Atomic effect algebras with compression bases

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Compression base effect algebras were recently introduced by Gudder [6]. They generalize sequential effect algebras [7] and compressible effect algebras [5]. The present paper focuses on atomic compression base effect algebras and the consequences of atoms being foci (so-called projections) of the compressions in the compression base. Part of our work generalizes results obtained in atomic sequential effect algebras by Tkadlec [11]. The notion of projection-atomicity is introduced and studied and several conditions that force a compression base effect algebra or the set of its projections to be Boolean are found. Finally, we apply some of these results to sequential effect algebras and strengthen a previously established result concerning a sufficient condition for them to be Boolean.

I. Introduction

The current framework for discussing the logical foundations of quantum mechanics is the algebraic structure of an effect algebra, which allows the study of measurements or observables that may be unsharp (see, e.g., [2]). Gudder and Greechie [7] discussed the notion of a sequential effect algebra (SEA)—an effect algebra on which a "sequential product" is defined. This sequential product satisfies a set of physically motivated axioms as it formalizes the case of sequentially performed measurements. The authors prove that the existence of a sequential product in an effect algebra is a restrictive condition, far from being met by all effect algebras.

Gudder [5] introduced the notion of a compression on an effect algebra and also of a compressible effect algebra. Although the important examples of effect algebras proves to be compressible, examples are also provided of noncompressible effect algebras.

As it turns out, the two notions (sequential effect algebra and compressible effect algebra) are somehow related, since the sequential product with a sharp element (of a SEA) defines a compression. Although the restrictions imposed by the existence of a sequential product seem stronger than those determined by compressibility, neither of the two notions is a generalization of the other, as an example of a noncompressible SEA shows [5]. However, in a later paper Gudder [6] introduced a common generalization of both SEA and compressible effect algebras, namely effect algebras having a compression base.

Tkadlec [11] proved various conditions for an atomic SEA or its set of sharp elements to be a Boolean algebra. In this paper we generalize some of these conditions to the case of effect algebras having a compression base, and also present some new ones for this more general framework. The role of the set of sharp elements of the SEA will be played by the orthomodular poset of foci (or projections) of the effect algebra's compression base.

In Sec. II, we recall some of the basic facts about effect algebras and their atoms. Section III is devoted to an introduction to compressions and their basic properties, as well as compression bases. As a particular case of compression base effect algebras, we briefly present sequential effect algebras. Sections IV and V contain results concerning mainly atomic compression base effect algebras. Sec. IV we establish some properties of atoms in effect algebras endowed with a compression base, mainly regarding coexistence and centrality. Then, in Sec. V, we introduce the notion of projection-atomicity which aims to be an analogue, in the framework of effect algebras with a compression base, for the property of an effect algebra of having sharp atoms—used in sequential effect algebras. Consequences of projection-atomicity are studied, some of which generalize results obtained in [11]. A few conditions for an atomic compression base effect algebra to be a Boolean algebra are established. Finally, we apply these results to the particular case of sequential effect algebras and find a sufficient condition for them to be Boolean algebras that strengthens previous results by Gudder and Greechie [7] and Tkadlec [11].

II. Basics about effect algebras

Definition 2.1 An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}, \mathbf{1} \in E$, $\oplus$ is a partial binary operation on E such that for a, b, c elements of E the following conditions hold:

(EA1) $a \oplus b = b \oplus a$ if $b \oplus a$ is defined;
(EA2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if $a \oplus (b \oplus c)$ is defined;
(EA3) for every $a \in E$, there is a unique $a' \in E$ such that $a \oplus a' = \mathbf{1}$ (*orthosupplement*);
(EA4) $a = \mathbf{0}$ whenever $a \oplus \mathbf{1}$ is defined (*zero-unit law*).

We usually write E rather than $(E, \oplus, \mathbf{0}, \mathbf{1})$, for simplicity. A partial ordering is defined on an effect algebra E by $a \leq b$ if there is an element $c \in E$ such that $a \oplus c = b$. If the element c exists, it is uniquely determined by $c = (a \oplus b')'$ and it is denoted by $b \ominus a$. For $a, b \in E$ with $a \leq b$ we denote $[a, b] = \{c \in E \colon a \leq c \leq b\}$. An *orthogonality* relation is defined by $a \perp b$ if $a \oplus b$ exists (i.e., $a \leq b'$). It is easy to check that $\mathbf{0}$ and $\mathbf{1}$ are the least and the greatest elements of E, respectively, that $a'' = a$, and that $a \leq b$ implies $b' \leq a'$. Also, $a \oplus \mathbf{0} = a$ for every $a \in E$ and a *cancellation law* holds: $a \oplus b \leq a \oplus c$ implies $b \leq c$ for every $a, b, c \in E$. (See, e.g., Foulis and Bennett [2], Dvurečenskij and Pulmannová [1]).

Let us consider the effect algebras E and E' and the mapping $J \colon E \to E'$. We denote $\mathrm{Ker}(J) = \{a \in E \colon J(a) = \mathbf{0}\}$. We call J *additive* if $a \perp b$ implies $J(a) \perp J(b)$ and $J(a \oplus b) = J(a) \oplus J(b)$. A subset F of the effect algebra E is a *sub-effect algebra* (denoted *sub-EA* in the following) if $\mathbf{0}, \mathbf{1} \in F$ and F is closed under operations $\oplus$ and $'$.

Definition 2.2 An element a of an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is called:

- *sharp* ($a \in E_{\mathrm{s}}$) if $a \wedge a' = \mathbf{0}$;
- *principal* if $b \oplus c \leq a$ whenever $b, c \leq a$ and $b \perp c$;
- *central* if a and a' are principal and for every $b \in E$ there are $b_1, b_2 \in E$ such that $b = b_1 \oplus b_2$ with $b_1 \leq a$ and $b_2 \leq a'$.

It is well known that central elements are principal and principal elements are sharp. The reverse implications need not hold.

Definition 2.3 An *orthoalgebra* is an effect algebra in which every element is sharp. An *orthomodular poset* (*OMP*) is an effect algebra in which every element is principal.

Definition 2.4 Let E be an effect algebra and let us denote by na the sum of n copies of an element $a \in E$, if it exists. We call E *Archimedean* if $\sup\{n \in \mathbb{N} \colon na \text{ is defined}\} < \infty$ for every nonzero element $a \in E$.

Let us remark that every orthoalgebra is Archimedean since no nonzero element is orthogonal to itself.

Definition 2.5 Let E be an effect algebra. A system $(a_i)_{i\in I}$ of elements of E is *orthogonal* if $\bigoplus_{i\in F} a_i$ is defined for every finite set $F \subset I$. A *majorant* of an orthogonal system $(a_i)_{i\in I}$ is an upper bound of $\{\bigoplus_{i\in F} a_i \colon F \subset I \text{ is finite}\}$. The *sum* of an orthogonal system is its least majorant (if it exists).

An effect algebra E is *orthocomplete* if every orthogonal system in E has a sum. An effect algebra E is *weakly orthocomplete* if every orthogonal system in E has a sum or no minimal majorant.

Definition 2.6 Let E be an effect algebra. Elements $a, b \in E$ *coexist* (denoted by $a \leftrightarrow b$) if there are $a_1, b_1, c \in E$ such that $a_1 \oplus b_1 \oplus c$ exists and $a = a_1 \oplus c$, $b = b_1 \oplus c$.

Definition 2.7 Let E be an effect algebra. A minimal non-zero element of E is called an *atom.* E is *atomic* if every non-zero element dominates an atom. E is *atomistic* if every non-zero element is the supremum of the atoms it dominates. E is *determined by atoms* if, for different $a, b \in E$, the sets of atoms dominated by a and b are different.

The relations between these notions are outlined in the following result.

Lemma 2.8 [11, Lemma 2.2] Every atomistic effect algebra is determined by atoms. Every effect algebra determined by atoms is atomic.

The converse implications do not hold [4, 11].

Proposition 2.9 [11, Corollary 2.6] Every atomic effect algebra in which each atom is sharp is an orthoalgebra.

III. Compression bases in effect algebras

In this section we will present a few basic facts about compression bases in effect algebras. For a detailed discussion, examples and more on their properties, the reader is refered to [6, 9].

Definition 3.1 Let E be an effect algebra. An additive map $J \colon E \to E$ is a *retraction* if $a \leq J(\mathbf{1})$ implies $J(a) = a$, $J(\mathbf{1})$ is then called the *focus* of J. A retraction is a *compression* if $J(a) = \mathbf{0}$ implies $a \leq J(\mathbf{1})'$. Retractions J, I on E are *supplementary* if $\mathrm{Ker}(J) = I(E)$ and $\mathrm{Ker}(I) = J(E)$, I is then called a *supplement* of J. An element p of E is called a *projection* if it is the focus of some retraction on E.

Let us remark that a retraction J is additive, hence it is order preserving. Therefore $J(a) = a$ implies $a \leq J(\mathbf{1})$. It is also easy to see that retractions are idempotent, which suggests they are generalizations of projection mappings (except that the latter are not additive).

Definition 3.2 Let E be an effect algebra. A sub-EA F of E is *normal* if, for every $a, b, c \in E$ such that $a \oplus b \oplus c$ exists in E and $a \oplus b, b \oplus c \in F$, it follows that $b \in F$.

Definition 3.3 Let E be an effect algebra. A system $(J_p)_{p\in P}$ of compressions on E indexed by a normal sub-EA P of E is called a *compression base* for E if the following conditions hold:

(1) Each compression J_p has the focus p.
(2) If $p, q, r \in P$ and $p \oplus q \oplus r$ is defined in E, then $J_{p\oplus r} \circ J_{r\oplus q} = J_r$.

Let us remark here the obvious fact that every effect algebra has a trivial compression base $\{J_{\mathbf{0}}, J_{\mathbf{1}}\}$ where $J_{\mathbf{0}}(a) = \mathbf{0}$, $J_{\mathbf{1}}(a) = a$ for every $a \in E$.

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are compression bases for E, then $\mathcal{J}_1 \cap \mathcal{J}_2$ is also a compression base for E. If $\{\mathcal{J}_\alpha\}$ is a chain of compression bases for E then $\bigcup_\alpha \mathcal{J}_\alpha$ is also a compression base for E. As a consequence, according to Zorn's lemma, every effect algebra has a maximal compression base. If J_p and $J_{p'}$ are compressions, they are contained in a maximal compression base.

Let us present a prototypical example of an effect algebra with a compression base. Consider H a Hilbert space and let $\mathcal{E}(H)$ be the set of all operators on H that are self-adjoint, positive and smaller than identity. It is well known (see, e.g., [2]) that $\mathcal{E}(H)$ can be organised as an effect algebra with the partial operation defined by $A \oplus B = A + B$ if $A + B \in \mathcal{E}(H)$, for all $A, B \in \mathcal{E}(H)$. The set of sharp elements of this effect algebra is $\mathcal{P}(H)$, the set of projection operators on H. For every $P \in \mathcal{P}(H)$, let us define $J_P \colon \mathcal{E}(H) \to \mathcal{E}(H)$ by $J_P(A) = PAP$ for every $A \in \mathcal{E}(H)$. Then $(J_P)_{P \in \mathcal{P}(H)}$ is a compression base for $\mathcal{E}(H)$. Clearly, the focus of each compression J_P is P, therefore the set of projections (in the sense of foci of compressions) of $\mathcal{E}(H)$ is just $\mathcal{P}(H)$.

Let us now summarize the properties of compressions that we intend to use in the sequel. They are direct consequences of the definition and of [5, Lemmas 3.1–3.3].

Lemma 3.4 *Let E be an effect algebra, J a compression on E with the focus p and let us denote $p \circ a = J(a)$ for every $a \in E$. Then, for every $a, b \in E$:* (1) *p, p' are principal and hence sharp;* (2) *$p \circ (a \oplus b) = (p \circ a) \oplus (p \circ b)$;* (3) *$p \circ a \le p \circ b$ whenever $a \le b$;* (4) *$p \circ \mathbf{0} = \mathbf{0}, p \circ \mathbf{1} = p$;* (5) *$p \circ a = a$ if $a \le p$;* (6) *$p \circ a \le p$; $p \circ a = p$ if and only if $p \le a$;* (7) *$p \circ a = \mathbf{0}$ if and only if $p \perp a$ ($a \le p'$).*

For an effect algebra E with a compression base $(J_p)_{p \in P}$ we denote:

- $p \circ a = J_p(a)$ for every $p \in P$ and $a \in E$;
- $p \mid q$ if $p, q \in P$ and $p \circ q = q \circ p$ (i.e., $J_p(q) = J_q(p)$);
- $C(p) = \{a \in E \colon a = J_p(a) \oplus J_{p'}(a)\}$ for every $p \in P$.

Lemma 3.5 *[6, Lemma 3.5] Let $(J_p)_{p \in P}$ be a compression base for the effect algebra E. Then P is an orthomodular poset and $J_{p'}$ is a supplement of J_p for every $p \in P$.*

Theorem 3.6 *[6, Theorem 3.6] Let E be an effect algebra with a compression base $(J_p)_{p \in P}$. For every $p, q \in P$, the following statements are equivalent:* (1) *$p \le q$;* (2) *$J_q \circ J_p = J_p$;* (3) *$q \circ p = p$;* (4) *$J_p \circ J_q = J_p$;* (5) *$p \circ q = p$.*

Theorem 3.7 *[6, see Theorem 4.2] Let E be an effect algebra with a compression base $(J_p)_{p \in P}$. For every $p, q \in P$, the following statements are equivalent:* (1) *$p \circ q = q \circ p$;* (2) *p and q coexist;* (3) *$p \in C(q)$.*

Let us now briefly present sequential effect algebras which will be regarded here as a particular case of effect algebras with a compression base. A detailed account regarding sequential effect algebras can be found in [7].

Definition 3.8 A *sequential product* on an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is a binary operation $\circ$ on E such that for every $a, b, c \in E$, the following conditions hold:

(S1) $a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$ if $b \oplus c$ exists;
(S2) $\mathbf{1} \circ a = a$;
(S3) if $a \circ b = \mathbf{0}$ then $a \mid b$ (where $a \mid b$ denotes $a \circ b = b \circ a$);
(S4) if $a \mid b$ then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$;
(S5) if $c \mid a, b$ then $c \mid a \circ b$ and $c \mid (a \oplus b)$ (if $a \oplus b$ exists).

An effect algebra endowed with a sequential product is called a *sequential effect algebra*.

The definition of sequential effect algebras was inspired by the so-called standard Hilbert space effect algebra which is exactly $\mathcal{E}(H)$, previously described. More precisely, on $\mathcal{E}(H)$ a sequential product is defined by $A \circ B = A^{1/2} B A^{1/2}$ for $A, B \in \mathcal{E}(H)$.

Theorem 3.9 *[6, Theorem 3.4] Let E be a sequential effect algebra. For every $p \in E_{\mathrm{s}}$, the mapping $J_p \colon E \to E$ defined by $J_p(a) = p \circ a$ is a compression with the focus p. The system $(J_p)_{p \in E_{\mathrm{s}}}$ is a maximal compression base for E.*

In view of the above theorem, it should be clear that the notation $p \circ a = J_p(a)$ (as well as $p \mid q$ for $p \circ q = q \circ p$) introduced for general effect algebras with a compression base is inspired by the particular case of sequential effect algebras. However, in the general case of an effect algebra E with a compression base $(J_p)_{p \in P}$, the (partial) operation $\circ \colon P \times E \to E$ defined by $p \circ a = J_p(a)$ need not be the restriction of a sequential product on E (see [5]).

IV. Atoms and centrality

Proposition 4.1 Let E be an effect algebra. If p is an atom in E that is the focus of a compression and $a \in E$ then $p \leq a$ or $p \leq a'$.

Proof: Since $p \circ a \leq p$ and p is an atom, either $p \circ a = \mathbf{0}$ or $p \circ a = p$. If $p \circ a = \mathbf{0}$ then, according to Lemma 3.4, $p \perp a$, hence $p \leq a'$. If $p \circ a = p$, then $p \circ a = p = p \circ \mathbf{1} = p \circ (a \oplus a') = (p \circ a) \oplus (p \circ a')$. Applying the cancellation law, $p \circ a' = \mathbf{0}$, hence, according to Lemma 3.4, $p \perp a'$ and therefore $p \leq a$.

Corollary 4.2 *Distinct atoms that are foci of compressions in an effect algebra are orthogonal.*

Corollary 4.3 *Let E be an effect algebra with a compression base $(J_p)_{p \in P}$. If $p, q \in P$ and p is an atom in E then $p \mid q$.*

Proof: According to Proposition 4.1, $p \leq q$ or $p \leq q'$. In the first case, according to Theorem 3.6, $p \circ q = p = q \circ p$, hence $p \mid q$. If $p \leq q'$, then $p \perp q$ and, according to Lemma 3.4, $p \circ q = \mathbf{0} = q \circ p$, hence $p \mid q$.

Proposition 4.4 Let E be an effect algebra with a compression base $(J_p)_{p \in P}$ and $p, q, r \in P$ such that $p \leq q \circ r$ and $p \mid r$. Then $p \leq r \circ q$.

Proof: According to Lemma 3.4, $p \leq q$. According to Lemma 3.4, Theorem 3.6, the assumption and Lemma 3.4 again, $p = p \circ (q \circ r) = J_p\big(J_q(r)\big) = J_p(r) = p \circ r = r \circ p \leq r \circ q$.

Proposition 4.5 Let E be an effect algebra with a compression base $(J_p)_{p \in P}$ and $p \in P$ be an atom in E. For every $q, r \in P$, $p \leq q \circ r$ if and only if $p \leq r \circ q$.

Proof: This is a straightforward consequence of Corollary 4.3 and Proposition 4.4.

Theorem 4.6 *Let E be an effect algebra with a compression base $(J_p)_{p \in P}$. If E is determined by atoms and every atom is in P then P is a Boolean algebra.*

Proof: Let $q, r \in P$. According to Proposition 4.5, $q \circ r$ and $r \circ q$ dominate the same set of atoms (since all atoms are in P). Since E is determined by atoms, this means $q \circ r = r \circ q$ and hence, according to Theorem 3.7, q, r coexist. According to Lemma 3.5, P is an OMP. Hence, P is an OMP with every pair of its elements coexistent and therefore a Boolean algebra (see, e.g., [8, Theorem 1.3.13]).

Let us remark that the conclusion of the above theorem cannot be improved to the statement that E is a Boolean algebra. The effect algebra in Example 5.12 satisfies the hypotheses (it is even atomistic), however it is not a Boolean algebra.

Lemma 4.7 *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. If $p \in P$ is an atom in E then $C(p) = E$.*

Proof: Let $a \in E$. According to Proposition 4.1, $p \le a$ or $p \le a'$.

If $p \le a$ (and therefore $a' \le p'$), then $J_p(a) = p$ and $J_{p'}(a') = a'$ and therefore $J_p(a) \oplus J_{p'}(a) = p \oplus J_{p'}(\mathbf{1} \ominus a') = p \oplus (p' \ominus a') = p \oplus (a \ominus p) = a$.

If $p \le a'$ (and thus $a \le p'$), then $J_{p'}(a) = a$, $J_p(a) = \mathbf{0}$ and thus $J_p(a) \oplus J_{p'}(a) = a$.

Remark 4.8 Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. The previous result implies that every atom $p \in P$ in E coexists with every element of E . Indeed, for every $a \in E = C(p)$, $a = J_p(a) \oplus J_{p'}(a)$. Since $J_p(a) \le p$, there is a $p_1 \in E$ such that $p = J_p(a) \oplus p_1$. Taking into account that $J_{p'}(a) \le p'$, it follows that the sum $J_{p'}(a) \oplus p = J_{p'}(a) \oplus J_p(a) \oplus p_1$ exists and therefore a and p coexist.

The following result that will be useful in the sequel can be deduced from [6, Lemma 4.1]. However, we will present a different proof for it.

Lemma 4.9 *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. An element $p \in P$ is a central element of E if and only if $C(p) = E$.*

Proof: If $C(p) = E$, then $a = J_p(a) \oplus J_{p'}(a)$, for every $a \in E$. According to Lemma 3.4, p, p' are principal, $J_p(a) \le p$ and $J_{p'}(a) \le p'$, hence p is a central element of E.

Conversely, let us suppose that p is a central element of E. For every $a \in E$ there are $a_1 \le p$, $a_2 \le p'$ such that $a = a_1 \oplus a_2$. Hence $J_p(a) = J_p(a_1 \oplus a_2) = J_p(a_1) \oplus J_p(a_2) = a_1 \oplus \mathbf{0} = a_1$ and similarly $J_{p'}(a) = a_2$. Thus $a = a_1 \oplus a_2 = J_p(a) \oplus J_{p'}(a)$ and it follows that $a \in C(p)$.

Remark 4.10 In particular, the above lemma holds if E is a sequential effect algebra endowed with the compression base $(J_p)_{p\in E_s}$, $J_p(a) = p \circ a$.

Theorem 4.11 *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. Every $p \in P$ that is an atom in E is central in E.*

Proof: Let $p \in P$ be an atom in E. According to Lemma 4.7, $C(p) = E$ and, according to Lemma 4.9, p is central in E.

V. Projection-atomic effect algebras

The following property is intended as a substitute, in the framework of atomic effect algebras having a compression base, for the property, in an effect algebra, of having all the atoms sharp.

Definition 5.1 An effect algebra E is *projection-atomic* if it is atomic and there is a compression base $(J_p)_{p\in P}$ of E such that P contains all atoms in E.

In view of the above definition, the result of Theorem 4.11 implies that atoms of a projection-atomic effect algebra are central. The converse also holds, as will be shown in the next remark.

Remark 5.2 Pulmannová [9, Example 3.4] proved that for every effect algebra E the center $\widetilde{C}(E)$ is a normal sub-EA and $(J_p)_{p\in\widetilde{C}(E)}$ with $J_p(a) = p \wedge a$ is a compression base. Hence, every atomic effect algebra with all atoms central is projection-atomic.

Proposition 5.3 *Every projection-atomic effect algebra is an orthoalgebra.*

Proof: Let E be a projection-atomic effect algebra. Then E is atomic and, according to Theorem 4.11, all its atoms are central, hence sharp. According to Proposition 2.9, E is an orthoalgebra.

The following properties of an effect algebra E will be useful in the sequel:

Definition 5.4 A subset M of an effect algebra E is *downward directed* if for every $a, b \in M$ there is an element $c \in M$ such that $c \leq a, b$.

An effect algebra E has the *maximality property* if $[\mathbf{0}, a] \cap [\mathbf{0}, b]$ has a maximal element for every $a, b \in E$.

An effect algebra E is *weakly distributive* if $a \wedge b = a \wedge b' = \mathbf{0}$ implies $a = \mathbf{0}$ for every $a, b \in E$.

Remark 5.5 The maximality property generalizes several important properties of effect algebras, e.g., every chain-finite, orthocomplete or lattice effect algebra has the maximality property. For details and more properties generalized by the maximality property see [12, Theorem 4.1] and [13, Theorem 3.1].

Theorem 5.6 *[10, Theorem 4.2] Every weakly distributive orthomodular poset with the maximality property is a Boolean algebra.*

Lemma 5.7 *Every projection-atomic effect algebra is weakly distributive.*

Proof: Let E be a projection-atomic effect algebra and $(J_p)_{p\in P}$ a compression base of E such that P contains all atoms in E. Suppose that E is not weakly distributive. Then there are $a, b \in E$ such that $a \neq \mathbf{0}$, $a \wedge b = \mathbf{0}$ and $a \wedge b' = \mathbf{0}$. Since E is projection-atomic, there is an atom $p \in P$ in E such that $p \leq a$. Then $p \nleq b$ and $p \nleq b'$, which contradicts to Proposition 4.1.

Lemma 5.8 *The set of upper bounds of a set of atoms in a projection-atomic effect algebra with the maximality property is downward directed.*

Proof: Let E be a projection-atomic effect algebra with a compression base $(J_p)_{p\in P}$ such that P contains the set of atoms of E, $A \subset P$ be a set of atoms, a, b be upper bounds of A. According to the maximality property, there is a maximal $c \leq a, b$. Let us suppose that c is not an upper bound of A and seek a contradiction. Then there is an atom $d \in A$ such that $d \nleq c$, hence, according to Proposition 4.1, $d \leq c'$ and therefore $d' \geq c$. Since $d \leq a, b$ and therefore $d' \geq a', b'$, $c \perp a', b'$ and d' is central (Theorem 4.11) and therefore principal, we obtain $d' \geq c \oplus a'$ and $d' \geq c \oplus b'$. Hence $d \leq (c \oplus a')' = a \ominus c$ and $d \leq (c \oplus b')' = b \ominus c$ and therefore $c \oplus d \leq a, b$—which contradicts the maximality of c.

Lemma 5.9 *Every element in a projection-atomic effect algebra is a minimal upper bound of the set of atoms it dominates. Every projection-atomic effect algebra with the maximality property is atomistic.*

Proof: Let E be a projection-atomic effect algebra, $a \in E$ and A_a be the set of atoms dominated by a. First, let us show that a is a minimal upper bound of A_a. Let us suppose that there is an upper bound $b < a$ of A_a and seek a contradiction. Then $a \ominus b \neq \mathbf{0}$ and since E is atomic, there is an atom $p \in A_a$ such that $p \leq a \ominus b$ and therefore $p \leq b'$. Since $p \leq b$ and E is an orthoalgebra (Proposition 5.3), we obtain $p \leq b \wedge b' = \mathbf{0}$—a contradiction.

If E has the maximality property then, according to Lemma 5.8, the set of upper bounds of A_a is downward directed, hence $a = \bigvee A_a$.

Lemma 5.10 *Every projection-atomic effect algebra with the maximality property is an orthomodular poset.*

Proof: Let E be a projection-atomic effect algebra with the maximality property, $a, b \in E$ with $a \perp b$ and A_a, A_b be the sets of atoms dominated by a and b respectively. According to Lemma 5.9, E is atomistic and therefore the set of upper bounds of $\{a, b\}$ is the set of upper bounds of $A_a \cup A_b$. According to Proposition 5.3, E is an orthoalgebra and therefore $a \oplus b$ is a minimal upper bound of $\{a, b\}$ ([2, Theorem 5.1]). According to Lemma 5.8, the set of upper bounds of $A_a \cup A_b$ is downward directed, hence $a \oplus b$ is the least upper bound of $\{a, b\}$. Hence $a \oplus b = a \vee b$ for orthogonal $a, b \in E$ and therefore E is an orthomodular poset (see [3, Theorem 2.12]).

Theorem 5.11 *Every projection-atomic effect algebra with the maximality property is a Boolean algebra.*

Proof: It follows from Lemma 5.7, Lemma 5.10 and Theorem 5.6.

We can replace the maximality property in Theorem 5.11 by various stronger properties (see Remark 5.5), e.g., by the orthocompleteness. It cannot be replaced by the weak orthocompleteness, as the following example based on Tkadlec [11, 13] shows.

Example 5.12 Let X_1, X_2, X_3, X_4 be infinite and mutually disjoint sets, $X = \bigcup_{i=1}^{4} X_i$,

$$E' = \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\},$$
$$E = \{(A \setminus F) \cup (F \setminus A) \colon A \in E' \text{ and } F \subset X \text{ is finite}\}.$$

For disjoint $A, B \in E$ we define $A \oplus B = A \cup B$. Then $(E, \oplus, \emptyset, X)$ is an orthomodular poset, the orthosupplement is the set theoretic complement in X and the partial ordering is the inclusion. E is atomic and the set of its atoms is $\big\{\{x\} \colon x \in X\big\}$. Let us put

$$P = \{F \subseteq X \colon F \text{ is finite or } X \setminus F \text{ is finite}\}$$

and for every $F \in P$ let us define $J_F \colon E \to E$ by $J_F(A) = F \cap A$ for every $A \in E$.

It is a straightforward verification that $\big(J_F\big)_{F \in P}$ is a compression base for E and that P contains all atoms, hence E is projection-atomic. E is weakly orthocomplete, because if an orthogonal system $(A_i)_{i \in I}$ has a minimal majorant $B \in E$ then $B = \bigcup_{i \in I} A_i$ is the sum of $(A_i)_{i \in I}$. Since all elements of $[\emptyset, X_2]$ are finite, $(X_1 \cup X_2) \wedge (X_2 \cup X_3)$ does not exist and therefore E is not a lattice (and hence not a Boolean algebra).

Let us remark that $P \neq E$—e.g., $X_1 \cup X_2 \in E \setminus P$.

Definition 5.13 A compression base $(J_p)_{p \in P}$ on an effect algebra E has the *projection cover property* [6] if for every element $a \in E$ there exists the least element $b \in P$ (the *projection cover* of a) with $b \geq a$.

Theorem 5.14 *Let E be a projection-atomic effect algebra. If a compression base on E for which all atoms are projections has the projection cover property, then E is a Boolean algebra.*

Proof: Let $(J_p)_{p\in P}$ be a compression base on E that has the projection cover property and such that all atoms are in P. According to [9, Theorem 5.1], P is an orthomodular lattice. Since P is atomic, it is atomistic (see, e.g., [8]). Since all atoms are mutually orthogonal (see Corollary 4.2), every two elements of P are compatible, and hence P is a Boolean algebra.

It remains to prove that $E = P$. Let $a \in E$ and let us denote A_a the set of atoms in E dominated by a and $P_a = \{p \in P\colon\ p \le a\}$. The set of projection upper bounds of a' is $P'_a = \{p' \in P\colon\ p \in P_a\}$ and, due to the projection cover property, there is a projection cover $\bigwedge P'_a \in P$ of a', hence $a \ge \bigvee P_a \in P$. Since a is a minimal upper bound of A_a (Lemma 5.9) and $\bigvee P_a$ is also an upper bound of A_a, it follows that $a = \bigvee P_a \in P$.

Corollary 5.15 *Every atomic sequential orthoalgebra is a Boolean algebra.*

Proof: According to Theorem 3.9, every sequential effect algebra E has a maximal compression base $(J_p)_{p\in E_{\mathrm{s}}}$. If E is an orthoalgebra then $E = E_{\mathrm{s}}$ and therefore every element of E is its own projection cover, hence, according to Theorem 5.14, E is a Boolean algebra.

Let us remark that the above corollary generalizes similar results obtained by Gudder and Greechie [7, Theorem 5.3] and Tkadlec [11, Theorems 5.4 and 5.6]. The first mentioned result assumes that the effect algebra is atomistic, the second assumes it has the maximality property and the third assumes it is determined by atoms.

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