac{\begin{vmatrix} \beta_0 & -\varepsilon_{0;2} \\ \beta_1 + \varepsilon_{0;2} & 2\,\alpha_3 + 2\,\beta_2 \end{vmatrix}}{\begin{vmatrix} \beta_0 & 2\alpha_0 \\ \beta_1 + \varepsilon_{0;2} & 2\alpha_1 + 2\beta_0 \end{vmatrix}}. \end{cases} \tag{60}$$

The equation corresponding to $k = 2$ and $n = 2$ establishes the sufficient condition

$$\begin{vmatrix} \varepsilon^{\ell}_{0;2} & \beta_0 & 2\alpha_0 \\ -2\,\alpha_3 - 2\,\beta_2 & \beta_1 + \varepsilon^{\ell}_{0;2} & 2\alpha_1 + 2\beta_0 \\ 0 & \beta_2 - 2\,\alpha_3 - 2\,\beta_2 & 2\,\alpha_2 + 2\,\beta_1 + \varepsilon^{\ell}_{0;2} \end{vmatrix} = 0, \tag{61}$$

where $\ell = 1, 2, 3$ refers to the three distinct simple roots $\varepsilon^{\ell}_{0;2}$, $\ell = 1, 2, 3$, if any, of the polynomial generated by the determinant (61). Hence, for each index $\ell = 1, 2, 3$, the differential equation

$$\begin{aligned} &\left(\alpha_0 + \alpha_1\, r + \alpha_2\, r^2 + \alpha_3\, r^3\right) \mathbf{P}''_{2;\ell}\,(r) + \left(\beta_0 + \beta_1\, r + \beta_2\, r^2\right) \mathbf{P}'_{2;\ell}\,(r) \\ &\quad + \left(\varepsilon^{\ell}_{0;2} - (2\alpha_3 + 2\beta_2)\, r\right) \mathbf{P}_{2;\ell}\,(r) = 0\,, \end{aligned} \tag{62}$$

has the polynomial solution (for $\ell = 1,\, 2,\, 3\,.$)

$$\mathbf{P}_{2;\ell}\,(r) = 1 + \frac{\begin{vmatrix} -\varepsilon^{\ell}_{0;2} & 2\alpha_0 \\ 2\,\alpha_3 + 2\,\beta_2 & 2\alpha_1 + 2\beta_0 \end{vmatrix}}{\begin{vmatrix} \beta_0 & 2\alpha_0 \\ \beta_1 + \varepsilon^{\ell}_{0;2} & 2\alpha_1 + 2\beta_0 \end{vmatrix}}\, r + \frac{\begin{vmatrix} \beta_0 & -\varepsilon^{\ell}_{0;2} \\ \beta_1 + \varepsilon^{\ell}_{0;2} & 2\,\alpha_3 + 2\,\beta_2 \end{vmatrix}}{\begin{vmatrix} \beta_0 & 2\alpha_0 \\ \beta_1 + \varepsilon^{\ell}_{0;2} & 2\alpha_1 + 2\beta_0 \end{vmatrix}}\, r^2, \tag{63}$$

The above constructive approach can be continued to generate higher-order polynomial solutions of an arbitrary degree.

Theorem 3.2. *Suppose the polynomial in $\varepsilon^{\ell}_{0;n}$ generated by the determinant (52) has $n+1$ distinct real roots arranged in ascending order $\varepsilon^{0}_{0;n} < \varepsilon^{1}_{0;n} < \varepsilon^{2}_{0;n} < \cdots < \varepsilon^{n}_{0;n}$, then, the eigenvalue problem*

$$\begin{aligned} &\left(\alpha_0 + \alpha_1\, r + \alpha_2\, r^2 + \alpha_3\, r^3\right) \frac{d^2\boldsymbol{P}_{n;\ell}}{dr^2} + \left(\beta_0 + \beta_1\, r + \beta_2\, r^2\right) \frac{d\boldsymbol{P}_{n;\ell}}{dr} \\ &\quad - n\big(\,(n-1)\,\alpha_3 + \beta_2\big)\, r\, \boldsymbol{P}_{n;\ell} = -\varepsilon^{\ell}_{0;n}\, \boldsymbol{P}_{n;\ell}, \end{aligned} \tag{64}$$

has a polynomial solution of the degree n, for $\ell = 1,\, 2,\, \ldots,\, n+1$.

This theorem is illustrated by Figure 1, for $n = 0, 1, 2, 3,$

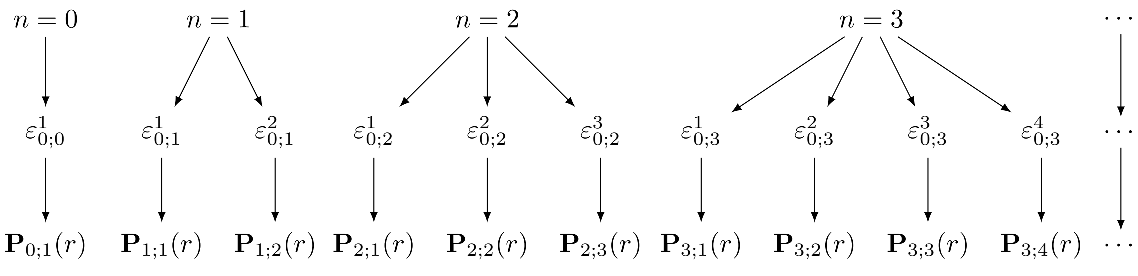

Figure 1. A graphical representation of Theorem 3.2

Open problem: *It is an open question to establish the condition(s) on the parameters so that the polynomial generated by the determinant (32) has simple and real distinct roots.*

4. The solutions in the neighbourhood of a singular point

4.1. Series Solution and infinite sequence of orthogonal polynomials $\{\mathcal{P}_k(\varepsilon_0)\}_{k=0}^{\infty}$

As mentioned earlier, if $\alpha_0 = 0$, there are seven subclasses characterized by the equation

$$r\,\left(\alpha_1 + \alpha_2\, r + \alpha_3 r^2\right) y'' + \left(\beta_0 + \beta_1\, r + \beta_2\, r^2\right) y' + \left(\varepsilon_0 + \varepsilon_1\, r\right) y = 0\,. \tag{65}$$

The classification of these seven equations along with their singularities and the associated domains are summarized in Table 3. From this Table, it is noted that if $\alpha_1 \neq 0$, there are four subclasses where the point $r = 0$ is a regular singular point, while if $\alpha_1 = 0$, the condition $\beta_0 = 0$ is necessary to ensure that $r = 0$ is a regular singular point for two additional subclasses and the last equation is a class where $r = 0$ is irregular singular point unless we reduce to Euler's type ($\alpha_1 = \alpha_2 = \beta_1 = \beta_0 = \varepsilon_0 = 0$).

In the neighbourhood of the regular singular point $r = 0$, the formal series solution $y(r) = r^s \sum_{k=0}^{\infty} \mathcal{C}_k r^k$ is then valid within the interval $(0, \zeta)$ where ζ is the nearest singular point obtained via the roots of the quadratic equation $\alpha_1 + \alpha_2\, r + \alpha_3 r^2 = 0$. Here, s are the roots of the indicial equation $\alpha_1\, s(s-1) + \beta_0\, s = 0$, i.e. $s_1 = 0$ and $s = 1 - \beta_0/\alpha_1$.

Using Frobenius method, it is straightforward to show that the coefficients $\{\mathcal{C}_k\}_{k=0}^{\infty}$ satisfy the three-term recurrence relation

$$\begin{aligned}(k+s+1)\big(\alpha_1(k+s)+\beta_0\big)\mathcal{C}_{k+1} &+ \big((k+s)[\alpha_2\,(k+s-1)+\beta_1]+\varepsilon_0\big)\mathcal{C}_k \\ &+ \big((k+s-1)[\alpha_3(k+s-2)+\beta_2]+\varepsilon_1\big)\mathcal{C}_{k-1} = 0\,,\end{aligned} \tag{66}$$

where $k = 1, 2, \ldots$. For

$$\begin{cases} \mathcal{C}_{-1} = 0, \\ \mathcal{C}_0 = 1, \\ \mathcal{C}_1 = -\dfrac{s(\alpha_2\,(s-1)+\beta_1)+\varepsilon_0}{(\alpha_1\, s+\beta_0)(s+1)} = -\dfrac{P_{1;s}(\varepsilon_0)}{\alpha_1\left(s+\frac{\beta_0}{\alpha_1}\right)(s+1)}, \end{cases}$$

this equation can be written as

$$\mathcal{C}_{k+2} = \lambda_0(k)\,\mathcal{C}_{k+1} + s_0(k)\,\mathcal{C}_k,$$

where

$$\begin{cases} \lambda_0(k) = -\dfrac{(\alpha_2\,(k+s)+\beta_1)\ (k+s+1)+\varepsilon_0}{(\alpha_1(k+s+1)+\beta_0)\,(k+s+2)}, \\ s_0(k) = -\dfrac{(\alpha_3(k+s-1)+\beta_2)\,(k+s)+\varepsilon_1}{(\alpha_1(k+s+1)+\beta_0)\,(k+s+2)}, \end{cases}$$

From this equation, we note that

$$\mathcal{C}_{k+3} = \lambda_1(k)\,\mathcal{C}_{k+1} + s_1(k)\,\mathcal{C}_k, \qquad \begin{cases} \lambda_1(k) = \lambda_0(k+1)\,\lambda_0(k) + s_0(k+1) \\ s_1(k) = \lambda_0(k+1)\,s_0(k), \end{cases}$$

$$\mathcal{C}_{k+4} = \lambda_2(k)\,\mathcal{C}_{k+1} + s_2(k)\,\mathcal{C}_k, \qquad \begin{cases} \lambda_2(k) = \lambda_1(k+1)\lambda_0(k) + s_1(k+1) \\ s_2(k) = \lambda_1(k+1)s_0(k), \end{cases}$$

$$\mathcal{C}_{k+5} = \lambda_3(k)\,\mathcal{C}_{k+1} + s_3(k)\,\mathcal{C}_k, \qquad \begin{cases} \lambda_3(k) = \lambda_2(k+1)\lambda_0(k) + s_2(k+1) \\ s_3(k) = \lambda_2(k+1)s_0(k), \end{cases}$$

and in general

$$\mathcal{C}_{k+m} = \lambda_{m-2}(k)\,\mathcal{C}_{k+1} + s_{m-2}(k)\,\mathcal{C}_k, \qquad \begin{cases} \lambda_m(k) = \lambda_{m-1}(k+1)\lambda_0(k) + s_{m-1}(k+1) \\ s_m(k) = \lambda_{m-1}(k+1)s_0(k), \end{cases}$$

and therefore

$$\mathcal{C}_2 = \frac{(s+1)(\alpha_2\, s+\beta_1)+\varepsilon_0}{(\alpha_1(s+1)+\beta_0)(s+2)}\left(\frac{s(\alpha_2\,(s-1)+\beta_1)+\varepsilon_0}{(\alpha_1\, s+\beta_0)(s+1)}\right) - \frac{s(\alpha_3(s-1)+\beta_2)+\varepsilon_1}{(\alpha_1(s+1)+\beta_0)(s+2)} = \frac{P_{2;s}(\varepsilon_0)}{\alpha_1^2\left(s+\frac{\beta_0}{\alpha_1}\right)_2(s+1)_2}. \tag{67}$$

DE	α_3	α_2	α_1	Condition	Roots of LPC	Domain definition
I	α_3	α_2	α_1	$a_2^2-4a_1a_3>0$	$r=0,\quad \xi_+\neq\xi_-$	$r\in(0,\min\xi_\pm)$ if $\xi_\pm>0$
						$r\in(\max\xi_\pm,0)$ if $\xi_\pm<0$
						$r\in(0,\xi_+)$ if $\xi_-<0<\xi_+, \lvert\xi_-\rvert>\xi_+$
						$r\in(\xi_-,0)$ if $\xi_-<0<\xi_+, \lvert\xi_-\rvert<\xi_+$
				$a_2^2-4a_1a_3=0$	$r=0,\quad \xi_+=\xi_-=\xi$	$r\in(0,\xi)$
	Differential Equation:			$r\,(\xi_1-r)(\xi_2-r)\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0$		
	Roots:			$r=0;\quad r=\xi_\pm\equiv(-\alpha_2\pm\sqrt{\alpha_2^2-4\alpha_1\alpha_3})/(2\alpha_3)$		
	Singularity:			$r=0,\xi_\pm,\infty$: Regular		
	Differential Equation:			$r\,(\xi-r)^2\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0$		
	Roots:			$r=0;\quad r=\xi\equiv-\alpha_2/(2\alpha_3)$		
	Singularity:			$r=0,\xi$: Regular; $r=\infty$: Irregular		
II	0	α_2	α_1		$r=0,\quad r=-\alpha_1/\alpha_2$	$r\in(0,-\alpha_1/\alpha_2)$ if $\alpha_1/\alpha_2<0$
						$r\in(-\alpha_1/\alpha_2,0)$ if $\alpha_1/\alpha_2>0$
	Differential Equation:			$r(\alpha_1+\alpha_2\,r)\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0$		
	Singularity:			$r=0,-\alpha_1/\alpha_2$: Regular; $r=\infty$: Irregular		
III	α_3	0	α_1	$a_1a_3<0$	$r=0,\pm\sqrt{-\alpha_1/\alpha_3}$	$r\in(0,\sqrt{-\alpha_1/\alpha_3})$
				$\alpha_1\alpha_3>0$	$r=0,$	$r\in(0,\infty)$
	Differential Equation:			$r\,(\alpha_3r^2+\alpha_1)\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0,\quad \alpha_3\alpha_1<0$		
	Singularity:			$r=0,\pm\sqrt{-\alpha_1/\alpha_3},\infty$: Regular		
	Differential Equation:			$r\,(\alpha_3r^2+\alpha_1)\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0,\quad \alpha_3\alpha_1>0$		
	Singularity:			$r=0$: Regular; $r=\infty$: Irregular		
IV	α_3	α_2	0	$\beta_0=0$	$r=0,\quad -\alpha_2/\alpha_3$	$r\in(0,-\alpha_2/\alpha_3)$ if $\alpha_2/\alpha_3<0$
						$r\in(-\alpha_2/\alpha_3,0)$ if $\alpha_2/\alpha_3>0$
	Differential Equation:			$r^2(\alpha_3r+\alpha_2)\,y''+r(\beta_1+\beta_2\,r)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0$		
	Singularity:			$r=0,-\alpha_2/\alpha_3$: Regular; $r=\infty$: Irregular		
V	0	0	α_1		$r=0$	$r\in(0,\infty)$
	Differential Equation:			$\alpha_1\,r\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0$		
	Singularity:			$r=0$: Regular; $r=\infty$: Irregular		
VI	0	α_2	0	$\beta_0=0$	$r=0$	$r\in(0,\infty)$
	Differential Equation:			$\alpha_2\,r^2\,y''+r(\beta_1+\beta_2\,r)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0$		
	Singularity:			$r=0$: Regular; $r=\infty$: Irregular		
VII	α_3	0	0		$r=0$	$r\in(0,\infty)$
	Differential Equation:			$\alpha_3\,r^3\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0$		
	Singulaity:			$r=0,\infty$: Irregular		

Table 3. Tabulating the seven different types of differential equations, which apply to Theorem 4.1.

initiated with

$$P_{2,s}(\varepsilon) = ((s+1)(\alpha_2\, s + \beta_1) + \varepsilon_0) P_{1;s}(\varepsilon_0) - (\alpha_1\, s + \beta_0)(s+1)(s(\alpha_3(s-1) + \beta_2) + \varepsilon_1)$$

Continuing with this process, it is straightforward to conclude that the series solution can be written as

$$y(r) = r^s \sum_{k=0}^{\infty} \mathcal{C}_k\, r^k = \sum_{k=0}^{\infty} (-1)^k \frac{P_{k;s}(\varepsilon_0)}{\alpha_1^k \left(\frac{\beta_0}{\alpha_1} + s\right)_k (1+s)_k}\, r^{k+s}, \tag{68}$$

where the k-degree polynomials of the parameter ε_0, namely $\{P_{k;s}(\varepsilon_0)\}_{k=0}^{\infty}$, satisfy the following three-term recurrence relation:

$$\begin{aligned} P_{k+1;s}(\varepsilon_0) = \big((k+s)\big[(k+s-1)\alpha_2 + \beta_1\big] + \varepsilon_0\big) P_{k;s}(\varepsilon_0) - (k+s)\big((k+s-1)\alpha_1 + \beta_0\big)\big((k+s-1) \\ \times \big[(k+s-2)\alpha_3 + \beta_2\big] + \varepsilon_1\big) P_{k-1;s}(\varepsilon_0), \end{aligned} \tag{69}$$

initiated with $P_{-1;s}(\varepsilon_0) = 0$ and $P_{0;s}(\varepsilon_0) = 1$.

For the classes I-IV in Table 3, including, of course, the classical Heun equation, $r = 0$ is a regular singular point with one of the exponents of singularities being $s = 0$, in which case, the coefficients $\{\mathcal{C}_k\}_{k=0}^{\infty}$ of the series solution $y(r) = \sum_{k=0}^{\infty} \mathcal{C}_k r^k$ satisfy the three-term recurrence relation

$$\big((k+1)(k\,\alpha_1 + \beta_0)\big)\mathcal{C}_{k+1} + \big(k\big((k-1)\alpha_2 + \beta_1\big) + \varepsilon_0\big)\mathcal{C}_k + \big((k-1)\big((k-2)\alpha_3 + \beta_2\big) + \varepsilon_1\big)\mathcal{C}_{k-1} = 0, \tag{70}$$

and we have the following general result concerning the series solutions of the equation (65):

Theorem 4.1. *In the neighbourhood of the regular singular point $r = 0$, the series solution $y(r) = \sum_{k=0}^{\infty} \mathcal{C}_k r^k$ of the differential equation (65), with $\alpha_1 \neq 0$, is explicitly given by*

$$y(r) = \sum_{k=0}^{\infty} (-1)^k \frac{\mathcal{P}_k(\varepsilon_0)}{k!\, \alpha_1^k \left(\dfrac{\beta_0}{\alpha_1}\right)_k} r^k, \tag{71}$$

where the infinite sequence $\{\mathcal{P}_k(\varepsilon_0)\}_{k=0}^{\infty}$ is evaluated using the three-term recurrence relation

$$\begin{aligned} \mathcal{P}_{k+1}(\varepsilon_0) = \big(k(k-1)\alpha_2 + k\beta_1 + \varepsilon_0\big)\mathcal{P}_k(\varepsilon_0) - k\big((k-1)\alpha_1 + \beta_0\big) \\ \times \big((k-1)(k-2)\alpha_3 + (k-1)\beta_2 + \varepsilon_1\big)\mathcal{P}_{k-1}(\varepsilon_0), \end{aligned} \tag{72}$$

where $\mathcal{P}_{-1}(\varepsilon_0) = 0$, and $\mathcal{P}_0(\varepsilon_0) = 1$.

Here, $(\alpha)_n$ refers to the Pochhammer symbol $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha-n+1) = \Gamma(\alpha+n)/\Gamma(\alpha)$ which is defined in terms of Gamma functions and satisfies the identity $(-n)_k = 0$ for any positive integers $k \geq n+1$. Equation (72) in Theorem follows directly by substituting the coefficients of (71) in the recurrence relation (65) and eliminates the common terms.

Corollary 4.2. *In the neighbourhood of the regular singular point $r = 0$, the series solution $y(r) = \sum_{k=0}^{\infty} \mathcal{C}_k r^k$ of the differential equation*

$$r\,(\alpha_1 + \alpha_3 r^2)\, y'' + (\beta_0 + \beta_1\, r + \beta_2\, r^2)\, y' + (\varepsilon_0 + \varepsilon_1\, r)\, y = 0, \tag{73}$$

is given, explicitly, by

$$y(r) = \sum_{k=0}^{\infty} (-1)^k \frac{\mathfrak{P}_k(\varepsilon_0)}{k!\, \alpha_1^k \left(\dfrac{\beta_0}{\alpha_1}\right)_k} r^k, \tag{74}$$

where

$$\mathfrak{P}_{k+1}(\varepsilon_0) = (k\,\beta_1 + \varepsilon_0)\,\mathfrak{P}_k(\varepsilon_0) - k\,((k-1)\alpha_1 + \beta_0) \times ((k-1)(k-2)\alpha_3 + (k-1)\beta_2 + \varepsilon_1)\,\mathfrak{P}_{k-1}(\varepsilon_0), \tag{75}$$

initiated with $\mathfrak{P}_{-1}(\varepsilon_0) = 0,\ \mathfrak{P}_0(\varepsilon_0) = 1$.

Corollary 4.3. *In the neighbourhood of the regular singular point $r=0$, the series solution $y(r)=\sum_{k=0}^\infty C_k r^k$ of the differential equation*

$$r\,(\alpha_1+\alpha_2\,r)\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0, \tag{76}$$

is given, explicitly, by

$$y(r)=\sum_{k=0}^\infty(-1)^k\,\frac{\mathcal{P}_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k}\,r^k, \tag{77}$$

where

$$\mathcal{P}_{k+1}(\varepsilon_0)=(k(k-1)\alpha_2+k\beta_1+\varepsilon_0)\mathcal{P}_k(\varepsilon_0)-k((k-1)\alpha_1+\beta_0)((k-1)\beta_2+\varepsilon_1)\mathcal{P}_{k-1}(\varepsilon_0), \tag{78}$$

initiated with $\mathcal{P}_{-1}(\varepsilon_0)=0,\ \mathcal{P}_0(\varepsilon_0)=1$.

Corollary 4.4. *In the neighbourhood of the regular singular point $r=0$, the series solution $y(x)=\sum_{k=0}^\infty C_k r^k$ of the differential equation*

$$\alpha_1\,r\,y''+(\beta_0+\beta_1\,r+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0, \tag{79}$$

is given, explicitly, by

$$y(r)=\sum_{k=0}^\infty(-1)^k\,\frac{\mathbb{P}_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k}\,r^k, \tag{80}$$

where

$$\mathbb{P}_{k+1}(\varepsilon_0)=(k\beta_1+\varepsilon_0)\mathbb{P}_k(\varepsilon_0)-k\,((k-1)\,\alpha_1+\beta_0)((k-1)\beta_2+\varepsilon_1)\mathbb{P}_{k-1}(\varepsilon_0), \tag{81}$$

initiated with $\mathbb{P}_{-1}(\varepsilon_0)=0,\ \mathbb{P}_0(\varepsilon_0)=0$.

Corollary 4.5. *In the neighbourhood of the regular singular point $x=0$, the series solution $y(x)=\sum_{k=0}^\infty C_k x^k$ of the differential equation*

$$\alpha_1 r\,y''+(\beta_0+\beta_2\,r^2)\,y'+(\varepsilon_0+\varepsilon_1\,r)\,y=0, \tag{82}$$

is given, explicitly, by

$$y(r)=\sum_{k=0}^\infty(-1)^k\,\frac{\mathbf{P}_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k}\,r^k, \tag{83}$$

where

$$\mathbf{P}_{k+1}(\varepsilon_0)=\varepsilon_0\mathbf{P}_k(\varepsilon_0)-k((k-1)\alpha_1+\beta_0)((k-1)\beta_2+\varepsilon_1)\mathbf{P}_{k-1}(\varepsilon_0), \tag{84}$$

initiated with $\mathbf{P}_{-1}(\varepsilon_0)=0,\mathbf{P}_1(\varepsilon_0)=1$.

Remark 4.6. If, in addition to $\alpha_0=0$, we also have $\alpha_1=0$, then $r=0$ is a regular singular point only if $\beta_0=0$, in which case the differential equation reduces to an equation that resembles Euler's equation, namely

$$r^2\,\big(\alpha_2+\alpha_3 r\big)\,y''+r\big(\beta_1+\beta_2\,r\big)\,y'+\big(\varepsilon_0+\varepsilon_1\,r\big)\,y=0. \tag{85}$$

The exponents of the singularity $r=0$ are

$$s_\pm=\big(\alpha_2-\beta_1\pm\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}\big)/(2\alpha_2).$$

From the relation (66), the coefficients of the formal series solution $y(r)=r^s\sum_{k=0}^\infty \mathcal{C}_k\,r^k$ satisfy the two-term recurrence relation ($k=1,2,\dots,\ \mathcal{C}_0=1$),

$$\mathcal{C}_k=-\tfrac{(k-1+s_\pm)(k-2+s_\pm)\alpha_3+(k-1+s_\pm)\beta_2+\varepsilon_1}{(k+s_\pm)(k-1+s_\pm)\alpha_2+(k+s_\pm)\beta_1+\varepsilon_0}\mathcal{C}_{k-1},=\prod_{j=1}^k(-1)^j\,\tfrac{(j-1+s_\pm)(j-2+s_\pm)\alpha_3+(j-1+s_\pm)\beta_2+\varepsilon_1}{(j+s_\pm)(j-1+s_\pm)\alpha_2+(j+s_\pm)\beta_1+\varepsilon_0}, \tag{86}$$

that allows to obtain a closed form of the series solution of (71) in terms of the generalized hypergeometric function as

$$\begin{aligned}y(r)=&r^{s_\pm}\,{}_3F_2\Big(1,s_\pm-\tfrac12+\tfrac{\beta_2}{2\alpha_3}-\tfrac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3},\ s_\pm+\tfrac12+\frac{\beta_2}{2\alpha_3}-\tfrac{\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3};\\ &s_\pm+\tfrac12+\tfrac{\beta_1}{2\alpha_2}-\tfrac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2},\,s_\pm+\tfrac12+\tfrac{\beta_1}{2\alpha_2}+\tfrac{\sqrt{(\alpha_2-\beta_1)^2-4\alpha_2\varepsilon_0}}{2\alpha_2};-\frac{\alpha_3}{\alpha_2}\,r\Big).\end{aligned} \tag{87}$$

4.2. Polynomial Solution and finite sequence of orthogonal polynomials

Theorem 4.7. *The necessary condition for the second-order linear differential equation (65) to have an n^{th}-degree polynomial solution $y_n(r) = \sum_{k=0}^{n} \mathcal{C}_k\, r^k$, $n = 0, 1, 2, \ldots$, in the neighbourhood of the regular singular point $r = 0$ with one of the indicial equation exponents $s = 0$, is*

$$\varepsilon_{1;n} = -n\,(n-1)\,\alpha_3 - n\,\beta_2\,, \quad n = 0, 1, 2, \ldots\,, \tag{88}$$

along with the sufficient condition, relating the remaining coefficients, given by the vanishing of the tridiagonal $(n+1)\times(n+1)$-determinant $\Delta_{n+1} \equiv 0$ given by

$$\Delta_{n+1} = \begin{vmatrix} S_0 & T_1 & & & & & \\ \gamma_1 & S_1 & T_2 & & & & \\ & \gamma_2 & S_2 & T_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma_{n-2} & S_{n-2} & T_{n-1} & \\ & & & & \gamma_{n-1} & S_{n-1} & T_n \\ & & & & & \gamma_n & S_n \end{vmatrix}\,, \tag{89}$$

where, for fixed n : $\varepsilon_{1;n} = -n\,(n-1)\,\alpha_3 - n\,\beta_2$,

$$\begin{cases} S_k = \varepsilon_{0;n} + k\big((k-1)\alpha_2 + \beta_1\big), \\ T_k = -k\big((k-1)\alpha_1 + \beta_0\big), \\ \gamma_k = -\varepsilon_{1;n} - (k-1)\big((k-2)\alpha_3 + \beta_2\big), \end{cases}$$

and all other entries are zeros. In this case, the polynomial solutions are given explicitly by

$$y_n(r) = \sum_{k=0}^{n} (-1)^k\, \frac{\mathcal{P}_k^n(\varepsilon_{0;n})}{k!\,\alpha_1^k \left(\dfrac{\beta_0}{\alpha_1}\right)_k}\, r^k\,, \tag{90}$$

where the <u>finite</u> orthogonal sequences $\{\mathcal{P}_k^n(\varepsilon_{0;n})\}_{k=0}^{n}$ are evaluated using the three-term recurrence relation

$$\mathcal{P}_{k+1}^n(\varepsilon_{0;n}) = \big(S_k + \varepsilon_{0;n}\big)\mathcal{P}_k^n(\varepsilon_{0;n}) - \gamma_k T_k \mathcal{P}_{k-1}^n(\varepsilon_{0;n})\,,$$

or, more explicitly,

$$\begin{aligned} \mathcal{P}_{k+1}^n(\varepsilon_{0;n}) &= \big(k(k-1)\alpha_2 + k\beta_1 + \varepsilon_{0;n}\big)\mathcal{P}_k^n(\varepsilon_{0;n}) + k(n-k+1)\big((k-1)\alpha_1 + \beta_0\big) \\ &\quad \times \big(\beta_2 + \alpha_3(k+n-2)\big)\mathcal{P}_{k-1}^n(\varepsilon_{0;n})\,, \end{aligned} \tag{91}$$

where $\mathcal{P}_{-1}^n(\varepsilon_{0;n}) = 0$, and $\mathcal{P}_0^n(\varepsilon_{0;n}) = 1$ for the non-negative integer n.

Expanding Δ_{k+1} with respect to the last column, it is clear that the determinant (89) satisfies a three-term recurrence relation

$$\begin{cases} \Delta_{k+1} = (S_k + \varepsilon_{0;n})\,\Delta_k - \gamma_k\,T_k\,\Delta_{k-1}, \\ \Delta_0 = 1, \quad \Delta_{-1} = 0, \quad k = 0, 1, \ldots, n\,, \end{cases} \tag{92}$$

that allow to compute the determinant Δ_k recursively in terms of lower-order determinants. We now show, by induction on k, that

$$\Delta_{k+1} = P_{k+1}(\varepsilon_{0;n}). \tag{93}$$

For $k = 0$, we find by (89) that $\Delta_1 = (S_0 + \varepsilon_{0;n})$ where the right hand side equals to $P_1^n(\varepsilon_{0;n})$ using (91). Next, suppose that $\Delta_j = P_j(\varepsilon_{0;n})$, for $j = 0, 1, 2, \cdots, k$, then from (91)

$$\mathcal{P}_{k+1}^n(\varepsilon_{0;n}) = \big(S_k + \varepsilon_{0;n}\big)\,\mathcal{P}_k^n(\varepsilon_{0;n}) - \gamma_k\,T_k\,\mathcal{P}_{k-1}^n(\varepsilon_{0;n}) = \big(S_k + \varepsilon_{0;n}\big)\,\Delta_k - \gamma_k\,T_k\,\Delta_{k-1} = \Delta_{k+1}$$

and the induction step is reached. These results can be represented by the graphical representation (Figure 2).

Some of the mathematical properties of the finite sequence of polynomials $\{\mathcal{P}_k^n(\varepsilon_{0;n})\}_{k=0}^{n}$ will be explored in later sections.

$$\Delta_{k+1} = \det \begin{pmatrix} S_0 & T_1 & 0 & \vdots & 0 \\ \gamma_1 & S_1 & T_2 & \vdots & 0 \\ 0 & \gamma_2 & S_2 & \vdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S_n \end{pmatrix} \quad \begin{matrix} \mathcal{P}_1^n & \mathcal{P}_2^n & \mathcal{P}_3^n & \mathcal{P}_n^n \end{matrix}$$

Figure 2. A demonstration of how the polynomials $\{\mathcal{P}_k^n(\varepsilon_{0;n})\}_{k=0}^n$ may be obtained from the $(k+1)$-determinant Δ_{k+1} for $k = 0, 1, 2, \ldots, n$.

Remark 4.8. For $\alpha_3 + \alpha_2 + \alpha_1 = 0$, the canonical form of Heun's equation can be deduced from (65) by means of the following substitutions:

$$y''(r) + \left(\frac{\frac{\beta_0+\beta_1+\beta_3}{\alpha_3-\alpha_1}}{r-1} + \frac{\frac{\beta_0}{\alpha_1}}{r} + \frac{\frac{\alpha_3^2\beta_0+\alpha_1\alpha_3\beta_1+\alpha_1^2\beta_2}{\alpha_1\alpha_3(\alpha_1-\alpha_3)}}{(r-\frac{\alpha_1}{\alpha_3})} \right) y'(r) + \frac{\frac{\varepsilon_1}{\alpha_3} r + \frac{\varepsilon_0}{\alpha_3}}{r\,(r-1)\left(r-\frac{\alpha_1}{\alpha_3}\right)}\, y(r) = 0. \tag{94}$$

or, simply in the standard form as

$$y''(r) + \left(\frac{\gamma}{r} + \frac{\delta}{r-1} + \frac{\varepsilon}{r-b} \right) y'(r) + \frac{\alpha\,\beta\; r - q}{r(r-1)(r-b)} y(r) = 0, \tag{95}$$

where

$$\begin{array}{ccccccc} \gamma & \delta & \varepsilon & \alpha & \beta & q & b \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \\ \dfrac{\beta_0}{\alpha_1} & \dfrac{\beta_2+\beta_1+\beta_0}{\alpha_3-\alpha_1} & \dfrac{\beta_2\alpha_1^2+\beta_1\alpha_1\alpha_3+\beta_0\alpha_3^2}{\alpha_3\alpha_1(\alpha_1-\alpha_3)} & \dfrac{\beta_2+(n-1)\alpha_3}{\alpha_3} & -n & -\dfrac{\varepsilon_0}{\alpha_3} & \dfrac{\alpha_1}{\alpha_3} \\ \Uparrow & \Uparrow & \Uparrow & \Uparrow & \Uparrow & \Uparrow & \\ \gamma & \delta & \varepsilon & \beta & \alpha & q & b \end{array}$$

where, in either case, it follows

$$\gamma + \delta + \varepsilon = \alpha + \beta + 1$$

that ensures the regularity of the singular point ∞. With these parameters, the Sturm-Liouville form of the differential equation (65) is

$$-\frac{d}{dr}\left(r^\gamma\,(r-1)^\delta\,(r-b)^\varepsilon\,\frac{dy}{dr} \right) + \alpha\,\beta\, r^\gamma\,(r-1)^{\delta-1}(r-b)^{\varepsilon-1} y = q\, r^{\gamma-1}(r-1)^{\delta-1}(r-b)^{\varepsilon-1}\, y \tag{96}$$

where, for $b \ge 1,\ \gamma \ge 0,\ \delta \ge 1,\ r \in (0,1)$.

Corollary 4.9. *The second-order linear differential equation*

$$r^2(\alpha_3\, r + \alpha_2) y''(r) + r\,(\beta_2\, r + \beta_1)\; y' + (-(n(n-1)\; \alpha_3 + n\; \beta_2)\, r + \varepsilon_0)\; y = 0, \tag{97}$$

where $r \in (-\alpha_2/\alpha_3, 0)$ if $\alpha_2\alpha_3 > 0$ or $r \in (0, \alpha_2/\alpha_3)$ if $\alpha_2\alpha_3 < 0$, has a polynomial solution of degree n subject to

$$\begin{cases} \prod\limits_{k=0}^{n} (\varepsilon_0 + k((k-1)\alpha_2 + \beta_1) = 0 \\ \Longrightarrow \\ \varepsilon_0 = -n\,(n-1)\,\alpha_2 - n\,\beta_1, \quad n = 0, 1, 2, \cdots . \end{cases} \tag{98}$$

In particular, the differential equation

$$r^2(\alpha_3 r+\alpha_2)\ y''(r)+r(\beta_2\, r+\beta_1)\ y'(r)-\Big((n\,(n-1)\ \alpha_3+n\ \beta_2)\,r+n\,(n-1)\ \alpha_2+n\ \beta_1\Big)\ y(r)=0, \tag{99}$$

has the polynomial solutions

$$y_n(r)=r^n,\quad n=0,1,2,\dots. \tag{100}$$

Proof. Follows immediately from Theorem 4.7 with $\alpha_1=\beta_0=0$.

■

5. Mathematical properties of the orthogonal polynomials $\{\mathcal{P}_k(\varepsilon_0)\}_{k=0}^{\infty}$

As pointed out by Theorem 4.1, in the neighbourhood of the singular point $r=0$ with an indicial exponent root zero, the series solution of the differential equation with four singular points, see (65),

$$r\left(\alpha_1+\alpha_2\, r+\alpha_3 r^2\right)y''+\left(\beta_0+\beta_1\, r+\beta_2\, r^2\right)y'+\left(\varepsilon_0+\varepsilon_1\, r\right)y=0\,.$$

can be written as

$$y(r)=\sum_{k=0}^{\infty}(-1)^k\ \frac{\mathcal{P}_k(\varepsilon_0)}{k!\,\alpha_1^k\left(\dfrac{\beta_0}{\alpha_1}\right)_k}r^k\,, \tag{101}$$

where the infinite sequence of polynomials $\{\mathcal{P}_k(\varepsilon_0)\}_{k=0}^{\infty}$ in the real variable ε_0 satisfies the three-term recurrence relation

$$\mathcal{P}_{k+1}(\varepsilon_0)=(\varepsilon_0-\mathcal{A}_k)\mathcal{P}_k(\varepsilon_0)-\mathcal{B}_k\mathcal{P}_{k-1}(\varepsilon_0), \tag{102}$$

initiated with

$$\mathcal{P}_{-1}(\varepsilon_0)=0,\quad \mathcal{P}_0(\varepsilon_0)=1,\ k=1,2,3,\cdots.$$

where

$$\begin{aligned}\mathcal{A}_k&=-k(k-1)\alpha_2-k\beta_1,\\ \mathcal{B}_k&=k\big((k-1)\alpha_1+\beta_0\big)\big((k-1)((k-2)\alpha_3+\beta_2)+\varepsilon_1\big).\end{aligned}$$

For $\mathcal{A}_k,\mathcal{B}_k\in\mathbb{R}$ and if $\mathcal{B}_k>0$, then according to Favard Theorem [25], see also [26, Theorem 2.14], there exists a positive Borel measure μ such that $\{\mathcal{P}_k\}_{k=0}^{\infty}$ is orthogonal with respect to the inner product

$$\langle P_k,P_{k'}\rangle=\int_{\mathbb{R}}P_k(\varepsilon_0)P_{p'}(\varepsilon_0)d\mu \tag{103}$$

such that

$$\int_{\mathbb{R}}P_k(\varepsilon_0)P_{k'}(\varepsilon_0)d\mu=p_kp_{k'}\delta_{kk'},\quad \int_{\mathbb{R}}d\mu=1, \tag{104}$$

where $\delta_{kk'}$ is the Kronecker symbol. In particular,

$$\int_{\mathbb{R}}\varepsilon_0^kP_{k'}(\varepsilon_0)d\mu=0\quad for\ all\ 0<k<k'. \tag{105}$$

The norm p_k can be found using the recurrence relations (102) by multiplying with ε_0^{k-1} and taking the integral over ε_0 with respect to μ that yields

$$\int_{\mathbb{R}}\varepsilon_0^k\mathcal{P}_k(\varepsilon_0)d\mu=\mathcal{B}_k\int_{\mathbb{R}}\varepsilon_0^{k-1}\mathcal{P}_{k-1}(\varepsilon_0)d\mu=\mathcal{B}_k\mathcal{B}_{k-1}\int_{\mathbb{R}}\varepsilon_0^{k-2}\mathcal{P}_{k-2}(\varepsilon_0)d\mu=\cdots=\left(\prod_{j=2}^{k}\mathcal{B}_j\right)\int_{\mathbb{R}}d\mu \tag{106}$$

and

$$\int_{\mathbb{R}} \mathcal{P}_k(\varepsilon_0)\mathcal{P}_{k'}(\varepsilon_0)\, d\mu = \frac{k!\,(\alpha_1\,\alpha_3)^k}{\beta_0\varepsilon_1}\left(\frac{\beta_0}{\alpha_1}\right)_k \times \left(\frac{-\alpha_3+\beta_2-\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}\right)_k$$
$$\times \left(\frac{-\alpha_3+\beta_2+\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}\right)_k \delta_{kk'}. \tag{107}$$

Using the recurrence relation (102), it also follows that

$$\int_{\mathbb{R}} \varepsilon_0[\mathcal{P}_k(\varepsilon_0)]^2 d\mu = -\frac{k((k-1)\alpha_2+\beta_1)k!\,(\alpha_1\,\alpha_3)^k}{\beta_0\varepsilon_1}\left(\frac{\beta_0}{\alpha_1}\right)_k \left(\frac{-\alpha_3+\beta_2-\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}\right)_k$$
$$\times \left(\frac{-\alpha_3+\beta_2+\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}\right)_k. \tag{108}$$

Further, for $k = 0, 1, 2, \cdots$,

$$\int_{\mathbb{R}} \varepsilon_0\mathcal{P}_{k+1}(\varepsilon_0)\mathcal{P}_k(\varepsilon_0) d\mu = \frac{(k+1)!\,(\alpha_1\,\alpha_3)^{k+1}}{\beta_0\varepsilon_1}$$
$$\times \left(\frac{\beta_0}{\alpha_1}\right)_{k+1} \left(\frac{-\alpha_3+\beta_2-\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}\right)_{k+1} \left(\frac{-\alpha_3+\beta_2+\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}\right)_{k+1}. \tag{109}$$

Other integrals can be evaluated similarly, for example $\int_{\mathbb{R}}[\varepsilon_0\mathcal{P}_k(\varepsilon_0)]^2 d\mu$ can be evaluated by multiplying (102) by $\varepsilon_0\mathcal{P}_k(\varepsilon_0)$ and integrate with respect to the measure μ using (107), (108), and (109) and we continue similarly for

$$\int_{\mathbb{R}} \varepsilon_0^m [\mathcal{P}_k(\varepsilon_0)]^2 d\mu, \qquad m = 0, 1, 2, \cdots .$$

The recurrence relations (102) for $x = \varepsilon_0$ and $y = \varepsilon_0'$ read

$$\mathcal{P}_{k+1}(x) = \big(x - \mathcal{A}_k\big)\mathcal{P}_k^n(x) - \mathcal{B}_k\mathcal{P}_{k-1}^n(x)\,,$$
$$\mathcal{P}_{k+1}(y) = \big(y - \mathcal{A}_k\big)\mathcal{P}_k^n(y) - \mathcal{B}_k\mathcal{P}_{k-1}^n(y),$$

respectively. By multiplying the first by $\mathcal{P}_k(y)$ and the second by $\mathcal{P}_k(x)$ and subtracting, the resulting equation becomes

$$(x-y)\mathcal{P}_k(y)\mathcal{P}_k(x) = Q_{k+1}(x,y) - \mathcal{B}_k\, Q_k(x,y) \tag{110}$$

where $Q_{k+1}(x, y) = \mathcal{P}_{k+1}(x)\mathcal{P}_k(y) - \mathcal{P}_k(x)\mathcal{P}_{k+1}(y)$. Thus, recursively over k, we have

$$\begin{aligned}(x-y)\mathcal{P}_k(x)\mathcal{P}_k(y) &= Q_{k+1}(x,y) - \mathcal{B}_k\, Q_k(x,y)\\ (x-y)\mathcal{P}_{k-1}(x)\mathcal{P}_{k-1}(y) &= Q_k(x,y) - \mathcal{B}_{k-1}\, Q_{k-1}(x,y)\\ &\vdots\\ (x-y)\mathcal{P}_0^n(x)\mathcal{P}_0^n(y) &= Q_1(x,y)\,,\end{aligned}$$

from which it is straightforward to obtain

$$(x-y)\Big[\mathcal{P}_k(x)\mathcal{P}_k(y) + \mathcal{B}_k\mathcal{P}_{k-1}(x)\mathcal{P}_{k-1}(y) + \mathcal{B}_k\mathcal{B}_{k-1}\mathcal{P}_{k-2}(x)\mathcal{P}_{k-2}(y)$$
$$+ \mathcal{B}_k\mathcal{B}_{k-1}\mathcal{B}_{k-2}\mathcal{P}_{k-3}(x)\mathcal{P}_{k-3}(y) + \cdots + \lambda_{k+1}\lambda_k\lambda_{k-1}\lambda_{k-2}\ldots\lambda_2\mathcal{P}_0(x)\mathcal{P}_0(y)\Big] = Q_{k+1}(\varepsilon_0, y)\,.$$

Dividing both sides by $(x-y)\mathcal{B}_k\mathcal{B}_{k-1}\mathcal{B}_{k-2}\ldots\mathcal{B}_2$ and summing over k results in

$$\sum_{j=0}^{k} \frac{\mathcal{P}_j(x)\mathcal{P}_j(y)}{\mathcal{B}_j\mathcal{B}_{j-1}\mathcal{B}_{j-2}\ldots\mathcal{B}_2} = (\mathcal{B}_k\mathcal{B}_{k-1}\mathcal{B}_{k-2}\ldots\mathcal{B}_2)^{-1} \times \frac{\mathcal{P}_{k+1}(x)\mathcal{P}_k(y) - \mathcal{P}_k^n(x)\mathcal{P}_{k+1}(y)}{x-y}\,.$$

(101) then follows using

$$\begin{aligned}\mathcal{B}_k\mathcal{B}_{k-1}\mathcal{B}_{k-2}\dots\mathcal{B}_2 &= \prod_{i=2}^{k}\mathcal{B}_i\\ &= \frac{k!\,(\alpha_1\,\alpha_3)^k}{\beta_0\varepsilon_1}\left(\frac{\beta_0}{\alpha_1}\right)_k \times \left(\frac{-\alpha_3+\beta_2-\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}\right)_k \times \left(\frac{-\alpha_3+\beta_2+\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}\right)_k\end{aligned}$$

and finally, we have, for $k \geq 0$, Christoffel-Darboux identities:

$$\sum_{j=0}^{k}\frac{\mathcal{P}_j(x)\mathcal{P}_j(y)}{j!\,(\alpha_1\,\alpha_3)^j\left(\frac{\beta_0}{\alpha_1}\right)_j(\xi_+)_j\,(\xi_-)_j} = \frac{\mathcal{P}_{k+1}(x)\mathcal{P}_k(y)-\mathcal{P}_k(x)\mathcal{P}_{k+1}(y)}{k!\,(\alpha_1\,\alpha_3)^k\left(\frac{\beta_0}{\alpha_1}\right)_k(\xi_+)_k\,(\xi_-)_k\,(x-y)}, \tag{111}$$

where

$$\xi_\pm = \frac{-\alpha_3+\beta_2\pm\sqrt{(\alpha_3-\beta_2)^2-4\alpha_3\varepsilon_1}}{2\alpha_3}$$

and by evaluating the limit of both sides as $y \to x$, its confluent form

$$\sum_{j=0}^{k}\frac{[\mathcal{P}_j(x)]^2}{j!\,(\alpha_1\,\alpha_3)^j\,(\xi_+)_j\,(\xi_-)_j} = \frac{\mathcal{P}'_{k+1}(x)\mathcal{P}_k(x)-\mathcal{P}'_k(x)\mathcal{P}_{k+1}(x)}{k!\,(\alpha_1\,\alpha_3)^k\left(\frac{\beta_0}{\alpha_1}\right)_k(\xi_+)_k\,(\xi_-)_k} \tag{112}$$

follows. Here, the prime refers to the derivative with respect to the variable x. As a direct consequence of the Christoffel-Darboux formula (112), all the zeros of the n-degree polynomial $\mathcal{P}_n(\varepsilon)$ are simple. To prove that they are also real, we note that the recurrence relation (102) can be written in a matrix form as

$$x\begin{pmatrix}\mathcal{P}_0(x)\\ \mathcal{P}_1(x)\\ \mathcal{P}_2(x)\\ \vdots\\ \mathcal{P}_{k-1}(x)\end{pmatrix} = \begin{pmatrix}\mathcal{A}_0 & 1 & 0 & \cdots & 0 & 0\\ \mathcal{B}_1 & \mathcal{A}_1 & 1 & \cdots & 0 & 0\\ 0 & \mathcal{B}_2 & \mathcal{A}_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \mathcal{B}_{k-1} & \mathcal{A}_{k-1}\end{pmatrix}\begin{pmatrix}\mathcal{P}_0(x)\\ \mathcal{P}_1(x)\\ \mathcal{P}_2(x)\\ \vdots\\ \mathcal{P}_{k-1}(x)\end{pmatrix} + \mathcal{P}_k(x)\begin{pmatrix}0\\0\\0\\ \vdots\\1\end{pmatrix} \tag{113}$$

Thus, if x_i is a zero of $\mathcal{P}_k(x)$, it is an eigenvalue of the given tridiagonal matrix. Since, by the hypothesis of (102), $\mathcal{B}_k > 0$ for all $k \geq 1$, the results of Arscott [24] confirm that (i) the zeros of $\mathcal{P}_{k-1}(x)$ and $\mathcal{P}_k(x)$ interlace – that is, between two consecutive zeros of either polynomial lies precisely one zero of the other (ii) at the zeros of $\mathcal{P}_k(x)$ the values of $\mathcal{P}_{k-1}(x)$ are alternately positive and negative, (iii) all the zeros of $\mathcal{P}_k(x)$ – i.e. all the eigenvalues of tridiagonal matrix are real and different.

6. Mathematical properties of the finite orthogonal polynomials $\{\mathcal{P}_k^n(\varepsilon_0)\}_{k=0}^n$

In this section, we shall study some of the mathematical properties of the orthogonal polynomials $\{\mathcal{P}_k^n(\varepsilon_{0;n})\}_{k=0}^n$. First, the zeros of the polynomial generated by the aforementioned determinant are all simple. This fact can be confirmed by establishing the Christoffel-Darboux formula. Denote $x = \varepsilon_{0;k}$ and $y = \varepsilon_{0;k'}$, where $k \neq k'$ and $k, k' = 0, 1, 2, \cdots, n-1$: For $x \neq y$

$$\sum_{j=0}^{k}\frac{\mathcal{P}_j^n(x)\mathcal{P}_j^n(y)}{j!(\alpha_1\alpha_3)^j(-n)_j\left(\frac{\beta_0}{\alpha_1}\right)_j\left(\frac{\beta_2}{\alpha_3}+n-1\right)_j} = \frac{\mathcal{P}_{k+1}^n(x)\mathcal{P}_k^n(y)-\mathcal{P}_k^n(x)\mathcal{P}_{k+1}^n(y)}{k!(\alpha_1\alpha_3)^k(-n)_k\left(\frac{\beta_0}{\alpha_1}\right)_k\left(\frac{\beta_2}{\alpha_3}+n-1\right)_k(x-y)}, \tag{114}$$

while, for the limit $y \to x$,

$$\sum_{j=0}^{k}\frac{\left(\mathcal{P}_j^n(x)\right)^2}{j!(\alpha_1\alpha_3)^j(-n)_j\left(\frac{\beta_0}{\alpha_1}\right)_j\left(\frac{\beta_2}{\alpha_3}+n-1\right)_j} = \frac{[\mathcal{P}_{k+1}^n(x)]'\mathcal{P}_k^n(x)-[\mathcal{P}_k^n(x)]'\mathcal{P}_{k+1}^n(x)}{k!(\alpha_1\alpha_3)^k(-n)_k\left(\frac{\beta_0}{\alpha_1}\right)_k\left(\frac{\beta_2}{\alpha_3}+n-1\right)_k}. \tag{115}$$

Here, the prime refers to the derivative with respect to the variable x. If $x = x_k$ is a zero of the polynomial $\mathcal{P}_k^n(x)$ with multiplicity > 1, then $\mathcal{P}_k^n(x_k) = 0$ and (115) yields the contradiction

$$0 < \sum_{j=0}^{k-1} \frac{\left(\mathcal{P}_j^n(x_i)\right)^2}{j!(\alpha_1\alpha_3)^j(-n)_j \left(\frac{\beta_0}{\alpha_1}\right)_j \left(\frac{\beta_2}{\alpha_3}+n-1\right)_j} = 0, \tag{116}$$

and the zeros of the polynomial $P_k^n(x)$, $k = 1, 2, \cdots n$ are distinct.

6.1. Norms of the orthogonal polynomials

Denote $\varepsilon_{0;n} = x$, the general theory of orthogonal polynomials [27] guarantees that the finite sequence of polynomials $\{\mathcal{P}_k(x)\}_{k=0}^n$ form a set of orthogonal polynomials for each n. This implies the existence of a certain weight function, $\mathcal{W}(x)$, which can be normalized as

$$\int d\mathcal{W} = 1\,, \tag{117}$$

for which

$$\int \mathcal{P}_k(x)\mathcal{P}_{k'}(x)d\mathcal{W} = p_k\, p_{k'}\, \delta_{kk'}\,, \qquad 0 \le k,\, k' \le n\,, \tag{118}$$

where p_k denotes the norms of polynomials $\mathcal{P}_k(x)$. These norms can be found from the recurrence relations (36) by multiplying with $x^{k-1}\mathcal{W}(x)$ and taking the integral over x yields the recurrence formula

$$\int x^k\, \mathcal{P}_k^n(x)\, \mathcal{W}(x)\, dx = -k(n-k+1)\big((k-1)\alpha_1+\beta_0\big) \times \big(\beta_2+\alpha_3(k+n-2)\big) \int x^{k-1}\mathcal{P}_{k-1}^n(x)\mathcal{W}(x)dx\,, \tag{119}$$

and thus

$$\int \mathcal{P}_k^n(x)\, x^k\, \mathcal{W}(x)\, dx = k!\, (\alpha_1\alpha_3)^k\, (-n)_k \left(\frac{\beta_0}{\alpha_1}\right)_k \left(\frac{\beta_2}{\alpha_3}+n-1\right)_k. \tag{120}$$

From which it follows

$$\int [\mathcal{P}_k^n(x)]^2\mathcal{W}(x)dx = p_k^2 = k!\, (\alpha_1\alpha_3)^k\, (-n)_k \left(\frac{\beta_0}{\alpha_1}\right)_k \left(\frac{\beta_2}{\alpha_3}+n-1\right)_k \tag{121}$$

for all $0 \le k \le n$.

Because of the Pochhammer identity $(-n)_k = 0$ for $k > n$, it follows from (71) that the norms of all polynomials $\mathcal{P}_k^n(x)$ with $k \ge n+1$ vanish. Thus

$$p_k = 0\,, \qquad k \ge n+1\,. \tag{122}$$

We may also note, using the recurrence relation, that

$$\int x[\mathcal{P}_k(x)]^2\mathcal{W}(x)dx = -k((k-1)\alpha_2+\beta_1)\; k!\, (\alpha_1\alpha_3)^k\, (-n)_k \left(\frac{\beta_0}{\alpha_1}\right)_k \left(\frac{\beta_2}{\alpha_3}+n-1\right)_k. \tag{123}$$

6.2. The zeros of the polynomials $\{\mathcal{P}_k^n(\varepsilon_{0;n})\}_{k=0}^n$

One of the important properties of the polynomials $\mathcal{P}_{n+1}^n(\varepsilon_{0;n})$ concerns their zeros. An argument provided by Arscott [24] proves that if the product $(\gamma_k \cdot T_k) > 0$ for all $k = 1, 2, \ldots, n$, then the polynomials that satisfy the tri-diagonal determinant (67) are real and simple. Let us denote that the roots of the polynomials $\mathcal{P}_{n+1}^n(\varepsilon_{0;n}) = 0$ by $\varepsilon_{0;n}^\ell$, $\ell = 0, 1, \ldots, n$ such that

$$\mathcal{P}_{n+1}^n(\varepsilon_{0;n}^\ell) = 0\,, \tag{124}$$

where

$$\varepsilon_{0;n}^0 < \varepsilon_{0;n}^1 < \cdots < \varepsilon_{0;n}^n\,.$$

In particular, since $\mathcal{P}^n_{n+1}(\varepsilon_{0;n})$ is of degree $n+1$ and all the roots are simple and different, it follows that

$$\mathcal{P}^n_{n+1}(\varepsilon_{0;n}) = \prod_{\ell=0}^{n}(\varepsilon_{0;n} - \varepsilon^{\ell}_{0;n})\,. \tag{125}$$

The 'discrete' weight function $\mathcal{W}$ can be computed numerically [28] using (118), (119) and (125) for the given n. Denote $\mathcal{P}_\ell(\varepsilon_0) = \mathcal{P}^n_\ell(\varepsilon_0)$, and let the roots of $\mathcal{P}^n_{n+1}(\varepsilon_{0;n}) = 0$ be $\varepsilon^j_{0;n}$ arranged in ascending order for $j = 0, 1, 2, \cdots, n$. The weights $\mathcal{W}_j, j = 0, 1, \cdots, n$, for the orthogonal polynomials $\{\mathcal{P}^n_\ell(\varepsilon_{0;n})\}^n_{k=0}$ can be computed by solving the linear system

$$\sum_{j=0}^{n} \mathcal{W}_j\, \mathcal{P}^n_\ell(\varepsilon^j_{0;n}) = 0 \tag{126}$$

for $\ell = 0, 1, \cdots, n$.

6.3. Factorization property

Another interesting property of the polynomials $\{\mathcal{P}^n_k(x)\}^n_{k=0}$, aside from being an orthogonal sequence, is that when the parameter n takes positive integer values, the polynomials exhibit a factorization property. Clearly, the factorization occurs because the third term in the recursion relation (36) vanishes when $k = n+1$, so that all subsequent polynomials have a common factor $\mathcal{P}^n_{n+1}(\zeta)$ called a *critical polynomial*. Indeed, all the polynomials $\mathcal{P}^n_{k+n+1}(x)$, beyond the critical polynomial $\mathcal{P}^n_{n+1}(x)$ are factored into the product

$$\mathcal{P}^n_{k+n+1}(x) = \mathcal{Q}^n_k(x)\,\mathcal{P}^n_{n+1}(x),\ \ k = 0,\,1,\,\ldots\,, \tag{127}$$

where the sequence $\{\mathcal{Q}^n_k(x)\}$ are polynomials of degree $k = 0, 1, \ldots$. Interestingly, the quotient polynomials $\{\mathcal{Q}^n_k(x)\}^\infty_{k=0}$ form an infinite sequence of orthogonal polynomials. To prove this claim, we substitute (128) into (36) and re-index the polynomials to eliminate the common factor $\mathcal{P}^n_{n+1}(\zeta)$ from both sides. The recurrence relation (36) then reduces to a three-term recurrence relation for the polynomials $\{\mathcal{Q}^n_k(\zeta)\}_{k\geq 0}$ that reads

$$\begin{aligned}\mathcal{Q}^n_k(x) &= \big((k+n)(k+n-1)\alpha_2 + (k+n)\beta_1 + x\big)\mathcal{Q}^n_{k-1}(x) - (k+n)(k-1)\big((k+n-1)\alpha_1 + \beta_0\big)\\ &\quad\times\big(\beta_2 + \alpha_3(k+2n-2)\big)\mathcal{Q}^n_{k-2}(x)\,,\end{aligned} \tag{128}$$

where $\mathcal{Q}^n_{-1}(\zeta) = 0$, and $\mathcal{Q}^n_0(\zeta) = 1$. Hence, the quotient polynomials $\mathcal{Q}^n_k(\zeta)$ also form a new sequence of orthogonal polynomials for each value of n. For example, if $n = 2$, the critical polynomial is

$$\begin{aligned}\mathcal{P}^2_3(x) &= x^3 + (2\alpha_2 + 3\beta_1)x^2 + 2\big((3\alpha_3 + 2\beta_2)\beta_0 + \beta_1(\alpha_2 + \beta_1) + \alpha_1(2\alpha_3 + \beta_2)\big)x\\ &\quad + 4\beta_0(\alpha_2 + \beta_1)(\alpha_3 + \beta_2)\,.\end{aligned} \tag{129}$$

and

$$\begin{aligned}\mathcal{P}^2_4(x) &= \big(x + 6\alpha_2 + 3\beta_1\big)\,\mathcal{P}^2_3(x)\,,\\ \mathcal{P}^2_5(x) &= \big(x^2 + (18\alpha_2 + 7\beta_1)x - 4\big((3\alpha_1 + \beta_0)(4\alpha_3 + \beta_2) - 3(2\alpha_2 + \beta_1)(3\alpha_2 + \beta_1)\big)\big)\,\mathcal{P}^2_3(x)\,,\\ \mathcal{P}^2_6(x) &= \Big(x^3 + (38\alpha_2 + 12\beta_1)x^2 + \big(432\alpha_2^2 + 290\alpha_2\beta_1 + 47\beta_1^2 - 2(124\alpha_1\alpha_3 + 33\alpha_3\beta_0 + 26\alpha_1\beta_2 + 7\beta_0\beta_2)\big)\,x\\ &\quad - 10\big(\beta_1(84\alpha_1\alpha_3 + 23\alpha_3\beta_0 - 6\beta_1^2 + 18\alpha_1\beta_2 + 5\beta_0\beta_2) + 2\alpha_2\big(31\alpha_3\beta_0 - 27\beta_1^2 + 7\beta_0\beta_2 + 12\alpha_1(9\alpha_3 + 2\beta_2)\big)\\ &\quad - 144\alpha_2^3 - 156\alpha_2^2\beta_1\big)\Big)\mathcal{P}^2_3(x)\,,\\ &\ \ \vdots\end{aligned}$$

from which we have

$$\begin{aligned}\mathcal{Q}_0(x) &= 1\,,\\ \mathcal{Q}_1(x) &= x + 6\alpha_2 + 3\beta_1\,,\\ \mathcal{Q}_2(x) &= x^2 + (18\alpha_2 + 7\beta_1)x - 4\big((3\alpha_1 + \beta_0)(4\alpha_3 + \beta_2) - 3(2\alpha_2 + \beta_1)(3\alpha_2 + \beta_1)\big)\,,\\ \mathcal{Q}_3(x) &= x^3 + (38\alpha_2 + 12\beta_1)x^2 + \big(432\alpha_2^2 + 290\alpha_2\beta_1 + 47\beta_1^2 - 2(124\alpha_1\alpha_3 + 33\alpha_3\beta_0 + 26\alpha_1\beta_2 + 7\beta_0\beta_2)\big)\,x\\ &\quad - 10\big(\beta_1(84\alpha_1\alpha_3 + 23\alpha_3\beta_0 - 6\beta_1^2 + 18\alpha_1\beta_2 + 5\beta_0\beta_2) + 2\alpha_2\big(31\alpha_3\beta_0 - 27\beta_1^2 + 7\beta_0\beta_2 + 12\alpha_1(9\alpha_3 + 2\beta_2)\big)\\ &\quad - 144\alpha_2^3 - 156\alpha_2^2\beta_1\big)\,,\\ &\ \ \vdots\end{aligned}$$

and so on. The Christoffel-Darboux formula for this infinite sequence of orthogonal polynomials reads

$$\sum_{j=0}^{k}\frac{\mathcal{Q}_j^n(x)\mathcal{Q}_j^n(y)}{j!\,(\alpha_1\alpha_3)^j(n+2)_j\left(\frac{\beta_0}{\alpha_1}+n+1\right)_j\left(\frac{\beta_2}{\alpha_3}+2n\right)_j}=\frac{\mathcal{Q}_{k+1}^n(x)\mathcal{Q}_k^n(y)-\mathcal{Q}_k^n(x)\mathcal{Q}_{k+1}^n(y)}{k!\,(\alpha_1\alpha_3)^k(n+2)_k\left(\frac{\beta_0}{\alpha_1}+n+1\right)_k\left(\frac{\beta_2}{\alpha_3}+2n\right)_k(x-y)}\,, \tag{130}$$

and as $y\to x$

$$\sum_{j=0}^{k}\frac{\left(\mathcal{Q}_j^n(x)\right)^2}{j!\,(\alpha_1\alpha_3)^j(n+2)_j\left(\frac{\beta_0}{\alpha_1}+n+1\right)_j\left(\frac{\beta_2}{\alpha_3}+2n\right)_j}=\frac{[\mathcal{Q}_{k+1}^n(x)]'\mathcal{Q}_k^n(x)-[\mathcal{Q}_k^n(x)]'\mathcal{Q}_{k+1}^n(x)}{k!\,(\alpha_1\alpha_3)^k(n+2)_k\left(\frac{\beta_0}{\alpha_1}+n+1\right)_k\left(\frac{\beta_2}{\alpha_3}+2n\right)_k}\,. \tag{131}$$

Theorem 6.1. *The norms of all polynomials $\mathcal{Q}_k^n(\xi)$ are given by*

$$\mathcal{G}_k^{\mathcal{Q}}=k!\,(\alpha_1\alpha_3)^k(n+2)_k\left(\frac{\beta_0}{\alpha_1}+n+1\right)_k\left(\frac{\beta_2}{\alpha_3}+2n\right)_k\,. \tag{132}$$

Proof. The proof follows by multiplying the recurrence relation (128) by $x^{k-2}\rho(x)$, with the normalized weight function $\int\rho(x)dx=1$, and integrating over x. This procedure yields a two-term recurrence relation

$$\mathcal{G}_k^{\mathcal{Q}}=k\,(k+n+1)\,((k+n)\alpha_1+\beta_0)\,(\beta_2+\alpha_3(k+2n-1)))\,\mathcal{G}_{k-1}^{\mathcal{Q}}\,,$$

where $\mathcal{G}_k^{\mathcal{Q}}=\int|\mathcal{Q}_k^n(x)|^2\rho(z)dz=\int x^kQ_k^n(x)\rho(x)dx$ with a solution given by (132). We see that, in general, the norm of the polynomials $Q_k^n(x)$ does not vanish. ∎

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References

[1] A. F. Nikiforov, V. B. Uvarov. *Special functions of mathematical physics.* Birkhäuser Verlag, Basel, 1988. https://doi.org/10.1007/978-1-4757-1595-8.

[2] E. J. Routh. On some properties of certain solutions of a differential equation of the second order. *Proceedings of the London Mathematical Society* **16**:245–262, 1884/85. https://doi.org/10.1112/plms/s1-16.1.245.

[3] N. Saad, R. L. Hall, H. Ciftci. Criterion for polynomial solutions to a class of linear differential equations of second order. *Journal of Physics A: Mathematical and General* **39**(43):13445–13454, 2006. https://doi.org/10.1088/0305-4470/39/43/004.

[4] N. Saad, R. L. Hall, V. A. Trenton. Polynomial solutions for a class of second-order linear differential equations. *Applied Mathematics and Computation* **226**:615–634, 2014. https://doi.org/10.1016/j.amc.2013.10.056.

[5] A. Ronveaux (ed.). *Heun's differential equations.* Oxford University Press, New York, 1995.

[6] S. Y. Slavyanov, W. Lay. *Special functions: a unified theory based on singularities.* Oxford University Press, Oxford, 2000. ISBN 0-19-850573-6.

[7] A. Decarreau, M.-C. Dumont-Lepage, P. Maroni, et al. Formes canoniques des équations confluentes de l'équation de Heun. *Annales de la Societé Scientifique de Bruxelles Série I Sciences Mathématiques, Astronomiques et Physiques* **92**(1-2):53–78, 1978.

[8] A. Decarreau, P. Maroni, A. Robert. Sur les équations confluentes de l'équation de Heun. *Annales de la Societé Scientifique de Bruxelles Série I Sciences Mathématiques, Astronomiques et Physiques* **92**(3):151–189, 1978.

[9] K. Heun. Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten. *Mathematische Annalen* **33**(2):161–179, 1888. https://doi.org/10.1007/BF01443849.

[10] F. Beukers, A. van der Waall. Lamé equations with algebraic solutions. *Journal of Differential Equations* **197**(1):1–25, 2004. https://doi.org/10.1016/j.jde.2003.10.017.

[11] A. Turbiner. On polynomial solutions of differential equations. *Journal of Mathematical Physics* **33**(12):3989–3993, 1992. https://doi.org/10.1063/1.529848.

[12] R. V. Craster, V. H. Hoàng. Applications of Fuchsian differential equations to free boundary problems. *Proceedings of the Royal Society A Mathematical, Physical and Engineering Sciences* **454**(1972):1241–1252, 1998. https://doi.org/10.1098/rspa.1998.0204.

[13] P. P. Fiziev. The Heun functions as a modern powerful tool for research in different scientific domains, 2015. arXiv:1512.04025v1.

[14] Mathematical physics. In U. Camcı, I. Semiz (eds.), *Proceedings of the 13th Regional Conference held in Antalya, October 27–31, 2010*, pp. 23–39. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013. https://doi.org/10.1142/8566.

[15] H. Ciftci, R. L. Hall, N. Saad, E. Dogu. Physical applications of second-order linear differential equations that admit polynomial solutions. *Journal of Physics A: Mathematical and Theoretical* **43**(41):415206, 14, 2010. https://doi.org/10.1088/1751-8113/43/41/415206.

[16] Y.-Z. Zhang. Exact polynomial solutions of second order differential equations and their applications. *Journal of Physics A: Mathematical and Theoretical* **45**(6):065206, 2012. https://doi.org/10.1088/1751-8113/45/6/065206.

[17] B.-H. Chen, Y. Wu, Q.-T. Xie. Heun functions and quasi-exactly solvable double-well potentials. *Journal of Physics A: Mathematical and Theoretical* **46**(3):035301, 2013. https://doi.org/10.1088/1751-8113/46/3/035301.

[18] F. Caruso, J. Martins, V. Oguri. Solving a two-electron quantum dot model in terms of polynomial solutions of a Biconfluent Heun equation. *Annals of Physics* **347**:130–140, 2014. https://doi.org/10.1016/j.aop.2014.04.023.

[19] A. V. Turbiner. One-dimensional quasi-exactly solvable Schrödinger equations. *Physics Reports A Review Section of Physics Letters* **642**:1–71, 2016. https://doi.org/10.1016/j.physrep.2016.06.002.

[20] H. Karayer, D. Demirhan, F. Büyükkılıç. Solution of Schrödinger equation for two different potentials using extended Nikiforov-Uvarov method and polynomial solutions of biconfluent Heun equation. *Journal of Mathematical Physics* **59**(5):053501, 2018. https://doi.org/10.1063/1.5022008.

[21] H. Ciftci, R. L. Hall, N. Saad. Asymptotic iteration method for eigenvalue problems. *Journal of Physics A: Mathematical and General* **36**(47):11807–11816, 2003. https://doi.org/10.1088/0305-4470/36/47/008.

[22] H. Scheffé. Linear differential equations with two-term recurrence formulas. *Journal of Mathematics and Physics* **21**(1-4):240–249, 1942. https://doi.org/10.1002/sapm1942211240.

[23] R. S. Irving. *Integers, polynomials, and rings*. Springer-Verlag, New York, 2004. https://doi.org/10.1007/b97633.

[24] F. M. Arscott. Latent roots of tri-diagonal matrices. *Edinburgh Mathematical Notes* **44**:5–7, 1961. https://doi.org/10.1017/S095018430000330X.

[25] J. Favard. Sur les polynômes de Tchebicheff. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* **200**:2052–2053, 1935.

[26] F. Marcellán, R. Álvarez Nodarse. On the "Favard theorem" and its extensions. *Journal of Computational and Applied Mathematics* **127**(1-2):231–254, 2001. https://doi.org/10.1016/S0377-0427(00)00497-0.

[27] M. E. H. Ismail. *Classical and quantum orthogonal polynomials in one variable*. Cambridge University Press, Cambridge, 2009.

[28] A. Krajewska, A. Ushveridze, Z. Walczak. Bender-Dunne orthogonal polynomials general theory. *Modern Physics Letters A* **12**(16):1131–1144, 1997. https://doi.org/10.1142/S0217732397001163.